Involutions of the Symmetric Group and Congruence B-Orbits (Extended Abstract)

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Abstract. We study the poset of Borel congruence classes of symmetric matrices ordered by containment of closures. We give a combinatorial description of this poset and calculate its rank function. We discuss the relation between this poset and the Bruhat poset of involutions of the symmetric group. Also we present the poset of Borel congruence classes of anti-symmetric matrices ordered by containment of closures. We show that there exists a bijection between the set of these classes and the set of involutions of the symmetric group. We give two formulas for the rank function of this poset.

Résumé Nous étudions l'ensemble ordonné des classes de congruence de matrices symétriques ordonnées par containment de leurs fermetures. Nous donnons une description combinatoire de cet ensemble et calculons sa fonction rang. Nous étudions les relations entre cet ensemble et l'ensemble des involutions du groupe symérique ordonné selon l'ordre de Bruhat. Nous montrons qu'il existe une bijection parmi l'ensemble ordonné de classes de congruences de Borel des matrices anti-symétriques et l'ensemble des involutions du groupe symétrique. On termine en donnant deux formules pour la fonction rang pour ce dernier ensemble.

Keywords: Bruhat poset, congruence orbit, involutions of the symmetric group

1   Introduction

A remarkable property of the Bruhat decomposition of $GL_n(\mathbb{C})$ (i.e. the decomposition of $GL_n(\mathbb{C})$ into double cosets $\{B_1\pi B_2\}$ where $\pi \in S_n$, $B_1, B_2 \in \mathbb{B}_n(\mathbb{C})$ – the subgroup of upper-triangular invertible matrices) is that the natural order on double cosets (defined by containment of closures) leads to the same poset as the combinatorially defined Bruhat order on permutations of S_n (for $\pi, \sigma \in S_n$, $\pi \leqslant \sigma$ if π is a subword of σ with respect to the reduced form in Coxeter generators). L. Renner introduced and developed the beautiful theory of Bruhat decomposition for not necessarily invertible matrices, see [10] and [9]. When the Borel group acts on all the matrices, the double cosets are in bijection with partial permutations which form a so called rook monoid R_n which is the finite monoid whose elements are the 0-1 matrices with at most one nonzero entry in each row and column. The group of invertible elements of R_n is isomorphic to the symmetric group S_n. Another efficient, combinatorial description of the Bruhat ordering on R_n and a useful, combinatorial formula for the length function on R_n are given by M. Can and L. Renner in [3].

The Bruhat poset of involutions of S_n was first studied by F. Incitti in [6] from a purely combinatorial point of view. He proved that this poset is graded, calculated the rank function and also showed several

other important properties of this poset.

In this extended abstract we present a geometric interpretation of this poset and its natural generalization, considering the action of the Borel subgroup on symmetric matrices by congruence. Denote by $\mathbb{B}_n(\mathbb{C})$ the Borel subgroup of $GL_n(\mathbb{C})$, i.e. the group of invertible upper-triangular $n \times n$ matrices over the complex numbers. Denote by $\mathbb{S}(n, \mathbb{C})$ the set of all complex symmetric $n \times n$ matrices. The congruence action of $B \in \mathbb{B}_n(\mathbb{C})$ on $S \in \mathbb{S}(n, \mathbb{C})$ is defined in the following way:

$$ S \longmapsto B^t S B \ . $$

The orbits of this action (to be precisely correct, we must say $S \mapsto \left(B^{-1}\right)^t S B^{-1}$ to get indeed a group action) are called the *congruence B-orbits*. It is known that the orbits of this action may be indexed by partial S_n-involutions (i.e. symmetric $n \times n$ matrices with at most one 1 in each row and in each column) (see [11]). Thus, if π is such a partial involution, we denote by $\mathcal{C}_\pi$ the corresponding congruence B-orbit of symmetric matrices. The poset of these orbits gives a natural extension of the Bruhat poset of regular (i.e. not partial) involutions of S_n. If we restrict this action to the set of invertible symmetric matrices we get a poset of orbits that is isomorphic to the Bruhat poset of involutions of S_n studied by F. Incitti.

Here, we give another view of the rank function of this poset, combining combinatorics with the geometric nature of it. The rank function equals to the dimension of the orbit variety. We define the combinatorial parameter $\mathfrak{D}$ which is an invariant of the orbit closure and give two combinatorial formulas for the rank function of the poset of partial involutions (Theorems 2 and 7). The result of Incitti that the Bruhat poset of involutions of S_n is graded and his formula for the rank function of this poset follow from our exposition (Corollary 1).

Also we present another graded poset of involutions of the symmetric group which also has the geometric nature, i.e. it can be described as a poset of matrix varieties ordered by containment of closures. Denote by $\mathbb{AS}(n, \mathbb{C})$ the set of all complex anti-symmetric $n \times n$ matrices. It is actually a vector space with respect to standard operations of addition and multiplication by complex scalars, also it is a Lie algebra usually denoted as $\mathfrak{so}$ with $[A, B] := AB - BA$. It is easy to see that $\mathbb{AS}(n, \mathbb{C})$ is closed under the congruence action. We consider the orbits of the congruence action of $\mathbb{B}_n(\mathbb{C})$ on $\mathbb{AS}(n, \mathbb{C})$.

The main points of this work are Proposition 2, Definition 8, Theorem **??** and Proposition 8. In Proposition 2 we show that the orbits of this action may be indexed by involutions of S_n. Then we consider the poset of these orbits ordered by containment of closures. In Definition 7 we introduce the parameter $\mathfrak{A}$ and then in Theorem 2 and Proposition 8 we give two different formulas for the rank function of the studied poset using the parameter $\mathfrak{A}$. This parameter is similar to the parameter $\mathfrak{D}$ introduced in [1] and it can be seen as a particular case of a certain unified approach to the calculation of the rank function for several "Bruhat-like" posets as we briefly discuss it at the last section of [1].

If we restrict this action on the set of invertible anti-symmetric matrices we get a poset of orbits that is isomorphic to the (reversed) Bruhat poset of involutions of S_n without fixed points which is a subposet of the poset studied by F. Incitti.

2 Preliminaries

2.1 Permutations and partial permutations. The Bruhat order

The Bruhat order on permutations of S_n is defined as follows: $\pi \leqslant \sigma$ if π is a subword of σ in Coxeter generators $s_1 = (1,2)$, $s_2 = (2,3)$,...,$s_{n-1} = (n-1,n)$. It it well studied from various points of view. The rank function is the length in Coxeter generators which is exactly the number of inversions in a permutation. A permutation matrix is a square matrix which has exactly one 1 in each row and each column while all other entries are zeros. A partial permutation is an injective map defined on a subset of $\{1, 2, .., n\}$. A partial permutation matrix is a square matrix which has at most one 1 at each row and each column and all other entries are zeros. So, if we delete the zero rows and columns from a partial permutation matrix we get a (regular) permutation matrix of smaller size, we will use this view later. See works of L. Renner [9] and [10] where the Bruhat order on partial permutations is introduced and studied.

2.2 Partial order on orbits

When an algebraic group acts on a set of matrices, the classical partial order on the set of all orbits is defined as follows:

$$\mathcal{O}_1 \leq \mathcal{O}_2 \iff \mathcal{O}_1 \subseteq \overline{\mathcal{O}_2}$$

where $\overline{S}$ is the (Zariski) closure of the set S.

Reminder 1 *Note that $\mathcal{O}_1 \subseteq \overline{\mathcal{O}_2} \implies \overline{\mathcal{O}_1} \subseteq \overline{\mathcal{O}_2}$ for any two sets $\mathcal{O}_1, \mathcal{O}_2$.*

Definition 1 *As usual, a monomial matrix is a matrix which has at most one non-zero entry in each its row and in each its column.*

3 Rank-control matrices

In this section we define the *rank control matrix* which will turn out to be a key corner in the identification of our poset. We start with the following definition:

Definition 2 *Let $X = (x_{ij})$ be an $n \times m$ matrix. For each $1 \leq k \leq n$ and $1 \leq l \leq m$, denote by $X_{k\ell}$ the upper-left $k \times \ell$ submatrix of X. We denote by $R(X)$ the $n \times m$ matrix whose entries are: $r_{k\ell} = rank\,(X_{k\ell})$ and call it the rank control matrix of X.*

It follows from the definitions that for each matrix X, the entries of $R(X)$ are nonnegative integers which do not decrease in rows and columns and each entry is not greater than its row and column number. If X is symmetric, then $R(X)$ is symmetric as well.

Reminder 2 *This rank-control matrix is similar to the one introduced by A. Melnikov [7] when she studied the poset (with respect to the covering relation given in Definition 2.2) of adjoint B-orbits of certain nilpotent strictly upper-triangular matrices.*

The rank control matrix is connected also to the work of Incitti [6] where regular involutions of S_n are discussed.

Proposition 1 *Let $X, Y \in GL_n(\mathbb{F})$ be such that $Y = LXB$ for some invertible lower-triangular matrix L and some matrix $B \in \mathbb{B}_n(\mathbb{C})$. Denote by $X_{k\ell}$ and $Y_{k\ell}$ the upper-left $k \times \ell$ submatrices of X and Y respectively. Then for all $1 \leqslant k, \ell \leqslant n$*

$$rank\,(X_{k\ell}) = rank\,(Y_{k\ell})\;.$$

Proof:

$$\begin{pmatrix} L_{kk} & 0_{k \times (n-k)} \\ * & * \end{pmatrix} \begin{pmatrix} X_{k\ell} & * \\ * & * \end{pmatrix} \begin{pmatrix} B_{\ell\ell} & * \\ 0_{(n-\ell) \times \ell} & * \end{pmatrix} = \begin{pmatrix} L_{kk} X_{k\ell} B_{\ell\ell} & * \\ * & * \end{pmatrix} ,$$

and therefore, $Y_{k\ell} = L_{kk} X_{k\ell} B_{\ell\ell}$. The matrices L_{kk} and $B_{\ell\ell}$ are invertible, which implies that $Y_{k\ell}$ and $X_{k\ell}$ have equal ranks. □

The rank control matrices of two permutations can be used to compare between them in the sense of Bruhat order. This is the reasoning for the next definition:

Definition 3 *Define the following order on $n \times m$ matrices with positive integer entries: Let $P = (p_{ij})$ and $Q = (q_{ij})$ be two such matrices.*
Then

$$P \leqslant_{\mathcal{R}} Q \iff p_{ij} \leqslant q_{ij} \text{ for all } i, j .$$

The following lemma appears in another form as Theorem 2.1.5 of [2].

Lemma 1 *Denote by $\leqslant_{B}$ the Bruhat order of S_n and let $\pi, \sigma \in S_n$. Then*

$$\pi \leqslant_{\mathcal{B}} \sigma \iff R(\pi) \geqslant_{\mathcal{R}} R(\sigma) .$$

In other words, the Bruhat order on permutations corresponds to the inverse order of their rank-control matrices. □

4 Partial permutations, Partial Involutions and Congruence B-Orbits

Definition 4 *A* partial permutation *is an $n \times n$ $(0, 1)$-matrix such that each row and each column contains at most one '1'.*

Definition 5 *If a partial permutation matrix is symmetric, then we call it a* partial involution.

The following easily verified lemma claims that partial permutations are completely characterized by their rank control matrices.

Lemma 2 *For two $n \times n$ partial permutation matrices π, σ we have*

$$R(\pi) = R(\sigma) \iff \pi = \sigma.$$

Proof: The statement of the lemma is implied by the following simple observation: let U be the $n \times n$ upper-triangular matrix with '1's on the main diagonal and in all upper triangle and let π be any partial permutation. Then

$$R(\pi) = U^t \pi U .$$

□ The following theorem can be found in [11] (Theorem 3.2). It is proved by performing a symmetric version of Gauss elimination process.

Theorem 1 *There exists a bijection between the set of congruence B-orbits of symmetric matrices over $\mathbb{C}$ and the set of partial involutions.*

5   A bijection between orbits and involutions

The following Proposition 2 is somewhat similar to Theorem 3.2 in [11].

Proposition 2 *There is a bijection between the set of congruence B-orbits of all anti-symmetric $n \times n$ matrices and the set of all involutions of S_n.*

Proof: The complete proof can be found in [4]. It is done by symmetric elimination process which starts with an anti-symmetric matrix and terminates with a certain monomial anti-symmetric matrix which has 1's in its upper triangle and -1's in its lower triangle. Such matrix is unique for the given orbit and there is a bihection between the set of such matrices and involutions of S_n. This bijection is illustrated in the Example 1. $\square$

Example 1 *The monomial anti-symmetric matrix* $\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ *corresponds to the involution*

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 3 & 1 & 2 & 6 \end{pmatrix} \in S_6$, *which can be written as the product of disjoint transpositions as* $(1,4)(2,5)$.
a

Observation 1 *The congruence B-orbits of invertible anti-symmetric $2n \times 2n$ matrices can be indexed by involutions of S_{2n} without fixed points.*

6   The Poset of Congruence B-Orbits of Symmetric Matrices

Here is a direct consequence of Lemma 2 and Proposition 1.

Proposition 3 *All the matrices of a fixed congruence B-Orbit share a comon rank-control matrix. In other words, if π is a partial S_n-involution, and C_π is the congruence B-orbit of symmetric matrices associated with π then*

$$\mathcal{C}_\pi = \{S \in \mathbb{S}(n, \mathbb{C}) \,|\, R(S) = R(\pi)\} \,.$$

The following lemma describes the orbits:

Lemma 3 *Let π be a partial involution and let $R(\pi)$ be its rank-control matrix. Then*

$$\overline{\mathcal{C}_\pi} = \{S \in \mathbb{S}(n, \mathbb{C}) \mid R(S) \leqslant_{\mathcal{R}} R(\pi)\} \,.$$

This lemma follows from Theorem 15.31 of [8]. Their exposition differs somewhat from ours as it deals with rectangular, not necessarily symmetric matrices but the differences can be easily overwhelmed by considering also equations of the form $a_{ij} = a_{ji}$ which are polynomial equations with regard to the entries of a matrix.

Reminder 3 *Over the fields $\mathbb{C}$ and $\mathbb{R}$ the closure in Lemma 5 may also be considered with respect to the metric topology.*

The next corollary follows from Lemma 5 and characterizes the order relation of the poset of B-orbits.
  Let π and σ be partial S_n-involutions. Then

$$\mathcal{C}_\pi \leqslant_\mathcal{O} \mathcal{C}_\sigma \iff R(\pi) \leqslant_\mathcal{R} R(\sigma)$$

  Explicit examples for $n = 3$ in the symmetric case can be found in [1] and for $n = 4$ in the antisymmetric case can be found in [4].

7   The Poset of Congruence B-Orbits of Anti-Symmetric Matrices

Here is a direct consequence of Proposition 1.

Proposition 4 *All the matrices of a fixed congruence B-Orbit have the same rank-control matrix. In other words, if $X \in \mathbb{AS}(n, \mathbb{C})$ and $\mathcal{A}_X$ is the congruence B-orbit of X, then*

$$\mathcal{A}_X = \{S \in \mathbb{AS}(n, \mathbb{C}) \,|\, R(S) = R(X)\} \,.$$

  Similarly to the symmetric case we give the proposition which describes the orbit closures in the anti-symmetric case. This proposition also follows from Theorem 15.31 given by E. Miller and B. Sturmfels, see [8, Chapter 15, page 301]:

Proposition 5 *Let X be an anti-symmetric matrix and let $R(X)$ be its rank-control matrix. Then*

$$\overline{\mathcal{A}_X} = \{S \in \mathbb{AS}(n, \mathbb{C}) \,|\, R(S) \leqslant_\mathcal{R} R(X)\} \,.$$

The next corollary characterizes the order relation of the poset of B-orbits.

Corollary 1 *Let $X, Y \in \mathbb{AS}(n, \mathbb{C})$. Then*

$$\mathcal{A}_X \leqslant_\mathcal{O} \mathcal{A}_Y \iff R(X) \leqslant_\mathcal{R} R(Y)$$

8   The Rank Function

Definition 6 *A poset P is called graded (or ranked) if for every $x, y \in P$, any two maximal chains from x to y have the same length.*

Proposition 6 *The poset of congruence B-orbits of symmetric matrices and the poset of congruence B-orbits of anti-symmetric matrices(with respect to the order $\leqslant_\mathcal{O}$) are graded posets with the rank function given by the dimension of the closure.*

This proposition is a particular case of the following fact. Let G be a connected, solvable group acting on an irreducible, affine variety X. Suppose that there are a finite number of orbits. Let O be the set of G-orbits on X. For $x, y \in O$ define $x \leqslant y$ if $x \subseteq \overline{y}$. Then O is a graded poset.
  This fact is given as an exercise in [10] (exercise 12, page 151) and can be proved using the proof of the theorem appearing of Section 8 of [9]. (Note that in our case the Borel group is solvable, the varieties of all symmetric and anti-symmetric matrices are irreducible because they are vector spaces and the number of orbits is finite since there are only finitely many partial permutation.)
  A natural problem is to find an algorithm which calculates the dimension of the orbit closure from the monomial matrix or from its rank-control matrix. Here we present such an algorithm.

Definition 7 *Let π be a partial involution matrix and let $R(\pi) = (r_{ij})$ be its rank-control matrix. Add an extra 0 row to $R(\pi)$, pushed one place to the left, i.e. assume that $r_{0k} = 0$ for each $0 \leqslant k < n$.*
   Denote

$$\mathfrak{D}(\pi) = \# \left\{ (i,j) \mid 1 \leqslant i \leqslant j \leqslant n \quad and \quad r_{ij} = r_{i-1,j-1} \right\}.$$

Definition 8 *Let $X \in \mathbb{AS}(n, \mathbb{C})$ and let $R(X) = (r_{ij})_{i,j=1}^n$ be the rank-control matrix of X. Add an extra 0 row to $R(X)$, pushed one place to the left, i.e. assume that $r_{0k} = 0$ for each $0 \leqslant k < n$. Denote*

$$\mathfrak{A}(X) = \# \left\{ (i,j) \mid 1 \leqslant i < j \leqslant n \quad and \quad r_{ij} = r_{i-1,j-1} \right\}.$$

The first parameter $\mathfrak{D}$ counts equalities in the diagonals of the upper triangle of the rank-control matrix including the main diagonal and the second parameter $\mathfrak{A}$ counts equalities in the diagonals of the upper triangle of the rank-control matrix without the main diagonal.

Theorem 2 *Let π be a partial S_n-involution. As above $\mathcal{C}_\pi$ denotes the orbit of symmetric matrices which corresponds to π. Then*

$$\dim \overline{\mathcal{C}_\pi} = \frac{n^2 + n}{2} - \mathfrak{D}(\pi).$$

Proof: Consider the vector space

$$\mathbb{C}^{n^2} = \left\{ [a_{ij}]_{i,j=1}^n = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} : a_{ij} \in \mathbb{C} \right\}.$$

Let X be some set of pairs of indexes, i.e. $X \subseteq \{(i,j) : 1 \leqslant i, j \leqslant n\}$. Define a subspace $W_X \subset \mathbb{C}^{n^2}$ of dimension $n^2 - |X|$ in the following way:

$$W_X = \left\{ [a_{i_t j_t}] : (i_t, j_t) \notin X \right\},$$

i.e. W_X is spanned by the elements of the standard basis of $\mathbb{C}^{n^2}$ which we index by all pairs of indices not belonging to X.

   Consider also the natural projection $p_X : \mathbb{C}^{n^2} \to W_X$. Since we consider elements of $\mathbb{C}^{n^2}$ as $n \times n$ matrices, we denote elements of W_X as matrices with empty boxes in the positions whose indexes are in X. For example, consider

$$\mathbb{C}^{3^2} = \left\{ [a_{ij}]_{i,j=1}^3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} : a_{ij} \in \mathbb{C} \right\}$$

and let $X = \{(2,3), (3,2), (3,3)\}$. Then

$$W_X = \left\{ [a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{31}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & \square \\ a_{31} & \square & \square \end{bmatrix} : a_{ij} \in \mathbb{C} \right\} \subset \mathbb{C}^{3^2}.$$

In this example the natural projection $p_X : \mathbb{C}^{3^2} \to W_X$ is

$$p_X \left([a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}] \right) = [a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{31}]$$

or in the matrix notation

$$p_X \left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \right) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & \square \\ a_{31} & \square & \square \end{bmatrix} .$$

By a *fragment* of an $n \times n$ matrix we mean the image of this matrix under the projection p_X with certain X.

Denote

$$V^{kn} = \overline{p_X(\overline{\mathcal{C}_\pi})}$$

where $X = \{(k+1, n), (n, k+1), (k+2, n), (n, k+2), ...(n, n)\}$.

The variety V^{kn} corresponds to the fragments of V with empty entries in the n-th row and column: the last non-empty entry in the n-th column is in the row number k, all further positions in the n-th row and column are empty.

Observation 2 *Let V be a variety in $\mathbb{C}^n$ which is described by the polynomial equations*

$$f_1 (x_1, ..., x_n) = 0, f_2 (x_1, ..., x_n) = 0, ..., f_k (x_1, ..., x_n) = 0$$

and let $p : \mathbb{C}^n \to \mathbb{C}^{n-k}$ be the natural projection

$$p (x_1, x_2, ..., x_{n-k}, ..., x_{n-1}, x_n) = (x_1, x_2, ..., x_{n-k}) .$$

Then

$$\overline{p(V)} = \{(x_1, x_2, ..., x_{n-k}) \in \mathbb{C}^{n-k} : f_{i_1} = 0, f_{i_2} = 0, ..., f_{i_t} = 0\}$$

where the equations $f_{i_j} = 0$ appearing here are only those which do not include the variables x_{n-k+1}, x_{n-k+2},...,x_n, i.e. only those f_i whose partial derivatives by with respect to the variables x_{n-k+1}, x_{n-k+2},...,x_n are zeros.

Observation 3 *Note that since V^{kn} and $V^{k-1,n}$ are projections of the same variety $\overline{\mathcal{C}_\pi}$ and V^{kn} has one more coordinate than $V^{k-1,n}$, there are only two possibilities for their dimensions: $\dim V^{kn} = \dim V^{k-1,n}$ or $\dim V^{kn} = \dim V^{k-1,n} + 1$.*

(This is true since the rank of the Jacobian matrix can change only by 1 when we delete the rows corresponding to the coordinates.)

Now, let us start the course of the proof, by induction on n. For $n = 1$ the statement is obviously true.

Let π_n be any partial S_n involution. Denote by π_{n-1} its upper-left $n-1 \times n-1$ submatrix (which is an S_{n-1} partial involution by itself). Denote by $R(\pi_n)$, $(R(\pi_{n-1}))$ the corresponding rank-control matrices.

By the induction hypothesis, $\dim \overline{\mathcal{C}_{\pi_{n-1}}} = \frac{n^2-n}{2} - \mathfrak{D}(\pi_{n-1})$. Now we add to π_{n-1} the n-th column and consider the n-th column of $R(\pi_n)$. (We also add the n-th row but since our matrices are symmetric it suffices to check the dimension when we add the n-th column.) We added n new coordinates to the variety $\overline{\mathcal{C}_{\pi_{n-1}}}$ and we have to show that

$$\dim \overline{\mathcal{C}_\pi} = \dim \overline{\mathcal{C}_{\pi_{n-1}}} + n - \# \{(i, n) \,|\, 1 \leqslant i \leqslant n \quad \text{and} \quad r_{in} = r_{i-1,n-1}\} , \qquad (*)$$

The equality $(*)$ implies the statement of our theorem since $\frac{n^2-n}{2} + n = \frac{n^2+n}{2}$ and

$$\mathfrak{D}(\pi) = \mathfrak{D}(\pi_{n-1}) + \# \{(i, n) \,|\, 1 \leqslant i \leqslant n \quad \text{and} \quad r_{in} = r_{i-1,n-1}\} .$$

Obviously, if $r_{1,n} = 0$, then $a_{1,n} = 0$ for any $A = (a_{ij})_{i,j=1}^n \in \overline{\mathcal{C}_\pi}$. This itself is a polynomial equation which decreases the dimension by 1.

If, on the other hand, $r_{1,n} = 1$, it means that the rank of the first row is maximal and therefore, no equation is involved. In other words, the dimension of the variety V^{1n} is one more than the dimension of the variety V^{0n}, corresponding to and they have equal dimensions when $r_{1,n} = 0$.

Now move down along the n-th column of $R(\pi_n)$. Again, by induction, this time on the number of rows, assume that for each $1 \leqslant i \leqslant k-1$ $dim V^{in} = dim V^{i-1,n}$ if and only if $r_{i-1,n-1} = r_{i,n}$ while $\dim V^{in} = \dim V^{i-1,n} + 1$ if and only if $r_{i-1,n-1} < r_{i,n}$.

First, let $r_{k-1,n-1} = r_{k,n} = c$. Consider a matrix $A = (a_{ij})_{i,j=1}^n \in \overline{\mathcal{C}_\pi}$ and its upper-left $(k-1) \times (n-1)$ submatrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} \\ a_{21} & a_{22} & \cdots & a_{2,n-1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k-1,1} & a_{k-1,2} & \cdots & a_{k-1,n-1} \end{bmatrix}.$$

Using the notation introduced in Proposition 1, we denote this submatrix as $A_{k-1,n-1}$.

If $c = 0$, then $rank A_{kn} = 0$, so A_{kn} is a zero matrix and thus $\dim V^{in} = \dim V^{i-1,n} = 0$.

Let $c \neq 0$. Since $rank (A_{k-1,n-1}) = c$, we can take c linearly independent columns $\begin{bmatrix} a_{1,j_1} \\ a_{2,j_1} \\ \cdots \\ a_{k-1,j_1} \end{bmatrix}, \dots ,$ $\begin{bmatrix} a_{1,j_c} \\ a_{2,j_c} \\ \cdots \\ a_{k-1,j_c} \end{bmatrix}$ which span its column space. Now take only the linearly independent rows of the $(k-1) \times c$

matrix $\begin{bmatrix} a_{1,j_1} & \cdots & a_{1,j_c} \\ a_{2,j_1} & \cdots & a_{2,j_c} \\ \cdots & \cdots & \cdots \\ a_{k-1,j_1} & \cdots & a_{k-1,j_c} \end{bmatrix}$ to get a nonsingular $c \times c$ matrix $T_c = \begin{bmatrix} a_{i_1,j_1} & \cdots & a_{i_1,j_c} \\ a_{i_2,j_1} & \cdots & a_{i_2,j_c} \\ \cdots & \cdots & \cdots \\ a_{i_c,j_1} & \cdots & a_{i_c,j_c} \end{bmatrix}.$

The equality $r_{k-1,n-1} = r_{k,n} = c \leqslant k-1$ implies that any $(c+1) \times (c+1)$ minor of the matrix A_{kn} is zero, in particular

$$\det \begin{bmatrix} a_{i_1,j_1} & \cdots & a_{i_1,j_c} & a_{i_1,n} \\ a_{i_2,j_1} & \cdots & a_{i_2,j_c} & a_{i_2,n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i_c,j_1} & \cdots & a_{i_c,j_c} & a_{i_c,n} \\ a_{k,j_1} & \cdots & a_{k,j_c} & a_{k,n} \end{bmatrix} = 0$$

which is a polynomial equation. This equation is algebraically independent of the similar equations obtained for $1 \leqslant i \leqslant k-1$ since it contains a "new" variable – the entry $a_{k,n}$. It indeed involves the entry $a_{k,n}$ since $\det T_c \neq 0$. This equation means that the variable $a_{k,n}$ is not independent of the coordinates of the variety $V^{k-1,n}$, and therefore $\dim V^{k-1,n} = \dim V^{kn}$.

Now let $r_{k-1,n-1} < r_{k,n} = c$. We have to show that in this case the variable a_{nk} is independent of the coordinates of $V^{k-1,n}$, in other words, we have to show that there is no new equation. Consider the

fragment $\begin{bmatrix} r_{k-1,n-1} & r_{k-1,n} \\ r_{k-1,n} & r_{k,n} \end{bmatrix}$. There are four possible cases:

$$\begin{bmatrix} r_{k-1,n-1} & r_{k-1,n} \\ r_{k-1,n} & r_{k,n} \end{bmatrix} = \begin{bmatrix} c-1 & c-1 \\ c-1 & c \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} c-2 & c-1 \\ c-1 & c \end{bmatrix} \quad \text{or}$$

$$\begin{bmatrix} c-1 & c \\ c-1 & c \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} c-1 & c-1 \\ c & c \end{bmatrix}.$$

The equality $r_{k,n} = c$ implies that each $(c+1) \times (c+1)$ minor of A_{kn} is equal to zero, but we shall see that each such equation is not new, i.e. it is implied by the equality $r_{k,n-1} = c-1$ or by the equality $r_{k-1,n} = c-1$. In the first three cases we decompose the $(c+1) \times (c+1)$ determinant $\det \begin{bmatrix} \dots & \dots \\ \dots & a_{k,n} \end{bmatrix}$ using the last column. Since in all these cases $r_{k,n-1} = c-1$, each $c \times c$ minor of this decomposition (i.e. each $c \times c$ minor of $A_{k,n-1}$) is zero and therefore, this determinant is zero. In the fourth case we get the same if we decompose the determinant using the last row instead of the last column: since $r_{k-1,n} = c-1$, all the $c \times c$ minors of this decomposition (i.e. all $c \times c$ minor of $A_{k-1,n}$) are zeros and thus, our $(c+1) \times (c+1)$ determinant equals to zero. So there is no algebraic dependence between a_{kn} and the coordinates of $V^{k-1,n}$. Therefore, $\dim V^{kn} = \dim V^{k-1,n} + 1$. The case $k = n$ is the same as other cases when $k \leqslant n-1$. The proof is completed. □

Theorem 3 *Let $\pi \in S_n$ be an involution. Denote by $\mathcal{A}_\pi$ the orbit of anti-symmetric matrices which corresponds to π. Then*

$$\dim \overline{\mathcal{A}_\pi} = \frac{n^2 - n}{2} - \mathfrak{A}(\pi).$$

The proof is similar to the proof of Theorem 2 and can be found in [4].

9   Another formula for the rank function.

9.1   The symmetric case.

Obviously, an $n \times n$ partial involution matrix π can be described uniquely by the pair $(\tilde{\pi}, \{i_1, ..., i_k\})$, where $n - k$ is the rank of the matrix π, $\tilde{\pi} \in S_{n-k}$ such that $\tilde{\pi}^2 = Id$ is the regular (not partial) involution of the symmetric group S_{n-k} and the integers $i_1, ..., i_k$ are the numbers of zero rows (columns) in the matrix π.

   The following theorem is a generalization of the formula for the rank function of the Bruhat poset of the involutions of S_n given by Incitti in [6]. It is indeed the rank function because we already know that the rank function is the dimension (Proposition 6) and the dimension is determined by the parameter $\mathfrak{D}$ (Theorem 2). Recall that for $\sigma \in S_n$, $inv(\sigma) = \#\{(i,j)|i < j \,\&\, \sigma(i) > \sigma(j)\}$ and $exc(\sigma) = \#\{i|\sigma(i) > i\}$.

Proposition 7 *Following Incitti, denote by $Invol(G)$ the set of all involutions in the group G. Then for a partial permutation $\pi = (\tilde{\pi}, \{i_1, ..., i_k\})$, where $\tilde{\pi} \in Invol(S_{n-k})$ and the integers $i_1, ..., i_k$ are the numbers of zero rows (columns) in the matrix π is:*

$$\mathfrak{D}(\pi) = \frac{exc(\tilde{\pi}) + inv(\tilde{\pi})}{2} + \sum_{t=1}^{k}(n+1-i_t)$$

In other words, $\mathfrak{D}(\pi)$ equals to the length of $\tilde{\pi}$ in the poset of the involutions of the group S_{n-k} plus the sum of the numbers of zero rows of the matrix π, where the numbers are taken in the opposite order, i.e. the n-th row is labeled by 1, the $(n-1)$-th row is labeled by 2,..., the first row is labeled by n.

Comment 1 *The Bruhat poset of regular (not partial) involutions of S_n is a graded poset with the rank function given by the formula*

$$\mathfrak{D}(\sigma) = \frac{exc(\sigma) + inv(\sigma)}{2} \ ,$$

where $\sigma \in Invol(S_n)$.

The proofs of Proposition 7 and Corollary 1 can be found in [1].

9.2   The anti-symmetric case.

Here we don't distinguish between an involution $\pi \in S_n$ and the monomial anti-symmetric matrix (with minuses in the lower triangle) associated to π by the bijection presented in Proposition 2.

Definition 9 *Let $\pi \in S_n$ be an involution. It is always possible to write it as product of disjoint transpositions*

$$\pi = (i_1, j_1)\,(i_2, j_2) \cdots (i_k, j_k)$$

in such a way that for all $1 \leqslant t \leqslant k$, $i_t < j_t$ and $i_1 < i_2 < \cdots < i_k$. Let us call it "the canonic form".
   *Denote by $\mathfrak{I}(\pi)$ the **number of inversions** in the word $i_1 j_1 i_2 j_2 \cdots i_k j_k$.*

Proposition 8 *Let $\pi \in S_n$ be an involution. Then*

$$\mathfrak{A}(\pi) = \mathfrak{I}(\pi) + \sum_{a \,:\, \pi(a)=a} (n-a)\,.$$

The proof can be found in [4].
   The proofs of Propositions 7 and 8 are done by induction and use Theorems 2 and 3 respectively.

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