Hyperplane Arrangements and Diagonal Harmonics

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Abstract. In 2003, Haglund's bounce statistic gave the first combinatorial interpretation of the q,t-Catalan numbers and the Hilbert series of diagonal harmonics. In this paper we propose a new combinatorial interpretation in terms of the affine Weyl group of type A. In particular, we define two statistics on affine permutations; one in terms of the Shi hyperplane arrangement, and one in terms of a new arrangement — which we call the Ish arrangement. We prove that our statistics are equivalent to the area' and bounce statistics of Haglund and Loehr. In this setting, we observe that bounce is naturally expressed as a statistic on the root lattice. We extend our statistics in two directions: to "extended" Shi arrangements and to the bounded chambers of these arrangements. This leads to a (conjectural) combinatorial interpretation for all integral powers of the Bergeron-Garsia nabla operator applied to elementary symmetric functions.

Résumé. En 2003, la statistique bounce de Haglund a donné la première interprétation combinatoire de la somme des nombres q,t-Catalan et de la série de Hilbert des harmoniques diagonaux. Dans cet article nous proposons une nouvelle interprétation combinatoire à partir du groupe de Weyl affine de type A. En particulier, nous définissons deux statistiques sur les permutations affines; l'une à partir de l'arrangement d'hyperplans Shi, et l'autre à partir d'un nouvel arrangement — que nous appelons l'arrangement Ish. Nous prouvons que nos statistiques sont équivalentes aux statistiques area' et bounce de Haglund et Loehr. Dans ce contexte, nous observons que bounce s'exprime naturellement comme une statistique sur le réseau des racines. Nous prolongeons nos statistiques dans deux directions: arrangements Shi "étendus", et chambres bornées associées. Cela conduit à une interprétation (conjecturale) combinatoire pour toutes les puissances entières de l'opérateur nabla de Bergeron-Garsia appliqué aux fonctions symétriques élémentaires.

Keywords: Shi arrangement, Ish arrangement, affine permutations, diagonal harmonics, Catalan numbers, nabla operator, parking functions

1 Introduction

1.1 Diagonal Harmonics

The symmetric group $\mathfrak{S}(n)$ acts on the polynomial ring $S = \mathbb{Q}[x_1, \ldots, x_n]$ by permuting variables. Newton showed that the subring of $\mathfrak{S}(n)$-invariant polynomials is generated by the algebraically independent

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power sum polynomials: $p_k = \sum_{i=1}^n x_i^k$ for $k = 1, 2, \ldots, n$. It is known that the coinvariant ring $R = S/(p_1, \ldots, p_n)$ is a graded version of the regular representation of $\mathfrak{S}(n)$, with Hilbert series

$$\sum_{i=0}^{n} \dim R_i \, q^i = \prod_{j=1}^{n} (1 + q + q^2 + \cdots + q^j) = [n]_q!.$$

The dual ring $S^* = \mathbb{Q}[\partial/\partial x_1, \ldots, \partial/\partial x_n]$ acts on S via the pairing $(\partial/\partial x_i)x_j = \delta_{ij}$, hence the coinvariant ring is isomorphic to the quotient $S^*/(p_1^*, \ldots, p_n^*)$, where $p_k^* = \sum_{i=1}^n (\partial/\partial x_i)^k$ for $k = 1, \ldots, n$. On the other hand, this quotient is naturally isomorphic to the submodule $H \subseteq S$ annihilated by the p_k^*:

$$H = \{f \in S : p_k^* \, f = 0 \text{ for all } k\}.$$

This H is called the ring of harmonic polynomials since, in particular, p_2^* is the standard Laplacian operator on S.

Now consider the ring $DS = \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ of polynomials in two sets of commuting variables, together with the diagonal action of $\mathfrak{S}(n)$, which permutes the x variables and the y variables *simultaneously*. Weyl [31] showed that the $\mathfrak{S}(n)$-invariant subring of DS is generated by the polarized power sums: $p_{k,\ell} = \sum_{i=1}^n x_i^k y_i^\ell$ for all $k + \ell > 0$. Hence the ring of diagonal coinvariants $DR = DS/(p_{k,\ell} : k + \ell > 0)$ is naturally isomorphic to the ring of diagonal harmonic polynomials:

$$DH = \{f \in DS : \sum_{i=1}^{n} (\partial/\partial x_i)^k (\partial/\partial y_i)^\ell \, f = 0 \text{ for all } k + \ell > 0\}.$$

The diagonal action preserves the bigrading of DS by x-degree and y-degree, hence DH is a bigraded $\mathfrak{S}(n)$-module. The bigraded Hilbert series

$$\mathcal{DH}(n; q, t) := \sum_{i,j=0}^{n} \dim(DH)_{i,j} \; q^i t^j \tag{1}$$

has beautiful and remarkable properties. The study of $\mathcal{DH}(n; q, t)$ was initiated by Garsia and Haiman (see [13]) and is today an active area of research.

1.2 Some Arrangements

Let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis for $\mathbb{R}^n$. Given $v \in \mathbb{R}^n$ and $k \in \mathbb{R}$, we will often use the notation "$v = k$" as shorthand for the set $\{x : (x, v) = k\} \subseteq \mathbb{R}^n$, where $(\cdot, \cdot)$ is the standard inner product. Consider the following three arrangements of hyperplanes, respectively called the Coxeter arrangement, Shi arrangement, and affine arrangement of type A_{n-1}:

$$\begin{aligned}
\mathsf{Cox}(n) &:= \{e_i - e_j = a : 1 \le i < j \le n, \, a = 0\}, \\
\mathsf{Shi}(n) &:= \{e_i - e_j = a : 1 \le i < j \le n, \, a \in \{0, 1\}\}, \\
\mathsf{Aff}(n) &:= \{e_i - e_j = a : 1 \le i < j \le n, \, a \in \mathbb{Z}\}.
\end{aligned}$$

Since all hyperplanes in this paper contain the line $e_1 + e_2 + \cdots + e_n$, we will typically restrict these arrangements to the $(n-1)$-dimensional space

$$\mathbb{R}_0^n := \{e_1 + e_2 + \cdots + e_n = 0\}.$$

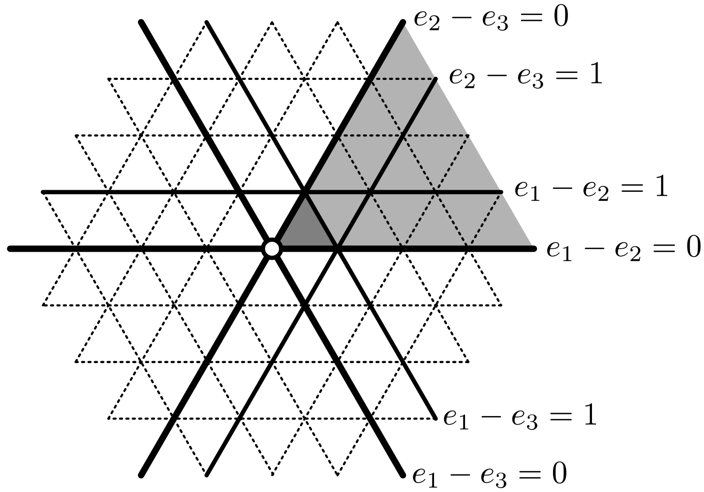

Fig. 1: Some arrangements in $\mathbb{R}^3_0$

If $\mathcal{A}$ is an arrangement in a space V then the connected components of the complement $V - \cup_{H\in\mathcal{A}} H$ are called chambers. We will refer to chambers of the Coxeter arrangement as cones; and refer to affine chambers as alcoves. Let $C_\circ$ denote the dominant cone, which satisfies the coordinate inequalities

$$e_1 > e_2 > \cdots > e_n,$$

and let $A_\circ$ denote the fundamental alcove, satisfying

$$e_1 > e_2 > \cdots > e_n > e_1 - 1.$$

Figure 1 displays the arrangements $\mathsf{Cox}(3)$, $\mathsf{Shi}(3)$, and $\mathsf{Aff}(3)$ in $\mathbb{R}^3_0$, with the dominant cone and fundamental alcove shaded. The Shi arrangement was introduced by Jian-Yi Shi (see [23, Chapter 7]) in his description of the Kazhdan-Lusztig cells for certain affine Weyl groups.

1.3 Symmetric Group

The symmetric group $\mathfrak{S}(n)$ has a faithful representation as a group of isometries of $\mathbb{R}^n_0$ generated by the set

$$S = \{s_1, s_2, \ldots, s_{n-1}\},$$

where s_i is the reflection in the hyperplane $e_i - e_{i+1} = 0$. The reflection s_i corresponds in $\mathfrak{S}(n)$ to the transposition of adjacent symbols $(i, i+1)$.

The symmetric group acts simply-transitively on the cones of the Coxeter arrangement $\mathsf{Cox}(n)$. By convention, let the dominant cone $C_\circ$ correspond to the identity permutation; then for any permutation $w \in \mathfrak{S}(n)$ the cone $wC_\circ$ satisfies

$$e_{w(1)} > e_{w(2)} > \cdots > e_{w(n)}.$$

1.4 Affine Symmetric Group

Now let s_n denote the reflection in the *affine* hyperplane $e_1 - e_n = 1$. The linear reflections $\{s_1, s_2, \ldots, s_{n-1}\}$ together with the affine reflection a_n generate the affine Weyl group of type $\tilde{A}_{n-1}$. This group acts simply-transitively on the set of alcoves, where the fundamental alcove $A_\circ$ corresponds to the identity element of the group. Note that $A_\circ$ is a (non-regular) simplex in $\mathbb{R}^n_0$ whose facets are supported by the reflecting hyperplanes of the generators $\{s_1, s_2, \ldots, s_n\}$.

Lusztig [21] introduced an affine version of the symmetric group, whose combinatorial properties were developed further by Björner and Brenti [6]. We define $\tilde{\mathfrak{S}}(n)$ as the group of infinite permutations $\tilde{w} : \mathbb{Z} \to \mathbb{Z}$ satisfying:

- $\tilde{w}(k+n) = \tilde{w}(k) + n$ for all $k \in \mathbb{Z}$,
- $\tilde{w}(1) + \tilde{w}(2) + \cdots + \tilde{w}(n) = \binom{n+1}{2}$.

The first property says that $\tilde{w}$ is periodic and the second fixes a frame of reference. The elements of $\tilde{\mathfrak{S}}(n)$ are called affine permutations, and $\tilde{\mathfrak{S}}(n)$ is the affine symmetric group. Following Björner and Brenti, we will usually express an affine permutation $\tilde{w} \in \tilde{\mathfrak{S}}(n)$ using the window notation:

$$\text{“}\tilde{w} = [\tilde{w}(1), \tilde{w}(2), \cdots, \tilde{w}(n)].\text{''}$$

For integers $i < j$ with $i \not\equiv j \pmod n$ we will write $((i,j)) : \mathbb{Z} \to \mathbb{Z}$ to denote the “affine tranposition” that swaps the elements in positions $i + kn$ and $j + kn$ for all $k \in \mathbb{Z}$. We could also write $((i,j)) = \prod_k (i + kn, j + kn)$. Lusztig proved that the correspondence $s_i \leftrightarrow ((i, i+1))$ defines an isomorphism between the affine symmetric group and the affine Weyl group of type A. Here the affine tranposition $((i,j))$ corresponds to the reflection in the affine hyperplane

$$e_{i-n(\lceil i/n \rceil - 1)} - e_{j-n(\lceil j/n \rceil -1)} = \left\lceil \frac{i}{n} \right\rceil - \left\lceil \frac{j}{n} \right\rceil . \tag{2}$$

In particular, note that the generator $s_i = ((i, i+1))$ corresponds to $e_i - e_{i+1} = 0$ for $1 \leq i \leq n-1$, and $s_n = ((n, n+1))$ corresponds to $e_1 - e_n = 1$.

1.5 The Ish Arrangement

Finally, we introduce a new hyperplane arrangement, called the Ish arrangement. Like the Shi arrangement, the Ish arrangement begins with the $\binom{n}{2}$ linear hyperplanes of the Coxeter arrangement and then adds another $\binom{n}{2}$ affine hyperplanes:

$$\mathsf{Ish}(n) := \mathsf{Cox}(n) \cup \{e_i - e_n = a : 1 \leq i \leq n-1,\ a \in \{1, \ldots, n-i\}\}.$$

Figure 2 displays the arrangements $\mathsf{Shi}(3)$ and $\mathsf{Ish}(3)$. Note that each has 16 chambers and 4 bounded chambers. There is an important reason for this: the arrangements $\mathsf{Shi}(n)$ and $\mathsf{Ish}(n)$ share the same *characteristic polynomial*, as we now show.

To avoid extra notation, we will use a non-standard definition of the characteristic polynomial. This definition is due to Crapo and Rota, and was applied extensively by Athanasiadis — see Stanley [29, Lecture 5] for details. Let $\mathcal{A}$ be an arrangement of *finitely many* hyperplanes in $\mathbb{R}^n$. Suppose further that

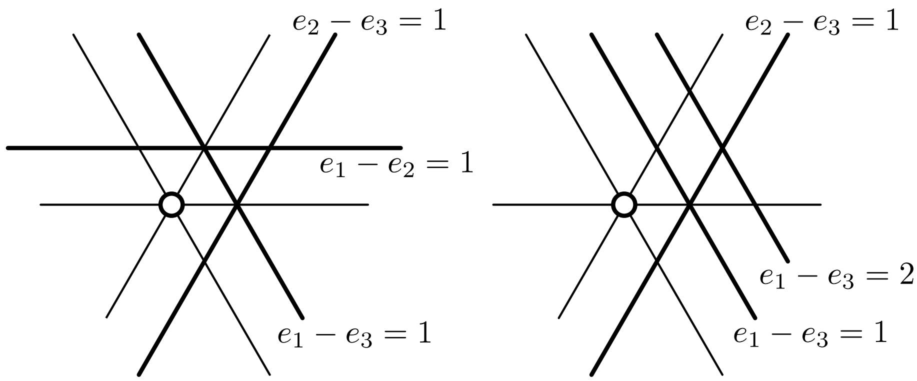

Fig. 2: The arrangements $\mathsf{Shi}(3)$ and $\mathsf{Ish}(3)$

each of these hyperplanes has an equation with integer coefficients. Then, given a (large) finite field $\mathbb{F}_q$ with q elements, we may consider the reduced arrangement $\mathcal{A}_q$ in $\mathbb{F}_q^n$. It turns out that (for all but finitely many q), the number of points of $\mathbb{F}_q^n$ **not** on any hyperplane of $\mathcal{A}_q$ is given by a polynomial in q, called the characteristic polynomial of $\mathcal{A}$:

$$\chi(\mathcal{A}, q) = \# \left(\mathbb{F}_q^n - \cup_{H \in \mathcal{A}_q} H\right) = q^n - \# \cup_{H \in \mathcal{A}_q} H.$$

The characteristic polynomial of the Shi arrangement is well known (cf. [29, Theorem 5.16]). Our new result is the following. (Proof omitted.)

Theorem 1. *The Shi arrangement and the Ish arrangement share the same characteristic polynomial, viz.*

$$\chi(\mathsf{Ish}(n), q) = q\,(q-n)^{n-1}.$$

The following is a standard result on real hyperplane arrangements. Let $\mathcal{A}$ be an arrangement in a real d-dimensional space V and suppose that the normals to $\mathcal{A}$ span a subspace $U \subseteq V$ of dimension k — called the rank of $\mathcal{A}$. If $k < d$ then $\mathcal{A}$ has no bounded chambers; its chambers that have bounded intersection with U are called relatively bounded.

Zaslavsky's Theorem (see, e.g., Theorem 2.5 of [29]). *Let $\mathcal{A}$ be a real arrangement with dimension d and rank k. Then:*

- *The number of chambers of $\mathcal{A}$ is $(-1)^d \chi(A, -1)$.*
- *The number of relatively bounded chambers of $\mathcal{A}$ is $(-1)^k \chi(A, 1)$.*

If we think of $\mathsf{Shi}(n)$ and $\mathsf{Ish}(n)$ as arrangements in the space $\mathbb{R}_0^n$, then $d = k = n-1$.

Corollary 1. *The arrangements $\mathsf{Shi}(n)$ and $\mathsf{Ish}(n)$ have the same number of chambers — i.e. $(n+1)^{n-1}$ — and the same number of bounded chambers — i.e. $(n-1)^{n-1}$.*

Open Problem. Find a bijective proof of the corollary.

In a recent joint paper with Rhoades [2], we have expanded on the relationship between the Shi and Ish arrangements. The paper shows that the relationship between these objects is deep and somewhat mysterious. In it we give nice combinatorial labels of the regions and show that these are equinumerous, but the problem of a bijective proof is still open.

2 Two Statistics on Shi Chambers

Now we define two statistics — called shi and ish — on the chambers of a Shi arrangement (more generally, on the elements of the group $\tilde{\mathfrak{S}}(n)$). The first statistic is well known and the second is new. Each statistic counts certain kind of "inversions" of an affine permutation. We begin by defining these.

2.1 Affine Inversions

Let w be an element of the (finite) symmetric group $\mathfrak{S}(n)$. If $w(i) > w(j)$ for indices $1 \leq i < j \leq n$ we say that the tranposition (i,j) is an inversion of w — equivalently, this means that the hyperplane $e_i - e_j = 0$ separates the cone $wC_\circ$ from the dominant cone $C_\circ$. The number of inversions of w is called its length.

In the affine symmetric group $\tilde{\mathfrak{S}}(n)$, there is again a correspondence between hyperplanes and transpositions. Recall that the affine transpositions $((i,j))$ and $((i',j'))$ coincide if $i' = i + kn$ and $j' = j + kn$ for some $k \in \mathbb{Z}$, in which case they represent the same hyperplane (2). Hence, each affine transposition has a standard representative in the set

$$\tilde{T} := \{((i,j)) : \ 1 \leq i \leq n, \ i < j, \ i \not\equiv j \mod n\} \subseteq \tilde{\mathfrak{S}}(n).$$

Given an affine permutation $\tilde{w} \in \tilde{\mathfrak{S}}(n)$ and an affine transposition $((i,j)) \in \tilde{T}$ such that $\tilde{w}(i) > \tilde{w}(j)$, we say that $((i,j))$ is an affine inversion of $\tilde{w}$ — equivalently, the hyperplane (2) separates the alcove $\tilde{w}A_\circ$ from the fundamental alcove $A_\circ$. Again, the (affine) length of $\tilde{w}$ is its number of affine inversions.

2.2 *The* shi *statistic*

Each chamber of the Shi arrangement contains a set of alcoves and among these is a unique alcove of minimum length — which we call the representing alcove of the chamber, or just a Shi alcove. This defines an injection from Shi chambers into the affine symmetric group. Figure 3 displays the representing alcoves for $\mathsf{Shi}(3)$, labeled by affine permutations. We have labeled the Shi hyperplanes with their corresponding affine transpositions,

$$\mathsf{Shi}(n) = \{((i,j)) : \quad 1 \leq i \leq n, \quad i < j < n + i\}.$$

Definition 2.1. Given a Shi chamber with representing alcove A, let $\mathsf{shi}(A)$ denote the number of Shi hyperplanes separating A from the fundamental alcove $A_\circ$. Equivalently, if $A = \tilde{w}A_\circ$ for affine permutation $\tilde{w} \in \mathfrak{S}(n)$, then $\mathsf{shi}(\tilde{w})$ is the number of affine inversions $((i,j))$ of $\tilde{w}$ satisfying $i < j < n + i$.

For example, consider the permutation $\tilde{w} = [1,5,0]$ in the figure. The inversions of $\tilde{w}$ are $((1,3)), ((2,3)), ((2,4)), ((2,6))$, and hence $\tilde{w}$ has length 4. However, only three of these — viz. $((1,3)), ((2,3)), ((2,4))$ — come from Shi hyperplanes, hence $\mathsf{shi}(\tilde{w}) = 3$.

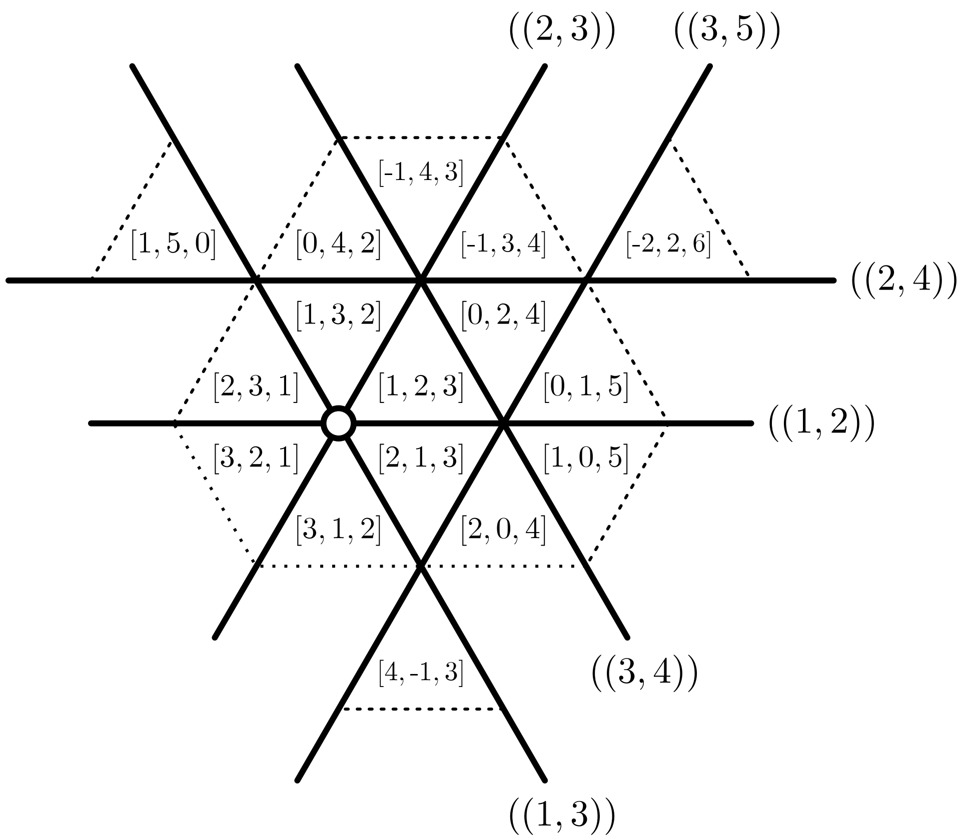

Fig. 3: Chambers of $\mathsf{Shi}(3)$ labeled by affine permutations

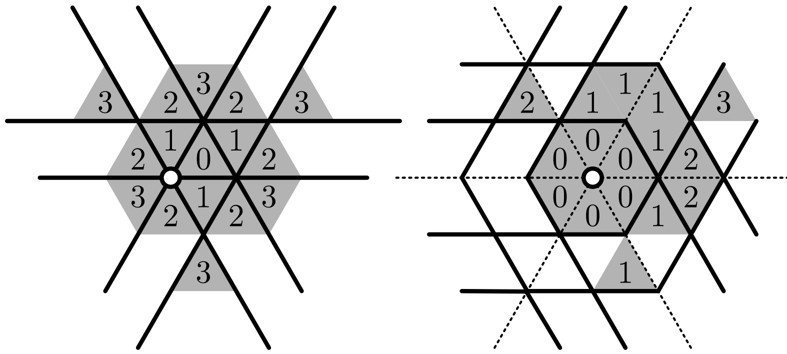

Fig. 4: The shi and ish statistics on the chambers of $\mathsf{Shi}(3)$

2.3 *The* ish *statistic*

To give a natural definition for our second statistic, we must discuss the coset space $\tilde{\mathfrak{S}}(n)/\mathfrak{S}(n)$. By abuse of notation, let $\mathfrak{S}(n)$ denote the subgroup of $\tilde{\mathfrak{S}}(n)$ generated by the subset

$$I = \{s_1, \ldots, s_{n-1}\} \subseteq \{s_1, \ldots, s_{n-1}, s_n\} = S.$$

In the language of Coxeter groups we say that $\mathfrak{S}(n)$ is a parabolic subgroup of $\tilde{\mathfrak{S}}(n)$. When $W = \tilde{\mathfrak{S}}(n)$ the standard notation for this is to write $\mathfrak{S}(n) = W_I$. Then each affine permutation $\tilde{w}$ has a canonical decomposition

$$\tilde{w} = w_I \tilde{w}^I,$$

where $w_I \in W_I$ is a finite permutation and $\tilde{w}^I \in W$ is the unique coset representative of minimum (affine) length. Combinatorially, $[\tilde{w}^I(1), \ldots, \tilde{w}^I(n)]$ is the increasing rearrangement of $[\tilde{w}(1), \ldots, \tilde{w}(n)]$ and w_I is the finite permutation needed to achieve the rearrangement. Geometrically, alcoves of the form $A = \tilde{w}^I A_\circ$ are contained in the dominant cone $C_\circ$; hence $\tilde{w}A_\circ = w_I A$ is contained in the cone $w_I C_\circ$.

We define the ish statistic in terms of minimal coset representatives.

Definition 2.2. Consider a Shi chamber with representing alcove A and suppose that $A = \tilde{w}A_\circ$. Its minimal coset representative $\tilde{w}^I A_\circ$ is an alcove in the dominant cone $C_\circ$. Let $\mathsf{ish}(A)$ denote the number of hyperplanes of the form $e_i - e_n = a$ (with $1 \leq i \leq n-1$ and $a \in \mathbb{Z}$) separating $\tilde{w}^I A_\circ$ from the fundamental alcove $A_\circ$. Equivaently, let $\mathsf{ish}(\tilde{w})$ denote the number of affine inversions of $\tilde{w}^I$ of the form $((n, j))$.

Two notes: In order to facilitate later generalization, we have defined ish in terms of *all* hyperplanes of the form $e_i - e_n = a$. In our current context, however, only the Ish hyperplanes (i.e. $a \in \{1, \ldots, n-i\}$) will contribute. We also emphasize the fact that ish **is a statistic on the (representing alcoves of) Shi chambers, not on the Ish chambers**. It seems that the chambers of the Ish arrangement are not so natural.

For example, consider the affine permutation $\tilde{w} = [\text{-}1, 4, 3]$, as shown in Figure 3. It is contained in the cone $[1, 3, 2]\, C_\circ$ and its increasing rearrangement is $[\text{-}1, 3, 4]$. Hence, it has parabolic decomposition

$$[\text{-}1, 4, 3] = \tilde{w} = w_I \tilde{w}^I = [1, 3, 2]\,[\text{-}1, 3, 4].$$

The inversions of $\tilde{w}^I = [\text{-}1, 3, 4]$ are $((2,4))$ and $((3,4))$, of which only the second is an Ish hyperplane; hence $\mathsf{ish}(\tilde{w}) = 1$. In Figure 4 we have displayed the shi and ish statistics for all chambers of $\mathsf{Shi}(3)$. (Note: to compute ish by hand, one may extend the Ish hyperplanes from the dominant cone to the other cones by reflection.) Their joint-distribution is recorded in the following table:

		ish			
		0	1	2	3
shi	0	1			
	1	2	1		
	2	2	3	1	
	3	1	2	2	1

2.4 Theorems and a Conjecture

We will make four assertions and then describe our state of knowledge about them (i.e. whether each is a Theorem or a Conjecture). We will use the following notation.

Recall from (1) that $\mathcal{DH}(n; q, t)$ denotes the bigraded Hilbert series of the ring of diagonal harmonic polynomials. Define

$$\mathsf{Shi}(n; q, t) := \sum_A q^{\mathsf{ish}(A)} t^{\binom{n}{2} - \mathsf{shi}(A)},$$

where the sum is taken over representing alcoves A for the chambers of the arrangement $\mathsf{Shi}(n)$. We say that an alcove is **positive** if it is contained in the dominant cone $C_\circ$ (i.e. if A is on the "positive" side of each generating hyperplane). Let $\mathsf{Shi}_+(n; q, t)$ denote the corresponding sum over positive Shi alcoves. Finally, consider the standard q-integer, q-factorial, and q-binomial coefficient:

$$\begin{aligned} [a]_q &= 1 + q + \cdots + q^{a-1}, \\ [a]_q! &= [a]_q [a-1]_q \cdots [2]_q [1]_q, \\ \begin{bmatrix} a \\ b \end{bmatrix}_q &= \frac{[a]_q!}{[a-b]_q! [b]_q!}. \end{aligned}$$

Assertions.

(1) $\mathsf{Shi}(n; q, t) = \mathcal{DH}(n; q, t)$, and hence is symmetric in q and t.

(2) $q^{\binom{n}{2}} \mathsf{Shi}(n; q, 1/q) = [n+1]_q^{n-1}$.

(3) $\mathsf{Shi}_+(n; q, t)$ is equal to Garsia and Haiman's q,t-Catalan number, and hence is symmetric in q and t.

(4) $q^{\binom{n}{2}} \mathsf{Shi}_+(n; q, 1/q) = \frac{1}{[n]_q} \begin{bmatrix} 2n \\ n-1 \end{bmatrix}_q$, the q-Catalan number.

In particular, note that $q^{\binom{n}{2}} \mathsf{Shi}_+(n; q, 1/q)$ is equal to the sum of $q^{\mathsf{shi}(A) + \mathsf{ish}(A)}$ over the positive Shi alcoves A. For $n = 3$ we may compute this sum using the data in Figure 4 to obtain

$$1 + q^2 + q^3 + q^4 + q^6 = \frac{[6]_q [5]_q}{[3]_q [2]_q} = \frac{1}{[3]_q} \begin{bmatrix} 6 \\ 2 \end{bmatrix}_q,$$

which is a q-Catalan number. One may check that the other three assertions are also true in the case $n = 3$.

We will now state the main theorem of this paper, but omit its proof.

Main Theorem. *There exists a natural bijection from the* $(n+1)^{n-1}$ *chambers of the Shi arrangement to parking functions which sends our statistics* $(\mathsf{ish}, \mathsf{shi} - \binom{n}{2})$ *to the statistics* $(\mathsf{bounce}, \mathsf{area}')$ *of Haglund and Loehr [16].*

This allows us to clarify the Assertions.

Status. Each of the following depends on the Main Theorem.

(1) **Conjecture.** This is equivalent to a conjecture of Haglund and Loehr [16] (known in a different form to Haiman). No combinatorial explanation of the q, t symmetry is known.

(2) **Theorem.** This is equivalent to a theorem of Loehr [17].

(3) **Theorem.** This follows from theorems of Garsia and Haglund [9, 10]. No combinatorial explanation of the q, t symmetry is known.

(4) **Theorem.** This is equivalent to a theorem of Haglund [14], which was later proved bijectively by Loehr [18].

Acknowledgements

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In January 2010 we learned that Mark Haiman had a result in 2007 (unpublished) equivalent to our Main Theorem, although he approached the topic from the opposite direction — from dilations of the fundamental alcove instead of hyperplane arrangements. This should appear in a forthcoming paper.

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