Mixed Statistics on 01-Fillings of Moon Polyominoes

William Y. C. Chen[1] and Andrew Y. Z. Wang[1] and Catherine H. Yan[2] and Alina F. Y. Zhao[1†]

[1]*Center for Combinatorics, LPMC-TJKLC*
Nankai University, Tianjin 300071, P. R. China
[2] *Department of Mathematics*
Texas A&M University, College Station, TX 77843, USA

Abstract. We establish a stronger symmetry between the numbers of northeast and southeast chains in the context of 01-fillings of moon polyominoes. Let $\mathcal{M}$ be a moon polyomino. Consider all the 01-fillings of $\mathcal{M}$ in which every row has at most one 1. We introduce four mixed statistics with respect to a bipartition of rows or columns of $\mathcal{M}$. More precisely, let S be a subset of rows of $\mathcal{M}$. For any filling M, the top-mixed (resp. bottom-mixed) statistic $\alpha(S;M)$ (resp. $\beta(S;M)$) is the sum of the number of northeast chains whose top (resp. bottom) cell is in S, together with the number of southeast chains whose top (resp. bottom) cell is in the complement of S. Similarly, we define the left-mixed and right-mixed statistics $\gamma(T;M)$ and $\delta(T;M)$, where T is a subset of the columns. Let $\lambda(A;M)$ be any of these four statistics $\alpha(S;M)$, $\beta(S;M)$, $\gamma(T;M)$ and $\delta(T;M)$. We show that the joint distribution of the pair $(\lambda(A;M), \lambda(M/A;M))$ is symmetric and independent of the subsets S, T. In particular, the pair of statistics $(\lambda(A;M), \lambda(M/A;M))$ is equidistributed with $(\mathrm{se}(M), \mathrm{ne}(M))$, where $\mathrm{se}(M)$ and $\mathrm{ne}(M)$ are the numbers of southeast chains and northeast chains of M, respectively.

Résumé. Nous établissons une symétrie plus forte entre les nombres de chaînes nord-est et sud-est dans le cadre des remplissages 01 des polyominos lune. Soit $\mathcal{M}$ un polyomino lune. Considérez tous les remplissages 01 de $\mathcal{M}$ dans lesquels chaque rangée contient au plus un 1. Nous présentons quatre statistiques mixtes sur les bipartitions des rangées et des colonnes de $\mathcal{M}$. Plus précisément, soit S un sous-ensemble de rangées de $\mathcal{M}$. Pour tout remplissage M, la statistique mixte du dessus (resp. du dessous) $\alpha(S;M)$ (resp. $\beta(S;M)$) est la somme du nombre de chaînes nord-est dont le dessus (resp. le dessous) est dans S, et du nombre de chaînes sud-est dont la cellule supérieure (resp. inférieure) est dans le complément de S. De même, nous définissons les statistiques mixtes à gauche et à droite $\gamma(T;M)$ et $\delta(T;M)$, où T est un sous-ensemble des colonnes. Soit $\lambda(A;M)$ une des quatre statistiques $\alpha(S;M)$, $\beta(S;M)$, $\gamma(T;M)$ et $\delta(T;M)$. Nous montrons que la distribution commune des paires $(\lambda(A;M), \lambda(M/A;M))$ est symétrique et indépendante des sous-ensembles S, T. En particulier, la paire de statistiques $(\lambda(A;M), \lambda(M/A;M))$ est équidistribuée avec $(\mathrm{se}(M), \mathrm{ne}(M))$, où $\mathrm{se}(M)$ et $\mathrm{ne}(M)$ sont les nombres de chaînes sud-est et nord-est de M respectivement.

Keywords: mixed statistic, polyomino, symmetric distribution.

[†]The first, the second, and the fourth authors were supported by the 973 Project, the PCSIRT Project of the Ministry of Education, and the National Science Foundation of China. The third author was supported in part by a US-NSF grant #DMS-0653846. The authors thank Peter McNamara and Alain Goupil for the french translation of the abstract.

1 Introduction

Recently it is observed that the numbers of crossings and nestings have a symmetric distribution over many families of combinatorial objects, such as matchings and set partitions. Recall that a matching of $[2n] = \{1, 2, \ldots, 2n\}$ is a partition of the set $[2n]$ with the property that each block has exactly two elements. It can be represented as a graph with vertices $1, 2, \ldots, 2n$ drawn on a horizontal line in increasing order, where two vertices i and j are connected by an edge if and only if $\{i, j\}$ is a block. We say that two edges (i_1, j_1) and (i_2, j_2) form a *crossing* if $i_1 < i_2 < j_1 < j_2$; they form a *nesting* if $i_1 < i_2 < j_2 < j_1$. The symmetry of the joint distribution of crossings and nestings follows from the bijections of de Sainte-Catherine, who also found the generating functions for the number of crossings and the number of nestings. Klazar [12] further studied the distribution of crossings and nestings over the set of matchings obtained from a given matching by successfully adding edges.

The symmetry between crossings and nestings was extended by Kasraoui and Zeng [11] to set partitions, and by Chen, Wu and Yan [3] to linked set partitions. Poznanović and Yan [15] determined the distribution of crossings and nestings over the set of partitions which are identical to a given partition π when restricted to the last n elements.

Many classical results on enumerative combinatorics can be put in the larger context of counting submatrices in fillings of certain polyominoes. For example, words and permutations can be represented as 01-fillings of rectangular boards, and general graphs can be represented as $\mathbb{N}$-fillings of arbitrary Ferrers shapes, as studied by [13, 6, 7]. Other polyominoes studied include stack polyominoes [9], and moon polyominoes [16, 10]. It is well-known that crossings and nestings in matchings and set partitions correspond to northeast chains and southeast chains of length 2 in a filling of polyominoes. The symmetry between crossings and nestings has been extended by Kasraoui [10] to 01-fillings of moon polyominoes where either every row has at most one 1, or every column has at most one 1. In both cases, the joint distribution of the numbers of northeast and southeast chains can be expressed as a product of p, q-Gaussian coefficients. Other known statistics on fillings of moon polyominoes are the length of the longest northeast/southeast chains [2, 13, 16], and the major index [4].

The main objective of this paper is to present a stronger symmetry between the numbers of northeast and southeast chains in the context of 01-fillings of moon polyominoes. Given a bipartition of the rows (or columns) of a moon polyomino, we define four statistics by considering mixed sets of northeast and southeast chains according to the bipartition. Let M be a 01-filling of a moon polyomino $\mathcal{M}$ with n rows and m columns. These statistics are the top-mixed and the bottom-mixed statistics $\alpha(S; M), \beta(S; M)$ with respect to a row-bipartition $(S, \bar{S})$, and the left-mixed and the right-mixed statistics $\gamma(T; M), \delta(T; M)$ with respect to a column-bipartition $(T, \bar{T})$. We show that for any of these four statistics $\lambda(A; M)$, namely, $\alpha(S; M), \beta(S; M)$ for $S \subseteq [n]$ and $\gamma(T; M), \delta(T; M)$ for $T \subseteq [m]$, the joint distribution of the pair $(\lambda(A; M), \lambda(\bar{A}; M))$ is symmetric and independent of the subsets S, T. Consequently, we have the equidistribution

$$\sum_M p^{\lambda(A;M)} q^{\lambda(\bar{A};M)} = \sum_M p^{\mathrm{se}(M)} q^{\mathrm{ne}(M)},$$

where M ranges over all 01-fillings of $\mathcal{M}$ with the property that either every row has at most one 1, or every column has at most one 1, and $\mathrm{se}(M)$ and $\mathrm{ne}(M)$ are the numbers of southeast and northeast chains of M, respectively.

The paper is organized as follows. Section 2 contains necessary notation and the statements of the main results. We present the proofs in Section 3, and show by bijections in Section 4 that these new statistics are invariant under a permutation of columns or rows on moon polyominoes.

2 Notation and the Main Results

A *polyomino* is a finite subset of $\mathbb{Z}^2$, where every element of $\mathbb{Z}^2$ is represented by a square cell. The polyomino is *convex* if its intersection with any column or row is connected. It is *intersection-free* if every two columns are comparable, i.e., the row-coordinates of one column form a subset of those of the other column. Equivalently, it is intersection-free if every two rows are comparable. A *moon polyomino* is a convex and intersection-free polyomino.

Given a moon polyomino $\mathcal{M}$, we assign 0 or 1 to each cell of $\mathcal{M}$ so that there is at most one 1 in each row. Throughout this paper we will simply use the term *filling* to denote such 01-fillings. We say that a cell is empty if it is assigned 0, and it is a 1-cell otherwise. Assume $\mathcal{M}$ has n rows and m columns. We label the rows $R_1, \dots, R_n$ from top to bottom, and the columns $C_1, \dots, C_m$ from left to right. Let $\mathbf{e} = (\varepsilon_1, \dots, \varepsilon_n) \in \{0,1\}^n$ and $\mathbf{s} = (s_1, \dots, s_m) \in \mathbb{N}^m$ with $\sum_{i=1}^n \varepsilon_i = \sum_{j=1}^m s_j$. We denote by $\mathbf{F}(\mathcal{M}, \mathbf{e}, \mathbf{s})$ the set of fillings M of $\mathcal{M}$ such that the row R_i has exactly ε_i many 1's, and the column C_j has exactly s_j many 1's, for $1 \le i \le n$ and $1 \le j \le m$. See Figure 1 for an illustration.

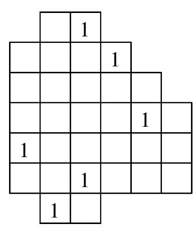

Fig. 1: A filling M with $\mathbf{e} = (1,1,0,1,1,1,1)$ and $\mathbf{s} = (1,1,2,1,1,0)$.

A *northeast* (resp. *southeast*) *chain* in a filling M of $\mathcal{M}$ is a set of two 1-cells such that one of them is strictly above (resp. below) and to the right of the other and the smallest rectangle containing them is contained in $\mathcal{M}$. Northeast (resp. southeast) chains will be called NE (resp. SE) chains. The number of NE (resp. SE) chains of M is denoted by $\mathrm{ne}(M)$ (resp. $\mathrm{se}(M)$). It is proved by Kasraoui [10] that $\mathrm{ne}(M)$ and $\mathrm{se}(M)$ have a symmetric joint distribution over $\mathbf{F}(\mathcal{M}, \mathbf{e}, \mathbf{s})$.

Theorem 2.1

$$\sum_{M \in \mathbf{F}(\mathcal{M},\mathbf{e},\mathbf{s})} p^{\mathrm{ne}(M)} q^{\mathrm{se}(M)} = \sum_{M \in \mathbf{F}(\mathcal{M},\mathbf{e},\mathbf{s})} p^{\mathrm{se}(M)} q^{\mathrm{ne}(M)} = \prod_{i=1}^{m} \begin{bmatrix} h_i \\ s_i \end{bmatrix}_{p,q}.$$

Let $\mathcal{R}$ be the set of rows of the moon polyomino $\mathcal{M}$. For $S \subseteq [n]$, let $\mathcal{R}(S) = \bigcup_{i \in S} R_i$. We say a 1-cell is an *S-cell* if it lies in $\mathcal{R}(S)$. An NE chain is called a *top S-NE chain* if its northeast 1-cell is an S-cell. Similarly, an SE chain is called a *top S-SE chain* if its northwest 1-cell is an S-cell. In other words, an NE/SE chain is a top S-NE/SE chain if the upper 1-cell of the chain is in $\mathcal{R}(S)$. Similarly, an NE/SE chain is a *bottom S-NE/SE chain* if the lower 1-cell of the chain is in $\mathcal{R}(S)$.

Let $\bar{S} = [n] \setminus S$ be the complement of S. Given a filling $M \in \mathbf{F}(\mathcal{M}, \mathbf{e}, \mathbf{s})$, we define the *top-mixed statistic* $\alpha(S; M)$ and the *bottom-mixed statistic* $\beta(S; M)$ *with respect to* S as

$$\begin{aligned} \alpha(S; M) &= \#\{\text{top } S\text{-NE chain of } M\} + \#\{\text{top } \bar{S}\text{-SE chain of } M\}, \\ \beta(S; M) &= \#\{\text{bottom } S\text{-NE chain of } M\} + \#\{\text{bottom } \bar{S}\text{-SE chain of } M\}. \end{aligned}$$

See Example 2.3 for some of these statistics on the filling M in Figure 1.

Let $F_S^t(p, q)$ and $F_S^b(p, q)$ be the bi-variate generating functions for the pairs $(\alpha(S; M), \alpha(\bar{S}; M))$ and $(\beta(S; M), \beta(\bar{S}; M))$ respectively, namely,

$$F_S^t(p, q) = \sum_{M \in \mathbf{F}(\mathcal{M}, \mathbf{e}, \mathbf{s})} p^{\alpha(S;M)} q^{\alpha(\bar{S};M)} \quad \text{and} \quad F_S^b(p, q) = \sum_{M \in \mathbf{F}(\mathcal{M}, \mathbf{e}, \mathbf{s})} p^{\beta(S;M)} q^{\beta(\bar{S};M)}.$$

Note that

$$(\alpha(\emptyset; M), \alpha([n]; M)) = (\beta(\emptyset; M), \beta([n]; M)) = (\mathrm{se}(M), \mathrm{ne}(M)).$$

Our first result is the following property.

Theorem 2.2 $F_S^t(p, q) = F_{S'}^t(p, q)$ *for any two subsets* S, S' *of* $[n]$*. In other words, the bi-variate generating function* $F_S^t(p, q)$ *does not depend on* S*. Consequently,*

$$F_S^t(p, q) = F_\emptyset^t(p, q) = \sum_{M \in \mathbf{F}(\mathcal{M}, \mathbf{e}, \mathbf{s})} p^{\mathrm{se}(M)} q^{\mathrm{ne}(M)}$$

is a symmetric function. The same statement holds for $F_S^b(p, q)$*.*

We can also define the mixed statistics with respect to a subset of columns. Let $\mathcal{C}$ be the set of columns of $\mathcal{M}$. For $T \subseteq [m]$, let $\mathcal{C}(T) = \bigcup_{j \in T} C_j$. An NE chain is called a *left* T*-NE chain* if the southwest 1-cell of the chain lies in $\mathcal{C}(T)$. Similarly, an SE chain is called a *left* T*-SE chain* if the northwest 1-cell of the chain lies in $\mathcal{C}(T)$. In other words, an NE/SE chain is a left T-NE/SE chain if its left 1-cell is in $\mathcal{C}(T)$. Similarly, an NE/SE chain is a *right* T*-NE/SE chain* if its right 1-cell is in $\mathcal{C}(T)$.

Let $\bar{T} = [m] \setminus T$ be the complement of T. For any filling M of $\mathbf{F}(\mathcal{M}, \mathbf{e}, \mathbf{s})$, we define the *left-mixed statistic* $\gamma(T; M)$ and the *right-mixed statistic* $\delta(T; M)$ *with respect to* T as

$$\begin{aligned} \gamma(T; M) &= \#\{\text{left } T\text{-NE chain of } M\} + \#\{\text{left } \bar{T}\text{-SE chain of } M\}, \\ \delta(T; M) &= \#\{\text{right } T\text{-NE chain of } M\} + \#\{\text{right } \bar{T}\text{-SE chain of } M\}. \end{aligned}$$

Example 2.3 Let M be the filling in Figure 1, where $\mathrm{ne}(M) = 6$ and $\mathrm{se}(M) = 1$. Let $S = \{2, 4\}$, i.e., $\mathcal{R}(S)$ contains the second and the fourth rows. Then

$$\alpha(S; M) = 5, \quad \alpha(\bar{S}; M) = 2, \quad \beta(S; M) = 1, \quad \beta(\bar{S}; M) = 6.$$

Let $T = \{1, 3\}$, i.e., $\mathcal{C}(T)$ contains the first and the third columns. Then

$$\gamma(T; M) = 4, \quad \gamma(\bar{T}; M) = 3, \quad \delta(T; M) = 2, \quad \delta(\bar{T}; M) = 5. \qquad \blacksquare$$

Let $G_T^l(p,q)$ and $G_T^r(p,q)$ be the bi-variate generating functions of the pairs $(\gamma(T;M),\gamma(\bar{T};M))$ and $(\delta(T;M),\delta(\bar{T};M))$ respectively, namely,

$$G_T^l(p,q)=\sum_{M\in\mathbf{F}(\mathcal{M},\mathbf{e},\mathbf{s})}p^{\gamma(T;M)}q^{\gamma(\bar{T};M)}\quad\text{and}\quad G_T^r(p,q)=\sum_{M\in\mathbf{F}(\mathcal{M},\mathbf{e},\mathbf{s})}p^{\delta(T;M)}q^{\delta(\bar{T};M)}.$$

Again note that

$$(\gamma(\emptyset;M),\gamma([m];M))=(\delta(\emptyset;M),\delta([m];M))=(\mathrm{se}(M),\mathrm{ne}(M)).$$

Our second result shows that the generating function $G_T^l(p,q)$ possesses a similar property as $F_S^t(p,q)$.

Theorem 2.4 *$G_T^l(p,q)=G_{T'}^l(p,q)$ for any two subsets T,T' of $[m]$. In other words, the bi-variate generating function $G_T^l(p,q)$ does not depend on T. Consequently,*

$$G_T^l(p,q)=G_\emptyset^l(p,q)=\sum_{M\in\mathbf{F}(\mathcal{M},\mathbf{e},\mathbf{s})}p^{\mathrm{se}(M)}q^{\mathrm{ne}(M)}$$

is a symmetric function. The same statement holds for $G_T^r(p,q)$.

We notice that the set $\mathbf{F}(\mathcal{M},\mathbf{e},\mathbf{s})$ appeared as $\mathcal{N}^r(T,\mathbf{m},A)$ in Kasraoui [10], where $\mathbf{m}$ is the column sum vector, and A is the set of empty rows, i.e., $A=\{i:\varepsilon_i=0\}$. Kasraoui also considered the set $\mathcal{N}^c(T,\mathbf{n},B)$ of fillings whose row sum is an arbitrary $\mathbb{N}$-vector $\mathbf{n}$ under the condition that there is at most one 1 in each column and where B is the set of empty columns. By a rotation of moon polyominoes, it is easily seen that Theorem 2.2 and Theorem 2.4 also hold for the set $\mathcal{N}^c(T,\mathbf{n},B)$, as well as for the set of fillings such that there is at most one 1 in each row and in each column.

As an interesting example, we explain how Theorems 2.2 and 2.4 specialize to permutations and words, which are in bijections with fillings of squares or rectangles. More precisely, a word $w=w_1w_2\cdots w_n$ on $[m]$ can be represented as a filling M on an $n\times m$ rectangle $\mathcal{M}$ in which the cell in row $n+1-i$ and column j is assigned the integer 1 if and only if $w_i=j$. In the word $w_1w_2\cdots w_n$, a pair (w_i,w_j) is an *inversion* if $i<j$ and $w_i>w_j$; we say that it is a *co-inversion* if $i<j$ and $w_i<w_j$, see also [14]. Denote by $\mathrm{inv}(w)$ the number of inversions of w, and by $\mathrm{coinv}(w)$ the number of co-inversions of w.

For $S\subseteq[n]$ and $T\subseteq[m]$, we have

$$\begin{aligned}
\alpha(S;w) &= \#\{(w_i,w_j):n+1-j\in S\text{ and }(w_i,w_j)\text{ is a co-inversion}\}\\
&\qquad +\#\{(w_i,w_j):n+1-j\notin S\text{ and }(w_i,w_j)\text{ is an inversion}\},\\
\beta(S;w) &= \#\{(w_i,w_j):n+1-i\in S\text{ and }(w_i,w_j)\text{ is a co-inversion}\}\\
&\qquad +\#\{(w_i,w_j):n+1-i\notin S\text{ and }(w_i,w_j)\text{ is an inversion}\}.\\
\gamma(T,w) &= \#\{(w_i,w_j):w_i\in T\text{ and }(w_i,w_j)\text{ is a co-inversion}\}\\
&\qquad +\#\{(w_i,w_j):w_j\notin T\text{ and }(w_i,w_j)\text{ is an inversion}\},\\
\delta(T,w) &= \#\{(w_i,w_j):w_j\in T\text{ and }(w_i,w_j)\text{ is a co-inversion}\}\\
&\qquad +\#\{(w_i,w_j):w_i\notin T\text{ and }(w_i,w_j)\text{ is an inversion}\}.
\end{aligned}$$

Let $W=\{1^{s_1},2^{s_2},\ldots,m^{s_m}\}$ be a multiset with $s_1+\cdots+s_m=n$, and $R(W)$ be the set of permutations, also called rearrangements, of the elements in W. Let $\lambda(A;w)$ denote any of the four statistics

$\alpha(S;w), \beta(S;w), \gamma(T;w), \delta(T;w)$. Theorems 2.2 and 2.4 imply that the bi-variate generating function for $(\lambda(A;w), \lambda(\bar{A};w))$ is symmetric and

$$\sum_{w\in R(W)} p^{\lambda(A;w)} q^{\lambda(\bar{A};w)} = \sum_{w\in R(W)} p^{\mathrm{inv}(w)} q^{\mathrm{coinv}(w)} = \begin{bmatrix} n \\ s_1, \ldots, s_m \end{bmatrix}_{p,q}, \tag{1}$$

where $\left[\begin{smallmatrix} n \\ s_1,\ldots,s_m \end{smallmatrix}\right]_{p,q}$ is the p,q-Gaussian coefficient with the p,q-integer $[r]_{p,q}$ given by $[r]_{p,q} = p^{r-1} + p^{r-2}q + \cdots + pq^{r-2} + q^{r-1}$.

3 Proof of the Main Results

It is sufficient to prove our results for $\alpha(S;M)$ and $\gamma(T;M)$ only, since conclusions for $\beta(S;M)$ and $\delta(T;M)$ can be obtained by reflecting the moon polyomino with respect to a horizontal line or a vertical line.

In Subsection 3.1, we recall Kasraoui's bijection Ψ from $\mathbf{F}(\mathcal{M}, \mathbf{e}, \mathbf{s})$ to sequences of compositions [10]. Kasraoui's construction is stated for the set $\mathcal{N}^c(T, \mathbf{n}, B)$. We shall modify the description to fit our notation. This bijection will be used in the proof of Lemma 3.2 which states that the pair of the top-mixed statistics $(\alpha(\{1\};M), \alpha(\overline{\{1\}};M))$ is equidistributed with $(\mathrm{se}(M), \mathrm{ne}(M))$. Theorem 2.2 follows from an iteration of Lemma 3.2. In Subsection 3.3 we prove Theorem 2.4. Again the crucial step is the observation that $(\gamma(\{1\};M), \gamma(\overline{\{1\}};M))$ has the same distribution as $(\mathrm{se}(M), \mathrm{ne}(M))$.

Due to the space limit, in this extended abstract we would just describe the main ideas and the construction of the bijections, and leave out the detailed proofs. A complete version of the present paper is available in [5].

3.1 Kasraoui's bijection Ψ

Assume the columns of $\mathcal{M}$ are $C_1, \ldots, C_m$ from left to right. Let $|C_i|$ be the length of the column C_i. Assume that k is the smallest index such that $|C_k| \geq |C_i|$ for all i. Define the left part of $\mathcal{M}$, denoted $L(\mathcal{M})$, to be the union $\cup_{1\leq i\leq k-1} C_i$, and the right part of $\mathcal{M}$, denoted $R(\mathcal{M})$, to be the union $\cup_{k\leq i\leq m} C_i$. Note that the columns of maximal length in $\mathcal{M}$ belong to $R(\mathcal{M})$.

We order the columns $C_1, \ldots, C_m$ by a total order $\prec$ as follows: $C_i \prec C_j$ if and only if

- $|C_i| < |C_j|$ or
- $|C_i| = |C_j|$, $C_i \in L(\mathcal{M})$ and $C_j \in R(\mathcal{M})$, or
- $|C_i| = |C_j|$, $C_i, C_j \in L(\mathcal{M})$ and C_i is on the left of C_j, or
- $|C_i| = |C_j|$, $C_i, C_j \in R(\mathcal{M})$ and C_i is on the right of C_j.

For every column $C_i \in L(\mathcal{M})$, we define the rectangle $\mathcal{M}(C_i)$ to be the largest rectangle that contains C_i as the leftmost column. For $C_i \in R(\mathcal{M})$, the rectangle $\mathcal{M}(C_i)$ is taken to be the largest rectangle that contains C_i as the rightmost column and does not contain any column $C_j \in L(\mathcal{M})$ such that $C_j \prec C_i$.

Given $M \in \mathbf{F}(\mathcal{M}, \mathbf{e}, \mathbf{s})$, we define a coloring of M by the following steps.

The coloring of the filling M

1. Color the cells of empty rows;

2. For each $C_i \in L(\mathcal{M})$, color the cells which are contained in the rectangle $\mathcal{M}(C_i)$ and on the right of any 1-cell in C_i.

3. For each $C_i \in R(\mathcal{M})$, color the cells which are contained in the rectangle $\mathcal{M}(C_i)$ and on the left of any 1-cell in C_i.

For $M \in \mathbf{F}(\mathcal{M}, \mathbf{e}, \mathbf{s})$, let a_i be the number of empty rows (i.e., $\{R_i : \varepsilon_i = 0\}$) that intersect the column C_i. Suppose that $C_{i_1} \prec C_{i_2} \prec \cdots \prec C_{i_m}$. For $j = 1, \ldots, m$, we define

$$h_{i_j} = |C_{i_j}| - a_{i_j} - (s_{i_1} + s_{i_2} + \cdots + s_{i_{j-1}}). \tag{2}$$

For positive integers n and k, denote by $\mathcal{C}_k(n)$ the set of compositions of n into k nonnegative parts, that is, $\mathcal{C}_k(n) = \{(\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{N}^k : \sum_{i=1}^k \lambda_i = n\}$. The bijection Ψ is constructed as follows.

The bijection $\Psi : \mathbf{F}(\mathcal{M}, \mathbf{e}, \mathbf{s}) \longrightarrow \mathcal{C}_{s_1+1}(h_1 - s_1) \times \mathcal{C}_{s_2+1}(h_2 - s_2) \times \cdots \times \mathcal{C}_{s_m+1}(h_m - s_m)$.

For each $M \in \mathbf{F}(\mathcal{M}, \mathbf{e}, \mathbf{s})$ with the coloring, $\Psi(M)$ is a sequence of compositions $(c^{(1)}, c^{(2)}, \ldots, c^{(m)})$, where

- $c^{(i)} = (0)$ if $s_i = 0$. Otherwise
- $c^{(i)} = (c_1^{(i)}, c_2^{(i)}, \ldots, c_{s_i+1}^{(i)})$ where
  - $c_1^{(i)}$ is the number of uncolored cells above the first 1-cell in the column C_i;
  - $c_k^{(i)}$ is the number of uncolored cells between the $(k-1)$-st and the k-th 1-cells in the column C_i, for $2 \le k \le s_i$;
  - $c_{s_i+1}^{(i)}$ is the number of uncolored cells below the last 1-cell in the column C_i.

The statistics $\mathrm{ne}(M)$ and $\mathrm{se}(M)$ can be written in terms of the compositions.

Theorem 3.1 *Let $M \in \mathbf{F}(\mathcal{M}, \mathbf{e}, \mathbf{s})$ and $\mathbf{c} = \Psi(M) = (c^{(1)}, c^{(2)}, \ldots, c^{(m)})$. Then*

$$\mathrm{ne}(M) = \sum_{C_i \in L(\mathcal{M})} \sum_{k=1}^{s_i} (c_1^{(i)} + c_2^{(i)} + \cdots + c_k^{(i)}) + \sum_{C_j \in R(\mathcal{M})} \sum_{k=1}^{s_j} (h_j - s_j - c_1^{(j)} - c_2^{(j)} - \cdots - c_k^{(j)}),$$

$$\mathrm{se}(M) = \sum_{C_i \in L(\mathcal{M})} \sum_{k=1}^{s_i} (h_i - s_i - c_1^{(i)} - c_2^{(i)} - \cdots - c_k^{(i)}) + \sum_{C_j \in R(\mathcal{M})} \sum_{k=1}^{s_j} (c_1^{(j)} + c_2^{(j)} + \cdots + c_k^{(j)}).$$

Summing over the sequences of compositions yields the symmetric generating function of $\mathrm{ne}(M)$ and $\mathrm{se}(M)$, c.f. Theorem 2.1.

3.2 Proof of Theorem 2.2

To prove Theorem 2.2 for the top-mixed statistic $\alpha(S; M)$, we first consider the special case when $\mathcal{R}(S)$ contains the first row only.

Lemma 3.2 *For $S = \{1\}$, we have*

$$F_{\{1\}}^t(p, q) = F_\emptyset^t(p, q) = \sum_{M \in \mathbf{F}(\mathcal{M}, \mathbf{e}, \mathbf{s})} p^{\mathrm{se}(M)} q^{\mathrm{ne}(M)}.$$

Proof: We assume that the first row is nonempty. Otherwise the identity is obvious. Given a filling $M \in \mathbf{F}(\mathcal{M}, \mathbf{e}, \mathbf{s})$, assume that the unique 1-cell of the first row lies in the column C_t. Let the upper polyomino $\mathcal{M}_u$ be the union of the rows that intersect C_t, and the lower polyomino $\mathcal{M}_d$ be the complement of $\mathcal{M}_u$, i.e., $\mathcal{M}_d = \mathcal{M} \setminus \mathcal{M}_u$. We aim to construct a bijection $\phi_\alpha : \mathbf{F}(\mathcal{M}, \mathbf{e}, \mathbf{s}) \to \mathbf{F}(\mathcal{M}, \mathbf{e}, \mathbf{s})$ such that for any filling M,

$$(\alpha(\{1\}; M), \alpha(\overline{\{1\}}; M)) = (\mathrm{se}(\phi_\alpha(M)), \mathrm{ne}(\phi_\alpha(M))),$$

and $\phi_\alpha(M)$ is identical to M on $\mathcal{M}_d$ (which depends on M).

Let $M_u = M \cap \mathcal{M}_u$ and $M_d = M \cap \mathcal{M}_d$. Let s'_i be the number of 1-cells of M in the column $C_i \cap \mathcal{M}_u$, and $\mathbf{s}' = (s'_1, \ldots, s'_m)$. Let $\mathbf{e}' = (\varepsilon_1, \ldots, \varepsilon_r)$, where r is the number of rows in $\mathcal{M}_u$. We shall define ϕ_α on $\mathbf{F}(\mathcal{M}_u, \mathbf{e}', \mathbf{s}')$ first.

Let $C'_i = C_i \cap \mathcal{M}_u$. Suppose that in $\mathcal{M}$ the columns intersecting with the first row are $C_a, \ldots, C_t, \ldots, C_b$ from left to right. Then $C_t = C'_t$, and in $\mathcal{M}_u$ the columns $C'_a, \ldots, C'_t, \ldots, C'_b$ intersect the first row. Assume that among them the ones with the same length as C'_t are $C'_u, \ldots, C'_t, \ldots, C'_v$ from left to right. Clearly, the columns $C'_u, \ldots, C'_t, \ldots, C'_v$ are those with maximal length and belong to $R(\mathcal{M}_u)$. Note that in M_u, the number of top $\{1\}$-NE chains is $\sum_{a \le i < t} s'_i$, while the number of top $\{1\}$-SE chains is $\sum_{t < i \le b} s'_i$. Let h'_i be given as in Eq. (2) for $\mathbf{F}(\mathcal{M}_u, \mathbf{e}', \mathbf{s}')$. Let $\mathbf{c} = \Psi(M_u) = (c^{(1)}, c^{(2)}, \ldots, c^{(m)})$. Then we can compute that

$$\begin{aligned} \alpha(\{1\}; M_u) \quad &= \quad \sum_{a \le i < u} s'_i + (h'_t - s'_t) + \sum_{C'_i \in L(\mathcal{M}_u)} \sum_{k=1}^{s'_i} (h'_i - s'_i - c_1^{(i)} - c_2^{(i)} - \cdots - c_k^{(i)}) \\ &\quad + \sum_{C'_j \in R(\mathcal{M}_u)} \sum_{k=1}^{s'_j} (c_1^{(j)} + c_2^{(j)} + \cdots + c_k^{(j)}) - \sum_{t < i \le b} s'_i. \end{aligned} \tag{3}$$

and

$$\begin{aligned} \alpha(\overline{\{1\}}; M_u) \quad &= \quad \sum_{t < i \le b} s'_i + \sum_{C'_i \in L(\mathcal{M}_u)} \sum_{k=1}^{s'_i} (c_1^{(i)} + c_2^{(i)} + \cdots + c_k^{(i)}) \\ &\quad + \sum_{C'_j \in R(\mathcal{M}_u)} \sum_{k=1}^{s'_j} (h'_j - s'_j - c_1^{(j)} - c_2^{(j)} - \cdots - c_k^{(j)}) - \sum_{a \le i < u} s'_i - (h'_t - s'_t). \end{aligned} \tag{4}$$

The fact that the 1-cell of the first row lies in the column C'_t implies that $c_1^{(t)} = 0$, and $c_1^{(i)} > 0$ for $a \le i < u$ or $t < i \le b$. We define the filling $\phi_\alpha(M_u)$ by setting $\phi_\alpha(M_u) = \Psi^{-1}(\tilde{\mathbf{c}})$, where $\tilde{\mathbf{c}}$ is obtained from $\mathbf{c}$ as follows:

$$\begin{cases} \tilde{c}^{(i)} = (c_1^{(i)} - 1, c_2^{(i)}, \ldots, c_{s_i}^{(i)}, c_{s_i+1}^{(i)} + 1), & \text{if } a \le i < u \text{ or } t < i \le b, \text{ and } s'_i \ne 0, \\ \tilde{c}^{(t)} = (c_2^{(t)}, c_3^{(t)}, \ldots, c_{s_t+1}^{(t)}, c_1^{(t)}), & \text{if } i = t, \\ \tilde{c}^{(i)} = c^{(i)}, & \text{for any other } i. \end{cases}$$

Comparing the formulas (3) and (4) with Theorem 3.1 for $\tilde{\mathbf{c}}$, one easily verifies that

$$(\alpha(\{1\}; M_u), \alpha(\overline{\{1\}}; M_u)) = (\mathrm{se}(\phi_\alpha(M_u)), \mathrm{ne}(\phi_\alpha(M_u))).$$

Now $\phi_\alpha(M)$ is obtained from M by replacing M_u with $\phi_\alpha(M_u)$. $\square$

Proposition 3.3 *Assume* $S = \{r_1, r_2, \ldots, r_s\} \subseteq [n]$ *with* $r_1 < r_2 < \cdots < r_s$. *Let* $S' = \{r_1, r_2, \ldots, r_{s-1}\}$. *Then* $F_S^t(p,q) = F_{S'}^t(p,q)$.

Proof: Let $X = \{R_i : 1 \le i < r_s\}$ be the set of rows above the row R_{r_s}, and Y be the set of remaining rows. Given a filling $M \in \mathbf{F}(\mathcal{M}, \mathbf{e}, \mathbf{s})$, let $\mathcal{T}(M)$ be the set of fillings $M' \in \mathbf{F}(\mathcal{M}, \mathbf{e}, \mathbf{s})$ that are identical to M in the rows of X. Construct a map $\theta_{r_s} : \mathcal{T}(M) \to \mathcal{T}(M)$ by setting $\theta_{r_s}(M)$ to be the filling obtained from M by replacing $M \cap Y$ with $\phi_\alpha(M \cap Y)$. Then it is a bijection with the property that

$$(\alpha(S; M), \alpha(\bar{S}; M)) = (\alpha(S'; \theta_{r_s}(M)), \alpha(\overline{S'}; \theta_{r_s}(M))). \tag{5}$$

$\square$

Proof of Theorem 2.2. Assume $S = \{r_1, r_2, \ldots, r_s\} \subseteq \mathcal{R}$ with $r_1 < r_2 < \cdots < r_s$. Let $\Theta_\alpha = \theta_{r_1} \circ \theta_{r_2} \circ \cdots \circ \theta_{r_s}$, where θ_r is defined in the proof of Prop. 3.3. Then Θ_α is a bijection on $\mathbf{F}(\mathcal{M}, \mathbf{e}, \mathbf{s})$ with the property that $(\alpha(S; M), \alpha(\bar{S}; M)) = (\mathrm{se}(\Theta_\alpha(M)), \mathrm{ne}(\Theta_\alpha(M)))$. The symmetry of $F_S^t(p,q)$ follows from Theorem 2.1. $\square$

3.3 Proof of Theorem 2.4

Theorem 2.4 is concerned with the left-mixed statistic $\gamma(T; M)$. The idea of the proof is similar to that of Theorem 2.2: we show that the statement is true when T contains the left-most column only. However, Kasraoui's bijection ϕ does not help here, since the columns and rows play different roles in the fillings. Instead, we give an algorithm which gradually maps the left-mixed statistics with respect to the first column to the pair $(\mathrm{ne}, \mathrm{se})$.

Lemma 3.4 *For* $T = \{1\}$, *we have*

$$G^l_{\{1\}}(p,q) = G^l_{\emptyset}(p,q) = \prod_{i=1}^{m} \begin{bmatrix} h_i \\ s_i \end{bmatrix}_{p,q}.$$

The proof is built on an involution ρ on the fillings of a rectangular shape $\mathcal{M}$.

An involution ρ on rectangular shapes.

Let $\mathcal{M}$ be an $n \times m$ rectangle, and M a filling of $\mathcal{M}$. Let C_1 be the left-most column of M, in which the 1-cells are in the $l_1, \ldots, l_k$ rows from top to bottom. Replace C_1 by the column C_1^r so that the 1's in C_1^r appear in the $l_1, \ldots, l_k$ rows from bottom to top. This is $\rho(M)$. Note that this map does not change the relative positions of those 1-cells that are not in C_1. It is easy to verify that $\rho(\rho(M)) = M$ and $(\gamma(\{1\}; M), \gamma(\overline{\{1\}}; M)) = (\mathrm{se}(\rho(M)), \mathrm{ne}(\rho(M)))$.

Proof: Given a general moon polyomino $\mathcal{M}$, assume that the rows intersecting the first column are $\{R_a, \ldots, R_b\}$. Let $\mathcal{M}_c$ be the union $R_a \cup \cdots \cup R_b$. Clearly, for any $M \in \mathbf{F}(\mathcal{M}, \mathbf{e}, \mathbf{s})$, a left $\{1\}$-NE (SE) chain consists of two 1-cells in $\mathcal{M}_c$. Let $C_i' = C_i \cap \mathcal{M}_c$ be the restriction of the column C_i on $\mathcal{M}_c$. Then $C_1' = C_1$ and $|C_1'| \ge |C_2'| \ge \cdots \ge |C_m'|$.

Suppose that

$$|C_1'| = |C_2'| = \cdots = |C_{j_1}'| > |C_{j_1+1}'| = |C_{j_1+2}'| = \cdots = |C_{j_2}'| > |C_{j_2+1}'| \cdots$$
$$\cdots = |C_{j_{k-1}}'| > |C_{j_{k-1}+1}'| = |C_{j_{k-1}+2}'| = \cdots = |C_{j_k}'| = |C_m'|.$$

Let B_i be the greatest rectangle contained in $\mathcal{M}_c$ whose right most column is C'_{j_i} $(1 \le i \le k)$, and $B'_i = B_i \cap B_{i+1}$ $(1 \le i \le k-1)$.

We define $\phi_\gamma \colon \mathbf{F}(\mathcal{M}, \mathbf{e}, \mathbf{s}) \to \mathbf{F}(\mathcal{M}, \mathbf{e}, \mathbf{s})$ by constructing a sequence of fillings $(M, M_k, \ldots, M_1)$ starting from M.

The map $\phi_\gamma \colon \mathbf{F}(\mathcal{M}, \mathbf{e}, \mathbf{s}) \to \mathbf{F}(\mathcal{M}, \mathbf{e}, \mathbf{s})$

Let $M \in \mathbf{F}(\mathcal{M}, \mathbf{e}, \mathbf{s})$.

1. The filling M_k is obtained from M by replacing $M \cap B_k$ with $\rho(M \cap B_k)$.
2. For i from $k-1$ to 1:
   (a) Define a filling N_i on B'_i by setting $N_i = \rho(M_{i+1} \cap B'_i)$. Let the filling M'_i be obtained from M_{i+1} by replacing $M_{i+1} \cap B'_i$ with N_i.
   (b) The filling M_i is obtained from M'_i by replacing $M'_i \cap B_i$ with $\rho(M'_i \cap B_i)$.
3. Set $\phi_\gamma(M) = M_1$.

Then ϕ_γ is a bijection satisfying the equation $(\gamma(\{1\}; M), \gamma(\overline{\{1\}}; M)) = (\mathrm{se}(\phi_\gamma(M)), \mathrm{ne}(\phi_\gamma(M)))$. $\square$

Proposition 3.5 *Assume* $T = \{c_1, c_2, \ldots, c_t\} \subseteq [m]$ *with* $c_1 < c_2 < \cdots < c_t$. *Let* $T' = \{c_1, c_2, \ldots, c_{t-1}\}$. *Then* $G^l_T(p, q) = G^l_{T'}(p, q)$.

The proof is similar to that of Prop. 3.3. Iterating Prop. 3.5 leads to Theorem 2.4.

4 Invariance Properties

The bi-variate generating function of $(\mathrm{ne}, \mathrm{se})$ (cf. Theorem 2.1) implies that the mixed statistics are invariant under any permutation of rows and/or columns. To be more specific, let $\mathcal{M}$ be a moon polyomino. For any moon polyomino $\mathcal{M}'$ obtained from $\mathcal{M}$ by permuting the rows and/or the columns of $\mathcal{M}$, we have

$$\begin{aligned}&\#\{M \in \mathbf{F}(\mathcal{M}, \mathbf{e}, \mathbf{s}) : \lambda(A; M) = i, \lambda(\bar{A}; M) = j\}\\ &\quad = \#\{M' \in \mathbf{F}(\mathcal{M}', \mathbf{e}', \mathbf{s}') : \lambda(A; M') = i, \lambda(\bar{A}; M') = j\}\end{aligned}$$

for any nonnegative integers i and j, where $\mathbf{e}'$ (resp. $\mathbf{s}'$) is the sequence obtained from $\mathbf{e}$ (resp. $\mathbf{s}$) in the same ways as the rows (resp. columns) of $\mathcal{M}'$ are obtained from the rows (resp. columns) of $\mathcal{M}$, and $\lambda(A; M)$ is any of the four statistics $\alpha(S; M), \beta(S; M), \gamma(T; M)$, and $\delta(T; M)$. In this section we present bijective proofs of such phenomena.

Let $\mathcal{M}$ be a general moon polyomino. Let $\mathcal{N}_l$ be the unique left-aligned moon polyomino whose sequence of row lengths is equal to $|R_1|, \ldots, |R_n|$ from top to bottom. In other words, $\mathcal{N}_l$ is the left-aligned polyomino obtained by rearranging the columns of $\mathcal{M}$ by length in weakly decreasing order from left to right. We shall use an algorithm developed in [4] that rearranges the columns of $\mathcal{M}$ to generate $\mathcal{N}_l$.

The algorithm α for rearranging $\mathcal{M}$:

Step 1 Set $\mathcal{M}' = \mathcal{M}$.

Step 2 If $\mathcal{M}'$ is left aligned, go to Step 4.

Step 3 If $\mathcal{M}'$ is not left-aligned, consider the largest rectangle $\mathcal{B}$ completely contained in $\mathcal{M}'$ that contains C_1, the leftmost column of $\mathcal{M}'$. Update $\mathcal{M}'$ by setting $\mathcal{M}'$ to be the polyomino obtained by moving the leftmost column of $\mathcal{B}$ to the right end. Go to Step 2.

Step 4 Set $\mathcal{N}_l = \mathcal{M}'$.

Based on the algorithm α, Chen et al. constructed a bijection $g = g_{\mathcal{M}} : \mathbf{F}(\mathcal{M}, \mathbf{e}, \mathbf{s}) \to \mathbf{F}(\mathcal{N}_l, \mathbf{e}, \mathbf{s}')$ such that $(\mathrm{se}(M), \mathrm{ne}(M)) = (\mathrm{se}(g(M)), \mathrm{ne}(g(M)))$, see [4, Section 5.3.2].

Combining $g_{\mathcal{M}}$ with the bijection Θ_α constructed in the proof of Theorem 2.2, we are led to the following invariance property.

Theorem 4.1 *Let $\mathcal{M}$ be a moon polyomino. For any moon polyomino $\mathcal{M}'$ obtained from $\mathcal{M}$ by permuting the columns of $\mathcal{M}$, the map*

$$\Phi_\alpha = \Theta_\alpha^{-1} \circ g_{\mathcal{M}'}^{-1} \circ g_{\mathcal{M}} \circ \Theta_\alpha : \mathbf{F}(\mathcal{M}, \mathbf{e}, \mathbf{s}) \to \mathbf{F}(\mathcal{M}', \mathbf{e}, \mathbf{s}') \tag{6}$$

is a bijection with the property that

$$(\alpha(S; M), \alpha(\bar{S}; M)) = (\alpha(S; M'), \alpha(\bar{S}; M')).$$

Similarly, let $\mathcal{N}_t$ be the top aligned polyomino obtained from $\mathcal{M}$ by rotating 90 degrees counterclockwise first, followed by applying the algorithm α, and finally rotating 90 degrees clockwise. Such operations enable us to establish a bijection $h = h_{\mathcal{M}}$ from $\mathbf{F}(\mathcal{M}, \mathbf{e}, \mathbf{s})$ to $\mathbf{F}(\mathcal{N}_t, \mathbf{e}', \mathbf{s})$ that keeps the statistics $(\mathrm{se}, \mathrm{ne})$. The exact description of $h_{\mathcal{M}}$ is given in [5]. Combining the bijection Θ_α with $h_{\mathcal{M}}$, we arrive at the second invariance property.

Theorem 4.2 *Let $\mathcal{M}$ be a moon polyomino. For any moon polyomino $\mathcal{M}'$ obtained from $\mathcal{M}$ by permuting the rows of $\mathcal{M}$, the map*

$$\Lambda_\alpha = \Theta_\alpha^{-1} \circ h_{\mathcal{M}'}^{-1} \circ h_{\mathcal{M}} \circ \Theta_\alpha : \mathbf{F}(\mathcal{M}, \mathbf{e}, \mathbf{s}) \to \mathbf{F}(\mathcal{M}', \mathbf{e}', \mathbf{s}) \tag{7}$$

is a bijection with the property that

$$(\alpha(S; M), \alpha(\bar{S}; M)) = (\alpha(S; M'), \alpha(\bar{S}; M')).$$

Similar statements hold for the statistics $\beta(S; M)$, $\gamma(T; M)$ and $\delta(T; M)$.

References

[1] D. Chebikin, Variations on descents and inversions in permutations, Electron. J. Combin. 15 (2008) R132.

[2] W. Y. C. Chen, E. Y. P. Deng, R. R. X. Du, R. P. Stanley and C. H. Yan, Crossings and nestings of matchings and partitions, Trans. Amer. Math. Soc. 359 (2007) 1555–1575.

[3] W. Y. C. Chen, S. Y. Wu and C. H. Yan, Linked partitions and linked cycles, European J. Combin. 29 (2008) 1408–1426.

[4] W. Y. C. Chen, S. Poznanović, C. H. Yan and A. L. B. Yang, Major index for 01-fillings of moon polyominoes, arXiv:math.CO/0902.3637.

[5] W. Y. C. Chen, A.Y.Z. Wang, C. H. Yan and A. F.Y. Zhao, Mixed statistics on 01-fillings of moon polyominoes, (full version), preprint. arXiv:math.CO/0908.0073.

[6] A. de Mier, On the symmetry of the distribution of crossings and nestings in graphs, Electron. J. Combin. 13 (2006), N21.

[7] A. de Mier, k-noncrossing and k-nonnesting graphs and fillings of Ferrers diagrams, Combinatorica 27 (2007) 699–720.

[8] M. de Sainte-Catherine, Couplages et Pfaffiens en Combinatoire, Physique et Informatique, Ph.D. Thesis, University of Bordeaux I, Talence, France, 1983.

[9] J. Jonsson, Generalized triangulations and diagonal-free subsets of stack polyominoes, J. Combin. Theory Ser. A 112 (2005), 117–142.

[10] A. Kasraoui, Ascents and descents in 01-fillings of moon polyominoes, European J. Combin., to appear.

[11] A. Kasraoui and J. Zeng, Distributions of crossings, nestings and alignments of two edges in matchings and partitions, Electron. J. Combin. 13 (2006) R33.

[12] M. Klazar, On identities concerning the numbers of crossings and nestings of two edges in matchings, SIAM J. Discrete Math. 20 (2006) 960–976.

[13] C. Krattenthaler, Growth diagrams, and increasing and decreasing chains in fillings of Ferrers shapes, Adv. in Appl. Math. 37 (2006), 404–431.

[14] A. Mendes, J. Remmel, Descents, inversions, and major indices in permutation groups, Discrete Math. 308 (2008) 2509–2524.

[15] S. Poznanović, C. H. Yan, Crossings and nestings of two edges in set partitions, SIAM J. Discrete Math., to appear.

[16] M. Rubey, Increasing and decreasing sequences in fillings of moon polyominoes, arXiv:math.CO/0604140.