Finite Eulerian posets which are binomial or Sheffer

Hoda Bidkhori

North Carolina State University, NC, USA

Abstract. In this paper we study finite Eulerian posets which are binomial or Sheffer. These important classes of posets are related to the theory of generating functions and to geometry. The results of this paper are organized as follows:

- We completely determine the structure of Eulerian binomial posets and, as a conclusion, we are able to classify factorial functions of Eulerian binomial posets;
- We give an almost complete classification of factorial functions of Eulerian Sheffer posets by dividing the original question into several cases;
- In most cases above, we completely determine the structure of Eulerian Sheffer posets, a result stronger than just classifying factorial functions of these Eulerian Sheffer posets.

We also study Eulerian triangular posets. This paper answers questions posed by R. Ehrenborg and M. Readdy. This research is also motivated by the work of R. Stanley about recognizing the *boolean lattice* by looking at smaller intervals.

Résumé. Nous étudions les ensembles partiellement ordonnés finis (EPO) qui sont soit binomiaux soit de type Sheffer (deux notions reliées aux séries génératrices et à la géométrie). Nos résultats sont les suivants:

- nous déterminons la structure des EPO Euleriens et binomiaux; nous classifions ainsi les fonctions factorielles de tous ces EPO;
- nous donnons une classification presque complète des fonctions factorielles des EPO Euleriens de type Sheffer;
- dans la plupart de ces cas, nous déterminons complètement la structure des EPO Euleriens et Sheffer, ce qui est plus fort que classifier leurs fonctions factorielles.

Nous étudions aussi les EPO Euleriens triangulaires. Cet article répond à des questions de R. Ehrenborg and M. Readdy. Il est aussi motivé par le travail de R. Stanley sur la reconnaissance du treillis booléen via l'étude des petits intervalles.

Keywords: Eulerian poset, binomial poset, Sheffer poset

1 Introduction

There are many theories which unify various aspects of enumerative combinatorics and generating functions. One such successful theory introduced by Doubilet, Rota and Stanley [3] is that of binomial posets. Classically, binomial posets are infinite posets with the property that every two intervals of the same length have the same number of maximal chains. Doubilet, Rota and Stanley show this chain regularity condition gives rise to universal families of generating functions. Ehrenborg and Readdy [5] and Reiner [9] independently generalized the notion of binomial posets to a larger class of posets called Sheffer posets or upper binomial posets.

Ehrenborg and Readdy [4] gave a complete classification of the factorial functions of infinite Eulerian binomial posets and infinite Eulerian Sheffer posets. Recall that infinite posets are those posets which contain an infinite chain. They posed the open question of characterizing the finite case. This paper deals with these questions.

A *Sheffer poset* is a graded poset such that the number of maximal chains $D(n)$ in an n-interval $[\hat{0}, y]$ depends only on $\rho(y) = n$, the rank of the element y, and the number $B(n)$ of maximal chains in an n-interval $[x, y]$, where $x \neq \hat{0}$, depends only on $\rho(x, y) = \rho(y) - \rho(x)$. The two functions $B(n)$ and $D(n)$ are called the *binomial factorial function* and *Sheffer factorial function*, respectively. *Binomial posets* are a special class of Sheffer posets. A binomial poset is a graded poset such that the number of maximal chains $B(n)$ in an n-interval $[x, y]$ depends only on $\rho(x, y) = \rho(y) - \rho(x)$.

Binomial posets were previously considered in [1], [3], [8], [11] and [13]. Ehrenborg and Readdy [5] used Sheffer posets and a generalization of R-labeling to study augmented r-signed permutations. Reiner [9] used them to derive generating functions counting signed permutations by two statistics.

A graded poset P is *Eulerian* if every non-singleton interval of P satisfies the *Euler-Poincaré* relation. (See Definition 2.1.) Eulerian posets form an important class of posets as there are many geometric examples such as the face lattices of convex polytopes, and more generally, the face posets of regular CW-spheres.

As we mentioned above, Ehrenborg and Readdy in [4] classify the factorial functions of infinite Eulerian binomial posets and infinite Eulerian Sheffer posets. Since we are concerned here with finite posets, we drop the requirement that binomial, Sheffer and triangular posets have an infinite chain. This paper studies the following natural questions, as suggested by Ehrenborg and Readdy in [4].

1. Which Eulerian posets are binomial?

2. Which Eulerian posets are Sheffer?

Stanley has proved that one can recognize *boolean lattices* by looking at smaller intervals (see [7], Lemma 8). Farley and Schmidt answer a similar question for *distributive lattices* in [6]. The project of studying Eulerian binomial posets and Eulerian Sheffer posets is also motivated by their work. In many cases we use the factorial function of smaller intervals to characterize the whole poset.

1.1 Acknowledgments

The author would like to thank Richard Stanley for suggesting this research problem. I also would like to thank Richard Ehrenborg, Margaret A. Readdy and Richard Stanley for helpful discussions and comments and thank Craig Desjardins for reading a draft of this paper. I am grateful to two anonymous referees for many helpful suggestions..

2 Background and Definition

We encourage readers to consult Chapter 3 of [12] for basic poset terminology. All the posets which are considered in this paper are finite.

We begin by recalling that a graded interval satisfies the *Euler-Poincaré relation* if it has the same number of elements of even rank as of odd rank.

Definition 2.1. 1. *A graded poset is* Eulerian *if every non-singleton interval satisfies the Euler-Poincaré relation. Equivalently, a poset P is Eulerian if its Möbius function satisfies $\mu(x, y) = (-1)^{\rho(x,y)}$ for all $x \leq y$ in P, where ρ denotes the rank function of P.*

2. *Consider a graded poset P with rank function ρ. If $\rho(x, y) = n$, then we call $[x, y]$ an n-*interval.

Definition 2.2. *A finite graded poset P with unique minimal element $\hat{0}$ and unique maximal element $\hat{1}$ is called a* (finite) binomial poset *if it satisfies the following condition:*

1. *For all $n \in \mathbb{N}$, $n \leq$ rank(P), any two n-intervals have the same number $B(n)$ of maximal chains. We call $B(n)$ the* factorial function *or* binomial factorial function *of the poset P.*

Definition 2.3. *A finite graded poset P with a unique minimal element $\hat{0}$ and a unique maximal element $\hat{1}$ is called a* (finite) Sheffer poset *if it satisfies the following two conditions:*

1. *Any pair of n-intervals $[\hat{0}, y]$ and $[\hat{0}, v]$ have the same number $D(n)$ of maximal chains.*

2. *Any pair of n-intervals $[x, y]$ and $[u, v]$ such that $x \neq \hat{0}$ and $u \neq \hat{0}$ have the same number $B(n)$ of maximal chains.*

Let us consider a Sheffer poset P. An interval $[\hat{0}, y]$, where $y \neq \hat{0}$, is called a *Sheffer interval* whereas an interval $[x, y]$ with $x \neq \hat{0}$ is called a *binomial interval*. The functions $B(n)$ and $D(n)$ are called the *binomial factorial function* and *Sheffer factorial function* of P, respectively. Next we define $A(n)$ and $C(n)$ to be the number of coatoms in a binomial interval of length n, respectively, a Sheffer interval of length n. The functions $A(n)$ and $C(n)$ are called the *atom function* and *coatom function* of P, respectively. The number of elements of rank k in a Sheffer interval of rank n is

$$\frac{D(n)}{D(k)B(n-k)}. \tag{1}$$

Moreover, for a binomial interval $[x, y]$ of rank n in a Sheffer poset, the number of elements of rank k is equal to

$$\frac{B(n)}{B(k)B(n-k)}. \tag{2}$$

The *dual suspension* of a poset P is defined in [4] as follows.

Definition 2.4. *Let P be a poset with $\hat{0}$. We define the* dual suspension *of P, denoted $\Sigma^*(P)$, to be the poset P with two new elements a_1 and a_2. $\Sigma^*(P)$ has the following order relation: $\hat{0} <_{\Sigma^*(P)} a_i <_{\Sigma^*(P)} y$, for all $y > \hat{0}$ in P and $i = 1, 2$. That is, the elements a_1 and a_2 are inserted between $\hat{0}$ and atoms of P. Clearly if P is Eulerian then so is $\Sigma^*(P)$. Moreover, if P is a binomial poset then $\Sigma^*(P)$ is a Sheffer poset with the factorial function $D_{\Sigma^*(P)}(n) = 2B(n-1)$, for $n \geq 2$.*

Definition 2.5. *Let P be a poset with $\hat{1}$. We define the* suspension *of P, denoted by $\Sigma(P)$, to be the poset P with two new elements a_1 and a_2 adjoined with the additional order relations that $y <_{\Sigma(P)} a_i <_{\Sigma(P)} \hat{1}$, for all $y < \hat{1}$ in P and $i = 1, 2$.*

The *dual* of the poset P, denoted P^*, is defined as follows: P^* has the same set of elements as P and the following order relation: $x <_{P^*} y$ if and only if $y <_P x$.

Definition 2.6. *The* boolean lattice *B_n of rank n is the poset of subsets of $[n] = \{1, \ldots, n\}$ ordered by inclusion.*

Definition 2.7. *The* butterfly poset *T_n of rank n consists of the elements of $\{\hat{0}\} \cup (D_{n-1} \times \{1, 2\}) \cup \{\hat{1}\}$, where $D_{n-1} \times \{1, 2\}$ is the direct product of the chain of length $n-1$, denoted by D_{n-1}, and the anti-chain of rank 2, with the order relation $(k, i) \prec (k+1, j)$ for all $i, j \in \{1, 2\}$. Also $\hat{0}$ and $\hat{1}$ are the unique minimal and maximal elements of this poset, respectively. Clearly, $T_n \cong \Sigma^*(T_{n-1})$.*

3 Finite Eulerian binomial posets

In this section, we classify the structure of finite Eulerian binomial posets.

First we provide some examples of finite binomial posets. See [4] for infinite versions of Examples 3.1 and 3.2.

Example 3.1. *The boolean lattice B_n of rank n is an Eulerian binomial poset with factorial function $B(k) = k!$ and atom function $A(k) = k$, $k \leq n$. Every interval of length k of this poset is isomorphic to B_k.*

Example 3.2. *The butterfly poset T_n of rank n is an Eulerian binomial poset with factorial function $B(k) = 2^{k-1}$ for $1 \leq k \leq n$ and atom function $A(k) = 2$, for $2 \leq k \leq n$, and $A(1) = 1$.*

It is not hard to see that in any n-interval of an Eulerian binomial poset P with factorial function $B(k)$ for $1 \leq k \leq n$, the Euler-Poincaré relation is stated as follows:

$$\sum_{k=0}^{n} (-1)^k \cdot \frac{B(n)}{B(k)B(n-k)} = 0. \tag{3}$$

The following is [4, Lemma 2.6].

Lemma 3.3. *Let P be a graded poset of odd rank such that every proper interval of P is Eulerian. Then P is an Eulerian poset.*

Lemma 3.4. *Let P be an Eulerian binomial poset of rank 3. Then the poset P and its factorial function $B(n)$ satisfy the following conditions:*

(i) *$B(2) = 2$ and $B(3) = 2q$, where q is a positive integer such that $q \geq 2$.*

(ii) *There is a list of integers $q_1, \ldots, q_r$, $q_i \geq 2$, such that $P \cong \boxplus_{i=1,\ldots,r} P_{q_i}$, where P_{q_i} is the face lattice of the q_i-gon.*

This result is [4, Example 2.5].

R. Ehrenborg and M. Readdy proved the following two propositions. See [4, Lemma 2.17 and Prop. 2.15].

Proposition 3.5. *Let P be a binomial poset of rank n with factorial function $B(k) = 2^{k-1}$ for $1 \leq k \leq n$. Then the poset P is isomorphic to the butterfly poset T_n.*

Proposition 3.6. *Let P be a binomial poset of rank n with factorial function $B(k) = k!$ for $1 \leq k \leq n$. Then the poset P is isomorphic to the boolean lattice B_n of rank n.*

The following is [4, Lemma 2.12].

Lemma 3.7. *Let $P^{'}$ and P be two Eulerian binomial posets of rank $2m+2$, $m \geq 2$, having atom functions $A^{'}(n)$ and $A(n)$, respectively, which agree for $n \leq 2m$. Then the following equality holds:*

$$\frac{1}{A(2m+1)}\left(1 - \frac{1}{A(2m+2)}\right) = \frac{1}{A'(2m+1)}\left(1 - \frac{1}{A'(2m+2)}\right). \tag{4}$$

Lemma 3.8. *Every Eulerian binomial poset P of rank 4 is isomorphic to either T_4 or B_4.*

In the following theorem we obtain the structure of Eulerian binomial posets of even rank.

Theorem 3.9. *Every Eulerian binomial poset of even rank $n = 2m \geq 4$ is isomorphic to either T_n or B_n (the butterfly poset of rank n or boolean lattice of rank n).*

Theorem 3.10. *Let P be an Eulerian binomial poset of odd rank $n = 2m + 1 \geq 5$. Then the poset P satisfies one of the following conditions:*

(i) *There is a positive integer k such that P is the k-summation of the boolean lattice of rank n. In other words, $P \cong \boxplus^k(B_n)$.*

(ii) *There is a positive integer k such that P is the k-summation of the butterfly poset of rank n. In other words, $P \cong \boxplus^k(T_n)$.*

Proof. We prove the theorem for two different cases $B(3) = 4$ and $B(3) = 6$. Lemma 3.8 implies that every interval of length 4 is isomorphic either to B_4 or T_4. Thus the factorial function $B(3)$ can only take the values 4 or 6 and therefore we are in one of these two cases.

1. $B(3) = 6$. In this case we claim that there is a positive integer k such that $P \cong \boxplus^k(B_n)$. In order to show that $P \cong \boxplus^k(B_n)$, we make the following construction. We remove $\hat{1}$ and $\hat{0}$ from P. The remaining poset is a disjoint union of connected components. Consider one of the obtained connected components and add a minimal element $\hat{0}$ and a maximal element $\hat{1}$ to it. Denote the resulting poset by Q. We wish to show that $Q \cong B_n$. This implies that $P \cong \boxplus^k(B_n)$. It is not hard to see that Q is an Eulerian binomial poset. The posets P and Q have the same factorial functions and atom functions up to rank $2m$. Hence $B_Q(k) = B_P(k)$ and $A_Q(k) = A_P(k)$, for $1 \leq k \leq 2m$. Therefore, Eq. (2) implies that the number of atoms and coatoms are the same in the poset Q. Denote this number by t. Let $x_1, \ldots, x_t$ and $a_1, \ldots, a_t$ be an ordering of the atoms and coatoms of Q, respectively. Also, let $c_1, \ldots, c_l$ be the set of elements of rank $2m - 1$ in Q. We show that $t = 2m + 1$, and this implies that $Q \cong B_{2m+1}$. For each element y of rank at least 2 in Q, let $S(y)$ be the set of atoms of Q that are below y. Set $A_i := S(a_i)$ for each element a_i of rank $2m$, $1 \leq i \leq t$, and also set $C_i := S(c_i)$ for each element c_i of rank $2m - 1$, $1 \leq i \leq l$. In order to show that $Q \cong B_n$, we prove the following.

(1) We show that $|A_i \cap A_j| = 2m - 1$ for $i \neq j$.

(2) We use part (1) to show that $t = 2m + 1$.

(1) We first show that $|A_i \cap A_j| = 2m - 1$ for $i \neq j$. By considering the factorial functions, Theorem 3.9 implies that the intervals $[\hat{0}, a_i]$ and $[x_j, \hat{1}]$ have the same factorial functions as B_{2m} and so they are isomorphic to B_{2m} for

$1 \leq i \leq t$ and $1 \leq j \leq t$. We conclude that any interval $[\hat{0}, c_k]$ of rank $2m - 1$ is isomorphic to B_{2m-1}. As a consequence, we have $|A_i| = |S(a_i)| = 2m$, $1 \leq i \leq t$ and also $|C_k| = |S(c_k)| = 2m - 1$, $1 \leq k \leq l$.

If there exist i and j such that $A_i \cap A_j \neq \emptyset$, where $1 \leq i, j \leq t$, we claim that $2m - 1 \leq |A_i \cap A_j| \leq 2m$. Consider an atom $x_k \in A_i \cap A_j$, $1 \leq k \leq t$. Theorem 3.9 implies that $[x_k, \hat{1}] \cong B_{2m}$. Thus, by considering properties of boolean lattices, there is an element c_h of rank $2m - 2$ in this interval which is covered by a_i and a_j, $1 \leq h \leq l$. Notice that c_h is an element of rank $2m-1$ in Q. Therefore, $|C_h| = 2m-1 \leq |A_i \cap A_j| \leq |A_i| = |S(a_i)| = 2m$.

We claim that for all distinct pairs i and j, $1 \leq i, j \leq t$, we have $A_i \cap A_j \neq \emptyset$. In order to show this claim, associate the graph G_Q to the poset Q as follows: $A_1, \ldots, A_t$ are vertices of this graph, and we connect vertices A_i and A_j if and only if $A_i \cap A_j \neq \emptyset$.

We will show that G_Q is a complete graph and so $|A_i \cap A_j| \neq 0$ for all $i \neq j$. Since $Q - \{\hat{0}, \hat{1}\}$ is connected, G_Q is also a connected graph. We show that if $\{A_i, A_j\}$ and $\{A_j, A_k\}$ are different edges of G_Q, $\{A_i, A_k\}$ is also an edge of G_Q. Since $\{A_i, A_j\}$ and $\{A_j, A_k\}$ are edges of G_Q, we have $|A_i \cap A_j| \geq 2m - 1$ as well as $|A_j \cap A_k| \geq 2m - 1$. On the other hand, since $|A_i| = |A_j| = |A_k| = 2m$, we conclude that $A_i \cap A_k \neq \emptyset$. Therefore $\{A_i, A_k\}$ is also an edge of G_Q. As a consequence, the connected graph G_Q is a complete graph. Thus $A_i \cap A_j \neq \emptyset$ and also $2m - 1 \leq |A_i \cap A_j| \leq 2m$ for $1 \leq i, j \leq t$ and $i \neq j$.

Now, we show that $|A_i \cap A_j| = 2m - 1$ for all $i \neq j$. We proceed by contradiction. Suppose this claim does not hold. Then there are different i and j such that $|A_i \cap A_j| = 2m$. We claim that in the case $|A_i \cap A_j| = 2m$, there are two elements of rank $2m - 1$ in Q such that they both are covered by coatoms a_i and a_j. To show this claim, consider an atom $x_f \in A_i \cap A_j$, so we have $[x_f, \hat{1}] \cong B_{2m}$. Hence, there is a unique element c_h of rank $2m-2$ in this interval which is covered by both a_i and a_j. By induction on m, Lemma 3.4, and the property that $|C_h| \leq |A_i \cap A_j| = 2m$, we conclude that $[\hat{0}, c_h]$ is isomorphic to B_{2m-1} and so $|C_h| = 2m - 1$. Therefore there is an atom $x_d \in A_i \cap A_j \setminus C_h$. Since the interval $[x_d, \hat{1}]$ is isomorphic to B_{2m}, there is an element $c_k \neq c_h$ of rank $2m - 1$ which is covered by coatoms a_i and a_j .

Since $|C_h| = |S(c_h)| = |C_k| = |S(c_k)| = 2m-1$ and C_k and C_h are both subsets of $A_i \cap A_j$, we conclude that there should be an atom $x_s \in C_k \cap C_h$. Therefore the interval $[x_s, \hat{1}]$ has two elements c_k and c_h of rank $2m - 2$ such that they both are covered by two elements a_i and a_j of rank $2m - 1$ in the interval $[x_s, \hat{1}]$. We know $[x_s, \hat{1}] \cong B_{2m}$ and there are no two elements of rank $2m - 2$ covered by two elements of rank $2m - 1$ in B_{2m}. This contradicts our assumption, and so $|A_i \cap A_j| = 2m - 1$ for pairs i and j of distinct elements.

In summary, we have:

(a) $|A_i| = 2m$ for $1 \leq i \leq t$,

(b) $|A_i \cap A_j| = 2m-1$ for all $1 \leq i < j \leq t$,

(c) $\bigcup_{i=1}^{t} A_i = \{x_1, \ldots, x_t\}$.

As a consequence, we have $t > 2m$.

(2) Now, we show that $t = 2m+1$. We are going to show that $t = 2m+1$. Without loss of generality, consider the three different sets $A_1 = S(a_1), A_2 = S(a_2)$ and $A_3 = S(a_3)$ associated with the three coatoms a_1, a_2 and a_3. We know that $|A_1| = |A_2| = |A_3| = 2m$ and $|A_1 \cap A_2| = |A_2 \cap A_3| = |A_1 \cap A_3| = 2m-1$. Without loss of generality, let us assume that $A_1 = \{x_1, x_2, \ldots, x_{2m-1}, y_1\}$ and $A_2 = \{x_1, x_2, \ldots, x_{2m-1}, y_2\}$ where $y_i \neq x_1, \ldots, x_{2m-1}$ for $i = 1, 2$. We have the following two different cases:

(a) A_3 does not contain y_1 and y_2.

(b) A_3 contains at least one of y_1 and y_2.

First we study the case, $A_3 = \{x_1, x_2, \ldots, x_{2m-1}, y_3\}$ where $y_3 \notin \{y_1, y_2, x_1, \ldots, x_{2m-1}\}$. Considering the $t-3$ other coatoms a_k, $4 \leq k \leq t$, there are different atoms y_k, $4 \leq k \leq t$, such that $y_k \notin \{y_1, y_2, y_3, x_1, \ldots, x_{2m-1}\}$ and $A_k = S(a_k) = \{x_1, x_2, \ldots, x_{2m-1}, y_k\}$. This implies that the number of atoms is $|\bigcup_{i=1}^{t} A_i| = t+2m-1$, which is a contradiction. So it must be the case that A_3 contains one of y_1 or y_2. In this case $|A_2 \cap A_3| = |A_1 \cap A_3| = 2m-1$ implies that $A_3 = \{x_1, x_2, \ldots, x_{2m-1}, y_1, y_2\} \setminus \{x_j\} \subset A_1 \cup A_2$ for some x_j. Since A_3 was chosen arbitrarily, it follows that for each A_k we have $A_k \subset A_1 \cup A_2$.
Therefore,

$$\bigcup_{i=1}^{t} A_k = \{x_1, \ldots, x_{2m-1}, y_1, y_2\}. \tag{5}$$

Thus the number of coatoms in the poset Q is $t = 2m+1$.
By Theorem 3.9, $B_Q(k) = k!$ for $1 \leq k \leq 2m$, therefore $B_Q(2m+1) = (2m+1)!$. By Proposition 3.6, Q is isomorphic to B_{2m+1} and so P is a union of copies of B_{2m+1} with their minimal elements and maximal elements identified. In other words, $P \cong \boxplus^k(B_{2m+1})$. It can be seen that P is binomial and Eulerian and the proof follows.

(ii) $B(3) = 4$. With the same argument as part (i), we remove $\hat{1}$ and $\hat{0}$ from P. The remaining poset is a disjoint union of connected components. We add a minimal element $\hat{0}$ and a maximal element $\hat{1}$ to each of these connected components. We show that the obtained posets are isomorphic to T_n. This implies that $P \cong \boxplus^k(T_n)$.

We construct the binomial poset Q by adding $\hat{1}$ and $\hat{0}$ to one of the connected components of $P - \{\hat{0}, \hat{1}\}$. We claim that Q is isomorphic to T_{2m+1}. Similar to part (i), let $a_1, \ldots, a_t$ and $x_1, \ldots, x_t$ denote coatoms and atoms of Q. We show that $t = 2$ which implies $Q \cong T_{2m+1}$.

Set $A_i = S(a_i)$. By Theorem 3.9, we have $|A_i| = 2$. It is easy to see that $\bigcup_{i=1}^{t} A_i = \{x_1, \ldots, x_t\}$. Define G_Q to be the graph with vertices $x_1, \ldots, x_t$ and edges $A_1, \ldots, A_t$. Since $Q \setminus \{\hat{0}, \hat{1}\}$ is connected, G_Q is also a connected graph. Since $[x_i, \hat{1}] \cong T_{2m}$, the degree of each vertex of G_Q is 2 and G_Q is the cycle of length t. Therefore if $t > 2$, we have $|A_i \cap A_j| = 1$ or 0, $1 \leq i < j \leq t$.

We claim that $t = 2$. Suppose this claim does not hold and $t > 2$. Consider an element c of rank 3 in Q. Lemma 3.4 and Theorem 3.9 imply that both intervals $[\hat{0}, c]$ and $[c, \hat{1}]$ are isomorphic to butterfly posets. Hence there are two coatoms above c, say a_k and a_l, and similarly there are two atoms below c, say x_h and x_s. Therefore, we have $A_k = A_l = \{x_h, x_s\}$. This is not possible when $t > 2$. As a consequence, $t = 2$ and all the A_i's have two elements and $|\bigcup_1^t A_i| = |\{x_1, \ldots, x_t\}| = 2 = t$.

Similar to part (i), $B_Q(k) = 2^{k-1}$ for $1 \leq k \leq 2m+1$. By Proposition 3.5, we conclude that Q is isomorphic to T_{2m+1}. Therefore, there is an integer $k > 0$ such that $P \cong \boxplus^k(T_n)$.

□

4 Finite Eulerian Sheffer Posets

In this section, we give an almost complete classification of the factorial functions and the structure of Eulerian Sheffer posets. We study Eulerian Sheffer posets of ranks $n = 3$ and 4 in Lemmas 3.4 and 4.2. By these two lemmas, we reduce the set of possible values of $B(3)$ to 4 or 6. In Section 4.1, Lemma 4.3 and Theorems 4.4, 4.9, 4.10 and 4.11 deal with Eulerian Sheffer posets with $B(3) = 6$. Finally in Section 4.2, Theorem 4.12 deals with Eulerian Sheffer posets with $B(3) = 4$.

It is clear that every binomial poset is also a Sheffer poset. Here is an other example of Sheffer posets, some of which appear in [4] and [9].

Example 4.1. *Let T be the poset with the elements $\hat{0}_1, \hat{0}_2, \hat{1}$ and the cover relations $\hat{0}_1 < \hat{1}$ and $\hat{0}_2 < \hat{1}$. Let T^n be the Cartesian product of n copies of the poset T. The poset $C_n = T^n \cup \{\hat{0}\}$ denotes the face lattice of an n-dimensional cube, also known as the* cubical lattice*. The cubical lattice is a Sheffer poset with $B(k) = k!$ for $1 \leq k \leq n$ and $D(k) = 2^{k-1}(k-1)!$ for $1 \leq k \leq n+1$.*

It is not hard to see that Lemma 3.4 also characterize the structure of Eulerian Sheffer posets of rank 3. Lemma 4.2 deals with Eulerian Sheffer posets of rank 4.

Lemma 4.2. *Let poset P be an Eulerian Sheffer poset of rank 4. Then one of the following conditions hold.*

1. $B(3) = 2b$, $D(3) = 4$, $D(4) = 4b$, *where* $b \geq 2$.
2. $B(3) = 8$, $D(3) = 3!$, $D(4) = 2^3 \cdot 3!$.
3. $B(3) = 10$, $D(3) = 3!$, $D(4) = 5!$.
4. $B(3) = 4$, $D(3) = 3!$, $D(4) = 2 \cdot 3!$.
5. $B(3) = 3!$, $D(3) = 3!$, $D(4) = 4!$.
6. $B(3) = 3!$, $D(3) = 4$, $D(4) = 2 \cdot 3!$.
7. $B(3) = 3!$, $D(3) = 10$, $D(4) = 5!$.
8. $B(3) = 3!$, $D(3) = 8$, $D(4) = 2^3 \cdot 3!$.
9. $B(3) = 4$, $D(3) = 2b$, $D(4) = 4b$ *where* $b \geq 2$.

4.1 Characterization of the factorial functions and structure of Eulerian Sheffer posets of rank $n \geq 5$ for which $B(3) = 3!$.

In this subsection. we mainly consider Eulerian Sheffer posets with $B(3) = 3!$. As a consequence of Lemma 4.2, we know that Eulerian Sheffer posets of rank $n \geq 4$ with $B(3) = 3!$ have the Sheffer factorial functions $D(3) = 4, 6, 8$ and 10. Lemma 4.3 shows that for any such poset of rank $n \geq 6$, the Sheffer factorial function $D(3)$ can only take the values $4, 6$ or 8.

In Subsections 4.1.1, 4.1.2 and 4.1.3, we consider posets with $B(3) = 6$ and different cases $D(3) = 4, 6$ and 8, respectively. The question of studying the finite Eulerian Sheffer posets of rank 5 with $B(3) = 6$ and $D(3) = 10$ remains open. There is such a poset, namely the face lattice of the 120-cell with Schläfli symbol $\{5, 3, 3\}$.

Lemma 4.3. *Let P be an Eulerian Sheffer poset of rank $n \geq 6$ with $B(3) = 3!$. Then $D(3)$ can take only the values $4, 6, 8$.*

4.1.1 Characterization of the factorial functions of Eulerian Sheffer posets of rank $n \geq 5$ for which $B(3) = 3!$ and $D(3) = 8$.

In this subsection, we study the factorial functions of Eulerian Sheffer posets of rank $n \geq 5$ for which $B(3) = 3!$ and $D(3) = 8$. Theorem 4.4 characterizes the factorial functions of such posets of even rank. However, the question of characterizing the factorial functions of Eulerian Sheffer posets of odd rank $n = 2m + 1 \geq 5$ with $B(3) = 3!$ and $D(3) = 8$ remains open.

Theorem 4.4. *Let P be an Eulerian Sheffer poset of even rank $n = 2m + 2 \geq 4$ with $B(3) = 3!$ and $D(3) = 8$. Then P has the same factorial functions as C_n, the cubical lattice of rank n, that is, $D(k) = 2^{k-1}(k-1)!$, $1 \leq k \leq n$ and $B(k) = k!$, $1 \leq k \leq n - 1$.*

In order to prove Theorem 4.4, we establish the following two lemmas.

Lemma 4.5. *Let Q be an Eulerian Sheffer poset of odd rank $2m + 1$, $m \geq 2$, with $B(3) = 3!$. Then the coatom function of Q must satisfy at least one of the following inequalities: $C(n) \neq 2(n - 1)$ for $2 \leq n \leq 2m$ and $C(2m + 1) \neq 4m + 1$.*

Lemma 4.5 implies the following.

Corollary 4.6. *Let P be an Eulerian Sheffer poset of rank $2m + 2$, $m \geq 2$, with $B(k) = k!$, for $1 \leq k \leq 2m$. Then the coatom function of P must satisfy at least one of the following inequalities: $C(n) \neq 2(n - 1)$, $2 \leq n \leq 2m$, $C(2m + 1) \neq 4m + 1$ and $C(2m + 2) \neq 4(2m + 1)$.*

Lemma 4.7. *Let Q be an Eulerian Sheffer poset of rank $2m+2$, $m \geq 2$, with $B(k) = k!$ for $1 \leq k \leq 2m$. Then the coatom function of Q must satisfy at least one of the following inequalities: $C(n) \neq 2(n - 1)$, $2 \leq n \leq 2m$, $C(2m + 1) \neq 4m - 1$ and $C(2m + 2) \neq \frac{4}{3}(2m + 1)$.*

The following lemma can be obtained by applying the proof of Lemma 4.8 in [4].

Lemma 4.8. *Let P and $P^{'}$ be two Eulerian Sheffer posets of rank $2m + 2$, $m \geq 2$, such that their binomial factorial functions and coatom functions agree up to rank $n \leq 2m$. That is, $B(n) = B^{'}(n)$ and $C(n) = C^{'}(n)$, where $m \geq 2$. Then the following equation holds:*

$$\frac{1}{C(2m+1)}\left(1 - \frac{1}{C(2m+2)}\right) = \frac{1}{C'(2m+1)}\left(1 - \frac{1}{C'(2m+2)}\right). \tag{6}$$

of Theorem 4.4. In order to prove the theorem, we inductively show that the Eulerian Sheffer poset P and C_{2m+2}, the cubical latice of rank $2m+2$, have the same coatom functions.

Let $C(k)$ and $C^{'}(k) = 2(k-1)$ respectively be the coatom functions of the Eulerian Sheffer poset P and C_{2m+2} for $2 \leq k \leq 2m+2$. We only need to show that $C(k) = C^{'}(k) = 2(k-1)$ for $2 \leq k \leq 2m+2$. We prove this claim by induction on m. By Lemma 4.2, an Eulerian Sheffer poset of even rank 4 with $B(3) = 3!$ and $D(3) = 8$ has the same factorial function as C_4. Therefore, $C(4) = C^{'}(4) = 6$ and the claim holds for $m = 1$. Suppose $m \geq 2$. By the induction hypothesis $C(k) = C^{'}(k) = 2(k-1)$ for $2 \leq k \leq 2m$. Set $F = C(2m+1)$ and $E = C(2m+2)$. Theorem 3.10 implies that $B(k) = k!$ for $1 \leq k \leq 2m$ and there is a positive integer α such that $B(2m+1) = \alpha(2m+1)!$. We know that $D(k) = 2^{k-1}(k-1)!$ for $1 \leq k \leq 2m$, so $D(2m+1) = F2^{2m-1}(2m-1)!$ and $D(2m+2) = EF2^{2m-1}(2m-1)!$. Since P is an Eulerian Sheffer poset, the Euler-Poincaré relation implies that

$$1 + \sum_{k=1}^{2m+2} \frac{(-1)^k D(2m+2)}{D(k)B(2m+2-k)} = 0. \tag{7}$$

By substituting the values of the factorial functions, we have

$$2 - E + \frac{EF}{2}\left[\frac{1}{2m} - \frac{1}{2m(2m+1)} + \frac{2^{2m}}{2m(2m+1)} - \frac{2^{2m}}{2\alpha m(2m+1)}\right] = 0. \tag{8}$$

Thus,

$$E\left(1 - F\left(\frac{2\alpha m + (\alpha-1)2^{2m}}{4\alpha m(2m+1)}\right)\right) = 2. \tag{9}$$

In case $\alpha \geq 2$, it is easy to verify that

$$\left(\frac{2\alpha m + (\alpha-1)2^{2m}}{4\alpha m(2m+1)}\right) > \frac{1}{2m}. \tag{10}$$

Since $F \geq A(2m) \geq 2m$, the left-hand side of Eq. (9) becomes negative in this case. Therefore, $\alpha = 1$ and the posets P and C_{2m+2} have the same binomial factorial functions. Since $2m+1 = A(2m+1) \leq C(2m+2) < \infty$, Lemma 4.8 implies that $4m-1 \leq C(2m+1) = F \leq 4m+1$. Since $\alpha = 1$, Eq. (9) implies that $2 - E + \frac{EF}{4m+2} = 0$. Thus E and F must satisfy one of the following cases:

(1) $F = 4m-1$ and $E = \frac{4}{3}(2m+1)$.

(2) $F = 4m$ and $E = 4m+2$.

(3) $F = 4m+1$ and $E = 4(2m+1)$.

As we have discussed in Corollary 4.6 and Lemma 4.7, the cases (1) and (3) are not possible. Case (2) occurs in the cubical lattice of rank $2m+2$, C_{2m+2}. Thus, the poset P has the same factorial functions as C_{2m+2}, as desired. □

Classification of the factorial functions of Eulerian Sheffer posets of odd rank $n = 2m+1 \geq 5$ with $B(3) = 6$ and $D(3) = 8$ remains open. Let α be a positive integer and set $Q_\alpha = \boxplus^\alpha(C_{2m+1})$. It can be seen that Q_α is an Eulerian Sheffer poset and it has the following factorial functions: $D(k) = 2^{k-1}(k-1)!$

for $1 \leq k \leq n-1$, $D(n) = \alpha \cdot 2^{n-1}(n-1)!$ and $B(k) = k!$ for $1 \leq k \leq n-1$. We ask the following question:

Question: Let P be an Eulerian Sheffer poset of odd rank $n = 2m+1 \geq 5$ with $B(3) = 6, D(3) = 8$. Is there a positive integer α such that the poset P has the same factorial functions as poset $Q_\alpha = \boxplus^\alpha(C_{2m+1})$, where C_{2m+1} is a cubical lattice of rank $2m+1$?

4.1.2 Characterization of the structure of Eulerian Sheffer posets of rank $n \geq 5$ for which $B(3) = 3!$, and $D(3) = 3! = 6$.

Theorem 4.9. *Let P be an Eulerian Sheffer poset of rank $n \geq 3$ with $B(3) = D(3) = 3! = 6$ for 3-intervals. P satisfies one of the following cases:*

(i) *There is an integer $k \geq 1$ such that $P \cong \boxplus^k(B_n)$, where n is odd.*

(ii) *$P \cong B_n$, where n is even.*

4.1.3 Characterization of the structure of Eulerian Sheffer posets of rank $n \geq 5$ for which $B(3) = 3!$ and $D(3) = 4$.

Let P be an Eulerian Sheffer poset of rank $n \geq 5$, with $B(3) = 3!$ and $D(3) = 4$. In this section we show that in the case $n = 2m+2$ the poset P satisfies $P \cong \Sigma^*(\boxplus^\alpha(B_{2m+1}))$ for some integer $\alpha \geq 1$ and in the case $n = 2m+1$, $P \cong \boxplus^\alpha(\Sigma^*(B_{2m}))$, for some integer $\alpha \geq 1$.

Theorem 4.10. *Let P be an Eulerian Sheffer poset of even rank $n = 2m+2 \geq 4$ with $B(3) = 3!$ and $D(3) = 4$. Then $P \cong \Sigma^*(\boxplus^\alpha(B_{2m+1}))$, where $\alpha = \frac{B(2m+1)}{(2m+1)!}$ is a positive integer for $n \geq 6$ and $\alpha = 1$ for $n = 4$. Consequently the poset P has the following binomial and Sheffer factorial functions.*

(i) $B(k) = k!$ *for* $1 \leq k \leq 2m$, *and* $B(2m+1) = \alpha(2m+1)!$,

(ii) $D(1) = 1$, $D(k) = 2(k-1)!$ *for* $2 \leq k \leq 2m+1$, *and* $D(2m+2) = 2\alpha(2m+1)!$.

Theorem 4.11. *Let P be an Eulerian Sheffer poset of odd rank $n = 2m+1 \geq 5$ with $B(3) = 6$ and $D(3) = 4$. Then $P \cong \boxplus^\alpha(\Sigma^*(B_{2m}))$ for some positive integer α.*

4.2 Characterization of the structure and factorial functions of Eulerian Sheffer posets of rank $n \geq 5$ with $B(3) = 4$.

In this section, we characterize Eulerian Sheffer posets of rank $n \geq 5$ with $B(3) = 4$. Let P be an Eulerian Sheffer poset of rank $n \geq 5$ with $B(3) = 4$. It can be seen that the poset P satisfies one of the cases:

1. P has the following binomial factorial function $B(k) = 2^{k-1}$, where $1 \leq k \leq n-1$;
2. n is even and there is a positive integer $\alpha > 1$ such that poset P has the binomial factorial function $B(k) = 2^{k-1}$ for $1 \leq k \leq n-2$ and $B(n-1) = \alpha \cdot 2^{n-2}$ for some positive integer α.

As a consequence of Theorems 3.11 and 3.12 in [4], we can characterize posets in the case (i). Theorem 4.12 deals with the case (ii).

Theorem 4.12. *Let P be an Eulerian Sheffer poset of even rank $n = 2m+2 > 4$ with the binomial factorial function $B(k) = 2^{k-1}$ for $1 \leq k \leq 2m$, and $B(2m+1) = \alpha \cdot 2^{2m}$, where $\alpha > 1$ is a positive integer. Then $P \cong \Sigma^*(\boxplus^\alpha(T_{2m+1}))$.*

References

[1] J. Backelin, Binomial posets with non-isomorphic intervals, math.CO/0508397, 22 August, 2005.

[2] A. Björner, V. Welker, Segre and Rees products of posets, with ring-theoretic applications, J. Pure Appl. Algebra 198 (2005) 43–55.

[3] P. Doubilet, G.-C.Rota, R. Stanley, On the foundations of combinatorial theory (VI). The idea of generating functions, in Sixth Berkeley Symp. on Math. Stat. and Prob., vol. 2: Probability Theory, Univ. of California, Berkeley, 1972, pp. 267–318.

[4] R. Ehrenborg and M. Readdy, Classification of the factorial functions of Eulerian binomial and Sheffer posets, J. Combin. Theory Ser. A 114 (2007) 339–359.

[5] R. Ehrenborg and M. Readdy, Sheffer posets and r-signed permutations, Ann. Sci. Math. Québec 19 (1995) 173–196.

[6] J. Farley and S. Schmidt, Posets that locally resemble distributive lattices, J. Combin. Theory Ser. A 92 (2000) 119–137.

[7] D.J. Grabiner, Posets in which every interval is a product of chains, and natural local actions of the symmetric group, Discrete Math. 199 (1999) 77–84.

[8] G. Hetyei, Matrices of formal power series associated to binomial posets. J. Algebraic Combin. 22 (2005) 65–104.

[9] V. Reiner, Upper binomial posets and signed permutation statistics, European J. Combin. 14 (1993) 581–588.

[10] R. Simion, R. Stanley, Flag-symmetry of the poset of shuffles and a local action of the symmetric group, Discrete Math. 204 (1999) 369–396.

[11] R. Stanley, Binomial posets, Möbius inversion, and permutation enumeration, J. Combin. Theory Ser. A 20 (1976) 336–356.

[12] R. Stanley, Enumerative Combinatorics, vol. I, Wadsworth and Brooks/Cole, Pacific Grove, 1986.

[13] R. Stanley, Flag-symmetric and locally rank-symmetric partially ordered sets, Electron. J. Combin. 3 (R6) (1996) 22.