On the degree-chromatic polynomial of a tree

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Abstract. The *degree chromatic polynomial* $P_m(G, k)$ of a graph G counts the number of k-colorings in which no vertex has m adjacent vertices of its same color. We prove Humpert and Martin's conjecture on the leading terms of the degree chromatic polynomial of a tree.

Résumé. Le *polynôme degré chromatique* $P_m(G, k)$ d'un graphe G compte le nombre de k-colorations dans lesquelles aucun sommet n'a m sommets adjacents de sa même couleur. On démontre la conjecture de Humpert et Martin sur les coefficients principaux du polynôme degré chromatique d'un arbre.

Keywords: chromatic polynomial, graph coloring, tree

George David Birkhoff defined the chromatic polynomial of a graph to attack the renowned four color problem. The chromatic polynomial $P(G, k)$ counts the k-colorings of a graph G in which no two adjacent vertices have the same color [Read(1968)].

Given a graph G, Humpert and Martin defined its m-chromatic polynomial $P_m(G, k)$ to be the number of k-colorings of G such that no vertex has m adjacent vertices of its same color. They proved this is indeed a polynomial. When $m = 1$, we recover the usual chromatic polynomial of the graph.

The chromatic polynomial is of the form

$$P(G, k) = k^n - ek^{n-1} + o(k^{n-1}),$$

where n is the number of vertices and e the number of edges of G. For $m > 1$ the formula is no longer true, but Humpert and Martin conjectured the following formula which we now prove:

Theorem 1 ([Humpert and Martin(2010), Humpert and Martin(2011)], Conjecture) *Let T be a tree with n vertices and let m be an integer with $1 < m < n$, then the following equation holds.*

$$P_m(T, k) = k^n - \sum_{v \in V(T)} \binom{d(v)}{m} k^{n-m} + o(k^{n-m})$$

Proof: For a given coloring of T, say vertices v_1 and v_2 are "friends" if they are adjacent and have the same color. For each v, let A_v be the set of colorings such that v has at least m friends. We want to find the number of colorings which are not in any A_v, and we will use the inclusion-exclusion principle. As the total number of k-colorings is k^n, we have

$$P_m(T, k) = k^n - \sum_{v \in V} |A_v| + \sum_{v_1, v_2 \in V} |A_{v_1} \cap A_{v_2}| - \dots$$

We first show that $|A_v| = \binom{d(v)}{m}k^{n-m} + o(k^{n-m})$. Let $A_v^{(l)}$ be the set of k-colorings such that v has exactly l friends. In order to obtain a coloring in $A_v^{(l)}$, we may choose the l friends in $\binom{d(v)}{l}$ ways, the color of v and its friends in k ways, the color of the remaining adjacent vertices to v in $(k-1)^{d(v)-l}$ ways, and the color of the rest of the vertices in $k^{n-1-d(v)}$ ways. Then

$$|A_v| = \sum_{l=m}^{n-1} |A_v^{(l)}| = \sum_{l=m}^{n-1} \binom{d(v)}{l}k^{n-d(v)}(k-1)^{d(v)-l}$$
$$= \binom{d(v)}{m}k^{n-m} + o(k^{n-m}).$$

To complete the proof, it is sufficient to see that for any set S of at least 2 vertices $\left|\bigcap_{v\in S} A_v\right| = o(k^{n-m})$; clearly we may assume $S = \{v_1, v_2\}$. Consider the following cases:

Case 1 v_1 and v_2 *are not adjacent.*

Split A_{v_1} into equivalence classes with the equivalence relation

$$\sigma_1 \sim \sigma_2 \Leftrightarrow \sigma_1(w) = \sigma_2(w) \text{ for all } w \neq v_2.$$

Note that each equivalence class C consists of k colorings, which only differ in the color of v_2. In addition, for each C at most $\frac{d(v_2)}{m}$ of its colorings are in A_{v_2}, as if $\sigma \in A_{v_2}$ there must be m vertices adjacent to v_2 with the color $\sigma(v_2)$. Therefore

$$|A_{v_1} \cap A_{v_2}| = \sum_C |C \cap A_{v_2}| \leq \sum_C \frac{d(v_2)}{m} = \frac{|A_{v_1}|}{k} \cdot \frac{d(v_2)}{m}.$$

It follows that $\frac{|A_{v_1}\cap A_{v_2}|}{|A_{v_1}|}$ goes to 0 as k goes to infinity, so $|A_{v_1} \cap A_{v_2}| = o(k^{n-m})$.

Case 2 v_1 and v_2 *are adjacent.*

Let W be the set of adjacent vertices to v_2 other than v_1. They are not adjacent to v_1 as T has no cycles. Split A_{v_1} into equivalence classes with the equivalence relation

$$\sigma_1 \sim \sigma_2 \Leftrightarrow \sigma_1(w) = \sigma_2(w) \text{ for all } w \notin W.$$

Each equivalence class C consists of $k^{|W|}$ colorings, which may only differ in the colors of the vertices in W. If v_1 and v_2 are friends in the colorings of C, then a coloring in $|C \cap A_{v_2}|$ must contain at least $m-1$ vertices in W of the same color as v_2. Therefore

$$|C \cap A_{v_2}| = \sum_{l=m-1}^{|W|} \binom{|W|}{l}(k-1)^{|W|-l} < \sum_{l=0}^{|W|}\binom{|W|}{l}k^{|W|-1} = 2^{|W|}k^{|W|-1}.$$

Notice that here we are using $m \geq 2$ so that $l \geq 1$. Otherwise, if v_1 and v_2 are not friends in the colorings of C, then

$$|C \cap A_{v_2}| = \sum_{l=m}^{|W|} \binom{|W|}{l}(k-1)^{|W|-l} < \sum_{l=0}^{|W|} \binom{|W|}{l}k^{|W|-1} = 2^{|W|}k^{|W|-1}.$$

Therefore

$$|A_{v_1} \cap A_{v_2}| = \sum_{C} |C \cap A_{v_2}| < \sum_{C} 2^{|W|}k^{|W|-1}$$

$$= \frac{|A_{v_1}|}{k^{|W|}} \cdot 2^{|W|}k^{|W|-1} = \frac{|A_{v_1}| \cdot 2^{|W|}}{k}$$

and $|A_{v_1} \cap A_{v_2}| = o(k^{n-m})$ follows as in the first case.

This completes the proof of the theorem. □

Acknowledgements

References

[Humpert and Martin(2010)] B. Humpert and J. L. Martin. The incidence Hopf algebra of graphs. *Preprint arXiv:1012.4786*, 2010.

[Humpert and Martin(2011)] B. Humpert and J. L. Martin. The incidence Hopf algebra of graphs. *DMTCS Proceedings*, 0(01), 2011. ISSN 1365-8050. URL http://www.dmtcs.org/dmtcs-ojs/index.php/proceedings/article/view/dmAO0146.

[Read(1968)] R. C. Read. An introduction to chromatic polynomials. *Journal of Combinatorial Theory*, 4(1):52–71, 1968.