*FPSAC 2011, Reykjavík, Iceland* DMTCS proc. **AO**, 2011, 897–902

# A $q$-analog of Ljunggren's binomial congruence

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**Abstract.** We prove a $q$-analog of a classical binomial congruence due to Ljunggren which states that

$$\binom{ap}{bp} \equiv \binom{a}{b}$$

modulo $p^3$ for primes $p \geqslant 5$. This congruence subsumes and builds on earlier congruences by Babbage, Wolstenholme and Glaisher for which we recall existing $q$-analogs. Our congruence generalizes an earlier result of Clark.

**Résumé.** Nous démontrons un $q$-analogue d'une congruence binomiale classique de Ljunggren qui stipule:

$$\binom{ap}{bp} \equiv \binom{a}{b}$$

modulo $p^3$ pour $p$ premier tel que $p \geqslant 5$. Cette congruence s'inspire d'une précédente congruence prouvée par Babbage, Wolstenholme et Glaisher pour laquelle nous présentons les $q$-analogues existantes. Notre congruence généralise un précédent résultat de Clark.

**Keywords:** $q$-analogs, binomial coefficients, binomial congruence

## 1 Introduction and notation

Recently, $q$-analogs of classical congruences have been studied by several authors including (Cla95), (And99), (SP07), (Pan07), (CP08), (Dil08). Here, we consider the classical congruence

$$\binom{ap}{bp} \equiv \binom{a}{b} \mod p^3 \tag{1}$$

which holds true for primes $p \geqslant 5$. This also appears as Problem 1.6 (d) in (Sta97). Congruence (1) was proved in 1952 by Ljunggren, see (Gra97), and subsequently generalized by Jacobsthal, see Remark 6.

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Let $[n]_q := 1 + q + \dots q^{n-1}$, $[n]_q! := [n]_q[n-1]_q \cdots [1]_q$ and

$$\binom{n}{k}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}$$

denote the usual $q$-analogs of numbers, factorials and binomial coefficients respectively. Observe that $[n]_1 = n$ so that in the case $q = 1$ we recover the usual factorials and binomial coefficients as well. Also, recall that the $q$-binomial coefficients are polynomials in $q$ with nonnegative integer coefficients. An introduction to these $q$-analogs can be found in (Sta97).

We establish the following $q$-analog of (1):

**Theorem 1** *For primes $p \geqslant 5$ and nonnegative integers $a, b$,*

$$\binom{ap}{bp}_q \equiv \binom{a}{b}_{q^{p^2}} - \binom{a}{b+1}\binom{b+1}{2}\frac{p^2-1}{12}(q^p-1)^2 \mod [p]_q^3. \tag{2}$$

The congruence (2) and similar ones to follow are to be understood over the ring of polynomials in $q$ with integer coefficients. We remark that $p^2 - 1$ is divisible by 12 for all primes $p \geqslant 5$.

Observe that (2) is indeed a $q$-analog of (1): as $q \to 1$ we recover (1).

**Example 2** Choosing $p = 13$, $a = 2$, and $b = 1$, we have

$$\binom{26}{13}_q = 1 + q^{169} - 14(q^{13}-1)^2 + (1 + q + \dots + q^{12})^3 f(q)$$

where $f(q) = 14 - 41q + 41q^2 - \dots + q^{132}$ is an irreducible polynomial with integer coefficients. Upon setting $q = 1$, we obtain $\binom{26}{13} \equiv 2$ modulo $13^3$.

Since our treatment very much parallels the classical case, we give a brief history of the congruence (1) in the next section before turning to the proof of Theorem 1.

## 2 A bit of history

A classical result of Wilson states that $(n-1)! + 1$ is divisible by $n$ if and only if $n$ is a prime number. "In attempting to discover some analogous expression which should be divisible by $n^2$, whenever $n$ is a prime, but not divisible if $n$ is a composite number", (Bab19), Babbage is led to the congruence

$$\binom{2p-1}{p-1} \equiv 1 \mod p^2 \tag{3}$$

for primes $p \geqslant 3$. In 1862 Wolstenholme, (Wol62), discovered (3) to hold modulo $p^3$, "for several cases, in testing numerically a result of certain investigations, and after some trouble succeeded in proving it to hold universally" for $p \geqslant 5$. To this end, he proves the fractional congruences

$$\sum_{i=1}^{p-1} \frac{1}{i} \equiv 0 \mod p^2, \tag{4}$$

$$\sum_{i=1}^{p-1} \frac{1}{i^2} \equiv 0 \mod p \tag{5}$$

for primes $p \geqslant 5$. Using (4) and (5) he then extends Babbage's congruence (3) to hold modulo $p^3$:

$$\binom{2p-1}{p-1} \equiv 1 \mod p^3 \tag{6}$$

for all primes $p \geqslant 5$. Note that (6) can be rewritten as $\binom{2p}{p} \equiv 2$ modulo $p^3$. The further generalization of (6) to (1), according to (Gra97), was found by Ljunggren in 1952. The case $b = 1$ of (1) was obtained by Glaisher, (Gla00), in 1900.

In fact, Wolstenholme's congruence (6) is central to the further generalization (1). This is just as true when considering the $q$-analogs of these congruences as we will see here in Lemma 5.

A $q$-analog of the congruence of Babbage has been found by Clark (Cla95) who proved that

$$\binom{ap}{bp}_q \equiv \binom{a}{b}_{q^{p^2}} \mod [p]_q^2. \tag{7}$$

We generalize this congruence to obtain the $q$-analog (2) of Ljunggren's congruence (1). A result similar to (7) has also been given by Andrews in (And99).

Our proof of the $q$-analog proceeds very closely to the history just outlined. Besides the $q$-analog (7) of Babbage's congruence (3) we will employ $q$-analogs of Wolstenholme's harmonic congruences (4) and (5) which were recently supplied by Shi and Pan, (SP07):

**Theorem 3** *For primes $p \geqslant 5$,*

$$\sum_{i=1}^{p-1} \frac{1}{[i]_q} \equiv -\frac{p-1}{2}(q-1) + \frac{p^2-1}{24}(q-1)^2[p]_q \mod [p]_q^2 \tag{8}$$

*as well as*

$$\sum_{i=1}^{p-1} \frac{1}{[i]_q^2} \equiv -\frac{(p-1)(p-5)}{12}(q-1)^2 \mod [p]_q. \tag{9}$$

This generalizes an earlier result (And99) of Andrews.

## 3 A $q$-analog of Ljunggren's congruence

In the classical case, the typical proof of Ljunggren's congruence (1) starts with the Chu-Vandermonde identity which has the following well-known $q$-analog:

**Theorem 4**

$$\binom{m+n}{k}_q = \sum_j \binom{m}{j}_q \binom{n}{k-j}_q q^{j(n-k+j)}.$$

We are now in a position to prove the $q$-analog of (1).

**Proof of Theorem 1:** As in (Cla95) we start with the identity

$$\binom{ap}{bp}_q = \sum_{c_1+\ldots+c_a=bp} \binom{p}{c_1}_q \binom{p}{c_2}_q \cdots \binom{p}{c_a}_q q^{p\sum_{1\leqslant i\leqslant a}(i-1)c_i - \sum_{1\leqslant i<j\leqslant a} c_i c_j} \tag{10}$$

which follows inductively from the $q$-analog of the Chu-Vandermonde identity given in Theorem 4. The summands which are not divisible by $[p]_q^2$ correspond to the $c_i$ taking only the values 0 and $p$. Since each such summand is determined by the indices $1 \leqslant j_1 < j_2 < \ldots < j_b \leqslant a$ for which $c_i = p$, the total contribution of these terms is

$$\sum_{1 \leqslant j_1 < \ldots < j_b \leqslant a} q^{p^2 \sum_{k=1}^{b}(j_k - 1) - p^2 \binom{b}{2}} = \sum_{0 \leqslant i_1 \leqslant \ldots \leqslant i_b \leqslant a-b} q^{p^2 \sum_{k=1}^{b} i_k} = \binom{a}{b}_{q^{p^2}}.$$

This completes the proof of (7) given in (Cla95).

To obtain (2) we now consider those summands in (10) which are divisible by $[p]_q^2$ but not divisible by $[p]_q^3$. These correspond to all but two of the $c_i$ taking values 0 or $p$. More precisely, such a summand is determined by indices $1 \leqslant j_1 < j_2 < \ldots < j_b < j_{b+1} \leqslant a$, two subindices $1 \leqslant k < \ell \leqslant b+1$, and $1 \leqslant d \leqslant p-1$ such that

$$c_i = \begin{cases} d \text{ for } i = j_k, \\ p - d \text{ for } i = j_\ell, \\ p \text{ for } i \in \{j_1, \ldots, j_{b+1}\} \backslash \{j_k, j_\ell\}, \\ 0 \text{ for } i \notin \{j_1, \ldots, j_{b+1}\}. \end{cases}$$

For each fixed choice of the $j_i$ and $k, \ell$ the contribution of the corresponding summands is

$$\sum_{d=1}^{p-1} \binom{p}{d}_q \binom{p}{p-d}_q q^{p \sum_{1 \leqslant i \leqslant a}(i-1)c_i - \sum_{1 \leqslant i < j \leqslant a} c_i c_j}$$

which, using that $q^p \equiv 1$ modulo $[p]_q$, reduces modulo $[p]_q^3$ to

$$\sum_{d=1}^{p-1} \binom{p}{d}_q \binom{p}{p-d}_q q^{d^2} = \binom{2p}{p}_q - [2]_{q^{p^2}}.$$

We conclude that

$$\binom{ap}{bp}_q \equiv \binom{a}{b}_{q^{p^2}} + \binom{a}{b+1}\binom{b+1}{2}\left(\binom{2p}{p}_q - [2]_{q^{p^2}}\right) \mod [p]_q^3. \tag{11}$$

The general result therefore follows from the special case $a = 2$, $b = 1$ which is separately proved next.
□

## 4 A $q$-analog of Wolstenholme's congruence

We have thus shown that, as in the classical case, the congruence (2) can be reduced, via (11), to the case $a = 2$, $b = 1$. The next result therefore is a $q$-analog of Wolstenholme's congruence (6).

**Lemma 5** *For primes $p \geqslant 5$,*

$$\binom{2p}{p}_q \equiv [2]_{q^{p^2}} - \frac{p^2-1}{12}(q^p - 1)^2 \mod [p]_q^3.$$

**Proof:** Using that $[an]_q = [a]_{q^n} [n]_q$ and $[n+m]_q = [n]_q + q^n [m]_q$ we compute

$$\binom{2p}{p}_q = \frac{[2p]_q [2p-1]_q \cdots [p+1]_q}{[p]_q [p-1]_q \cdots [1]_q} = \frac{[2]_{q^p}}{[p-1]_q!} \prod_{k=1}^{p-1} \left( [p]_q + q^p [p-k]_q \right)$$

which modulo $[p]_q^3$ reduces to (note that $[p-1]_q!$ is relatively prime to $[p]_q^3$)

$$[2]_{q^p} \left( q^{(p-1)p} + q^{(p-2)p} \sum_{1 \leqslant i \leqslant p-1} \frac{[p]_q}{[i]_q} + q^{(p-3)p} \sum_{1 \leqslant i < j \leqslant p-1} \frac{[p]_q [p]_q}{[i]_q [j]_q} \right). \tag{12}$$

Combining the results (8) and (9) of Shi and Pan, (SP07), given in Theorem 3, we deduce that for primes $p \geqslant 5$,

$$\sum_{1 \leqslant i < j \leqslant p-1} \frac{1}{[i]_q [j]_q} \equiv \frac{(p-1)(p-2)}{6} (q-1)^2 \mod [p]_q. \tag{13}$$

Together with (8) this allows us to rewrite (12) modulo $[p]_q^3$ as

$$[2]_{q^p} \left( q^{(p-1)p} + q^{(p-2)p} \left( -\frac{p-1}{2}(q^p - 1) + \frac{p^2-1}{24}(q^p-1)^2 \right) + \right.$$
$$\left. + q^{(p-3)p} \frac{(p-1)(p-2)}{6} (q^p - 1)^2 \right).$$

Using the binomial expansion

$$q^{mp} = ((q^p - 1) + 1)^m = \sum_k \binom{m}{k} (q^p - 1)^k$$

to reduce the terms $q^{mp}$ as well as $[2]_{q^p} = 1 + q^p$ modulo the appropriate power of $[p]_q$ we obtain

$$\binom{2p}{p}_q \equiv 2 + p(q^p - 1) + \frac{(p-1)(5p-1)}{12}(q^p - 1)^2 \mod [p]_q^3.$$

Since

$$[2]_{q^{p^2}} \equiv 2 + p(q^p - 1) + \frac{(p-1)p}{2}(q^p-1)^2 \mod [p]_q^3$$

the result follows. □

**Remark 6** Jacobsthal, see (Gra97), generalized the congruence (1) to hold modulo $p^{3+r}$ where $r$ is the $p$-adic valuation of

$$ab(a-b)\binom{a}{b} = 2a \binom{a}{b+1} \binom{b+1}{2}.$$

It would be interesting to see if this generalization has a nice analog in the $q$-world.

## Acknowledgements

Most parts of this paper have been written during a visit of the author at Grinnell College. The author wishes to thank Marc Chamberland for his encouraging and helpful support. Partial support of grant NSF-DMS 0713836 is also thankfully acknowledged.

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