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Projective invariants of vector configurations

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Abstract. We investigate the Zariski closure of the projective equivalence class of a matrix. New results are presented regarding the matrices in this variety and their matroids, and we give equations for the variety. We also discuss the K-polynomial of the closure of a projective equivalence class, and two other geometric invariants that can be obtained from this.

Résumé. Nous enquêtons sur l'adhérence Zariski de la classe d'équivalence projective d'une matrice. Des résultats nouveaux sont présentés sur les matrices dans cette variété et leur matroïdes, et nous donnons équations pour la variété. Nous discutons également le K-polynôme de l'adhérence de la classe d'équivalence projective, et deux autres invariants géométriques qui peuvent être obtenus à partir de cela.

Keywords: projective equivalence, matroids, orbit closure

1 Introduction

Let $v \in \mathbb{A}^{r\times n}$ be a matrix with entries in a field $\mathbf{k}$ of characteristic zero. The columns of v are thought of as a vector configuration $(v_1, \ldots, v_n)$. The group $\mathrm{GL}_r = \mathrm{GL}_r(\mathbf{k})$ acts on the left of $\mathbb{A}^{r\times n}$ and the algebraic torus $T = (\mathbf{k}^\times)^n \subset \mathrm{GL}_n(\mathbf{k}^n)$ acts on the right, by $(g,t)\cdot v = gvt^{-1}$. The configurations that are **projectively equivalent** to v are those in the orbit

$$\mathrm{GL}_r vT = \{gvt : g \in \mathrm{GL}_r, t \in T\}.$$

The Zariski closure of this orbit is an irreducible subvariety of $\mathbb{A}^{r\times n}$, which we denote by $\overline{\mathrm{GL}_r vT}$ and refer to as a **matrix orbit closure**. The ideal of $\overline{\mathrm{GL}_r vT}$, denoted I_v, is contained in the coordinate ring of $\mathbb{A}^{r\times n}$, denoted $R = \mathbf{k}[x_{i,j} : 1 \le i \le r, 1 \le j \le n]$.

Broadly, the goal of our work is to thoroughly understand matrix orbit closures and their relations to the matroid of v. Recall that the **matroid** of a matrix $v \in \mathbb{A}^{r\times n}$, denoted $M(v)$, is the simplicial complex on $[n]$ whose faces index independent subsets of columns of v.

Our work is motivated by attempting to explain similarities observed in two related objects, both of which are orbit closures. Let $(\mathbb{A}^{r\times n})^{\mathrm{nz}}$ denote the matrices in $\mathbb{A}^{r\times n}$ with *n*o *z*ero columns, and $(\mathbb{A}^{r\times n})^{\mathrm{fr}}$

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the matrices of *f*ull *r*ank. These are principal T and GL_r bundles over, respectively, $(\mathbb{P}^{r-1})^n$ and the Grassmannian $\mathbb{G}_r(n)$ of r dimensional subspaces of $\mathbf{k}^n$. These spaces fit into the following diagram:

$$\begin{tikzcd} & \mathbb{A}^{r\times n} & \\ (\mathbb{A}^{r\times n})^{\mathrm{nz}} \arrow[ur, hook, "j"] \arrow[d, "\rho"] & & (\mathbb{A}^{r\times n})^{\mathrm{fr}} \arrow[ul, hook', "i"'] \arrow[d, "\pi"] \\ (\mathbb{P}^{r-1})^n & & \mathbb{G}_r(n) \end{tikzcd} \tag{1}$$

The GL_r-orbit closures in $(\mathbb{P}^{r-1})^n$ and T-orbit closures in $\mathbb{G}_r(n)$ are closely related to our matrix orbit closures.

One way to study orbits in these variaties is to form geometric invariant theory quotients of these spaces, as was done for $(\mathbb{P}^{r-1})^n$ by Mumford. Another way is to form the Chow quotient by these actions, which is the Zariski closure of the sufficiently general GL_r-orbits (respectively, T-orbits) in the Chow variety of $(\mathbb{P}^{r-1})^n$ (resp., $\mathbb{G}_r(n)$). This was done by [Kapranov(1993)], and later studied by [Keel and Tevelev(2006)], and [Hacking et al.(2006)]. Kapranov refines the Gel′fand–MacPherson correspondance to show that the there is an isomorphism of Chow quotients

$$\mathrm{GL}_r\backslash\backslash(\mathbb{P}^{r-1})^n \approx \mathrm{GL}_r \times T\backslash\backslash\mathbb{A}^{r\times n} \approx \mathbb{G}_r(n)//T.$$

Hence, the Chow quotient is a natural way to complete the diagram above into a hexagon.

Certain particular questions we would like to answer about these orbit closures follow.

1. What is the class of $\overline{\mathrm{GL}_r vT}$ in $\mathrm{GL}_r \times T$-equivariant K-theory of $\mathbb{A}^{r\times n}$? This class can be thought of as a polynomial, called the K-polynomial of $\overline{\mathrm{GL}_r vT}$, of the form

$$\sum_{\lambda\in\Lambda,\mathbf{b}\in\mathbb{N}^n} d_{\lambda,\mathbf{b}}(v)s_\lambda(u^{-1})t^{\mathbf{b}} \in \mathbb{Z}[u_1^{-1},\ldots,u_r^{-1},t_1,\ldots,t_n]^{\mathfrak{S}_r},$$

where $d_{\lambda,\mathbf{b}}(v)$ are integers depending on v, s_λ is the Schur polynomial in its arguments and Λ is the set of partitions. Ideally we would like an explicit formula for $d_{\lambda,\mathbf{b}}(v)$.

2. What is the GL_r-module decomposition of the space of $(1,1,\ldots,1)$-weight functions on a GL_r-orbit closure in $(\mathbb{P}^{r-1})^n$? This is related to the problem of when symmetrizations of decomposable tensors are zero, a problem studied by Dias da Silva's school. The K-polynomial of $\overline{\mathrm{GL}_r vT}$ can be used to answer this question.

3. What is the class of a T-orbit closure $\overline{\pi(v)T}$ in the T-equivariant or ordinary cohomology of the Grassmannian $\mathbb{G}_r(n)$? This question was studied by Kapranov, Klyachko, Speyer. The answer, once again, is determined by the K-polynomial of $\overline{\mathrm{GL}_r vT}$.

In this extended abstract we announce several new results on these orbit closures. Specifically, we characterize the matrices in $\overline{\mathrm{GL}_r vT}$ and their matroids. We also describe a set of equations whose vanishing locus is precisely $\overline{\mathrm{GL}_r vT}$. We then offer our main conjecture on the coefficients $d_{\lambda,\mathbf{b}}(v)$. Following this, we present several results pertaining to the second and third questions above. Namely, we characterize the support of the space of $(1,1,\ldots,1)$-weight functions on a GL_r-orbit closure in $(\mathbb{P}^{r-1})^n$ in terms of the matroid of v. Lastly, a new formula is given for the cohomology class of a generic T-orbit class in the Grassmannian.

2 Geometry of orbit closures

In this section we discuss the geometry of the closures $\overline{\mathrm{GL}_r vT}$ with respect to the $\mathrm{GL}_r \times T$ orbits they comprise.

Proposition 2.1 *The closure of a* $\mathrm{GL}_r \times T$*-orbit in* $\mathbb{A}^{r\times n}$ *is an irreducible affine variety. If* v *has a connected matroid of full rank then* $\dim(\overline{\mathrm{GL}_r vT}) = r^2 + n - 1$*; otherwise the dimension is less.*

A matroid is said to be connected if it cannot be written as the direct sum (simplicial complex join) of two matroids. The hypothesis above is quite tame, as even the highly degenerate matrix

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

has a connected matroid.

Recall that π is the projection of the GL_r-bundle $(\mathbb{A}^{r\times n})^{\mathrm{fr}} \to \mathbb{G}_r(n)$. Consider the case that $v \in (\mathbb{A}^{r\times n})^{\mathrm{fr}}$. It is well known that $\overline{\pi(v)T}$ is the toric variety associated to the matroid polytope of $M(v)$. The **matroid (base) polytope** $P(M(v))$ of $M(v)$ is the convex hull in $\mathbf{R}^n$ of the indicator vectors of bases of $M(v)$. The T-orbits in $\overline{\pi(v)T}$ are in bijection with the faces of this polytope. We can give a combinatorial description of the faces of the matroid polytope as follows [Ardila and Klivans(2006), Proposition 2]. Let $S_\bullet$ be a flag of subsets

$$\emptyset = S_0 \subset S_1 \subset \cdots \subset S_k \subset S_{k+1} = [n].$$

Every face of $P(M(v))$ is of the form $P(M(v)_{S_\bullet})$ where

$$M(v)_{S_\bullet} = \bigoplus_{i=1}^{k+1} (M(v)|S_i)/S_{i-1}.$$

A realization of this result in terms of torus orbit closures is obtained as follows: Rescale column $i \in S_j \setminus S_{j-1}$ of v by s^{j-1}. Projecting this matrix into $\mathbb{G}_r(n)$ we obtain a subspace $\pi(v)\lambda(s)$, where $\lambda(s)$ is a one-parameter subgroup of $T(\mathbf{k}(\!(s)\!))$. Taking the limit $\lim_{s\to 0}\pi(v)\lambda(s)$ yields a point of $\overline{\pi(v)T}$ with matroid $M(v)_{S_\bullet}$. Since every T-orbit in $\overline{\pi(v)T}$ is of this form, we get a geometric proof of [Ardila and Klivans(2006), Proposition 2].

The pullback $\pi^{-1}(\lim_{s\to 0}\pi(v)\lambda(s))$ is the $\mathrm{GL}_r \times T$-orbit of a full rank matrix in $\overline{\mathrm{GL}_r vT}$ whose matroid is $M(v)_{S_\bullet}$. We call any such matrix a **projection of** v **along the flag** $S_\bullet$.

Example 2.2 *Consider the matrix*

$$v = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \in \mathbb{A}^{3\times 6}.$$

The matrices below are projections of v along the flags $\emptyset \subset \{1\} \subset \{1,2,3,4\} \subset \{1,2,3,4,5,6\}$ and $\emptyset \subset \{1,2,4\} \subset \{1,2,3,4,5,6\}$, respectively.

$$\begin{bmatrix}1&0&0&0&0&0\\0&1&0&1&0&0\\0&0&1&0&0&0\end{bmatrix} \qquad \begin{bmatrix}1&0&0&1&0&0\\0&1&0&1&0&0\\0&0&1&0&1&1\end{bmatrix}$$

The next result shows that all elements of $\overline{\mathrm{GL}_r vT}$ are obtained by projecting v along some flag and applying some element $g \in \mathrm{End}(\mathbf{k}^r)$ on the left.

Proposition 2.3 *Suppose that v has rank r and $w \in \overline{\mathrm{GL}_r vT}$ is a matrix of rank less than r. Then there is a matrix $w' \in \overline{\mathrm{GL}_r vT}$ whose rank is that of v, and $w = gw'$ for some singular $g \in \mathrm{End}(\mathbf{k}^r)$.*

Corollary 2.4 *If $w \in \overline{\mathrm{GL}_r vT}$ then there is a flag of sets $S_\bullet$ such that the matroid of w is a quotient of*

$$\bigoplus_{i=1}^{k+1} (M(v)|S_i)/S_{i-1}.$$

Conversely, every quotient of such a matroid occurs as the matroid of some $w \in \overline{\mathrm{GL}_r vT}$.

3 The ideal of an orbit closure

Let $I_v \subseteq R$ be the ideal of $\overline{\mathrm{GL}_r vT}$. Being the ideal of the image of a rational map, I_v can be computed as the kernel of the associated ring map, as follows. Let $\mathbf{k}[z,t]$ denote the polynomial ring in variable $z_{i,j}$, $1 \le i,j \le r$ and $t_1,\dots,t_n$, where we think of z as the r-by-r matrix of variables $z_{i,j}$ and t as the list of variables $(t_1,\dots,t_n)$. Notice that $\mathbf{k}[z,t]$ is the affine coordinate ring of $\mathrm{End}(\mathbf{k}^r) \times \mathbf{k}^n$, and that $\overline{\mathrm{GL}_r vT}$ is the closure of the subspace $\mathrm{End}(\mathbf{k}^r) v \mathbf{k}^n \subset \mathbb{A}^{r\times n}$. Define the ring homomorphism $\varphi_v : R \to \mathbf{k}[z,t]$ that maps $x_{i,j}$ to the (i,j) entry of zvt.

Proposition 3.1 *The kernel of the ring homomorphism $\varphi_v : R \to \mathbf{k}[z,t]$ is the ideal of $\overline{\mathrm{GL}_r vT}$ in R.*

We now give generators for I_v, up to radical; that is, we give the polynomial conditions for a matrix to lie in $\overline{\mathrm{GL}_r vT}$. We need the notion of Gale duality. For $v \in \mathbb{A}^{r\times n}$, its **Gale dual** is any $v^\perp \in \mathbb{A}^{(n-\mathrm{rk}(v))\times n}$ whose rows form a basis for the (right) kernel of v. Thus, the Gale dual is determined up to the action of $\mathrm{GL}_{n-\mathrm{rk}(v)}$ on $\mathbb{A}^{(n-\mathrm{rk}(v))}$. If v has full rank then Gale duality really is a duality: $\mathrm{GL}_r(v^\perp)^\perp = \mathrm{GL}_r v$.

Example 3.2 *Here is a configuration v, its Gale dual $v^\perp$, and its double dual $(v^\perp)^\perp$.*

$$v = \begin{bmatrix}1&0&1&1\\0&1&1&1\\1&1&2&2\end{bmatrix}, \qquad v^\perp = \begin{bmatrix}1&1&-1&0\\1&1&0&-1\end{bmatrix}, \qquad (v^\perp)^\perp = \begin{bmatrix}1&0&1&1\\0&1&1&1\end{bmatrix}.$$

For any $w = (w_1,\dots,w_n) \in \mathrm{GL}_r vT$, the vectors

$$w_1 \otimes v_1^\perp, \quad w_2 \otimes v_2^\perp, \quad \dots, \quad w_n \otimes v_n^\perp$$

are linearly dependent. This can be seen by expanding a linear combination in the standard basis of $\mathbf{k}^r \otimes \mathbf{k}^{n-r}$. By continuity this holds for any $u \in \overline{\mathrm{GL}_r vT}$. More is true:

Proposition 3.3 ([Kapranov(1993)]) *Suppose that $w \in \mathbb{A}^{r\times n}$ has a connected matroid and full rank. If the collection of tensors*

$$w_1 \otimes v_1^\perp, \quad w_2 \otimes v_2^\perp, \quad \ldots, \quad w_n \otimes v_n^\perp$$

forms a circuit in $\mathbf{k}^r \otimes \mathbf{k}^{n-r}$ then $w \in \mathrm{GL}_r vT$.

For a subset J of $[n]$, let v_J be the submatrix of v on the columns indexed by J, so that the rank $\mathrm{rk}(J)$ in the matroid of v is the dimension of the span of these columns in $\mathbf{k}^r$. The Gale dual of v_J is not $(v^\perp)_J$, but it is a projection of this configuration. This fact is matroidally manifested by the equality $(M|J)^* = M^*/J^c$ where J^c is the complement of J in the ground set of M.

Theorem 3.4 *For any $v \in \mathbb{A}^{r\times n}$, a matrix w is in $\overline{\mathrm{GL}_r vT}$ if and only if, for every $J = \{j_1, \ldots, j_\ell\} \subset [n]$ the tensors*

$$\{w_{j_i} \otimes (v_J^\perp)_i : i = 1, \ldots, \ell\}, \tag{2}$$

are linearly dependent.

Theorem 3.4 immediately gives equations for the ideal I_v. Let x denote the matrix of variables $x_{i,j}$, and let x_j denote the j-th column $(x_{1,j}, \ldots, x_{r,j})^t$ of x. For each subset $J = \{j_1, \ldots, j_\ell\} \subset [n]$ we form the matrix $x_J \otimes v_J^\perp$, whose columns are the tensors $x_{j_i} \otimes (v_J^\perp)_i \in R^r \otimes \mathbf{k}^{n-\mathrm{rk}(v_J)}$.

Theorem 3.5 *The size $|J|$ minors of the matrices $x_J \otimes v_J^\perp$, $J \subset [n]$, generates the ideal I_v, up to radical.*

Example 3.6 *Consider the matrix with rational entries*

$$v = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \in \mathbb{A}^{3\times 6}.$$

The generators of Theorem 3.5 form a radical ideal (we checked this with `Macaulay2`), and hence they generate I_v exactly. There are 121 generators in I_v, whose degrees range from 3 to 6. Under the term order that reads the matrix x from right-to-left, bottom-to-top, our generators form a Gröbner basis for I_v.

It is worth pointing out that these generators are not all binomials in the minors of $x = (x_{i,j})$, unlike what one might expect from experience with the case of torus orbit closures in Grassmannians.

4 K-polynomials

In this section we consider the class of $\overline{\mathrm{GL}_r vT}$ in the $\mathrm{GL}_r \times T$-equivariant K-theory of $\mathbb{A}^{r\times n}$. In notation we make an effort to follow the book [Miller and Sturmfels(2005)].

4.1 Definitions

Recall that $R = \mathbf{k}[x_{i,j} : 1 \le i \le r, 1 \le j \le n]$ and we may regard $\mathbb{A}^{r\times n}$ as $\mathrm{Spec}\, R$. R is graded by $\mathbb{Z}^r \times \mathbb{Z}^n$, the degree of $x_{i,j}$ being $(b_j, -a_i)$, where $a_1, \ldots, a_r, b_1, \ldots, b_n$ are the standard basis vectors of $\mathbb{Z}^r \times \mathbb{Z}^n$. The grading group should be thought of as the weight lattice of the maximal torus in $\mathrm{GL}_r \times T$ given by (the diagonal torus of GL_r) $\times\, T$.

A finitely generated graded R-module $M = \bigoplus_{(\mathbf{a},\mathbf{b})\in\mathbb{Z}^r\times\mathbb{Z}^n} M_{(\mathbf{a},\mathbf{b})}$ has **Hilbert series**

$$\mathrm{Hilb}(M) = \sum_{(\mathbf{a},\mathbf{b})\in\mathbb{Z}^r\times\mathbb{Z}^n} \dim_{\mathbf{k}}(M_{(\mathbf{a},\mathbf{b})}) u^{\mathbf{a}} t^{\mathbf{b}} \in \mathbb{Z}[[u_1^{\pm 1}, \ldots, u_r^{\pm 1}, t_1^{\pm 1}, \ldots, t_n^{\pm 1}]]$$

By [Miller and Sturmfels(2005), Theorem 8.20], this can be written in the form

$$\frac{\mathcal{K}(M;u,t)}{\prod_{i=1}^{r}\prod_{j=1}^{n}(1-t_j/u_i)},$$

the numerator being a Laurent polynomial that we refer to as the K**-polynomial** of M. In the case that M is the coordinate ring of an affine variety, then we call the K-polynomial of the coordinate ring the K-polynomial of the variety.

The ring R has the action of $\mathrm{GL}_r \times T$ given by $((g,t)\cdot f)(v) = f(g^{-1}vt)$. The decomposition of R into its various graded pieces $R_{(\mathbf{a},\mathbf{b})}$ is a refinement of the irreducible decomposition of R as a $\mathrm{GL}_r \times T$-module; it is precisely the refinement into weight spaces. If an ideal $I \subset R$ is $\mathrm{GL}_r \times T$ invariant, then the K-polynomial of R/I is a symmetric Laurent polynomial in the u's. This follows by decomposing I into irreducible $\mathrm{GL}_r \times T$-modules and seeing that $\mathrm{Hilb}(R/I)$ is literally the character of this $\mathrm{GL}_r \times T$-module. Multiplying by the symmetric polynomial $\prod_{j=1}^{n}(1-t_j/u_i)$ yields the K-polynomial, which is thus symmetric. We may thus uniquely write the K-polynomial of a matrix orbit closure as

$$\mathcal{K}(R/I_v;u,t) = \sum_{\lambda\in\Lambda,\mathbf{b}\in\mathbb{N}^n} d_{\lambda,\mathbf{b}}(v)s_\lambda(u^{-1})t^{\mathbf{b}},$$

for some integers $d_{\lambda,\mathbf{b}}(v)$. Here, Λ denotes the set of all partitions and s_λ denotes the Schur polynomial in its arguments.

The $\mathrm{GL}_r \times T$-equivariant K-theory ring of $\mathbb{A}^{r\times n}$ is the ring of possible K-polynomials of graded R-modules, namely

$$\begin{aligned} K^0_{\mathrm{GL}_r\times T^n}(\mathbb{A}^{r\times n}) &= \mathbb{Z}[[u_1,\dots,u_r,t_1,\dots,t_n]][u_1^{-1},\dots,u_r^{-1},t_1^{-1},\dots,t_n^{-1}]^{\mathfrak{S}_r} \\ &= \mathbb{Z}[[e_1(u),\dots,e_r(u),t_1,\dots,t_n]][e_r(u)^{-1},t_1^{-1},\dots,t_n^{-1}], \end{aligned}$$

where $\mathfrak{S}_r$ acts on the u variables. The class in this ring of a sheaf $\mathcal{E}$ on $\mathbb{A}^{r\times n}$ is the K-polynomial of its module of global sections. In particular, the class of the structure sheaf of a $\mathrm{GL}_r \times T$-invariant algebraic subset of $\mathbb{A}^{r\times n}$ with defining ideal I is the K-polynomial of R/I. Note that $K^0_{\mathrm{GL}_r\times T^n}(\mathbb{A}^{r\times n})$ equals the equivariant K-theory ring of a point, since $\mathbb{A}^{r\times n}$ is a vector bundle over a point.

4.2 K-polynomials of orbit closures

The K-polynomial of $\overline{\mathrm{GL}_r vT}$ varies with v. In this section we offer a very strong conjecture stating that it does not vary too wildly.

Conjecture 4.1 *Suppose that $v \in \mathbb{A}^{r\times n}$ has a full dimensional orbit. Then, the K-polynomial of $\overline{\mathrm{GL}_r vT}$ is determined by the labelled matroid of v.*

We offer a few comments on why one might find this conjecure surprising, and then say a few words on how one might go about proving it.

Every matrix in $\mathbb{A}^{r\times n}$ has an associated matroid. The set of matrices with a fixed matroid is called a matroid stratum of $\mathbb{A}^{r\times n}$. A matroid stratum can contain arbitrarily complicated singularities, a result due in various forms to various authors (most commonly cited is Mnëv; also Bokowski–Sturmfels, Lafforgue, Richter-Geber). These results are collectively and uncarefully referred to as the "universality theorem" for matroid realization spaces. The K-polynomials of full matroid strata in $\mathbb{A}^{r\times n}$ are considered by [Fehér et al.(2010)].

In light of the universality theorem it is entirely plausible that when one considers a very singular point v of a matroid stratum, $\mathcal{K}(R/I_v;u,t)$ might deviate from an expected generic answer. The conjecture above posits that $\mathcal{K}(R/I_v;u,t)$ is constant as v ranges over even the most complicated matroid stratum.

The strategy of the proof is allows: One starts by relating the K-polynomial of $\overline{\mathrm{GL}_r vT}$ to the K-polynomial of the torus orbit closure $\overline{\pi(v)T} \subset \mathbb{G}_r(n)$. The latter can be computed explicitly using the polyhedral combinatorics of the matroid polytope of v. The class of $\overline{\pi(v)T}$ in the T-equivariant K-theory of $\mathbb{G}_r(n)$ is considered by [Fink and Speyer(2010)], where the connection to polyhedral combinatorics is made precise.

The technically difficult step of the proof arises from the singular locus of the orbit closure. Indeed, in joint work with Dave Anderson, we can show that if $\overline{\mathrm{GL}_r vT}$ has *rational singularities* then Conjecture 4.1 is true. Following a strategy of [Weyman(2003)], this amounts to proving that a certain vector bundle on $\overline{\pi(v)T}$ has vanishing higher cohomology. This last portion of the proof of the conjecture is work in progress.

Applying [Fink and Speyer(2010), Proposition 3.3] we conclude Corollary 4.2(a), below.

Recall that a matroid base polytope subdivision is a polyhedral subdivision of $P(M(v))$, the matroid base polytope of $M(v)$, where each facet in the subdivision is a matroid base polytope. Recall that a function f from the set of labelled matroids to an abelian group is said to behave valuatively if whenever $P(M) = \bigcup_i P(M_i)$ is a matroid base polytope subdivision,

$$f(M) = \sum_i f(M_i) - \sum_{(i,j)} M_{i,j} + \sum_{(i,j,k)} M_{i,j,k} - \dots$$

Here $M_{i_1,\dots,i_k}$ is the matroid whose polytope is the intersection $P(M_{i_1})\cap\cdots\cap P(M_{i_k})$. See [Ardila et al.(2010)] for more on such subdivisions. From [Fink and Speyer(2010), Proposition 4.3] we conclude part (b) of the following.

Corollary 4.2 *Suppose that Conjecture 4.1 is true.*

Then, given any rank r matroid M on n elements, realizable over $\mathbf{k}$ or not, one may construct in a uniform way an element $z(M) \in \mathbb{Z}[u_1^{\pm 1},\dots,u_r^{\pm 1},t_1^{\pm 1},\dots,t_n^{\pm 1}]^{\mathfrak{S}_r}$ that has the following properties:

(a) When M is realizable over $\mathbf{k}$ by a matrix $v \in \mathbb{A}^{r\times n}$, the polynomial $z(M)$ is the K-polynomial of $\overline{\mathrm{GL}_r vT}$.

(b) For v with a full dimensional orbit, and M of rank r, the function $z(M)$ behaves valuatively on matroid base polytope subdivisions.

5 The tensor module

In this section we investigate the second of our desiderata from the introduction. Given $v \in (\mathbb{A}^{r\times n})^{\mathrm{nz}}$ we form the $\mathbf{k}$-vector space $G(v) \subset (\mathbf{k}^r)^{\otimes n}$ that is spanned by the tensors

$$\{gv_1 \otimes \cdots \otimes gv_n : g \in \mathrm{GL}_r\}.$$

We dub this the **tensor module** of v. We begin by relating $G(v)$ to the line bundle $\mathcal{O}(1,\dots,1)$ on $(\mathbb{P}^{r-1})^n$, which is the external tensor product of the $\mathcal{O}(1)$'s on each factor.

Recall that j is the inclusion of the matrices with non-zero columns into $\mathbb{A}^{r\times n}$. The inverse image $j^{-1}\overline{\mathrm{GL}_r vT}$ is the intersection of $\overline{\mathrm{GL}_r vT}$ with $(\mathbb{A}^{r\times n})^{\mathrm{nz}}$ and the projection of this to $(\mathbb{P}^{r-1})^n$ is the GL_r-orbit closure of $\rho(v)$.

Proposition 5.1 *For $v \in (\mathbb{A}^{r\times n})^{\mathrm{nz}}$, $G(v)$ is the $\mathbf{k}$-vector space dual to the global sections of $\mathcal{O}(1,\dots,1)|_{\overline{\mathrm{GL}_r\rho(v)}}$. This isomorphism is GL_r-equivariant and hence the character of $G(v)$, as a GL_r-module, is the coefficient of $t_1\cdots t_n$ in*

$$\frac{\mathcal{K}(R/I_v; u^{-1}, t)}{\prod_{i\le r, j\le n}(1-u_i t_j)}.$$

Thus, if Conjecture 4.1 is true, then the irreducible decomposition of $G(v)$ is determined by the matroid of v.

Our primary interest in the tensor invariant stems from its connection to the rank partition of the matroid of v. Let $M = M(v)$ denote the matroid of v. The **rank partition** of M is the sequence $\rho(M) = (\rho_1, \rho_2, \rho_3, \dots)$ determined by the condition that

$$\rho_1 + \rho_2 + \cdots + \rho_k$$

is the size of the largest union of k independent sets in M.

Theorem 5.2 (Dias da Silva) *The rank partition of M is a partition. If M has no loops (which is true if $v \in (\mathbb{A}^{r\times n})^{\mathrm{nz}}$), then there is a set partition of the ground set of M into independent sets of sizes $\lambda = (\lambda_1 \ge \lambda_2 \ge \dots \lambda_\ell) \vdash n$ if and only if $\lambda \le \rho(M)$ (dominance order).*

A result of Gamas on the vanishing of symmetrized tensors yields the following description of the support of $G(v)$.

Theorem 5.3 *The tensor module $G(v)$ has an irreducible submodule of highest weight λ if and only if $\lambda \ge \rho(M)'$.*

Let $S \subset [n]$ be a subset and $M|_S$ the restriction of the matroid of v to S. In the Hilbert series of R/I_v, the coefficient of $s_\lambda(u^{-1})\prod_{i\in S} t_i$ is a positive integer if and only if $\lambda \ge \rho(M|_S)'$.

Using Schur–Weyl duality one can relate $G(v)$ to the smallest $\mathfrak{S}_n$-representation in $(\mathbf{k}^r)^{\otimes n}$ containing $v_1\otimes\cdots\otimes v_n$. Recall that Schur-Weyl duality asserts that GL_r and $\mathfrak{S}_n$ generate $\mathrm{End}_{\mathfrak{S}_r}((\mathbf{k}^r)^{\otimes n})$ and $\mathrm{End}_{\mathrm{GL}_r}((\mathbf{k}^r)^{\otimes n})$, respectively.

Proposition 5.4 *The tensor module $G(v)$ is Schur-Weyl dual to $\mathfrak{S}(v)$. This is to say,*

$$\mathrm{Hom}_{\mathrm{GL}_r}(G(v), (\mathbf{k}^r)^{\otimes n}) \approx \mathfrak{S}(v) \qquad \mathrm{Hom}_{\mathfrak{S}_n}(\mathfrak{S}(v), (\mathbf{k}^r)^{\otimes n}) \approx G(v),$$

as $\mathfrak{S}_n$ and GL_r modules, respectively.

This is useful in the proof of the next result which characterizes the coefficient of a hook shaped Schur polynomial $s_\lambda(u^{-1})t^{\mathbf{b}}$ in the multigraded Hilbert series of R/I_v. For an element $\mathbf{b}\in\mathbb{N}^n$ let $\underline{\mathbf{b}} \in \{0,1\}^n$ denote its support, which we conflate with the subset of $[n]$ it indexes, and let $|\mathbf{b}|$ be the sum $\sum_i b_i$. We let $\lambda_{m,k}$ denote the hook shape that is a partition of m and has k parts.

Recall that a broken circuit of a matroid M with ground set $[n]$ is a circuit with its smallest element deleted. An independent set of M is said to be a no broken circuit set (NBC) if it contains no broken circuits.

Proposition 5.5 *The coefficient of $s_{\lambda_{|\mathbf{b}|,k}}(u^{-1})t^{\mathbf{b}}$ in the Hilbert series of R/I_v is equal to the number of NBC bases of the truncation of $M(v)|_{\underline{\mathbf{b}}}$ to rank k, if this matroid has rank at most k. The coefficient is zero otherwise.*

Taking $\mathbf{b}$ to be the all ones vector yields the multiplicty of a hook shaped irreducible in $G(v)$.

Example 5.6 *Consider the matrix*

$$v = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \in \mathbb{A}^{3\times 6}.$$

The coefficient of $s_{(4,1,1)}(u)$ in the character of $G(v)$ is the number of no broken circuit bases of $M(v)$, which is 7. The NBC bases of $M(v)$ are

$$123, 125, 126, 134, 145, 146, 156,$$

where we write ijk for $\{i, j, k\}$.

6 Cohomology classes of T-orbit closures in Grassmannians

We now turn to equivariant cohomology, the third of our objects of interest from the introduction. In fact, instead of working in topological G-equivariant cohomology of a variety X with the action of a linear algebraic group G, we will work in the Chow cohomology ring of G-invariant algebraic cocycles of X modulo rational equivalence. This is of no consequence since all the varieties in question have sufficiently nice stratifications that the two notions coincide [Fulton(1998), Example 19.1.11].

We pause briefly to say how this is related to our matrix orbit closures and their K-polynomials. By Grothendieck-Riemann-Roch, the $\mathrm{GL}_r \times T$-equivariant cohomology class of an invariant algebraic subset $Z \subset \mathbb{A}^{r\times n}$ is obtained from its K-polynomial as the lowest degree term of

$$\mathcal{K}(R/I; \exp(u), \exp(t)) \in \mathbb{Z}[[u_1, \dots, u_r, t_1, \dots, t_n]]^{\mathfrak{S}_r}.$$

where I is the radical ideal that defines Z. That is, replace each u_i and t_j in the K-polynomial by its formal exponential and take the lowest degree term of the resulting power series. We denote this polynomial $[Z]_{\mathrm{GL}_r\times T}$ and call it the **equivariant cohomology class** of Z. This is an element of the $\mathrm{GL}_r \times T$-equivariant cohomology ring of $\mathbb{A}^{r\times n}$, namely

$$H^*_{\mathrm{GL}_r\times T}(\mathbb{A}^{r\times n}) = \mathbb{Z}[u_1, \dots, u_r, t_1, \dots, t_n]^{\mathfrak{S}_r}$$

(see [Brion(1997)]). This defines a map $K^0_{T\times\mathrm{GL}_r}(\mathbb{A}^{r\times n}) \to H^*_{T\times\mathrm{GL}_r}(\mathbb{A}^{r\times n})$. The remaining spaces in (1) also have maps from equivariant K-theory to equivariant cohomology, and these maps form a natural transformation between the K-theory and cohomology of diagram (1). The maps ρ and π induce isomorphisms at the level of equivariant K-theory and cohomology, and the inclusions j and i give rise to surjections. The kernel of the restriction map

$$H^*_{\mathrm{GL}_r\times T}(\mathbb{A}^{r\times n}) \to H^*_T(\mathbb{G}_r(n))$$

induced by i, is the ideal

$$I_{\mathrm{fr}} = \Big(\sum_{a+b=k} e_a(-t)h_b(u) : k > n-r \Big) \subset H^*_{\mathrm{GL}_r \times T}(\mathbb{A}^{r\times n}).$$

On $\mathbb{G}_r(n)$ we have two equivariant universal vector bundles: S, the tautological bundle or universal subbundle, whose fiber over $x \in \mathbb{G}_r(n)$ is the subspace $x \subseteq \mathbf{C}^n$; and Q, the universal quotient bundle, with fiber $\mathbf{C}^n/x$. Grothendieck-Riemann-Roch allows the Chern classes of these bundles to be extracted from their K-classes. This comes out nicely for S, which has $[S] = e_1(u)$, whence one extracts $c_k(S) = (-1)^k e_k(u)$. The exact sequence of vector bundles

$$0 \to S \to \mathbf{C}^n \to Q \to 0$$

is transformed to the Whitney sum formula

$$\prod_{i=1}^{n}(1-t_i) = c(\mathbf{C}^n) = c(Q)c(S), \tag{3}$$

from which one can solve for the $c_k(Q)$ in terms of the $c_k(S)$. Then the generators of I_{fr} can be understood to impose the vanishing of $c_k(Q)$ for $k > n-r = \operatorname{rank} Q$.

Our main result on cohomology classes is the following.

Theorem 6.1 *Given $v \in \mathbb{A}^{r\times n}$ whose matroid is uniform, the class of $\overline{\pi(v)T}$ in T-equivariant cohomology is*

$$[\overline{\pi(v)T}] = \sum_{\lambda\,:\,\lambda \text{ fits in } (n-r-1)^{r-1}} s_\lambda(S^\vee)s_{(\lambda^\vee)'}(Q) \tag{4}$$

where $\lambda^\vee$ is the complement of λ within the $(r-1)\times(n-r-1)$ rectangle, and by a symmetric function of a vector bundle we mean the symmetric function of its Chern roots.

This is proved using the equivariant localization theory of Goresky–Kottwitz–MacPherson.

There is an isomorphism of $\mathbb{Z}$-modules

$$\mathbb{Z} \otimes_{H^*_T(\mathrm{pt})} - : H^*_T(\mathbb{G}_r(n)) \to H^*(\mathbb{G}_r(n)).$$

The image of the cohomology class $s_\lambda(S^\vee) \in H^*_T(\mathbb{G}_r(n))$ under this map is the class of the Schubert variety indexed by λ, denoted σ_λ (we take this Schubert variety to have codimension $|\lambda|$, so that σ_λ is the in $|\lambda|$-th graded piece of the cohomology).

Passing from equivariant cohomology to ordinary cohomology yields the following description of the class of $\overline{\pi(v)T}$.

Corollary 6.2 *Given $v \in \mathbb{A}^{r\times n}$ whose matroid is uniform, the class of $\overline{\pi(v)T}$ in cohomology is*

$$[\overline{\pi(v)T}] = \sum_{\lambda\,:\,\lambda \text{ fits in } (n-r-1)^{r-1}} \sigma_\lambda \sigma_{(\lambda^\vee)'}.$$

This stands in contrast to a formula of Klyachko for this cohomology class.

Theorem 6.3 (Klyachko) *Given $v \in \mathbb{A}^{r\times n}$ whose matroid is uniform, the class of $\overline{\pi(v)T}$ in cohomology is*

$$[\overline{\pi(v)T}] = \sum_{\lambda \vdash (n-r-1)(r-1)} \left(\sum_{k=0}^{r} (-1)^k \binom{n}{k} s_\lambda(1^{r-k}) \right) \sigma_\lambda.$$

7 Case study: $r = 2$

Our results take a particularly simple and explicit form when $r = 2$. In this section we sumarize our results in this setting.

As before, we let x denote the r-by-n matrix of variables $x_{i,j}$.

Theorem 7.1 *Suppose that v has a uniform matroid. The prime ideal I_v of $\overline{\mathrm{GL}_2 vT}$ is generated by the quartics*

$$p_{ab}(v)p_{cd}(v)p_{ac}(x)p_{bd}(x) - p_{ac}(v)p_{bd}(v)p_{ab}(x)p_{bc}(x), \quad 1 \leq a < b < c < d \leq n, \tag{5}$$

where $p_{ij}(\cdot)$ denotes the 2-by-2 minor of the submatrix of its argument with columns i and j. The quotient R/I_v is a Cohen–Macaulay ring.

The following result comes from resolving R/I_v by the Eagon–Northcott complex.

Theorem 7.2 *Suppose that v has a uniform matroid. The K-polynomial of R/I_v is*

$$\mathcal{K}(R/I_v; u, t) = 1 - \sum_{\substack{\lambda=(\lambda_1 \geq \lambda_2) \\ 2 \leq \lambda_2,\ \lambda_1+\lambda_2 \leq n}} (-1)^{|\lambda|} s_\lambda(1,1) s_\lambda(u^{-1}) e_{|\lambda|}(t)$$

Here $s_\lambda(u^{-1})$ is the Schur polynomial of shape λ in u_1^{-1} and u_2^{-1}, and $e_m(t)$ is the m-th elementary symmetric polynomial in its arguments.

We next consider the tensor module $G(v)$.

Theorem 7.3 *Suppose that $v \in \mathbb{A}^{2\times n}$ has a uniform matroid. The character of $G(v)$ is*

$$s_{(n,0)}(u) + \sum_{\substack{\lambda=(\lambda_1 \geq \lambda_2) \\ 1 \leq \lambda_2,\ \lambda_1+\lambda_2 = n}} s_\lambda(1,1) s_\lambda(u)$$

Notice that the coefficient of $s_{(n-1,1)}(u)$ is the $n-1$, the number of NBC bases of the uniform matroid of rank 2 on n elements.

Finally, we consider the cohomology class of $\overline{\pi(v)T}$.

Theorem 7.4 *Suppose that $v \in \mathbb{A}^{2\times n}$ has a uniform matroid. The cohomology class of $\overline{\pi(v)T}$ is*

$$\sum_{\lambda \vdash n-3} s_\lambda(1,1)\sigma_\lambda = \sum_{k+\ell=n-3} \sigma_{(k)}\sigma_{(\ell)}.$$

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