

# Products of Geck-Rouquier conjugacy classes and the Hecke algebra of composed permutations

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**Abstract.** We show the  $q$ -analog of a well-known result of Farahat and Higman: in the center of the Iwahori-Hecke algebra  $\mathcal{H}_{n,q}$ , if  $(a_{\lambda\mu}^\nu(n, q))_\nu$  is the set of structure constants involved in the product of two Geck-Rouquier conjugacy classes  $\Gamma_{\lambda,n}$  and  $\Gamma_{\mu,n}$ , then each coefficient  $a_{\lambda\mu}^\nu(n, q)$  depend on  $n$  and  $q$  in a polynomial way. Our proof relies on the construction of a projective limit of the Hecke algebras; this projective limit is inspired by the Ivanov-Kerov algebra of partial permutations.

**Résumé.** Nous démontrons le  $q$ -analogue d'un résultat bien connu de Farahat et Higman : dans le centre de l'algèbre d'Iwahori-Hecke  $\mathcal{H}_{n,q}$ , si  $(a_{\lambda\mu}^\nu(n, q))_\nu$  est l'ensemble des constantes de structure mises en jeu dans le produit de deux classes de conjugaison de Geck-Rouquier  $\Gamma_{\lambda,n}$  et  $\Gamma_{\mu,n}$ , alors chaque coefficient  $a_{\lambda\mu}^\nu(n, q)$  dépend de façon polynomiale de  $n$  et de  $q$ . Notre preuve repose sur la construction d'une limite projective des algèbres d'Hecke ; cette limite projective est inspirée de l'algèbre d'Ivanov-Kerov des permutations partielles.

**Keywords:** Iwahori-Hecke algebras, Geck-Rouquier conjugacy classes, symmetric functions.

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In this paper, we answer a question asked in [FW09] that concerns the products of Geck-Rouquier conjugacy classes in the Hecke algebras  $\mathcal{H}_{n,q}$ . If  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$  is a partition with  $|\lambda| + \ell(\lambda) \leq n$ , we consider the completed partition

$$\lambda \rightarrow n = (\lambda_1 + 1, \lambda_2 + 1, \dots, \lambda_r + 1, 1^{n-|\lambda|-\ell(\lambda)}),$$

and we denote by  $C_{\lambda,n} = C_{\lambda \rightarrow n}$  the corresponding conjugacy class, that is to say, the sum of all permutations with cycle type  $\lambda \rightarrow n$  in the center of the symmetric group algebra  $\mathbb{C}\mathfrak{S}_n$ . Notice that in particular,  $C_{\lambda,n} = 0$  if  $|\lambda| + \ell(\lambda) > n$ . It is known since [FH59] that the products of completed conjugacy classes write as

$$C_{\lambda,n} * C_{\mu,n} = \sum_{|\nu| \leq |\lambda| + |\mu|} a_{\lambda\mu}^\nu(n) C_{\nu,n},$$

where the structure constants  $a_{\lambda\mu}^\nu(n)$  depend on  $n$  in a polynomial way. In [GR97], some deformations  $\Gamma_\lambda$  of the conjugacy classes  $C_\lambda$  are constructed. These central elements form a basis of the center  $\mathcal{L}_{n,q}$  of the Iwahori-Hecke algebra  $\mathcal{H}_{n,q}$ , and they are characterized by the two following properties, see [Fra99]:

1. The element  $\Gamma_\lambda$  is central and specializes to  $C_\lambda$  for  $q = 1$ .
2. The difference  $\Gamma_\lambda - C_\lambda$  involves no permutation of minimal length in its conjugacy class.

As before,  $\Gamma_{\lambda,n} = \Gamma_{\lambda \rightarrow n}$  if  $|\lambda| + \ell(\lambda) \leq n$ , and 0 otherwise. Our main result is the following:

**Theorem 1** *In the center of the Hecke algebra  $\mathcal{H}_{n,q}$ , the products of completed Geck-Rouquier conjugacy classes write as*

$$\Gamma_{\lambda,n} * \Gamma_{\mu,n} = \sum_{|\nu| \leq |\lambda| + |\mu|} a_{\lambda\mu}^\nu(n, q) \Gamma_{\nu,n},$$

and the structure constants  $a_{\lambda\mu}^\nu(n, q)$  are in  $\mathbb{Q}[n, q, q^{-1}]$ .

The first part of Theorem 1 — that is to say, that elements  $\Gamma_{\nu,n}$  involved in the product satisfy the inequality  $|\nu| \leq |\lambda| + |\mu|$  — was already in [FW09, Theorem 1.1], and the polynomial dependance of the coefficients  $a_{\lambda\mu}^\nu(n, q)$  was Conjecture 3.1; our paper is devoted to a proof of this conjecture. We shall combine two main arguments:

- We construct a projective limit  $\mathcal{D}_{\infty,q}$  of the Hecke algebras, which is essentially a  $q$ -version of the algebra of Ivanov and Kerov, see [IK99]. We perform *generic computations* inside various subalgebras of  $\mathcal{D}_{\infty,q}$ , and we project then these calculations on the algebras  $\mathcal{H}_{n,q}$  and their centers.
- The centers of the Hecke algebras admit numerous bases, and these bases are related one to another in the same way as the bases of the symmetric function algebra  $\Lambda$ . This allows to separate the dependance on  $q$  and the dependance on  $n$  of the coefficients  $a_{\lambda\mu}^\nu(n, q)$ .

Before we start, let us fix some notations. If  $n$  is a non-negative integer,  $\mathfrak{P}_n$  is the set of partitions of  $n$ ,  $\mathfrak{C}_n$  is the set of compositions of  $n$ , and  $\mathfrak{S}_n$  is the set of permutations of the interval  $\llbracket 1, n \rrbracket$ . The **type** of a permutation  $\sigma \in \mathfrak{S}_n$  is the partition  $\lambda = t(\sigma)$  obtained by ordering the sizes of the orbits of  $\sigma$ ; for instance,  $t(24513) = (3, 2)$ . The **code** of a composition  $c \in \mathfrak{C}_n$  is the complementary in  $\llbracket 1, n \rrbracket$  of the set of descents of  $c$ ; for instance, the code of  $(3, 2, 3)$  is  $\{1, 2, 4, 6, 7\}$ . Finally, we denote by  $\mathcal{Z}_n = Z(\mathbb{C}\mathfrak{S}_n)$  the center of the algebra  $\mathbb{C}\mathfrak{S}_n$ ; the conjugacy classes  $C_\lambda$  form a linear basis of  $\mathcal{Z}_n$  when  $\lambda$  runs over  $\mathfrak{P}_n$ .

## 1 Partial permutations and the Ivanov-Kerov algebra

Since our argument is essentially inspired by the construction of [IK99], let us recall it briefly. A **partial permutation** of order  $n$  is a pair  $(\sigma, S)$  where  $S$  is a subset of  $\llbracket 1, n \rrbracket$ , and  $\sigma$  is a permutation in  $\mathfrak{S}(S)$ . Alternatively, one may see a partial permutation as a permutation  $\sigma$  in  $\mathfrak{S}_n$  together with a subset containing the non-trivial orbits of  $\sigma$ . The product of two partial permutations is

$$(\sigma, S)(\tau, T) = (\sigma\tau, S \cup T),$$

and this operation yield a semigroup whose complex algebra is denoted by  $\mathcal{B}_n$ . There is a natural projection  $\text{pr}_n : \mathcal{B}_n \rightarrow \mathbb{C}\mathfrak{S}_n$  that consists in forgetting the support of a partial permutation, and also natural compatible maps

$$\phi_{N,n} : (\sigma, S) \in \mathcal{B}_N \mapsto \begin{cases} (\sigma, S) \in \mathcal{B}_n & \text{if } S \subset \llbracket 1, n \rrbracket, \\ 0 & \text{otherwise,} \end{cases}$$

whence a projective limit  $\mathcal{B}_\infty = \varprojlim \mathcal{B}_n$  with respect to this system  $(\phi_{N,n})_{N \geq n}$  and in the category of filtered algebras. Now, one can lift the conjugacy classes  $C_\lambda$  to the algebras of partial permutations. Indeed, the symmetric group  $\mathfrak{S}_n$  acts on  $\mathcal{B}_n$  by

$$\sigma \cdot (\tau, S) = (\sigma\tau\sigma^{-1}, \sigma(S)),$$

and a linear basis of the invariant subalgebra  $\mathcal{A}_n = (\mathcal{B}_n)^{\mathfrak{S}_n}$  is labelled by the partitions  $\lambda$  of size less than or equal to  $n$ :

$$\mathcal{A}_n = \bigoplus_{|\lambda| \leq n} \mathbb{C}A_{\lambda,n}, \quad \text{where } A_{\lambda,n} = \sum_{\substack{|S|=|\lambda| \\ \sigma \in \mathfrak{S}(S), t(\sigma)=\lambda}} (\sigma, S).$$

Since the actions  $\mathfrak{S}_n \curvearrowright \mathcal{B}_n$  are compatible with the morphisms  $\phi_{N,n}$ , the inverse limit  $\mathcal{A}_\infty = (\mathcal{B}_\infty)^{\mathfrak{S}_\infty}$  of the invariant subalgebras has a basis  $(A_\lambda)_\lambda$  indexed by all partitions  $\lambda \in \mathfrak{P} = \bigsqcup_{n \in \mathbb{N}} \mathfrak{P}_n$ , and such that  $\phi_{\infty,n}(A_\lambda) = A_{\lambda,n}$  (with by convention  $A_{\lambda,n} = 0$  if  $|\lambda| > n$ ). As a consequence, if  $(a'_{\lambda\mu})_{\lambda,\mu,\nu}$  is the family of structure constants of the **Ivanov-Kerov algebra**<sup>(i)</sup>  $\mathcal{A}_\infty$  in the basis  $(A_\lambda)_{\lambda \in \mathfrak{P}}$ , then

$$\forall n, A_{\lambda,n} * A_{\mu,n} = \sum_{\nu} a'_{\lambda\mu} A_{\nu,n},$$

with  $A_{\lambda,n} = 0$  if  $|\lambda| \geq n$ . Moreover, it is not difficult to see that  $a'_{\lambda\mu} \neq 0$  implies  $|\nu| \leq |\lambda| + |\mu|$ , and also  $|\nu| - \ell(\nu) \leq |\lambda| - \ell(\lambda) + |\mu| - \ell(\mu)$ , cf. [IK99, §10], for the study of the filtrations of  $\mathcal{A}_\infty$ . Now,  $\text{pr}_n(\mathcal{A}_n) = \mathcal{L}_n$ , and more precisely,

$$\text{pr}_n(A_{\lambda,n}) = \binom{n - |\lambda| + m_1(\lambda)}{m_1(\lambda)} C_{\lambda-1,n}.$$

where  $\lambda - 1 = (\lambda_1 - 1, \dots, \lambda_s - 1)$  if  $\lambda = (\lambda_1, \dots, \lambda_s \geq 2, 1, \dots, 1)$ . The result of Farahat and Higman follows immediately, and we shall try to mimic this construction in the context of Iwahori-Hecke algebras.

## 2 Composed permutations and their Hecke algebra

We recall that the **Iwahori-Hecke algebra** of type A and order  $n$  is the quantized version of the symmetric group algebra defined over  $\mathbb{C}(q)$  by

$$\mathcal{H}_{n,q} = \left\langle S_1, \dots, S_{n-1} \left| \begin{array}{l} \text{braid relations: } \forall i, S_i S_{i+1} S_i = S_{i+1} S_i S_{i+1} \\ \text{commutation relations: } \forall |j-i| > 1, S_i S_j = S_j S_i \\ \text{quadratic relations: } \forall i, (S_i)^2 = (q-1) S_i + q \end{array} \right. \right\rangle.$$

When  $q = 1$ , we recover the symmetric group algebra  $\mathbb{C}\mathfrak{S}_n$ . If  $\omega \in \mathfrak{S}_n$ , let us denote by  $T_\omega$  the product  $S_{i_1} S_{i_2} \cdots S_{i_r}$ , where  $\omega = s_{i_1} s_{i_2} \cdots s_{i_r}$  is any reduced expression of  $\omega$  in elementary transpositions  $s_i = (i, i + 1)$ . Then, it is well known that the elements  $T_\omega$  do not depend on the choice of reduced expressions, and that they form a  $\mathbb{C}(q)$ -linear basis of  $\mathcal{H}_{n,q}$ , see [Mat99].

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<sup>(i)</sup> It can be shown that  $\mathcal{A}_\infty$  is isomorphic to the algebra of shifted symmetric polynomials, see Theorem 9.1 in [IK99].

In order to construct a *projective* limit of the algebras  $\mathcal{H}_{n,q}$ , it is very tempting to mimic the construction of Ivanov and Kerov, and therefore to build an Hecke algebra of partial permutations. Unfortunately, this is not possible; let us explain why by considering for instance the transposition  $\sigma = 1432$  in  $\mathfrak{S}_4$ . The possible supports for  $\sigma$  are  $\{2, 4\}$ ,  $\{1, 2, 4\}$ ,  $\{2, 3, 4\}$  and  $\{1, 2, 3, 4\}$ . However,

$$\sigma = s_2 s_3 s_2,$$

and the support of  $s_2$  (respectively, of  $s_3$ ) contains at least  $\{2, 3\}$  (resp.,  $\{3, 4\}$ ). So, if we take account of the Coxeter structure of  $\mathfrak{S}_4$  — and it should obviously be the case in the context of Hecke algebras — then the only valid supports for  $\sigma$  are the connected ones, namely,  $\{2, 3, 4\}$  and  $\{1, 2, 3, 4\}$ . This problem leads to consider *composed permutations* instead of *partial permutations*. If  $c$  is a composition of  $n$ , let us denote by  $\pi(c)$  the corresponding set partition of  $\llbracket 1, n \rrbracket$ , i.e., the set partition whose parts are the intervals  $\llbracket 1, c_1 \rrbracket$ ,  $\llbracket c_1 + 1, c_1 + c_2 \rrbracket$ , etc. A **composed permutation** of order  $n$  is a pair  $(\sigma, c)$  with  $\sigma \in \mathfrak{S}_n$  and  $c$  composition in  $\mathfrak{C}_n$  such that  $\pi(c)$  is coarser than the set partition of orbits of  $\sigma$ . For instance,  $(32154867, (5, 3))$  is a composed permutation of order 8; we shall also write this  $32154|867$ . The product of two composed permutations is then defined by

$$(\sigma, c)(\tau, d) = (\sigma\tau, c \vee d),$$

where  $c \vee d$  is the finest composition of  $n$  such that  $\pi(c \vee d) \geq \pi(c) \vee \pi(d)$  in the lattice of set partitions. For instance,

$$321|54|867 \times 12|435|687 = 42153|768.$$

One obtains so a semigroup of composed permutations; its complex semigroup algebra will be denoted by<sup>(ii)</sup>  $\mathcal{D}_n$ , and the dimension of  $\mathcal{D}_n$  is the number of composed permutations of order  $n$ .

Now, let us describe an Hecke version  $\mathcal{D}_{n,q}$  of the algebra  $\mathcal{D}_n$ . As for  $\mathcal{H}_{n,q}$ , one introduces generators  $(S_i)_{1 \leq i \leq n-1}$  corresponding to the elementary transpositions  $s_i$ , but one has also to introduce generators  $(I_i)_{1 \leq i \leq n-1}$  that allow to join the parts of the composition of a composed permutation. Hence, the **Iwahori-Hecke algebra of composed permutations** is defined (over the ground field  $\mathbb{C}(q)$ ) by  $\mathcal{D}_{n,q} = \langle S_1, \dots, S_{n-1}, I_1, \dots, I_{n-1} \rangle$  and the following relations:

$$\begin{aligned} \forall i, \quad S_i S_{i+1} S_i &= S_{i+1} S_i S_{i+1} \\ \forall |j - i| > 1, \quad S_i S_j &= S_j S_i \\ \forall i, \quad (S_i)^2 &= (q - 1) S_i + q I_i \\ \forall i, j, \quad S_i I_j &= I_j S_i \\ \forall i, j, \quad I_i I_j &= I_j I_i \\ \forall i, \quad S_i I_i &= S_i \\ \forall i, \quad (I_i)^2 &= I_i \end{aligned}$$

The generators  $S_i$  correspond to the composed permutations  $1|2| \dots |i - 1|i + 1, i|i + 2| \dots |n$ , and the generators  $I_i$  correspond to the composed permutations  $1|2| \dots |i - 1|i, i + 1|i + 2| \dots |n$ .

<sup>(ii)</sup> If one considers pairs  $(\sigma, \pi)$  where  $\pi$  is any set partition of  $\llbracket 1, n \rrbracket$  coarser than  $\text{orb}(\sigma)$  (and not necessarily a set partition associated to a composition), then one obtains an algebra of *split permutations* whose subalgebra of invariants is related to the connected Hurwitz numbers  $H_{n,g}(\lambda)$ .

**Proposition 2** *The algebra  $\mathcal{D}_{n,q}$  specializes to the algebra of composed permutations  $\mathcal{D}_n$  when  $q = 1$ ; to the Iwahori-Hecke algebra  $\mathcal{H}_{n,q}$  when  $I_1 = I_2 = \dots = I_{n-1} = 1$ ; and to the algebra  $\mathcal{D}_{m,q}$  of lower order  $m < n$  when  $I_m = I_{m+1} = \dots = I_{n-1} = 0$  and  $S_m = S_{m+1} = \dots = S_{n-1} = 0$ .*

In the following, we shall denote by  $\text{pr}_n$  the specialization  $\mathcal{D}_{n,q} \rightarrow \mathcal{H}_{n,q}$ ; it generalizes the map  $\mathcal{D}_n \rightarrow \mathbb{C}\mathfrak{S}_n$  of the first section. The first part of Proposition 2 is actually the only one that is non trivial, and it will be a consequence of Theorem 3. If  $\omega$  is a permutation with reduced expression  $\omega = s_{i_1} s_{i_2} \dots s_{i_r}$ , we denote as before by  $T_\omega$  the product  $S_{i_1} S_{i_2} \dots S_{i_r}$  in  $\mathcal{D}_{n,q}$ . On the other hand, if  $c$  is a composition of  $\llbracket 1, n \rrbracket$ , we denote by  $I_c$  the product of the generators  $I_j$  with  $j$  in the code of  $c$  (so for instance,  $I_{(3,2,3)} = I_1 I_2 I_4 I_6 I_7$  in  $\mathcal{D}_{8,q}$ ). These elements are central idempotents, and  $I_c$  correspond to the composed permutation  $(\text{id}, c)$ . Finally, if  $(\sigma, c)$  is a composed permutation,  $T_{\sigma,c}$  is the product  $T_\sigma I_c$ .

**Theorem 3** *In  $\mathcal{D}_{n,q}$ , the products  $T_\sigma$  do not depend on the choice of reduced expressions, and the products  $T_{\sigma,c}$  form a linear basis of  $\mathcal{D}_{n,q}$  when  $(\sigma, c)$  runs over composed permutations of order  $n$ . There is an isomorphism of  $\mathbb{C}(q)$ -algebras between*

$$\mathcal{D}_{n,q} \quad \text{and} \quad \bigoplus_{c \in \mathfrak{C}_n} \mathcal{H}_{c,q},$$

where  $\mathcal{H}_{c,q}$  is the Young subalgebra  $\mathcal{H}_{c_1,q} \otimes \mathcal{H}_{c_2,q} \otimes \dots \otimes \mathcal{H}_{c_r,q}$  of  $\mathcal{H}_{n,q}$ .

**Proof:** If  $\sigma \in \mathfrak{S}_n$ , the Matsumoto theorem ensures that it is always possible to go from a reduced expression  $s_{i_1} s_{i_2} \dots s_{i_r}$  to another reduced expression  $s_{j_1} s_{j_2} \dots s_{j_r}$  by braid moves  $s_i s_{i+1} s_i \leftrightarrow s_{i+1} s_i s_{i+1}$  and commutations  $s_i s_j \leftrightarrow s_j s_i$  when  $|j - i| > 1$ . Since the corresponding products of  $S_i$  in  $\mathcal{D}_{n,q}$  are preserved by these substitutions, a product  $T_\sigma$  in  $\mathcal{D}_{n,q}$  does not depend on the choice of a reduced expression. Now, let us consider an arbitrary product  $\Pi$  of generators  $S_i$  and  $I_j$  (in any order). As the elements  $I_j$  are central idempotents, it is always possible to reduce the product to

$$\Pi = S_{i_1} S_{i_2} \dots S_{i_p} I_c$$

with  $c$  composition of  $n$  — here,  $s_{i_1} s_{i_2} \dots s_{i_p}$  is *a priori* not a reduced expression. Moreover, since  $S_i I_i = S_i$ , we can suppose that the code of  $c$  contains  $\{i_1, \dots, i_p\}$ . Now, suppose that  $\sigma = s_{i_1} s_{i_2} \dots s_{i_p}$  is not a reduced expression. Then, by using braid moves and commutations, we can transform the expression in one with two consecutive letters that are identical, that is to say that if  $j_k = j_{k+1}$ ,

$$\sigma = s_{j_1} \dots s_{j_k} s_{j_{k+1}} \dots s_{j_p} = s_{j_1} \dots s_{j_{k-1}} s_{j_{k+2}} \dots s_{j_p}.$$

We apply the same moves to the  $S_i$  in  $\mathcal{D}_{n,q}$  and we obtain  $\Pi = S_{j_1} \dots S_{j_k} S_{j_{k+1}} \dots S_{j_p} I_c$ ; notice that the code of  $c$  still contains  $\{j_1, \dots, j_p\} = \{i_1, \dots, i_p\}$ . By using the quadratic relation in  $\mathcal{D}_{n,q}$ , we conclude that if  $j_k = j_{k+1}$ ,

$$\begin{aligned} \Pi &= (q - 1) S_{j_1} \dots S_{j_{k-1}} S_{j_k} S_{j_{k+2}} \dots S_{j_p} I_c + q S_{j_1} \dots S_{j_{k-1}} I_{j_k} S_{j_{k+2}} \dots S_{j_p} I_c \\ &= (q - 1) S_{j_1} \dots S_{j_{k-1}} S_{j_k} S_{j_{k+2}} \dots S_{j_p} I_c + q S_{j_1} \dots S_{j_{k-1}} S_{j_{k+2}} \dots S_{j_p} I_c \end{aligned}$$

because  $I_{j_k} I_c = I_c$ . Consequently, by induction on  $p$ , any product  $\Pi$  is a  $\mathbb{Z}[q]$ -linear combination of products  $T_{\tau,c}$  (and with the same composition  $c$  for all the terms of the linear combination). So, the

reduced products  $T_{\sigma,c}$  span linearly  $\mathcal{D}_{n,q}$  when  $(\sigma, c)$  runs over composed permutations of order  $n$ . If  $c$  is in  $\mathfrak{C}_n$ , we define a morphism of  $\mathbb{C}(q)$ -algebras from  $\mathcal{D}_{n,q}$  to  $\mathcal{H}_{c,q}$  by

$$\psi_c(S_i) = \begin{cases} S_i & \text{if } i \text{ is in the code of } c, \\ 0 & \text{otherwise,} \end{cases} \quad ; \quad \psi_c(I_i) = \begin{cases} 1 & \text{if } i \text{ is in the code of } c, \\ 0 & \text{otherwise.} \end{cases}$$

The elements  $\psi_c(S_i)$  and  $\psi_c(I_i)$  satisfy in  $\mathcal{H}_{c,q}$  the relations of the generators  $S_i$  and  $I_i$  in  $\mathcal{D}_{n,q}$ . So, there is indeed such a morphism of algebras  $\psi_c : \mathcal{D}_{n,q} \rightarrow \mathcal{H}_{c,q}$ , and one has in fact  $\psi_c(T_{\sigma,b}) = T_\sigma$  if  $\pi(b) \leq \pi(c)$ , and 0 otherwise. Let us consider the direct sum of algebras  $\mathcal{H}_{\mathfrak{C}_n,q} = \bigoplus_{c \in \mathfrak{C}_n} \mathcal{H}_{c,q}$ , and the direct sum of morphisms  $\psi = \bigoplus_{c \in \mathfrak{C}_n} \psi_c$ . We denote the basis vectors  $[0, 0, \dots, (T_\sigma \in \mathcal{H}_{c,q}), \dots, 0]$  of  $\mathcal{H}_{\mathfrak{C}_n,q}$  by  $T_{\sigma \in \mathcal{H}_{c,q}}$ ; in particular,

$$\psi(T_{\sigma,c}) = \sum_{d \geq c} T_{\sigma \in \mathcal{H}_{d,q}}$$

for any composed permutation  $(\sigma, c)$ . As a consequence, the map  $\psi$  is surjective, because

$$\psi \left( \sum_{d \geq c} \mu(c, d) T_{\sigma,c} \right) = T_{\sigma \in \mathcal{H}_{c,q}}$$

where  $\mu(c, d) = \mu(\pi(c), \pi(d)) = (-1)^{\ell(c) - \ell(d)}$  is the Möbius function of the hypercube lattice of compositions. If  $\sigma$  is a permutation, we denote by  $\text{orb}(\sigma)$  the set partition whose parts are the orbits of  $\sigma$ . Since the families  $(T_{\sigma,c})_{\text{orb}(\sigma) \leq \pi(c)}$  and  $(T_{\sigma \in \mathcal{H}_{c,q}})_{\text{orb}(\sigma) \leq \pi(c)}$  have the same cardinality  $\dim \mathcal{D}_n$ , we conclude that  $(T_{\sigma,c})_{\text{orb}(\sigma) \leq \pi(c)}$  is a  $\mathbb{C}(q)$ -linear basis of  $\mathcal{D}_{n,q}$  and that  $\psi$  is an isomorphism of  $\mathbb{C}(q)$ -algebras.  $\square$

Notice that the second part of Theorem 3 is the  $q$ -analog of Corollary 3.2 in [IK99]. To conclude this part, we have to build the inverse limit  $\mathcal{D}_{\infty,q} = \varprojlim \mathcal{D}_{n,q}$ , but this is easy thanks to the specializations evoked in the third part of Proposition 2. Hence, if  $\phi_{N,n} : \mathcal{D}_{N,q} \rightarrow \mathcal{D}_{n,q}$  is the map that sends the generators  $I_{i \geq n}$  and  $S_{i \geq n}$  to zero and that preserves the other generators, then  $(\phi_{N,n})_{N \geq n}$  is a system of compatible maps, and these maps behave well with respect to the filtration  $\deg T_{\sigma,c} = |\text{code}(c)|$ . Consequently, there is a projective limit  $\mathcal{D}_{\infty,q}$  whose elements are the infinite linear combinations of  $T_{\sigma,c}$ , with  $\sigma$  finite permutation in  $\mathfrak{S}_\infty$  and  $c$  infinite composition compatible with  $\sigma$  and with almost all its parts of size 1.

It is not true that two elements  $x$  and  $y$  in  $\mathcal{D}_{\infty,q}$  are equal if and only if their projections  $\text{pr}_n(\phi_{\infty,n}(x))$  and  $\text{pr}_n(\phi_{\infty,n}(y))$  are equal for all  $n$ : for instance,

$$T[21|34|5|6|\dots] = S_1 I_1 I_3 \quad \text{and} \quad T[2134|5|6|\dots] = S_1 I_1 I_2 I_3$$

have the same projections in all the Hecke algebras (namely,  $S_1$  if  $n \geq 4$  and 0 otherwise), but they are not equal. However, the result is true if we consider only the subalgebras  $\mathcal{D}'_{n,q} \subset \mathcal{D}_{n,q}$  spanned by the  $T_{\sigma,c}$  with  $c = (k, 1, \dots, 1)$  — then,  $\sigma$  may be considered as a partial permutation of  $[[1, k]]$ .

**Proposition 4** *For any  $n$ , the vector space  $\mathcal{D}'_{n,q}$  spanned by the  $T_{\sigma,c}$  with  $c = (k, 1^{n-k})$  is a subalgebra of  $\mathcal{D}_{n,q}$ . In the inverse limit  $\mathcal{D}'_{\infty,q} \subset \mathcal{D}_{\infty,q}$ , the projections  $\text{pr}_{\infty,n} = \text{pr}_n \circ \phi_{\infty,n}$  separate the vectors:*

$$\forall x, y \in \mathcal{D}'_{\infty,q}, \quad (\forall n, \text{pr}_{\infty,n}(x) = \text{pr}_{\infty,n}(y)) \iff (x = y).$$

**Proof:** The supremum of two compositions  $(k, 1^{n-k})$  and  $(l, 1^{n-l})$  is  $(m, 1^{n-m})$  with  $m = \max(k, l)$ ; consequently,  $\mathcal{D}'_{n,q}$  is indeed a subalgebra of  $\mathcal{D}_{n,q}$ . Any element  $x$  of the projective limit  $\mathcal{D}'_{\infty,q}$  writes uniquely as

$$x = \sum_{k=0}^{\infty} \sum_{\sigma \in \mathfrak{S}_k} a_{\sigma,k}(x) T_{\sigma,(k,1^\infty)}.$$

Suppose that  $x$  and  $y$  have the same projections, and let us fix a permutation  $\sigma$ . There is a minimal integer  $k$  such that  $\sigma \in \mathfrak{S}_k$ , and  $a_{\sigma,k}(x)$  is the coefficient of  $T_\sigma$  in  $\text{pr}_{\infty,k}(x)$ ; consequently,  $a_{\sigma,k}(x) = a_{\sigma,k}(y)$ . Now,  $a_{\sigma,k}(x) + a_{\sigma,k+1}(x)$  is the coefficient of  $T_\sigma$  in  $\text{pr}_{\infty,k+1}(x)$ , so one has also  $a_{\sigma,k}(x) + a_{\sigma,k+1}(x) = a_{\sigma,k}(y) + a_{\sigma,k+1}(y)$ , and  $a_{\sigma,k+1}(x) = a_{\sigma,k+1}(y)$ . By using the same argument and by induction on  $l$ , we conclude that  $a_{\sigma,k+l}(x) = a_{\sigma,k+l}(y)$  for every  $l$ , and therefore  $x = y$ . We have then proved that the projections separate the vectors in  $\mathcal{D}'_{\infty,q}$ .  $\square$

### 3 Bases of the center of the Hecke algebra

In the following,  $\mathcal{Z}_{n,q}$  is the center of  $\mathcal{H}_{n,q}$ . We have already given a characterization of the **Geck-Rouquier central elements**  $\Gamma_\lambda$ , and they form a linear basis of  $\mathcal{Z}_{n,q}$  when  $\lambda$  runs over  $\mathfrak{P}_n$ . Let us write down explicitly this basis when  $n = 3$ :

$$\Gamma_3 = T_{231} + T_{312} + (q-1)q^{-1}T_{321} \quad ; \quad \Gamma_{2,1} = T_{213} + T_{132} + q^{-1}T_{321} \quad ; \quad \Gamma_{1,1,1} = T_{123}$$

The first significative example of Geck-Rouquier element is actually when  $n = 4$ . Thus, if one considers

$$\begin{aligned} \Gamma_{3,1} = & T_{1342} + T_{1423} + T_{2314} + T_{3124} + q^{-1}(T_{2431} + T_{4132} + T_{3214} + T_{4213}) \\ & + (q-1)q^{-1}(T_{1432} + T_{3214}) + (q-1)q^{-2}(T_{3421} + T_{4312} + 2T_{4231}) + (q-1)^2q^{-3}T_{4321}, \end{aligned}$$

the terms with coefficient 1 are the four minimal 3-cycles in  $\mathfrak{S}_4$ ; the terms whose coefficients specialize to 1 when  $q = 1$  are the eight 3-cycles in  $\mathfrak{S}_4$ ; and the other terms are not minimal in their conjugacy classes, and their coefficients vanish when  $q = 1$ .

It is really unclear how one can lift these elements to the Hecke algebras of composed permutations; fortunately, the center  $\mathcal{Z}_{n,q}$  admits other linear bases that are easier to pull back from  $\mathcal{H}_{n,q}$  to  $\mathcal{D}_{n,q}$ . In [Las06], seven different bases for  $\mathcal{Z}_{n,q}$  are studied<sup>(iii)</sup>, and it is shown that up to diagonal matrices that depend on  $q$  in a polynomial way, the transition matrices between these bases are the same as the transition matrices between the usual bases of the algebra of symmetric functions. We shall only need the **norm basis**  $N_\lambda$ , whose properties are recalled in Proposition 5. If  $c$  is a composition of  $n$  and  $\mathfrak{S}_c$  is the corresponding Young subgroup of  $\mathfrak{S}_n$ , it is well-known that each coset in  $\mathfrak{S}_n/\mathfrak{S}_c$  or  $\mathfrak{S}_c \backslash \mathfrak{S}_n$  has a unique representative  $\omega$  of minimal length which is called the **distinguished representative** — this fact is even true for parabolic double cosets. In what follows, we rather work with right cosets, and the distinguished representatives of  $\mathfrak{S}_c \backslash \mathfrak{S}_n$  are precisely the permutation words whose recoils are contained in the set of descents of  $c$ . So for instance, if  $c = (2, 3)$ , then

$$\mathfrak{S}_{(2,3)} \backslash \mathfrak{S}_5 = \{12345, 13245, 13425, 13452, 31245, 31425, 31452, 34125, 34152, 34512\} = 12 \sqcup \sqcup 345.$$

<sup>(iii)</sup> One can also consult [Jon90] and [Fra99].

**Proposition 5** [Las06, Theorem 7] *If  $c$  is a composition of  $n$ , let us denote by  $N_c$  the element*

$$\sum_{\omega \in \mathfrak{S}_c \setminus \mathfrak{S}_n} q^{-\ell(\omega)} T_{\omega^{-1}} T_{\omega}$$

*in the Hecke algebra  $\mathcal{H}_{n,q}$ . Then,  $N_c$  does not depend on the order of the parts of  $c$ , and the  $N_{\lambda}$  form a linear basis of  $\mathcal{Z}_{n,q}$  when  $\lambda$  runs over  $\mathfrak{P}_n$  — in particular, the norms  $N_c$  are central elements. Moreover,*

$$(\Gamma_{\lambda})_{\lambda \in \mathfrak{P}_n} = D \cdot M2E \cdot (N_{\mu})_{\mu \in \mathfrak{P}_n},$$

*where  $M2E$  is the transition matrix between monomial functions  $m_{\lambda}$  and elementary functions  $e_{\mu}$ , and  $D$  is the diagonal matrix with coefficients  $(q/(q-1))^{n-\ell(\lambda)}$ .*

So for instance,  $\Gamma_3 = q^2 (q-1)^{-2} (3N_3 - 3N_{2,1} + N_{1,1,1})$ , because  $m_3 = 3e_3 - 3e_{2,1} + e_{1,1,1}$ . Let us write down explicitly the norm basis when  $n = 3$ :

$$\begin{aligned} N_3 &= T_{123} & ; & & N_{2,1} &= 3T_{123} + (q-1)q^{-1}(T_{213} + T_{132}) + (q-1)q^{-2}T_{321} \\ N_{1,1,1} &= 6T_{123} + 3(q-1)q^{-1}(T_{213} + T_{132}) + (q-1)^2q^{-2}(T_{231} + T_{312}) + (q^3-1)q^{-3}T_{321} \end{aligned}$$

We shall see hereafter that these norms have natural preimages by the projections  $\text{pr}_n$  and  $\text{pr}_{\infty,n}$ .

## 4 Generic norms and the Hecke-Ivanov-Kerov algebra

Let us fix some notations. If  $c$  is a composition of size  $|c|$  less than  $n$ , then  $c \uparrow n$  is the composition  $(c_1, \dots, c_r, n - |c|)$ ,  $J_c = I_1 I_2 \cdots I_{|c|-1}$ , and

$$M_{c,n} = \sum_{\omega \in \mathfrak{S}_{c \uparrow n} \setminus \mathfrak{S}_n} q^{-\ell(\omega)} T_{\omega^{-1}} T_{\omega} J_c,$$

the products  $T_{\omega}$  being considered as elements of  $\mathcal{D}_{n,q}$ . So,  $M_{c,n}$  is an element of  $\mathcal{D}_{n,q}$ , and we set  $M_{c,n} = 0$  if  $|c| > n$ .

**Proposition 6** *For any  $N, n$  and any composition  $c$ ,  $\phi_{N,n}(M_{c,N}) = M_{c,n}$ , and  $\text{pr}_n(M_{c,n}) = N_{c \uparrow n}$  if  $|c| \leq n$ , and 0 otherwise. On the other hand,  $M_{c,n}$  is always in  $\mathcal{D}'_{n,q}$ .*

**Proof:** Because of the description of distinguished representatives of right cosets by positions of recoils, if  $|c| \leq n$ , then the sum  $M_{c,n}$  is over permutation words  $\omega$  with recoils in the set of descents of  $c$  (notice that we include  $|c|$  in the set of descents of  $c$ ). Let us denote by  $R_{c,n}$  this set of words, and suppose that  $|c| \leq n-1$ . If  $\omega \in R_{c,n}$  is such that  $\omega(n) \neq n$ , then  $T_{\omega}$  involves  $S_{n-1}$ , so the image by  $\phi_{n,n-1}$  of the corresponding term in  $M_{c,n}$  is zero. On the other hand, if  $\omega(n) = n$ , then any reduced decomposition of  $T_{\omega}$  does not involve  $S_{n-1}$ , so the corresponding term in  $M_{c,n}$  is preserved by  $\phi_{n,n-1}$ . Consequently,  $\phi_{n,n-1}(M_{c,n})$  is a sum with the same terms as  $M_{c,n}$ , but with  $\omega$  running over  $R_{c,n-1}$ ; so, we have proved that  $\phi_{n,n-1}(M_{c,n}) = M_{c,n-1}$  when  $|c| \leq n-1$ . The other cases are much easier: thus, if  $|c| = n$ , then



$M_{c,n-1} = 0$ , and  $\phi_{n,n-1}(M_{c,n})$  is also zero because  $\phi_{n,n-1}(J_c) = 0$ . And if  $|c| > n$ , then  $M_{c,n}$  and  $M_{c,n-1}$  are both equal to zero, and again  $\phi_{n,n-1}(M_{c,n}) = M_{c,n-1}$ . Since

$$\phi_{N,n} = \phi_{n+1,n} \circ \phi_{n+2,n+1} \circ \cdots \circ \phi_{N,N-1},$$

we have proved the first part of the proposition, and the second part is really obvious.

Now, let us show that  $M_{c,n}$  is in  $\mathcal{D}'_{n,q}$ . Notice that the result is trivial if  $|c| > n$ , and also if  $|c| = n$ , because we have then  $J_c = I_{(n)}$ , and therefore  $d = (n)$  for any composed permutation  $(\sigma, d)$  involved in  $M_{c,|c|}$ . Suppose then that  $|c| \leq n - 1$ . Because of the description of  $\mathfrak{S}_d \backslash \mathfrak{S}_{|d|}$  as a shuffle product, any distinguished representative  $\omega$  of  $\mathfrak{S}_{c \uparrow n} \backslash \mathfrak{S}_n$  is the shuffle of a distinguished representative  $\omega_c$  of  $\mathfrak{S}_c \backslash \mathfrak{S}_{|c|}$  with the word  $|c| + 1, |c| + 2, \dots, n$ . For instance, 5613724 is the distinguished representative of a right  $\mathfrak{S}_{(2,2,3)}$ -coset, and it is a shuffle of 567 with the distinguished representative 1324 of a right  $\mathfrak{S}_{(2,2)}$ -coset. Let us denote by  $s_{i_1} \cdots s_{i_r}$  a reduced expression of  $\omega_c$ , and by  $j_{|c|+1}, \dots, j_n$  the positions of  $|c| + 1, \dots, n$  in  $\omega$ . Then, it is not difficult to see that

$$s_{i_1} \cdots s_{i_r} \times (s_{|c|} s_{|c|-1} \cdots s_{j_{|c|+1}}) (s_{|c|+1} s_{|c|} \cdots s_{j_{|c|+2}}) \cdots (s_{n-1} s_{n-2} \cdots s_{j_n})$$

is a reduced expression for  $\omega$ ; for instance,  $s_2$  is the reduced expression of 1324, and

$$s_2 \times (s_4 s_3 s_2 s_1) (s_5 s_4 s_3 s_2) (s_6 s_5)$$

is a reduced expression of 5613724. From this, we deduce that  $T_\omega J_c = T_{\omega, (k, 1^{n-k})}$ , where  $k$  is the highest integer in  $[|c| + 1, n]$  such that  $j_k < k$  — we take  $k = |c|$  if  $\omega = \omega_c$ . Then, the multiplication by  $T_{\omega^{-1}}$  cannot fatten the composition anymore, so  $T_{\omega^{-1}} T_\omega J_c$  is a linear combination of  $T_{\tau, (k, 1^{n-k})}$ , and we have proved that  $M_{n,c}$  is indeed in  $\mathcal{D}'_{n,q}$ .  $\square$

From the previous proof, it is now clear that if we consider the infinite sum  $M_c = \sum q^{-\ell(\omega)} T_{\omega^{-1}} T_\omega J_c$  over permutation words  $\omega \in \mathfrak{S}_\infty$  with their recoils in the set of descents of  $c$ , then  $M_c$  is the unique element of  $\mathcal{D}_{\infty,q}$  such that  $\phi_{\infty,n}(M_c) = M_{c,n}$  for any positive integer  $n$ , and also the unique element of  $\mathcal{D}'_{\infty,q}$  such that  $\text{pr}_{\infty,n}(M_c) = N_{c \uparrow n}$  for any positive integer  $n$  (with by convention  $N_{c \uparrow n} = 0$  if  $|c| > n$ ). In particular,  $M_c$  does not depend on the order of the parts of  $c$ , because this is true for the  $N_{c \uparrow n}$  and the projections separate the vectors in  $\mathcal{D}'_{\infty,q}$ . Consequently, we shall consider only elements  $M_\lambda$  labelled by partitions  $\lambda$  of arbitrary size, and call them **generic norms**. For instance:

$$M_{(2),3} = T_{12|3} + 2T_{123} + (1 - q^{-1})(T_{132} + T_{213}) + (q^{-1} - q^{-2})T_{321}$$

In what follows, if  $i < n$ , we denote by  $(S_i)^{-1}$  the element of  $\mathcal{D}_{n,q}$  equal to:

$$(S_i)^{-1} = q^{-1} S_i + (q^{-1} - 1) I_i$$

The product  $S_i (S_i)^{-1} = (S_i)^{-1} S_i$  equals  $I_i$  in  $\mathcal{D}_{n,q}$ , and by the specialization  $\text{pr}_n : \mathcal{D}_{n,q} \rightarrow \mathcal{H}_{n,q}$ , one recovers  $S_i (S_i)^{-1} = 1$  in the Hecke algebra  $\mathcal{H}_{n,q}$ .

**Theorem 7** *The  $M_\lambda$  span linearly the subalgebra  $\mathcal{C}_{\infty,q} \subset \mathcal{D}'_{\infty,q}$  that consists in elements  $x \in \mathcal{D}'_{\infty,q}$  such that  $I_i x = S_i x (S_i)^{-1}$  for every  $i$ . In particular, any product  $M_\lambda * M_\mu$  is a linear combination of  $M_\nu$ , and moreover, the terms  $M_\nu$  involved in the product satisfy the inequality  $|\nu| \leq |\lambda| + |\mu|$ .*

**Proof:** If  $I_i x = S_i x (S_i)^{-1}$  and  $I_i y = S_i y (S_i)^{-1}$ , then

$$I_i xy = I_i x I_i y = S_i x (S_i)^{-1} S_i y (S_i)^{-1} = S_i x I_i y (S_i)^{-1} = S_i xy (S_i)^{-1},$$

so the elements that “commute” with  $S_i$  in  $\mathcal{D}_{\infty,q}$  form a subalgebra. As an intersection,  $\mathcal{C}_{\infty,q}$  is also a subalgebra of  $\mathcal{D}_{\infty,q}$ ; let us see why it is spanned by the generic norms. If  $\mathcal{D}'_{\infty,q,i}$  is the subspace of  $\mathcal{D}_{\infty,q}$  spanned by the  $T_{\sigma,c}$  with  $c = (k, 1^\infty) \vee (1^{i-1}, 2, 1^\infty)$ , then the projections separate the vectors in this subspace — this is the same proof as in Proposition 4. For  $\lambda \in \mathfrak{P}$ ,  $I_i M_\lambda$  and  $S_i M_\lambda (S_i)^{-1}$  belong to  $\mathcal{D}'_{\infty,q,i}$ , and they have the same projections in  $\mathcal{H}_{n,q}$ , because  $\text{pr}_{\infty,n}(M_\lambda)$  is a norm and in particular a central element. Consequently,  $I_i M_\lambda = S_i M_\lambda (S_i)^{-1}$ , and the  $M_\lambda$  are indeed in  $\mathcal{C}_{\infty,q}$ . Now, if we consider an element  $x \in \mathcal{C}_{\infty,q}$ , then for  $i < n$ ,  $\text{pr}_n(x) = S_i \text{pr}_n(x) (S_i)^{-1}$ , so  $\text{pr}_n(x)$  is in  $\mathcal{X}_{n,q}$  and is a linear combination of norms:

$$\forall n \in \mathbb{N}, \text{pr}_n(x) = \sum_{\lambda \in \mathfrak{P}_n} a_\lambda(x) N_\lambda$$

Since the same holds for any difference  $x - \sum b_\lambda M_\lambda$ , we can construct by induction on  $n$  an infinite linear combination  $S_\infty$  of  $M_\lambda$  that has the same projections as  $x$ :

$$\begin{aligned} \text{pr}_1(x) = \sum_{|\lambda|=1} b_\lambda N_\lambda &\Rightarrow \text{pr}_1\left(x - \sum_{|\lambda|=1} b_\lambda M_\lambda\right) = 0, \quad S_1 = \sum_{|\lambda|=1} b_\lambda M_\lambda \\ \text{pr}_2(x - S_1) = \sum_{|\lambda|=2} b_\lambda N_\lambda &\Rightarrow \text{pr}_{1,2}\left(x - \sum_{|\lambda|\leq 2} b_\lambda M_\lambda\right) = 0, \quad S_2 = \sum_{|\lambda|\leq 2} b_\lambda M_\lambda \\ &\vdots \\ \text{pr}_{n+1}(x - S_n) = \sum_{|\lambda|=n+1} b_\lambda N_\lambda &\Rightarrow S_{n+1} = S_n + \sum_{|\lambda|=n+1} b_\lambda M_\lambda = \sum_{|\lambda|\leq n+1} b_\lambda M_\lambda \end{aligned}$$

Then,  $S_\infty = \sum_{\lambda \in \mathfrak{P}} b_\lambda M_\lambda$  is in  $\mathcal{D}'_{\infty,q}$  and has the same projections as  $x$ , so  $S_\infty = x$ . In particular, since  $\mathcal{C}_{\infty,q}$  is a subalgebra, a product  $M_\lambda * M_\mu$  is in  $\mathcal{C}_{\infty,q}$  and is an *a priori* infinite linear combination of  $M_\nu$ :

$$\forall \lambda, \mu, M_\lambda * M_\mu = \sum g'_{\lambda\mu} M_\nu$$

Since the norms  $N_\lambda$  are defined over  $\mathbb{Z}[q, q^{-1}]$ , by projection on the Hecke algebras  $\mathcal{H}_{n,q}$ , one sees that the  $g'_{\lambda\mu}$  are also in  $\mathbb{Z}[q, q^{-1}]$  — in fact, they are *symmetric* polynomials in  $q$  and  $q^{-1}$ . It remains to be shown that the previous sum is in fact over partitions  $|\nu|$  with  $|\nu| \leq |\lambda| + |\mu|$ ; we shall see why this is true in the last paragraph<sup>(iv)</sup>. □

For example,  $M_1 * M_1 = M_1 + (q + 1 + q^{-1}) M_{1,1} - (q + 2 + q^{-1}) M_2$ , and from this generic identity one deduces the expression of any product  $(N_{(1)\uparrow n})^2$ , e.g.,

$$N_{1,1}^2 = (q + 2 + q^{-1}) (N_{1,1} - N_2) \quad ; \quad N_{3,1}^2 = N_{3,1} + (q + 1 + q^{-1}) N_{2,1,1} - (q + 2 + q^{-1}) N_{2,2}.$$

Let us denote by  $\mathcal{A}_{\infty,q}$  the subspace of  $\mathcal{C}_{\infty,q}$  whose elements are *finite* linear combinations of generic norms; this is in fact a subalgebra, which we call the **Hecke-Ivanov-Kerov algebra** since it plays the same role for Iwahori-Hecke algebras as  $\mathcal{A}_\infty$  for symmetric group algebras.

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<sup>(iv)</sup> Unfortunately, we did not succeed in proving this result with adequate filtrations on  $\mathcal{D}_{\infty,q}$  or  $\mathcal{D}'_{\infty,q}$ .

## 5 Completion of partitions and symmetric functions

The proof of Theorem 1 and of the last part of Theorem 7 relies now on a rather elementary property of the transition matrices  $M2E$  and  $E2M$ . By convention, we set  $e_{\lambda \uparrow n} = 0$  if  $|\lambda| > n$ , and  $m_{\lambda \rightarrow n} = 0$  if  $|\lambda| + \ell(\lambda) > n$ . Then:

**Proposition 8** *There exists polynomials  $P_{\lambda\mu}(n) \in \mathbb{Q}[n]$  and  $Q_{\lambda\mu}(n) \in \mathbb{Q}[n]$  such that*

$$\forall \lambda, n, \quad m_{\lambda \rightarrow n} = \sum_{\mu' \leq_d \lambda} P_{\lambda\mu}(n) e_{\mu \uparrow n} \quad \text{and} \quad e_{\lambda \uparrow n} = \sum_{\mu \leq_d \lambda'} Q_{\lambda\mu}(n) m_{\mu \rightarrow n},$$

where  $\mu \leq_d \lambda$  is the domination relation on partitions.

This fact follows from the study of the Kotska matrix elements  $K_{\lambda, \mu \rightarrow n}$ , see [Mac95, §1.6, in particular the example 4. (c)]. It can also be shown directly by expanding  $e_{\lambda \uparrow n}$  on a sufficient number of variables and collecting the monomials; this simpler proof explains the appearance of binomial coefficients  $\binom{n}{k}$ . For instance,

$$\begin{aligned} m_{2,1 \rightarrow n} &= e_{2,1 \uparrow n} - 3e_{3 \uparrow n} - (n-3)e_{1,1 \uparrow n} + (2n-8)e_{2 \uparrow n} + (2n-5)e_{1 \uparrow n} - n(n-4)e_{\uparrow n}, \\ e_{2,1 \uparrow n} &= \frac{n(n-1)(n-2)}{2} m_{\rightarrow n} + \frac{(n-2)(3n-7)}{2} m_{1 \rightarrow n} + (3n-10)m_{1,1 \rightarrow n} + 3m_{1,1,1 \rightarrow n} \\ &\quad + (n-3)m_{2 \rightarrow n} + m_{2,1 \rightarrow n}. \end{aligned}$$

In the following,  $N_{\lambda,n} = N_{\lambda \uparrow n}$  if  $|\lambda| \leq n$ , and 0 otherwise. Because of the existence of the projective limits  $M_\lambda$ , we know that  $N_{\lambda,n} * N_{\mu,n} = \sum_{\nu} g_{\lambda\mu}^\nu N_{\nu,n}$ , where the sum is not restricted. But on the other hand, by using Proposition 5 and the second identity in Proposition 8, one sees that

$$N_{\lambda,n} * N_{\mu,n} = \sum_{|\rho| \leq |\lambda|, |\sigma| \leq |\mu|} h_{\lambda\mu}^{\rho\sigma}(n) \Gamma_{\rho,n} * \Gamma_{\sigma,n}, \quad \text{with the } h_{\lambda\mu}^{\rho\sigma}(n) \in \mathbb{Q}[n, q, q^{-1}].$$

Because of the result of Francis and Wang, the latter sum may be written as  $\sum_{|\tau| \leq |\lambda| + |\mu|} i_{\lambda\mu}^\tau(n) \Gamma_{\tau,n}$ , and by using the first identity of Proposition 8, one has finally

$$N_{\lambda,n} * N_{\mu,n} = \sum_{|\nu| \leq |\lambda| + |\mu|} j_{\lambda\mu}^\nu(n) N_{\nu,n}, \quad \text{with the } j_{\lambda\mu}^\nu(n) \in \mathbb{Q}[n, q, q^{-1}].$$

From this, it can be shown that the first sum  $\sum_{\nu} g_{\lambda\mu}^\nu N_{\nu,n}$  is in fact restricted on partitions  $|\nu|$  such that  $|\nu| \leq |\lambda| + |\mu|$ , and because the projections separate the vectors of  $\mathcal{D}'_{\infty,q}$ , this implies that  $M_\lambda * M_\mu = \sum_{|\nu| \leq |\lambda| + |\mu|} g_{\lambda\mu}^\nu M_\nu$ , so the last part of Theorem 7 is proved. Finally, by reversing the argument, one sees that the  $a_{\lambda\mu}^\nu(n, q)$  are in  $\mathbb{Q}[n](q)$ :

$$\begin{aligned} \Gamma_{\lambda,n} * \Gamma_{\mu,n} &= (q/(q-1))^{| \lambda | + | \mu |} \sum_{\rho, \sigma} P_{\lambda\rho}(n) P_{\mu\sigma}(n) N_{\rho,n} * N_{\sigma,n} \\ &= (q/(q-1))^{| \lambda | + | \mu |} \sum_{\rho, \sigma, \tau} P_{\lambda\rho}(n) P_{\mu\sigma}(n) g_{\rho\sigma}^\tau N_{\tau,n} \\ &= \sum_{\rho, \sigma, \tau, \nu} (q/(q-1))^{| \lambda | + | \mu | - | \nu |} P_{\lambda\rho}(n) P_{\mu\sigma}(n) g_{\rho\sigma}^\tau(q) Q_{\tau\nu}(n) \Gamma_{\nu,n} = \sum_{\nu} a_{\lambda\mu}^\nu(n, q) \Gamma_{\nu,n} \end{aligned}$$

with  $a_{\lambda\mu}^{\nu}(n, q) = (q/(q-1))^{|^{\lambda}|+|^{\mu}|-|^{\nu}|} (P^{\otimes 2}(n) g(q) Q(n))_{\lambda\mu}^{\nu}$  in tensor notation. And since the  $\Gamma_{\lambda}$  are known to be defined over  $\mathbb{Z}[q, q^{-1}]$ , the coefficients  $a_{\lambda\mu}^{\nu}(n, q) \in \mathbb{Q}[n](q)$  are in fact<sup>(v)</sup> in  $\mathbb{Q}[n, q, q^{-1}]$ . Using this technique, one can for instance show that

$$(\Gamma_{(1),n})^2 = \frac{n(n-1)}{2} q \Gamma_{(0),n} + (n-1)(q-1) \Gamma_{(1),n} + (q+q^{-1}) \Gamma_{(1,1),n} + (q+1+q^{-1}) \Gamma_{(2),n},$$

and this is because  $m_{1 \rightarrow n} = e_{1 \uparrow n} - n e_{\uparrow n}$  and  $e_{1 \uparrow n} = n m_{\rightarrow n} + m_{1 \rightarrow n}$ . Let us conclude by two remarks. First, the reader may have noticed that we did not construct generic conjugacy classes  $F_{\lambda} \in \mathcal{A}_{\infty, q}$  such that  $\text{pr}_{\infty, n}(F_{\lambda}) = \Gamma_{\lambda, n}$ ; since the Geck-Rouquier elements themselves are difficult to describe, we had little hope to obtain simple generic versions of these  $\Gamma_{\lambda}$ . Secondly, the Ivanov-Kerov projective limits of other group algebras — e.g., the algebras of the finite reductive Lie groups  $\text{GL}(n, \mathbb{F}_q)$ ,  $\text{U}(n, \mathbb{F}_{q^2})$ , etc. — have not yet been studied. It seems to be an interesting open question.

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<sup>(v)</sup> They are even in  $\mathbb{Q}_{\mathbb{Z}}[n] \otimes \mathbb{Z}[q, q^{-1}]$ , where  $\mathbb{Q}_{\mathbb{Z}}[n]$  is the  $\mathbb{Z}$ -module of polynomials with rational coefficients and integer values on integers; indeed, the matrices  $M2E$  and  $E2M$  have integer entries. It is well known that  $\mathbb{Q}_{\mathbb{Z}}[n]$  is spanned over  $\mathbb{Z}$  by the binomials  $\binom{n}{k}$ .