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The cluster and dual canonical bases of $\mathbb{Z}[x_{11}, \ldots, x_{33}]$ are equal

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Abstract. The polynomial ring $\mathbb{Z}[x_{11}, \ldots, x_{33}]$ has a basis called the dual canonical basis whose quantization facilitates the study of representations of the quantum group $U_q(\mathfrak{sl}_3(\mathbb{C}))$. On the other hand, $\mathbb{Z}[x_{11}, \ldots, x_{33}]$ inherits a basis from the cluster monomial basis of a geometric model of the type D_4 cluster algebra. We prove that these two bases are equal. This extends work of Skandera and proves a conjecture of Fomin and Zelevinsky. This also provides an explicit factorization of the dual canonical basis elements of $\mathbb{Z}[x_{11}, \ldots, x_{33}]$ into irreducible polynomials.

Résumé. L'anneau de polynômes $\mathbb{Z}[x_{11}, \ldots, x_{33}]$ a une base appelée base duale canonique, et dont une quantification facilite l'étude des représentations du groupe quantique $U_q(\mathfrak{sl}_3(\mathbb{C}))$. D'autre part, $\mathbb{Z}[x_{11}, \ldots, x_{33}]$ admet une base issue de la base des monômes d'amas de l'algèbre amassée géométrique de type D_4. Nous montrons que ces deux bases sont égales. Ceci prolonge les travaux de Skandera et démontre une conjecture de Fomin et Zelevinsky. Ceci fournit également une factorisation explicite en polynômes irréductibles des éléments de la base duale canonique de $\mathbb{Z}[x_{11}, \ldots, x_{33}]$.

Keywords: cluster algebra, dual canonical basis

1 Introduction

For $n \geq 0$, let $\mathcal{A}_n$ denote the polynomial ring $\mathbb{Z}[x_{11}, \ldots, x_{nn}]$ in the n^2 commuting variables $(x_{ij})_{1 \leq i,j \leq n}$. The algebra $\mathcal{A}_n$ has an obvious $\mathbb{Z}$-basis of monomials in the variables x_{ij}, which we call the *natural* basis. In addition to the natural basis, the ring $\mathcal{A}_n$ has many other interesting bases such as a bitableau basis defined by Mead and popularized by Désarménien, Kung, and Rota [2] having applications in invariant theory and the dual canonical basis of Lusztig [8] and Kashiwara [5] whose quantization facilitates the study of representations of the quantum group $U_q(\mathfrak{sl}_n(\mathbb{C}))$. Given two bases of $\mathcal{A}_n$, it is natural to compare them by examining the corresponding transition matrix. For example, in [9] it is shown that these latter two bases are related via a transition matrix which may be taken to be unitriangular (i.e., upper triangular with 1's on the main diagonal) with respect to an appropriate ordering of basis elements.

Cluster algebras are a certain class of commutative rings introduced by Fomin and Zelevinsky [3] to study total positivity and dual canonical bases. Any cluster algebra comes equipped with a distinguished set of generators called *cluster variables* which are grouped into finite overlapping subsets called *clusters*, all of which have the same cardinality. The cluster algebras with a finite number of clusters have a classification similar to the Cartan-Killing classification of finite-dimensional simple complex Lie algebras [4]. In this classification, it turns out that the cluster algebra of type D_4 is a localization of the ring $\mathcal{A}_3$ (see for example [10]) and the ring $\mathcal{A}_3$ inherits a $\mathbb{Z}$-basis consisting of cluster monomials. We call this basis the *cluster basis*. Fomin and Zelevinsky conjectured that the cluster basis and the dual canonical basis of $\mathcal{A}_3$ are equal, and Skandera showed that any two of the natural, cluster, and dual canonical bases of $\mathcal{A}_3$ are related via a unitriangular transition matrix when basis elements are ordered appropriately [10]. In this paper we strengthen Skandera's result and prove Fomin and Zelevinsky's conjecture with the following result (definitions will be postponed until Section 2).

Theorem 1.1 *The dual canonical and cluster basis of $\mathcal{A}_3$ are equal.*

Since each of the cluster and frozen variables of $\mathcal{A}_3$ are irreducible polynomials, this result can be viewed as giving a complete factorization of the dual canonical basis elements of $\mathcal{A}_3$ into irreducibles. Because the natural $GL_3(\mathbb{C})$ action on $\mathbb{C} \otimes_{\mathbb{Z}} \mathcal{A}_3$ is multiplicative, this could aid in constructing representing matrices for this action with respect to the dual canonical basis.

Theorem 1.1 will turn out to be the classical $q = 1$ specialization of a result (Theorem 3.6) comparing two bases of a noncommutative quantization $\mathcal{A}_3^{(q)}$ of the polynomial ring $\mathbb{A}_3$. In Section 2 we define the cluster basis of the classical ring $\mathcal{A}_3$. In Section 3 we introduce the quantum polynomial ring $\mathcal{A}_n^{(q)}$ together with its dual canonical basis and a quantum analogue of the cluster basis of $\mathcal{A}_3$. In Section 4 we comment on possible extensions of the results in this extended abstract.

2 The Cluster Basis of $\mathbb{Z}[x_{11}, \ldots, x_{33}]$

We shall not find it necessary to use a great deal of the general theory of cluster algebras to define and study the cluster basis of $\mathcal{A}_3$. Rather, we simply will define a collection of 16 polynomials in $\mathcal{A}_3$ to be cluster variables and associate to each of them a certain decorated octogon, define an additional 5 polynomials to be frozen variables, define (extended) clusters in terms of noncrossing conditions on decorated octogons, and define cluster monomials to be products of elements of an extended cluster.

For any two subsets $I, J \subseteq 3$ of equal size, define the $(I, J)-$minor $\Delta_{I,J}(x)$ of $x = (x_{i,j})_{1 \leq i,j \leq 3}$ to be the determinant of the submatrix of x with row set I and column set J. Define additionally two more polynomials, the 132- and 213-Kazhdan-Lusztig immanants of x, by

$$\mathrm{Imm}_{132}(x) = x_{11}x_{23}x_{32} - x_{12}x_{23}x_{31} - x_{13}x_{21}x_{32} + x_{13}x_{22}x_{31}$$

and

$$\mathrm{Imm}_{213}(x) = x_{12}x_{21}x_{33} - x_{12}x_{23}x_{31} - x_{13}x_{21}x_{32} + x_{13}x_{22}x_{31}.$$

The *cluster variables* are the 16 elements of $\mathcal{A}_3$ shown in Figure 2.1 [10, p. 3], with the associated decorated octogons. Every octogon is decorated with either a pair of parallel nonintersecting nondiameters or a diameter colored one of two colors, red or blue.

A *centrally symmetric modified triangulation* of the octogon is a maximal collection of the above octogon decorations without crossings, where we adopt the convention that identical diameters of different colors do not cross and distinct diameters of the same color do not cross. Every centrally symmetric modified triangulation of the octogon consists of four decorations, and a *cluster* is the associated four element set of polynomials corresponding to the decorations in such a triangulation. There are 50 centrally symmetric modified triangulations of the octogon, and hence 50 clusters. Four examples of centrally symmetric modified triangulations are shown in Figure 2.2. The corresponding clusters are, from left to

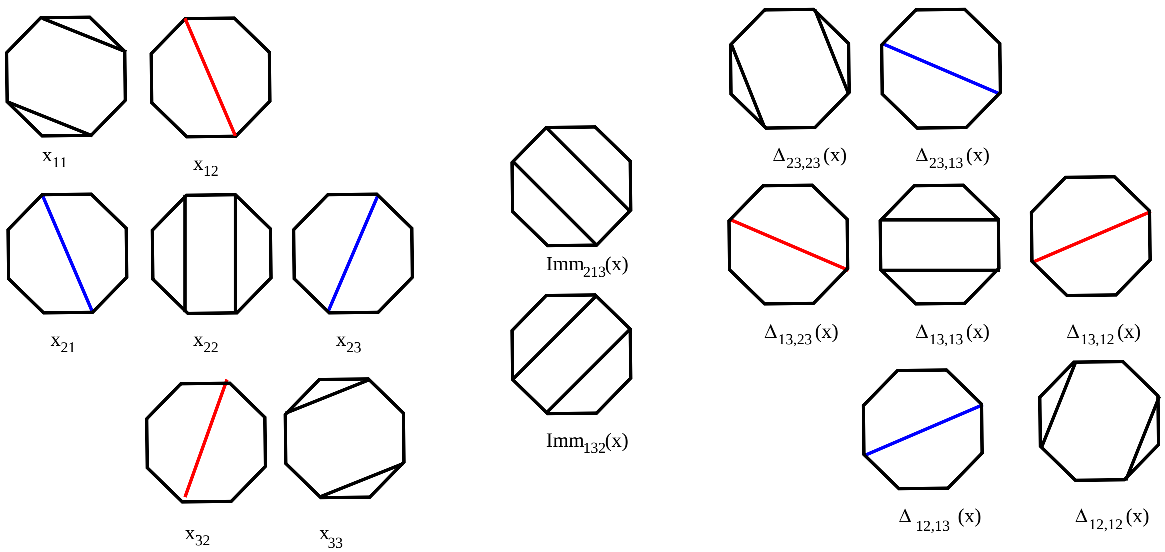

Fig. 2.1: Cluster variables in $\mathbb{Z}[x_{11}, \ldots, x_{33}]$

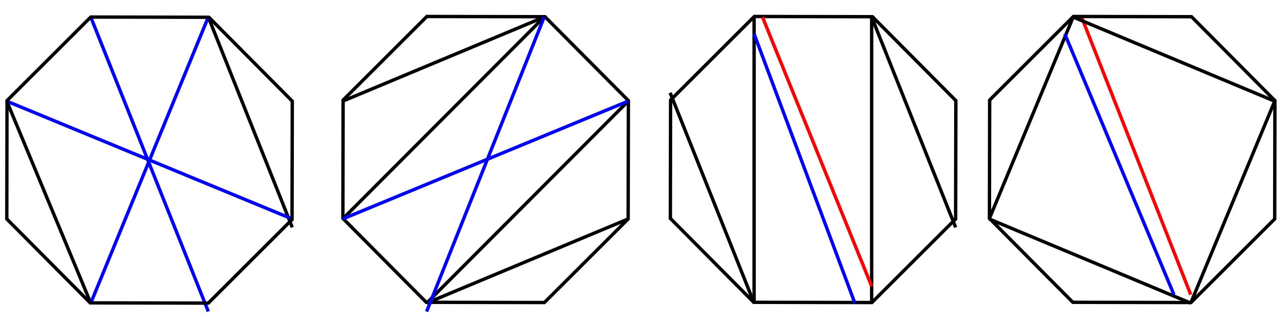

Fig. 2.2: Four centrally symmetric modified triangulations corresponding to clusters

right, $\{x_{21}, x_{23}, \Delta_{23,13}(x), \Delta_{23,23}(x)\}$, $\{x_{23}, x_{33}, \Delta_{12,13}(x), \mathrm{Imm}_{132}(x)\}$, $\{x_{12}, x_{21}, x_{22}, \Delta_{23,23}(x)\}$, and $\{x_{11}, x_{12}, x_{21}, \Delta_{12,12}(x)\}$.

We define additionally a set $\mathcal{F}$ consisting of the five polynomials

$$\mathcal{F} := \{x_{13}, \Delta_{12,23}(x), \Delta_{123,123}(x) = \det(x), \Delta_{23,12}(x), x_{31}\}.$$

Elements in $\mathcal{F}$ are called *frozen variables* and the union of $\mathcal{F}$ with any cluster is an *extended cluster*. A *cluster monomial* is a product of the form $z_1^{a_1} \cdots z_9^{a_9}$, where $\{z_1, \dots, z_9\}$ is an extended cluster and the a_i are nonnegative integers. Observe that the same cluster monomial can arise from different extended clusters. The *cluster basis* of $\mathcal{A}_3$ is the set of all possible cluster monomials.

Skandera [10] develops a bijection ϕ between the cluster basis and the set $\mathrm{Mat}_3(\mathbb{N})$ as follows. For any cluster or frozen variable z, let $\phi(z)$ be the lexicographically greatest matrix $A = (a_{ij})$ for which the monomial $\prod x_{ij}^{a_{ij}}$ appears with nonzero coefficient in the expansion of z in the natural basis. Given an arbitrary cluster monomial $z_1^{a_1} \cdots z_9^{a_9}$, extend the definition of ϕ via

$$\phi(z_1^{a_1} \cdots z_9^{a_9}) := a_1\phi(z_1) + \cdots + a_9\phi(z_9).$$

The fact that ϕ is a bijection [10] implies that the set of cluster monomials is related to the natural basis of $\mathcal{A}_3$ via a unitriangular, integer transition matrix, and thus is actually a $\mathbb{Z}$-basis for $\mathcal{A}_3$ (the fact that the cluster monomials form a basis for $\mathcal{A}_3$ is also a consequence of more theory of finite type cluster algebras).

Example 2.1 *Consider the cluster corresponding to the leftmost centrally symmetric modified triangulation in Figure 2.2, i.e.* $\{x_{21}, x_{23}, \Delta_{23,13}(x), \Delta_{23,23}(x)\}$. *An example of a cluster monomial drawn from the corresponding extended cluster is*

$$z := x_{21}^7 x_{23}^0 \Delta_{23,13}(x)^2 \Delta_{23,23}(x)^1 x_{13}^0 \Delta_{12,23}(x)^2 \Delta_{123,123}(x)^0 \Delta_{23,12}(x)^0 x_{31}^7.$$

We have that

$$\phi(z) = 7\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \cdots = \begin{pmatrix} 0 & 2 & 0 \\ 9 & 1 & 2 \\ 7 & 0 & 3 \end{pmatrix}.$$

3 The Quantum Polynomial Ring

For $n \geq 0$, define the *quantum polynomial ring* $\mathcal{A}_n^{(q)}$ to be the unital associative (noncommutative) $\mathbb{Z}[q^{\pm 1/2}]$-algebra generated by the n^2 variables $x = (x_{ij})_{1 \leq i,j \leq n}$ and subject to the relations

$$x_{ik}x_{il} = qx_{il}x_{ik} \tag{1}$$
$$x_{ik}x_{jk} = qx_{jk}x_{ik} \tag{2}$$
$$x_{il}x_{jk} = x_{jk}x_{il} \tag{3}$$
$$x_{ik}x_{jl} = x_{jl}x_{ik} + (q - q^{-1})x_{il}x_{jk}, \tag{4}$$

where $i < k$ and $k < l$. It follows from these relations that the specialization of $\mathcal{A}_n^{(q)}$ to $q = 1$ recovers the classical polynomial ring $\mathcal{A}_n$. The center of $\mathcal{A}_n^{(q)}$ is generated by the *quantum determinant* $\det_q(x) :=$

$\sum_{w\in S_n}(-q)^{\ell(w)}x_{1,w(1)}\cdots x_{n,w(n)}$. Factoring the extension $\mathbb{C}\otimes_{\mathbb{Z}}\mathcal{A}_n^{(q)}$ by the ideal $(\det_q(x)-1)$ yields the quantum coordinate ring $\mathcal{O}_q(SL_n(\mathbb{C}))$ of the special linear group. Given two ring elements $f,g\in\mathcal{A}_n^{(q)}$, we say that f is a *q-shift* of g if there is a number a so that $f=q^a g$.

The natural basis of $\mathcal{A}_n$ lifts to a $\mathbb{Z}[q^{\pm 1/2}]$-basis of the quantum polynomial ring $\mathcal{A}_n^{(q)}$ given by $\{X^A := x_{11}^{a_{11}}\cdots x_{nn}^{a_{nn}} \mid A=(a_{ij})\in \mathrm{Mat}_n(\mathbb{N})\}$, where the terms in the product are in lexicographical order (see, for example, [12]). We call this basis the *quantum natural basis* (QNB). We will find it convenient to work with a $\mathbb{Z}[q^{\pm 1/2}]$-basis for $\mathcal{A}_n^{(q)}$ whose elements are q-shifts of QNB elements. Following [12], for any matrix $A=(a_{ij})\in\mathrm{Mat}_n(\mathbb{N})$, define the number $e(A):=-\frac{1}{2}\sum_i\sum_{j<k}(a_{ij}a_{ik}+a_{ji}a_{ki})$ and the quantum polynomial $X(A):=q^{e(A)}X^A\in\mathcal{A}_n^{(q)}$. The set $\{X(A)\,|\,A\in\mathrm{Mat}_n(\mathbb{N})\}$ is also a $\mathbb{Z}[q^{\pm 1/2}]$-basis of $\mathcal{A}_n^{(q)}$, called the *modified quantum natrual basis* (MQNB).

As with the classical polynomial ring $\mathcal{A}_n$, the quantum ring $\mathcal{A}_n^{(q)}$ admits a natural $\mathbb{N}$-grading by degree. Finer than this grading is an $\mathbb{N}^n\times\mathbb{N}^n$-grading, where the $(r_1,\dots,r_n)\times(c_1,\dots,c_n)$-graded piece is the $\mathbb{Z}[q^{\pm 1/2}]$-linear span of all MQNB elements $X(A)$ for matrices $A\in\mathrm{Mat}_n(\mathbb{N})$ with row vector $row(A)=(r_1,\dots,r_n)$ and column vector $col(A)=(c_1,\dots,c_n)$. It is routine to check from (1)-(4) that this grading is well-defined.

The ring $\mathcal{A}_n^{(q)}$ is equipped with an involutive bar antiautomorphism defined by the $\mathbb{Z}$-linear extension of $\overline{q^{1/2}}=q^{-1/2}$ and $\overline{x_{ij}}=x_{ij}$. It follows readily from relations (1)-(4) that $\bar{\cdot}$ is well-defined. Observe that the bar involution specializes to the identity map at $q=1$. The *dual canonical basis* (DCB) of $\mathcal{A}_n^{(q)}$ arises naturally when attempting to find bases of $\mathcal{A}_n^{(q)}$ consisting of bar invariant polynomials.

Define a partial order $\le_{Br}$ on $\mathrm{Mat}_n(\mathbb{N})$ called *Bruhat order* by letting $\le_{Br}$ be the transitive closure of $A\prec_{Br}B$ if B can be obtained from $A=(a_{ij})_{1\le i,j\le n}$ by a 2×2 submatrix transformation of the form

$$\begin{pmatrix} a_{ik} & a_{il}\\ a_{jk} & a_{jl}\end{pmatrix}\mapsto\begin{pmatrix} a_{ik}-1 & a_{il}+1\\ a_{jk}+1 & a_{jl}-1\end{pmatrix},$$

for $i<j$ and $k<l$ with $a_{ik},a_{ji}>0$. Observe that the restriction of $\le_{Br}$ to the set of permutation matrices is isomorphic to the ordinary (strong) Bruhat order on the symmetric group S_n. Observe also that matrix transposition and antitransposition are automorphisms of the poset $(\mathrm{Mat}_n(\mathbb{N}),\le_{Br})$.

Theorem 3.1 *There exists a unique* $\mathbb{Z}[q^{\pm 1/2}]$*-basis*

$$\{b(A)\,|\,A\in\mathrm{Mat}_n(\mathbb{N})\}$$

of $\mathcal{A}_n^{(q)}$ *where* $b(A)$ *is homogeneous with respect to the* $\mathbb{N}^n\times\mathbb{N}^n$*-grading of* $\mathcal{A}_n^{(q)}$ *with degree* $row(A)\times col(A)$ *and the b(A) satisfy*
(1) (Bar invariance) $\overline{b(A)}=b(A)$ *for all* $A\in\mathrm{Mat}_n(\mathbb{N})$*, and*
(2) (Triangularity) For all $A\in\mathrm{Mat}_n(\mathbb{N})$*, the basis element* $b(A)$ *expands in the MQNB as*

$$b(A)=X(A)+\sum_{B>_{Br}A}\beta_{A,B}(q^{1/2})X(B),$$

where the $\beta_{A,B}$ *are polynomials in* $q^{1/2}\mathbb{Z}[q^{1/2}]$.

This basis is called the dual canonical basis.

Proof: If one replaces Bruhat order with the dual of lexicographical order on $\mathrm{Mat}_n(\mathbb{N})$, this result is [12, Theorem 3.2]. However, as noted in the first paragraph of the proof of [12, Corollary 3.4], one has that the coefficient $\beta_{A,B}(q^{1/2})$ in the expansion

$$b(A) = X(A) + \sum_{B <_{lex} A} \beta_{A,B}(q^{1/2}) X(B),$$

is nonzero only if $A >_{Br} B$. □

While the DCB is important in the study of the representation theory of the quantum group $U_q(\mathfrak{sl}_n(\mathbb{C}))$[5] [8] , the lack of an elementary formula for the expansion of the $b(A)$ in the MQNB can make computations involving the DCB difficult. Setting $q = 1$ in the DCB elements yields a basis of the classical polynomial ring $\mathcal{A}_n$, also called the dual canonical basis.

Due to the triangularity condition (2) of Theorem 3.1, we will frequently need to analyze expansions of quantum ring elements in the (M)QNB. To find these expansions, we use relations (1)-(4) to put arbitrary ring elements in lexicographical order. While the somewhat exotic relation (4) can make for complicated expansions, if we are only interested in the Bruhat leading term the situation improves. Given any product $m \in \mathcal{A}_n^{(q)}$ of the generators x_{ij} and a ground ring element $\beta(q) \in \mathbb{Z}[q^{\pm 1/2}]$, define the *content* $C(\beta(q)m)$ of $\beta(q)m$ to be the $n \times n$ matrix whose (i, j)-entry is equal to the number of occurances of x_{ij} in m.

Lemma 3.2 *(Leading Lemma) Let $f = m + m_1 + \cdots + m_r \in \mathcal{A}_n^{(q)}$ be an element of $\mathcal{A}_n^{(q)}$ such that the m_k are monomials and $C(m) <_{Br} C(m_k)$ for all k. Write $m = \beta(q)x_{i_1 j_1} \cdots x_{i_r j_r}$, where $\beta(q) \in \mathbb{Z}[q^{\pm 1/2}]$.*

The expansion of f in the QNB has unique term $X^{C(m)}$ with $C(m)$ Bruhat minimal among the contents of the terms in the QNB expansion of f and the coefficient of $X^{C(m)}$ in this expansion is

$$q^{-y}\beta(q),$$

where y is given by

$$y = |\{(k < \ell) \,|\, i_k = i_\ell, j_k > j_\ell\}| + |\{(k < \ell) \,|\, j_k = j_\ell, i_k > i_\ell\}|.$$

Proof: Observe that the application of relations (1)-(3) to the ring element f do not change the contents of the monomial constituents of f. Moreover, the application of relation (4) to any monomial m_0 in $\mathcal{A}_n^{(q)}$ yields a sum $m_0' + m_0''$, where m_0' has the same content and coefficient as m_0 and m_0'' has content which is greater in Bruhat order than the content of m_0. The value of y follows from the exponents of q which appear in the quasicommutativity relations (1)-(3). □

In the classical setting $q = 1$, Skandera [11] discovered an explicit formula for dual canonical basis elements of $\mathcal{A}_n$ which involves certain polynomials called *immanants*. Given a permutation $w \in S_m$ and an $m \times m$ matrix $y = (y_{ij})_{1 \leq i,j \leq m}$ with entries drawn from the set $\{x_{ij} \,|\, 1 \leq i, j \leq n\}$, define the *w-KL immanant* of y to be

$$\mathrm{Imm}_w(y) := \sum_{v \in S_m} Q_{v,w}(1) y_{1,v(1)} \cdots y_{m,v(m)}.$$

Here $Q_{v,w}(q)$ is the *inverse Kazhdan-Lusztig polynomial* corresponding to the permutations v and w (see [6] or [1]). It can be shown that the KL immanant $\mathrm{Imm}_1(y)$ corresponding to the identity permutation $1 \in S_m$ is equal to the determinant $\det(y)$.

Any (weak) composition $\alpha \models m$ with n parts induces a function $[m] \to [n]$, also denoted α, which maps the interval $(\alpha_1 + \cdots + \alpha_{i-1}, \alpha_1 + \cdots \alpha_i]$ onto i for all i. We also have the associated parabolic subgroup $S_\alpha \cong S_{\alpha_1} \times \cdots \times S_{\alpha_n}$ of S_m which stabilizes all of the above intervals. Given a pair $\alpha, \beta \models m$ of compositions of m both having n parts, we define the *generalized submatrix* $x_{\alpha,\beta}$ of x to be the $m \times m$ matrix satisfying $(x_{\alpha,\beta})_{ij} := x_{\alpha(i),\beta(j)}$ for all $1 \leq i, j \leq m$. Let $\Lambda_m(\alpha, \beta)$ denote the set of Bruhat maximal permutations in the set of double cosets $S_\alpha \backslash S_m / S_\beta$. Skandera's work [11, Section 2] implies that the dual canonical basis of $\mathcal{A}_n$ is equal to the set

$$\bigcup_{m \geq 0} \bigcup_{\alpha,\beta} \{\mathrm{Imm}_w(x_{\alpha,\beta}) \,|\, w \in \Lambda_m(\alpha, \beta)\}. \tag{5}$$

The lack of an elementary description of the inverse KL polynomials is the most difficult part in using Skandera's formula to write down DCB elements.

Returning to the quantum setting, for two subsets $I, J \subseteq [n]$ with $|I| = |J|$, the *quantum minor* $\Delta^{(q)}_{I,J}(x) \in \mathcal{A}^{(q)}_n$ is the quantum determinant of the submatrix of x with row set I and column set J. Restricting to the case $n = 3$, we define the *quantum 132- and 213-KL immanants*, denoted $\mathrm{Imm}^{(q)}_{132}(x)$ and $\mathrm{Imm}^{(q)}_{213}(x)$, to be the elements of $\mathcal{A}^{(q)}_3$ given by

$$\mathrm{Imm}^{(q)}_{132}(x) = x_{11}x_{23}x_{32} - qx_{12}x_{23}x_{31} - qx_{13}x_{21}x_{32} + q^2x_{13}x_{22}x_{31}$$

and

$$\mathrm{Imm}^{(q)}_{213}(x) = x_{12}x_{21}x_{33} - qx_{12}x_{23}x_{31} - qx_{13}x_{21}x_{32} + q^2x_{13}x_{22}x_{31}.$$

Define *quantum cluster* and *quantum frozen variables* to be the polynomials obtained by replacing every minor in the classical quantum or frozen variable definition by its corresponding quantum minor and the classical polynomials $\mathrm{Imm}_{132}(x)$ and $\mathrm{Imm}_{213}(x)$ by their quantum counterparts. Define a quantum (extended) cluster to be the set of quantum (frozen and) cluster variables corresponding to polynomials in a classical (extended) cluster.

To define the quantum cluster monomials, fix a total order $\{z'_1 < z'_2 < \cdots < z'_{21}\}$ on the union of the quantum cluster and frozen variables. A *quantum cluster monomial* is any product of the form $z_1^{a_1} \cdots z_9^{a_9} \in \mathcal{A}^{(q)}_3$, where $\{z_1 < \cdots < z_9\}$ is an ordered quantum extended cluster and the a_i are nonnegative integers. Skandera's map ϕ yields a bijection (also denoted ϕ) between the set of quantum cluster monomials and $\mathrm{Mat}_3(\mathbb{N})$. This bijection, combined with the Leading Lemma, implies that the transition matrix between the set of quantum cluster monomials and the QNB is triangular with units on the diagonal with respect to any order of basis elements which is obtained from a linear extension of Bruhat order. Therefore, the set of all quantum cluster monomials is a $\mathbb{Z}[q^{\pm 1/2}]$-basis for $\mathcal{A}^{(q)}_3$, called the *quantum cluster basis* (QCB). Observe also that every QCB element is homogeneous with respect to the $\mathbb{N}^3 \times \mathbb{N}^3$-grading of the ring $\mathcal{A}^{(q)}_3$.

It is natural to ask how much the QCB depends on the initial choice of total order $\{z'_1 < z'_2 < \cdots < z'_{21}\}$ on the union of the quantum cluster and frozen variables. While it is the case that different choices of the total order $\{z'_1 < z'_2 < \cdots < z'_{21}\}$ can lead to different quantum cluster monomials, we will show in Observation 3.5 that these ring elements differ only up to a q-shift (which may depend not only on the order chosen, but also on the quantum cluster monomial in question). Thus, the QCB is independent of this choice of order 'up to q-shift'.

Our computational work with the ring $\mathcal{A}_n^{(q)}$ will be economized by means of a collection of algebra maps. Define maps τ and α on the generators of $\mathcal{A}_n^{(q)}$ by the formulas $\tau(x_{ij}) = x_{ji}$ and $\alpha(x_{ij}) = x_{(n-j+1)(n-i+1)}$. It is routine to check from the relations (1)-(4) that τ extends to an involutive $\mathbb{Z}[q^{\pm 1/2}]$-algebra automorphism $\tau : \mathcal{A}_n^{(q)} \to \mathcal{A}_n^{(q)}$ and that α extends to an involutive $\mathbb{Z}[q^{\pm 1/2}]$-algebra antiautomorphism $\alpha : \mathcal{A}_n^{(q)} \to \mathcal{A}_n^{(q)}$. The maps τ and α will be called the *transposition* and *antitransposition* maps, respectively, because they act on the matrix $x = (x_{ij})$ of generators by transposition and antitransposition. In addition, for any two subsets $I, J \subseteq [n]$, we can form the subalgebra $\mathcal{A}_n^{(q)}(I, J)$ of $\mathcal{A}_n^{(q)}$ generated by $\{x_{ij} \mid i \in I, j \in J\}$. Writing $I = \{i_1 < \cdots < i_r\}$ and $J = \{j_1 < \cdots < j_s\}$, we have a $\mathbb{Z}[q^{\pm 1/2}]$-algebra isomorphism $c_{I,J} : \mathcal{A}_n^{(q)}(I, J) \to \mathcal{A}_n^{(q)}([r], [s])$ given by $c_{I,J} : x_{i_a, j_b} \mapsto x_{a,b}$. The map $c_{I,J}$ will be called the *compression map* corresponding to I and J because it acts on the matrix x of generators by compression into the northwest corner. The set of quantum cluster and frozen variables is closed under taking images under τ, α, and the compression maps.

Observation 3.3 *Let z be a quantum cluster or frozen variable. Then, $\alpha(z)$ and $\tau(z)$ are quantum cluster or frozen variables with $\phi(\alpha(z)) = \phi(z)^{T'}$ and $\phi(\tau(z)) = \phi(z)^T$, where $\cdot^T$ denotes matrix transposition and $\cdot^{T'}$ denotes matrix antitransposition. Moreover, if the row support of $\phi(z)$ is contained in $I \subseteq [3]$ and the column support of $\phi(z)$ is contained in $J \subseteq [3]$, then the image of z under the compression map $c_{I,J}$ corresponding to I and J is a quantum cluster or frozen variable whose image under ϕ is obtained by compressing the nonzero rows and columns of $\phi(z)$ to the northwest.*

The proof of this observation is a direct computation. For example, the quantum cluster variable $z = \Delta_{23,13}^{(q)}(x)$ satisfies $\phi(z) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, which has row support $\{2, 3\}$ and column support $\{1, 3\}$. The image of z under the compression map $c_{23,13}$ is $y = \Delta_{12,12}^{(q)}(x)$, which satisfies $\phi(y) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. The 7 equivalence classes of quantum cluster variables under the maps τ and α are shown in Figure 3.1. The actions of τ, α, and the compression maps remain well-defined on the level of quantum clusters. The transposition, antitransposition, and compression maps act on MQNB elements in a simple way.

Observation 3.4 *Let $A \in \mathrm{Mat}_n(\mathbb{N})$. We have the following formulae involving the MQNB, retaining notation from Observation 3.3:*
(1) $\tau(X(A)) = X(A^T)$
(2) $\alpha(X(A)) = X(A^{T'})$.
Moreover, if the row support of A is contained in I and the column support of A is contained in J for subsets $I, J \subseteq [n]$, then
(3) $c_{I,J}(X(A)) = X(C_{I,J}(A))$,
where $C_{I,J}(A)$ is obtained by compressing the entries in A with row indicies in I and column indicies in J to the northwest.

Proof: (3) is trivial. To verify (1) and (2), one applies the maps τ and α to $X(A)$ and uses the defining relations (3.1)-(3.3) to get the desired result. $\square$

It is natural to ask to what extent the choice of total order $\{z'_1 < \cdots < z'_{21}\}$ on the union of the quantum cluster and frozen variables affects the quantum cluster monomials. The effects of this choice turn out to

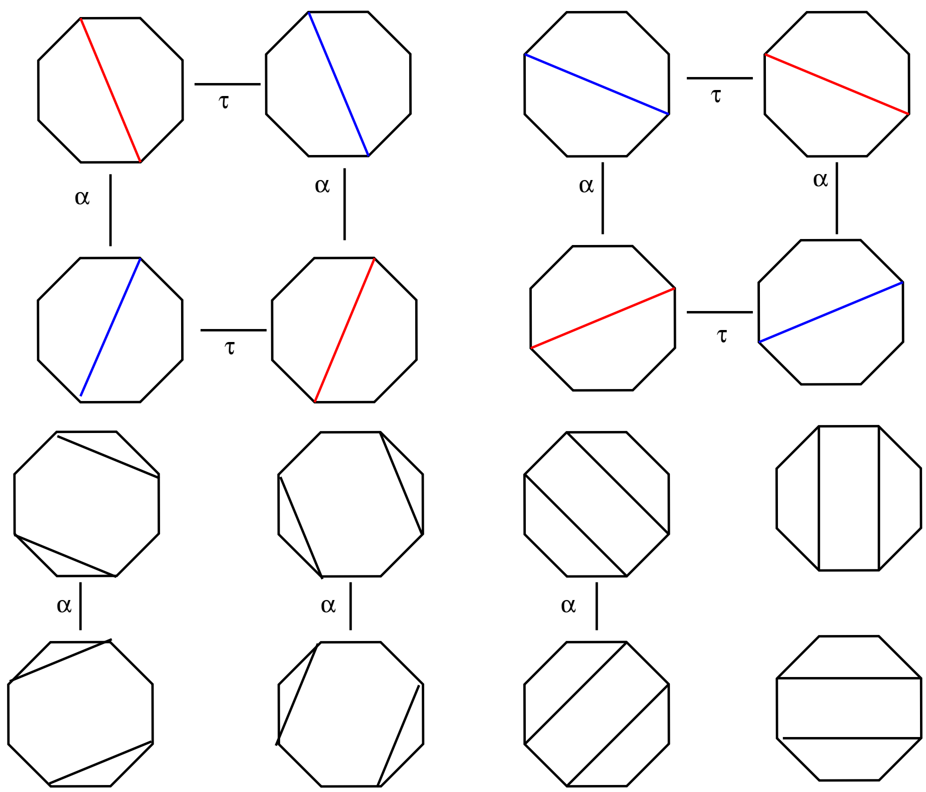

Fig. 3.1: Equivalence classes of cluster variables under α and τ

be quite benign. More precisely, we observe that different choices of total orders only affect the QCB by q-shifts. Two ring elements $f, g \in \mathcal{A}_n^{(q)}$ are said to *quasicommute* if $fg = q^a gf$ for some a.

Observation 3.5 *Let z, z' be a pair of polynomials which appear in the same quantum extended cluster. Then, z and z' quasicommute and moreover $zz' = q^a z'z$ for some $a \in \mathbb{Z}$.*

The proof of the above observation is a straightforward, albeit tedious calculation using the relations (1)-(4). It follows from Lemma 5.1 of [12] that any quantum frozen variable quasicommutes with any quantum frozen or cluster variable, so it is enough to show that cluster variables appearing in the same (nonextended) cluster pairwise quasicommute. By use of the transposition map τ, the antitransposition map α, and the compression maps, we need only check this observation on 7 pairs of quantum cluster variables. The resulting identities are as follows.

$$\begin{array}{cc}
\Delta^{(q)}_{12,12}(x)\Delta^{(q)}_{13,23}(x) = q\Delta^{(q)}_{13,23}(x)\Delta^{(q)}_{12,12}(x) & \mathrm{Imm}^{(q)}_{132}(x)\Delta^{(q)}_{13,12}(x) = \Delta^{(q)}_{13,12}(x)\mathrm{Imm}^{(q)}_{132}(x) \\
\Delta^{(q)}_{12,12}(x)\mathrm{Imm}^{(q)}_{132}(x) = q^2\mathrm{Imm}^{(q)}_{132}(x)\Delta^{(q)}_{12,12}(x) & \Delta^{(q)}_{13,12}(x)\Delta^{(q)}_{12,13}(x) = \Delta^{(q)}_{12,13}(x)\Delta^{(q)}_{13,12}(x) \\
\Delta^{(q)}_{13,12}(x)\Delta^{(q)}_{13,23}(x) = q\Delta^{(q)}_{13,23}(x)\Delta^{(q)}_{13,12}(x) & x_{11}x_{12} = qx_{12}x_{11} \\
x_{12}x_{21} = x_{21}x_{12} &
\end{array}$$

Theorem 1.1 follows from specializing the following result at $q = 1$.

Theorem 3.6 *Every DCB element of $\mathcal{A}_3^{(q)}$ is a q-shift of a unique QCB element of $\mathcal{A}_3^{(q)}$.*

Proof: (Sketch) Fixed a quantum cluster monomial $z_1^{a_1} \cdots z_9^{a_9}$ arising from an ordered quantum extended cluster $\{z_1 < \cdots < z_9\}$. We show that $z_1^{a_1} \cdots z_9^{a_9}$ has a q-shift which satisfies the bar invariance condition (1) of Theorem 3.1 and that this same q-shift also satisfies the triangularity condition (2) of Theorem 3.1.

To check that a q-shift of $z_1^{a_1} \cdots z_9^{a_9}$ is invariant under the bar involution, one first shows by direct computation that every quantum frozen and cluster variable is invariant under the bar involution. Using the transposition map τ, the antitransposition map α, and the compression maps $c_{I,J}$ reduces our computations here to checking that the four elements $x_{11}, \Delta^{(q)}_{12,12}(x), \mathrm{Imm}^{(q)}_{132}(x), \Delta^{(q)}_{123,123}(x) \in \mathcal{A}_3^{(q)}$ are bar invariant. Therefore, $\overline{z_i} = z_i$ for $1 \leq i \leq 9$. By Observation 3.5, we have that for all $1 \leq i < j \leq 9$, there exists $b_{ij} \in \mathbb{Z}$ so that $z_i z_j = q^{b_{ij}} z_j z_i$. Therefore, we have

$$\begin{aligned}
\overline{z_1^{a_1} \cdots z_9^{a_9}} &= z_9^{a_9} \cdots z_1^{a_1} \\
&= q^c z_1^{a_1} \cdots z_9^{a_9},
\end{aligned}$$

where the exponent c is given by

$$c = -\sum_{1 \leq i < j \leq 9} a_i a_j b_{ij}.$$

It follows that the ring element $q^{-c/2} z_1^{a_1} \cdots z_9^{a_9}$ is invariant under the bar involution. We now must check that $q^{-c/2} z_1^{a_1} \cdots z_9^{a_9}$ also satisfies the triangularity condition of Theorem 3.1. This verification is omitted from this extended abstract. □

Example 3.1 *Consider the quantum analogue*

$$z := x_{21}^7 \Delta_{23,13}^{(q)}(x)^2 \Delta_{23,23}^{(q)}(x)^1 \Delta_{12,23}^{(q)}(x)^2 x_{31}^7$$

of the cluster monomial of Example 2.1. By Theorem 3.6, the ring element z is a q-shift of a DCB element. Computing the $q = 1$ specialization of this DCB element using Skandera's characterization of the DCB of $\mathcal{A}_3$ would involve computing inverse Kazhdan-Lusztig polynomials corresponding to pairs of elements in the symmetric group on 24 letters.

4 Future Directions

In this paper, by means of a series of computations, we have proven that the dual canonical basis and the cluster monomial basis of the classical polynomial ring $\mathcal{A}_3$ are equal by showing that they have quantizations which differ by a q-shift. In doing so, we discovered how DCB elements for $\mathcal{A}_3$ and $\mathcal{A}_3^{(q)}$ decompose into irreducibles and found an easy way to write down any DCB element of these rings - up to a q-shift, just choose a decorated octogon and write down some monomial in the elements of the related extended (quantum) cluster. It is natural to ask how much of this can be extended to rings $\mathcal{A}_n$ and $\mathcal{A}_n^{(q)}$ for $n > 3$. It turns out that there are obstructions to finding such results from both the theory of cluster monomial bases and dual canonical bases.

For $n > 3$ there is a known cluster algebra structure on a subalgebra of $\mathcal{A}_n$ which gives rise to a linearly independent set of cluster monomials. Unfortunately, these cluster monomials do not span $\mathcal{A}_n$ for $n > 3$. Moreover, for $n > 3$ this cluster algebra is of infinite type, i.e., it has infinitely many clusters. Since these clusters are not given at the outset but rather are determined by a 'mutation' procedure starting with some initial cluster and 'mutation matrix' (see [3]), this would seem to make the cluster monomials in these algebras difficult to work with.

Leaving aside the present lack of a cluster algebra structure on $\mathcal{A}_n$, one can still ask how dual canonical basis elements of $\mathcal{A}_n$ and its quantization $\mathcal{A}_n^{(q)}$ factor. By Theorem 3.6 and the fact that quantum cluster monomials are arbitrary products of the ring elements in some quantum extended cluster, we have the following result in $\mathcal{A}_3^{(q)}$.

Corollary 4.1 *Let b be any element in the dual canonical basis of $\mathcal{A}_3^{(q)}$. Then, a q-shift of b^k is in the dual canonical basis of $\mathcal{A}_3^{(q)}$ for any $k \geq 0$.*

For n large, Leclerc [7] has shown that there exist elements b of the DCB of $\mathcal{A}_n^{(q)}$ such that b^2 is not a q-shift of a DCB element of $\mathcal{A}_n^{(q)}$ (so-called *imaginary* vectors). In light of the construction of cluster monomials, Leclerc's result is troubling if one wants to find a cluster-style interpretation of the factorization of all of the DCB elements of $\mathcal{A}_n^{(q)}$ for $n > 3$.

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