

Computing Node Polynomials for Plane Curves

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Abstract. According to the Göttsche conjecture (now a theorem), the degree $N^{d,\delta}$ of the Severi variety of plane curves of degree d with δ nodes is given by a polynomial in d , provided d is large enough. These “node polynomials” $N_\delta(d)$ were determined by Vainsencher and Kleiman–Piene for $\delta \leq 6$ and $\delta \leq 8$, respectively. Building on ideas of Fomin and Mikhalkin, we develop an explicit algorithm for computing all node polynomials, and use it to compute $N_\delta(d)$ for $\delta \leq 14$. Furthermore, we improve the threshold of polynomiality and verify Göttsche’s conjecture on the optimal threshold up to $\delta \leq 14$. We also determine the first 9 coefficients of $N_\delta(d)$, for general δ , settling and extending a 1994 conjecture of Di Francesco and Itzykson.

Résumé. Selon la Conjecture de Göttsche (maintenant un Théorème), le degré $N^{d,\delta}$ de la variété de Severi des courbes planes de degré d avec δ noeuds est donné par un polynôme en d , pour d assez grand. Ces *polynômes de noeuds* $N_\delta(d)$ ont été déterminés par Vainsencher et Kleiman–Piene pour $\delta \leq 6$ et $\delta \leq 8$, respectivement. S’appuyant sur les idées de Fomin et Mikhalkin, nous développons un algorithme explicite permettant de calculer tous les polynômes de noeuds, et l’utilisons pour calculer $N_\delta(d)$, pour $\delta \leq 14$. De plus, nous améliorons le seuil de polynomialité et vérifions la Conjecture de Göttsche sur le seuil optimal jusqu’à $\delta \leq 14$. Nous déterminons aussi les 9 premiers coefficients de $N_\delta(d)$, pour un δ quelconque, confirmant et étendant la Conjecture de Di Francesco et Itzykson de 1994.

Keywords: Severi degree, curve enumeration, plane curve, node polynomial, labeled floor diagram.

1 Introduction and Main Results

Node Polynomials

Counting algebraic plane curves is a very old problem. In 1848, J. Steiner determined that the number of curves of degree d with 1 node through $\frac{d(d+3)}{2} - 1$ generic points in the complex projective plane \mathbb{P}^2 is $3(d-1)^2$. Much effort has since been put forth towards answering the following question:

*How many (possibly reducible) degree d nodal curves with
 δ nodes pass through $\frac{d(d+3)}{2} - \delta$ generic points in \mathbb{P}^2 ?*

The answer to this question is the *Severi degree* $N^{d,\delta}$, the degree of the corresponding Severi variety. In 1994, P. Di Francesco and C. Itzykson [DFI95] conjectured that $N^{d,\delta}$ is given by a polynomial in d (assuming δ is fixed and d is sufficiently large). It is not hard to see that, if such a polynomial exists, it has to be of degree 2δ .

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Recently, S. Fomin and G. Mikhalkin [FM, Theorem 5.1] established the polynomiality of $N^{d,\delta}$ using tropical geometry and floor decompositions. More precisely, they showed that there exists, for every $\delta \geq 1$, a *node polynomial* $N_\delta(d)$ which satisfies $N^{d,\delta} = N_\delta(d)$ for all $d \geq 2\delta$. (The $\delta = 0$ case is trivial as $N^{d,0} = 1$ for all $d \geq 1$.)

For $\delta = 1, 2, 3$, the polynomiality of the Severi degrees and the formulas for $N_\delta(d)$ were determined in the 19th century. For $\delta = 4, 5, 6$, this was only achieved by I. Vainsencher [Vai95] in 1995. In 2001, S. Kleiman and R. Piene [KP04] settled the cases $\delta = 7, 8$. Earlier, L. Göttsche [Göt98] conjectured a more detailed (still not entirely explicit) description of these polynomials for counting curves on arbitrary projective algebraic surfaces.

Main Results

In this paper we develop, building on ideas of S. Fomin and G. Mikhalkin [FM], an explicit algorithm for computing the node polynomials $N_\delta(d)$ for an arbitrary δ . This algorithm is then used to calculate the node polynomials for all $\delta \leq 14$.

Theorem 1.1 *The node polynomials $N_\delta(d)$, for $\delta \leq 14$, are as listed in [Blo10, Appendix A].*

A list of all $N_\delta(d)$ for $\delta \leq 14$ is implicitly given in Theorem 3.1 of this paper using generating functions. P. Di Francesco and C. Itzykson [DFI95] conjectured the first seven terms of the node polynomial $N_\delta(d)$, for arbitrary δ . We confirm and extend their assertion. The first two terms already appeared in [KP04].

Theorem 1.2 *The first nine coefficients of $N_\delta(d)$ are given by*

$$\begin{aligned}
N_\delta(d) = & \frac{3^\delta}{\delta!} \left[d^{2\delta} - 2\delta d^{2\delta-1} - \frac{\delta(\delta-4)}{3} d^{2\delta-2} + \frac{\delta(\delta-1)(20\delta-13)}{6} d^{2\delta-3} + \right. \\
& - \frac{\delta(\delta-1)(69\delta^2-85\delta+92)}{54} d^{2\delta-4} - \frac{\delta(\delta-1)(\delta-2)(702\delta^2-629\delta-286)}{270} d^{2\delta-5} + \\
& + \frac{\delta(\delta-1)(\delta-2)(6028\delta^3-15476\delta^2+11701\delta+4425)}{3240} d^{2\delta-6} + \\
& + \frac{\delta(\delta-1)(\delta-2)(\delta-3)(13628\delta^3-6089\delta^2-29572\delta-24485)}{11340} d^{2\delta-7} + \\
& \left. - \frac{\delta(\delta-1)(\delta-2)(\delta-3)(282855\delta^4-931146\delta^3+417490\delta^2+425202\delta+1141616)}{204120} d^{2\delta-8} + \dots \right]. \tag{1.1}
\end{aligned}$$

Let $d^*(\delta)$ denote the *polynomiality threshold* for Severi degrees, i.e., the smallest positive integer $d^* = d^*(\delta)$ such that $N_\delta(d) = N^{d,\delta}$ for $d \geq d^*$. As mentioned above S. Fomin and G. Mikhalkin showed that $d^* \leq 2\delta$. We improve this as follows:

Theorem 1.3 *For $\delta \geq 1$, we have $d^*(\delta) \leq \delta$.*

In other words, $N^{d,\delta} = N_\delta(d)$ provided $d \geq \delta \geq 1$. L. Göttsche [Göt98, Conjecture 4.1] conjectured that $d^* \leq \lceil \frac{\delta}{2} \rceil + 1$ for $\delta \geq 1$. This was verified for $\delta \leq 8$ by S. Kleiman and R. Piene [KP04]. By direct computation we can push it further.

Proposition 1.4 *For $3 \leq \delta \leq 14$, we have $d^*(\delta) = \lceil \frac{\delta}{2} \rceil + 1$.*

That is, Göttsche's threshold is correct and sharp for $3 \leq \delta \leq 14$. For $\delta = 1, 2$ it is easy to see that $d^*(1) = 1$ and $d^*(2) = 1$.

P. Di Francesco and C. Itzykson [DFI95] hypothesized that $d^*(\delta) \leq \left\lceil \frac{3}{2} + \sqrt{2\delta + \frac{1}{4}} \right\rceil$ (which is equivalent to $\delta \leq \frac{(d^*-1)(d^*-2)}{2}$). However, our computations show that this fails for $\delta = 13$ as $d^*(13) = 8$.

The main techniques of this paper are combinatorial. By the celebrated Correspondence Theorem of G. Mikhalkin [Mik05, Theorem 1] one can replace the algebraic curve count by an enumeration of certain *tropical curves*. E. Brugallé and G. Mikhalkin [BM07, BM09] introduced some purely combinatorial gadgets, called (*marked*) *labeled floor diagrams* (see Section 2), which, if counted correctly, are equinumerous to these tropical curves. Recently, S. Fomin and G. Mikhalkin [FM] enhanced Brugallé’s and Mikhalkin’s definition and introduced a *template decomposition* of labeled floor diagrams which is crucial in the proofs of all results in this paper, as is the reformulation of algebraic plane curve counts in terms of labeled floor diagrams (see Theorem 2.5).

This paper is organized as follows: In Section 2 we review labeled floor diagrams, their markings, and their relationship with the enumeration of plane algebraic curves. The proofs of Theorems 1.1 and 1.2 are algorithmic in nature and involve a computer computation. We describe both algorithms in detail in Sections 3 and 5, respectively. The first algorithm computes the node polynomials $N_\delta(d)$ for arbitrary δ , the second determines a prescribed number of leading terms of $N_\delta(d)$. The latter algorithm relies on the polynomiality of solutions of certain polynomial difference equations: This polynomiality has been verified for pertinent values of δ (see Section 5). Proposition 1.4 is proved by comparison of the numerical values of $N_\delta(d)$ and $N^{d,\delta}$ for various d and δ (see Appendices A and B of [Blo]). Theorem 1.3 is discussed in Section 4. For complete proofs of all statements see [Blo].

Additional Comments

In principle, once polynomiality of the Severi degrees $N^{d,\delta}$ is established with some threshold, one could use the Caporaso-Harris recursion [CH98] to compute the node polynomials using simple interpolation. This method, together with the threshold proved in Section 4 of this paper, can in principle be used to compute $N_\delta(d)$ for larger values of δ , and also to increase the upper bound in Proposition 1.4.

The *Gromov-Witten invariant* $N_{d,g}$ enumerates irreducible plane curves of degree d and genus g through $3d + g - 1$ generic points in \mathbb{P}^2 . Algorithm 1 (with minor adjustments, cf. Theorem 2.5(2)) can be used to directly compute $N_{d,g}$, without resorting to a recursion involving relative Gromov-Witten invariants à la Caporaso–Harris [CH98].

By extending ideas of S. Fomin and G. Mikhalkin [FM] and of the present paper, we can obtain polynomiality results for *relative Severi degrees*, associated with counting curves satisfying given tangency conditions to a fixed line. This will be discussed in the forthcoming paper [Blo10].

A. Gathmann, H. Markwig and the author [BGM] define Psi-floor diagrams which enumerate plane curves which satisfy point and tangency conditions, and conditions given by *Psi-classes*. We prove a Caporaso-Harris type recursion for Psi-floor diagrams, and show that relative descendant Gromov-Witten invariants equal their tropical counterparts.

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2 Labeled Floor Diagrams

Labeled floor diagrams are combinatorial gadgets which, if counted correctly, enumerate plane curves with certain prescribed properties. E. Brugallé and G. Mikhalkin introduced them in [BM07] (in slightly different notation) and studied them further in [BM09]. To keep this paper self-contained and to fix

notation we review them and their markings following [FM] where the framework that best suits our purposes was introduced.

Definition 2.1 A labeled floor diagram \mathcal{D} on a vertex set $\{1, \dots, d\}$ is a directed graph (possibly with multiple edges) with positive integer edge weights $w(e)$ satisfying:

1. The edge directions respect the order of the vertices, i.e., for each edge $i \rightarrow j$ of \mathcal{D} we have $i < j$.
2. (Divergence Condition) For each vertex j of \mathcal{D} , we have

$$\text{div}(j) \stackrel{\text{def}}{=} \sum_{\substack{\text{edges } e \\ j \xrightarrow{e} k}} w(e) - \sum_{\substack{\text{edges } e \\ i \xrightarrow{e} j}} w(e) \leq 1. \tag{2.1}$$

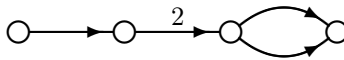
This means that at every vertex of \mathcal{D} the total weight of the outgoing edges is larger by at most 1 than the total weight of the incoming edges.

The *degree* of a labeled floor diagram \mathcal{D} is the number of its vertices. It is *connected* if its underlying graph is. Note that in [FM] labeled floor diagrams are required to be connected. If \mathcal{D} is connected its *genus* is the genus of the underlying graph (or the first Betti number of the underlying topological space). The *cogenus* of a connected labeled floor diagram \mathcal{D} of degree d and genus g is given by $\delta(\mathcal{D}) = \frac{(d-1)(d-2)}{2} - g$. If \mathcal{D} is not connected, let d_1, d_2, \dots and $\delta_1, \delta_2, \dots$ be the degrees and cogenera, respectively, of its connected components. Then the *cogenus* of \mathcal{D} is $\sum_j \delta_j + \sum_{j < j'} d_j d_{j'}$. Via the correspondence between algebraic curves and labeled floor diagrams ([FM, Theorem 3.9]) these notions correspond literally to the respective analogues for algebraic curves. Connectedness corresponds to irreducibility. Lastly, a labeled floor diagram \mathcal{D} has *multiplicity*⁽ⁱ⁾

$$\mu(\mathcal{D}) = \prod_{\text{edges } e} w(e)^2. \tag{2.2}$$

We draw labeled floor diagrams using the convention that vertices in increasing order are arranged left to right. Edge weights of 1 are omitted.

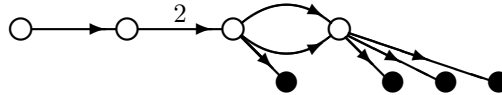
Example 2.2 An example of a labeled floor diagram of degree $d = 4$, genus $g = 1$, cogenus $\delta = 2$, divergences $1, 1, 0, -2$, and multiplicity $\mu = 4$ is drawn below.



To enumerate algebraic curves via labeled floor diagrams we need the notion of markings of such diagrams.

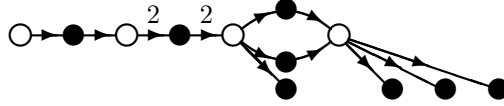
Definition 2.3 A marking of a labeled floor diagram \mathcal{D} is defined by the following three step process which we illustrate in the case of Example 2.2.

Step 1: For each vertex j of \mathcal{D} create $1 - \text{div}(j)$ many new vertices and connect them to j with new edges directed away from j .

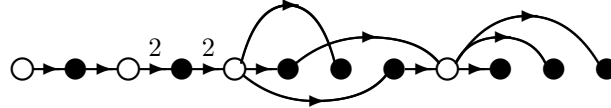


⁽ⁱ⁾ If floor diagrams are viewed as floor contractions of tropical plane curves this corresponds to the notion of multiplicity of tropical plane curves.

Step 2: Subdivide each edge of the original labeled floor diagram \mathcal{D} into two directed edges by introducing a new vertex for each edge. The new edges inherit their weights and orientations. Call the resulting graph $\tilde{\mathcal{D}}$.



Step 3: Linearly order the vertices of $\tilde{\mathcal{D}}$ extending the order of the vertices of the original labeled floor diagram \mathcal{D} such that, as before, each edge is directed from a smaller vertex to a larger vertex.



The extended graph $\tilde{\mathcal{D}}$ together with the linear order on its vertices is called a marked floor diagram, or a marking of the original labeled floor diagram \mathcal{D} .

We want to count marked floor diagrams up to equivalence. Two markings $\tilde{\mathcal{D}}_1, \tilde{\mathcal{D}}_2$ of a labeled floor diagram \mathcal{D} are *equivalent* if there exists an automorphism of weighted graphs which preserves the vertices of \mathcal{D} and maps $\tilde{\mathcal{D}}_1$ to $\tilde{\mathcal{D}}_2$. The *number of markings* $\nu(\mathcal{D})$ is the number of marked floor diagrams $\tilde{\mathcal{D}}$ up to equivalence.

Example 2.4 The labeled floor diagram \mathcal{D} of Example 2.2 has $\nu(\mathcal{D}) = 7$ markings (up to equivalence): In step 3 the extra 1-valent vertex connected to the third white vertex from the left can be inserted in three ways between the third and fourth white vertex (up to equivalence) and in four ways right of the fourth white vertex (again up to equivalence).

Now we can make precise how to rephrase the initial question of this paper in terms of combinatorics of labeled floor diagrams.

Theorem 2.5 (Corollary 1.9 of [FM]) The Severi degree $N^{d,\delta}$, i.e., the number of (possibly reducible) nodal curves in \mathbb{P}^2 of degree d with δ nodes through $\frac{d(d+3)}{2} - \delta$ generic points, is equal to

$$N^{d,\delta} = \sum_{\mathcal{D}} \mu(\mathcal{D}) \nu(\mathcal{D}), \tag{2.3}$$

where \mathcal{D} runs over all (possibly disconnected) labeled floor diagrams of degree d and cogenus δ .

3 Computing Node Polynomials

In this section we give an explicit algorithm that symbolically computes the node polynomials $N_\delta(d)$, for given $\delta \geq 1$. (As $N^{d,0} = 1$ for $d \geq 1$, we put $N_0(d) = 1$.) An implementation of this algorithm was used to prove Theorem 1.1 and Proposition 1.4. We mostly follow the notation in [FM, Section 5]. First, we rephrase Theorem 1.1 in more compact notation. For $\delta \leq 8$ one recovers [KP04, Theorem 3.1].

Theorem 3.1 The node polynomials $N_\delta(d)$, for $\delta \leq 14$, are given by the generating function $\sum_{\delta \geq 0} N_\delta(d)x^\delta$ via the transformation

$$\sum_{\delta \geq 0} N_\delta(d)x^\delta = \exp \left(\sum_{\delta \geq 0} Q_\delta(d)x^\delta \right), \tag{3.1}$$

where

$$\begin{aligned}
Q_0(d) &= 1, \\
Q_1(d) &= 3(d-1)^2, \\
Q_2(d) &= \frac{-3}{2}(d-1)(14d-25), \\
Q_3(d) &= \frac{1}{3}(690d^2 - 2364d + 1899), \\
Q_4(d) &= \frac{1}{4}(-12060d^2 + 47835d - 45207), \\
Q_5(d) &= \frac{1}{5}(217728d^2 - 965646d + 1031823), \\
Q_6(d) &= \frac{1}{6}(-4010328d^2 + 19451628d - 22907925), \\
Q_7(d) &= \frac{1}{7}(74884932d^2 - 391230216d + 499072374), \\
Q_8(d) &= \frac{1}{8}(-1412380980d^2 + 7860785643d - 10727554959), \\
Q_9(d) &= \frac{1}{9}(26842726680d^2 - 157836614730d + 228307435911), \\
Q_{10}(d) &= \frac{1}{10}(-513240952752d^2 + 3167809665372d - 4822190211285), \\
Q_{11}(d) &= \frac{1}{11}(9861407170992d^2 - 63560584231524d + 101248067530602), \\
Q_{12}(d) &= \frac{1}{12}(-190244562607008d^2 + 1275088266948600d - 2115732543025293), \\
Q_{13}(d) &= \frac{1}{13}(3682665360521280d^2 - 25576895657724768d + 44039919476860362), \\
Q_{14}(d) &= \frac{1}{14}(-71494333556133600d^2 + 513017995615177680d - 913759995239314452).
\end{aligned}$$

In particular, all $Q_\delta(d)$, for $1 \leq \delta \leq 14$, are quadratic.

L. Göttsche [Göt98] conjectured that all $Q_\delta(d)$ are quadratic. This theorem proves his conjecture for $\delta \leq 14$.

The basic idea of the algorithm (see [FM, Section 5]) is to decompose labeled floor diagrams into smaller building blocks. These gadgets will be crucial in the proofs of all theorems in this paper.

Definition 3.2 A template Γ is a directed graph (with possibly multiple edges) on vertices $\{0, \dots, l\}$, for $l \geq 1$, and edge weights $w(e) \in \mathbb{Z}_{>0}$, satisfying:

1. If $i \rightarrow j$ is an edge then $i < j$.
2. Every edge $i \xrightarrow{e} i+1$ has weight $w(e) \geq 2$. (No “short edges.”)
3. For each vertex j , $1 \leq j \leq l-1$, there is an edge “covering” it, i.e., there exists an edge $i \rightarrow k$ with $i < j < k$.

Every template Γ comes with some numerical data associated with it. Its *length* $l(\Gamma)$ is the number of vertices minus 1. The product of squares of the edge weights is its *multiplicity* $\mu(\Gamma)$. Its *cogenus* $\delta(\Gamma)$ is

$$\delta(\Gamma) = \sum_{i \xrightarrow{e} j} \left[(j-i)w(e) - 1 \right]. \quad (3.2)$$

For $1 \leq j \leq l(\Gamma)$ let $\varkappa_j = \varkappa_j(\Gamma)$ denote the sum of the weights of edges $i \rightarrow k$ with $i < j \leq k$ and define

$$k_{\min}(\Gamma) = \max_{1 \leq j \leq l} (\varkappa_j - j + 1). \quad (3.3)$$

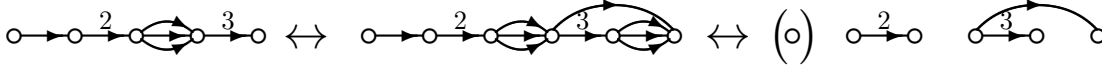
This makes $k_{\min}(\Gamma)$ the smallest positive integer k such that Γ can appear in a floor diagram on $\{1, 2, \dots\}$ with left-most vertex k . Lastly, set

$$\varepsilon(\Gamma) = \begin{cases} 1 & \text{if all edges arriving at } l \text{ have weight 1,} \\ 0 & \text{otherwise.} \end{cases} \quad (3.4)$$

For a list of all templates with $\delta \leq 2$ see [FM, Figure 10].

A labeled floor diagram \mathcal{D} with d vertices decomposes into an ordered collection $(\Gamma_1, \dots, \Gamma_m)$ of templates as follows: First, add an additional vertex $d + 1$ ($> d$) to \mathcal{D} along with, for every vertex j of \mathcal{D} , $1 - \text{div}(j)$ new edges of weight 1 from j to the new vertex $d + 1$. The resulting floor diagram \mathcal{D}' has divergence 1 at every vertex coming from \mathcal{D} . Now remove all *short edges* from \mathcal{D}' , that is, all edges of weight 1 between consecutive vertices. The result is an ordered collection of templates $(\Gamma_1, \dots, \Gamma_m)$, listed left to right, and it is not hard to see that $\sum \delta(\Gamma_i) = \delta(\mathcal{D})$. This process is reversible once we record the smallest vertex k_i of each template Γ_i (see Example 3.3).

Example 3.3 An example of the decomposition of a labeled floor diagram into templates is illustrated below. Here, $k_1 = 2$ and $k_2 = 4$.



To each template Γ we associate a polynomial that records the number of “markings of Γ .” For $k \in \mathbb{Z}_{>0}$ let $\Gamma^{(k)}$ denote the graph obtained from Γ by first adding $k + i - 1 - \varkappa_i$ short edges connecting $i - 1$ to i , for $1 \leq i \leq l(\Gamma)$, and then subdividing each edge of the resulting graph by introducing one new vertex for each edge. By [FM, Lemma 5.6] the number of linear extensions (up to equivalence) of the vertex poset of the graph $\Gamma^{(k)}$ extending the vertex order of Γ is a polynomial in k , if $k \geq k_{\min}(\Gamma)$, which we denote by $P(\Gamma, k)$ (see [FM, Figure 10]). The number of markings of a labeled floor diagram \mathcal{D} decomposing into templates $(\Gamma_1, \dots, \Gamma_m)$ is then

$$\nu(\mathcal{D}) = \prod_{i=1}^m P(\Gamma_i, k_i), \quad (3.5)$$

where k_i is the smallest vertex of Γ_i in \mathcal{D} . The algorithm is based on

Theorem 3.4 ([FM], (5.13)) The Severi degree $N^{d, \delta}$, for $d, \delta \geq 1$, is given by the template decomposition formula

$$\sum_{(\Gamma_1, \dots, \Gamma_m)} \prod_{i=1}^m \mu(\Gamma_i) \sum_{k_m = k_{\min}(\Gamma_m)}^{d - l(\Gamma_m) + \varepsilon(\Gamma_m)} P(\Gamma_m, k_m) \cdots \sum_{k_1 = k_{\min}(\Gamma_1)}^{k_2 - l(\Gamma_1)} P(\Gamma_1, k_1), \quad (3.6)$$

where the first sum is over all ordered collections of templates $(\Gamma_1, \dots, \Gamma_m)$, for all $m \geq 1$, with $\sum_{i=1}^m \delta(\Gamma_i) = \delta$, and the sums indexed by k_i , for $1 \leq i < m$, are over $k_{\min}(\Gamma_i) \leq k_i \leq k_{i+1} - l(\Gamma_i)$.

Expression (3.6) can be evaluated symbolically, using the following two lemmata. The first is Faulhaber’s formula [Knu93] from 1631 for discrete integration of polynomials. The second treats lower limits of iterated discrete integrals and its proof is straightforward. Here B_j denotes the j th Bernoulli number with the convention that $B_1 = +\frac{1}{2}$.

Lemma 3.5 ([Knu93]) Let $f(k) = \sum_{i=0}^d c_i k^i$ be a polynomial in k . Then, for $n \geq 0$,

$$F(n) \stackrel{\text{def}}{=} \sum_{k=0}^n f(k) = \sum_{s=0}^d \frac{c_s}{s+1} \sum_{j=0}^s \binom{s+1}{j} B_j n^{s+1-j}. \quad (3.7)$$

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Data: The cogenus  $\delta$ .
Result: The node polynomial  $N_\delta(d)$ .
begin
  Generate all templates  $\Gamma$  with  $\delta(\Gamma) \leq \delta$ ;
   $N_\delta(d) \leftarrow 0$ ;
  forall the ordered collections of templates  $\tilde{\Gamma} = (\Gamma_1, \dots, \Gamma_m)$  with  $\sum_{i=1}^m \delta(\Gamma_i) = \delta$  do
     $i \leftarrow 1$ ;
     $Q_1 \leftarrow 1$ ;
    while  $i \leq m$  do
       $a_i \leftarrow \max(k_{\min}(\Gamma_i), k_{\min}(\Gamma_{i-1}) + l(\Gamma_{i-1}), \dots, k_{\min}(\Gamma_1) + l(\Gamma_1) + \dots + l(\Gamma_{i-1}))$ ;
    end
    while  $i \leq m - 1$  do
       $Q_{i+1}(k_{i+1}) \leftarrow \sum_{k_i=a_i}^{k_{i+1}-l(\Gamma_i)} P(\Gamma_i, k_i) Q_i(k_i)$ ;
       $i \leftarrow i + 1$ ;
    end
     $Q^{\tilde{\Gamma}}(d) \leftarrow \sum_{k_m=a_m}^{d-l(\Gamma_m)+\varepsilon(\Gamma_m)} P(\Gamma_m, k_m) Q_m(k_m)$ ;
     $Q^{\tilde{\Gamma}}(d) \leftarrow \prod_{i=1}^m \mu(\Gamma_i) \cdot Q^{\tilde{\Gamma}}(d)$ ;
     $N_\delta(d) \leftarrow N_\delta(d) + Q^{\tilde{\Gamma}}(d)$ ;
  end
end

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Algorithm 1: Algorithm to compute node polynomials.

In particular, $\deg(F) = \deg(f) + 1$.

Lemma 3.6 Let $f(k_1)$ and $g(k_2)$ be polynomials in k_1 and k_2 , respectively, and let $a_1, b_1, a_2, b_2 \in \mathbb{Z}_{\geq 0}$. Furthermore, let $F(k_2) = \sum_{k_1=a_1}^{k_2-b_1} f(k_1)$ be a discrete anti-derivative of $f(k_1)$, where $k_2 \geq a_1 + b_1$. Then, for $n \geq \max(a_1 + b_1 + b_2, a_2 + b_2)$,

$$\sum_{k_2=a_2}^{n-b_2} g(k_2) \sum_{k_1=a_1}^{k_2-b_1} f(k_1) = \sum_{k_2=\max(a_1+b_1, a_2)}^{n-b_2} g(k_2) F(k_2). \quad (3.8)$$

Using these results Algorithm 1 can be used to compute node polynomials $N_\delta(d)$ for an arbitrary number of nodes δ . The first step, the template enumeration, is explained in [Blo, Section 3].

Proof of Correctness of Algorithm 1.: The algorithm is a direct implementation of Theorem 3.4. The m -fold discrete integral is evaluated symbolically, one sum at a time, using Faulhaber's formula (Lemma 3.5). The lower limit a_i of the i th sum is given by an iterated application of Lemma 3.6. \square

As Algorithm 1 is stated its termination in reasonable time is hopeless for $\delta \geq 8$ or 9. The novelty of this section, together with an explicit formulation, is how to implement the algorithm efficiently. This is explained in Remark 3.7.

Remark 3.7 The running time of the algorithm can be improved vastly as follows: As the limits of summation in (3.6) only depend on $k_{\min}(\Gamma_i)$, $l(\Gamma_i)$ and $\varepsilon(\Gamma_m)$, we can replace the template polynomials $P(\Gamma_i, k_i)$ by $\sum P(\Gamma_i, k_i)$, where the sum is over all templates Γ_i with prescribed $(k_{\min}, l, \varepsilon)$. After this transformation the first sum in (3.6) is over all combinations of those tuples. This reduces the computation

drastically as, for example, the 167885753 templates of cogenus 14 make up only 343 equivalence classes. Also, in (3.6) we can distribute the template multiplicities $\mu(\Gamma_i)$ and replace $P(\Gamma_i, k_i)$ by $\mu(\Gamma_i)P(\Gamma_i, k_i)$ and thereby eliminate $\prod \mu(\Gamma_i)$. Another speed-up is to compute all discrete integrals of monomials using Lemma 3.5 in advance.

The generation of the templates is the bottleneck of the algorithm. Their number grows rapidly with δ as can be seen from Figure 1. However, their generation can be parallelized easily (see [Blo]).

Algorithm 1 has been implemented in Maple. Computing $N_{14}(d)$ on a machine with two quad-core Intel(R) Xeon(R) CPU L5420 @ 2.50GHz, 6144 KB cache, and 24 GB RAM took about 70 days.

Remark 3.8 We can use Algorithm 1 to compute the values of the Severi degrees $N^{d,\delta}$ for prescribed values of d and δ . After we specify a degree d and a number of nodes δ all sums in our algorithm become finite and can be evaluated numerically. See [Blo, Appendix B] for all values of $N^{d,\delta}$ for $0 \leq \delta \leq 14$ and $1 \leq d \leq 13$.

δ	# of templates	δ	# of templates	δ	# of templates
1	2	6	1711	11	2233572
2	7	7	7135	12	9423100
3	26	8	29913	13	39769731
4	102	9	125775	14	167885753
5	414	10	529755		

Fig. 1: The number of templates with cogenus $\delta \leq 14$.

4 Threshold Values

S. Fomin and G. Mikhalkin [FM, Theorem 5.1] proved polynomiality of Severi degrees $N^{d,\delta}$ in d , for fixed δ , if d is sufficiently large. More precisely, they showed that $N_\delta(d) = N^{d,\delta}$ for $d \geq 2\delta$. Here we show that their threshold can be improved to $d \geq \delta$ (Theorem 1.3).

We need the following elementary observation about robustness of discrete anti-derivatives of polynomials whose continuous counterpart is the well known fact that $\int_{a-1}^{a-1} f(x)dx = 0$.

Lemma 4.1 For a polynomial $f(k)$ and $a \in \mathbb{Z}_{>0}$ let $F(n) = \sum_{k=a}^n f(k)$ be the polynomial in n uniquely determined by large enough values of n . ($F(n)$ is a polynomial by Lemma 3.5.) Then $F(a-1) = 0$. In particular, $\sum_{k=a}^n f(k)$ is a polynomial in n , for $n \geq a-1$.

The lemma is non-trivial as, in general, $F(a-2) \neq 0$.

Proof of Theorem 1.3 (Sketch): This follows from Equation (3.6) and repeated application of Lemma 3.6 and Lemma 4.1 as $d \geq \delta$ simultaneously implies

$$\begin{aligned}
 d &\geq l(\Gamma_m) - \varepsilon(\Gamma_m) + k_{\min}(\Gamma_m) - 1, \\
 d &\geq l(\Gamma_m) - \varepsilon(\Gamma_m) + l(\Gamma_{m-1}) + k_{\min}(\Gamma_{m-1}) - 2, \\
 &\vdots \\
 d &\geq l(\Gamma_m) - \varepsilon(\Gamma_m) + l(\Gamma_{m-1}) + \dots + l(\Gamma_1) + k_{\min}(\Gamma_1) - m,
 \end{aligned}
 \tag{4.1}$$

for all collections of templates $(\Gamma_1, \dots, \Gamma_m)$ with $\sum_{i=1}^m \delta(\Gamma_i) = \delta$. For details see [Blo]. □

5 Coefficients of Node Polynomials

The goal of this section is to present an algorithm for the computation of the coefficients of $N_\delta(d)$, for general δ . The algorithm can be used to prove Theorem 1.2 and thereby confirm and extend a conjecture of P. Di Francesco and C. Itzykson in [DFI95] where they conjectured the 7 terms of $N_\delta(d)$ of largest degree.

Our algorithm should be able to find formulas for arbitrarily many coefficients of $N_\delta(d)$. We prove correctness of our algorithm in this section. The algorithm rests on the polynomiality of solutions of certain polynomial difference equations (see [Blo, (5.7)]).

First, we fix some notation building on terminology of Section 3. By Remark 3.7 we can replace the polynomials $P(\Gamma, k)$ in (3.6) by the product $\mu(\Gamma)P(\Gamma, k)$, thereby removing the product $\prod \mu(\Gamma_i)$ of the template multiplicities. In this section we write $P^*(\Gamma, k)$ for $\mu(\Gamma)P(\Gamma, k)$. For integers $i \geq 0$ and $a \geq 0$ let $M_i(a)$ denote the matrix of the linear map

$$f(k) \mapsto \sum_{\Gamma: \delta(\Gamma)=i} \sum_{k=k_{\min}(\Gamma)}^{n-l(\Gamma)} P^*(\Gamma, k) \cdot f(k), \quad (5.1)$$

where $f(k) = c_0 k^a + c_1 k^{a-1} + \dots$, a polynomial of degree a , is mapped to the polynomial $M_i(a)(f(k)) = d_0 n^{a+i+1} + d_1 n^{a+i} + \dots$ in n . (By Lemma 3.5 and the proof of Lemma 5.1 the image has degree $a+i+1$.) Hence $M_i(a)\mathbf{c} = \mathbf{d}$. Similarly, define $M_i^{\text{end}}(a)$ to be the matrix of the linear map

$$f(k) \mapsto \sum_{\Gamma: \delta(\Gamma)=i} \sum_{k=k_{\min}(\Gamma)}^{n-l(\Gamma)+\varepsilon(\Gamma)} P^*(\Gamma, k) \cdot f(k). \quad (5.2)$$

Later we will consider square sub-matrices of $M_i(a)$ and $M_i^{\text{end}}(a)$ by restriction to the first few rows and columns which will be denoted $M_i(a)$ and $M_i^{\text{end}}(a)$ as well. Note that $M_i(a)$ and $M_i^{\text{end}}(a)$ are lower triangular. The following observation is key to our algorithm.

Lemma 5.1 *The first $a+i$ rows of $M_i(a)$ and $M_i^{\text{end}}(a)$ are independent of the lower limits of summation in (5.1) and (5.2), respectively.*

The basic idea of the algorithm is that templates with higher cogenera do not contribute to higher degree terms of the node polynomial. With this in mind we define, for each finite collection $(\Gamma_1, \dots, \Gamma_m)$ of templates, its *type* $\tau = (\tau_2, \tau_3, \dots)$, where τ_i is the number of templates in $(\Gamma_1, \dots, \Gamma_m)$ with cogenus equal to i , for $i \geq 2$. Note that we do not record the number of templates with cogenus equal to 1.

To collect the contributions of all collections of templates with a given type τ , let $\tau = (\tau_2, \tau_3, \dots)$ and fix $\delta \geq \sum_{j \geq 2} \tau_j$ (so that there exist template collections $(\Gamma_1, \dots, \Gamma_m)$ of type τ with $\sum \delta(\Gamma_j) = \delta$). We define two (column) vectors $C_\tau(\delta)$ and $C_\tau^{\text{end}}(\delta)$ as the coefficient vectors, listed in decreasing order, of the polynomials

$$\sum_{(\Gamma_1, \dots, \Gamma_m)} \sum_{k_m=k_{\min}(\Gamma_m)}^{n-l(\Gamma_m)} P^*(\Gamma_m, k_m) \cdots \sum_{k_1=k_{\min}(\Gamma_1)}^{k_2-l(\Gamma_1)} P^*(\Gamma_1, k_1) \quad (5.3)$$

and

$$\sum_{(\Gamma_1, \dots, \Gamma_m)} \sum_{k_m=k_{\min}(\Gamma_m)}^{n-l(\Gamma_m)+\varepsilon(\Gamma)} P^*(\Gamma_m, k_m) \sum_{k_{m-1}=k_{\min}(\Gamma_{m-1})}^{k_m-l(\Gamma_{m-1})} \cdots \sum_{k_1=k_{\min}(\Gamma_1)}^{k_2-l(\Gamma_1)} P^*(\Gamma_1, k_1) \quad (5.4)$$

Data: A positive integer N .
Result: The coefficient vector C of the first N coefficients of $N_\delta(d)$.

```

begin
  Compute all templates  $\Gamma$  with  $\delta(\Gamma) \leq N$ ;
  forall the types  $\tau$  with  $\text{def}(\tau) < N$  do
    Compute initial values  $C_\tau(\delta_0(\tau))$  using (5.3), with  $\delta_0(\tau)$  as in Proposition 5.3;
    Solve recursion (5.5) for first  $N - \text{def}(\tau)$  coordinates of  $C_\tau(\delta)$ ;
    Set
      
$$C_\tau^{\text{end}}(\delta) \leftarrow \sum_{i:\tau_i \neq 0} M_i^{\text{end}}(2\delta - i - 1 - \text{def}(\tau)) C_{\tau \downarrow i}(\delta - i) \\ + M_1^{\text{end}}(2\delta - 2 - \text{def}(\tau)) C_\tau(\delta - 1);$$

    end
     $C \leftarrow 0$ ;
    forall the types  $\tau$  with  $\text{def}(\tau) < N$  do
      Shift the entries of  $C_\tau^{\text{end}}(\delta)$  down by  $\text{def}(\tau)$ ;
       $C \leftarrow C + \text{shifted } C_\tau^{\text{end}}(\delta)$ ;
    end
end

```

Algorithm 2: Computation of the leading coefficients of the node polynomial.

in the indeterminate n , where the respective first sums are over all ordered collections of templates of type τ .

Before we can state the main recursion we need two more notations. For a type $\tau = (\tau_2, \tau_3, \dots)$ and $i \geq 2$ with $\tau_i > 0$ define a new type $\tau \downarrow_i$ via $(\tau \downarrow_i)_i = \tau_i - 1$ and $(\tau \downarrow_i)_j = \tau_j$ for $j \neq i$. Furthermore, let $\text{def}(\tau) = \sum_{j \geq 2} (j-1)\tau_j$ be the *defect* of τ . The following lemma justifies this terminology. Its proof is elementary and can be found in [Blo].

Lemma 5.2 *The polynomials (5.3) and (5.4) are of degree $2\delta - \text{def}(\tau)$.*

The last lemma makes precise which collections of templates contribute to which coefficients of $N_\delta(d)$. Namely, the first N coefficients of $N_\delta(d)$ of largest degree depend only on collections of templates with types τ such that $\text{def}(\tau) < N$. The following recursion is the heart of the algorithm.

Proposition 5.3 *For every type τ and integer δ large enough, it holds that*

$$C_\tau(\delta) = \sum_{i:\tau_i \neq 0} M_i(2\delta - i - 1 - \text{def}(\tau)) C_{\tau \downarrow i}(\delta - i) \\ + M_1(2\delta - 2 - \text{def}(\tau)) C_\tau(\delta - 1). \quad (5.5)$$

More precisely, if we restrict all matrices M_i to be square of size $N - \text{def}(\tau)$ and all C_τ to be vectors of length $N - \text{def}(\tau)$, then recursion (5.5) holds for

$$\delta \geq \max \left(\left\lceil \frac{N+1}{2} \right\rceil, \sum_{j \geq 2} j\tau_j \right). \quad (5.6)$$

We propose Algorithm 2 for the computation of the coefficients of the node polynomial $N_\delta(d)$. Due to spacial constrains we explain the step which requires a solution of recursion (5.5) in [Blo].

As in Section 3 (Remark 3.7), Algorithm 2 can be improved significantly by summing the template polynomials $P(\Gamma, k)$ for templates Γ with fixed $(k_{\min}(\Gamma), l(\Gamma), \varepsilon(\Gamma))$ in advance. Algorithm 2 has been implemented in Maple. Once the templates are known the bottleneck of the algorithm is the initial value computation. With an improved implementation this should become faster than the template enumeration. Hence we expect Algorithm 2 to be able to compute the first 14 terms of $N_\delta(d)$ in reasonable time.

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