Normal polytopes

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Abstract. In Section 1 we overview combinatorial results on normal polytopes, old and new. These polytopes represent central objects of study in the contemporary discrete convex geometry, on the crossroads of combinatorics, commutative algebra, and algebraic geometry. In Sections 2 and 3 we describe two very different possible ways of advancing the theory of normal polytopes to next essential level, involving arithmetic and topological aspects.

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1 Normal polytopes: old and new

All our polytopes are assumed to be convex.

Let $P \subset \mathbb{R}^d$ be a lattice polytope and denote by L the affine lattice in $\mathbb{Z}^d$, generated by the lattice points in P; i. e., $L = v + \sum_{x,y \in P \cap \mathbb{Z}^d} \mathbb{Z}(x - y) \subset \mathbb{Z}^d$, where v is some (equivalently, any) lattice point in P. Observe, $P \cap L = P \cap \mathbb{Z}^d$. Here is the central definition:

(a) P is *integrally closed* if the following condition is satisfied:

$$c \in \mathbb{N}, \ z \in cP \cap \mathbb{Z}^d \quad \Longrightarrow \quad \exists x_1, \ldots, x_c \in P \cap \mathbb{Z}^d \quad x_1 + \cdots + x_c = z.$$

(b) P is *normal* if the following condition is satisfied:

$$c \in \mathbb{N}, \ z \in cP \cap L \quad \Longrightarrow \quad \exists x_1, \ldots, x_c \in P \cap L \quad x_1 + \cdots + x_c = z.$$

The normality property is invariant under affine-lattice isomorphisms of lattice polytopes, and the property of being integrally closed is invariant under an affine change of coordinates, leaving the lattice structure $\mathbb{Z}^d \subset \mathbb{R}^d$ invariant.

A lattice polytope $P \subset \mathbb{R}^d$ is integrally closed if and only if it is normal and L is a direct summand of $\mathbb{Z}^d$. Obvious examples of normal but not integrally closed polytopes are the s. c. empty lattice simplices of large volume. No classification of such simplices is known in dimensions ≥ 5, the main difficulty being the lack of satisfactory characterization of their lattice widths; see [13, 20].

A normal polytope $P \subset \mathbb{R}^d$ can be made into a full-dimensional integrally closed polytope by changing the lattice of reference $\mathbb{Z}^d$ to L and the ambient Euclidean space $\mathbb{R}^d$ to the subspace $\mathbb{R}L$. In particular,

normal and integrally closed polytopes refer to same isomorphism classes of lattice polytopes. In the combinatorial literature the difference between 'normal' and 'integrally closed' is sometimes blurred.

Normal/integrally closed polytopes enjoy popularity in algebraic combinatorics and they have been showcased on recent workshops ([1, 2]). These polytopes represent the homogeneous case of the Hilbert bases of finite positive rational cones and the connection to algebraic geometry is that they define projectively normal embeddings of toric varieties. There are many challenges of number theoretic, ring theoretic, homological, and K-theoretic nature, concerning the associated objects: Ehrhart series', rational cones, toric rings, and toric varieties; see [7].

If a lattice polytope is covered by (in particular, subdivided into) integrally closed polytopes then it is integrally closed as well. The simplest integrally closed polytopes one can think of are *unimodular simplices*, i. e., the lattice simplices $\Delta = conv(x_1, \ldots, x_k) \subset \mathbb{R}^d$, $\dim \Delta = k - 1$, with $x_1 - x_j, \ldots, x_{j-1} - x_j, x_{j+1} - x_j, \ldots, x_k - x_j$ a part of a basis of $\mathbb{Z}^d$ for some (equivalently, every) j.

Unimodular simplices are the smallest 'atoms' in the world of normal polytopes. But the latter is not built out exclusively of these atoms: not all 4-dimensional integrally closed polytopes are triangulated into unimodular simplices [9, Prop. 1.2.4], and not all 5-dimensional integrally closed polytopes are covered by unimodular simplices [5] – contrary to what had been conjectured before [19]. Further 'negative' results, such as [4] and [8] (the latter disproving a conjecture from [10]), contributed to the current thinking in the area that there is no succinct geometric characterization of the normality property. One could even conjecture that in higher dimensions the situation gets as bad as it can; see Section 2 for details.

'Positive' results in the field mostly concern special classes of lattice polytopes that are normal, or have unimodular triangulations or unimodular covers. Knudsen-Mumford's classical theorem ([7, Sect. 3B], [14, Chap. III],) says that every lattice polytope P has a multiple cP for some $c \in \mathbb{N}$ that is triangulated into unimodular simplices. The existence of a dimensionally uniform lower bound for such c seems to be a very hard problem. More recently, it was shown in [6] that there exists a dimensionally uniform exponential lower bound for unimodularly covered multiple polytopes. By improving one crucial step in [6], von Thaden was able to cut down the bound to a degree 6 polynomial function in the dimension [7, Sect. 3C], [22].

The results above on multiple polytopes yield no new examples of normal polytopes, though. In fact, an easy argument ensures that for any lattice d-polytope P the multiples cP, $c \geq d - 1$, are integrally closed [9, Prop. 1.3.3], [11]. One should remark that there is no algebraic obstruction to the existence of (even quadratic regular) unimodular triangulations for all multiples cP, $c \geq d - 1$: the nice homological properties that the corresponding toric rings would have (according to Sturmfels' theory [21]) should such triangulations existed, are all present [9].

Lattice polytopes with long edges of independent length are considered in [12], where it is shown that if the edges of a lattice d-polytope P have lattice lengths $\geq 2d^2(d+1)$ then P is integrally closed. In the special case when P is a simplex one can do better: P is covered by lattice parallelepipeds, provided the edges of P have lattice lengths $\geq d(d+1)$. Lattice parallelepipeds are the simplest integrally closed polytopes after unimodular simplices.

Currently, one problem attracts much attention in the field. Namely, Oda's question asks whether all smooth polytopes are integrally closed. A lattice polytope $P \subset \mathbb{R}^d$ is called smooth if the primitive edge vectors at every vertex of P define a part of a basis of $\mathbb{Z}^d$. Smooth polytopes correspond to projective embeddings of smooth projective toric varieties. Oda's question remains (as of writing this) wide open – so far no smooth polytope just without a unimodular triangulation has been found. The research was triggered by a faulty attempt in the mid 1990s to answer the question in the positive. The Frobenius

splitting techniques that was used then was recently revived in Payne's work, leading to new classes of normal (and even Koszul) polytopes, associated to root systems of various types [17].

For further classes of polytopes, associated to root systems and admitting unimodulaar triangulations as certificate of normality, see [3, 15, 16, 18].

In the next two sections we describe two very different possible ways of advancing the theory of normal polytopes to next essential level.

2 Carathéodory rank

What follows next is the homogeneous special case of a more general story that concerns rational positive cones and their Hilbert bases.

An arithmetic version of a unimodular cover is the *integral Carathéodory property (ICP):* a lattice d-polytope P has (ICP) if for every natural number c and every lattice point $z \in cP$ there exist lattice points $x_1, \ldots, x_{d+1} \in P$ and integers $a_1, \ldots, a_{d+1} \geq 0$ such that $z = a_1 x_1 + \cdots + a_{d+1} x_{d+1}$ and $a_1 + \cdots + a_{d+1} = c$.

For a lattice polytope P its *Carathéodory rank,* denoted by $CR(P)$, is the smallest natural number k such that for every natural number c and every lattice point $z \in cP$ there exist lattice points $x_1, \ldots, x_k \in P$ and integers $a_1, \ldots, a_k \geq 0$ such that $z = a_1 x_1 + \cdots + a_k x_k$ and $a_1 + \cdots + a_k = c$.

Sebö has shown [19] that $CR(P) \leq 2 \dim P$ for arbitrary integrally closed polytope P. If P has a unimodular cover then it has (ICP) too, i. e., $CR(P) = \dim P + 1$. It is known that (ICP) implies 'integrally closed' [5]. The converse is not true: there are integrally closed 5-polytopes without (ICP) [8]. That (ICP) and the existence of a unimodular cover are different conditions was discovered only recently: there are 5-polytopes with (ICP) but without unimodular cover [4].

We conjecture that Sëbo's estimate for Carathéodory rank is asymptotically sharp:

$$\lim_{d \to \infty} d^{-1} \max CR(P) = 2,$$

where, for each fixed d, P runs over the integrally closed d-polytopes. This conjecture, in particular, says that there are essentially new types of counterexamples to (ICP) in higher dimensions, not obtained by trivial extensions of counterexamples in lower dimensions.

3 Do normal polytopes model quantum states?

The method, by which counterexamples to the unimodular cover property and (ICP) were found, was to check s. c. *tight polytopes* for these properties. An integrally closed polytope $P \subset \mathbb{R}^d$ is called tight if, whatever lattice point $x \in P \cap \mathbb{Z}^d$ we choose, the convex hull of $(P \cap \mathbb{Z}^d) \setminus \{x\}$ in $\mathbb{R}^d$ is not an integrally closed polytope. This moves center stage the descending sequences of lattice integrally closed polytopes in $\mathbb{R}^d$ of type

$$P_1 \supset P_2 \supset \cdots \supset P_k, \qquad \# (P_i \cap \mathbb{Z}^d) = \# (P_{i+1} \cap \mathbb{Z}^d) + 1, i = 1, \ldots, k-1.$$

That such a sequence may halt at all at a positive dimensional polytope, or equivalently, that there exist nontrivial tight polytopes is already something not quite obvious. This phenomenon shows up in dimensions ≥ 4. There are no tight polygons, and the existence of tight 3-polytopes is not known.

Consider the poset $\mathsf{Pol}(d)$ of lattice integrally closed polytopes in $\mathbb{R}^d$, where the order relation is generated by the elementary relations of type $P < Q$, $\#\left(Q \cap \mathbb{Z}^d\right) = \#\left(P \cap \mathbb{Z}^d\right) + 1$. Minimal elements of $\mathsf{Pol}(d)$ are exactly the tight polytopes in $\mathbb{R}^d$. Informally, the poset $\mathsf{Pol}(d)$ offers a global picture of the interaction of polytopal shapes in $\mathbb{R}^d$ with the integer lattice $\mathbb{Z}^d$.

Here is a list of a several interesting questions one can ask about $\mathsf{Pol}(d)$: Do there exist maximal elements in $\mathsf{Pol}(d)$? What is the homotopy type of $\mathsf{Pol}(d)$? Is it contractible? Does it have isolated points?

We suggest the following game: think of the chains in $\mathsf{Pol}(d)$ as quantum processes, individual polytopes as quantum states, and elementary relations $P < Q$ as quantum jumps. The vocabulary can be extended to accommodate such terminology as energy and time (both discrete), observables, entanglement, tunneling, uncertainty principle, fluctuations. Real fun starts when one thinks of the potentially nontrivial homotopy groups of $\mathsf{Pol}(d)$ as a force that permeates all of the geometric realization space of $\mathsf{Pol}(d)$ and keeps the world of integrally closed polytopes from collapsing into a point.

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