FPSAC 2010, San Francisco, USA DMTCS proc. **AN**, 2010, 653–664

Affine structures and a tableau model for E_6 crystals

Brant Jones[1†] and Anne Schilling[1‡]

[1] *Department of Mathematics, One Shields Avenue, University of California, Davis, CA 95616*

Abstract.

We provide the unique affine crystal structure for type $E_6^{(1)}$ Kirillov–Reshetikhin crystals corresponding to the multiples of fundamental weights $s\Lambda_1$, $s\Lambda_2$, and $s\Lambda_6$ for all $s \geq 1$ (in Bourbaki's labeling of the Dynkin nodes, where 2 is the adjoint node). Our methods introduce a generalized tableaux model for classical highest weight crystals of type E and use the order three automorphism of the affine $E_6^{(1)}$ Dynkin diagram. In addition, we provide a conjecture for the affine crystal structure of type $E_7^{(1)}$ Kirillov–Reshetikhin crystals corresponding to the adjoint node.

Résumé.

Nous donnons l'unique structure cristalline affine pour les cristaux de Kirillov–Reshetikhin de type $E_6^{(1)}$ correspondant aux multiples des poids fondamentaux $s\Lambda_1, s\Lambda_2$ et $s\Lambda_6$ pour tout $s \geq 1$ (dans l'étiquetage de Bourbaki des noeuds de Dynkin, où 2 est le noeud adjoint). Pour ceci, nous introduisons un modèle de tableaux généralisés pour les cristaux classiques du plus haut poids de type E et nous employons l'automorphisme d'ordre trois du diagramme de Dynkin du type $E_6^{(1)}$. En outre, nous fournissons une conjecture pour la structure affine pour les cristaux de Kirillov–Reshetikhin de type $E_7^{(1)}$ correspondant au noeud adjoint.

Keywords: Affine crystals, Kirillov–Reshetikhin crystals, type E_6

This document is an extended abstract of Jones and Schilling (2009). Please see the full paper for complete proofs.

1 Introduction

Let $\mathfrak{g}$ be an affine Kac–Moody algebra and $U_q'(\mathfrak{g})$ be the associated quantized affine algebra. Kirillov–Reshetikhin modules are finite dimensional $U_q'(\mathfrak{g})$-modules labeled by a node r of the Dynkin diagram together with a nonnegative integer s. It is expected that each Kirillov–Reshetikhin module has a crystal basis.

[†] BJ was partially supported by NSF grant DMS-0636297.

[‡] AS was partially supported by the NSF grants DMS–0501101, DMS–0652641, and DMS–0652652.

In this paper, we provide the unique affine crystal structure for the Kirillov–Reshetikhin crystals $B^{r,s}$ of type $E_6^{(1)}$ for the Dynkin nodes $r = 1, 2$, and 6 in the Bourbaki labeling, where node 2 corresponds to the adjoint node (see Figure 1). In addition, we provide a conjecture for the affine crystal structure for type $E_7^{(1)}$ Kirillov–Reshetikhin crystals of level s corresponding to the adjoint node.

Our construction of the affine crystals uses the classical decomposition given by Chari (2001) together with a promotion operator. Combinatorial models of all Kirillov–Reshetikhin crystals of nonexceptional types were constructed using promotion and similarity methods in Schilling (2008); Okado and Schilling (2008); Fourier et al. (2009) and perfectness was proven in Fourier et al. (2010). Affine crystals of type $E_6^{(1)}$ and $E_7^{(1)}$ of level 1 corresponding to minuscule coweights ($r = 1, 6$) have also been studied in Magyar (2006) using the Littelmann path model. Hernandez and Nakajima (2006) gave a construction of the Kirillov–Reshetihkin crystals $B^{r,1}$ for all r for type $E_6^{(1)}$ and most nodes r in type $E_7^{(1)}$.

For nonexceptional types, the classical crystals appearing in the decomposition can be described using Kashiwara–Nakashima tableaux Kashiwara and Nakashima (1994). We provide a similar construction for general types (see Theorem 2.6). This involves the explicit construction of the highest weight crystals $B(\Lambda_i)$ corresponding to fundamental weights Λ_i using the Lenart–Postnikov Lenart and Postnikov (2008) model and the notion of pairwise weakly increasing columns (see Definition 2.1).

This paper is structured as follows. In Section 2, the fundamental crystals $B(\Lambda_1)$ and $B(\Lambda_6)$ are constructed explicitly for type E_6 and it is shown that all other highest weight crystals $B(\lambda)$ of type E_6 can be constructed from these. In Section 2.4, a generalized tableaux model is given for $B(\lambda)$ for general types. These results are used to construct the affine crystals in Section 3. Our main results are stated in Theorem 3.10 and Conjecture 3.11.

2 A tableau model for finite-dimensional highest weight crystals

In this section, we describe a model for the classical highest weight crystals in type E. In Section 2.1, we introduce our notation and give the axiomatic definition of a crystal. The tensor product rule for crystals is reviewed in Section 2.2. In Section 2.3, we give an explicit construction of the highest weight crystals associated to the fundamental weights in types E_6 and E_7. In Section 2.4, we give a generalized tableaux model to realize all of the highest weight crystals in these types. The generalized tableaux are type-independent, and can be viewed as an extension of the Kashiwara–Nakashima tableaux Kashiwara and Nakashima (1994) to type E. For a general introduction to crystals we refer to Hong and Kang (2002).

2.1 Axiomatic definition of crystals

Denote by $\mathfrak{g}$ a symmetrizable Kac-Moody algebra, P the weight lattice, I the index set for the vertices of the Dynkin diagram of $\mathfrak{g}$, $\{\alpha_i \in P \mid i \in I\}$ the simple roots, and $\{\alpha_i^\vee \in P^* \mid i \in I\}$ the simple coroots. Let $U_q(\mathfrak{g})$ be the quantized universal enveloping algebra of $\mathfrak{g}$. A $U_q(\mathfrak{g})$-*crystal* Kashiwara (1995) is a nonempty set B equipped with maps $\mathrm{wt} : B \to P$ and $e_i, f_i : B \to B \cup \{\mathbf{0}\}$ for all $i \in I$, satisfying

$$\begin{aligned} f_i(b) = b' &\Leftrightarrow e_i(b') = b \text{ if } b, b' \in B \\ \mathrm{wt}(f_i(b)) &= \mathrm{wt}(b) - \alpha_i \text{ if } f_i(b) \in B \\ \langle \alpha_i^\vee, \mathrm{wt}(b) \rangle &= \varphi_i(b) - \varepsilon_i(b). \end{aligned}$$

Here, we have

$$\varepsilon_i(b) = \max\{n \geq 0 \mid e_i^n(b) \neq \mathbf{0}\}$$
$$\varphi_i(b) = \max\{n \geq 0 \mid f_i^n(b) \neq \mathbf{0}\}$$

for $b \in B$, and we denote $\langle \alpha_i^\vee, \mathrm{wt}(b)\rangle$ by $\mathrm{wt}_i(b)$. A $U_q(\mathfrak{g})$-crystal B can be viewed as a directed edge-colored graph called the *crystal graph* whose vertices are the elements of B, with a directed edge from b to b' labeled $i \in I$, if and only if $f_i(b) = b'$. Given $i \in I$ and $b \in B$, the *i-string through b* consists of the nodes $\{f_i^m(b) : 0 \leq m \leq \varphi_i(b)\} \cup \{e_i^m(b) : 0 < m \leq \varepsilon_i(b)\}$.

Let $\{\Lambda_i \mid i \in I\}$ be the fundamental weights of $\mathfrak{g}$. For every $b \in B$ define $\varphi(b) = \sum_{i\in I} \varphi_i(b)\Lambda_i$ and $\varepsilon(b) = \sum_{i\in I} \varepsilon_i(b)\Lambda_i$. An element $b \in B$ is called *highest weight* if $e_i(b) = \mathbf{0}$ for all $i \in I$. We say that B is a *highest weight crystal* of highest weight λ if it has a unique highest weight element of weight λ. For a dominant weight λ, we let $B(\lambda)$ denote the unique highest-weight crystal with highest weight λ.

An *isomorphism* of crystals is a bijection $\Psi : B \cup \{\mathbf{0}\} \to B' \cup \{\mathbf{0}\}$ such that $\Psi(\mathbf{0}) = \mathbf{0}$, $\varepsilon(\Psi(b)) = \varepsilon(b)$, $\varphi(\Psi(b)) = \varphi(b)$, $f_i\Psi(b) = \Psi(f_i(b))$, and $\Psi(e_i(c)) = e_i\Psi(c)$ for all $b, c \in B$, $\Psi(b), \Psi(c) \in B'$ where $f_i(b) = c$.

When $\widetilde{\lambda}$ is a weight in an affine type, we call

$$\langle \widetilde{\lambda}, c\rangle = \sum_{i\in I} a_i^\vee \langle \widetilde{\lambda}, \alpha_i^\vee\rangle \tag{1}$$

the *level* of $\widetilde{\lambda}$, where c is the canonical central element and $\widetilde{\lambda} = \sum_{i\in I} \lambda_i\Lambda_i$ is the affine weight. In our work, we will often compute the 0-weight $\lambda_0\Lambda_0$ at level 0 for a node b in a classical crystal from the classical weight $\lambda = \sum_{i\in I\setminus\{0\}} \lambda_i\Lambda_i = \mathrm{wt}(b)$ by setting $\langle \lambda_0\Lambda_0 + \lambda, c\rangle = 0$ and solving for λ_0.

When $\mathfrak{g}$ is a finite-dimensional Lie algebra, every integrable $U_q(\mathfrak{g})$-module decomposes as a direct sum of highest weight modules. On the level of crystals, this implies that every crystal graph B corresponding to an integrable module is a union of connected components, and each connected component is the crystal graph of a highest weight module. We denote this by $B = \bigoplus B(\lambda)$ for some set of dominant weights λ, and we call these $B(\lambda)$ the *components* of the crystal.

Suppose that $\mathfrak{g}$ is a symmetrizable Kac–Moody algebra and let $U_q'(\mathfrak{g})$ be the corresponding quantum algebra without derivation. The goal of this work is to study crystals $B^{r,s}$ that correspond to certain finite dimensional $U_q'(\mathfrak{g})$-modules known as Kirillov–Reshetikhin modules. Here, r is a node of the Dynkin diagram and s is a nonnegative integer. The existence of the crystals $B^{r,s}$ that we study follows from results in (Kang et al., 1992, Proposition 3.4.4) for $r = 1, 6$ and (Kang et al., 1992, Proposition 3.4.5) for $r = 2$, while the classical decomposition of these crystals is given in Chari (2001).

2.2 Tensor products of crystals

Let $B_1, B_2, \ldots, B_L$ be $U_q(\mathfrak{g})$-crystals. The Cartesian product $B_1 \times B_2 \times \cdots \times B_L$ has the structure of a $U_q(\mathfrak{g})$-crystal using the so-called signature rule. The resulting crystal is denoted $B = B_1 \otimes B_2 \otimes \cdots \otimes B_L$ and its elements $(b_1, \ldots, b_L)$ are written $b_1 \otimes \cdots \otimes b_L$ where $b_j \in B_j$. The reader is warned that our convention is opposite to that of Kashiwara Kashiwara (1995). Fix $i \in I$ and $b = b_1 \otimes \cdots \otimes b_L \in B$. The *$i$-signature* of b is the word consisting of the symbols $+$ and $-$ given by

$$\underbrace{-\cdots-}_{\varphi_i(b_1)\text{ times}}\ \underbrace{+\cdots+}_{\varepsilon_i(b_1)\text{ times}}\ \cdots\ \underbrace{-\cdots-}_{\varphi_i(b_L)\text{ times}}\ \underbrace{+\cdots+}_{\varepsilon_i(b_L)\text{ times}}.$$

The *reduced i-signature* of b is the subword of the i-signature of b, given by the repeated removal of adjacent symbols $+-$ (in that order); it has the form

$$\underbrace{-\cdots-}_{\varphi_i \text{ times}} \quad \underbrace{+\cdots+}_{\varepsilon_i \text{ times}}.$$

If $\varphi_i = 0$ then $f_i(b) = \mathbf{0}$; otherwise

$$f_i(b_1 \otimes \cdots \otimes b_L) = b_1 \otimes \cdots \otimes b_{j-1} \otimes f_i(b_j) \otimes \cdots \otimes b_L$$

where the rightmost symbol $-$ in the reduced i-signature of b comes from b_j. Similarly, if $\varepsilon_i = 0$ then $e_i(b) = \mathbf{0}$; otherwise

$$e_i(b_1 \otimes \cdots \otimes b_L) = b_1 \otimes \cdots \otimes b_{j-1} \otimes e_i(b_j) \otimes \cdots \otimes b_L$$

where the leftmost symbol $+$ in the reduced i-signature of b comes from b_j. It is not hard to verify that this defines the structure of a $U_q(\mathfrak{g})$-crystal with $\varphi_i(b) = \varphi_i$ and $\varepsilon_i(b) = \varepsilon_i$ in the above notation, and weight function

$$\mathrm{wt}(b_1 \otimes \cdots \otimes b_L) = \sum_{j=1}^{L} \mathrm{wt}(b_j).$$

2.3 Fundamental crystals for type E_6 and E_7

Let $I = \{1, 2, 3, 4, 5, 6\}$ denote the classical index set for E_6. We number the nodes of the affine Dynkin diagram as in Figure 1.

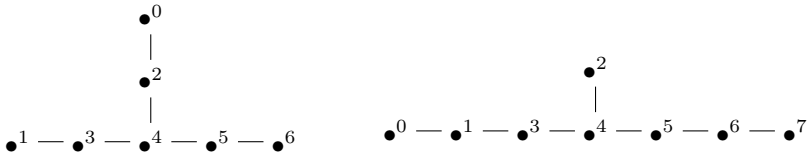

Fig. 1: Affine $E_6^{(1)}$ and $E_7^{(1)}$ Dynkin diagrams

Classical highest-weight crystals $B(\lambda)$ for E_6 can be realized by the Lenart–Postnikov alcove path model described in Lenart and Postnikov (2008). We implemented this model in `Sage` and have recorded the crystal $B(\Lambda_1)$ in Figure 2. This crystal has 27 nodes.

To describe our labeling of the nodes, observe that all of the i-strings in $B(\Lambda_1)$ have length 1 for each $i \in I$. Therefore, the crystal admits a transitive action of the Weyl group. Also, it is straightforward to verify that all of the nodes in $B(\Lambda_1)$ are determined by weight. For our work in Section 3, we also compute the 0-weight at level 0 of a node b in any classical crystal from the classical weight as described in Remark 3.4.

Thus, we label the nodes of $B(\Lambda_1)$ by weight, which is equivalent to recording which i-arrows come in and out of b. The i-arrows into b are recorded with an overline to indicate that they contribute negative weight, while the i-arrows out of b contribute positive weight.

By the symmetry of the Dynkin diagram, we have that $B(\Lambda_6)$ also has 27 nodes and is dual to $B(\Lambda_1)$ in the sense that its crystal graph is obtained from $B(\Lambda_1)$ by reversing all of the arrows. Reversing the arrows requires us to label the nodes of $B(\Lambda_6)$ by the weight that is the negative of the weight of the corresponding node in $B(\Lambda_1)$. Moreover, observe that $B(\Lambda_1)$ contains no pair of nodes with weights μ, $-\mu$, respectively. Hence, we can unambiguously label any node of $B(\Lambda_1) \cup B(\Lambda_6)$ by weight.

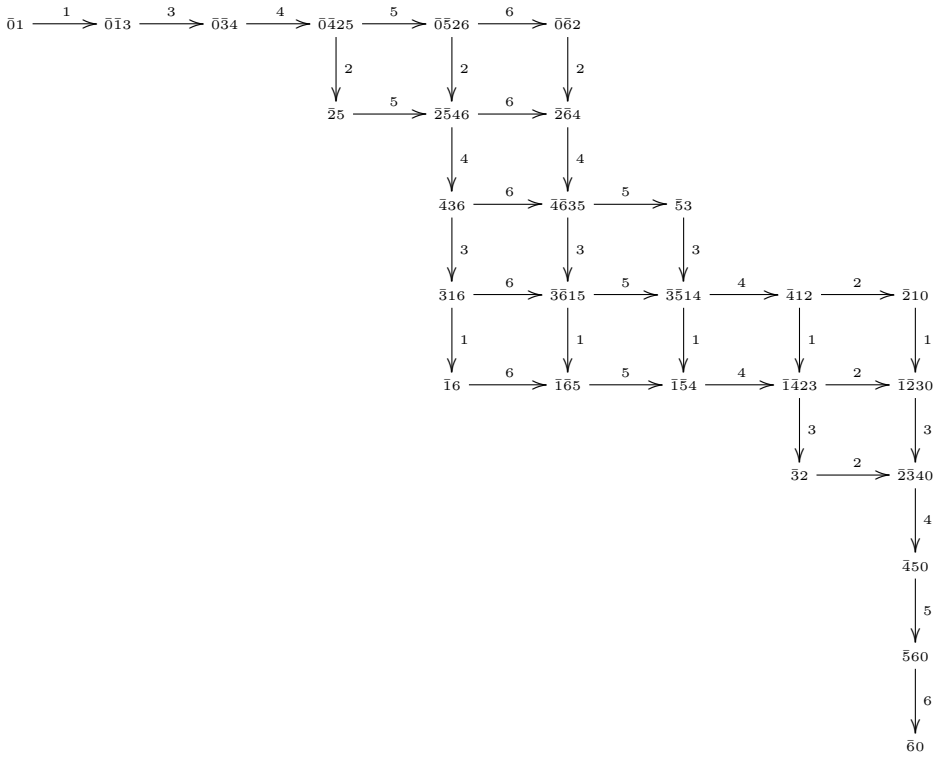

Fig. 2: Crystal graph for $B(\Lambda_1)$ of type E_6

It is straightforward to show using characters that every classical highest-weight representation $B(\Lambda_i)$ for $i \in I$ can be realized as a component of some tensor product of $B(\Lambda_1)$ and $B(\Lambda_6)$ factors. On the level of crystals, the tensor products $B(\Lambda_1)^{\otimes k}$, $B(\Lambda_6)^{\otimes k}$ and $B(\Lambda_6) \otimes B(\Lambda_1)$ are defined for all k by the tensor product rule of Section 2.2. Therefore, we can realize the other classical fundamental crystals $B(\Lambda_i)$ as shown in Table 1. There are additional realizations for these crystals obtained by dualizing.

There is a similar construction for the fundamental crystals for type E_7. The highest weight crystal $B(\Lambda_7)$ has 56 nodes and these nodes all have distinct weights. Also, $\varphi_i(b) \leq 1$ and $\varepsilon_i(b) \leq 1$ for all $i \in \{1, 2, \ldots, 7\}$ and $b \in B(\Lambda_7)$. Using character calculations, we can show that every classical highest-weight representation $B(\Lambda_i)$ appears in some tensor product of $B(\Lambda_7)$ factors.

Tab. 1: Fundamental realizations for E_6

	Generator	in	Dimension
$B(\Lambda_2)$	$2\bar{1}\bar{0} \otimes \bar{0}1$	$B(\Lambda_6) \otimes B(\Lambda_1)$	78
$B(\Lambda_3)$	$\bar{0}\bar{1}3 \otimes \bar{0}1$	$B(\Lambda_1)^{\otimes 2}$	351
$B(\Lambda_4)$	$\bar{0}\bar{3}4 \otimes \bar{0}\bar{1}3 \otimes \bar{0}1$	$B(\Lambda_1)^{\otimes 3}$	2925
$B(\Lambda_5)$	$5\bar{6}\bar{0} \otimes 6\bar{0}$	$B(\Lambda_6)^{\otimes 2}$	351

2.4 Generalized tableaux

In this section, we describe how to realize the crystal $B(\Lambda_{i_1} + \Lambda_{i_2} + \cdots + \Lambda_{i_k})$ inside the tensor product $B(\Lambda_{i_1}) \otimes B(\Lambda_{i_2}) \otimes \cdots \otimes B(\Lambda_{i_k})$, where the Λ_i are all fundamental, or more generally dominant weights. Our arguments use only abstract crystal properties, so the results in this section apply to any finite type.

If b is the unique highest weight node in $B(\lambda)$ and c is the unique highest weight node in $B(\mu)$, then $B(\lambda+\mu)$ is generated by $b \otimes c \in B(\lambda) \otimes B(\mu)$. Iterating this procedure provides a recursive description of any highest-weight crystal embedded in a tensor product of crystals. Our goal is to give a non-recursive description of the nodes of $B(\Lambda_{i_1} + \Lambda_{i_2} + \cdots + \Lambda_{i_k})$ for any collection of fundamental weights Λ_i.

For an ordered set of dominant weights $(\mu_1, \mu_2, \ldots, \mu_k)$ and for each permutation w in the symmetric group S_k, define

$$B_w(\mu_1, \ldots, \mu_k) = B(\mu_{w(1)}) \otimes B(\mu_{w(2)}) \otimes \cdots \otimes B(\mu_{w(k)})$$

so $B_e(\mu_1, \ldots, \mu_k)$ is $B(\mu_1) \otimes \cdots \otimes B(\mu_k)$ where $e \in S_k$ is the identity.

Definition 2.1 *Let $(\mu_1, \mu_2, \ldots, \mu_k)$ be dominant weights. Then, we say that*

$$b_1 \otimes b_2 \otimes \cdots \otimes b_k \in B(\mu_1) \otimes B(\mu_2) \otimes \cdots \otimes B(\mu_k)$$

is pairwise weakly increasing *if*

$$b_j \otimes b_{j+1} \in B(\mu_j + \mu_{j+1}) \subset B(\mu_j) \otimes B(\mu_{j+1})$$

for each $1 \leq j < k$.

Next, we fix an isomorphism of crystals

$$\Phi_w^{(\mu_1,\ldots,\mu_k)} : B_w(\mu_1, \ldots, \mu_k) \to B_e(\mu_1, \ldots, \mu_k)$$

for each $w \in S_k$. Observe that each choice of $\Phi_w^{(\mu_1,\ldots,\mu_k)}$ corresponds to a choice for the image of each of the highest-weight nodes in $B_w(\mu_1, \ldots, \mu_k)$.

Let b_j^* denote the unique highest weight node of the jth factor $B(\mu_j)$. Since we are fixing the dominant weights $(\mu_1, \ldots, \mu_k)$, we will sometimes drop the notation $(\mu_1, \ldots, \mu_k)$ from B_w and Φ_w.

Definition 2.2 *Let w be a permutation and choose j to be the maximal integer such that w that fixes $\{1, 2, \ldots, j\}$. We say that $\Phi_w^{(\mu_1,\ldots,\mu_k)}$ is a* lazy isomorphism *if the image of every highest weight node of the form*

$$b_1 \otimes b_2 \otimes \cdots \otimes b_j \otimes b_{j+1}^* \otimes \cdots \otimes b_k^*$$

under $\Phi_w^{(\mu_1,\ldots,\mu_k)}$ is equal to

$$b_1 \otimes b_2 \otimes \cdots \otimes b_j \otimes b^*_{w^{-1}(j+1)} \otimes \cdots \otimes b^*_{w^{-1}(k)} \ .$$

We want to choose our isomorphisms $\Phi_w^{(\mu_1,\ldots,\mu_k)}$ to be lazy, but our results do not otherwise depend upon the choice of $\Phi_w^{(\mu_1,\ldots,\mu_k)}$.

Definition 2.3 *Let T be any subset of S_k, and $\{\Phi_w^{(\mu_1,\ldots,\mu_k)}\}_{w\in T}$ be a collection of lazy isomorphisms. We define $I^{(\mu_1,\ldots,\mu_k)}(T)$ to be*

$$\bigcap_{w\in T} \Phi_w^{(\mu_1,\ldots,\mu_k)}(\{\textit{pairwise weakly increasing nodes of } B_w(\mu_1,\ldots,\mu_k)\ \}) \subset B_e(\mu_1,\ldots,\mu_k).$$

Proposition 2.4 *Let T be any subset of S_k. Then, whenever $b \in I^{(\mu_1,\ldots,\mu_k)}(T)$ we have $e_i(b), f_i(b) \in I^{(\mu_1,\ldots,\mu_k)}(T)$.*

Corollary 2.5 *For any subset T of S_k, we have that $I^{(\mu_1,\ldots,\mu_k)}(T)$ is a direct sum of highest weight crystals $\bigoplus_\lambda B(\lambda)$ for some collection of weights λ.*

Proof: Proposition 2.4 implies that whenever $b \in I^{(\mu_1,\ldots,\mu_k)}(T)$, the entire connected component of the crystal graph containing b is in $I^{(\mu_1,\ldots,\mu_k)}(T)$. □

Theorem 2.6 *Fix a sequence $(\mu_1,\ldots,\mu_k)$ of dominant weights. Then,*

$$I^{(\mu_1,\ldots,\mu_k)}(S_k) \cong B(\mu_1+\mu_2+\ldots+\mu_k).$$

Proof: Let b_j^* be the unique highest weight node of B_j with highest weight μ_j for each $j = 1,\ldots,k$. Then $b^* = b_1^* \otimes b_2^* \otimes \cdots \otimes b_k^*$ generates $B(\mu_1+\ldots+\mu_k)$ and this node lies in $I^{(\mu_1,\ldots,\mu_k)}(S_k)$.

The proof proceeds to show that b^* is the only highest weight node of $I^{(\mu_1,\ldots,\mu_k)}(S_k)$ using calculations involving the tensor product rule. □

Remark 2.7 *The condition that there is a unique highest weight element that we used in the proof of Theorem 2.6 is equivalent to the hypothesis of (Kashiwara and Nakashima, 1994, Proposition 2.2.1) from which the desired conclusion also follows.*

Remark 2.8 *Because we only require a constant amount of data to check the pairwise weakly increasing condition for each pair of tensor factors, Theorem 2.6 and its refinements will allow us to formulate arguments that apply to all highest-weight crystals simultaneously, regardless of the number of tensor factors.*

When we are considering a specific highest-weight crystal, it may be computationally easier to generate $B(\mu_1+\cdots+\mu_k)$ by simply applying f_i operations to the highest-weight node in all possible ways.

We say that any node of $I^{(\mu_1,\ldots,\mu_k)}(S_k)$ is *weakly increasing*. It turns out that we can often take T to be much smaller than S_k by starting with $T = \{e\}$ and adding permutations to T until $I^{(\mu_1,\ldots,\mu_k)}(T)$ contains a unique highest weight node. In particular, the next result shows that we can take $T = \{e\}$ when we are considering a linear combination of two distinct fundamental weights.

Lemma 2.9 *Let Λ_{i_1} and Λ_{i_2} be distinct fundamental weights, and $k_1, k_2 \in \mathbb{Z}_{\geq 0}$ with $k = k_1 + k_2$. Then, the nodes of*

$$B(k_1\Lambda_{i_1} + k_2\Lambda_{i_2}) \subset B(\Lambda_{i_1})^{\otimes k_1} \otimes B(\Lambda_{i_2})^{\otimes k_2}$$

are precisely the pairwise weakly increasing tensor products $b_1 \otimes b_2 \otimes \cdots \otimes b_k$ of $B(\Lambda_{i_1})^{\otimes k_1} \otimes B(\Lambda_{i_2})^{\otimes k_2}$.

All of the crystals in our work have classical decompositions that have been given by Chari (2001). These crystals satisfy the requirement of Lemma 2.9 that at most two fundamental weights appear. On the other hand, there exist examples showing that no ordering of the factors in $B(\Lambda_2) \otimes B(\Lambda_1) \otimes B(\Lambda_6)$ in type E_6 admits an analogous weakly increasing condition that is defined using only pairwise comparisons.

We now restrict to type E_6. Lemma 2.9 implies that we have a non-recursive description of all $B(k\Lambda_i)$ determined by the finite information in $B(2\Lambda_i)$. In the case of particular fundamental representations, we can be more specific about how to test for the weakly increasing condition.

Proposition 2.10 *We have that $b_1 \otimes b_2 \in B(2\Lambda_1) \subset B(\Lambda_1)^{\otimes 2}$ if and only if b_2 can be reached from b_1 by a sequence of f_i operations in $B(\Lambda_1)$.*

Proof: This is a finite computation on $B(2\Lambda_1)$. □

The crystal graph for $B(\Lambda_1)$ of Figure 2 can be viewed as a poset. Then Proposition 2.10 implies in particular that incomparable pairs in $B(\Lambda_1)$ are not weakly increasing.

There are 78 nodes in $B(\Lambda_2)$. We construct $B(\Lambda_2)$ as the highest weight crystal graph generated by $2\bar{1}\bar{0} \otimes \bar{0}1$ inside $B(\Lambda_6) \otimes B(\Lambda_1)$. Note that we only need to use the nodes in the "top half" of Figure 2 and their duals. There are 2430 nodes in $B(2\Lambda_2)$.

Proposition 2.11 *We have that*

$$(b_1 \otimes c_1) \otimes (b_2 \otimes c_2) \in B(2\Lambda_2) \subset (B(\Lambda_6) \otimes B(\Lambda_1))^{\otimes 2}$$

if and only if

(1) b_2 can be reached from b_1 by f_i operations in $B(\Lambda_6)$, and c_2 can be reached from c_1 by f_i operations in $B(\Lambda_1)$, and

(2) Whenever c_1 is dual to b_2, we have that there is a path of f_i operations from $(b_1 \otimes c_1)$ to $(b_2 \otimes c_2)$ of length at least 1 (so in particular, the elements are not equal) in $B(\Lambda_2)$.

Proof: This is a finite computation on $B(2\Lambda_2)$. □

3 Affine structures

In this section, we study the affine crystals of type $E_6^{(1)}$. We introduce the method of promotion to obtain a combinatorial affine crystal structure in Section 3.1 and the notion of composition graphs in Section 3.2. It is shown in Theorem 3.7 that order three twisted isomorphisms yield regular affine crystals. This is used to construct $B^{r,s}$ of type $E_6^{(1)}$ for the minuscule nodes $r = 1, 6$ and the adjoint node $r = 2$. We summarize these results in Section 3.3 along with a conjecture for $B^{1,s}$ of type $E_7^{(1)}$.

3.1 Combinatorial affine crystals and twisted isomorphisms

The following concept is fundamental to this work.

Definition 3.1 *Let $\widetilde{C}$ be an affine Dynkin diagram and C the associated finite Dynkin diagram (obtained by removing node 0) with index set I. Let $\dot{p}$ be an automorphism of $\widetilde{C}$, and B be a classical crystal of type C. We say that $\dot{p}$* induces a twisted isomorphism of crystals *if there exists a bijection of crystals $p : B \cup \{\mathbf{0}\} \to B' \cup \{\mathbf{0}\}$ satisfying*

$$p(b) = \mathbf{0} \text{ if and only if } b = \mathbf{0}, \text{ and} \tag{2}$$

$$p \circ f_i(b) = f_{\dot{p}(i)} \circ p(b) \text{ and } p \circ e_i(b) = e_{\dot{p}(i)} \circ p(b) \tag{3}$$

for all $i \in I \setminus \{\dot{p}^{-1}(0)\}$ and all $b \in B$.

We frequently abuse notation and denote B' by $p(B)$ even though the isomorphism $p : B \to p(B)$ may not be unique.

If we are given two classical crystals B and B', and there exists a Dynkin diagram automorphism $\dot{p}$ that induces a twisted isomorphism between B and B', then we say that B and B' are twisted-isomorphic.

Definition 3.2 *Let B be a directed graph with edges labeled by I. Then B is called* regular *if for any 2-subset $J \subset I$, we have that the restriction of B to its J-arrows is a classical rank two crystal.*

Definition 3.3 *Let B be a classical crystal with index set I. Suppose $\widetilde{B}$ is a labeled directed graph on the same nodes as B and with the same I-arrows, but with an additional set of 0-arrows. If $\widetilde{B}$ is regular with respect to $I \cup \{0\}$, then we say that $\widetilde{B}$ is a* combinatorial affine structure *for B.*

Remark 3.4 *Although we do not assume that $\widetilde{B}$ is a crystal graph for a $U'_q(\mathfrak{g})$-module, Kashiwara (2002, 2005) has shown that the crystals of such modules must be regular and have weights at level 0. Therefore, we compute the 0-weight $\lambda_0\Lambda_0$ of the nodes b in a classical crystal from the classical weight $\lambda = \sum_{i\in I} \lambda_i\Lambda_i = \mathrm{wt}(b)$ using the formula given in Equation (1) (recall that I in this section is the index set of the Dynkin diagram without 0).*

Remark 3.5 *Here are some consequences of Definitions 3.1 and 3.3.*

(1) Any crystal $p(B)$ induced by $\dot{p}$ is just a classical crystal that is isomorphic to B up to relabeling. In particular, any graph automorphism $\dot{p}$ induces at least one twisted isomorphism p: If we view B as an edge-labeled directed graph, the image of p is given on the same nodes as B by relabeling all of the arrows according to $\dot{p}$. On the other hand, it is important to emphasize that there is no canonical labeling for the nodes of $p(B)$. Also, some crystal graphs may have additional symmetry which lead to multiple twisted isomorphisms of crystals associated with a single graph automorphism $\dot{p}$.

(2) For $b \in B$, we have $\varphi(p(b)) = \sum_{i\in I} \varphi_{\dot{p}^{-1}(i)}(b)\Lambda_i$ and $\varepsilon(p(b)) = \sum_{i\in I} \varepsilon_{\dot{p}^{-1}(i)}(b)\Lambda_i$. In addition, we can compute the 0-weight of any node in B by Remark 3.4. Therefore, $\dot{p}$ permutes all of the affine weights, in the sense that

$$\mathrm{wt}_i(b) = \mathrm{wt}_{\dot{p}(i)}(p(b)) \quad \text{for all } b \in B \text{ and } i \in I \cup \{0\} \ .$$

(3) Since the node $\dot{p}(0)$ becomes the affine node in $p(B)$, it is sometimes possible to define a combinatorial affine structure for B "by promotion." Namely, we define f_0 on B to be $p^{-1} \circ f_{\dot{p}(0)} \circ p$. Note that in order for this to succeed, we must take the additional step of identifying the image $p(B)$ with a canonically labeled classical crystal so that we can infer the $f_{\dot{p}(0)}$ edges.

Example 3.6 *The E_6 Dynkin diagram automorphism of order two that interchanges nodes 1 and 6 induces the dual map between $B(\Lambda_1)$ and $B(\Lambda_6)$.*

The Dynkin diagram of $E_6^{(1)}$ has an automorphism of order three that we can use to construct combinatorial affine structures by promotion.

Theorem 3.7 *Let B be a classical E_6 crystal. Suppose there exists a bijection $p : B \to B$ that is a twisted isomorphism satisfying $p \circ f_1 = f_6 \circ p$, and suppose that p has order three. Then, there exists a combinatorial affine structure on B. This structure is given by defining f_0 to be $p^2 \circ f_1 \circ p$.*

Proof: If we apply p on the left and right of $pf_1 = f_6p$, we obtain $ppf_1p = pf_6pp$. Since p has order three, this is

$$p^{-1}f_1p = pf_6p^{-1}. \tag{4}$$

Because p is a bijection on B, we may define 0-arrows on B by the map $p^{-1}f_1p$. By the hypotheses, p must be induced by the unique Dynkin diagram automorphism $\dot{p}$ of order three that sends node 0 to 1.

To verify that this affine structure satisfies Definition 3.3, we need to check that restricting B to $\{0, i\}$-arrows is a crystal for all $i \in I$. Each of these restrictions corresponds to a rank 2 classical crystal, and Stembridge has given local rules in Stembridge (2003) that characterize such classical crystals in simply laced types. These rules depend only on calculations involving $\varphi_i(b)$ and $\varepsilon_i(b)$ at each node $b \in B$, and these quantities are preserved by twisted isomorphism.

Hence, we obtain a combinatorial affine structure for B. □

From now on, we use the notation p to denote a twisted isomorphism induced by $\dot{p}$ sending

$$0 \mapsto 1 \mapsto 6 \mapsto 0, 2 \mapsto 3 \mapsto 5 \mapsto 2, 4 \mapsto 4.$$

Also, we let $\dot{p}$ act on the affine weight lattice as in Remark 3.5(2).

3.2 Composition graphs

Let $I = \{1, 2, \ldots, 6\}$ be the index set for the Dynkin diagram of E_6, and $\widetilde{I} = I \cup \{0\}$ be the index set of $E_6^{(1)}$. Suppose $J \subset I$. Consider a classical crystal B of the form $\bigoplus B(k\Lambda)$ where Λ is a fundamental weight and we sum over some collection of nonnegative integers k. Let $H^J(B)$ denote the $(I \setminus J)$-highest weight nodes of B. We will study affine crystals with B as underlying classical crystal. For a given such affine crystal, let $H^{J;0}(B)$ be the $(\widetilde{I} \setminus J)$-highest weight nodes. Using the level 0 hypothesis of Remark 3.4, we can prove properties of $H^{J;0}(B)$ for any given affine crystal with B as underlying classical crystal.

Our general strategy to define a twisted isomorphism p on a classical crystal B is to first define p on $H^J(B)$, and then extend this definition to the rest of B using Equation (3). To accomplish this, we introduce the following model for the nodes in $H^J(B)$ and $H^{J;0}(B)$.

Definition 3.8 *Fix $J \subset I$ and form directed graphs G_J and $G_{J;0}$ as follows.*

We construct the vertices of G_J and $G_{J;0}$ iteratively, beginning with all of the $(I \setminus J)$-highest weight nodes of $B(\Lambda)$. Then, we add all of the vertices $b \in B(\Lambda)$ such that

$$\{i \in I : \varepsilon_i(b) > 0\} \subset J \cup \{i \in I : \text{there exists } b' \in G_J \text{ with } b \otimes b' \text{ pairwise weakly increasing and } \varphi_i(b') > 0\}$$

to G_J. Moreover, if b also satisfies the property that there exists $b' \in G_{J;0}$ with $b \otimes b'$ pairwise weakly increasing and $\mathrm{wt}_0(b') > 0$ whenever $\mathrm{wt}_0(b) < 0$, then we add b to $G_{J;0}$. We repeat this construction until no new vertices are added. This process eventually terminates since $B(\Lambda)$ is finite.

The edges of G_J and $G_{J;0}$ are determined by the pairwise weakly increasing condition described in Definition 2.1. Note that some nodes may have loops. We call G_J and $G_{J;0}$ the complete composition graph *for J and $J;0$, respectively.*

Lemma 3.9 *Every element of $H^J(B)$ and $H^{J;0}(B)$ is a pairwise weakly increasing tensor product of vertices that form a directed path in G_J, respectively $G_{J;0}$, where the element in $B(0) \subset H^J(B)$ is identified with the empty tensor product.*

3.3 Further results

Using composition graphs and the tableau model, we are able to prove the following result which gives an affine structure for the Kirillov–Reshetikhin crystal $B^{2,s}$.

Theorem 3.10 *There exists a unique twisted isomorphism $p : \bigoplus_{k=0}^{s} B(k\Lambda_2) \to \bigoplus_{k=0}^{s} B(k\Lambda_2)$ of order three. This isomorphism sends an $I \setminus \{6\}$-highest weight node b from component k to the unique $I \setminus \{1\}$-highest weight node b' in component $(s-k) + (\mathrm{wt}_2(b) + \mathrm{wt}_3(b) + \mathrm{wt}_5(b))$ satisfying $\mathrm{wt}_{\dot{p}(i)}(b') = \mathrm{wt}_i(b)$ for each $i \in \{2, 3, 5\}$.*

We also obtain analogous results for $B^{1,s}$ and $B^{6,s}$. Furthermore, we provide a conjecture for the adjoint crystal $B^{1,s}$ in type $E_7^{(1)}$.

Conjecture 3.11 *Define $p : \bigoplus_{k=0}^{s} B(k\Lambda_1) \to \bigoplus_{k=0}^{s} B(k\Lambda_1)$ on the $I \setminus \{7\}$-highest weight nodes by sending $b \in B(k\Lambda_1)$ to the unique $I \setminus \{7\}$-highest weight node b' in component $(s-k) + (\mathrm{wt}_1(b) + \mathrm{wt}_2(b) + \mathrm{wt}_6(b))$ satisfying $\mathrm{wt}_{\dot{p}(i)}(b') = \mathrm{wt}_i(b)$ for each $i \in \{1, 2, 6\}$.*

Let $f_0 = p \circ f_7 \circ p$. Then f_0 commutes with f_7 so we obtain a combinatorial affine structure on $\bigoplus_{k=0}^{s} B(k\Lambda_1)$, which is isomorphic to $B^{1,s}$ of type $E_7^{(1)}$.

We have verified this conjecture for $s \leq 2$.

Acknowledgements

We thank Daniel Bump, Jesus DeLoera, Matthias Koeppe and Masato Okado for helpful conversations about this work.

This research was driven by computer exploration using the open-source mathematical software `Sage` and its algebraic combinatorics features developed by the `Sage-Combinat` community; we are grateful to Nicolas M. Thiéry for all his support.

References

V. Chari. On the fermionic formula and the Kirillov-Reshetikhin conjecture. *Internat. Math. Res. Notices*, (12):629–654, 2001. ISSN 1073-7928.

G. Fourier, M. Okado, and A. Schilling. Kirillov-Reshetikhin crystals for nonexceptional types. *Advances in Mathematics*, 222:1080–1116, 2009.

G. Fourier, M. Okado, and A. Schilling. Perfectness of Kirillov–Reshetikhin crystals for nonexceptional types. *Contemp. Math.*, 506:127–143, 2010.

D. Hernandez and H. Nakajima. Level 0 monomial crystals. *Nagoya Math. J.*, 184:85–153, 2006. ISSN 0027-7630. URL `http://projecteuclid.org/getRecord?id=euclid.nmj/1167159343`.

J. Hong and S.-J. Kang. *Introduction to quantum groups and crystal bases*, volume 42 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2002. ISBN 0-8218-2874-6.

B. Jones and A. Schilling. Affine structures and a tableau model for E_6 crystals. `arXiv:0909.2442`, 2009.

S.-J. Kang, M. Kashiwara, K. C. Misra, T. Miwa, T. Nakashima, and A. Nakayashiki. Perfect crystals of quantum affine Lie algebras. *Duke Math. J.*, 68(3):499–607, 1992. ISSN 0012-7094.

M. Kashiwara. On crystal bases. In *Representations of groups (Banff, AB, 1994)*, volume 16 of *CMS Conf. Proc.*, pages 155–197. Amer. Math. Soc., Providence, RI, 1995.

M. Kashiwara. On level-zero representations of quantized affine algebras. *Duke Math. J.*, 112(1):117–175, 2002. ISSN 0012-7094.

M. Kashiwara. Level zero fundamental representations over quantized affine algebras and Demazure modules. *Publ. Res. Inst. Math. Sci.*, 41(1):223–250, 2005. ISSN 0034-5318.

M. Kashiwara and T. Nakashima. Crystal graphs for representations of the q-analogue of classical Lie algebras. *J. Algebra*, 165(2):295–345, 1994. ISSN 0021-8693.

C. Lenart and A. Postnikov. A combinatorial model for crystals of Kac-Moody algebras. *Trans. Amer. Math. Soc.*, 360(8):4349–4381, 2008. ISSN 0002-9947.

P. Magyar. Littelmann paths for the basic representation of an affine Lie algebra. *J. Algebra*, 305(2): 1037–1054, 2006. ISSN 0021-8693.

M. Okado and A. Schilling. Existence of Kirillov-Reshetikhin crystals for nonexceptional types. *Represent. Theory*, 12:186–207, 2008. ISSN 1088-4165.

A. Schilling. Combinatorial structure of Kirillov-Reshetikhin crystals of type $D_n^{(1)}, B_n^{(1)}, A_{2n-1}^{(2)}$. *J. Algebra*, 319(7):2938–2962, 2008. ISSN 0021-8693.

J. R. Stembridge. A local characterization of simply-laced crystals. *Trans. Amer. Math. Soc.*, 355(12): 4807–4823 (electronic), 2003. ISSN 0002-9947.