

# Schubert complexes and degeneracy loci

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**Abstract.** The classical Thom–Porteous formula expresses the homology class of the degeneracy locus of a generic map between two vector bundles as an alternating sum of Schur polynomials. A proof of this formula was given by Pragacz by expressing this alternating sum as the Euler characteristic of a Schur complex, which gives an explanation for the signs. Fulton later generalized this formula to the situation of flags of vector bundles by using alternating sums of Schubert polynomials. Building on the Schubert functors of Kraśkiewicz and Pragacz, we introduce Schubert complexes and show that Fulton’s alternating sum can be realized as the Euler characteristic of this complex, thereby providing a conceptual proof for why an alternating sum appears.

**Résumé.** La formule classique de Thom–Porteous exprime la classe d’homologie du locus de la dégénérescence d’une fonction générique entre deux fibrés vectoriels comme une somme alternée des polynômes de Schur. Une preuve de cette formule a été donnée par Pragacz en exprimant ce alternant somme comme la caractéristique d’Euler d’un complexe de Schur, ce qui donne une explication pour les signes. Fulton puis généralisée cette formule à la situation des drapeaux de fibrés vectoriels à l’aide alternant des sommes de polynômes de Schubert. S’appuyant sur le Schubert foncteurs de Kraśkiewicz et Pragacz, nous introduisons les complexes de Schubert et montrent que la somme alternée de Fulton peuvent être réalisées en tant que Euler caractéristique de ce complexe, fournissant ainsi une preuve conceptuelle pour lesquelles une somme alternée apparaît.

**Keywords:** Schubert polynomials, Schubert complexes, degeneracy loci, balanced labelings, Thom–Porteous formula

## 1 Introduction

Let  $X$  be a smooth variety, and let  $\varphi: E \rightarrow F$  be a map of vector bundles over  $X$ , with ranks  $e$  and  $f$  respectively. Given a number  $k \leq \min(e, f)$ , let  $D_k(\varphi)$  be the degeneracy locus of points  $x$  where the rank of  $\varphi$  restricted to the fiber of  $x$  is at most  $k$ . Then  $\text{codim } D_k(\varphi) \leq (e - k)(f - k)$ , and in the case of equality, the Thom–Porteous formula gives an expression for the homology class of  $D_k(\varphi)$  in the Chow groups of  $X$  in terms of the Chern classes of  $E$  and  $F$  using super Schur polynomials. Also in the case of equality, the Schur complex associated with the rectangular partition  $(f - k) \times (e - k)$  [ABW] of  $\varphi$  is a linear locally free resolution for a coherent sheaf whose support is  $D_k(\varphi)$ . Interpreted appropriately, the Euler characteristic of this complex recovers the Thom–Porteous formula. Hence the complex provides a “linear approximation” of the syzygies of  $D_k(\varphi)$ .

The situation was generalized by Fulton as follows. We provide the additional data of a flag of subbundles  $E_\bullet$  for  $E$  and a flag of quotient bundles  $F_\bullet$  for  $F$ , and we can define degeneracy loci for an array of

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numbers which specifies the ranks of maps  $E_p \rightarrow F_q$ . In particular, the rank functions which give rise to irreducible degeneracy loci are indexed by permutations in a natural way. Under the right codimension assumptions, one can express its homology class as a substitution of a double Schubert polynomial with the Chern classes of the quotients  $E_i/E_{i-1}$  and the kernels  $\ker(F_j \rightarrow F_{j-1})$ . The motivation for this work was to complete the analogy of this situation with the previous one by constructing “Schubert complexes” which would be acyclic whenever the degeneracy loci has the right codimension.

Building on the constructions for Schubert functors by Kraśkiewicz and Pragacz of [KP], we construct these complexes over an arbitrary (commutative) ring  $R$  from the data of two free  $R$ -modules  $M_0, M_1$ , with given flags of submodules, respectively, quotient modules, and a map  $\partial: M_0 \rightarrow M_1$ . We show that they are generically acyclic (in the sense of [BE]) and that in general they are acyclic when a certain ideal defined in terms of minors of  $\partial$  has the right depth, i.e., they are “depth-sensitive.” This allows us to extend the construction to an arbitrary variety (or more generally, an arbitrary scheme). We will stick to the language of varieties, however the results can be generalized as necessary. Again, the complexes are linear and provide a “linear approximation” to the syzygies of Fulton’s degeneracy loci. We remark here that as a special case of Fulton’s degeneracy loci, one gets Schubert varieties inside of arbitrary partial flag varieties.

Our main result is that in the situation of Fulton’s theorem, the complex is acyclic and the Euler characteristic provides the formula in the same sense as above. A majority of the hard work goes into proving that our constructed complexes are acyclic under the appropriate depth assumption. Our proof uses techniques from commutative algebra, algebraic geometry, and combinatorics, and will appear in the full version of this paper. In the present article, we offer a short sketch of the proof.

Using the work of Fomin, Greene, Reiner, and Shimozono [FGRS], we can also construct explicit bases for the terms of the Schubert complex in the case that  $M_0$  and  $M_1$  are free. This basis naturally extends their notion of “balanced labelings” and their generating function gives an alternative expression for double Schubert polynomials. Furthermore, the complex naturally affords a representation of the Lie superalgebra of upper triangular matrices (with respect to the given flags) in  $\text{Hom}(M_0, M_1)$ , and its supercharacter is the double Schubert polynomial.

The article is structured as follows. In Section 2 we recall some facts about double Schubert polynomials and balanced labelings. We introduce balanced super labelings and explain their relationship with the double Schubert polynomials. In Section 3 we extend the construction for Schubert functors to the  $\mathbf{Z}/2$ -graded setting and describe a basis for them naturally indexed by the balanced super labelings. In Section 4 we construct the Schubert complex from this  $\mathbf{Z}/2$ -graded Schubert functor. We mention the relevant facts and sketch the idea of a proof that these complexes are generically acyclic, and that in general the acyclicity of the complex is controlled by depth of a Schubert determinantal ideal. We also give some examples of Schubert complexes. Finally, in Section 5, we relate the acyclicity of the Schubert complexes to a degeneracy locus formula of Fulton.

## 2 Double Schubert polynomials.

### 2.1 Preliminaries.

Let  $\Sigma_n$  be the permutation group on the set  $\{1, \dots, n\}$ . Let  $s_i$  denote the transposition which switches  $i$  and  $i + 1$ . Then  $\Sigma_n$  is generated by  $\{s_1, \dots, s_{n-1}\}$ , and for  $w \in \Sigma_n$ , we define the **length** of  $w$  to be the least number of  $\ell(w)$  such that  $w = s_{i_1} \cdots s_{i_{\ell(w)}}$ . Such a minimal expression is a **reduced decomposition**

for  $w$ . We can also write  $\ell(w) = \#\{i < j \mid w(i) > w(j)\}$ . There is a unique word  $w_0$  with maximal length, which is the permutation defined by  $w_0(i) = n + 1 - i$ .

We will use two partial orders on  $\Sigma_n$ . The **weak Bruhat order**, denoted by  $u \leq_W w$ , holds if some reduced decomposition of  $u$  is the suffix of some reduced decomposition of  $w$ . We denote the **strong Bruhat order** by  $u \leq w$ , which holds if some reduced decomposition of  $w$  contains a subword that is a reduced decomposition of  $u$ . It follows from the definition that  $u \leq w$  if and only if  $u^{-1} \leq w^{-1}$ . For a permutation  $w$ , let  $r_w(p, q) = \#\{i \leq p \mid w(i) \leq q\}$  be its **rank function**. Then  $u \leq w$  if and only if  $r_u(p, q) \geq r_w(p, q)$  for all  $p$  and  $q$  (the inequality on rank functions is reversed).

Given a polynomial (with arbitrary coefficient ring) in the variables  $\{x_i\}_{i \geq 1}$ , let  $\partial_i$  be the **divided difference operator**

$$(\partial_i P)(x_1, x_2, \dots) = \frac{P(\dots, x_{i-1}, x_i, x_{i+1}, \dots) - P(\dots, x_{i-1}, x_{i+1}, x_i, \dots)}{x_i - x_{i+1}}. \tag{2.1}$$

The operators  $\partial_i$  satisfy the braid relations:  $\partial_i \partial_j = \partial_j \partial_i$  when  $|i - j| > 1$  and  $\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}$ .

For the long word  $w_0 \in \Sigma_n$ , set  $\mathfrak{S}_{w_0}(x, y) = \prod_{i+j \leq n} (x_i - y_j)$ . In general, if  $\ell(ws_i) = \ell(w) - 1$ , we set  $\mathfrak{S}_{ws_i}(x, y) = \partial_i \mathfrak{S}_w(x, y)$ , where we interpret  $\mathfrak{S}_w(x, y)$  as a polynomial in the variables  $\{x_i\}_{i \geq 1}$  with coefficients in the ring  $\mathbf{Z}[y_1, y_2, \dots]$ . These polynomials are the **double Schubert polynomials**, and are well-defined thanks to the braid relations. They also enjoy the following stability property: if we embed  $\Sigma_n$  into  $\Sigma_{n+1}$  by identifying permutations of  $\Sigma_n$  with permutations of  $\Sigma_{n+1}$  which fix  $n + 1$ , then the polynomial  $\mathfrak{S}_w(x, y)$  is the same whether we regard  $w$  as an element of  $\Sigma_n$  or  $\Sigma_{n+1}$ .

Define the **single Schubert polynomials** by  $\mathfrak{S}_w(x) = \mathfrak{S}_w(x, 0)$ . We will use the identity [Man, Proposition 2.4.7]

$$\mathfrak{S}_w(x, y) = \sum_{u \leq_W w} \mathfrak{S}_u(x) \mathfrak{S}_{uw^{-1}}(-y). \tag{2.2}$$

## 2.2 Balanced super labelings.

For the rest of this article, we fix a totally ordered alphabet  $\dots < 3' < 2' < 1' < 1 < 2 < 3 < \dots$ .

For a permutation  $w$ , define its **diagram**  $D(w) = \{(i, w(j)) \mid i < j, w(i) > w(j)\}$ . Let  $T$  be a labeling of  $D(w)$ . The **hook** of a box  $b \in D(w)$  is the set of boxes in the same column below it, and the set of boxes in the same row to the right of it (including itself). A hook is **balanced** (with respect to  $T$ ) if it satisfies the following property: when the entries are rearranged so that they are weakly increasing going from the top right end to the bottom left end, the label in the corner stays the same. A labeling is **balanced** if all of the hooks are balanced. Call a labeling  $T$  of  $D(w)$  with entries in our alphabet a **balanced super labeling (BSL)** if it is balanced, column-strict (no repetitions in any column) with respect to the unmarked alphabet, row-strict with respect to the marked alphabet, and satisfies  $j' \leq T(i, j) \leq i$  for all  $i$  and  $j$  (this last condition will be referred to as the **flag conditions**).

Given a BSL  $T$  of  $D(w)$ , let  $f_T(i)$ , respectively  $f_T(i')$ , be the number of occurrences of  $i$ , respectively  $i'$ . Define a monomial

$$m(T) = x_1^{f_T(1)} \dots x_{n-1}^{f_T(n-1)} (-y_1)^{f_T(1')} \dots (-y_{n-1})^{f_T((n-1)')}. \tag{2.3}$$

Using [FGRS, Lemma 4.7, Theorem 4.8], we can prove the following.

**Theorem 2.4** For every permutation  $w$ ,

$$\mathfrak{S}_w(x, y) = \sum_T m(T),$$

where the sum is over all BSL  $T$  of  $D(w)$ .

**Remark 2.5** Given a labeling  $T$  of  $D(w)$ , let  $T^*$  denote the labeling of  $D(w^{-1})$  obtained by transposing  $T$  and performing the swap  $i \leftrightarrow i'$ . The operation  $T \mapsto T^*$  gives a concrete realization of the symmetry  $\mathfrak{S}_w(-y, -x) = \mathfrak{S}_{w^{-1}}(x, y)$  [Man, Corollary 2.4.2].

**Example 2.6** We list the BSL for the permutation  $w = 321$ .

$$\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 1' & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 2' \\ \hline 2 & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1' & 2' \\ \hline 1 & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 1' \\ \hline 2 & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1' & 1 \\ \hline 1' & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1' & 2' \\ \hline 2 & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1' & 2' \\ \hline 1' & \\ \hline \end{array}$$

In this case,  $\mathfrak{S}_{321}(x, y) = (x_1 - y_1)(x_1 - y_2)(x_2 - y_1)$ .

### 3 Double Schubert functions.

#### 3.1 Super linear algebra preliminaries.

Let  $V = V_0 \oplus V_1$  be a free super module over a commutative ring  $R$  with  $V_0 = \langle e_1, \dots, e_n \rangle$  and  $V_1 = \langle e'_1, \dots, e'_m \rangle$ , and let  $\mathfrak{gl}(m|n) = \mathfrak{gl}(V)$  be the Lie superalgebra of endomorphisms of  $V$ . Let  $\mathfrak{b}(m|n) \subset \mathfrak{gl}(m|n)$  be the standard Borel subalgebra of upper triangular matrices with respect to the ordered basis  $\langle e'_m, \dots, e'_1, e_1, \dots, e_n \rangle$ . In the case  $m = n$ , we will write  $\mathfrak{b}(n) = \mathfrak{b}(n|n)$ , and if it is clear from context, we will drop the  $n$  and simply write  $\mathfrak{b}$ . Also, let  $\mathfrak{b}(n)_0 = \mathfrak{gl}(V)_0 \cap \mathfrak{b}(n)$  be the even degree elements in  $\mathfrak{b}(n)$ , and again, we will usually denote this by simply  $\mathfrak{b}_0$ . We also write  $\mathfrak{h}(n) \subset \mathfrak{b}(n)$  for the Cartan subalgebra of diagonal matrices (this is a Lie algebra concentrated in degree 0). Let  $\varepsilon'_n, \dots, \varepsilon'_1, \varepsilon_1, \dots, \varepsilon_n$  be the dual basis vectors to the standard basis of  $\mathfrak{h}(n)$ . For notation, write  $(a_n, \dots, a_1 | b_1, \dots, b_n)$  for  $\sum_{i=1}^n (a_i \varepsilon'_i + b_i \varepsilon_i)$ . The even and odd roots of  $\mathfrak{b}(n)$  are  $\Phi_0 = \{\varepsilon'_j - \varepsilon'_i, \varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n\}$  and  $\Phi_1 = \{\varepsilon'_i - \varepsilon_j \mid 1 \leq i, j \leq n\}$ , respectively. The even and odd simple roots are  $\Delta_0 = \{\varepsilon'_{i+1} - \varepsilon'_i, \varepsilon_i - \varepsilon_{i+1} \mid i = 1, \dots, n - 1\}$  and  $\Delta_1 = \{\varepsilon'_1 - \varepsilon_1\}$ .

Given a highest weight representation  $W$  of  $\mathfrak{b}(n)$ , we have a weight decomposition  $W = \bigoplus_{\lambda} W_{\lambda}$  as a representation of  $\mathfrak{h}(n)$ . Let  $\Lambda$  be the highest weight of  $W$ . Then every weight  $\lambda$  appearing in the weight decomposition can be written in the form  $\Lambda - \sum n_{\alpha} \alpha$  where  $\alpha$  ranges over the simple roots of  $\mathfrak{b}(n)$  and  $n_{\alpha} \in \mathbf{Z}_{\geq 0}$ . For such a  $\lambda$ , set  $\omega(\lambda) = (-1)^{\sum n_{\alpha} \deg \alpha}$ . Then we define the **character** and **supercharacter** of  $W$  as

$$\text{ch } W = \sum_{\lambda} (\dim W_{\lambda}) e^{\lambda}, \quad \text{sch } W = \sum_{\lambda} \omega(\lambda) (\dim W_{\lambda}) e^{\lambda}. \tag{3.1}$$

We recall the  $\mathbf{Z}/2$ -graded analogues of the symmetric and exterior powers. Let  $F = F_0 \oplus F_1$  be a free  $R$ -supermodule. Let  $D$  denote the divided power functor. Then  $\bigwedge^i F$  and  $D^i F$  are  $\mathbf{Z}$ -graded modules with terms given by

$$\left(\bigwedge^i F\right)_d = \bigwedge^{i-d} F_0 \otimes \text{Sym}^d F_1, \quad (D^i F)_d = D^{i-d} F_0 \otimes \bigwedge^d F_1. \tag{3.2}$$

We can define a coassociative  $\mathbf{Z}$ -graded comultiplication  $\Delta: D^{i+j}F \rightarrow D^iF \otimes D^jF$  as follows. On degree  $d$ , pick  $0 \leq a \leq i$  and  $0 \leq b \leq j$  such that  $a + b = d$ . Then we have the composition  $\Delta_{a,b}$

$$\begin{aligned} (D^{i+j}F)_d &= D^{i+j-a-b}F_0 \otimes \bigwedge^{a+b} F_1 \\ &\xrightarrow{\Delta \otimes \Delta} D^{i-a}F_0 \otimes D^{j-b}F_0 \otimes \bigwedge^a F_1 \otimes \bigwedge^b F_1 \\ &\cong D^{i-a}F_0 \otimes \bigwedge^a F_1 \otimes D^{j-b}F_0 \otimes \bigwedge^b F_1 = (D^iF)_a \otimes (D^jF)_b, \end{aligned} \tag{3.3}$$

where  $\Delta$  is the usual symmetrization map, and we define  $\Delta$  on the degree  $d$  part to be  $\sum_{a+b=d} \Delta_{a,b}$ .

Similarly, we can define an associative  $\mathbf{Z}$ -graded multiplication  $m: \bigwedge^i F \otimes \bigwedge^j F \rightarrow \bigwedge^{i+j} F$  as follows. For degrees  $a$  and  $b$ , we have

$$\begin{aligned} \left(\bigwedge^i F\right)_a \otimes \left(\bigwedge^j F\right)_b &= \bigwedge^{i-a} F_0 \otimes \text{Sym}^a F_1 \otimes \bigwedge^{j-b} F_0 \otimes \text{Sym}^b F_1 \\ &\cong \bigwedge^{i-a} F_0 \otimes \bigwedge^{j-b} F_0 \otimes \text{Sym}^a F_1 \otimes \text{Sym}^b F_1 \\ &\xrightarrow{m \otimes m} \bigwedge^{i+j-a-b} F_0 \otimes \text{Sym}^{a+b} F_1 = \left(\bigwedge^{i+j} F\right)_{a+b}, \end{aligned} \tag{3.4}$$

where  $m$  is the usual exterior multiplication.

### 3.2 Constructions.

Define a flag of  $\mathbf{Z}/2$ -graded submodules

$$V^\bullet : V^{-n} \subset \dots \subset V^{-1} \subset V^1 \subset \dots \subset V^n \tag{3.5}$$

such that  $V^{-1}$  consists of all of the odd elements of  $V^n$ . We will say that the flag is **split** if each term and each quotient is a free module. Fix a permutation  $w \in \Sigma_n$ . Let  $r_k = r_k(w)$ , respectively  $c_j = c_j(w)$ , be the number of boxes in the  $k$ th row, respectively  $j$ th column, of  $D(w)$ . Define  $\chi_{k,j}$  to be 1 if  $(k, j) \in D(w)$  and 0 otherwise. Consider the map

$$\begin{aligned} \bigotimes_{k=1}^{n-1} D^{r_k} V^k &\xrightarrow{\otimes \Delta} \bigotimes_{k=1}^{n-1} \bigotimes_{j=1}^{n-1} D^{\chi_{k,j}} V^k \cong \bigotimes_{j=1}^{n-1} \bigotimes_{k=1}^{n-1} D^{\chi_{k,j}} V^k \\ &\xrightarrow{\otimes m} \bigotimes_{j=1}^{n-1} \bigwedge^{c_j} V^{w^{-1}(j)} \xrightarrow{\otimes \pi} \bigotimes_{j=1}^{n-1} \bigwedge (V^{w^{-1}(j)} / V^{-j-1}), \end{aligned} \tag{3.6}$$

where  $\otimes \Delta$  denotes the product of symmetrization operations,  $\otimes m$  denotes the product of exterior multiplications, and  $\otimes \pi$  denotes the product of projection maps. Then its image  $\mathcal{S}_w(V^\bullet)$  is the  $\mathbf{Z}/2$ -**graded Schubert functor**, or **double Schubert functor**. By convention, the empty tensor product is  $R$ , so that if  $w$  is the identity permutation, then  $\mathcal{S}_w(V^\bullet) = R$  with the trivial action of  $\mathfrak{b}(n)$ .

This definition is clearly functorial: given an even map of flags  $f: V^\bullet \rightarrow W^\bullet$ , i.e.,  $f(V^k) \subset W^k$  for  $-n \leq k \leq n$ , we have an induced map  $f: \mathcal{S}_w(V^\bullet) \rightarrow \mathcal{S}_w(W^\bullet)$ .

We will focus on the case when  $V^{-i} = \langle e'_n, e'_{n-1}, \dots, e'_i \rangle$  and  $V^i = V^{-1} + \langle e_1, e_2, \dots, e_i \rangle$ , so that  $\mathcal{S}_w = \mathcal{S}_w(V^\bullet)$  is a  $\mathfrak{b}(n)$ -module.

**Remark 3.7** One could dually define the double Schubert functor as the image of (dual) exterior powers mapping to symmetric powers. The dual exterior powers are as in our definition, except that divided powers replace symmetric powers. However, we have chosen this definition to be consistent with [KP]. This will be especially convenient in Section 4.1.

Here is a combinatorial description of the map (3.6). The elements of  $\bigotimes_{k=1}^{n-1} D^{r_k} V^k$  can be thought of as labelings of  $D = D(w)$  such that in row  $k$ , only the labels  $n', (n-1)', \dots, 1', 1, \dots, k$  are used, such that there is at most one use of  $i'$  in a given row, and such that the entries in each row are ordered in the usual way (i.e.,  $n' < (n-1)' < \dots < 1' < 1 < \dots < k$ ). Let  $\Sigma_D$  be the permutation group of  $D$ . We say that  $\sigma \in \Sigma_D$  is **row-preserving** if each box and its image under  $\sigma$  are in the same row. Denote the set of row-preserving permutations as  $\text{Row}(D)$ . Let  $T$  be a labeling of  $D$  that is row-strict with respect to the marked letters. Let  $\text{Row}(D)_T$  be the subgroup of  $\text{Row}(D)$  that leaves  $T$  fixed, and let  $\text{Row}(D)^T$  be the set of cosets  $\text{Row}(D)/\text{Row}(D)_T$ . Given  $\sigma \in \text{Row}(D)^T$ , and considering the boxes as ordered from left to right, let  $\alpha(T, \sigma)_k$  be the number of inversions of  $\sigma$  among the *marked letters* in the  $k$ th row, and define  $\alpha(T, \sigma) = \sum_{k=1}^{n-1} \alpha(T, \sigma)_k$ . Note that this number is independent of the representative chosen since  $T$  is row strict with respect to the marked letters. Then the comultiplication sends  $T$  to  $\sum_{\sigma \in \text{Row}(D)^T} (-1)^{\alpha(T, \sigma)} \sigma T$  where  $\sigma T$  is the result of permuting the labels of  $T$  according to  $\sigma$ .

For the multiplication map, we can interpret the columns as being alternating in the *unmarked letters* and symmetric in the *marked letters*. We write  $m(T)$  for the image of  $T$  under this equivalence relation. Therefore, the map (3.6) can be defined as

$$T \mapsto \sum_{\sigma \in \text{Row}(D)^T} (-1)^{\alpha(T, \sigma)} m(\sigma T). \tag{3.8}$$

### 3.3 A basis and a filtration.

**Theorem 3.9** Assume that the flag  $V^\bullet$  is split. The images of the BSLs under (3.6) form a basis over  $R$  for  $\mathcal{S}_w$ . By convention, the empty diagram has exactly one labeling.

**Corollary 3.10** Identify  $x_i = -e^{\varepsilon_i}$  and  $y_i = -e^{\varepsilon'_i}$  for  $1 \leq i \leq n$ . Then

$$\text{ch } \mathcal{S}_w = \mathfrak{S}_w(-x, y), \quad \text{sch } \mathcal{S}_w = \mathfrak{S}_w(x, y).$$

**Corollary 3.11** Choose an ordering of the set of permutations below  $w$  in the weak Bruhat order:  $1 = v_1 \prec v_2 \prec \dots \prec v_N = w$  such that  $v_i \prec v_{i+1}$  implies that  $\ell(v_i) \leq \ell(v_{i+1})$ . Then there exists a  $\mathfrak{b}$ -equivariant filtration

$$0 = F_0 \subset F_1 \subset \dots \subset F_N = \mathcal{S}_w$$

such that

$$F_i/F_{i-1} \cong \mathcal{S}'_{v_i} \otimes \mathcal{S}''_{wv_i^{-1}}$$

as  $\mathfrak{b}_0$ -modules.

Our proof does not establish how one can write the image of an arbitrary labeling as a linear combination of the images of the BSLs. Such a straightening algorithm is preferred, but we have not been successful in finding one, so we leave this task as an open problem.

**Problem 3.12** Find an algorithm for writing the image of an arbitrary labeling of  $D(w)$  as a linear combination of the images of the BSLs of  $D(w)$ .

## 4 Schubert complexes.

Now we can use the above machinery to define Schubert complexes. We start with the data of two flags  $F_0^\bullet : 0 = F_0^0 \subset F_0^1 \subset \dots \subset F_0^n = F_0$  and  $F_1^\bullet : 0 = F_1^{-n-1} \subset F_1^{-n} \subset F_1^{-n+1} \subset \dots \subset F_1^{-1} = F_1$ , and a map  $\partial : F_0 \rightarrow F_1$  between them. Given the flag for  $F_0$ , we pick an ordered basis  $\{e_1, \dots, e_n\}$  for it such that  $e_i \in F_0^i \setminus F_0^{i-1}$ . Similarly, we pick an ordered basis  $\{e'_1, \dots, e'_n\}$  for  $F_1$  such that  $e'_i \in F_1^{-i} \setminus F_1^{-i-1}$ . Given these bases, we can represent  $\partial$  as a matrix. This matrix representation will be relevant for the definition of certain ideals later.

Equivalently, we can give  $F_1^\bullet$  as a quotient flag  $F_1 = G^n \twoheadrightarrow G^{n-1} \twoheadrightarrow \dots \twoheadrightarrow G^1 \twoheadrightarrow G^0 = 0$ , so that the correspondence is given by  $F_1^{-i} = \ker(G^n \twoheadrightarrow G^{i-1})$ . Note that  $F_1^{-i}/F_1^{-i-1} = \ker(G^i \twoheadrightarrow G^{i-1})$ . We assume that each quotient has rank 1. Then we form a flagged supermodule  $F$  with even part  $F_0$  and odd part  $F_1$ . The formation of symmetric and exterior products commutes with the differential  $\partial$ , so we can form the **Schubert complex**  $\mathcal{S}_w(F)$  for a permutation  $w \in \Sigma_n$ .

**Proposition 4.1** The  $i$ th term of  $\mathcal{S}_w(F)$  has a natural filtration whose associated graded is

$$\bigoplus_{\substack{v \leq w \\ \ell(v)=i}} \mathcal{S}_v(F_0) \otimes \mathcal{S}_{wv^{-1}}(F_1).$$

**Proof:** This is a consequence of Corollary 3.11. □

### 4.1 The Kraśkiewicz–Pragacz filtration.

In order to prove properties of  $\mathcal{S}_w$ , we will construct a filtration of subcomplexes, which is based on the filtration of the single Schubert functors introduced by Kraśkiewicz and Pragacz [KP].

Let  $w \in \Sigma_n$  be a nonidentity permutation. Consider the set of pairs  $(\alpha, \beta)$  such that  $\alpha < \beta$  and  $w(\alpha) > w(\beta)$ . Choose  $(\alpha, \beta)$  to be maximal with respect to the lexicographic ordering. Let  $k_1 < \dots < k_k$  be the numbers such that  $k_t < \alpha$  and  $w(k_t) < w(\beta)$ , and such that  $k_t < i < \alpha$  implies that  $w(i) \notin \{w(k_t), w(k_t) + 1, \dots, w(\beta)\}$ . Then we have the following identity of double Schubert polynomials

$$\mathfrak{S}_w = \mathfrak{S}_v \cdot (x_\alpha - y_{w(\beta)}) + \sum_{t=1}^k \mathfrak{S}_{\psi_t}, \tag{4.2}$$

where  $v = wt_{\alpha, \beta}$  and  $\psi_t = wt_{\alpha, \beta} t_{k_t, \alpha}$ . Here  $t_{i,j}$  denotes the transposition which switches  $i$  and  $j$ . See, for example, [Man, Exercise 2.7.3]. The formula in (4.2) will be called a **maximal transition** for  $w$ . Define the **index** of a permutation  $u$  to be the number  $\sum (k-1) \#\{j > k \mid u(k) > u(j)\}$ . Note that the index of  $\psi_t$  is smaller than the index of  $w$ .

**Theorem 4.3** *Let  $V^\bullet$  be a split flag as in (3.5). Given a permutation  $w \in \Sigma_n$ , let (4.2) be the maximal transition for  $w$ . Then there exists a functorial  $\mathfrak{b}$ -equivariant filtration*

$$0 = F_0 \subset F_1 \subset \cdots \subset F_k \subset F' \subset F = \mathcal{S}_w(V^\bullet)$$

*such that  $F/F' \cong \mathcal{S}_v(V^\bullet) \otimes V^\alpha/V^{\alpha-1}$ ,  $F'/F_k \cong \mathcal{S}_v(V^\bullet) \otimes V^{-w(\beta)}/V^{-w(\beta)+1}$ , and  $F_t/F_{t-1} \cong \mathcal{S}_{\psi_t}(V^\bullet)$  for  $t = 1, \dots, k$ .*

**Corollary 4.4** *Let  $\partial: F_0 \rightarrow F_1$  be a map. With the notation as in Theorem 4.3, there is a functorial  $\mathfrak{b}$ -equivariant filtration of complexes*

$$0 = C_0 \subset C_1 \subset \cdots \subset C_k \subset C' \subset C = \mathcal{S}_w(\partial)$$

*such that  $C/C' \cong \mathcal{S}_v(\partial)[-1] \otimes F_0^\alpha/F_0^{\alpha-1}$ ,  $C'/C_k \cong \mathcal{S}_v(\partial) \otimes F_1^{-w(\beta)}/F_1^{-w(\beta)+1}$ , and  $C_t/C_{t-1} \cong \mathcal{S}_{\psi_t}(\partial)$  for  $t = 1, \dots, k$ .*

**Proof:** The filtration of Theorem 4.3 respects the differentials since everything is defined in terms of multilinear operations. The grading shift of  $C/C'$  follows from the fact that the  $F_0$  terms have homological degree 1. □

### 4.2 Generic acyclicity of Schubert complexes.

Given a matrix  $\partial$  and a permutation  $w$ , let  $I_w(\partial)$  be the ideal generated by the  $(r_w(p, q)+1) \times (r_w(p, q)+1)$  minors of the upper left  $p \times q$  submatrix of  $\partial$ . It is clear that  $I_v \subseteq I_w$  if and only if  $v \leq w$ . In the case that  $\partial$  is a generic matrix of variables over some coefficient ring  $R$ , let  $X(w)$  be the variety defined by  $I_w(\partial) \subset R[\partial_{i,j}]$ . We refer to the ideals  $I_w(\partial)$  as **Schubert determinantal ideals**, and the varieties  $X(w)$  as **matrix Schubert varieties**. These ideals are prime and have codimension  $\ell(w)$  [MS, Chapter 15].

Our main result is the following.

**Theorem 4.5** *Let  $A = K[\partial_{i,j}]$  be a polynomial ring over a field  $K$ , and let  $\partial: F_0 \rightarrow F_1$  be a generic map of variables between two free  $A$ -modules.*

- (a) *The Schubert complex  $\mathcal{S}_w(\partial)$  is acyclic, and resolves a Cohen–Macaulay module  $M$  of codimension  $\ell(w)$  supported in  $I_{w^{-1}}(\partial) \subseteq A$ .*
- (b) *The restriction of  $M$  to  $X(w^{-1})$  is a line bundle outside of a certain codimension 2 subset.*
- (c) *The Schubert complex defined over the integers is acyclic.*

**Proof:** (Sketch). From Corollary 4.4, we get a short exact sequence

$$0 \rightarrow H_1(C) \rightarrow H_0(\mathcal{S}_v(\partial)) \otimes F_0^\alpha/F_0^{\alpha-1} \xrightarrow{\delta} H_0(C') \rightarrow H_0(C) \rightarrow 0, \tag{4.6}$$

so we have to show that  $\delta$  is injective, and that the support of  $H_0(C) = M$  is  $P = I_{w^{-1}}(\partial)$ .

The short exact sequence

$$0 \rightarrow C_k \rightarrow C' \rightarrow \mathcal{S}_v(\partial) \otimes \langle e'_{w(\beta)} \rangle \rightarrow 0$$

induces the sequence

$$0 \rightarrow H_0(C_k) \rightarrow H_0(C') \rightarrow H_0(\mathcal{S}_v(\partial)) \otimes \langle e'_{w(\beta)} \rangle \rightarrow 0.$$

By induction on the filtration in Corollary 4.4, the support of  $H_0(C_k)$  is in the union of the  $X(\psi_t^{-1})$ , and hence does not contain  $X(w^{-1})$ . So localizing at  $P$ , we get an isomorphism

$$H_0(C')_P \cong H_0(\mathcal{S}_v(\partial))_P \otimes \langle e'_{w(\beta)} \rangle.$$

So we can restrict this isomorphism to  $X(w^{-1})$ . Localizing (4.6) at  $P$  and then restricting  $X(w^{-1})$ , we get a surjection

$$H_0(\mathcal{S}_v(\partial))_P \otimes \langle e'_{w(\beta)} \rangle \rightarrow H_0(C)_P \rightarrow 0.$$

By induction, the first term has length 1 over the generic point of  $X(w^{-1})$ , so the length of  $H_0(C)_P$  is either 0 or 1.

The idea for using this partial information is to carry our situation to the complete flag variety and to use its K-theory to show that  $\text{length}(H_0(C)_P) - \text{length}(H_1(C)_P) = 1$ . Some more analysis of the K-theory gives us the other statements which allow us to complete the induction step.  $\square$

**Corollary 4.7** *Let  $X$  be an equidimensional Cohen–Macaulay variety, and let  $\partial: E \rightarrow F$  be a map of vector bundles on  $X$ . Let  $E_1 \subset \cdots \subset E_n = E$  and  $F^{-n} \subset \cdots \subset F^{-1} = F$  be split flags of subbundles. Let  $w \in \Sigma_n$  be a permutation, and define the degeneracy locus*

$$D_w(\partial) = \{x \in X \mid \text{rank}(\partial_x: E_p(x) \rightarrow F/F^{-q-1}(x)) \leq r_w(p, q)\},$$

where the ideal sheaf of  $D_w(\partial)$  is locally generated by the minors given by the rank conditions. Suppose that  $D_w(\partial)$  has codimension  $\ell(w)$ .

(a) *The Schubert complex  $\mathcal{S}_w(\partial)$  is acyclic, and the support of its cokernel  $\mathcal{L}$  is  $D_w(\partial)$ .*

(b) *The degeneracy locus  $D_w(\partial)$  is Cohen–Macaulay.*

(c) *The restriction of  $\mathcal{L}$  to  $D_w(\partial)$  is a line bundle outside of a certain codimension 2 subset.*

### 4.3 Examples.

Here is a combinatorial description of the differentials in the Schubert complex for a flagged isomorphism. We will work with just the tensor product complex  $\bigotimes_{k=1}^{n-1} D^{r_k(w)}(F)$ . Then the basis elements of its terms are row-strict labelings. The differential sends such a labeling to the signed sum of all possible ways to change a single unmarked letter to the corresponding marked letter. If  $T'$  is obtained from  $T$  by marking a letter in the  $i$ th row, then the sign on  $T'$  is  $(-1)^n$ , where  $n$  is the number of unmarked letters of  $T$  in the first  $i - 1$  rows.

**Example 4.8** Consider the permutation  $w = 1423$ . Then  $D(w) = \{(2, 2), (2, 3)\}$ , and we denote the generic map by  $e_1 \mapsto ae'_1 + be'_2 + ce'_3$  and  $e_2 \mapsto de'_1 + ee'_2 + fe'_3$  (the images of  $e_3$  and  $e_4$  are irrelevant,

and it is also irrelevant to map to  $e'_4$ ) instead of a flagged isomorphism. The cokernel  $M$  is Cohen–Macaulay of codimension 2 over  $A = K[a, b, c, d, e, f]$ :

$$0 \rightarrow A^3 \xrightarrow{\begin{pmatrix} e & b & 0 \\ 0 & e & b \\ d & a & 0 \\ 0 & d & a \\ 0 & f & c \\ f & c & 0 \end{pmatrix}} A^6 \xrightarrow{\begin{pmatrix} d & a & -e & -b & 0 & 0 \\ 0 & 0 & -f & -c & a & d \\ -f & -c & 0 & 0 & b & e \end{pmatrix}} A^3 \rightarrow M \rightarrow 0$$

**Example 4.9** Consider the permutation  $w = 2413$ . Then  $D(w) = \{(1, 1), (2, 1), (2, 3)\}$ , and denote the generic matrix by  $e_1 \mapsto ae'_1 + be'_2 + ce'_3$  and  $e_2 \mapsto de'_1 + ee'_2 + fe'_3$  (the images of  $e_3$  and  $e_4$  are irrelevant, and it is also irrelevant to map to  $e'_4$ ). The cokernel  $M$  is Cohen–Macaulay of codimension 3 over  $A = K[a, b, c, d, e, f]$ :

$$0 \rightarrow A^2 \xrightarrow{\begin{pmatrix} -d & -a \\ -e & -b \\ -f & -c \\ 0 & -d \\ a & 0 \\ 0 & a \end{pmatrix}} A^6 \xrightarrow{\begin{pmatrix} 0 & 0 & 0 & a & 0 & d \\ e & -d & 0 & b & 0 & e \\ f & 0 & -d & c & 0 & f \\ a & 0 & 0 & 0 & d & a \\ 0 & a & 0 & 0 & e & b \\ 0 & 0 & a & 0 & f & c \end{pmatrix}} A^6 \xrightarrow{\begin{pmatrix} -b & a & 0 & -e & d & 0 \\ -c & 0 & a & -f & 0 & d \end{pmatrix}} A^2 \rightarrow M \rightarrow 0$$

## 5 Degeneracy loci.

### 5.1 A formula of Fulton.

Suppose we are given a map  $\partial: E \rightarrow F$  of vector bundles of rank  $n$  on a variety  $X$ , together with a flag of subbundles  $E_1 \subset E_2 \subset \dots \subset E_n = E$  and a flag of quotient bundles  $F = F_n \twoheadrightarrow F_{n-1} \twoheadrightarrow \dots \twoheadrightarrow F_1$  such that  $\text{rank } E_i = \text{rank } F_i = i$ . We assume that the quotient flags  $E_i/E_{i+1}$  are locally free. For a permutation  $w$ , define

$$D_w(\partial) = \{x \in X \mid \text{rank}(\partial_x: E_p(x) \rightarrow F_q(x)) \leq r_w(p, q)\}.$$

Then  $\text{codim } D_w(\partial) \leq \ell(w)$ . Define Chern classes  $x_i = -c_1(E_i/E_{i-1})$  and  $y_i = -c_1(\ker(F_i \twoheadrightarrow F_{i-1}))$ .

**Theorem 5.1 (Fulton)** *Suppose that  $X$  is an equidimensional Cohen–Macaulay variety and  $D_w(\partial)$  has codimension  $\ell(w)$ . Then the identity*

$$[D_w(\partial)] = \mathfrak{S}_w(x, y) \cap [X]$$

holds in the Chow group  $A_{\dim(D_w(\partial))}(X)$ .

See [F1, §8] for a more general statement which does not enforce a codimension requirement on  $D_w(\partial)$  or assume that  $X$  is Cohen–Macaulay.

We will only deal with the case when  $X$  is smooth. The general case can be reduced to this case using a “universal construction” (see [F2, Chapter 14]). So suppose that  $X$  is smooth. Let  $A_*(X) =$

$\bigoplus_{k \geq 0} A_k(X)$  be the direct sum of its Chow groups, and  $\text{Gr K}(X)$  be the associated graded of the topological filtration of its Grothendieck group (see [F2, Example 15.1.5]). Let  $\varphi: A_*(X) \rightarrow \text{Gr K}(X)$  be the functorial morphism of graded rings which for a subvariety  $V \subseteq X$  sends  $[V]$  to  $[\mathcal{O}_V]$ . If  $\mathcal{F}$  is a coherent sheaf whose support has dimension at most  $k$ , then we have  $\varphi(Z_k(\mathcal{F})) = [\mathcal{F}]$  where

$$Z_k(\mathcal{F}) = \sum_{\dim V=k} m_V(\mathcal{F})[V],$$

and  $m_V(\mathcal{F})$  is the length of the stalk of  $\mathcal{F}$  at the generic point of  $V$ . In order to state the connection between the Schubert complex and Fulton’s formula, we will need the following lemma which was observed in [Pra, Appendix 6].

**Lemma 5.2** *Let  $D$  be an irreducible closed subvariety of a smooth variety  $X$ . Let  $C_\bullet$  be a finite complex of vector bundles on  $X$  and let  $P \in A^{\text{codim } D}(X)$ . If*

$$\text{supp } C_\bullet = X \setminus \{x \in X \mid (C_\bullet)|_x \text{ is an exact complex}\}$$

*is contained in  $D$ , and  $\varphi(P \cap [X]) = [C_\bullet]$ , then  $c[D] = P \cap [X]$  for some  $c \in \mathbf{Q}$ .*

We will use Lemma 5.2 with  $D = D_w(\partial)$ ,  $C_\bullet = \mathcal{S}_w(\partial)$ , and  $P = \mathfrak{S}_w(x, y)$  using the notation from the beginning of this section. We know that  $\text{supp } C_\bullet \subseteq D$  and that the codimension of  $D$  is  $\ell(w) = \deg P$  by Corollary 4.7. So we need to check that  $\varphi(P \cap [X]) = [C_\bullet]$ .

For a line bundle  $L$  corresponding to an irreducible divisor  $D$ , we have  $c_1(L) \cap [X] = [D]$ , and hence

$$\varphi(c_1(L) \cap [X]) = 1 - [L^\vee]$$

by the short exact sequence

$$0 \rightarrow \mathcal{L}(-D) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_D \rightarrow 0.$$

So the same formula holds for all  $L$  by linearity, and  $\varphi(x_i) = 1 - [E_i/E_{i-1}]$  while  $\varphi(y_j) = 1 - [\ker(F_j \rightarrow F_{j-1})]$ . Let  $a$  and  $b$  be a new set of variables. We have  $\mathfrak{S}_w(a, b) = \sum_{u \leq_w w} \mathfrak{S}_u(a) \mathfrak{S}_{uw^{-1}}(-b)$ . Doing the transformation  $a_i \mapsto x_i - 1$  and  $b_j \mapsto y_j - 1$ , we get  $\varphi(\mathfrak{S}_w(a, b)) = \sum_{u \leq_w w} (-1)^{\ell(u)} \mathfrak{S}_u(E) \mathfrak{S}_{uw^{-1}}(F)$ . By Proposition 4.1, this sum is  $[\mathcal{S}_w(\partial)]$  (the change from  $uw^{-1}$  to  $wu^{-1}$  is a consequence of the fact that  $F_1$  in Proposition 4.1 contains only odd elements). So it is enough to show that the substitution  $a_i \mapsto a_i + 1, b_j \mapsto b_j + 1$  leaves the expression  $\mathfrak{S}_w(a, b)$  invariant. This is clearly true for  $\mathfrak{S}_{w_0}(x, y) = \prod_{i+j \leq n} (x_i - y_j)$ , and holds for an arbitrary permutation because the divided difference operators (see (2.1)) applied to a substitution invariant function yield a substitution invariant function.

So it remains to show that the constant given by Lemma 5.2 is 1. This follows from Corollary 4.7(c).

### 5.2 Some remarks.

First we point out that the above can be applied to partial flags, but we have kept to complete flags for simplicity of notation.

A permutation  $w \in \Sigma_n$  is **Grassmannian** if it has at most one descent, i.e., there exists  $r$  such that  $w(1) < w(2) < \dots < w(r) > w(r+1) < \dots < w(n)$ . Suppose that  $w$  is **bigrassmannian**, which means that  $w$  and  $w^{-1}$  are Grassmannian permutations. This is equivalent to saying that  $D(w)$  is a rectangle. In this case, the double Schubert polynomial  $\mathfrak{S}_w(x, y)$  is a super Schur polynomial for the

partition  $D(w)$ . The degeneracy locus  $D_w(\partial)$  can then be described by a single rank condition between the map  $\partial E \rightarrow F$ , so the degeneracy locus formula of Fulton specializes to the Thom–Porteous formula mentioned in the introduction. So in principle, the action of  $\mathfrak{b}$  on  $\mathcal{S}_w(\partial)$  should extend to an action of a general linear superalgebra, but it is not clear why this should be true without appealing to Schur polynomials.

We have seen that the modules which are the cokernels of generic Schubert complexes have linear minimal free resolutions. These modules can then be thought of as a sort of “linear approximation” to the ideal which defines the matrix Schubert varieties, which in general have rich and complicated minimal free resolutions.

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