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A topological interpretation of the cyclotomic polynomial

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Abstract. We interpret the coefficients of the cyclotomic polynomial in terms of simplicial homology.

Résumé. Nous donnons une interprétation des coefficients du polynôme cyclotomique en utilisant l'homologie simpliciale.

Keywords: Cyclotomic polynomial, higher-dimensional tree, matroid duality, oriented matroid, simplicial matroid, simplicial homology

1 Introduction

This paper studies the cyclotomic polynomial $\Phi_n(x)$, which is defined as the minimal polynomial over $\mathbb{Q}$ for any primitive n^{th} root of unity ζ in $\mathbb{C}$. It is monic, irreducible, and has degree given by the Euler phi function $\phi(n)$, with formula

$$\Phi_n(x) = \prod_{j \in (\mathbb{Z}/n\mathbb{Z})^\times} (x - \zeta^j).$$

The equation $x^n - 1 = \prod_{d|n} \Phi_d(x)$ gives a recurrence showing that all coefficients of $\Phi_n(x)$ lie in $\mathbb{Z}$.

Although well-studied, the coefficients of $\Phi_n(x)$ are mysterious [2, 10, 11, 14, 15, 17, 29]. We offer here two interpretations for their magnitudes, as orders of cyclic groups. In the first interpretation (Corollary 5 below) this group is a quotient of the free abelian group $\mathbb{Z}[\zeta]$ by a certain full rank sublattice.

The second interpretation is topological, given by Theorem 1 below, as the torsion in the homology of a certain simplicial complex associated with a squarefree integer $n = p_1 \cdots p_d$. These simplicial complexes originally arose in the work of Bolker [6], reappeared in the work of Kalai [13] and Adin [1] on higher-dimensional matrix-tree theorems, and were shown to be connected with cyclotomic extensions in work of J. Martin and the second author [18]. We review these simplicial complexes briefly here in order to state the result; see Section 4 for more details.

Given a positive integer p, let K_p denote a 0-dimensional abstract simplicial complex having p vertices(i), which we will label by the residues

$$\{0 \bmod p,\ 1 \bmod p,\ \ldots,\ (p-1) \bmod p\}$$

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(i) Note that here K_p does *not* refer to a complete graph on p vertices; we hope that this causes no confusion.

for reasons that will become clear in a moment. Given primes $p_1, \dots, p_d$, let

$$K_{p_1,\dots,p_d} := K_{p_1} * \cdots * K_{p_d}$$

be the *simplicial join*, [21, §62], of $K_{p_1}, \dots, K_{p_d}$. This is a pure $(d-1)$-dimensional abstract simplicial complex, that may be thought of as the *complete d-partite complex* on vertex sets K_{p_1} through K_{p_d} of sizes $p_1, \dots, p_d$. The *facets* (maximal simplices) of $K_{p_1,\dots,p_d}$ are labelled by sequences of residues $(j_1 \bmod p_1, \dots, j_d \bmod p_d)$. Denoting the squarefree product $p_1 \cdots p_d$ by n, the Chinese Remainder Theorem isomorphism

$$\mathbb{Z}/p_1\mathbb{Z} \ \times \cdots \times \ \mathbb{Z}/p_d\mathbb{Z} \ \xrightarrow{\Xi} \ \mathbb{Z}/n\mathbb{Z} \tag{1}$$

allows one to label such a facet by a residue $j \bmod n$; call this facet $F_{j \bmod n}$. Then for any subset $A \subseteq \{0, 1, \dots, \phi(n)\}$, let K_A denote the subcomplex of $K_{p_1,\dots,p_d}$ which is generated by the facets $\{F_{j \bmod n}\}$ as j runs through the following set of residues:

$$A \cup \{\phi(n)+1,\ \phi(n)+2,\ \dots,\ n-1,\ n\}.$$

Our first main result interprets the magnitudes of the coefficients of $\Phi_n(x)$. Let $\tilde{H}_i(-;\mathbb{Z})$ denote reduced simplicial homology with coefficients in $\mathbb{Z}$.

Theorem 1 *For a squarefree positive integer $n = p_1 \cdots p_d$, with cyclotomic polynomial $\Phi_n(x) = \sum_{j=0}^{\phi(n)} c_j x^j$, one has*

$$\tilde{H}_i(K_{\{j\}};\mathbb{Z}) = \begin{cases} \mathbb{Z}/c_j\mathbb{Z} & \text{if } i = d-2, \\ \mathbb{Z} & \text{if both } i = d-1 \text{ and } c_j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

We furthermore interpret topologically the signs of the coefficients in $\Phi_n(x)$. For this, we use oriented simplicial homology, and orient the facet $F_{j \bmod n}$ having $j \equiv j_i \bmod p_i$ for $i = 1, 2, \dots, d$ as

$$[F_j] = [F_{j \bmod n}] = [j_1 \bmod p_1, \dots j_d \bmod p_d].$$

Theorem 2 *Fix a squarefree positive integer $n = p_1 \cdots p_d$ with cyclotomic polynomial $\Phi_n(x) = \sum_{j=0}^{\phi(n)} c_j x^j$. Then for any $j \neq j'$ such that $c_j, c_{j'} \neq 0$, one has $\tilde{H}_{d-1}(K_{\{j,j'\}};\mathbb{Z}) \cong \mathbb{Z}$, and any nonzero $(d-1)$-cycle $z = \sum_\ell b_\ell [F_\ell]$ in this homology group will have $b_j, b_{j'} \neq 0$, with*

$$\frac{c_j}{c_{j'}} = -\frac{b_{j'}}{b_j}.$$

In particular, $c_j, c_{j'}$ have the same sign if and only if $b_j, b_{j'}$ have opposite signs.

Example 3 We illustrate these theorems for $n = 15$. Here $d = 2, p_1 = 3, p_2 = 5$, and $\phi(n) = 2 \cdot 4 = 8$. The cyclotomic polynomial is

$$\begin{aligned}\Phi_{15}(x) &= 1 - x + x^3 - x^4 + x^5 - x^7 + x^8 \\ &= (+1) \cdot (x^0 + x^3 + x^5 + x^8) + (-1) \cdot (x^1 + x^4 + x^7) + 0 \cdot (x^2 + x^6).\end{aligned}$$

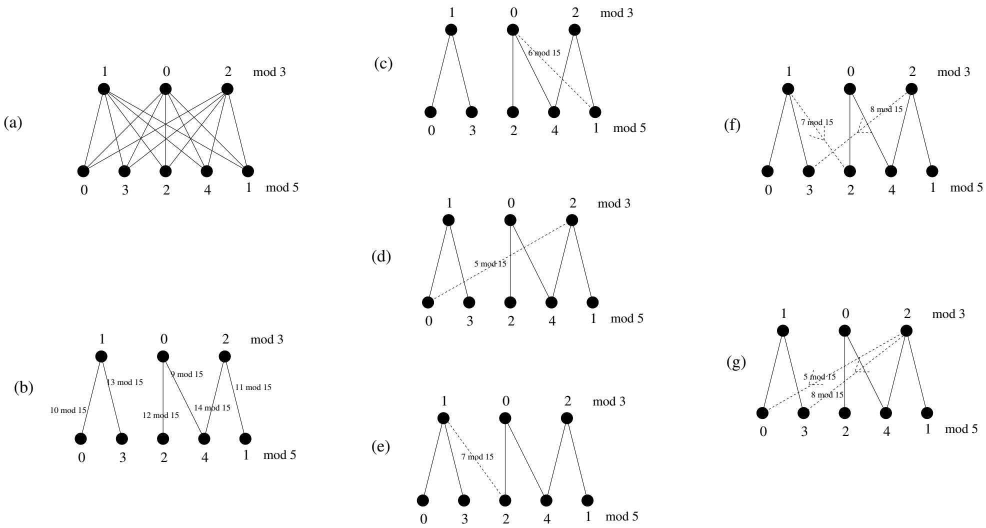

Fig. 1: The case of $\Phi_{15}(x)$

The complex $K_{p_1,p_2} = K_{3,5}$ is a complete bipartite graph with vertex sets labelled as in Figure 1(a). The subcomplex $K_\varnothing$ generated by the edges $F_{j \bmod 15}$ with $j \in \{\phi(n)+1, \phi(n)+2, \ldots, n-1\} = \{9, 10, 11, 12, 13, 14\}$ is the subgraph shown in Figure 1(b).

To see why the coefficient $c_6 = 0$ in $\Phi_{15}(x)$, one adds the edge $F_{6 \bmod 15}$ to the graph $K_\varnothing$, obtaining the graph $K_{\{6\}}$, shown in Figure 1(c), which has

$$\tilde{H}_0(K_{\{6\}};\mathbb{Z}) = \mathbb{Z} = \mathbb{Z}/0\mathbb{Z}$$
$$\tilde{H}_1(K_{\{6\}};\mathbb{Z}) = \mathbb{Z}.$$

To see why the coefficients $c_5 = +1$ or $c_7 = -1$ have magnitude 1, one adds the edge $F_{5 \bmod 15}$ or $F_{7 \bmod 15}$ to the graph $K_\varnothing$, obtaining the graphs $K_{\{5\}}$ or $K_{\{7\}}$ shown in Figures 1(d) and 1(e), which have

$$\begin{array}{rcccl} \tilde{H}_0(K_{\{5\}};\mathbb{Z}) & = & 0 & = & \mathbb{Z}/(+1)\mathbb{Z} \\ \tilde{H}_0(K_{\{7\}};\mathbb{Z}) & = & 0 & = & \mathbb{Z}/(-1)\mathbb{Z}. \end{array}$$

To understand the signs of the coefficients, note first that, by convention, $\Phi_{15}(x)$ is monic, so the coefficient $c_8 = c_{\phi(n)} = +1$. Therefore any other coefficient c_j should have sign

$$\operatorname{sgn}(c_j) = \frac{\operatorname{sgn}(c_j)}{\operatorname{sgn}(c_8)} = -\frac{\operatorname{sgn}(b_8)}{\operatorname{sgn}(b_j)}$$

where $z = \sum_i b_i[F_i]$ is a nontrivial cycle in $K_{\{j,8\}}$, in which the edge $[F_j]$ is directed from the vertex $(j_1 \bmod 3)$ toward the vertex $(j_2 \bmod 5)$. As shown in Figures 1(f) and 1(g), the nontrivial cycle in $K_{\{7,8\}}$ has $[F_7], [F_8]$ oriented in the *same* direction, explaining why $c_7 = -1$, while the nontrivial cycle in $K_{\{5,8\}}$ has $[F_5], [F_8]$ oriented in the *opposite* direction, explaining why $c_5 = +1$.

The remainder of the paper is structured as follows. Section 2 describes our first interpretation for the cyclotomic polynomial, which applies much more generally to any monic polynomial in $\mathbb{Z}[x]$. Section 3 reviews some facts, underlying the main results, about duality of matroids, Plücker coordinates, and oriented matroids. Section 4 recalls results and establishes terminology on Kalai's higher dimensional spanning trees in a simplicial complex. Section 5 discusses further properties of the simplicial complex $K_{p_1,\ldots,p_d}$ whose subcomplexes appear in Theorem 1 and 2. Section 6 proves these theorems. We end with Section 7, where we discuss known properties of $\Phi_n(x)$ that manifest themselves topologically.

2 Coefficients of monic polynomials in $\mathbb{Z}[x]$

Our goal here is the first interpretation for the coefficients of $\Phi_n(x)$, which applies more generally to the coefficients of *any* monic polynomial $f(x)$ in $\mathbb{Z}[x]$. Recall that when $f(x)$ is of degree r, one has an isomorphism of $\mathbb{Z}$-modules

$$\begin{array}{rcl} \mathbb{Z}^r & \longrightarrow & \mathbb{Z}[x]/(f(x)) \\ (a_0, a_1, \ldots, a_{r-1}) & \longmapsto & \displaystyle\sum_{j=0}^{r-1} a_j \overline{x}^j. \end{array}$$

As notation, given a subset A of some abelian group, let $\mathbb{Z}A$ denote the collection of all $\mathbb{Z}$-linear combinations of elements of A.

Proposition 4 *For a monic polynomial $f(x) = \sum_{j=0}^{r} c_j x^j$ of degree r in $\mathbb{Z}[x]$, one has an isomorphism of abelian groups*

$$\big(\mathbb{Z}[x]/(f)\big) \,/\, \mathbb{Z}A \quad \cong \quad \mathbb{Z}/c_j\mathbb{Z}$$

where A is the subset of size r given as $\{\overline{1}, \overline{x}, \overline{x^2}, \ldots, \overline{x^r}\} \setminus \{\overline{x^j}\}$.

Proof: Consider the matrix in $\mathbb{Z}^{r\times(r+1)}$

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & -c_0 \\ 0 & 1 & 0 & \cdots & 0 & -c_1 \\ 0 & 0 & 1 & \cdots & 0 & -c_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -c_{r-1} \end{bmatrix}$$

whose columns express the elements of $\{\overline{1}, \overline{x}, \overline{x^2}, \ldots, \overline{x^r}\}$ uniquely in the $\mathbb{Z}$-basis $\{\overline{1}, \overline{x}, \overline{x^2}, \ldots, \overline{x^{r-1}}\}$ for $\mathbb{Z}[x]/(f)$. The $r \times r$ submatrix obtained by restricting this matrix to the columns indexed by A is equivalent by row and column permutations to a upper triangular matrix with diagonal entries $(1, 1, \ldots, 1, -c_j)$. Hence $\big(\mathbb{Z}[x]/(f)\big) \,/\, \mathbb{Z}A \cong \mathbb{Z}/c_j\mathbb{Z}$. $\square$

The special case where $f(x)$ is the cyclotomic polynomial $\Phi_n(x)$ leads to the following considerations. Fix once and for all a primitive n^{th} root of unity ζ.

Corollary 5 *The cyclotomic polynomial $\Phi_n(x) = \sum_{j=0}^{\phi(n)} c_j x^j$ has*

$$\mathbb{Z}[\zeta]/\mathbb{Z}A \cong \mathbb{Z}/c_j\mathbb{Z}$$

where $A = \{1, \zeta, \zeta^2, \ldots, \zeta^{\phi(n)}\} \setminus \{\zeta^j\}$.

Proof: Apply the previous proposition with $f(x) = \Phi_n(x)$ and $r = \phi(n)$, noting that the ring map $\mathbb{Z}[x] \to \mathbb{Z}[\zeta]$ sending x to ζ will also send x^j to ζ^j, and induce an isomorphism $\mathbb{Z}[x]/(\Phi_n(x)) \to \mathbb{Z}[\zeta]$. $\square$

For later use (see the proof of Theorem 20), we note here that the set

$$P_n := \{\zeta^m\}_{m \in (\mathbb{Z}/n\mathbb{Z})^\times}$$

of all *primitive* n^{th} roots of unity within $\mathbb{Z}[\zeta]$ forms a $\mathbb{Z}$-basis whenever n is squarefree. This is a sharpening of an observation of Johnsen [12], who noted that P_n forms a $\mathbb{Q}$-basis of $\mathbb{Q}[\zeta]$ in the same situation.

Proposition 6 *When n is squarefree, the collection P_n of all primitive n^{th} roots of unity forms a $\mathbb{Z}$-basis for $\mathbb{Z}[\zeta]$.*

Proof: The result is easy when n is prime and can be deduced from the Chinese Remainder Theorem in the general case. See [22] for details. $\square$

3 Duality of matroids or Plücker coordinates

We will need a version of the linear algebraic duality between Plücker coordinates for complementary Grassmannians $G(r, \mathbb{F}^n), G(n-r, \mathbb{F}^n)$, or equivalently, the duality between bases and cobases in coordinatized matroids.

Proposition 7 *Let $0 \leq r \leq n$. Let M and $M^\perp$ be matrices in $\mathbb{F}^{r \times n}$ and $\mathbb{F}^{(n-r)\times n}$, respectively, both of maximal rank, with the following property:* $\ker M$ *is equal to the row space of $M^\perp$, or equivalently,* $\ker M^\perp$ *is the row space of M. Then*

(i) there exists a scalar α in $\mathbb{F}^\times$ having the following property: for every $(n-r)$-subset T of $[n]$, with complementary set T^c,

$$\det\left(M\big|_{T^c}\right) = \pm\alpha \cdot \det\left(M^\perp\big|_T\right)$$

where $A\big|_J$ denotes the restriction of a matrix A to the subset of columns indexed by J, and the $\pm$ sign depends upon the set T.

(ii) if one furthermore assumes that $\mathbb{F} = \mathbb{Q}$, that M and $M^\perp$ have entries in $\mathbb{Z}$, and that there exists at least one $(n-r)$-subset T_0 for which $M\big|_{T_0^c}, M^\perp\big|_{T_0}$ are both invertible over $\mathbb{Z}$, then the scalar α above equals ± 1, and one has for every $(n-r)$-subset T,

$$\text{coker}\left(M\big|_{T^c}\right) \cong \text{coker}\left(M^\perp\big|_T\right).$$

Here, we are thinking of $\text{coker } M$ *as signifying a map between powers of $\mathbb{Z}$.*

Proof: Both assertions can be reduced via row and column operations to the case where M takes the form $[I_r|A]$ for some r-by-$(n-r)$ matrix A, where they are easier to verify. See [22] for details. $\square$

The proof of Theorem 2 will ultimately rely on the following statement about duality of *oriented matroids* for vectors in a vector space over an ordered field $\mathbb{F}$, such as $\mathbb{F} = \mathbb{Q}$.

Proposition 8 *Let $\mathbb{F}$ be an ordered field, M and let $M^\perp$ be matrices in $\mathbb{F}^{r\times n}$ and $\mathbb{F}^{(n-r)\times n}$ as in Proposition 7, that is, both of maximal rank, with $\ker M$ perpendicular to the row space of $M^\perp$. Let the vectors v_ℓ in $\mathbb{F}^r$ and $v_\ell^\perp$ in $\mathbb{F}^{n-r}$ be the ℓ^{th} columns of M and $M^\perp$. Let A be an $(r+1)$-subset of $\{1,2,\ldots,n\}$ such that the matrix $M\big|_A$ in $\mathbb{F}^{r\times(r+1)}$ has full rank r, with*

$$\sum_{\ell\in A} c_\ell v_\ell = 0 \tag{2}$$

the unique dependence among its columns, up to scaling. Then for any pair of nonzero coefficients $c_j, c_{j'} \neq 0$, the matrix $M^\perp\big|_{A^c\cup\{j,j'\}}$ in $\mathbb{F}^{(n-r)\times(n-r+1)}$ has full rank $n-r$, and the unique dependence among its columns, up to scaling,

$$\sum_{\ell\in A^c\cup\{j,j'\}} b_\ell v_\ell^\perp = 0, \tag{3}$$

will have both $b_j, b_{j'} \neq 0$, with

$$\frac{c_j}{c_{j'}} = -\frac{b_{j'}}{b_j}.$$

In particular, $c_j, c_{j'}$ have the same sign if and only if $b_j, b_{j'}$ have opposite signs.

Proof: The main observation here is that vectors in the row space of $M^\perp$ are *covectors* for $\{v_\ell^\perp\}$. See [22] for details.

□

4 Simplicial spanning trees

For a collection of subsets S of some vertex set V, let $\langle S\rangle$ denote the (abstract) simplicial complex S on V generated by S, that is, $\langle S\rangle \subset 2^V$ consists of all subsets of V contained in at least one subset from S. We recall the notion of a simplicial spanning tree in S, following Adin [1], Duval, Klivans and Martin [8], Kalai [13], and Maxwell [19].

Definition 9 Let S be the collection of facets of a pure k-dimensional (abstract) simplicial complex. Say that $R \subset S$ is an *S-spanning tree* if

(i) $\langle R\rangle$ contains the entire $(k-1)$-skeleton of $\langle S\rangle$,

(ii) $\tilde{H}_k(\langle R\rangle;\mathbb{Z}) = 0, and$

(iii) $\tilde{H}_{k-1}(\langle R\rangle;\mathbb{Z})$ is finite.

We point out here three well-known features of this definition.

Proposition 10 *Fix the collection of facets S of a pure k-dimensional simplicial complex.*

(i) Condition (i) in Definition 9 is equivalent to $\tilde{H}_k(\langle S\rangle,\langle R\rangle;\mathbb{Z}) = \mathbb{Z}^{|S\setminus R|}$.

(ii) Condition (ii) in Definition 9 is equivalent to $\tilde{H}_k(\langle R\rangle;\mathbb{Q}) = 0$.

(iii) All S-spanning trees R have the same cardinality, namely

$$|R| = |S| - \mathrm{rank}_{\mathbb{Z}} \tilde{H}_k(\langle S \rangle; \mathbb{Z}). \tag{4}$$

Proof: See [22] for details. □

The following key observation essentially goes back to work of Kalai [13, Lemma 2].

Proposition 11 *Fix a vertex set V and a collection of k-dimensional simplices S. Consider a collection of $(k+1)$-dimensional faces T of cardinality*

$$|T| := \mathrm{rank}_{\mathbb{Z}} \tilde{H}_k(\langle S \rangle; \mathbb{Z})$$

for which $T \cup \langle S \rangle$ forms a simplicial complex K, that is, all boundaries of faces in T lie in $\langle S \rangle$.

Then the following two assertions hold for any choice of an S-spanning tree R.

(i) The $|T| \times |T|$ matrix ∂ that represents the relative simplicial boundary map

$$\begin{array}{ccc} C_{k+1}(K, \langle R \rangle; \mathbb{Z}) & \rightarrow & C_k(K, \langle R \rangle; \mathbb{Z}) \\ \| & & \| \\ \mathbb{Z}^{|T|} & & \mathbb{Z}^{|S \setminus R|} \end{array}$$

is nonsingular if and only if $\tilde{H}_{k+1}(K; \mathbb{Q}) = 0$.

(ii) When the matrix ∂ is nonsingular, then $\mathrm{coker}(\partial) = \tilde{H}_k(K, \langle R \rangle; \mathbb{Z})$.

Proof: See [22] for details. □

Definition 12 Given a collection of k-simplices S, and an S-spanning tree R, say[(ii)] that R is *torsion-free* if Condition (iii) in Definition 9 is strengthened to the vanishing condition

(iv) $\tilde{H}_{k-1}(\langle R \rangle; \mathbb{Z}) = 0$.

Example 13 For example, when $\langle R \rangle$ is a contractible subcomplex of $\langle S \rangle$ then it satisfies Condition (ii) of Definition 9 as well as the vanishing condition (iv). If it furthermore satisfies Condition (i) of Definition 9, then R becomes a torsion-free S-spanning tree.

A frequent combinatorial setting where this occurs (such as in Proposition 15 below) is when S is the set of facets of a (pure) *shellable* [3] simplicial complex, and R is the subset of facets which are not fully attached along their entire boundaries during the shelling process.

Proposition 14 *Using the hypotheses and notation of Proposition 11, if one assumes in addition that R is torsion-free, assertion (ii) of Proposition 11 becomes the following assertion about (non-relative) homology:*

(ii) When the matrix ∂ is nonsingular, then $\mathrm{coker}(\partial) = \tilde{H}_k(K; \mathbb{Z})$

Proof: When R is torsion-free, the long exact sequence for the pair $(K, \langle R \rangle)$ shows that $\tilde{H}_k(K; \mathbb{Z}) \cong \tilde{H}_k(K, \langle R \rangle; \mathbb{Z})$. □

(ii) This condition on an S-spanning tree also plays an important role in [9] by Duval, Klivans and Martin.

5 More on the complete d-partite complex

It is well-known and easy to see that for a positive integer n having prime factorization $n = p_1^{e_1} \cdots p_d^{e_d}$ with $e_i \geq 1$, one always has $\Phi_n(x) = \Phi_{p_1 \cdots p_d}(x^{n/p_1 \cdots p_d})$. Thus it suffices to interpret the coefficients of cyclotomic polynomials for squarefree n.

In this section, we fix such a squarefree $n = p_1 \cdots p_d$, and discuss further properties of the simplicial complexes $K_{p_1,\ldots,p_d}$, defined in Section 1, appearing in Theorems 1 and 2.

Proposition 15 *The $(d-2)$-dimensional skeleton of $K_{p_1,\ldots,p_d}$ is shellable, with*

$$\tilde{H}_{d-2}(K_{p_1,\ldots,p_d};\mathbb{Z}) = \mathbb{Z}^{n-\phi(n)}.$$

Proof: To show that the $(d-2)$-skeleton is shellable, we note the following three facts: (i) zero-dimensional complexes are all trivially shellable, (ii) joins of shellable complexes are shellable [24, Sec. 2], and (iii) skeleta of (pure) shellable simplicial complexes are shellable [5, Corollary 10.12]. Having shown that this skeleton is shellable, it therefore has only top homology; see, for example [3, Appendix]. This homology is free abelian, with rank the absolute value of its reduced Euler characteristic, namely

$$\left|\sum_{i\geq -1}(-1)^i \mathrm{rank}_{\mathbb{Z}}(C_i)\right| = \left|\sum_{i\geq -1}(-1)^i \sum_{\substack{I\subsetneq\{1,2,\ldots,d\}\\ |I|=i+1}} \prod_{i\in I} p_i\right| = \left|\sum_{I\subsetneq\{1,2,\ldots,d\}} (-1)^{|I|-1}\prod_{i\in I} p_i\right|$$
$$= |(p_1-1)\cdots(p_d-1) - p_1\cdots p_d| = |\phi(n) - n|.$$

□

As noted in the introduction, the Chinese Remainder Theorem isomorphism (1) identifies elements of $\mathbb{Z}/n\mathbb{Z}$ with the $(d-1)$-dimensional simplices of $K_{p_1,\ldots,p_d}$. Lower dimensional faces of $K_{p_1,\ldots,p_d}$ can also be identified as cosets of subgroups within $\mathbb{Z}/n\mathbb{Z}$, but we will use this identification sparingly in this paper. For the sake of writing down oriented simplicial boundary maps, choose the following orientation on the simplices of $K_{p_1,\ldots,p_d}$, consistent with the orientation of facets preceding Theorem 2: choose the oriented $(\ell-1)$-simplex $[j_{i_1} \bmod p_{i_1}, \ldots, j_{i_\ell} \bmod p_{i_\ell}]$ with $i_1 < \ldots < i_\ell$ as a basis element of $C_{\ell-1}(K_{p_1,\ldots,p_d};\mathbb{Z})$. The following simple observation was the crux of the results in [18].

Proposition 16 *If one identifies the indexing set $\mathbb{Z}/n\mathbb{Z}$ for the columns of the boundary map*

$$C_{d-1}(K_{p_1,\ldots,p_d};\mathbb{Z}) \to C_{d-2}(K_{p_1,\ldots,p_d};\mathbb{Z}) \tag{5}$$

with the set $\mu_n := \{\zeta^j\}_{j\in\mathbb{Z}/n\mathbb{Z}}$ of all n^{th} roots of unity, then every row of this boundary map represents a $\mathbb{Q}$-linear dependence on μ_n.

Proof: A row in this boundary map is indexed by an oriented $(d-2)$-face, which has the form

$$[j_1 \bmod p_1, \ldots, \widehat{j_k \bmod p_k}, \ldots, j_d \bmod p_d]$$

for some $j_k \in \{0,1,\ldots,p_k-1\}$ and $1 \leq k \leq d$. This row will contain mostly zeroes. Its non-zero entries are all $(-1)^{k-1}$, and lie in the columns indexed by those ζ^j having $j \equiv j_i \bmod p_i$ for $i \neq k$, and $j \bmod p_k$ arbitrary. These exponents j are exactly those lying in one coset of the subgroup $p_1\cdots\hat{p_k}\cdots p_d\mathbb{Z}/n\mathbb{Z}$ within $\mathbb{Z}/n\mathbb{Z}$. Summing ζ^j over j in such a coset gives zero. □

Example 17 Let $n = 15$ as in Example 3, and consider the matrix for the simplicial boundary map $C_1(K_{3,5};\mathbb{Z}) \to C_0(K_{3,5};\mathbb{Z})$. One of its rows is indexed by the 0-face $[2 \bmod 5]$ and this row has exactly three nonzero entries, all equal to $(-1)^0 = +1$. To see these signs, we rewrite $[2 \bmod 5]$ in three ways, all of which involve deleting the first entry out of two in an oriented 1-face:

$$[2 \bmod 5] = [\widehat{0 \bmod 3}, 2 \bmod 5] = [\widehat{1 \bmod 3}, 2 \bmod 5] = [\widehat{2 \bmod 3}, 2 \bmod 5].$$

The columns corresponding to these three 1-faces are indexed by the roots of unity ζ^{12}, ζ^7, and ζ^2, respectively. Summing these up with coefficients of positive one, we get

$$1 \cdot \zeta^{12} + 1 \cdot \zeta^7 + 1 \cdot \zeta^2 = \zeta^2(\zeta^{10} + \zeta^5 + 1),$$

which is the sum of ζ^j over j lying in a coset of $5\mathbb{Z}/15\mathbb{Z}$, and hence is zero.

Definition 18 Assume that n is squarefree and let T denote any set of $n - \phi(n)$ columns of the boundary map (5). Identify the complementary set T^c of $\phi(n)$ columns with a subset of the n^{th} roots-of-unity μ_n. Create a subcomplex of $K_{p_1,\ldots,p_d}$ by including its entire $(d-2)$-skeleton and attaching the subset of $(d-1)$-faces indexed by T. We denote this subcomplex as $K[T]$.

With this definition in mind, we will make use of an interesting feature of this labelling of the boundary map and the set P_n of primitive n^{th} roots of unity, noted already in [18, Remark 5]. For this next result, we let P_n^c denote the $(n - \phi(n))$-element subset of μ_n indexed by the n^{th} roots of unity which are not primitive.

Proposition 19 *Let n be a squarefree integer and P_n^c be as above. Then the subcomplex $K[{P_n}^c]$ of $K_{p_1,\ldots,p_d}$ is contractible.*

Proof: Observe that the primitive roots in $\mathbb{Z}/n\mathbb{Z}$ are exactly those elements which do not vanish modulo p_i for $i = 1, \ldots, d$. Tracing through the labelling of the $(d-1)$-faces via Ξ, we obtain the description

$$K[{P_n}^c] = \bigcup_{i=0}^{d} \mathrm{star}_{K_{p_1,\ldots,p_d}}(0 \bmod p_i),$$

where $\mathrm{star}_\Delta(v)$ denotes the *simplicial star* of the vertex v inside a simplicial complex Δ. Furthermore, each intersection of these stars is nonempty and contractible, because it is the star of another face: for $I \subset [d]$,

$$\bigcap_{i\in I} \mathrm{star}_{K_{p_1,\ldots,p_d}}(0 \bmod p_i) \quad = \quad \mathrm{star}_{K_{p_1,\ldots,p_d}}\left(\{0 \bmod p_i\}_{i\in I}\right).$$

A standard nerve lemma [4, Theorem 10.6] then shows that $K[{P_n}^c]$ itself is contractible. □

Theorem 20 *Let n be a squarefree integer and T be a subset of μ_n of size $n - \phi(n)$. Let $K[T]$ be the subcomplex of $K_{p_1,\ldots,p_d}$ of Definition 18. Then*

$$\tilde{H}_i(K[T];\mathbb{Z}) \cong \begin{cases} \mathbb{Z}[\zeta]/\mathbb{Z}T^c & \text{if } i = d-2, \\ \mathbb{Z} & \text{if both } i = d-1 \text{ and } \mathrm{rank}_{\mathbb{Z}}(\mathbb{Z}T^c) < \phi(n), \\ 0 & \text{otherwise.} \end{cases}$$

where $\mathbb{Z}T^c$ is the sublattice $\mathbb{Z}$-spanned by the roots-of-unity $T^c \subset \mu_n$.

Proof: Choose any $\mathbb{Z}$-basis for $\mathbb{Z}[\zeta]$. Let M in $\mathbb{Z}^{\phi(n)\times n}$ be the matrix that expresses the n^{th} roots of unity μ_n in this basis. We construct a particular matrix $M^\perp$ to accompany M as in Proposition 7 part (ii). Consider the collection S of all $(d-2)$-faces in the complete d-partite complex $K_{p_1,\ldots,p_d}$. The complex $\langle S\rangle$ generated by S is therefore the $(d-2)$-skeleton of $K_{p_1,\ldots,p_d}$. Proposition 15 implies that $\langle S\rangle$ is shellable, and that it has $\mathrm{rank}_\mathbb{Z}\tilde{H}_{d-2}(\langle S\rangle;\mathbb{Z}) = n-\phi(n)$. Therefore, we are in the situation of Example 13, implying that there exists a torsion-free S-spanning tree R, and any such R will have $|S\setminus R| = n-\phi(n)$.

Our candidate for the matrix $M^\perp$ in $\mathbb{Z}^{(n-\phi(n))\times n}$ is the restriction of the boundary map from (5) to its rows indexed by $S\setminus R$. Proposition 16 shows that the rows of $M^\perp$ are all perpendicular to the rows of M. Now choose T, T^c so that T^c indexes the set P_n of primitive n^{th} roots of unity. Proposition 6 implies that the maximal minor $M\big|_{T^c}$ of M is invertible over $\mathbb{Z}$, while Proposition 19 implies that the maximal minor $M^\perp\big|_T$ of $M^\perp$ is invertible over $\mathbb{Z}$. Thus $M, M^\perp$ satisfy the hypotheses of Proposition 7 part (ii), and combining this with Proposition 14 gives the assertion of the theorem. □

6 Proof of Theorems 1 and 2

We are now in a position to prove Theorems 1 and 2.

Proof of Theorem 1: Let $T^c = \{1,\zeta,\zeta^2,\ldots,\zeta^{\phi(n)}\}\setminus\{\zeta^j\}$ so that we have the equality of complexes $K[T] = K[\{\zeta^{\phi(n)+1},\zeta^{\phi(n)+2},\ldots,\zeta^{n-1}\}\cup\{j\}] = K_j$. The theorem then follows from Theorem 20 and Corollary 5. □

Proof of Theorem 2: We prove Theorem 2 by applying Proposition 8 to the matrices $M, M^\perp$ in the proof of Theorem 20, with $A = \{1,\zeta,\zeta^2,\ldots,\zeta^{\phi(n)}\}$. The dependence (2) among the columns of $M|_A$ has the same coefficients (up to scaling) as the cyclotomic polynomial, and the dependence (3) among the columns of $M^\perp\big|_{A^c\cup\{j,j'\}}$ has the same coefficients (up to scaling) as a nonzero cycle $z=\sum_\ell b_\ell[F_\ell]$ in $\tilde{H}_{d-1}(K_{\{j,j'\}};\mathbb{Z})$. □

7 Concordance with known properties of $\Phi_n(x)$

Here are some results about $\Phi_n(x)$ that manifest themselves topologically. See [22] for details.

1. The two maps $\mathbb{Z}/n\mathbb{Z}\to\mathbb{Z}/n\mathbb{Z}$ that send $\overline{j}$ to $\overline{-j}$ and send $\overline{j}$ to $\overline{j+1}$ generate a dihedral group of simplicial automorphisms of $K_{p_1,p_2,\ldots,p_d}$. One such automorphism sends the subcomplex $K_{\{j\}}$ to $K_{\{\phi(n)-j\}}$, and the subcomplex $K_{\{j,\phi(n)\}}$ to $K_{\{0,\phi(n)-j\}}$, explaining the symmetry $c_j = c_{\phi(n)-j}$ in $\Phi_n(x)$.

2. The fact that $\Phi_{2n}(x) = \Phi_n(-x)$ when n is odd manifests itself topologically as follows: the subcomplex $K_{\{j\}}$ whose homology interprets the coefficient of x^j for $\Phi_{2n}(x)$ is homotopy-equivalent to the suspension of the corresponding complex for $\Phi_n(x)$. Furthermore, there is a similar suspension relation between the complexes that predict the coefficients' signs.

3. When $d = 2$ so $n = p_1 p_2$ is the product of only two primes, all the subcomplexes $K_{\{j\}}$ of K_{p_1,p_2} are graphs. Hence their $(d-2)$-dimensional homology is torsion-free. It follows that the only nonzero coefficients of $\Phi_n(x)$ are ± 1, agreeing with a well-known old observation of Migotti [20]. The explicit expansion of $\Phi_{p_1 p_2}(x)$ is given in Elder [10], Lam and Leung [16], and Lenstra [17].

4. In contrast to above, when $d \geq 3$ and the p_i's are odd primes, $\Phi_n(x)$ often has coefficients with absolute value ≥ 2. For example, $\Phi_{105}(x)$ has coefficient -2 on x^7 and x^{41}. The 2-dimensional subcomplexes $K_{\{7\}}$ and $K_{\{41\}}$, whose 1-homology equals $\mathbb{Z}/2\mathbb{Z}$, turn out to be surprisingly nontrivial. For example, neither one can be collapsed down to a real projective plane.

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