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Enumerating (2+2)-free posets by the number of minimal elements and other statistics

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Abstract. A poset is said to be $(\mathbf{2}+\mathbf{2})$-free if it does not contain an induced subposet that is isomorphic to $\mathbf{2}+\mathbf{2}$, the union of two disjoint 2-element chains. In a recent paper, Bousquet-Mélou et al. found, using so called ascent sequences, the generating function for the number of $(\mathbf{2}+\mathbf{2})$-free posets: $P(t) = \sum_{n\geq 0}\prod_{i=1}^{n}\left(1-(1-t)^i\right)$. We extend this result by finding the generating function for $(\mathbf{2}+\mathbf{2})$-free posets when four statistics are taken into account, one of which is the number of minimal elements in a poset. We also show that in a special case when only minimal elements are of interest, our rather involved generating function can be rewritten in the form $P(t,z) = \sum_{n,k\geq 0} p_{n,k}t^n z^k = 1+\sum_{n\geq 0}\frac{zt}{(1-zt)^{n+1}}\prod_{i=1}^{n}(1-(1-t)^i)$ where $p_{n,k}$ equals the number of $(\mathbf{2}+\mathbf{2})$-free posets of size n with k minimal elements.

Résumé. Un poset sera dit $(\mathbf{2}+\mathbf{2})$-libre s'il ne contient aucun sous-poset isomorphe à $\mathbf{2}+\mathbf{2}$, l'union disjointe de deux chaînes à deux éléments. Dans un article récent, Bousquet-Mélou et al. ont trouvé, à l'aide de "suites de montées", la fonction génératrice des nombres de posets $(\mathbf{2}+\mathbf{2})$-libres: c'est $P(t) = \sum_{n\geq 0}\prod_{i=1}^{n}\left(1-(1-t)^i\right)$. Nous étendons ce résultat en trouvant la fonction génératrice des posets $(\mathbf{2}+\mathbf{2})$-libres rendant compte de quatre statistiques, dont le nombre d'éléments minimaux du poset. Nous montrons aussi que lorsqu'on ne s'intéresse qu'au nombre d'éléments minimaux, notre fonction génératrice assez compliquée peut être simplifiée en $P(t,z) = \sum_{n,k\geq 0} p_{n,k}t^n z^k = 1+\sum_{n\geq 0}\frac{zt}{(1-zt)^{n+1}}\prod_{i=1}^{n}(1-(1-t)^i)$, où $p_{n,k}$ est le nombre de posets $(\mathbf{2}+\mathbf{2})$-libres de taille n avec k éléments minimaux.

Keywords: (2+2)-free posets, minimal elements, generating function

1 Introduction

A poset is said to be $(\mathbf{2}+\mathbf{2})$-*free* if it does not contain an induced subposet that is isomorphic to $\mathbf{2}+\mathbf{2}$, the union of two disjoint 2-element chains. We let $\mathcal{P}$ denote the set of $(\mathbf{2}+\mathbf{2})$-free posets. Fishburn [7] showed that a poset is $(\mathbf{2}+\mathbf{2})$-free precisely when it is isomorphic to an *interval order*. Bousquet-Mélou et al. [1] showed that the generating function for the number p_n of $(\mathbf{2}+\mathbf{2})$-free posets on n elements is

$$P(t) = \sum_{n\geq 0} p_n\, t^n = \sum_{n\geq 0}\prod_{i=1}^{n}\left(1-(1-t)^i\right). \tag{1}$$

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In fact, El-Zahar [4] and Khamis [9] used a recursive description of $(\mathbf{2}+\mathbf{2})$-free posets, different from that of [1], to derive a pair of functional equations that define the series $P(t)$. However, they did not solve these equations. Haxell, McDonald and Thomasson [8] provided an algorithm, based on a complicated recurrence relation, to produce the first numbers p_n. Moreover, the above series was proved by Zagier [12] to count certain involutions introduced by Stoimenow [10]. Bousquet-Mélou et al. [1] gave a bijection between $(\mathbf{2}+\mathbf{2})$-free posets and the involutions, as well as a certain class of restricted permutations and so called *ascent sequences*. Given an integer sequence $(x_1,\ldots,x_i)$, the number of *ascents* of this sequence is

$$\mathrm{asc}(x_1,\ldots,x_i)=|\{\,1\le j<i\,:\,x_j<x_{j+1}\,\}|.$$

A sequence $(x_1,\ldots,x_n)\in\mathbb{N}^n$ an *ascent sequence of length* n if it satisfies $x_1=0$ and $x_i\in[0,1+\mathrm{asc}(x_1,\ldots,x_{i-1})]$ for all $2\le i\le n$. For instance, (0, 1, 0, 2, 3, 1, 0, 0, 2) is an ascent sequence. We let $\mathcal{A}$ denote the set of all ascent sequences (we assume the empty word to be an ascent sequence).

Amongst other results concerning $(\mathbf{2}+\mathbf{2})$-free posets [5, 6], the following characterization plays an important role in [1]: a poset is $(\mathbf{2}+\mathbf{2})$-free if and only if the collection of strict principal down-sets (for an element, a down-set is the set of its predecessors) can be linearly ordered by inclusion [6]. Here for any poset $\mathcal{P}=(P,<_p)$ and $x\in P$, the strict principal down set of x, $D(x)$, in $\mathcal{P}$ is the set of all $y\in P$ such that $y<_p x$. The *trivial down-set* is the empty set. Thus if $\mathcal{P}$ is a $(\mathbf{2}+\mathbf{2})$-free poset, we can write $D(P)=\{D(x):x\in P\}$ as

$$D(P)=\{D_0,D_1,\ldots,D_k\}$$

where $\emptyset=D_0\subset D_1\subset\cdots\subset D_k$. In such a situation, we say that $x\in P$ has level i if $D(x)=D_i$.

Bousquet-Mélou et al. [1] described a decomposition of a $(\mathbf{2}+\mathbf{2})$-free poset removing at each step a maximal element located on the lowest level, together with certain relations. Recording the levels from which we just removed a maximal element, and reading the obtained sequence backwards after removing all the elements, one obtains an ascent sequence. This gives a bijection between $(\mathbf{2}+\mathbf{2})$-free posets and ascent sequences. We note that in the process of decomposing a $(\mathbf{2}+\mathbf{2})$-free poset, element by element, at some point, the current poset will be a (possibly 1-element) antichain. The statistic lds is defined as the size of the (maximum) antichain in the last sentence, which is the size of the down-set of the last removed element that has a non-trivial down-set. By definition, the value of lds on an antichain is 0 (there are no non-trivial down-sets there). We refer to [1, Section 3] for the detailed description of the decomposition, as it is rather space-consuming to state here.

Bousquet-Mélou et al. [1] studied a more general generating function $F(t,u,v)$ of $(\mathbf{2}+\mathbf{2})$-free posets, which are counted by size="number of elements" (variable t), levels="*number of levels*" defined below (variable u), and minmax="level of minimum maximal element" (variable v). The first few terms of $F(t,u,v)$ are

$$F(t,u,v)=1+t+(1+uv)t^2+(1+2uv+u+u^2v^2)t^3+O(t^4).$$

An explicit form of $F(t,u,v)$ can be obtained from [1, Lemma 13] and [1, Proposition 14]. The main result of this paper, Theorem 4, is an explicit form of the generating function $G(t,u,v,z,x)$ for a generalization of $F(t,u,v)$, when two more statistics are taken into account — min="number of minimal elements" in a poset (variable z) and lds="size of *non-trivial last down-set*" (variable x). That is, we shall study the following generating function:

$$G(t,u,v,z,x)=\sum_{p\in\mathcal{P}}t^{\mathrm{size}(p)}u^{\mathrm{levels}(p)}v^{\mathrm{minmax}(p)}z^{\mathrm{min}(p)}x^{\mathrm{lds}(p)}.$$

Reduction of the main problem to considering ascent sequences. The basic idea used by Bousquet-Mélou et al. [1] to find the generating function $F(t,u,v)$ was to reduce the problem to counting ascent sequences using their bijection between $(\mathbf{2}+\mathbf{2})$-free posets and ascent sequences. We follow a similar strategy to find $G(t,u,v,z,x)$. That is, we define the following statistics on an ascent sequence: length="the number of elements in the sequence," last="the rightmost element of the sequence," zeros="the number of 0's in the sequence," run="the number of elements in the leftmost run of 0's"="the number of 0's to the left of the leftmost non-zero element." By definition, if there are no non-zero elements in an ascent sequence, the value of run is 0.

Lemma 1 *The function $G(t,u,v,z,x)$ defined above can alternatively be defined on ascent sequences as*

$$G(t,u,v,z,x)=\sum_{w\in\mathcal{A}} t^{\mathrm{length}(w)}u^{\mathrm{asc}(w)}v^{\mathrm{last}(w)}z^{\mathrm{zeros}(w)}x^{\mathrm{run}(w)}=\sum_{n,a,\ell,m,r\geq 0} G_{n,a,\ell,m,r}t^n u^a v^\ell z^m x^r.$$

Proof: To prove the statement we need to show equidistribution of the statistics involved. All but one case follow from the results in [1]. More precisely, we can use the bijection from $(\mathbf{2}+\mathbf{2})$-free posets to ascent sequences presented in [1] which sends $\mathrm{size}\to\mathrm{length}$, $\mathrm{levels}\to\mathrm{asc}$, $\mathrm{minmax}\to\mathrm{last}$, and $\min\to\mathrm{zeros}$.

The fact that lds goes to run follows from the definition of the statistics and the idea of the bijection in [1] described above. Indeed, while recording levels of just removed elements, after we removed the element, say e, whose down-set gives lds, we will be left with incomparable elements located on level 0, which gives in the corresponding ascent sequence the initial run of 0's followed by 1 corresponding to e located on level 1. □

Note that $G(t,u,v,1,1)=F(t,u,v)$ as studied in [1].

Organization of the paper. In Section 2 we find explicitly the function $G=G(t,u,v,z,x)$ using ascent sequences (see Theorem 4). In Section 3 we show that in a special case when only minimal elements are of interest, a rather involved generating function $G(t,u,v,z,x)$ can be rewritten in the form

$$P(t,z)=\sum_{n,k\geq 0} p_{n,k}t^n z^k = 1+\sum_{n\geq 0}\frac{zt}{(1-zt)^{n+1}}\prod_{i=1}^{n}(1-(1-t)^i)$$

where $p_{n,k}$ equals the number of $(\mathbf{2}+\mathbf{2})$-free posets of size n with k minimal elements. We shall see that our expression for $P(t,z)$ cannot be directly derived from $G(t,u,v,z,x)$ by substituting 1 for the variables u,v, and x.

2 Main results

For $r\geq 1$, let $G_r(t,u,v,z)$ denote the coefficient of x^r in $G(t,u,v,z,x)$. Thus $G_r(t,u,v,z)$ is the generating function of those ascent sequences that begin with $r\geq 1$ 0's followed by 1. We let $G^r_{a,l,m,n}$ denote the number of ascent sequences of length n which begin with r 0's followed by 1, have a ascents, the last letter ℓ, and a total of m zeros. We then let

$$G_r := G_r(t,u,v,z)=\sum_{a,\ell,m\geq 0,n\geq r+1} G^r_{a,l,m,n}t^n u^a v^\ell z^m. \tag{2}$$

Clearly, since the sequence $0\dots0$ has no ascents and no initial run of 0's (by definition), we have that the generating function for such sequences is

$$1 + tz + (tz)^2 + \cdots = \frac{1}{1-tz}$$

where 1 corresponds to the empty word. Thus, we have the following relation between G and G_r:

$$G = \frac{1}{1-tz} + \sum_{r\geq 1} G_r\, x^r. \tag{3}$$

Lemma 2 *For $r \geq 1$, the generating function $G_r(t,u,v,z)$ satisfies*

$$(v-1-tv(1-u))G_r = (v-1)t^{r+1}uvz^r + t((v-1)z-v)G_r(t,u,1,z) + tuv^2G_r(t,uv,1,z). \tag{4}$$

Proof:

Our proof follows the same steps as in Lemma 13 in [1]. Fix $r \geq 1$. Let $x' = (x_1,\dots,x_{n-1})$ be an ascent sequence beginning with r 0's followed by 1, with a ascents and m zeros where $x_{n-1} = \ell$. Then $x = (x_1,\dots,x_{n-1},i)$ is an ascent sequence if and only if $i \in [0,a+1]$. Clearly x also begins with r 0's followed by 1. Now, if $i=0$, the sequence x has a ascents and $m+1$ zeros. If $1 \leq i \leq \ell$, x has a ascents and m zeros. Finally if $i \in [\ell+1, a+1]$, then x has $a+1$ ascents and m zeros. Counting the sequence $0\dots01$ with r 0's separately, we have

$$\begin{aligned}
G_r &= t^{r+1}u^1v^1z^r + \sum_{\substack{a,\ell,m\geq 0\\ n\geq r+1}} G^r_{a,\ell,m,n}t^{n+1}\left(u^av^0z^{m+1} + \sum_{i=1}^{\ell}u^av^iz^m + \sum_{i=\ell+1}^{a+1}u^{a+1}v^iz^m\right)\\
&= t^{r+1}uvz^r + t\sum_{\substack{a,\ell,m\geq 0\\ n\geq r+1}} G_{a,\ell,m,n}t^nu^az^m\left(z + \frac{v^{\ell+1}-v}{v-1} + u\frac{v^{a+2}-v^{\ell+1}}{v-1}\right)\\
&= t^{r+1}uvz^r + tzG_r(t,u,1,z) + tv\frac{G_r - G_r(t,u,1,z)}{v-1} + tuv\frac{vG_r(t,uv,1,z)-G_r}{v-1}.
\end{aligned}$$

The result follows. □

Next just like in Subsection 6.2 of [1], we use the kernel method to proceed. Setting $(v-1-tv(1-u)) = 0$ and solving for v, we obtain that the substitution $v = 1/(1+t(u-1))$ will kill the left-hand side of (4). We can then solve for $G_r(t,u,1,z)$ to obtain that

$$G_r(t,u,1,z) = \frac{(1-u)t^{r+1}uz^r + uG_r\left(t, \frac{u}{1+t(u-1)}, 1, z\right)}{(1+zt(u-1))(1+t(u-1))}. \tag{5}$$

Next we define

$$\begin{aligned}
\delta_k &= u-(1-t)^k(u-1) \text{ and} &(6)\\
\gamma_k &= u-(1-zt)(1-t)^{k-1}(u-1) &(7)
\end{aligned}$$

for $k \geq 1$. We also set $\delta_0 = \gamma_0 = 1$. Observe that $\delta_1 = u - (1-t)(u-1) = 1 + t(u-1)$ and $\gamma_1 = u - (1-zt)(u-1) = 1 + zt(u-1)$. Thus we can rewrite (5) as

$$G_r(t,u,1,z) = \frac{t^{r+1}z^r u(1-u)}{\delta_1\gamma_1} + \frac{u}{\delta_1\gamma_1} G_r(t, \frac{u}{\delta_1}, 1, z). \tag{8}$$

For any function of $f(u)$, we shall write $f(u)|_{u=\frac{u}{\delta_k}}$ for $f(u/\delta_k)$. It is then easy to check that

1. $(u-1)|_{u=\frac{u}{\delta_k}} = \dfrac{(1-t)^k(u-1)}{\delta_k}$,
2. $\delta_s|_{u=\frac{u}{\delta_k}} = \dfrac{\delta_{s+k}}{\delta_k}$,
3. $\gamma_s|_{u=\frac{u}{\delta_k}} = \dfrac{\gamma_{s+k}}{\delta_k}$, and
4. $\dfrac{u}{\delta_s}|_{u=\frac{u}{\delta_k}} = \dfrac{u}{\delta_{s+k}}$.

Using these relations, one can iterate the recursion (8) to prove by induction that for all $n \geq 1$,

$$\begin{aligned} G_r(t,u,1,z) &= \frac{t^{r+1}z^r u(1-u)}{\delta_1\gamma_1} + \left(t^{r+1}z^r u(1-u) \sum_{s=2}^{2^n-1} \frac{u^s(1-t)^s}{\delta_s\delta_{s+1}\prod_{i=1}^{s+1}\gamma_i} \right) + \\ &\quad \frac{u^{2^n}}{\delta_{2^n}\prod_{i=1}^{2^n}\gamma_i} G_r(t, \frac{u}{\delta_{2^n}}, 1, z). \end{aligned} \tag{9}$$

Since $\delta_0 = 1$, it follows that as a power series in u,

$$G_r(t,u,1,z) = t^{r+1}z^r u(1-u) \sum_{s\geq 0} \frac{u^s(1-t)^s}{\delta_s\delta_{s+1}\prod_{i=1}^{s+1}\gamma_i}. \tag{10}$$

We have used Mathematica to compute that

$$\begin{aligned} &G_1(t,u,1,z) = uzt^2 + \left(uz + u^2z + uz^2\right)t^3 \\ &+ \left(uz + 3u^2z + u^3z + uz^2 + 3u^2z^2 + uz^3\right)t^4 \\ &+ \left(uz + 6u^2z + 7u^3z + u^4z + uz^2 + 8u^2z^2 + 7u^3z^2 + uz^3 + 5u^2z^3 + uz^4\right)t^5 + O[t]^6. \end{aligned}$$

For example, the coefficient of t^4u^2, $3z + 3z^2$ makes sense as there are 3 ascent sequences of length 4 with 2 ascents and 1 zero, namely, 0112, 0121, and 0122, while there are 3 ascent sequences of length 4 with 2 ascents and 2 zeros, namely, 0101, 0102, and 0120 (there are no other ascents sequences of length 4 with 2 ascents).

Note that we can rewrite (4) as

$$G_r(t,u,v,z) = \frac{t^{r+1}z^r uv(1-v)}{v\delta_1 - 1} + \frac{t(z(v-1)-v)}{v\delta_1 - 1} G_r(t,u,1,z) + \frac{uv^2t}{v\delta_1 - 1} G_r(t,uv,1,z). \tag{11}$$

For $s \geq 1$, we let

$$\begin{aligned} \bar{\delta}_s &= \delta_s|_{u=uv} = uv - (1-t)^s(uv-1) \text{ and} \\ \bar{\gamma}_s &= \gamma_s|_{u=uv} = uv - (1-zt)(1-t)^{s-1}(uv-1) \end{aligned}$$

and set $\bar{\delta}_0 = \bar{\gamma}_0 = 1$. Then using (11) and (10), we have the following theorem.

Theorem 3 *For all $r \geq 1$,*

$$\begin{aligned} G_r(t,u,v,z) &= \frac{t^{r+1}z^r u}{v\delta_1 - 1}\left(v(v-1) + t(1-u)(z(v-1)-v)\sum_{s\geq 0}\frac{u^s(1-t)^s}{\delta_s\delta_{s+1}\prod_{i=1}^{s+1}\gamma_i} \right. \\ & \left. + uv^3 t(1-uv)\sum_{s\geq 0}\frac{(uv)^s(1-t)^s}{\bar{\delta}_s\bar{\delta}_{s+1}\prod_{i=1}^{s+1}\bar{\gamma}_i}\right) \qquad (12) \end{aligned}$$

It is easy to see from Theorem 3 that

$$G_r(t,u,v,z) = t^{r-1}z^{r-1}G_1(t,u,v,z). \qquad (13)$$

This is also easy to see combinatorially since every ascent sequence counted by $G_r(t,u,v,z)$ is of the form $0^{r-1}a$ where a is an ascent sequence a counted by $G_1(t,u,v,z)$.

We have used Mathematica to compute that

$$\begin{aligned} & G_1(t,u,v,z) = uvzt^2 + \left(uvz + u^2v^2z + uz^2\right)t^3 \\ & + \left(uvz + u^2vz + 2u^2v^2z + u^3v^3z + uz^2 + u^2z^2 + u^2vz^2 + u^2v^2z^2 + uz^3\right)t^4 \\ & + \left(uvz + 3u^2vz + u^3vz + 3u^2v^2z + 2u^3v^2z + 4u^3v^3z + u^4v^4z + uz^2 + 3u^2z^2 + u^3z^2 + 3u^2vz^2\right. \\ & \left. + u^3vz^2 + 2u^2v^2z^2 + 2u^3v^2z^2 + 3u^3v^3z^2 + uz^3 + 3u^2z^3 + u^2vz^3 + u^2v^2z^3 + uz^4\right)t^5 + O[t]^6. \end{aligned}$$

For example, the coefficient of t^4u is $zv + z^2 + z^3$ which makes sense since the sequences counted by the terms are 0111, 0110, and 0100, respectively.

Note that

$$\begin{aligned} G(t,u,v,z,x) &= \frac{1}{(1-tz)} + \sum_{r\geq 1} G_r(t,u,v,z)x^r \\ &= \frac{1}{(1-tz)} + \sum_{r\geq 1} t^{r-1}z^{r-1}G_1(t,u,v,z)x^r \\ &= \frac{1}{(1-tz)} + \frac{1}{1-tzx}xG_1(t,u,v,z) \end{aligned}$$

Thus we have the following theorem.

Theorem 4

$$G(t,u,v,z,x) = \frac{1}{(1-tz)} + \frac{t^2zxu}{(1-tzx)(v\delta_1-1)}\Bigg(v(v-1) + t(1-u)(z(v-1)-v)\sum_{s\geq 0}\frac{u^s(1-t)^s}{\delta_s\delta_{s+1}\prod_{i=1}^{s+1}\gamma_i} + uv^3t(1-uv)\sum_{s\geq 0}\frac{(uv)^s(1-t)^s}{\bar{\delta}_s\bar{\delta}_{s+1}\prod_{i=1}^{s+1}\bar{\gamma}_i}\Bigg). \quad (14)$$

Again, we have used Mathematica to compute the first few terms of this series:

$$\begin{aligned}
&G(t,u,v,z,x) = 1 + zt + \left(uvxz + z^2\right)t^2 + \left(uvxz + u^2v^2xz + uxz^2 + uvx^2z^2 + z^3\right)t^3\\
&+ \left(uvxz + u^2vxz + 2u^2v^2xz + u^3v^3xz + uxz^2 + u^2xz^2 + u^2vxz^2\right.\\
&\left.+u^2v^2xz^2 + uvx^2z^2 + u^2v^2x^2z^2 + uxz^3 + ux^2z^3 + uvx^3z^3 + z^4\right)t^4\\
&\left(uvxz + 3u^2vxz + u^3vxz + 3u^2v^2xz + 2u^3v^2xz + 4u^3v^3xz + u^4v^4xz\right.\\
&+uxz^2 + 3u^2xz^2 + u^3xz^2 + 3u^2vxz^2 + u^3vxz^2 + 2u^2v^2xz^2 + 2u^3v^2xz^2 + 3u^3v^3xz^2\\
&+uvx^2z^2 + u^2vx^2z^2 + 2u^2v^2x^2z^2 + u^3v^3x^2z^2 + uxz^3 + 3u^2xz^3 + u^2vxz^3 + u^2v^2xz^3\\
&+ux^2z^3 + u^2x^2z^3 + u^2vx^2z^3 + u^2v^2x^2z^3 + uvx^3z^3 + u^2v^2x^3z^3 + uxz^4\\
&\left.+ux^2z^4 + ux^3z^4 + uvx^4z^4 + z^5\right)t^5 + O[t]^6.
\end{aligned}$$

One can check that, for instance, the 3 sequences corresponding to the term $3u^2v^2xzt^5$ are 01112, 01122 and 01222.

3 Counting $(\mathbf{2}+\mathbf{2})$-free posets by size and number of minimal elements

In this section, we shall compute the generating function of $(\mathbf{2}+\mathbf{2})$-free posets by size and the number of minimal elements which is equivalent to finding the generating function for ascent sequences by length and the number of zeros.

For $n \geq 1$, let $H_{a,b,\ell,n}$ denote the number of ascent sequences of length n with a ascents and b zeros which have last letter ℓ. Then we first wish to compute

$$H(u,z,v,t) = \sum_{n\geq 1, a,b,\ell\geq 0} H_{a,b,\ell,n}u^az^bv^\ell t^n. \quad (15)$$

Using the same reasoning as in the previous section, we see that

$$\begin{aligned}
H(u,z,v,t) &= tz + \sum_{\substack{a,b,\ell\geq 0\\ n\geq 1}} H_{a,b,\ell,n}t^{n+1}\left(u^av^0z^{b+1} + \sum_{i=1}^{\ell}u^av^iz^b + \sum_{i=\ell+1}^{a+1}u^{a+1}v^iz^b\right)\\
&= tz + t\sum_{\substack{a,b,\ell\geq 0\\ n\geq r+1}} H_{a,b\ell,n}t^nu^az^b\left(z + \frac{v^{\ell+1}-v}{v-1} + u\frac{v^{a+2}-v^{\ell+1}}{v-1}\right)\\
&= tz + \frac{tv(1-u)}{v-1}H(u,v,z,t) + \frac{t(z(v-1)-v)}{v-1}H(u,1,z,t) + \frac{tuv^2}{v-1}H(uv,1,z,t).
\end{aligned}$$

Thus we have the following lemma.

Lemma 5

$$(v-1-tv(1-u))H(u,v,z,t) = tz(v-1)+t(z(v-1)-v)H(u,1,z,t)+tuv^2H(uv,1,z,t). \quad (16)$$

Setting $(v-1-t(1-u))=0$, we see that the substitution $v=1+t(u-1)=\delta_1$ kills the left-hand side of (16). We can then solve for $H(u,1,z,t)$ to obtain the recursion

$$H(u,1,z,t) = \frac{zt(1-u)}{\gamma_1} + \frac{u}{\delta_1\gamma_1}H(uv,1,z,t). \quad (17)$$

By iterating (17), we can prove by induction that for all $n \geq 1$,

$$H(u,1,z,t) = \frac{zt(1-u)}{\gamma_1} + \left(\sum_{s=1}^{2^n-1} \frac{zt(1-u)u^s(1-t)^s}{\delta_s \prod_{i=1}^{s+1}\gamma_i}\right) + \frac{u^{2^n}}{\delta_{2^n}\prod_{i=1}^{2^n}\gamma_i}H(\frac{u}{\delta_{2^n}},1,z,t). \quad (18)$$

Since $\delta_0=1$, we can rewrite (18) as

$$H(u,1,z,t) = \left(\sum_{s=0}^{2^n-1} \frac{zt(1-u)u^s(1-t)^s}{\delta_s \prod_{i=1}^{s+1}\gamma_i}\right) + \frac{u^{2^n}}{\delta_{2^n}\prod_{i=1}^{2^n}\gamma_i}H(\frac{u}{\delta_{2^n}},1,z,t). \quad (19)$$

Thus as a power series in u, we can conclude the following.

Theorem 6

$$H(u,1,z,t) = \sum_{s=0}^{\infty} \frac{zt(1-u)u^s(1-t)^s}{\delta_s \prod_{i=1}^{s+1}\gamma_i}. \quad (20)$$

We would like to set $u=1$ in the power series $\sum_{s=0}^{\infty}\frac{zt(1-u)u^s(1-t)^s}{\delta_s\prod_{i=1}^{s+1}\gamma_i}$, but the factor $(1-u)$ in the series does not allow us to do that in this form. Thus our next step is to rewrite the series in a form where it is obvious that we can set $u=1$ in the series. To that end, observe that for $k \geq 1$,

$$\delta_k = u-(1-t)^k(u-1) = 1+u-1-(1-t)^k(u-1) = 1-(1-t)^k-1)(u-1)$$

so that

$$\frac{1}{\delta_k} = \sum_{n\geq 0}((1-t)^k-1)^n(u-1)^n \sum_{n\geq 0}(u-1)^n = \sum_{m=0}^{n}(-1)^{n-m}\binom{n}{m}(1-t)^{km}. \quad (21)$$

Substituting (21) into (20), we see that

$$\begin{aligned} H(u,1,z,t) &= \frac{zt(1-u)}{\gamma_1} + \sum_{k\geq 1} \frac{zt(1-u)u^k(1-t)^k}{\prod_{i=1}^{k+1}\gamma_i} \sum_{n\geq 0}(u-1)^n \sum_{m=0}^{n}(-1)^{n-m}\binom{n}{m}(1-t)^{km} \\ &= \frac{zt(1-u)}{\gamma_1} + \sum_{n\geq 0}\sum_{m=0}^{n}(-1)^{n-m-1}\binom{n}{m}(u-1)^{n-m} zt \sum_{k\geq 1}\frac{(u-1)^{m+1}u^k(1-t)^{k(m+1)}}{\prod_{i=1}^{k+1}\gamma_i} \\ &= \frac{zt(1-u)}{\gamma_1} + \sum_{n\geq 0}\sum_{m=0}^{n}(-1)^{n-m-1}\binom{n}{m}(u-1)^{n-m}\frac{zt}{(1-zt)^{m+1}} \times \\ &\quad \sum_{k\geq 1}\frac{(u-1)^{m+1}(1-zt)^{m+1}u^k(1-t)^{k(m+1)}}{\prod_{i=1}^{k+1}\gamma_i}. \end{aligned}$$

Next we need to study the series

$$\sum_{k\geq 1}\frac{(u-1)^{m+1}(1-zt)^{m+1}u^k(1-t)^{k(m+1)}}{\prod_{i=1}^{k+1}\gamma_i}$$

where $m \geq 0$. We can rewrite this series in the form

$$-\frac{(u-1)^{m+1}(1-zt)^{m+1}}{\gamma_1} + \sum_{k\geq 0}\frac{(u-1)^{m+1}(1-zt)^{m+1}u^k(1-t)^{k(m+1)}}{\prod_{i=1}^{k+1}\gamma_i}.$$

We let

$$\psi_{m+1}(u) = \sum_{k\geq 0}\frac{(u-1)^{m+1}(1-zt)^{m+1}u^k(1-t)^{k(m+1)}}{\prod_{i=1}^{k+1}\gamma_i}. \tag{22}$$

We shall show that $\psi_{m+1}(u)$ is in fact a polynomial for all $m \geq 0$. First, we claim that $\psi_{m+1}(u)$ salsifies the following recursion:

$$\psi_{m+1}(u) = \frac{(u-1)^{m+1}(1-zt)^{m+1}}{\gamma_1} + \frac{u\delta_1^m}{\gamma_1}\psi_{m+1}\left(\frac{u}{\delta_1}\right). \tag{23}$$

That is, one can easily iterate (23) to prove by induction that for all $n \geq 1$,

$$\psi_{m+1}(u) = \left(\sum_{s=0}^{2^n-1}\frac{(u-1)^{m+1}(1-zt)^{m+1}u^s(1-t)^{s(m+1)}}{\prod_{i=1}^{s+1}\gamma_i}\right) + \frac{u^{2n}(\delta_{2^n})^m}{\prod_{i=1}^{2^n}\gamma_i}\psi_{m+1}(\frac{u}{\delta_{2^n}}). \tag{24}$$

Hence it follows that if $\psi_{m+1}(u)$ satisfies the recursion (23), then $\psi_{m+1}(u)$ is given by the power series in (22). However, it is routine to check that the polynomial

$$\phi_{m+1}(u) = -\sum_{j=0}^{m}(u-1)^j(1-zt)^j u^{m-j}\prod_{i=j+1}^{m}(1-((1-t)^i) \tag{25}$$

satisfies the recursion that

$$\gamma_1 \phi_{m+1}(u) = (u-1)^{m+1}(1-zt)^{m+1} + u\delta_1^m \phi_{m+1}\left(\frac{u}{\delta_1}\right). \tag{26}$$

Thus we have proved the following lemma.

Lemma 7

$$\begin{aligned} \psi_{m+1}(u) &= \sum_{k\geq 0} \frac{(u-1)^{m+1}(1-zt)^{m+1}u^k(1-t)^{k(m+1)}}{\prod_{i=1}^{k+1}\gamma_i} \\ &= -\sum_{j=0}^{m}(u-1)^j(1-zt)^j u^{m-j} \prod_{i=j+1}^{m}(1-((1-t)^i). \end{aligned} \tag{27}$$

It thus follows that

$$\begin{aligned} H(u,1,z,t) &= \frac{zt(1-u)}{\gamma_1} + \sum_{n\geq 0}\sum_{m=0}^{n}(-1)^{n-m-1}\binom{n}{m}(u-1)^{n-m}\frac{zt}{(1-zt)^{m+1}} \times \\ & -\frac{(u-1)^{m+1}(1-zt)^{m+1}}{\gamma_1} - \sum_{j=0}^{m}(u-1)^j(1-zt)^j u^{m-j} \prod_{i=j+1}^{m}(1-((1-t)^i). \end{aligned}$$

There is no problem in setting $u=1$ in this expression to obtain that

$$H(1,1,z,t) = \sum_{n\geq 0}\frac{zt}{(1-zt)^{n+1}}\prod_{i=1}^{n}(1-(1-t)^i). \tag{28}$$

Clearly our definitions ensure that $1+H(1,1,z,t) = P(t,z)$ as defined in the introduction so that we have the following theorem.

Theorem 8

$$P(t,z) = \sum_{n,k\geq 0} p_{n,k}t^n z^k = 1 + \sum_{n\geq 0}\frac{zt}{(1-zt)^{n+1}}\prod_{i=1}^{n}(1-(1-t)^i). \tag{29}$$

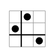
For example, we have used Mathematica to compute the first few terms of $P(t,z)$ as

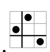
$$\begin{aligned} &P(t,z) = 1 + zt + \left(z+z^2\right)t^2 + \left(2z+2z^2+z^3\right)t^3 + \left(5z+6z^2+3z^3+z^4\right)t^4 \\ &+ \left(15z+21z^2+12z^3+4z^4+z^5\right)t^5 + \left(53z+84z^2+54z^3+20z^4+5z^5+z^6\right)t^6 + O[t]^7. \end{aligned}$$

Next we observe that one can easily derive the ordinary generating function for the number of $(\mathbf{2}+\mathbf{2})$-free posets or, equivalently, for the number of ascent sequences proved by Bousquet-Mélou et al. [1] from Theorem 8. That is, for any sequence of natural numbers $a = a_1 \ldots a_n$, let $a^+ = (a_1+1)\ldots(a_n+1)$ be the result of adding one from each element of the sequence. Moreover, if all the elements of $a = a_1 \ldots a_n$ are positive, then we let $a^- = (a_1-1)\ldots(a_n-1)$ be the result of subtracting one to each element of the sequence. It is easy to see that if $a = a_1 \dots a_n$ is an ascent sequence, then $0a^+$ is also an ascent sequence. Vice versa, if $b = 0a$ is an ascent sequence with only one zero where $a = a_1 \dots a_n$, then a^- is an ascent sequence. It follows that the number of ascent sequences of length n is equal to the number of ascent sequences of length $n+1$ which have only one zero. Hence

$$\begin{aligned} P(t) &= \sum_{n\geq 0} p_n t^n = \frac{1}{t}\frac{\partial P(t,z)}{\partial z}\Big|_{z=0} \\ &= \sum_{n\geq 0}\prod_{i=1}^{n}(1-(1-t)^i). \end{aligned}$$

Results in [1, 2, 3] show that $(\mathbf{2}+\mathbf{2})$-free posets of size n with k minimal elements are in bijection with the following objects. (See [1, 2, 3] for the precise definitions.)

- ascent sequences of length n with k zeros;
- permutations of length n *avoiding*  whose *leftmost-decreasing run* is of size k;
- *regular linearized chord diagrams* on $2n$ points with *initial run of openers* of size k;
- *upper triangular matrices* whose non-negative integer entries sum up to n, each row and column contains a non-zero element, and the sum of entries in the first row is k.

Thus (29) provides generating functions for  -avoiding permutations by the size of the leftmost-decreasing run, for regular linearized chord diagrams by the size of the initial run of openers, and for the upper triangular matrices by the sum of entries in the first row. Moreover, Theorem 4, together with bijections in [1, 2, 3] can be used to enumerate the permutations, diagrams, and matrices subject to 4 statistics.

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