Connections Between a Family of Recursive Polynomials and Parking Function Theory

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Abstract. In a 2010 paper Haglund, Morse, and Zabrocki studied the family of polynomials $\nabla C_{p_1} \dots C_{p_k} 1$, where $p = (p_1, \dots, p_k)$ is a composition, ∇ is the Bergeron-Garsia Macdonald operator and the C_a are certain slightly modified Hall-Littlewood vertex operators. They conjecture that these polynomials enumerate a composition indexed family of parking functions by area, dinv and an appropriate quasi-symmetric function. This refinement of the nearly decade old "Shuffle Conjecture," when combined with properties of the Hall-Littlewood operators can be shown to imply the existence of certain bijections between these families of parking functions. In previous work to appear in her PhD thesis, the author has shown that the existence of these bijections follows from some relatively simple properties of a certain family of polynomials in one variable x with coefficients in $\mathbb{N}[q]$. In this paper we introduce those polynomials, explain their connection to the conjecture of Haglund, Morse, and Zabrocki, and explore some of their surprising properties, both proven and conjectured.

Résumé. Dans un article de 2010, Haglund, Morse et Zabrocki étudient la famille de polynômes $\nabla C_{p_1} \cdots C_{p_k} 1$ où $(p_1, \dots, p_k)$ est une composition, ∇ est l'opérateur de Bergeron-Garsia et les C_a sont des opérateurs "vertex" de Hall-Littlewood légèrement altérés. Il posent la conjecture que ces polynômes donnent l'énumération d'une famille de fonctions "parking", indexées par des compositions, par aire, le "dinv" et une fonction quasi-symmétrique associée. Cette conjecture raffine la conjecture "Shuffle", qui est agée de presque dix ans. On peut montrer, a partir de cette conjecture, que les propriétés des operateurs de Hall-Littlewood, impliquent l'existence de certaines bijéctions entre ces familles de fonctions "parking". Dans un précédent travail , qui fait partie de sa thèse de doctorat, l'auteur montre que l'éxistence de ces bijéctions découle de certaines propriétés relativement simples d'une famille de polynômes à une vaiable x, avec coefficients dans $\mathbb{N}[q]$. Dans cet article, on introduit ces polynômes, on explique leur connexion avec la conjecture de Haglund, Morse et Zabrocki, et on explore certaines de leur proprieétés surprenantes, qu'elles soient prouvées ou seulement conjecturées.

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1 Introduction

We begin with a simple family of polynomials on n variables, call them $\{P_W(X_n; q)\}$, constructed recursively and indexed by a sequence $W = (w_1, \dots, w_n)$. We place several minor restrictions on this sequence, which we refer to hereafter as a *schedule*:

[†] work accomplished with NSF support.

- $w_1 = 1$ and $w_2 = 2$;
- $w_3 \in \{1, 2\}$; and
- (slow growth.) $w_i \leq w_{i-1} + 1$.

Then we begin by defining

$$P_{(1,2)}(X_2; q) = qx_1 + x_2.$$

To recursively construct the remaining members of the family, we define an operator

$$B_{n,w}P(X_{n-1}; q) = \frac{1}{1-q}((x_n - q^w)P(x_1, x_2, \dots, x_{n-1}; q) \tag{1}$$

$$+ (1 - x_n)P(x_1, x_2, \dots, x_{n-w-1}, qx_{n-w}, \dots, qx_{n-1}; q)) \tag{2}$$

Finally we simply define

$$P_{(w_1,\dots,w_n)}(X_n; q) = B_{n,w_n}P_{(w_1,\dots,w_{n-1})}(X_{n-1}; q).$$

Example 1.

$$P_{(1,2,2,3)}(X_4; q) = B_{4,3}(B_{3,2}(qx_1 + x_2)) \tag{3}$$

$$= B_{4,3}(\frac{1}{1-q}\big((x_3 - q^2)(qx_1 + x_2) + (1 - x_3)(q(qx_1) + qx_2)\big) \tag{4}$$

$$= B_{4,3}((qx_1 + x_2)(q + x_3)) \tag{5}$$

$$= (qx_1 + x_2)(q^2 + q^3 + q^2x_3 + qx_4 + x_3x_4 + qx_3x_4) \tag{6}$$

In fact, our primary interest lies with a specialization of these polynomials

$$Q_W(x; q) = P_W(X_n, q)|_{x_1=\dots=x_n=x}$$

which experimentally satisfies some surprising properties. In particular, we conjecture that

Conjecture 1. *For any schedule* $W = (w_1, \dots, w_n)$*,*

$$(1 - q/x)Q_W(x; q) + x^{n-1}(1 - qx)Q_W(1/x; q) = (1 + x^{n-1})(1 - q)\prod_{i=1}^{n}[w_i]_q. \tag{7}$$

We will refer hereafter to (7) as the "functional equation," and if the conjecture holds for a given schedule W, we will say the schedule *satisfies the functional equation*.

Example 2. Using our above calculation for $P_{(1,2,2,3)}(X_4; q)$ and some minor simplifications, we have that $Q_{(1,2,2,3)} = (1 + q)^2x(q^2 + qx + x^2)$. Then notice that:

$$\begin{aligned}
&(1 - q/x)Q_{(1,2,2,3)}(x; q) + x^3(1 - qx)Q_{(1,2,2,3)}(1/x; q) \\
&\quad = (1 - q/x)(1 + q)^2x(q^2 + qx + x^2) + x^3(1 - qx)((1 + q)^2(1/x)(q^2 + q(1/x) + (1/x^2))) \\
&\quad = (1 + q)^2((x - q)(q^2 + qx + x^2) + (1 - qx)(q^2x^2 + qx + 1)) \\
&\quad = 1 + 2q + q^2 - q^3 - 2q^4 - q^5 + (1 + 2q + q^2 - q^3 - 2q^4 - q^5)x^3
\end{aligned}$$

Why should we care about these polynomials or Conjecture 1? The answer to this question lies in an intriguing conjecture about the parking functions in [Haglund et al.(2011)]; to state it in full requires some background, which we give in the next section.

2 Parking Functions

We begin with some necessary definitions.

Definition 1 (Parking Function). A two line array

$$PF = \begin{bmatrix} c_1 & c_2 & \dots & c_n \\ d_1 & d_2 & \dots & d_n \end{bmatrix}$$

is a *parking function* exactly when

- The first line is a permutation of $\{1, 2, \dots, n\}$;
- (Dyck Path Condition.) $d_1 = 0$ and when $i > 1$, $d_i \leq d_{i-1} + 1$; and
- (Increasing Column Condition.) if $d_i = d_{i-1} + 1$, then $c_i > c_{i-1}$.

We say that car c_i is in diagonal d_i, where the 0^{th} diagonal is also called the main diagonal. Two easily defined statistics on the parking functions of historical interest are the area and dinv.

$$\text{area}(PF) = \sum_{i=1}^{n} d_i.$$

Using χ for the truth function,

$$\text{dinv}(PF) = \sum_{i<j} \chi(d_i = d_j \text{ and } c_i < c_j) + \chi(d_i = d_j + 1 \text{ and } c_i > c_j).$$

Definition 2 (Reading Word). The *reading word* of a parking function $(\text{word}(PF))$ is the list of its cars $(c_{\sigma_1}, \dots, c_{\sigma_n})$, where

- $d_{\sigma_i} \geq d_{\sigma_{i+1}}$ and
- if $d_{\sigma_i} = d_{\sigma_{i+1}}$, then $\sigma_i > \sigma_{i+1}$.

Thus the reading word lists the cars from highest to lowest diagonal, with cars in a given diagonal given from back to front.

Then we may define another common statistic on the parking functions, the i-descent set:

$$\text{ides}(PF) = \text{des}(\text{word}(PF)^{-1}).$$

In [Haglund et al.(2011)] Haglund, Morse, and Zabrocki introduce a further statistic, "$\text{comp}(PF)$", the composition of a parking function. Let $(z_1, \dots, z_k)$ give, in increasing order, the indices such that $d_{z_i} = 0$. Then

$$\text{comp}(PF) = (z_1, z_2 - z_1, \dots, z_k - z_{k-1}, n - z_k + 1)$$

Example 3. Let

$$PF = \begin{bmatrix} 1 & 2 & 5 & 3 & 4 \\ 0 & 1 & 2 & 1 & 0 \end{bmatrix}.$$

Then

$$\text{area}(PF) = 4 \text{ and } \text{dinv}(PF) = 3.$$

Moreover, the reading word of PF is $(5, 3, 2, 4, 1)$ and thus

$$\text{ides}(PF) = \text{des}((5, 3, 2, 4, 1)) = \{1, 2, 4\}.$$

Finally, notice that

$$\text{comp}(PF) = (4, 1).$$

With these definitions in hand, we can consider a number of conjectures about the parking functions and the special operator called nabla (∇), introduced in [Bergeron and Garsia(1999)].

Definition 3 (nabla). Let $\tilde{H}_\mu[X; q, t]$ represent the Macdonald polynomial basis element indexed by $\mu = [\mu_1, \cdots, \mu_k]$ and use $\mu' = [\mu'_1, \cdots, \mu'_{k'}]$ for the conjugate of μ . Then nabla is an eigenoperator for the Macdonalds defined by the following:

$$\nabla \tilde{H}_\mu[X; q, t] = t^{\sum (i-1)\mu_i} q^{\sum (i-1)\mu'_i} \tilde{H}_\mu[X; q, t].$$

A decade old conjecture in [Haglund et al.(2005)] about the parking functions can be expressed in terms of nabla:

Conjecture 2 (Shuffle Conjecture). *Using Q_S for the Gessel quasisymmetric function indexed by S and PF_n for the parking functions with n cars,*

$$\nabla e_n = \sum_{PF \in PF_n} t^{\text{area}(PF)} q^{\text{dinv}(PF)} Q_{\text{ides}(PF)}.$$

The conjecture motivating our current work is in fact a specialization of the shuffle conjecture in [Haglund et al.(2011)]. Using the brackets [ ] to indicate plethystic substitution, as in [Garsia and Procesi(1992)], set for any symmetric function $F[X]$

$$C_a F[X] = \left(\frac{-1}{q} \right)^{a-1} \sum_{k \geq 0} F \left[X + \frac{1-q}{q} z \right] \Big|_{z^k} h_{a+k}[X].$$

The C operator has several useful properties. Among them:

1. Using $p \models n$ for p is a composition of n, the identity, as shown in [Haglund et al.(2011)]:

$$e_n = \sum_{(p_1, \cdots, p_s) \models n} C_{p_1} C_{p_2} \ldots C_{p_s} 1.$$

2. Using $\mu \vdash n$ to indicate that μ is a partition of n, the collection $\left\{ C_{\mu_1} \cdots C_{\mu_n} 1 \right\}_{\mu \vdash n}$ is a basis for the symmetric functions of degree n.

3. The C operators obey the following commutativity law: For $a+1 \leq b$,

$$q(C_a C_b + C_{b-1} C_{a+1}) = C_b C_a + C_{a+1} C_{b-1}. \tag{8}$$

Note that to simplify notation, for a composition $p = (p_1, \cdots, p_k)$, we use the convention

$$C_p F[X] = C_{p_1} \dots C_{p_k} F[X].$$

Finally we may state the conjecture of principal importance to the subject at hand, as stated by Haglund, Morse and Zabrocki in [Haglund et al.(2011)].

Conjecture 3. *For p a composition,*

$$\nabla C_p 1 = \sum_{\mathrm{comp}(PF)=p} t^{area(PF)} q^{dinv(PF)} Q_{ides(PF)}. \tag{9}$$

There is a natural way to divide the proof of Conjecture 3 into two parts, namely:

1. Proving the equality in (9) when p is a partition.
2. Reducing the compositional case of (9) to the partitional case. (i. e. showing that both sides of (9) satisfy the same identities implied by successive applications of the commutativity relations, stated in (8).)

In a manner we will make more precise in the following sections, the polynomial conjecture with which we began, Conjecture 1, implies the latter of these two.

3 Our Polynomials and Parking Functions

If we use $\mathcal{F}_p$ for the family of parking functions with composition p, Conjecture 3, combined with the commutativity property in (8) suggest a combinatorial bijection.

Conjecture 4. *For $a \leq b-1$, there exists a bijection f*

$$f : \mathcal{F}_{(a,b)} \cup \mathcal{F}_{(b-1,a+1)} \leftrightarrow \mathcal{F}_{(b,a)} \cup \mathcal{F}_{(a+1,b-1)}$$

with the following properties:

1. *f increases the $dinv$ by exactly one*
2. *f does not change the diagonal containing any car, just the relative order of cars within their original diagonal.*
3. *f preserves the $area$ and the $ides$*

Note that the second property of f is not a direct consequence of Haglund, Morse and Zabrocki's conjecture; however, this additional conjecture has been tested experimentally for $n < 15$. It's inclusion in this conjecture is significant; a priori if our goal is to reduce the proof of Conjecture 3 to the partition case, we should be considering a map on parking functions with compositions of any length. When we include this additional restriction on our map, however, we are able to restrict our attention to parking functions with two parts, although we omit the details here for reasons of brevity.

One of the most significant breakthroughs that allows our reduction of Conjecture 4 to Conjecture 1 is that we showed that it is enough to find a map without checking property 3. (In particular, notice that property 2 is stronger than the restriction that f leaves the area fixed.)

Imagine we partition $\{1, 2, \ldots, n\}$ into disjoint sets $S_0 \sqcup \cdots \sqcup S_t$. Next, consider the family of parking functions, call it $\mathcal{F}(S_0, \ldots, S_t)$ with the cars in S_i in the i^{th} diagonal. (This family has previously been studied in [Haglund(2008)].) Since, as just mentioned, we need only consider parking functions with two parts, in particular, assume S_0 contains exactly two elements. This implies that in the corresponding two line arrays there will be only two diagonal numbers equal to 0. Say for one of our parking functions PF we have $d_0 = 0$ and $d_i = 0$, the we will let $\operatorname{top}(PF) = n + 1 - i$ give the number of cars weakly to the right of car c_i. This allows us to consider the following sum:

$$R_{S_0,\ldots,S_t}(x, q) = \sum_{PF \in \mathcal{F}(S_0,\ldots,S_t)} q^{\operatorname{dinv}(PF)} x^{\operatorname{top}(PF)}.$$

Or main breakthrough in [Hicks()] is a proof that the following conjecture, (which has now been verified for all $n < 15$) implies Conjecture 4.

Conjecture 5. *If $S_0 \sqcup \cdots \sqcup S_t = [n]$, $|S_0| = 2$, and $a \leq b - 1$ then*

$$R_{S_0,\ldots,S_t}(x, q)|_{x^b + x^{a+1}} = q \left(R_{S_0,\ldots,S_t}(x, q)|_{x^a + x^{b-1}} \right)$$

Now that we have an idea of the polynomials we would like to study, we may explain their connection to the polynomials that appeared in our introduction.

Theorem 1. *Let $S_0 \sqcup \cdots \sqcup S_t = [n]$ and $|S_0| = 2$ be one of our disjoint unions, then there exists a schedule W such that*

$$R_{S_0,\ldots,S_t}(x, q) = Q_W(x, q). \tag{10}$$

Moreover, the converse is also the case; that is, given a schedule W there exists a (not necessarily unique) disjoint union $S_0 \sqcup \cdots \sqcup S_t = [n]$ with $|S_0| = 2$ such that (10) holds.

Thus Conjecture 5 is equivalent to the following:

Conjecture 6. *For every schedule W and $a \leq b - 1$*

$$Q_W(x, q)|_{x^b + x^{a+1}} = q\ Q_W(x, q)|_{x^a + x^{b-1}}$$

Succinctly, our goal in the remainder of this paper is to study Conjecture 6 and some interesting properties of $Q_W(x, q)$.

4 Polynomial Properties

We begin this section with another way of generating $P_W(X_n; q)$, in particular one which allows us to directly find the coefficient of any given monomial in the x_i's.

Theorem 2. *Let $W = (w_1, \ldots, w_n)$ and $S \subset [n]$ contain exactly one of 1 or 2. Let*

$$m_i = \begin{cases} 0 & \text{if } i = 1 \text{ and } 2 \in S \\ 1 & \text{if } i = 1 \text{ and } 1 \in S \text{ or } i = 2 \\ \#(S \cap \{i-1, i-2, \ldots, i-w_i\}) & \text{if } i > 2 \end{cases} .$$

Then the coefficient of $\prod_{i \in S} x_i$ in $P_W(X_n; q)$ is nonzero if and only if

- *For all i in $S \backslash \{2\}$, $m_i \geq 1$*
- *For all i not in S, $w_i - m_i \geq 1$*

In this case, the coefficient of $\prod_{i \in S} x_i$ in $P_W(X_n; q)$ is exactly

$$A_S(q) = \left(\prod_{i \in S} [m_i]_q \right) \left(\prod_{i \notin S} q^{m_i} [w_i - m_i]_q \right)$$

Proof: By construction this is the case for $W = (1, 2)$. Working by induction, let $S \subset [n-1]$. We begin by applying B_{n,w_n} to a monomial $A_S(q) \prod_{i \in S} x_i$. Assume that $m_n = \#(S \cap \{n-1, n-2, \ldots, n-w_n\})$.

$$B_{n,w_n}\left(A_S(q) \prod_{i \in S} x_i \right) = \frac{1}{1-q} \left((x_n - q^w) A_S(q) \prod_{i \in S} x_i + (1 - x_n) q^{m_n} A_S(q) \prod_{i \in S} x_i \right) \tag{11}$$

$$= \left(x_n \left(\frac{1 - q^{m_n}}{1-q} \right) + \left(\frac{q^{m_n} - q^{w_n}}{1-q} \right) \right) A_S(q) \prod_{i \in S} x_i \tag{12}$$

$$= \left(x_n [m_n]_q + q^{m_n} [w_n - m_n]_q \right) A_S(q) \prod_{i \in S} x_i \tag{13}$$

Assuming the statement holds for $(w_1, w_2, \ldots, w_{n-1})$, we may inductively replace $A_S(q)$.

$$B_{n,w_n}\left(A_S(q) \prod_{i \in S} x_i \right) = \left(x_n [m_n]_q + q^{m_n} [w_n - m_n]_q \right) \left(\prod_{i \in S} [m_i]_q \right) \left(\prod_{i \notin S} q^{m_i} [w_i - m_i]_q \right) \prod_{i \in S} x_i \tag{14}$$

$$= \left(\prod_{i \in S \cup \{n\}} [m_i]_q \right) \left(\prod_{i \notin S \cup \{n\}} q^{m_i} [w_i - m_i]_q \right) \prod_{i \in S \cup \{n\}} x_i \tag{15}$$

$$+ \left(\prod_{i \in S} [m_i]_q \right) \left(\prod_{i \notin S} q^{m_i} [w_i - m_i]_q \right) \prod_{i \in S} x_i \tag{16}$$

and thus we have proved the required equality for S and $S \cup \{n\}$ and $W = (w_1, w_2, \ldots, w_n)$. $\square$

Using the previous theorem, we can conclude the following about the relationship between$A_S(q)$ and $A_{S^c}(q)$:

Theorem 3. *Let $W = (w_1, \ldots, w_n)$ and $S \subset [n]$ contain exactly one of 1 or 2. Then*

$$A_{S^c}(1/q) = q^{n-(\sum w_i)} A_S(q).$$

Proof: Here, when we consider the set S^c, we use m_i^c in place of m_i for ease of notation. Notice that by definition, $m_i^c = w_i - m_i$. Furthermore, recall that

$$[n]_q\big|_{q\to 1/q} = \frac{[n]_q}{q^{n-1}}.$$

Then

$$A_{S^c}(1/q) = \left[\left(\prod_{i\in S^c}[m_i^c]_q\right)\left(\prod_{i\notin S^c} q^{m_i^c}[w_i - m_i^c]_q\right)\right]_{q\to 1/q} \tag{17}$$

$$= \left[\left(\prod_{i\notin S}[w_i - m_i]_q\right)\left(\prod_{i\in S} q^{w_i - m_i}[m_i]_q\right)\right]_{q\to 1/q} \tag{18}$$

$$= \left(\prod_{i\notin S} q^{m_i - w_i + 1}[w_i - m_i]_q\right)\left(\prod_{i\in S} q^{1-w_i}[m_i]_q\right) \tag{19}$$

$$= q^{n-(\sum w_i)}\left(\prod_{i\notin S} q^{m_i}[w_i - m_i]_q\right)\left(\prod_{i\in S}[m_i]_q\right) \tag{20}$$

$$= q^{n-(\sum w_i)} A_S(q). \tag{21}$$

□

Corollary 1. *Let $W = (w_1, \ldots, w_n)$. Then*

$$P_W(X_n; q) = q^{n-(\sum_i w_i)}\left(\prod_i x_i\right) P_W\left(\frac{1}{x_1}, \ldots, \frac{1}{x_n}; \frac{1}{q}\right).$$

To simplify notation, say that

$$Q_W(x,q) = \sum_s A_s(q)x^s.$$

Corollary 2. *For all $1 \le s \le n/2$,*

$$A_s(q) + A_{n-s-1}(q) = q(A_{s+1}(q) + A_{n-s}(q)) \tag{22}$$

if and only if $R_s(q) = A_{s+1}(q) + A_{n-s}(q)$ is palindromic, with

$$R_s(q) = q^{n-(\sum w_i)-1} R_s(1/q). \tag{23}$$

Before we begin the proof, we observe that (22) is equivalent to the equations in Conjecture 6 with a suitable re-indexing, which will reappear again below.

Proof: By Corollary 1,

$$A_s(q) = q^{n-(\sum w_i)} A_s(1/q).$$

Then

$$A_s(q) + A_{n-s-1}(q) = q(A_{s+1}(q) + A_{n-s}(q)) \tag{24}$$

if and only if

$$q^{n-(\sum w_i)}(A_{s+1}(1/q) + A_{n-s}(1/q)) = q(A_{s+1}(q) + A_{n-s}(q)), \tag{25}$$

as required. □

5 The Functional Equation

Thus far, we have explained our interest in the polynomials $Q_W(x, q)$ and thus by extension $P_W(X_n; q)$, but not our interest in the functional equation. In fact, schedules which satisfy the functional equation satisfy Conjecture 6.

Theorem 4. *If a schedule W satisfies the functional equation, then it follows that for all $1 \leq s \leq n/2$,*

$$A_s(q) + A_{n-s-1}(q) = q(A_{s+1}(q) + A_{n-s}(q)). \tag{26}$$

Proof: Rewriting the left hand side of our functional equation,

$$(1 - q/x)Q_W(x; q) + x^{n-1}(1 - qx)Q_W(1/x; q) \tag{27}$$

$$= \sum_{s=1}^{n-1} A_s(q)x^s - \sum_{s=1}^{n-1} qA_s(q)x^{s-1} + \sum_{s=1}^{n-1} A_s(q)x^{n-s-1} - \sum_{s=1}^{n-1} qA_s(q)x^{n-s} \tag{28}$$

(29)

If a schedule satisfies the functional equation, then in (27) x^s must have vanishing coefficient when $1 \leq s \leq n - 2$. This happens exactly when

$$A_s(q) - qA_{s+1}(q) + A_{n-s-1}(q) - qA_{n-s}(q) = 0 \tag{30}$$

as required. □

We end with a number of preliminary results about schedules which satisfy the functional equation. Working by computer, Eugene Rodemich has shown that all schedules of length less than 15 satisfy the functional equation ([Rodemich(2011)].) A crucial, if innocuous sounding statement is the following.

Theorem 5. *Let $W = (w_1, \ldots, w_n)$ be a schedule with $j > 1$ such that $w_j = 1$. Then if $(w_1, \ldots, w_{j-1})$ and $(w_1, \ldots, w_{j-2})$ satisfy the functional equation, W satisfies the functional equation.*

The importance of this result is evident when we consider two schedules, $W = (w_1, \ldots, w_n)$ and $\overline{W} = (w_1, \ldots, w_{n-1}, 1)$. Once we conclude that the latter satisfies the functional equation, we may prove that W does as well by showing that

$$0 =(1 - q/x)(Q_W(x; q) - [w_n]_q Q_{\overline{W}}(x; q)) + x^{n-1}(1 - qx)(Q_W(1/x; q) - [w_n]_q Q_{\overline{W}}(1/x; q)), \quad (31)$$

which in practice is a much easier task. Moreover, we have the following:

Theorem 6. *Let* $W = (w_1, \ldots, w_n)$*,* $\overline{W} = (w_1, \ldots, w_{n-1}, 1)$*, and*

$$R_W(x, q) = Q_W(x; q) - [w_n]_q Q_{\overline{W}}(x; q).$$

Then we have the factorization

$$R_W(x, q) = (1 - x)(1 - qx)S_W(x; q).$$

Moreover, if $\overline{W}$ *satisfies the functional equation,* W *satisfies the functional equation if and only if*

$$S_W(x; q) = x^{n-1}S_W(1/x; q).$$

Several of our remaining results consider both W and $\overline{W}$.

Theorem 7. *If* $W = (1, 2, w_3 \ldots, w_m, s, s+1, \ldots, s+a-1)$ *and* $\overline{W}$ *both satisfy the functional equation, then*

$$W = (1, 2, w_3 \ldots, w_m, s, s + 1, \ldots, s + a - 1, a)$$

also satisfies the functional equation.

Moreover, we have that

Theorem 8. *Let* $W = (1, 2, w_3 \ldots, w_n)$ *and* $\overline{W}$ *satisfy the functional equation. Then* $(1, 2, v, w_3, \ldots, w_4)$ *also satisfies the functional equation for* $v = 2, 2$ *and* $v = 2, 3$*.*

This theorem gives us several infinite families that we may conclude satisfy the functional equation explicitly, including two easily described families:

Corollary 3. *Schedules of the form* $W = (1, 2, 2, 3, 2, 3, \ldots, 2, 3)$ *and* $W = (1, 2, 2, 2, \ldots, 2)$ *satisfy the functional equation.*

As in several of the above examples, it is often easier to conclude that a schedule satisfies the functional equation if we first assume some smaller schedules satisfy the functional equation. We formalize this idea by stating the following:

If a schedule $W = (w_1, \ldots, w_n)$ can be shown to satisfy the functional equation under the assumpation that schedules of length less than n satisfy the functional equation, say that the schedule *inductively* satisfies the functional equation. Notice that showing all schedules inductively satisfy the functional equation is enough to prove Conjecture 1. We then have the obvious corollary:

Corollary 4. *Schedules containing multiples 1's and schedules beginning with* $(1, 2, 2, 2, 2, w_6, \ldots, w_n)$ *or* $(1, 2, 2, 3, 2, w_6, \ldots, w_n)$ *inductively satisfy the functional equation.*

Additionally we have the following:

Theorem 9. *Schedules beginning with* $(1,2,2,2,3,2,w_7,\ldots,w_n)$, $(1,2,2,3,3,2,w_7,\ldots,w_n)$, $(1,2,2,3,4,2,w_7,\ldots,w_n)$, $(1,2,2,2,3,3,2,w_8,\ldots,w_n)$, $(1,2,2,2,3,4,2,w_8,\ldots,w_n)$, *and* $(1,2,2,3,3,3,2,w_8,\ldots,w_n)$ *inductively satisfy the functional equation.*

Note that when combined with the previous corollary, considering the slow growth restriction on our schedules, this theorem substantially reduces the family of schedules not known to inductively satisfy the functional equation.

Finally, several other infinite families have also been shown explicitly to satisfy the functional equation, for example:

Theorem 10. $(1,2,2,3,4,\ldots,k)$ *satisfies the functional equation for any value of* k.

Proof: In fact,

$$Q_{(1,2,2,3,4,\ldots,k)}(x;q) = \sum_{m=1}^{k}(1+q)q^{k-m}[k-1]_q!x^m,$$

which can be explicitly shown to satisfy the required property. □

References

[Bergeron and Garsia(1999)] F. Bergeron and A. M. Garsia. Science fiction and Macdonald's polynomials. In *Algebraic methods and q-special functions (Montréal, QC, 1996)*, volume 22 of *CRM Proc. Lecture Notes*, pages 1–52. Amer. Math. Soc., Providence, RI, 1999.

[Garsia and Procesi(1992)] A. M. Garsia and C. Procesi. On certain graded S_n-modules and the q-Kostka polynomials. *Adv. Math.*, 94(1):82–138, 1992. ISSN 0001-8708. doi: 10.1016/0001-8708(92)90034-I. URL http://dx.doi.org/10.1016/0001-8708(92)90034-I.

[Haglund(2008)] J. Haglund. *The q,t-Catalan numbers and the space of diagonal harmonics*, volume 41 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2008. ISBN 978-0-8218-4411-3; 0-8218-4411-3. With an appendix on the combinatorics of Macdonald polynomials.

[Haglund et al.(2005)] J. Haglund, M. Haiman, N. Loehr, J. B. Remmel, and A. Ulyanov. A combinatorial formula for the character of the diagonal coinvariants. *Duke Math. J.*, 126(2):195–232, 2005. ISSN 0012-7094. doi: 10.1215/S0012-7094-04-12621-1. URL http://dx.doi.org/10.1215/S0012-7094-04-12621-1.

[Haglund et al.(2011)] J. Haglund, J. Morse, and M. Zabrocki. A compositional shuffle conjecture specifying touch points of the dyck path. *Candian Journal Mathematics*, 2011. to appear.

[Hicks()] A. Hicks. PhD thesis, University of California-San Diego. in progress.

[Rodemich(2011)] E. Rodemich. private communication, 2011.