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Pattern avoidance in alternating permutations and tableaux (extended abstract)

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Abstract. We give bijective proofs of pattern-avoidance results for a class of permutations generalizing alternating permutations. The bijections employed include a modified form of the RSK insertion algorithm and recursive bijections based on generating trees. As special cases, we show that the sets $A_{2n}(1234)$ and $A_{2n}(2143)$ are in bijection with standard Young tableaux of shape $\langle 3^n \rangle$.

Alternating permutations may be viewed as the reading words of standard Young tableaux of a certain skew shape. In the last section of the paper, we study pattern avoidance in the reading words of standard Young tableaux of any skew shape. We show bijectively that the number of standard Young tableaux of shape λ/μ whose reading words avoid 213 is a natural μ-analogue of the Catalan numbers. Similar results for the patterns 132, 231 and 312.

Résumé. Nous présentons des preuves bijectives de résultats pour une classe de permutations à motifs exclus qui généralisent les permutations alternantes. Les bijections utilisées reposent sur une modification de l'algorithme d'insertion "RSK" et des bijections récursives basées sur des arbres de génération. Comme cas particuliers, nous montrons que les ensembles $A_{2n}(1234)$ et $A_{2n}(2143)$ sont en bijection avec les tableaux standards de Young de la forme $\langle 3^n \rangle$.

Une permutation alternante peut être considérée comme le mot de lecture de certain skew tableau. Dans la dernière section de l'article, nous étudions l'évitement des motifs dans les mots de lecture de skew tableaux genéraux. Nous montrons bijectivement que le nombre de tableaux standards de forme λ/μ dont les mots de lecture évitent 213 est un μ-analogue naturel des nombres de Catalan. Des résultats analogues sont valables pour les motifs 132, 231 et 312.

Resumen. Presentamos pruebas biyectivas de resultados de "evasión de patrones" para una clase de permutaciones que generalizan permutaciones alternantes. Las biyecciónes utilizadas incluyen una modificación del algoritmo de inserción de RSK y una biyección recursiva basada en árboles generatrices. Mostramos, como casos especiales, que los conjuntos $A_{2n}(1234)$ y $A_{2n}(2143)$ están en biyección con los tableaux de Young estándares de forma $\langle 3^n \rangle$.

Las permutaciones alternantes pueden ser entendidas como palabras de lectura de tableaux de Young estándares de cierta forma sezgada. En la ultima sección del articulo, expandimos nuestro estudio al considerar evasión de patrones en las palabras de lectura de tableaux de Young estándares de cualquier forma sezgada. Mostramos biyectivamente que el número de tableaux de Young estándares de forma λ/μ cuyas palabras de lectura evitan 213 es un μ-anólogo de los números de Catalán y resultados similares para los patrones 132, 231 y 312.

Keywords: Alternating permutations, permutation patterns, RSK, generating trees, Young tableaux

1 Introduction

A classical problem asks for the number of permutations that avoid a certain permutation pattern. This problem has received a great deal of attention (see *e.g.*, [12, 3]) and has led to a number of interesting variations including the enumeration of special classes of pattern-avoiding permutations (*e.g.*, involutions [12] and derangements [9]). One such variation, first studied by Mansour in [8], is the enumeration of *alternating* permutations avoiding a given pattern or collection of patterns. Alternating permutations have the intriguing property [8, 15, 4] that for any pattern of length three, the number of alternating permutations of a given length avoiding that pattern is given by a Catalan number. This property is doubly interesting because it is shared by the class of all permutations. This coincidence suggests that pattern avoidance in alternating permutations and in usual permutations may be closely related and so motivates the study of pattern avoidance in alternating permutations.

In this paper, we extend the study of pattern avoidance in alternating permutations to patterns of length four. In particular, we show that the number of alternating permutations of length $2n$ avoiding either of the patterns 1234 or 2143 is $\frac{2\cdot(3n)!}{n!(n+1)!(n+2)!}$. This is the first enumeration of a set of pattern-avoiding alternating permutations for a single pattern of length four. In the case of 1234, we give a direct bijective proof using a variation of RSK, while in the case of 2143 we give a recursive generating tree bijection.

Most of our bijections work in a more general setting in which we replace alternating permutations with the set $\mathcal{L}_{n,k}$ of reading words of standard Young tableaux of certain nice skew shapes. (These permutations are enumerated with no pattern restriction in [1].) Inspired by the idea of permutations as reading words of tableaux, we give an enumeration of standard skew Young tableaux of *any* fixed shape whose reading words avoid certain patterns. In particular, this provides a uniform argument to enumerate permutations in S_n and permutations in $\mathcal{L}_{n,k}$ that avoid either 132 or 213. That such a bijection should exist is far from obvious, and it raises the possibility that there is substantially more to be said in this area. In the remainder of this introduction, we provide a more detailed summary of results.

Both the set of all permutations of a given length and the set of alternating permutations of a given length can be expressed as the set of reading words of the standard Young tableaux of a particular skew shape (essentially a difference of two staircases). We define a class $\mathcal{L}_{n,k} \subseteq S_{nk}$ of permutations such that $\mathcal{L}_{n,1} = S_n$ is the set of all permutations of length n, $\mathcal{L}_{n,2}$ is the set of alternating permutations of length $2n$, and for each k $\mathcal{L}_{n,k}$ is the set of reading words of the standard Young tableaux of a certain skew shape. In Section 2, we provide definitions of all the most important objects in this paper. In Sections 3 and 4, we use bijective proofs to derive enumerative pattern avoidance results for $\mathcal{L}_{n,k}$. In Section 3 we give a simple bijection between elements of $\mathcal{L}_{n,k}$ with no $(k+1)$-term increasing subsequence and standard Young tableaux of rectangular shape $\langle k^n \rangle$. In Section 4 we exhibit two bijections between the elements of $\mathcal{L}_{n,k}$ with no $(k+2)$-term increasing subsequence and standard Young tableaux of rectangular shape $\langle (k+1)^n \rangle$, one of which is a modified version of the famous RSK bijection and the other of which is a generating tree approach that also yields an enumeration of alternating permutations avoiding 2143.

In Section 5, we broaden our study to arbitrary skew shapes and so initiate the study of pattern avoidance in reading words of skew tableaux of any shape. In Section 5.1, we show bijectively that the number of tableaux of shape λ/μ (under a technical restriction on the possible shapes that sacrifices no generality – see Note 2) whose reading words avoid 213 can be easily computed from the shape. Notably, the resulting value does not depend on λ and is in fact a natural μ-generalization of the Catalan numbers. Replacing 213 with $132, 231$ or 312 leads to similar results.

For a complete version of this extended abstract, see [6] and [5].

2 Definitions

A *permutation* w of length n is a word containing each of the elements of $[n] = \{1, 2, \ldots, n\}$ exactly once. The set of permutations of length n is denoted S_n. Given a word $w = w_1 w_2 \cdots w_n$ and a permutation $p = p_1 \cdots p_k \in S_k$, we say that w *contains the pattern* p if there exists a set of indices $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ such that the subsequence $w_{i_1} w_{i_2} \cdots w_{i_k}$ of w is order-isomorphic to p, *i.e.*, $w_{i_\ell} < w_{i_m}$ if and only if $p_\ell < p_m$. Otherwise, w is said to *avoid* p. Given a pattern p and a set S of permutations, we denote by $S(p)$ the set of elements of S that avoid p. For example, $S_n(123)$ is the set of permutations of length n avoiding the pattern 123, *i.e.*, the set of permutations with no three-term increasing subsequence.

A permutation $w = w_1 w_2 \cdots w_n$ is *alternating* if $w_1 < w_2 > w_3 < w_4 > \ldots$. (Note that in the terminology of [13], these "up-down" permutations are *reverse alternating* while alternating permutations are "down-up" permutations. Luckily, this convention doesn't matter: any pattern result on either set can be translated into a result on the other via *complementation*, *i.e.*, by considering w^c such that $w^c_i = n + 1 - w_i$. Then results for the pattern 123 would be replaced by results for 321 and so on.) We denote by A_n the set of alternating permutations of length n.

For $n, k \geq 1$, let $\mathcal{L}_{n,k}$ be the set of permutations $w = w_{1,1} w_{1,2} \cdots w_{1,k} w_{2,1} \cdots w_{n,k}$ in S_{nk} that satisfy the conditions

L1. $w_{i,j} < w_{i,j+1}$ for all $1 \leq i \leq n$, $1 \leq j \leq k-1$, and

L2. $w_{i,j+1} > w_{i+1,j}$ for all $1 \leq i \leq n-1$, $1 \leq j \leq k-1$.

Note in particular that $\mathcal{L}_{n,1} = S_n$ (we have no restrictions in this case) and $\mathcal{L}_{n,2} = A_{2n}$. For any k and n, $\mathcal{L}_{n,k}(12 \cdots k) = 0$. Thus, for monotone pattern-avoidance in $\mathcal{L}_{n,k}$ we should consider patterns of length $k+1$ or longer. The set $\mathcal{L}_{n,k}$ has been enumerated by Baryshnikov and Romik [1], and the formulas that result are quite simple for small values of k.

Note 1 *If $w = w_{1,1} \cdots w_{n,k} \in S_{nk}$ satisfies L1 and also avoids $12 \cdots (k+1)(k+2)$ then it automatically satisfies L2, since a violation $w_{i,j+1} < w_{i+1,j}$ of L2 leads immediately to a $(k+2)$-term increasing subsequence $w_{i,1} < \ldots < w_{i,j+1} < w_{i+1,j} < \ldots < w_{i+1,k}$. In particular, we can also describe $\mathcal{L}_{n,k}(1 \cdots (k+2))$ (respectively, $\mathcal{L}_{n,k}(1 \cdots (k+1))$) as the set of permutations in $S_{nk}(1 \cdots (k+2))$ (respectively, $S_{nk}(1 \cdots (k+1))$) whose descent set is (or in fact, is contained in) $\{k, 2k, \ldots, (n-1)k\}$.*

A *partition* is a weakly decreasing, finite sequence of nonnegative integers. We consider two partitions that differ only in the number of trailing zeroes to be the same. We write partitions in sequence notation, as $\langle \lambda_1, \lambda_2, \ldots, \lambda_n \rangle$, or to save space, with exponential notation instead of repetition of equal elements. Thus, the partition $\langle 5, 5, 3, 3, 2, 1 \rangle$ may be abbreviated $\langle 5^2, 3^2, 2, 1 \rangle$. If the sum of the entries of λ is equal to m then we write $\lambda \vdash m$.

Given a partition $\lambda = \langle \lambda_1, \lambda_2, \ldots \rangle$, the *Young diagram* of shape λ is the left-justified array of $\lambda_1 + \ldots + \lambda_n$ boxes with λ_1 in the first row, λ_2 in the second row, and so on. We will identify each partition with its Young diagram and speak of them interchangeably. A *skew Young diagram* λ/μ is the diagram that results when we remove the boxes of μ from those of λ, when both are arranged so that their first rows and first columns coincide. If λ/μ is a skew Young diagram with n boxes, a *standard Young tableau* of shape λ/μ is a filling of the boxes of λ/μ with $[n]$ so that each element appears in exactly one box, and entries increase along rows and columns. We identify boxes in a (skew) Young diagram using matrix coordinates, so the box in the first row and second column is numbered $(1, 2)$. We denote by $\mathrm{sh}(T)$ the

				2	3	6
			1	5	9	
		4	11	12		
	8	13	15			
7	10	14				

Fig. 1: A standard skew Young tableau (in English notation, *i.e.*, with the first row on top) whose reading word is the permutation 7 10 14 8 13 15 4 11 12 1 5 9 2 3 6 $\in \mathcal{L}_{5,3}$.

shape of the standard Young tableau T, by $\mathrm{SYT}(\lambda)$ the set of standard Young tableaux of shape λ and by $f^\lambda = |\,\mathrm{SYT}(\lambda)|$ the size of this set.

Given a standard Young tableau T, the *reading word* of T is the permutation that consists of the entries of the last row read from left to right, then the next-to-last row, and so on. For example, the reading words of the tableaux of shape $\langle n, n-1, \ldots, 2, 1\rangle / \langle n-1, n-2, \ldots, 1\rangle$ are all of S_n, and similarly $\mathcal{L}_{n,k}$ is equal to the set of reading words of standard skew Young tableaux of shape $\langle n+k-1, n+k-2, \ldots, k\rangle / \langle n-1, n-2, \ldots, 1\rangle$, as illustrated in Figure 1. The other "usual" reading order, from right to left then top to bottom in English notation, is simply the reverse of our reading order. Consequently, any pattern-avoidance result in our case carries over to the other reading order by taking the *reverse* of all permutations and patterns involved, *i.e.*, by replacing $w = w_1 \ldots w_n$ with $w^r = w_n \cdots w_1$.

We make note of two operations on Young diagrams and tableaux. Given a partition λ, the *conjugate partition* λ' is defined so that the ith row of λ' has the same length as the ith column of λ for all i. Similarly, the conjugate of a skew Young diagram λ/μ is defined by $(\lambda/\mu)' = \lambda'/\mu'$. Given a standard skew Young tableau T of shape λ/μ, the conjugate tableau T' of shape $(\lambda/\mu)'$ is defined to have the entry a in box (i, j) if and only if T has the entry a in box (j, i). Geometrically, all these operations can be described as "reflection through the main diagonal." Given a skew Young diagram λ/μ, rotation by $180°$ gives a new diagram $(\lambda/\mu)^*$. Given a tableaux T with n boxes, we can form T^*, the *rotated-complement* of T, by rotating T by $180°$ and replacing the entry i with $n + 1 - i$ for each i. Observe that the reading word of T^* is exactly the reverse-complement of the reading word of T.

The *Schensted insertion algorithm*, or equivalently the *RSK correspondence*, is an extremely powerful tool relating permutations to pairs of standard Young tableaux. For a description of the bijection and a proof of its correctness and some of its properties, we refer the reader to [14, Chapter 7]. Our use of notation follows that source, so in particular we denote by $T \leftarrow i$ the tableau that results when we (row-) insert i into the tableau T. Particular properties of RSK will be quoted as needed.

3 The pattern $12 \cdots (k+1)$

In this section we give the simplest of the bijections in this paper.

Proposition 3.1 *There is a bijection between $\mathcal{L}_{n,k}(12 \cdots (k+1))$ and the set of standard Young tableaux of shape $\langle k^n \rangle$.*

We have $f^{\langle n\rangle} = f^{\langle 1^n\rangle} = 1$ and $f^{\langle n,n\rangle} = f^{\langle 2^n\rangle} = \frac{1}{n+1}\binom{2n}{n} = C_n$, the nth Catalan number. By the hook-length formula [14, 11] we have

$$f^{\langle k^n\rangle} = \frac{(kn)!\cdot 1!\cdot 2!\cdots(k-1)!}{n!\cdot(n+1)!\cdots(n+k-1)!}.$$

So Proposition 3.1 says $|\mathcal{L}_{n,k}(1\cdots(k+1))| = f^{\langle k^n\rangle}$. For $k=1$, this is the uninspiring result $|S_n(12)| = 1$. For $k=2$, it tells us $|A_{2n}(123)| = C_n$, a result that Stanley [15] attributes to Deutsch and Reifegerste.

Proof idea: The bijection is to identify the permutation $w = w_{1,1}\cdots w_{n,k} \in \mathcal{L}_{n,k}(12\cdots k(k+1))$ with the tableau $T \in \mathrm{SYT}(\langle k^n\rangle)$ given by $T_{i,j} = w_{n+1-i,j}$. It is not difficult to verify that the conditions on w correspond precisely to the conditions that T be a standard Young tableau and vice-versa. □

Both directions of this bijection are more commonly seen with other names. The map that sends $w \mapsto T$ is actually the Schensted insertion algorithm used in the RSK correspondence. (For any $w \in \mathcal{L}_{n,k}(1\cdots(k+1))$, the recording tableau is the tableau whose first row contains the $\{1,\ldots,k\}$, second row contains $\{k+1,\ldots,2k\}$, and so on.) The map that sends $T \mapsto w$ is the reading-word map as defined in Section 2.

4 The pattern $12\cdots(k+2)$

There are several nice proofs of the equality $|S_n(123)| = C_n$ including a clever application of the RSK algorithm [14, Problem 6.19(ee)]. In this section, we give two bijective proofs of the following generalization of this result:

Theorem 4.1 *There is a bijection between $\mathcal{L}_{n,k}(12\cdots(k+2))$ and the set of standard Young tableaux of shape $\langle(k+1)^n\rangle$ and so*

$$|\mathcal{L}_{n,k}(12\cdots(k+2))| = f^{\langle(k+1)^n\rangle}.$$

For $k=1$ this is a rederivation of the equality $|S_n(123)| = C_n$ while for $k=2$ it implies

Corollary 4.2 *We have $|A_{2n}(1234)| = f^{\langle 3^n\rangle} = \dfrac{2(3n)!}{n!(n+1)!(n+2)!}$ for all $n \geq 0$.*

We believe this to be the first computation of any expression of the form $A_{2n}(\pi)$ or $A_{2n+1}(\pi)$ for $\pi \in S_4$. One can derive the complementary result for $|A_{2n+1}(1234)|$ using similar methods.

The first of our two bijections makes use of a modification of Schensted insertion, and the key idea for the modification appears in [10] (in the context of *doubly-alternating permutations*). The second bijection makes use of generating trees; its proof involves a number of technical results that we omit in this extended abstract.

4.1 A bijection using a modified version of RSK

In this section, we prove Theorem 4.1 using a modification of the RSK insertion algorithm. Recall that the RSK is a bijection between S_n and pairs (P,Q) of standard Young tableaux such that $\mathrm{sh}(P) = \mathrm{sh}(Q) \vdash n$ with the following properties:

Theorem 4.3 ([14, 7.11.2(b)]) *If P is a standard Young tableau and $j < k$ then the insertion path of j in $P \leftarrow j$ lies strictly to the left of the insertion path of k in $(P \leftarrow j) \leftarrow k$, and the latter insertion path does not extend below the former.*

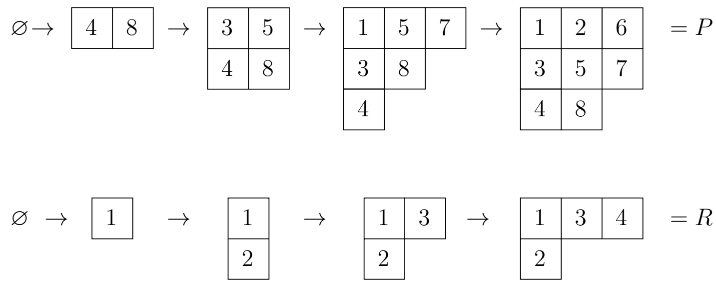

Fig. 2: An application of our modified version of RSK to the permutation $48351726 \in \mathcal{L}_{4,2}(1234)$. Note that only every other insertion step is shown in the construction of P.

Theorem 4.4 ([14, 7.23.11]) *If $w \in S_n$ and $w \xrightarrow{RSK} (P, Q)$ with $\mathrm{sh}(P) = \mathrm{sh}(Q) = \lambda$, then λ_1 is the length of the longest increasing subsequence in w.*

Now we describe a bijection from $\mathcal{L}_{n,k}(12 \cdots (k+2))$ to pairs (P, R) of standard Young tableau such that P has nk boxes, R has n boxes, and the shape of R can be rotated $180°$ and joined to the shape of P to form a rectangle of shape $\langle (k+1)^n \rangle$. (In other words, $\mathrm{sh}(P)'_i + \mathrm{sh}(R)'_{k+2-i} = n$ for $1 \leq i \leq k+1$.) Observe that the set of such pairs of tableaux is in natural bijection with the set of standard Young tableaux of shape $\langle (k+1)^n \rangle$: given a tableau of shape $\langle (k+1)^n \rangle$, break off the portion of the tableau filled with $nk+1, \ldots, n(k+1)$, rotate it $180°$ and replace each value i that appears in it with $nk + n + 1 - i$.

Given a permutation $w = w_{1,1} w_{1,2} \cdots w_{1,k} w_{2,1} \cdots w_{n,k}$, let $P_0 = \varnothing$ and for $1 \leq i \leq n$ let $P_i = (\cdots((P_{i-1} \leftarrow w_{i,1}) \leftarrow w_{i,2}) \cdots) \leftarrow w_{i,k}$. Define $P = P_n$, so P is the usual RSK insertion tableau for w. Define R as follows: set $R_0 = \varnothing$ and $\lambda_i = \mathrm{sh}(P_i)$. Observe that by Theorem 4.3, $\lambda_i / \lambda_{i-1}$ is a horizontal strip of size k and that by Theorem 4.4, $\lambda_i / \lambda_{i-1}$ stretches no further right than the $(k+1)$th column. Thus there is a unique j such that $\lambda_i / \lambda_{i-1}$ has boxes in the ℓth column for all $\ell \in [k+1] \setminus \{j\}$. Let R_i be the shape that arises from R_{i-1} by adding a box filled with i in the $(k+2-j)$th column, and define $R = R_n$. This map is illustrated in Figure 2.

Proposition 4.5 *The algorithm just described is a bijection between $\mathcal{L}_{n,k}(12 \cdots (k+2))$ and pairs (P, R) of standard Young tableaux such that P has nk boxes, R has n boxes, and $\mathrm{sh}(R)$ can be rotated and joined to $\mathrm{sh}(P)$ to form a rectangle of shape $\langle (k+1)^n \rangle$.*

Proof: By construction, P is a standard Young tableau with nk boxes and R is a shape with n boxes filled with $[n]$ such that we may rotate R by $180°$ and join it to P in order to get a rectangle of shape $\langle (k+1)^n \rangle$. Moreover, we have from standard properties of RSK that each P_i is of partition shape and by construction that the corresponding R_i may be rotated $180°$ and joined to P_i to form a rectangle, so each of the R_i (including R itself) is a partition shape. Finally, the unique box in R_i but not in R_{i-1} is filled with i, which is larger than the entry in any box in R_{i-1}, so R is a standard Young tableau.

We have left to show that this process is a bijection, *i.e.*, we need that this map is invertible and that its inverse takes pairs of tableaux of the given sort to permutations with the appropriate restrictions. Invertibility is straightforward, since from a pair (P, R) of standard Young tableaux of appropriate shapes we can construct a pair of standard Young tableaux (P, Q) of the same shape such that $w \mapsto (P, R)$ under our algorithm exactly when $w \xrightarrow{\mathrm{RSK}} (P, Q)$: if R has entry i in column $k+2-j$, place the entries

$ki-k+1, ki-k+2, \ldots, ki$ respectively into columns $1, \ldots, j-1, j+1, \ldots, k+1$ of Q. Moreover, by Theorem 4.3 we have that the preimage under RSK of this pair (P, Q) must consist of n runs of k elements each in increasing order, *i.e.*, it must satisfy L1, and by Theorem 4.4 it must have no increasing subsequence of length $k+2$. Then by the remarks in Section 2 following the definition of $\mathcal{L}_{n,k}$ we have that the preimage satisfies L2 as well. This completes the proof. □

4.2 A second approach using generating trees

Given a sequence $\{\Sigma_n\}_{n\geq 1}$ of nonempty sets with $|\Sigma_1| = 1$, a *generating tree* for this sequence is a rooted, labeled tree such that the vertices at level n are the elements of Σ_n and the label of each vertex determines the multiset of labels of its children. In other words, a generating tree is one particular type of recursive structure in which heredity is determined by some local data. We are particularly interested in generating trees for which the labels are (much) simpler than the objects they are labeling. In this case, we may easily describe a generating tree by giving the label L_1 of the *root vertex* (the element of Σ_1) and the *succession rule* $L \mapsto S$ that gives the set S of labels of the children in terms of the label L of the parent. Generating trees have proven to be an effective tool for finding bijections between different classes of pattern-avoiding permutations (see, *e.g.*, [16, 2]). In this section, we describe how generating trees can be used to give a second proof of Theorem 4.1 and to enumerate 2143-avoiding alternating permutations.

4.2.1 A tree for $\mathcal{L}_{n,k}$

There is a natural generating tree structure on $\bigcup_{n\geq 1} \mathcal{L}_{n,k}$: given a permutation $v \in \mathcal{L}_{n,k}$, its children are precisely the permutations $w \in \mathcal{L}_{n+1,k}$ such that the prefix of w of length nk is order-isomorphic to v. Since pattern containment is transitive, the subset $\bigcup_{n\geq 1} \mathcal{L}_{n,k}(p)$ of these permutations that avoid the pattern (or set of patterns) p is the set of vertices of a connected subtree. We now consider this restricted tree for the pattern $p = 12\cdots(k+2)$.

Given a permutation $w = w_1 w_2 \cdots w_{nk} \in \mathcal{L}_{n,k}(1\cdots(k+2))$, we associate a label $(a_2, \ldots, a_{k+1})$, where a_j is the smallest entry of w that is the largest entry in a j-term increasing subsequence, or $nk+1$ if there is no such entry. (Note that a_j could equivalently be defined as the last-occurring entry of w that is the largest term in a j-term increasing subsequence of w but is not the largest term in a $(j+1)$-term increasing subsequence.) Thus, for example, the unique permutation $12\cdots k \in \mathcal{L}_{1,k}(1\cdots(k+2))$ has label $(2, \ldots, k+1)$, while the permutation $136245 \in \mathcal{L}_{2,3}(12345)$ has label $(2, 4, 5)$.

Some relations among label entries are straightforward. For example, observe that if $(a_2, \ldots, a_{k+1})$ is the label of any permutation $u \in \mathcal{L}_{n,k}(1\cdots(k+2))$ then $2 \leq a_2 < \ldots < a_{k+1} \leq nk+1$. The following result (whose proof, which consists of many technical details and little insight, is omitted) characterizes the labels of children based on the labels of a parent.

Proposition 4.6 *Suppose that $u \in \mathcal{L}_{n,k}(12\cdots(k+2))$ has label $(a_2, \ldots, a_{k+1})$. Then for any k-tuple $(b_2, \ldots, b_{k+1})$ such that*

$$2 \leq b_2 < b_3 < \ldots < b_{k+1} \leq (n+1)k+1 \qquad \text{and} \qquad b_j \leq a_j + j - 1 \qquad \text{for all } j,$$

there is a unique child $w \in \mathcal{L}_{n+1,k}(12\cdots(k+2))$ of u with label $(b_2, \ldots, b_{k+1})$, and u has no other children.

4.2.2 A tree for Young tableaux

There is a natural generating tree on the set $\bigcup_{n\geq 1} \mathrm{SYT}(\langle (k+1)^n \rangle)$ of rectangular standard Young tableaux with $k+1$ columns: let a tableau T be the child of a tableau S if S is order-isomorphic to T with its first row removed.

Given a tableau $S \in \mathrm{SYT}(\langle (k+1)^n \rangle)$ with first row $(1, s_2, s_3, \ldots, s_{k+1})$, assign to it the label $(s_2, \ldots, s_{k+1})$. Thus, for example, that the unique tableau in $\mathrm{SYT}(\langle k+1 \rangle)$ has label $(2, 3, \ldots, k+1)$, while the tableau

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 5 \\ \hline 3 & 6 & 7 & 8 \\ \hline \end{array} \in \mathrm{SYT}(\langle 4, 4 \rangle)$$

has label $(2, 4, 5)$. It's easy to see that if $(s_2, \ldots, s_{k+1})$ is the label of a tableau $T \in \mathrm{SYT}(\langle (k+1)^n \rangle)$ then $2 \leq s_2 < s_3 < \ldots < s_{k+1} \leq n(k+1) - (n-1) = nk+1$. Without too much effort, one can also show the following result:

Proposition 4.7 *Suppose that $S \in \mathrm{SYT}(\langle (k+1)^n \rangle)$ has label $(s_2, \ldots, s_{k+1})$. Then for any k-tuple $(t_2, \ldots, t_{k+1})$ such that*

$$2 \leq t_2 < t_3 < \ldots < t_{k+1} \leq (n+1)k+1 \qquad \textit{and} \qquad t_j \leq s_j + j - 1 \qquad \textit{for all } j,$$

there is a unique child $T \in \mathrm{SYT}(\langle (k+1)^{n+1} \rangle)$ of S with label $(t_2, \ldots, t_{k+1})$, and S has no other children.

Theorem 4.1 follows immediately from Propositions 4.6 and 4.7.

4.2.3 A tree for 2143-avoiding alternating permutations

If, as in [5], we restrict our focus to alternating permutations (*i.e.*, to $A_{2n} = \mathcal{L}_{n,2}$), brute-force computations suggest that there may be several patterns $p \in S_4$ such that $|A_{2n}(p)| = |A_{2n}(1234)|$ for all n. In this section we use generating trees to show that 2143 is one such pattern.

Given any permutation $w \in S_n$ and any $c \in [n+1]$, denote by $w \circ c$ the unique permutation in S_{n+1} whose last entry is c and whose first n entries are order-isomorphic to w. If $w = w_1 w_2 \cdots w_{2n} \in A_{2n}(2143)$, say that a value $c \in [2n+1]$ is *active* for w if $w \circ c$ avoids 2143. To each $w \in A_{2n}(2143)$, assign the label (a, b) where $a = w_{2n-1} + 1$ and b is equal to the number of values in $[n+1]$ that are active for w. The following result shows that with this labeling, the generating tree for $\bigcup_{n\geq 1} A_{2n}(2143)$ obeys a simple succession rule.

Proposition 4.8 *Suppose that $u \in A_{2n}(2143)$ has label (a, b). Then for any ordered pair (x, y) such that*

$$2 \leq x \leq a+1 \qquad \textit{and} \qquad x < y \leq b+2,$$

there is a unique child $w \in A_{2n+2}(2143)$ with label (x, y), and u has no other children.

One can easily verify that these conditions are equivalent to those of Propositions 4.6 and 4.7 in the case $k = 2$. Therefore, we may conclude with the following result.

Theorem 4.9 *For all $n \geq 1$ we have*

$$|A_{2n}(1234)| = |A_{2n}(2143)| = \frac{2 \cdot (3n)!}{n! \cdot (n+1)! \cdot (n+2)!}.$$

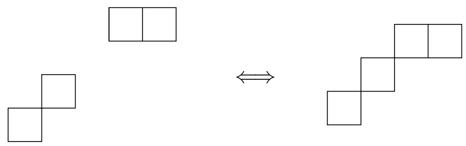

Fig. 3: Moving separated components gives a new shape but leaves the set of reading words of tableaux unchanged.

5 Pattern avoidance in reading words of tableaux of skew shapes

So far, we have considered permutations that arise as the reading words of standard skew Young tableaux of particular nice shapes. In this section, we expand our study to include pattern avoidance in the reading words of standard Young tableaux of *any* skew shape. As is the case for pattern avoidance in other settings, it is relatively simple to handle the case of small patterns (in our case, patterns of length three or less), but it appears to be quite difficult to prove exact results for larger patterns.

As we have seen, this new type of pattern avoidance encompasses pattern avoidance for the set of all permutations via the shape $\langle n, n-1 \ldots, 1\rangle / \langle n-1, n-2, \ldots, 1\rangle$, for alternating permutations via the shape $\langle n+1, n, \ldots, 2\rangle / \langle n-1, n-2, \ldots, 1\rangle$ and three other similar shapes, and for $\mathcal{L}_{n,k}$ for any k via the shape illustrated in Figure 1; it also incorporates other natural problems such as the enumeration of pattern-avoiding permutations with prescribed descent set (when the skew shape is a ribbon). Thus, on one hand the strength of our results is constrained by what is tractable to prove in these circumstances, while on the other hand any result we are able to prove in this context applies quite broadly.

Note 2 *We make the following general assumption on our Young diagrams: we will only ever be interested in diagrams λ/μ such that the inner (north-west) boundary of λ/μ contains the entire outer (south-east) boundary of μ. For example, the shape $\langle 4, 2, 1\rangle / \langle 2, 1\rangle$ meets this condition, while the shape $\langle 5, 2, 2, 1\rangle / \langle 3, 2, 1\rangle$ does not.*

Observe that imposing this restriction does not affect the universe of possible enumerative results: for a shape λ/μ failing this condition we can find a new shape λ'/μ' that passes it and has an identical set of reading words by moving the various disconnected components of λ/μ on the plane. For example, for $\lambda/\mu = \langle 5, 2, 2, 1\rangle / \langle 3, 2, 1\rangle$ we have $\lambda'/\mu' = \langle 4, 2, 1\rangle / \langle 2, 1\rangle$ – just slide disconnected sections of the tableau together until they share a corner. This example is illustrated in Figure 3.

5.1 The patterns 213 and 132

The equality $|S_n(213)| = |S_n(132)| = C_n$ is a simple recursive result. In [8] it was shown that $|A_{2n}(132)| = |A_{2n+1}(132)| = C_n$ (and so by reverse-complementation also $|A_{2n}(213)| = C_n$), and a bijective proof of this fact with implications for multiple-pattern avoidance was given in [7]. Here we extend this result to the reading words of tableaux of any fixed shape.

Theorem 5.1 *The number of tableaux of skew shape λ/μ whose reading words avoid the pattern 213 is equal to the number of partitions whose Young diagram is contained in that of μ (subject to Note 2).*

Note that this is a natural μ-generalization of the Catalan numbers: the outer boundaries of shapes contained in $\langle n-1, n-2, \ldots, 1\rangle$ are essentially Dyck paths of length $2n$ missing their first and last steps.

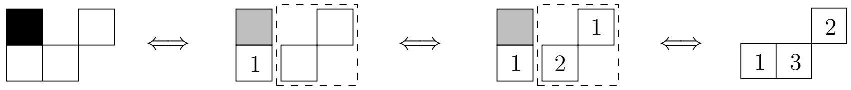

Fig. 4: Our bijection applied to the pair $(\langle 3,2\rangle/\langle 2\rangle, \langle 1\rangle)$ to generate a standard Young tableau.

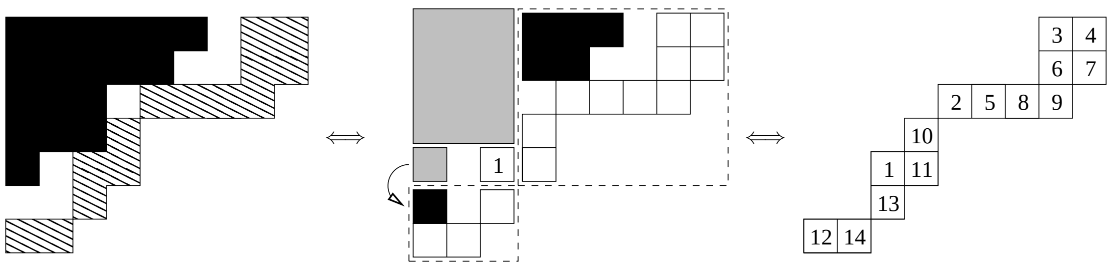

Fig. 5: A partial example: an application of our bijection to generate a standard Young tableau from the pair $(\langle 9,9,8,4,4,3,2\rangle/\langle 7,7,4,3,2,2\rangle, \langle 6,5,3,3,1\rangle)$.

Proof idea: We begin with a warm-up and demonstrate the claim in the case that μ is empty. In this case, the Proposition states that there is a unique standard Young tableau of a given shape $\lambda = \langle \lambda_1, \lambda_2, \ldots\rangle$ whose reading word avoids the pattern 213. In order to show this, we note that the reading word of every straight (*i.e.*, non-skew) tableau ends with an increasing run of length λ_1 and that the first entry of this run is 1. Since the reading word is 213-avoiding, each entry following the 1 must be smaller than every entry preceding the 1 and so this run consists of the values from 1 to λ_1. Applying the same argument to the remainder of the tableau (now with the minimal element $\lambda_1 + 1$), we see that the only possible filling is the one we get by filling the first row of the tableau with the smallest possible entries, then the second row with the smallest remaining entries, and so on. On the other hand, the reading word of the tableau just described is easily seen to be 213-avoiding, so we have our result in this case.

For the general case we give a recursive bijection. We recommend that the reader consult Figures 4 and 5 to most easily understand what follows.

Suppose we have a tableau T of shape λ/μ with entry 1 in position (i,j), an inner corner. Divide T into two pieces, one consisting of rows 1 through i with the box (i,j) removed, the other consisting of rows numbered $i+1, i+2$, etc. Let T_1 be the tableau order-isomorphic to the first part and let T_2 be the tableau order-isomorphic to the second part. Let $\nu = \langle \nu_1, \ldots, \nu_i\rangle$ be the result of applying this construction recursively to T_1 and let $\iota = \langle \iota_1, \iota_2, \ldots\rangle$ be the result of applying this construction recursively to T_2. Then the partition τ associated to T is given by $\tau = \langle \nu_1 + j, \ldots, \nu_i + j, \iota_1, \iota_2, \ldots\rangle$. That is, τ consists of all boxes (k,l) with $k < i$ and $l \leq j$ together with the result of applying our process to the right of this rectangle and the result of applying it below the rectangle, with the latter piece shifted up one row. By construction, τ is partition whose Young diagram fits inside μ.

To invert this process, start with a pair $(\lambda/\mu, \tau)$ of a skew and a non-skew shape such that τ fits inside μ. Let i be the largest index such that $\tau_{i-1} > \mu_i$, or let $i = 1$ if no such index exists. We divide τ and λ/μ into two pieces. For τ, we first remove the rectangle of shape $\langle (\mu_i+1)^{i-1}\rangle$, leaving a partition to the right of the rectangle of shape $\nu_1 = \langle \tau_1 - \mu_i - 1, \tau_2 - \mu_i - 1, \ldots, \tau_{i-1} - \mu_i - 1\rangle$

and a second partition below the rectangle of shape $\nu_2 = \langle \tau_i, \tau_{i+1}, \ldots \rangle$. For λ/μ, we begin by filling the box $(i, \mu_i + 1)$ with the entry 1. Then we take the boxes to the right of this entry as one skew shape $\alpha_1/\beta_1 = \langle \lambda_1 - \mu_i - 1, \lambda_2 - \mu_i - 1, \ldots, \lambda_i - \mu_i - 1 \rangle / \langle \mu_1 - \mu_i - 1, \mu_2 - \mu_i - 1, \ldots, \mu_{i-1} - \mu_i - 1 \rangle$ and the boxes below it as our second skew shape $\alpha_2/\beta_2 = \langle \lambda_{i+1}, \lambda_{i+2}, \ldots \rangle / \langle \mu_{i+1}, \mu_{i+2}, \ldots \rangle$. Note that ν_2 fits inside β_2 and that ν_1 fits inside β_1 by the choice of i. Thus we may apply this construction recursively with the pairs $(\alpha_1/\beta_1, \nu_1)$ and $(\alpha_2/\beta_2, \nu_2)$, filling α_1/β_1 with the values $2, \ldots, |s_1| + 1$ and filling α_2/β_2 with the values $|\alpha_1/\beta_1| + 2, \ldots, |\lambda/\mu| = |\alpha_1/\beta_1| + |\alpha_2/\beta_2| + 1$. (Observe that this coincides with what we did in the first paragraph for $\mu = \varnothing$.)

One can prove by a simple inductive argument that these maps are mutually-inverse bijections between the sets in question. □

Corollary 5.2 *We have that* $|\mathcal{L}_{n,k}(213)| = C_n$ *for all* $n, k \geq 1$.

Note that knowing the number of tableaux of each shape whose reading words avoid 213 automatically allows us to calculate for any shape the number of tableaux of that shape whose reading words avoid 132: if $\lambda = \langle \lambda_1, \ldots, \lambda_k \rangle$ and μ is contained in λ, the operation $T \mapsto T^*$ of rotation and complementation is a bijection between tableaux of shape λ/μ and tableaux of shape $\langle \lambda_1 - \mu_k, \lambda_1 - \mu_{k-1}, \ldots, \lambda_1 - \mu_1 \rangle / \langle \lambda_1 - \lambda_k, \lambda_1 - \lambda_{k-1}, \ldots, \lambda_1 - \lambda_2 \rangle$. Moreover, the reading word of T^* is the reversed-complement of the reading word of T, so the reading word of T avoids 132 if and only if the reading word of T^* avoids 213. This argument establishes the following corollary of Theorem 5.1:

Corollary 5.3 *The number of tableaux of skew shape* λ/μ *whose reading words avoid the pattern* 132 *is equal to the number of partitions whose Young diagram is contained in that of the partition* $\langle \lambda_1 - \lambda_k, \lambda_1 - \lambda_{k-1}, \ldots, \lambda_1 - \lambda_2 \rangle$.

Corollary 5.4 *We have that* $|\mathcal{L}_{n,k}(132)| = C_n$ *for all* $n, k \geq 1$.

5.2 The patterns 312 *and* 231

If the shape λ/μ contains a square, every tableau of that shape contains as a sub-tableau four entries

a	b
c	d

with $a < b < d$ and $a < c < d$, and the reading word of every such tableau is of the form $\ldots cd \ldots ab \ldots$. But any such permutation contains both an instance dab of the pattern 312 an instance cda of the pattern 231. Thus, the number of tableaux of shape λ/μ whose reading words avoid 312 or 231 is zero unless λ/μ contains no square, *i.e.*, unless λ/μ is contained in a ribbon. In this case, for a tableau T of shape λ/μ with reading word w we have that the reading word of the conjugate tableau T' is exactly the reverse w^r of w. Since w avoids 312 if and only if w^r avoids 213, we may apply Theorem 5.1 to deduce the following result.

Proposition 5.5 *If skew shape* λ/μ *is contained in a ribbon then the number of tableaux of shape* λ/μ *whose reading words avoid the pattern* 312 *is equal to the number of partitions whose Young diagram is contained in that of* μ. *Otherwise, the number of such tableaux is* 0.

Analogous arguments give the following result.

Corollary 5.6 *If skew shape λ/μ is contained in a ribbon then the number of tableaux of shape λ/μ whose reading words avoid the pattern 231 is equal to the number of partitions whose Young diagram is contained in that of the partition $\langle \lambda_1 - \lambda_k, \lambda_1 - \lambda_{k-1}, \ldots, \lambda_1 - \lambda_2 \rangle$. Otherwise, the number of such tableaux is 0.*

In the special case of $\mathcal{L}_{n,k}$ this says that for $k \geq 3$ and $n \geq 2$ we have $\mathcal{L}_{n,k}(231) = \mathcal{L}_{n,k}(312) = \varnothing$ while for $1 \leq k \leq 2$ we have that $|\mathcal{L}_{n,k}(231)|$ and $|\mathcal{L}_{n,k}(312)|$ are Catalan numbers [4, 15].

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