Linear coefficients of Kerov's polynomials: bijective proof and refinement of Zagier's result

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Abstract. We look at the number of permutations β of $[N]$ with m cycles such that $(1\ 2\ \ldots\ N)\beta^{-1}$ is a long cycle. These numbers appear as coefficients of linear monomials in Kerov's and Stanley's character polynomials. D. Zagier, using algebraic methods, found an unexpected connection with Stirling numbers of size $N + 1$. We present the first combinatorial proof of his result, introducing a new bijection between partitioned maps and thorn trees. Moreover, we obtain a finer result, which takes the type of the permutations into account.

Résumé. Nous étudions le nombre de permutations β de $[N]$ avec m cycles telles que $(1\ 2\ \ldots\ N)\beta^{-1}$ a un seul cycle. Ces nombres apparaissent en tant que coefficients des monômes linéaires des polynômes de Kerov et de Stanley. À l'aide de méthodes algébriques, D. Zagier a trouvé une connexion inattendue avec les nombres de Stirling de taille N+1. Nous présentons ici la première preuve combinatoire de son résultat, en introduisant une nouvelle bijection entre des cartes partitionnées et des arbres épineux. De plus, nous obtenons un résultat plus fin, prenant en compte le type des permutations.

Keywords: Kerov's Character Polynomials, Bicolored Maps, Long Cycle Factorization

1   Introduction

The question of the number of factorizations of the long cycle $(1\ 2\ \ldots\ N)$ into two permutations with given number of cycles has already been studied via algebraic or combinatorial[(i)] methods (see [Adr98, SV08]). In these papers, the authors obtain nice generating series for these numbers. Note that the combinatorial approach has been refined to state a result on the number of factorizations of the long cycle $(1\ 2\ \ldots\ N)$ in two permutations with given type (see [MV09]).

Unfortunately, in all these results, extracting one coefficient of the generating series gives complicated formulas. The case where one of the two factors has to be also a long cycle is particularly interesting. Indeed, the number $B'(N, m)$ of permutations β of $[N]$ with m cycles, such that $(1\ 2\ \ldots\ N)\beta^{-1}$ is a long cycle, is known to be the coefficient of some linear monomial in Kerov's and Stanley's character polynomials (see [Bia03, Theorem 6.1] and [Sta03, Fér10]). These polynomials express the character

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[(i)] It can be reformulated in terms of unicellular bicolored maps with given number of vertices, see paragraph 2.1.

value of the irreducible representation of the symmetric group indexed by a Young diagram λ on a cycle of fixed length in terms of some coordinates of λ.

A very simple formula for these numbers was found by D. Zagier [Zag95, Application 3] (see also [Sta09, Corollary 3.3]):

Theorem 1.1 (Zagier, 1995) *Let $m \leq N$ be two positive integers such that $m \equiv N[2]$. Then*

$$\frac{N(N+1)}{2} B'(N, m) = s(N+1, m), \tag{1}$$

where $s(N+1, m)$ is the unsigned Stirling number of the first kind.

It is well-known that $s(N+1, m)$ is the number of permutations of $[N+1]$ with m cycles. As former proofs of this result are purely algebraic, R. Stanley [Sta09] asked for a combinatorial proof of Theorem 1.1. This paper presents the first bijective approach proving this formula. We even prove a finer result, which takes the type[(ii)] of the permutations into consideration and not only their number of cycles. To state it, we need to introduce a few notations. First, we refine the numbers $s(N+1, m)$ and $B'(N, m)$: if $\lambda \vdash n$ (*i.e.* λ is a partition of n), let $A(\lambda)$ (resp. $B(\lambda)$) be the number of permutations $\beta \in S_n$ of type λ (resp. with the additional condition that $(1\ 2\ \ldots\ N)\beta^{-1}$ is a long cycle). Then, as Stanley's result deals with permutations of $[N]$ and $[N+1]$, we need operators on partitions which modify their size, but not their length. If μ (resp. λ) has at least one part $i+1$ (resp. i), let $\mu^{\downarrow(i+1)}$ (resp. $\lambda^{\uparrow(i)}$) be the partition obtained from μ (resp. λ) by erasing a part $i+1$ (resp. i) and adding a part i (resp. $i+1$). For instance, using exponential notations (see [Mac95, chapter 1, section 1]), $(1^2 3^1 4^2)^{\downarrow(4)} = 1^2 3^2 4^1$ and $(2^2 3^2 4)^{\uparrow(2)} = 2^1 3^3 4^1$.

Now we can state our main theorem, which implies immediately Theorem 1.1:

Theorem 1.2 (Main result) *For each partition $\mu \vdash N+1$ of length p with $p \equiv N[2]$, one has:*

$$\frac{N+1}{2} \sum_{\lambda = \mu^{\downarrow(i+1)}, i>0} i\, m_i(\lambda)\, B(\lambda) = A(\mu) = \frac{(N+1)!}{z_\mu}, \tag{2}$$

where $m_i(\lambda)$ is the number of parts i in λ and $z_\mu = \prod_i i^{m_i(\mu)} m_i(\mu)!$.

To be comprehensive on the subject, we mention that G. Boccara found an integral formula for $B(\lambda)$ (see [Boc80]), but there does not seem to be any direct link with our result.

As in paper [MV09], the first step (section 2) of our proof of Theorem 1.2 consists in a change of basis in the ring of symmetric functions in order to show the equivalence with the following statement:

Theorem 1.3 *Let λ be a partition of N of length p. Choose randomly (with uniform probability) a set-partition π of $\{1, \ldots, N\}$ of type λ and then (again with uniform probability) a permutation β in S_π (that means that each cycle of β is contained in a block of π). Then the probability for $(1\ 2\ \ldots\ n)\beta^{-1}$ to be a long cycle is exactly $1/(N-p+1)$.*

Once again, such a simple formula is surprising. We give a bijective proof in sections 3, 4 and 5.

Remark 1 *Theorem 1.2, written for all $\mu \vdash N+1$, gives the collection of numbers $B(\lambda)$ as solution of a sparse triangular system. Indeed, if we endow the set of partitions of N with the lexicographic order, Theorem 1.2, written for $\mu = \lambda^{\uparrow(\lambda_1)}$, gives $B(\lambda)$ in terms of the quantities $A(\mu)$ and $B(\nu)$ with $\nu > \lambda$.*

[(ii)] The type of a permutation is the sequence of the lengths of its cycles, sorted in increasing order.

2   Link between Theorems 1.2 and 1.3

2.1   Black-partitioned maps

By definition, a map is a graph drawn on a two-dimensional oriented surface (up to deformation), i.e. a graph with a cyclic order on the incident edges to each vertex.

As usual, a couple of permutations (α, β) in S_N can be represented as a bicolored map with N edges labeled with integers from 1 to N. In this identification, $\alpha(i)$ (resp. $\beta(i)$) is the edge following i when turning around its white (resp. black) extremity. White (resp. black) vertices correspond to cycles of α (resp. β). The condition $\alpha \cdot \beta = (1\ 2\ \ldots\ N)$ (which we will assume from now on) means that the map is unicellular (i.e. if we remove the edges of the maps from the oriented surface, the resulting surface is homeomorphic to an open disc) and that the positions of the labels are determined by the choice of the edge labeled by 1 (which can be seen as a *root*). In this case, the couple of permutations is entirely determined by β.

Therefore, if $\lambda \vdash N$, the quantity $A(\lambda)$ (resp. $B(\lambda)$) is the number of rooted unicellular maps (resp. *star* maps, that means that we make the additional assumption that the map has only one white vertex) with black vertices' degree distribution λ.
As in the papers [SV08] and [MV09], our combinatorial construction deals with maps with additional information:

Definition 2.1 *A black-partitioned (rooted unicellular) map is a rooted unicellular map with a set partition π of its black vertices. We call degree of a block of π the sum of the degrees of the vertices in π. The* type *of a black-partitioned map is its blocks' degree distribution.*

In terms of permutations, a black-partitioned map consists in a couple (α, β) in S_N with the condition $\alpha\beta = (1\ 2\ \ldots\ N)$ and a set partition π of $\{1, \ldots, N\}$ coarser than the set partition in orbits under the action of β (in other words, if i and j lie in the same cycle of β, they must be in the same part of π).

Example 1 *Let $\beta = (1)(25)(37)(4)(6)$, $\alpha = (1234567)\beta^{-1} = (1267453)$, and π be the partition $\{\{1,3,6,7\}; \{2,5\}; \{4\}\}$. Associating the triangle, circle and square shape to the blocks, (β, π) is the black-partitioned star map pictured on figure 1.*
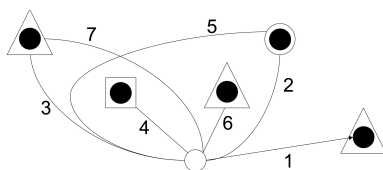

Fig. 1: The black-partitioned map defined in example 1

We denote by $C(\lambda)$ (resp. $D(\lambda)$) the number of black-partitioned maps (resp. black-partitioned star maps) of partition type λ. Equivalently, $C(\lambda)$ (resp. $D(\lambda)$) is the number of couples (β, π) as above such that π is a partition of type λ (resp. and $(1\ 2\ \ldots\ N)\beta^{-1}$ is a long cycle). Quantities A and C (resp. B and

D) are linked by the following equations (whose proofs are identical to the one of [MV09, Proposition 1], see also [Mac95, Chapter 1, equation (6.9)])

$$\sum_{\mu \vdash N+1} C(\mu) \operatorname{Aut}(\mu) m_\mu = \sum_{\nu \vdash N+1} A(\nu) p_\nu; \tag{3}$$

$$\sum_{\lambda \vdash N} D(\lambda) \operatorname{Aut}(\lambda) m_\lambda = \sum_{\pi \vdash N} B(\pi) p_\pi, \tag{4}$$

where $m_\bullet$ and $p_\bullet$ denote the monomial and power sum basis of the ring of symmetric functions.

2.2   Permuted star thorn trees and Morales'-Vassilieva's bijection

The main tool of this article is to encode black-partitioned maps into star thorn trees, which have a very simple combinatorial structure. Note that they are a particular case of the notion of thorn trees, introduced by A. Morales and the second author in [MV09].

Definition 2.2 (star thorn tree) *An* (ordered rooted bicolored) star thorn tree *of size N is a tree with a white root vertex, p black vertices and $N - p$ thorns connected to the white vertex (the order in which they are connected matters) and $N - p$ thorns connected to the black vertices. A thorn is an edge connected to only one vertex. We call* type *of a star thorn tree its black vertices' degree distribution (taking the thorns into account). If μ is an integer partition, we denote by $\widetilde{ST}(\mu)$ the number of star thorn trees of type μ.*

An example is given on Figure 2 (for the moment, please do not pay attention to the labels). The interest of this object lies in the following theorem, which corresponds to the case $\lambda = (N)$ of [MV09, Theorem 2] (note that the proof is entirely bijective).

Theorem 2.1 *Let $\mu \vdash N$ be a partition of length p. One has:*

$$C(\mu) = (N - p)! \cdot \widetilde{ST}(\mu). \tag{5}$$

The right-hand side of (5) is the number of couples (τ, σ) where:

- τ is a star thorn tree of type μ.

- σ is a permutation of $[N - p]$, which happens to be exactly the number of thorns with a white (resp. black) extremity in τ. So σ may be seen as a bijection between the thorns with a white extremity and thorns with a black extremity.

We call such a couple a permuted (star) thorn tree. By definition, the type of (τ, σ) is the type of τ. Examples of graphical representations are given on Figure 2: we put symbols on edges and thorns with the following rule. Two thorns have the same label if they are associated by σ and except from that rule, all labels are different (the chosen symbols and their order do not matter, we call that a symbolic labeling).

It is easy to transform a permuted thorn tree (τ, σ) where τ has type $\lambda \vdash N$ to a permuted thorn tree (τ', σ') where τ' has type $\mu = \lambda^{\uparrow(i)}$. We just add a thorn anywhere on the white vertex ($N + 1$ possible places) and a thorn anywhere on a black vertex of degree i (there are i possible places on each of the $m_i(\lambda)$ black vertices of degree i). Then we choose σ' to be the extension of σ associating the two new thorns. This procedure is invertible if we remember which thorn of black extremity is the new one (it
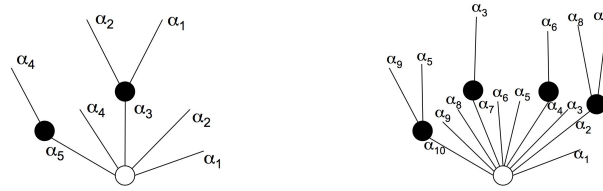
Fig. 2: Example of two permuted star thorn trees $(\tau_{\text{ex}}^1, \sigma_{\text{ex}}^1)$ of type $1^1 2^1$ and $(\tau_{\text{ex}}^2, \sigma_{\text{ex}}^2)$ of type $2^2 3^2$

must be on a black vertex of degree $i+1$, so there are $i \cdot m_{i+1}(\mu)$ choices). This leads immediately to the following relation:

$$\widetilde{ST}(\mu) \cdot (N+1-p)! \cdot i \cdot m_{i+1}(\mu) = (N+1) \cdot i \cdot m_i(\lambda) \cdot \widetilde{ST}(\lambda) \cdot (N-p)!. \tag{6}$$

2.3 Reduction of the main theorem

Proposition 2.2 *For any partition $\lambda \vdash N$ of length p, one has:*

$$D(\lambda) = \frac{1}{N-p+1}(N-p)!\widetilde{ST}(\lambda).$$

Sections 3, 4 and 5 are devoted to the proof. It is easy to see, with the definition of subsection 2.1 and the bijection of subsection 2.2, that this proposition is a reformulation of Theorem 1.3.

Lemma 2.3 *Proposition 2.2 implies Theorem 1.2.*

Proof: We fix a partition $\mu \vdash N+1$ of length $p < N+1$ and sum equation (6) on $\lambda = \mu^{\downarrow(i+1)}$:

$$\widetilde{ST}(\mu) \cdot (N+1-p)! \cdot (N+1-p) = (N+1) \sum_{\lambda = \mu^{\downarrow(i+1)}, i>0} i \cdot m_i(\lambda) \cdot \widetilde{ST}(\lambda) \cdot (N-p)!.$$

Using Morales'-Vassilieva's bijection and Proposition 2.2, this equality becomes:

$$C(\mu) \cdot (N+1-p) = (N+1) \sum_{\lambda = \mu^{\downarrow(i+1)}, i>0} i \cdot m_i(\lambda) \cdot D(\lambda) \cdot (N+1-p).$$

Hence, $\displaystyle \sum_{\substack{\mu \vdash N+1 \\ \mu \neq 1^{(N+1)}}} C(\mu) \operatorname{Aut}(\mu) m_\mu = (N+1) \sum_{\substack{\mu \vdash N+1 \\ \mu \neq 1^{(N+1)}}} \sum_{\substack{i>0 \\ \lambda = \mu^{\downarrow(i+1)}}} i \cdot m_i(\lambda) \operatorname{Aut}(\mu) D(\lambda) m_\mu;$

$$= (N+1) \sum_{\lambda \vdash N} \operatorname{Aut}(\lambda) D(\lambda) \left(\sum_{\substack{i>0 \\ \mu = \lambda^{\uparrow(i)}}} i \cdot m_{i+1}(\mu) m_\mu \right).$$

The last equality has been obtained by changing the order of summation and using the trivial fact that, if $\mu = \lambda^{\uparrow(i)}$, one has $\operatorname{Aut}(\mu) \cdot m_i(\lambda) = \operatorname{Aut}(\lambda) \cdot m_{i+1}(\mu)$. Now, observing that the expression in the bracket can be written $\Delta(m_\lambda)$, where Δ is the differential operator $\sum_i x_i^2 \partial/\partial x_i$, one has:

$$\sum_{\mu \vdash N+1} C(\mu) \operatorname{Aut}(\mu) m_\mu - (N+1)! m_{1^{N+1}} = (N+1) \cdot \Delta\left(\sum_{\lambda \vdash N} \operatorname{Aut}(\lambda) D(\lambda) m_\lambda \right).$$

Let us rewrite this equality in the power sum basis. The expansion of the two summations in this basis are given by equations (3) and (4). Furthermore, one has: $\Delta(p_\pi) = \sum_i i \cdot m_i(\pi) p_{\pi^{\uparrow(i)}}$. Indeed, the one-part case ($\Delta(p_k) = k \cdot p_{k+1}$) is trivial and the general case follows because Δ is a derivation. We also need the power-sum expansion of $(N+1)! m_{1^{N+1}}$, which is (see [Mac95, Chapter I, equation (2.14')]):

$$(N+1)! m_{1^{N+1}} = (N+1)! e_{N+1} = (N+1)! \sum_{\nu \vdash N+1} \frac{(-1)^{N+1-\ell(\nu)}}{z_\nu} p_\nu = \sum_{\nu \vdash N+1} A(\nu)(-1)^{N+1-\ell(\nu)} p_\nu,$$

where e_{N+1} is the $N+1$-th elementary function. Putting everything together, we have:

$$\sum_{\nu \vdash N+1} A(\nu) p_\nu + \sum_{\nu \vdash N+1} A(\nu)(-1)^{N-\ell(\nu)} p_\nu = (N+1) \sum_\pi B(\pi) \sum_{\rho = \pi^{\uparrow(i)}, i>0} i \cdot m_i(\pi) p_\rho.$$

If we identify the coefficients of p_μ in both sides, we obtain exactly Theorem 1.2   □

Remark 2 *Using Remark 1, the converse statement of Lemma 2.3 can be proved the same way.*

3   Mapping black-partitioned star maps to permuted thorn trees

The following sections provide a combinatorial proof of Proposition 2.2. We proceed in a three step fashion. First we define a mapping Ψ from the set of black-partitioned star maps of type λ (counted by $D(\lambda)$) to a set of permuted star thorn trees of the same type and show it is injective. As a final step, we compute the cardinality of the image set of Ψ and show it is exactly $(1/(N-p+1))\,(N-p)! \widetilde{ST}(\lambda)$. Although there are some related ideas, Ψ is not the restriction of the bijection of paper [MV09].

3.1   Labeled thorn tree

Let (β, π) be a black-partitioned star map. We first construct a labeled star thorn tree $\overline{\tau}$:

(i) Let $(\alpha_k)_{(1 \leq k \leq N)}$ be integers such that $\alpha_1 = 1$ and the long cycle $\alpha = (1\ 2\ \ldots\ N)\beta^{-1}$ is equal to $(\alpha_1 \alpha_2 \alpha_3 \ldots \alpha_N)$. The root of $\overline{\tau}$ is a white vertex with N descending edges labeled from right to left with $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_N$ (α_1 is the rightmost descending edge and α_N the leftmost).

(ii) Let m_i be the maximum element of block π_i. For $k = 1 \ldots N$, if $\alpha_k = \beta(m_i)$ for some i, we draw a black vertex at the other end of the descending edge labeled with α_k. Otherwise the descending edge is a thorn.

   Remark 3 *As $\alpha_N = \alpha^{-1}(1) = \beta(N)$ the leftmost descending edge isn't a thorn and is labeled with $\beta(N)$ (N is necessarily the maximum element of the block containing it).*

(iii) For $i = 1 \ldots p$, let $(\beta_1^u \ldots \beta_{l_u}^u)_{1 \leq u \leq c}$ be the c cycles included in block π_i such that $\beta_{l_u}^u$ is the maximum element of cycle u. (We have $\Sigma_u l_u = |\pi_i|$). We also order these cycles according to their maximum, *i.e.* we assume that $\beta_{l_c}^c < \beta_{l_{c-1}}^{c-1} < \ldots < \beta_{l_1}^1 = m_i$. As a direct consequence, $\beta_1^1 = \beta(m_i)$.
   We connect $|\pi_i| - 1$ thorns to the black vertex linked to the root by the edge $\beta(m_i)$. Moving around the vertex counter-clockwise and starting right after edge $\beta(m_i)$, we label the thorn with $\beta_2^1, \beta_3^1, \ldots, \beta_{l_1}^1, \beta_1^2, \ldots, \beta_{l_2}^2, \beta_{l_3}^3, \ldots \beta_{l_c}^c$. Then $\overline{\tau}$ is the resulting thorn tree.

Remark 4 *Moving around a black vertex clockwise starting with the thorn right after the edge (in clockwise order), a new cycle of β begins whenever we meet a left-to-right maximum of the labels.*

Remark 5 *As the long cycle α and the repartition of the cycles of β in the various blocks appear explicitly in $\overline{\tau}$, one can recover the black-partitioned star map from it.*

The idea behind this construction is to add a root to the map (α, β), select one edge per block, cut all other edges into two thorns and merge the vertices corresponding to the same black block together. Step (i) tells us where to place the root, step (ii) which edges we select and step (iii) how to merge vertices (in maps unlike in graphs, one has to do some choices to merge vertices).

Example 2 *Let us take the black-partitioned star map of example 1. Following construction rules (i) and (ii), one has $m_\triangle = 7$, $m_\bigcirc = 5$, $m_\square = 4$ and the descending edges indexed by $\beta(m_\triangle) = 3$, $\beta(m_\bigcirc) = 2$ and $\beta(m_\square) = 4$ connect a black vertex to the white root. Other descending edges from the root are thorns. Using (iii), we add labeled thorns to the black vertices to get the labeled thorn tree depicted on Figure 3. Focusing on the one connected to the root through the edge 3, we have $(\beta_1^1 \beta_{l_1}^1)(\beta_{l_2}^2)(\beta_{l_3}^3) = (37)(6)(1)$. Reading the labels clockwise around this vertex, we get $1, 6, 7, 3$. The three cycles can be simply recovered looking at the left-to-right maxima 1, 6 and 7.*
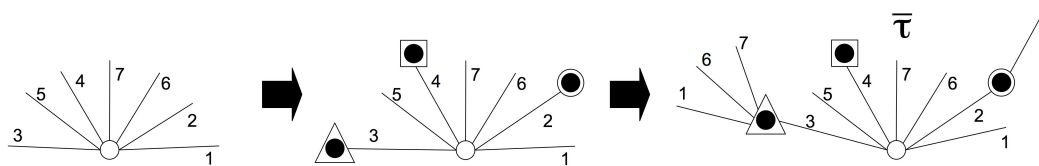

Fig. 3: Labeled thorn tree associated to the black-partitioned star map of Figure 1

3.2 Permuted thorn tree

Using $\overline{\tau}$, we call τ the star thorn tree obtained by removing labels and σ the permutation that associates to a white thorn in τ the black thorn with the same label in $\overline{\tau}$.

Finally, we define: $\Psi(\beta, \pi) = (\tau, \sigma)$.

Example 3 *Following up with example 1, we get the permuted thorn tree $(\tau_{ex}^3, \sigma_{ex}^3)$ drawn on Figure 4. Graphically we use the same convention as in section 2 to represent σ.*
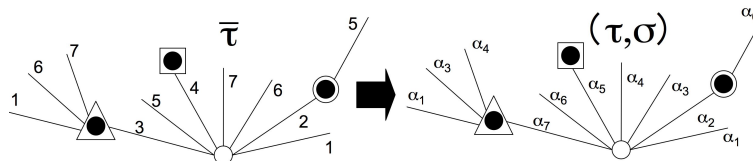

Fig. 4: Permuted thorn tree $(\tau_{ex}^3, \sigma_{ex}^3)$ associated to the black-partitioned star map of Figure 1

4   Injectivity and reverse mapping

Assume $(\tau, \sigma) = \Psi(\beta, \pi)$ for some black partitioned star map (β, π). We show that (β, π) is actually uniquely determined by (τ, σ).

As a first step, we recover the labeled thorn tree $\overline{\tau}$. Let us draw the permuted thorn tree (τ, σ) as explained in subsection 2.2. We show by induction that there is at most one possible integer value for each symbolic label.

(i) By construction, the label α_1 of the right-most edge or thorn descending from the root is necessarily 1.

(ii) Assume that for $i \in [N-1]$, we have identified the symbols of values $1, 2, \ldots$, and i. We look at the edge or thorn with label i connected to a black vertex b. In this step, we determine which symbol corresponds to $\beta(i)$.

Recall that, when we move around b clockwise finishing with the edge (in this step, we will always turn in this sense), a new cycle begins whenever we meet a left-to-right maximum. So, to find $\beta(i)$, one has to know whether i is a left-to-right maximum or not.

If all values of labels of thorns before i haven't already been retrieved, then i is not a left-to-right maximum. Indeed, the remaining label values are $i + 1, \ldots, N$ and at least one thorn's label on the left of i lies in this interval. Following our construction, necessarily $\beta(i)$ corresponds to the symbolic label of the thorn right at the left of i.

If all the thorns' label values on the left of i have already been retrieved (or there are no thorns at all), then i is a left-to-right maximum. According to the construction of $\overline{\tau}$, $\beta(i)$ corresponds necessarily to the symbolic label of the thorn preceding the next left-to-right maximum. But one can determine which thorn (or edge) corresponds to the next left-to-right maximum: it is the first thorn (or edge) e without a label value retrieved so far (again moving around the black vertex from left to right). Indeed, all the value retrieved so far are less than i and those not retrieved greater than i. Therefore $\beta(i)$ is the thorn right at the left of e. If all the values of the labels of the thorns connected to b have already been retrieved then i is the maximum element of the corresponding block and $\beta(i)$ corresponds to the symbolic label of the edge connecting this black vertex to the root.

(iii) Consider the element (thorn of edge) of white extremity with the symbolic label corresponding to $\beta(i)$. The next element (turning around the root in counter-clockwise order) has necessarily label $\alpha(\beta(i)) = i + 1$.

As a result, the knowledge of the thorn or edge with label i uniquely determines the edge or thorn with label $i + 1$.

Applying the previous procedure up to $i = N - 1$ we see that $\overline{\tau}$ is uniquely determined by (τ, σ) and so is (β, π) (see Remark 5).

Example 4 *Take as an example the permuted thorn tree $(\tau_{ex}^1, \sigma_{ex}^1)$ drawn on the left-hand side of Figure 2, the procedure goes as described on figure 5. First, we identify $\alpha_1 = 1$. Then, as there is a non (value) labeled thorn (α_2) on the left of the thorn connected to a black vertex with label value 1, necessarily 1 is not a left-to-right maximum and α_2 is the label of the thorn right on the left of 1, that is α_2. Then as α_3 follows $\alpha_2 = \beta(1)$ around the white root, we have $\alpha_3 = \alpha(\beta(1)) = 2$.*

We apply the procedure up to the full retrieval of the edges' and thorns' labels. We find $\alpha_2 = 3$, $\alpha_4 = 4$, $\alpha_5 = 5$. Finally, we have $\alpha = (13245)$, $\beta = (213)(4)(5)$, $\pi = \{\{1, 2, 3\}; \{4, 5\}\}$ as shown on figure 5.
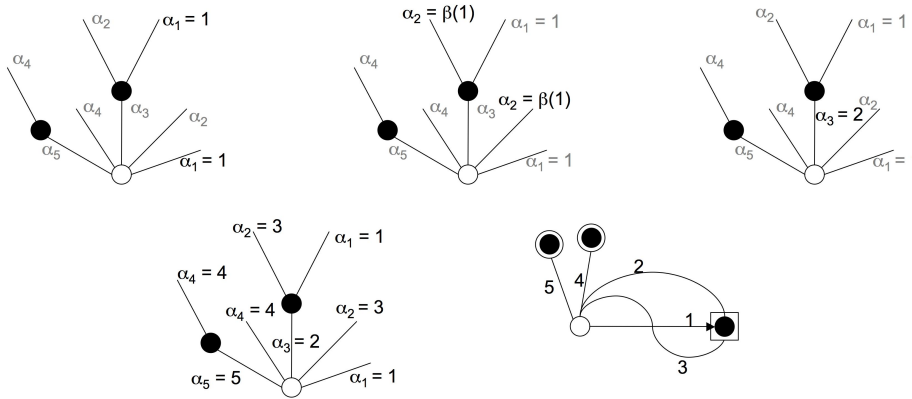
Fig. 5: Reconstruction of the map

5 Characterization and size of the image set $\Im(\Psi)$

5.1 A necessary and sufficient condition to belong to $\Im(\Psi)$

Why Ψ is not surjective? Let us fix a permuted star thorn tree (τ, σ). We can try to apply to it the procedure of section 4 and we distinguish two cases:

- it can happen, for some $i < N$, when one wants to give the label $i + 1$ to the edge following $\beta(i)$ (step (iii)), that this edge has already a label j. If so, the procedure fails and (τ, σ) is not in $\Im(\Psi)$.

- if this never happens, the procedure ends with a labeled thorn tree $\bar{\tau}$. In this case, one can find the unique black-partitioned star map M corresponding to $\bar{\tau}$ and by construction $\Psi(M) = (\tau, \sigma)$.

For instance, if we take as (τ, σ) the couple $(\tau_{\mathrm{ex}}^2, \sigma_{\mathrm{ex}}^2)$ on the right of Figure 2, the procedure gives successively : $\alpha_1 = 1$, $\alpha_9 = 2$, $\alpha_{10} = 3$, $\alpha_6 = 4$, $\alpha_5 = 5$ and then we should choose $\alpha_1 = 6$, but this is impossible because we already have $\alpha_1 = 1$.

Lemma 5.1 *If the procedure fails, the label j of the edge that should get a second label $i + 1$ is always 1.*

Proof: If $j > 1$, this means that $\beta(j - 1) = \beta(i)$. Let us distinguish two cases.
If $j - 1$ is a left-to-right maximum, the label i must be at the right of $\beta(j - 1)$ and not a left-to-right maximum. But this is impossible because all thorns at the left of $\beta(j - 1)$ (including $\beta(j - 1)$) have labels smaller than j.
If $j - 1$ is not a left-to-right maximum, the label $j - 1$ must be at the right of $\beta(j - 1) = \beta(i)$ and i is a left-to-right maximum. Then $\beta(i)$ is before the next left-to-right maximum. So the edge at the right of $\beta(i)$ has a label greater than i and can not be $j - 1$. $\qquad\square$

An auxiliary oriented graph Remark 3 gives a necessary condition for (τ, σ) to be in $\Im(\Psi)$: its leftmost edge leaving the root must be a real edge e_0, and not a thorn. From now on, we call this property $(P1)$: note that, among all permuted thorn tree of a given type $\lambda \vdash N$ of length p, exactly p over N have this property. When $(P1)$ is satisfied, we denote π_0 the black extremity of e_0. The lemma above shows

$$\pi_2 \quad \pi_1$$
$$\downarrow \quad \quad \quad \quad \pi_2 \rightarrow \pi_1 \rightarrow \pi_0$$
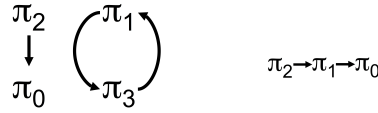$$\pi_0 \quad \pi_3$$

Fig. 6: Two examples of auxiliary graphs.

that the procedure fails if and only if e_0 is chosen as $\beta(i)$ for some $i < N$. But this can not happen at any time. Indeed, the following lemma is a direct consequence from step (ii) of the inverting procedure:

Lemma 5.2 *A real edge (i.e. which is not a thorn) e can be chosen as $\beta(i)$ only if the edge and all thorns leaving the corresponding black vertex have labels smaller or equal to i. If this happen, we say that the black vertex is completed at step i.*

Corollary 5.3 *Let e be a real edge of black extremity $\pi \neq \pi_0$. Let us denote e' the element (edge or thorn) right at the left of e on the white vertex. Let π' be the black extremity of the element e'' associated to e' (i.e. e' itself if it is an edge, its image by σ else). Then π' can not be completed before π.*

Proof: If π' is completed at step i, by Lemma 5.2, element e'' has a label $j \leq i$. As e' has the same label, this implies that e has label $\beta(j-1)$ or in other words, that π is completed at step $j - 1 < i$. $\quad\quad\square$

When applied for every black vertex $\pi \neq \pi_0$, this corollary gives some partial information on the order in which the black vertices can be completed. We will summarize this in an oriented graph $G(\tau, \sigma)$: its vertices are the black vertices of τ and its edges are $\pi \rightarrow \pi'$, where π and π' are in the situation of the corollary above. This graph has one edge leaving each of its vertex, except for π_0. As examples, the graphs corresponding to $(\tau_{\mathrm{ex}}^2, \sigma_{\mathrm{ex}}^2)$ and to $(\tau_{\mathrm{ex}}^3, \sigma_{\mathrm{ex}}^3)$ (see Figures 2 and 4) are drawn on Figure 6.

The graph $G(\tau, \sigma)$ gives all the information we need! Can we decide, using only $G(\tau, \sigma)$, whether (τ, σ) belongs to $\Im(\Psi)$ or not? There are two cases, in which the answer is obviously yes:

1. Let us suppose that $G(\tau, \sigma)$ is an oriented tree of root π_0 (all edges are oriented towards the root). In this case, we say that (τ, σ) has property $(P2)$. Then, the vertex π_0 can be completed only when all other vertices have been completed, *i.e.* when all edges and thorns have already a label. That means that e_0 can be chosen as $\beta(i)$ only for $i = N$. Therefore, in this case, the procedure always succeeds and (τ, σ) belongs to $\Im(\Psi)$. This is the case of $(\tau_{\mathrm{ex}}^3, \sigma_{\mathrm{ex}}^3)$.

2. Let us suppose that $G(\tau, \sigma)$ contains an oriented cycle (eventually a loop). Then all the vertices of this cycle can never be completed. Therefore, the procedure always fails in this case and (τ, σ) does not belong to $\Im(\Psi)$. This is the case of $(\tau_{\mathrm{ex}}^2, \sigma_{\mathrm{ex}}^2)$.

In fact, we are always in one of these two cases (the proof of the following lemma is left to the reader):

Lemma 5.4 *Let G be an oriented graph whose vertices have out-degree 1, except for one vertex v_0 which has out-degree 0. Then G is either an oriented tree of root v_0 or contains an oriented cycle.*

Finally, one has the following result:

Proposition 5.5 *The mapping Ψ defines a bijection:*

$$\left\{ \begin{array}{c} \text{black-partitioned star maps} \\ \text{of type } \lambda \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{permuted star thorn trees of type } \lambda \\ \text{with properties } (P1) \text{ and } (P2) \end{array} \right\}. \tag{7}$$

5.2   *Proportion of permuted thorn trees* (τ, σ) *in* $\Im(\Psi)$

To finish the proof of Proposition 2.2, one just has to compute the size of the right-hand side of (7):

Proposition 5.6 *Let λ be a partition of N of length p. Denote by $P(\lambda)$ (resp. $P'(\lambda)$) the proportion of couples (τ, σ) with properties $(P1)$ and $(P2)$ among all the permuted thorn trees of type λ (resp. among permuted thorn trees of type λ with property $(P1)$) of type λ. Then, one has:*

$$P'(\lambda) = \frac{N}{p(N - p + 1)} \text{ and, hence, } P(\lambda) = \frac{1}{N - p + 1}.$$

Proof (by induction on p): The case $p = 1$ is easy: as $G(\tau, \sigma)$ has only one vertex and no edges, it is always a tree. Therefore, for any $N \geq 1$, one has $P'((N)) = 1$.
Suppose that the result is true for any λ of length $p - 1$ and fix a partition $\mu \vdash N$ of length $p > 1$. Consider the permuted thorn trees (τ, σ) of type μ, verifying $(P1)$, with a marked black vertex $\overline{\pi} \neq \pi_0$: as there are always $p - 1$ choices for the marked vertex, the proportion of these objects verifying $(P2)$ is still $P'(\mu)$. Let us now split this set, depending on the degrees (in τ) of the marked vertex and of the end of the edge leaving $\overline{\pi}$ in the graph $G(\tau, \sigma)$. The proportion of marked star thorn trees of type μ (with property $(P1)$) whose marked vertex has degree k_0 is $m_{k_0}(\mu)/p$. We denote $k = \deg_\tau(\overline{\pi})$ and $\mu' = \mu \backslash k$ (*i.e.* the partition obtained from μ by deleting one part k).

- In $k - 1$ cases over $N - 1$, this second extremity is also $\overline{\pi}$. So $G(\tau, \sigma)$ contains a loop and (τ, σ) does not fulfill $(P2)$.

- For every j, in $j \cdot m_j(\mu')$ cases over $N - 1$, this second extremity is a vertex $\pi' \neq \overline{\pi}$ of degree j (in τ). But one has an easy bijection φ:

$$\left\{ \begin{array}{c} (\tau, \sigma) \text{ of type } \mu \text{ verifying } (P1) \\ \text{with a marked black vertex } \overline{\pi} \neq \pi_0 \\ \text{of size } k \text{ such that } \overline{\pi} \to_{G(\tau,\sigma)} \pi' \\ \text{with } \pi' \neq \overline{\pi} \text{ of size } j \end{array} \right\} \simeq \left\{ \begin{array}{c} (\tau', \sigma') \text{ of type } \mu^{\downarrow(j,k)} := \mu\backslash(j,k) \cup (j + k - 1) \\ \text{verifying } (P1) \text{ with the edge or one of the first } j - 1 \\ \text{thorns of a black vertex of size } j + k - 1 \text{ marked} \\ (\text{always } j \cdot m_{j+k-1}(\mu^{\downarrow(j,k)}) \text{ choices}) \end{array} \right\}$$
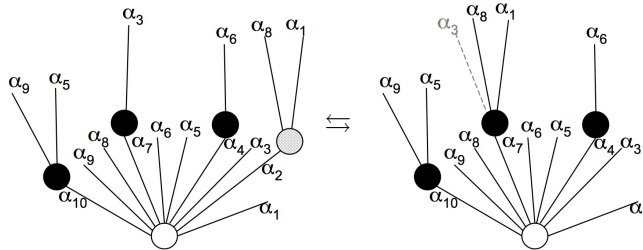
From left-to-right: erase the marked black vertex $\overline{\pi}$ with its edge $e_{\overline{\pi}}$ and move its thorns to the black vertex π' (at the right of its own thorns). Choose as marked the element (edge of thorn) with a black extremity with the same symbolic label as the element right at the left of $e_{\overline{\pi}}$.
From right to left: look at the white thorn corresponding to the marked thorn $\overline{e}$ (if the marked element is an edge, just take the edge itself). Then add a new edge with a black vertex just at the right

of this thorn (or edge). Finally, move the $k-1$ right-most thorns of the black extremity of $\overline{e}$ to this new black vertex. The marked black vertex is the new one.

This bijection keeps property (P2). Indeed, if $\varphi(\tau, \sigma, \overline{\pi}) = (\tau', \sigma', \overline{e})$, the graph $G(\tau', \sigma')$ is obtained from $G(\tau, \sigma)$ by contracting the edge of origin $\overline{\pi}$. Therefore, the proportion of couples having property $(P2)$ on the left-hand side is the same as on the right-hand side. But, as $\mu^{\downarrow(j,k)}$ has length $p-1$ and size $N-1$, by induction hypothesis, this proportion is:

$$\frac{N-1}{(p-1)\big((N-1)-(p-1)+1\big)}.$$

We can now put the different cases together to compute $P'(\mu)$:

$$P'(\mu) = \sum_k \frac{m_k(\mu)}{p} \left(\sum_j \frac{j \cdot m_j(\mu')}{N-1} \cdot \frac{N-1}{(p-1)\big(N-p+1\big)} \right) = \frac{N}{p\big(N-p+1\big)}.$$

The last equality is obtained by a straight-forward computation and ends the proof of Proposition 5.6 and, therefore, of Proposition 2.2.   □

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