

# The down operator and expansions of near rectangular $k$ -Schur functions<sup>†</sup>

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**Abstract.** We prove that the Lam-Shimozono “down operator” on the affine Weyl group induces a derivation of the affine Fomin-Stanley subalgebra of the affine nilCoxeter algebra. We use this to verify a conjecture of Berg, Bergeron, Pon and Zabrocki describing the expansion of  $k$ -Schur functions of “near rectangles” in the affine nilCoxeter algebra. Consequently, we obtain a combinatorial interpretation of the corresponding  $k$ -Littlewood–Richardson coefficients.

**Résumé.** Nous montrons que l'opérateur “down”, défini par Lam et Shimozono sur le groupe de Weyl affine, induit une dérivation de la sous-algèbre affine de Fomin-Stanley de l'algèbre affine de nilCoxeter. Nous employons cette dérivation pour vérifier une conjecture de Berg, Bergeron, Pon et Zabrocki sur l'expansion des  $k$ -fonctions de Schur indexées par les partitions qui sont “presque rectangles”. Par conséquent, nous obtenons une interprétation combinatoire des  $k$ -coefficients de Littlewood–Richardson correspondants.

**Keywords:** symmetric functions,  $k$ -Schur functions, affine Schubert calculus, dual graded graphs

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## 1 Introduction

$k$ -Schur functions were first introduced by Lapointe, Lascoux and Morse [13] in the study of Macdonald polynomials. Since then, their study has flourished (see for instance [9, 12, 10, 14, 15, 16]). In particular they have been realized as Schubert classes for the homology of the affine Grassmannian. This was done by identifying the algebra of  $k$ -Schur functions with the affine Fomin-Stanley subalgebra of the affine nilCoxeter algebra  $\mathbb{A}$  [8]. A natural question is to ask for the expansion of a  $k$ -Schur function in terms of the standard basis of  $\mathbb{A}$ , which is indexed by affine permutations.

An important related problem is to describe the multiplicative structure constants of the  $k$ -Schur functions, called the  $k$ -Littlewood–Richardson coefficients due to the similarity with the classical problem of multiplying Schur functions. Lam [7] pointed out that the  $k$ -Littlewood–Richardson coefficients are the same coefficients that appear in the expansion of a  $k$ -Schur function in the standard basis of  $\mathbb{A}$  (see Section 4.1). Hence, results that give such expansions also give information about the  $k$ -Littlewood–Richardson coefficients. This paper is one such example; others are [7, 1, 3, 2].

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<sup>†</sup>This manuscript has been shortened to fit the guidelines for submission. All substantial proofs have been omitted. A longer version will appear on the arXiv.

In early 2011, Berg, Bergeron, Pon and Zabrocki conjectured an expansion for  $k$ -Schur functions indexed by a  $k$ -rectangle  $R$  minus its unique removable cell. Their conjecture combined ideas coming from two groups: Pon's [21] description of the generators of the affine Fomin-Stanley subalgebra for arbitrary affine type; and Berg, Bergeron, Thomas and Zabrocki's [3] expansion of  $s_R^{(k)}$ .

This paper initiates the study of operators on the affine nilCoxeter algebra that stabilize the affine Fomin-Stanley subalgebra. We study one particular family of operators introduced by Lam and Shimozono [12] and prove that they are derivations of the affine Fomin-Stanley subalgebra (Theorem 3.5). As an application, we prove the conjecture of Berg, Bergeron, Pon and Zabrocki and provide a combinatorial interpretation for the corresponding  $k$ -Littlewood–Richardson coefficients (Theorem 4.6). Further properties of such operators and their applications to  $k$ -Schur functions will be developed in a companion article.

## 2 $k$ -Combinatorics

In this section, we recall the required terminology associated to the affine type  $A$  root system, the affine Weyl group, the connection with bounded partitions and core partitions and the definition of  $k$ -Schur functions. We work with the affine type  $A$  root system  $A_k^{(1)}$ . Much of this introduction is borrowed from [2] which in turn was borrowed from [26].

### 2.1 Affine symmetric group

$I = \{0, 1, \dots, k\}$  will denote the set of nodes of the corresponding Dynkin diagram. We say two nodes  $i, j \in I$  are adjacent if  $i - j = \pm 1 \pmod{k+1}$ .

We let  $W$  denote the *affine symmetric group* with generators  $s_i$  for  $i \in I$ , and relations  $s_i^2 = 1$ ,  $s_i s_j = s_j s_i$ , when  $i$  and  $j$  are not adjacent, and  $s_i s_j s_i = s_j s_i s_j$  when  $i$  and  $j$  are adjacent. An element of the affine symmetric group may be expressed as a word in the generators  $s_i$ . Given the relations above, an element of the affine symmetric group may have multiple *reduced words*, words of minimal length which express that element. The *length* of  $w$ , denoted  $\ell(w)$ , is the number of generators in any reduced word of  $w$ .

The *Bruhat order* on affine symmetric group elements is a partial order where  $v < w$  if there is a reduced word for  $v$  that is a subword of a reduced word for  $w$ . If  $v < w$  and  $\ell(v) = \ell(w) - 1$ , we write  $v \prec w$ . There is another order on  $W$ , called the *left weak order*, which is defined by the covering relation  $v \prec w$  if  $w = s_i v$  for some  $i$  and  $\ell(v) = \ell(w) - 1$ .

For  $j \in I$ , we denote by  $W_j$  the subgroup of  $W$  generated by the elements  $s_i$  with  $i \neq j$ . We denote by  $W^j$  the set of minimal length representatives of the cosets  $W/W_j$ .

### 2.2 Roots and weights

Associated to the affine Dynkin diagram of type  $A_k^{(1)}$  we have a root datum, which consists of a free  $\mathbb{Z}$ -module  $\mathfrak{h}$ , its dual lattice  $\mathfrak{h}^* = \text{Hom}(\mathfrak{h}, \mathbb{Z})$ , a pairing  $\langle \cdot, \cdot \rangle : \mathfrak{h} \times \mathfrak{h}^* \rightarrow \mathbb{Z}$  given by  $\langle \mu, \lambda \rangle = \lambda(\mu)$ , and sets of linearly independent elements  $\{\alpha_i \mid i \in I\} \subset \mathfrak{h}^*$  and  $\{\alpha_i^\vee \mid i \in I\} \subset \mathfrak{h}$  such that

$$\langle \alpha_i^\vee, \alpha_j \rangle = \begin{cases} 2 & \text{if } i = j; \\ -1 & \text{if } i \text{ and } j \text{ are adjacent;} \\ 0 & \text{else.} \end{cases} \quad (1)$$

The  $\alpha_i$  are known as *simple roots*, and the  $\alpha_i^\vee$  are *simple coroots*. The spaces  $\mathfrak{h}_{\mathbb{R}} = \mathfrak{h} \otimes \mathbb{R}$  and  $\mathfrak{h}_{\mathbb{R}}^* = \mathfrak{h}^* \otimes \mathbb{R}$  are the *coroot* and *root spaces*, respectively.

Given a simple root  $\alpha_i$ , we have actions of  $W$  on  $\mathfrak{h}_{\mathbb{R}}$  and  $\mathfrak{h}_{\mathbb{R}}^*$  defined by the action of the generators of  $W$  as

$$s_i(\lambda) = \lambda - \langle \alpha_i^\vee, \lambda \rangle \alpha_i \quad \text{for } i \in I, \lambda \in \mathfrak{h}_{\mathbb{R}}^*; \tag{2}$$

$$s_i(\mu) = \mu - \langle \mu, \alpha_i \rangle \alpha_i^\vee \quad \text{for } i \in I, \mu \in \mathfrak{h}_{\mathbb{R}}. \tag{3}$$

The action of  $W$  satisfies

$$\langle w(\mu), w(\lambda) \rangle = \langle \mu, \lambda \rangle \tag{4}$$

for all  $\mu \in \mathfrak{h}_{\mathbb{R}}$ ,  $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$  and  $w \in W$ .

The set of *real roots* is  $\Phi_{\text{re}} = W \cdot \{\alpha_i \mid i \in I\}$ . Given a real root  $\alpha = w(\alpha_i)$ , we have an *associated coroot*  $\alpha^\vee = w(\alpha_i^\vee)$  and an *associated reflection*  $s_\alpha = ws_iw^{-1}$  (these are well-defined, and independent of the choice of  $w$  and  $i$ ). For a Bruhat covering  $v < w$ , there exists a unique root  $\alpha_{v,w}$  satisfying the equation  $v^{-1}w = s_{\alpha_{v,w}}$ . We denote by  $\alpha_{v,w}^\vee$  the coroot corresponding to the root  $\alpha_{v,w}$ .

The *fundamental weights* are the elements  $\Lambda_i \in \mathfrak{h}_{\mathbb{R}}^*$  satisfying  $\langle \alpha_j^\vee, \Lambda_i \rangle = \delta_{ij}$  for  $i, j \in I$ . They generate the *weight lattice*  $P = \bigoplus_{i \in I} \mathbb{Z}\Lambda_i$ . We let  $P^+ = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0}\Lambda_i$  denote the *dominant weights*.

### 2.3 $k$ -bounded partitions, $(k + 1)$ -cores and affine Grassmannian elements

Let  $\lambda$  be a partition. To each box  $(i, j)$  (row  $i$ , column  $j$ ) of the Young diagram of  $\lambda$ , we associate its *residue* defined by  $c_{(i,j)} = (j - i) \bmod (k + 1)$ . We let  $\mathcal{P}^{(k)}$  denote the set of  $k$ -bounded partitions, namely the partitions  $\lambda = (\lambda_1, \lambda_2, \dots)$  whose first part  $\lambda_1$  is at most  $k$ .

A  $p$ -*core* is a partition that has no removable rim hooks of length  $p$ . Lapointe and Morse [15, Theorem 7] showed that the set  $\mathcal{P}^{(k)}$  bijects with the set of  $(k + 1)$ -cores. Following their notation, we let  $c(\lambda)$  denote the  $(k + 1)$ -core corresponding to the partition  $\lambda$ , and  $\mathfrak{p}(\mu)$  denote the  $k$ -bounded partition corresponding to the  $(k + 1)$ -core  $\mu$ . We will also use  $\mathcal{C}^{(k+1)}$  to represent the set of all  $(k + 1)$ -cores.

$W$  acts on  $\mathcal{C}^{(k+1)}$ . Specifically, if  $\lambda$  is a  $(k + 1)$ -core then

$$s_i\lambda = \begin{cases} \lambda \cup \{\text{addable residue } i \text{ cells}\} & \text{if } \lambda \text{ has an addable cell of residue } i, \\ \lambda \setminus \{\text{removable residue } i \text{ cells}\} & \text{if } \lambda \text{ has a removable cell of residue } i, \\ \lambda & \text{otherwise.} \end{cases}$$

The *affine Grassmannian elements* are the elements of  $W^0$ . These are naturally identified with  $(k + 1)$ -cores in the following way: to a core  $\lambda \in \mathcal{C}^{(k+1)}$ , we associate the unique element  $w \in W^0$  for which  $w\emptyset = \lambda$ . For a  $k$ -bounded partition  $\mu$ , we let  $w_\mu$  denote the element of  $W^0$  which satisfies  $w_\mu\emptyset = c(\mu)$ . More details on this can be found in [4].

**Example 2.1** The diagram of the 4-core  $\lambda = (5, 2, 1)$  augmented with its residues, together with the diagrams of the 4-cores  $s_1\lambda$  and  $s_0\lambda$ :

$$\lambda = \begin{array}{|c|c|c|c|c|} \hline 0 & 1 & 2 & 3 & 0 \\ \hline 3 & 0 & & & \\ \hline 2 & & & & \\ \hline \end{array} \quad s_1\lambda = \begin{array}{|c|c|c|c|c|c|} \hline 0 & 1 & 2 & 3 & 0 & 1 \\ \hline 3 & 0 & 1 & & & \\ \hline 2 & & & & & \\ \hline 1 & & & & & \\ \hline \end{array} \quad s_0\lambda = \begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 3 \\ \hline 3 & & & \\ \hline 2 & & & \\ \hline \end{array} .$$

### 2.4 $k$ -Schur functions in non-commutative variables

The nilCoxeter algebra  $\mathbb{A}$  may be defined via generators and relations with generators  $\mathbf{u}_i$  for  $i \in I$ , and relations  $\mathbf{u}_i^2 = 0$ ,  $\mathbf{u}_i\mathbf{u}_j = \mathbf{u}_j\mathbf{u}_i$  when  $i$  and  $j$  are not adjacent and  $\mathbf{u}_i\mathbf{u}_j\mathbf{u}_i = \mathbf{u}_j\mathbf{u}_i\mathbf{u}_j$  when  $i$  and  $j$  are adjacent. Since the braid relations are exactly those of the corresponding affine symmetric group, we may index nilCoxeter elements by elements of the affine symmetric group, e.g.,  $\mathbf{u}_w = \mathbf{u}_{i_1}\mathbf{u}_{i_2} \cdots \mathbf{u}_{i_k}$ , whenever  $s_{i_1}s_{i_2} \cdots s_{i_k}$  is a reduced word for  $w$ .

**Definition 2.2** For a subset  $S \subset I$ , one defines a cyclically decreasing word  $w_S \in W$  to be the unique element of  $W$  for which any (equivalently all) reduced words  $s_{i_1} \dots s_{i_m}$  of  $w_S$  satisfy:

1. each letter from  $I$  appears at most once in  $\{i_1, \dots, i_m\}$ ;
2. if  $j, j+1 \in S$ , then  $j+1$  appears before  $j$  in  $i_1, \dots, i_m$  (where the indices are taken modulo  $k+1$ ).

Furthermore, we let  $\mathbf{u}_S = \mathbf{u}_{w_S}$  and

$$\mathbf{h}_i = \sum_{\substack{S \subset I \\ |S|=i}} \mathbf{u}_S \in \mathbb{A}.$$

The elements  $\mathbf{h}_i$  are analogues of the  $i^{\text{th}}$  complete homogeneous symmetric functions.

**Example 2.3** Let  $k = 3$ . The cyclically decreasing elements of length 2 in the alphabet  $\{\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$  are  $\mathbf{u}_2\mathbf{u}_1, \mathbf{u}_1\mathbf{u}_0, \mathbf{u}_0\mathbf{u}_3, \mathbf{u}_3\mathbf{u}_2, \mathbf{u}_0\mathbf{u}_2$ , and  $\mathbf{u}_1\mathbf{u}_3$ . Thus,

$$\mathbf{h}_2 = \mathbf{u}_2\mathbf{u}_1 + \mathbf{u}_1\mathbf{u}_0 + \mathbf{u}_0\mathbf{u}_3 + \mathbf{u}_3\mathbf{u}_2 + \mathbf{u}_0\mathbf{u}_2 + \mathbf{u}_1\mathbf{u}_3.$$

**Theorem 2.4 (Lam [8])** The elements  $\{\mathbf{h}_i\}_{i \leq k}$  commute and freely generate a subalgebra  $\mathbb{B}$  of  $\mathbb{A}$  called the affine Fomin-Stanley subalgebra. Consequently,

$$\mathbb{B} \cong \Lambda_{(k)} := \mathbb{Q}[h_1, \dots, h_k],$$

where  $h_i$  denotes the  $i^{\text{th}}$  complete homogeneous symmetric function.

The  $k$ -Schur functions in non-commutative variables are then the images of the  $k$ -Schur functions of Lapointe, Lascoux and Morse [13] under this identification. We take instead the following equivalent definition (see [8, Definition 6.5] and [11, Theorem 4.6]).

**Definition 2.5** The  $k$ -Schur function (in non-commutative variables) corresponding to a  $k$ -bounded partition  $\lambda$  is the unique element  $s_\lambda^{(k)} = \sum_w c_w \mathbf{u}_w$  of  $\mathbb{B}$  satisfying:

$$c_{w_\lambda} = 1; \tag{5}$$

$$c_w = 0 \text{ for all other } w \in W^0. \tag{6}$$

## 3 The Lam-Shimozono up and down operators

In [12], Lam and Shimozono studied two graded graphs whose vertex set is the affine Weyl group  $W$ , from which one constructs two closely-related operators defined on the group algebra of  $W$ . In this section, we recall the construction of these operators and then develop some properties of the corresponding induced operators on the nilCoxeter algebra  $\mathbb{A}$ .

### 3.1 Dual graded graphs

In [5] and [6], Fomin introduced the notion of *dual graded graphs*, generalizing the notion of *differential posets* in [25]. A *graded graph* is a triple  $\Gamma = (V, \rho, m, E)$  where  $V$  is a set of vertices,  $\rho$  is a rank function on  $V$ ,  $E$  is a multiset of edges  $(x, y)$  for  $x, y \in V$  where  $\rho(y) = \rho(x) + 1$ , and every edge has multiplicity  $m(x, y) \in \mathbb{Z}_{\geq 0}$ . The set of vertices of the same rank is called a *level*.

$\Gamma$  is *locally finite* if every  $v \in V$  has finite degree, and we assume this condition for all graphs in this paper. For a graded graph  $\Gamma$ , the linear down and up operators  $D, U : \mathbb{Z}V \rightarrow \mathbb{Z}V$  are defined as follows.

$$D_{\Gamma}(v) = \sum_{(u,v) \in E} m(u, v)u \quad U_{\Gamma}(v) = \sum_{(v,u) \in E} m(v, u)u$$

In other words,  $D$  (respectively  $U$ ) maps a vertex  $v$  to a linear combination of its neighbors in the level immediately below (respectively above)  $v$  where the coefficients are the multiplicities of the edges.

A pair of graded graphs  $(\Gamma, \Gamma')$  is called *dual* if they have the same set of vertices and same rank function, but possibly different edges and multiplicities, and satisfies the following (Heisenberg) commutation relation

$$D_{\Gamma'}U_{\Gamma} - U_{\Gamma}D_{\Gamma'} = r\text{Id}$$

for a fixed  $r \in \mathbb{Z}_{\geq 0}$ , called the *differential coefficient*.

One can find many examples of dual graded graphs in [5] and [6], such as the Young, Fibonacci, and Pascal lattices, the graphs of Ferrers shapes and shifted shapes, and many more.

### 3.2 The Lam-Shimozono dual graded graphs in affine type $A$

In [12], Lam and Shimozono introduced pairs of dual graded graphs for arbitrary Kac-Moody algebras. Here, we specialize to the case of affine type  $A_k^{(1)}$ .

Following [12], we define two graded graph structures on  $W$ . The first constructs a graph with an edge from  $v$  to  $w$  whenever we have a weak cover  $v \prec w$ . We denote this graph by  $\Gamma_w$  (because its edges depend on weak Bruhat order). The second construction uses strong order. We fix a dominant integral weight  $\Lambda \in P^+$  and let  $\Gamma_s(\Lambda)$  be the graph that has  $\langle \alpha_{v,w}^{\vee}, \Lambda \rangle$  edges between  $v$  and  $w$  whenever  $v \prec w$ .

The up and down operators for the dual graded graphs  $\Gamma_w$  and  $\Gamma_s(\Lambda)$  induce operators on  $\mathbb{A}$ . Specifically, define  $U$  using the up operator on  $\Gamma_w$ ,

$$U(\mathbf{u}_w) = \sum_{v \prec w} \mathbf{u}_v,$$

and define  $D_{\Lambda}$  using the down operator on  $\Gamma_s(\Lambda)$ ,

$$D_{\Lambda}(\mathbf{u}_w) = \sum_{v \prec w} \langle \alpha_{v,w}^{\vee}, \Lambda \rangle \mathbf{u}_v.$$

It is clear from the definition and the bilinearity of the pairing  $\langle \cdot, \cdot \rangle$  that  $D_{\Lambda_i + \Lambda_j} = D_{\Lambda_i} + D_{\Lambda_j}$ . With this in mind, we will assume throughout this paper that  $\Lambda$  is a fundamental weight.

**Remark 3.1** Note that the operator  $U$  can be realized as left-multiplication by  $\mathbf{h}_1$  on  $\mathbb{A}$ . With this in mind, we define more generally  $U_i(\mathbf{u}) := \mathbf{h}_i \mathbf{u}$  for  $\mathbf{u} \in \mathbb{A}$ .

**Remark 3.2** Our notation differs slightly from that of [12]. Lam and Shimozono defined the operators  $U$  and  $D$  as operators on the opposite graphs;  $D$  was defined on the weak order graph, and  $U$  was defined on the strong order graph.

**Theorem 3.3 (Lam, Shimozono [12], Theorem 2.3)** The graphs  $\Gamma_w$  and  $\Gamma_s(\Lambda)$  are dual graded graphs with differential coefficient 1. In other words,  $D_\Lambda U - U D_\Lambda = Id$ .

### 3.3 Properties of the Lam-Shimozono down operator

In this section we further develop properties of the operator  $D_\Lambda$ . Our first observation is a generalization of the Heisenberg relation in Theorem (3.3).

**Theorem 3.4** Let  $\Lambda$  be a fundamental weight. For all  $w \in W$ ,

$$D_\Lambda(\mathbf{h}_i \mathbf{u}_w) = \mathbf{h}_{i-1} \mathbf{u}_w + \mathbf{h}_i D_\Lambda(\mathbf{u}_w).$$

In particular,  $D_\Lambda(\mathbf{h}_i) = \mathbf{h}_{i-1}$  and

$$D_\Lambda \circ U_i - U_i \circ D_\Lambda = U_{i-1}.$$

Next, we study the restrictions of the operators  $D_\Lambda$  to the affine Fomin-Stanley subalgebra  $\mathbb{B}$ . The following theorem implies that although the operators  $D_\Lambda$ , for distinct fundamental weights  $\Lambda$ , are distinct on  $\mathbb{A}$ , their restrictions to the affine Fomin-Stanley subalgebra  $\mathbb{B}$  coincide. In fact, the action of  $D_\Lambda$  on  $\mathbb{B}$  is determined by the conditions that  $D_\Lambda$  is a derivation and  $D_\Lambda(\mathbf{h}_i) = \mathbf{h}_{i-1}$ .

**Theorem 3.5** Let  $\Lambda$  be a fundamental weight.  $D_\Lambda$  is a derivation on the affine Fomin-Stanley subalgebra  $\mathbb{B}$ . Explicitly, for  $x, y \in \mathbb{B}$ ,

$$D_\Lambda(xy) = D_\Lambda(x)y + xD_\Lambda(y).$$

In particular,  $D_\Lambda$  stabilizes  $\mathbb{B}$ ; that is,  $D_\Lambda(\mathbb{B}) \subset \mathbb{B}$ .

Finally, we describe the coefficients of the operator combinatorially. The next result shows that it suffices to know the value of  $D_\Lambda$  on the elements in  $W^j$ . In the case that  $j = 0$ , this says that it suffices to know the values of  $D_\Lambda$  on the affine Grassmannian elements.

**Theorem 3.6** Suppose  $w \in W^j$  and  $v \in W_j$ . Then

$$D_{\Lambda_j}(\mathbf{u}_{wv}) = D_{\Lambda_j}(\mathbf{u}_w)\mathbf{u}_v.$$

We now give a combinatorial formula to apply the down operator to the elements of  $W^j$ . This generalizes the description of the coefficients given in [10].

**Theorem 3.7** Suppose  $w \in W^j$ . Then

$$D_{\Lambda_j}(\mathbf{u}_w) = \sum_{y < w} c_y^{w,j} \mathbf{u}_y,$$

where  $c_y^{w,j}$  is the number of addable  $(i_\ell - j)$ -cells of the  $(k+1)$ -core  $s_{i_{\ell-1}-j} \cdots s_{i_1-j} \emptyset$ , where  $s_{i_m} \cdots s_{i_2} s_{i_1}$  is a reduced expression for  $w$  and  $s_{i_m} \cdots s_{i_\ell} \cdots s_{i_1}$  is a reduced expression for  $y$ .

These previous two theorems combine to give a combinatorial method for calculating the down operator on any basis element  $\mathbf{u}_w$ . We illustrate this in the following example.

**Example 3.8** Fix  $k = 3$ . We calculate  $D_{\Lambda_0}(\mathbf{u}_2\mathbf{u}_3\mathbf{u}_0\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3\mathbf{u}_0\mathbf{u}_2)$ . By Theorem 3.6,

$$D_{\Lambda_0}(\mathbf{u}_2\mathbf{u}_3\mathbf{u}_0\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3\mathbf{u}_0\mathbf{u}_2) = D_{\Lambda_0}(\mathbf{u}_2\mathbf{u}_3\mathbf{u}_0\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3\mathbf{u}_0)\mathbf{u}_2$$

since  $s_2s_3s_0s_1s_2s_3s_0 \in W^0$ . Hence, it suffices to calculate  $D_{\Lambda_0}(\mathbf{u}_2\mathbf{u}_3\mathbf{u}_0\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3\mathbf{u}_0)$ .

By Theorem 3.7, the coefficient of  $\mathbf{u}_2\mathbf{u}_3\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3\mathbf{u}_0$  in  $D_{\Lambda_0}(\mathbf{u}_2\mathbf{u}_3\mathbf{u}_0\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3\mathbf{u}_0)$  is the number of addable 0-cells in the 4-core  $s_1s_2s_3s_0 \cdot \emptyset = (2, 1, 1, 1)$ , which is 2 (as indicated by the shaded cells in Figure 1).

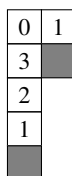


Fig. 1: The addable 0-cells in the 4-core  $(2, 1, 1, 1)$ .

Similarly, one computes all the other coefficients:

$$D_{\Lambda_0}(\mathbf{u}_2\mathbf{u}_3\mathbf{u}_0\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3\mathbf{u}_0) = 3\mathbf{u}_3\mathbf{u}_0\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3\mathbf{u}_0 + 2\mathbf{u}_2\mathbf{u}_0\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3\mathbf{u}_0 + 2\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3\mathbf{u}_1\mathbf{u}_2\mathbf{u}_0 + \mathbf{u}_2\mathbf{u}_3\mathbf{u}_0\mathbf{u}_1\mathbf{u}_3\mathbf{u}_0 + \mathbf{u}_2\mathbf{u}_3\mathbf{u}_0\mathbf{u}_1\mathbf{u}_2\mathbf{u}_0 + \mathbf{u}_2\mathbf{u}_3\mathbf{u}_0\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3.$$

## 4 Expansions of $k$ -Schur functions and $k$ -Littlewood–Richardson coefficients for “near” rectangles

This section describes the connection between expansions of  $k$ -Schur functions in the standard basis of  $\mathbb{A}$  and the  $k$ -Littlewood–Richardson rule. We then recall the expansions of the  $k$ -Schur functions for  $k$ -rectangles, from which we deduce expansions of the  $k$ -Schur functions for the “near” rectangles.

### 4.1 Expansion of $s_\lambda^{(k)}$ and the $k$ -Littlewood–Richardson coefficients

An important problem in the theory of  $k$ -Schur functions is to understand the multiplicative structure coefficients  $c_{\lambda,\mu}^{\nu,(k)}$ , called the  $k$ -Littlewood–Richardson coefficients:

$$s_\lambda^{(k)}s_\mu^{(k)} = \sum_{\nu} c_{\lambda,\mu}^{\nu,(k)}s_\nu^{(k)}.$$

Another difficult problem is determining an expansion for  $s_\lambda^{(k)}$  in terms of the natural basis  $\{\mathbf{u}_w\}_{w \in W}$  of  $\mathbb{A}$ . In other words, to find the coefficients  $d_\lambda^w$  in the expansion:

$$s_\lambda^{(k)} = \sum_{w \in W} d_\lambda^w \mathbf{u}_w.$$

Lam [7] proved that these two problems are actually equivalent. We reformulate his theorem as follows.

**Theorem 4.1** [7, Proposition 42] *The coefficient  $c_{\lambda,\mu}^{\nu,(k)}$  is nonzero only if  $w_\mu$  is less than  $w_\nu$  in left weak order, and in this case  $c_{\lambda,\mu}^{\nu,(k)} = d_\lambda^{w_\nu w_\mu^{-1}}$ .*

The main application in this paper of the down operator is to give the coefficients  $d_\lambda^w$  via explicit combinatorics when  $\lambda$  is a “near” rectangle. From this viewpoint our result gives a combinatorial description of the corresponding  $k$ -Littlewood–Richardson coefficients. A previous result of [3] will be reviewed in the next section. It contains the combinatorics of the coefficients that appear in the expansion of a  $k$ -Schur function corresponding to a rectangle and is needed to prove our main result.

### 4.2 Expansions of rectangular $k$ -Schur functions

In [3], Berg, Bergeron, Thomas and Zabrocki gave a combinatorial formula for the expansion of the  $k$ -Schur function  $s_R^{(k)}$  indexed by a  $k$ -rectangle  $R$ . We recall their result here; it will be a stepping stone for our main result.

Let  $\nu$  and  $\mu$  be  $k$ -bounded partitions. For the skew shape  $\nu/\mu$ , let  $\mathbf{word}(\nu/\mu) \in W$  be the word formed by the residues of the cells in  $\nu/\mu$ , reading each row from right to left and taking the rows from bottom to top. See Example 4.3.

**Theorem 4.2 (Berg, Bergeron, Thomas, Zabrocki [3])** *Suppose  $R = (c^r)$  with  $c + r = k + 1$ . The  $k$ -Schur function  $s_R^{(k)}$  in non-commutative variables has the expansion:*

$$s_R^{(k)} = \sum_{\lambda \subset R} \mathbf{u}_{\mathbf{word}((R \cup \lambda)/\lambda)},$$

where  $\mathbf{u}_{\mathbf{word}((R \cup \lambda)/\lambda)}$  is the monomial in the generators  $\mathbf{u}_i$  corresponding to  $\mathbf{word}((R \cup \lambda)/\lambda)$ .

**Example 4.3** *Let  $R = (3, 3)$  and  $k = 4$ . Then  $s_R^{(4)}$  is the sum of all the monomials in  $\mathbf{u}_i$  corresponding to the reading words of the skew-partitions  $(R \cup \lambda)/\lambda$ , where  $\lambda$  is a partition contained inside the rectangle  $R$ , as shown:*

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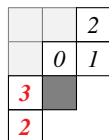


### 4.3 $k$ -Schur functions for “near” rectangles

**Proposition 4.4** Suppose  $R = (c^r)$  with  $c+r = k+1$  and let  $S = (c^{r-1}, c-1)$  be the partition obtained from  $R$  by removing its bottom-right corner. Let  $\Lambda$  be a fundamental weight. Then  $D_\Lambda(s_R^{(k)}) = s_S^{(k)}$ .

For  $\lambda \subset R$  and a cell  $x \in \lambda$ , we let  $\mathbf{word}(R, \lambda, x)$  denote the word corresponding to the diagram  $(R \cup \lambda_x)/\lambda$ , where  $\lambda_x$  denotes the diagram with the cell  $x$  removed.

**Example 4.5** Let  $k = 4$ , let  $R = (3, 3)$ ,  $\lambda = (2, 1) \subset R$  and  $x = (1, 2) \in \lambda$ . Then  $\mathbf{word}(R, \lambda, x) = s_2 s_3 s_1 s_0 s_2$ .



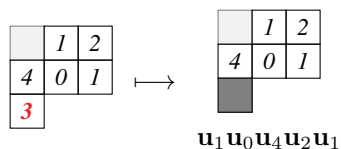
**Theorem 4.6**

$$s_{(c^{r-1}, c-1)}^{(k)} = \sum_{\lambda \subset R} \sum_{x \in \lambda} \mathbf{u}_{\mathbf{word}(R, \lambda, x)}$$

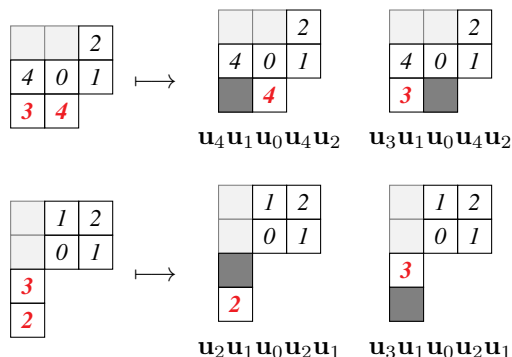
**Proof:** This follows from Theorem 3.7 and the application of Proposition 4.4 with the fundamental weight  $\Lambda_r$ . □

**Example 4.7** Let  $k = 4$  and  $\lambda = (3, 2)$ . Using Example 4.3, we can realize  $s_{3,2}^{(4)}$  as  $D_{\Lambda_3}(s_{3,3}^{(4)})$ .  $D_{\Lambda_3}$  acts on the pictures by deleting a bold letter from a term in the expansion of  $s_{3,3}^{(4)}$ . In particular, the first diagram of  $s_{3,3}^{(4)}$  has no bold letters, so it does not contribute any terms to  $s_{3,2}^{(4)}$ . The second diagram gives a term:

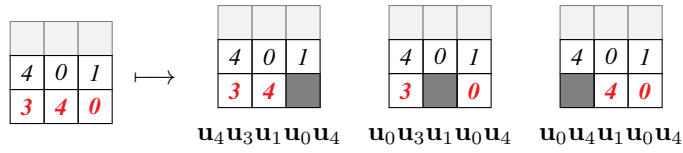
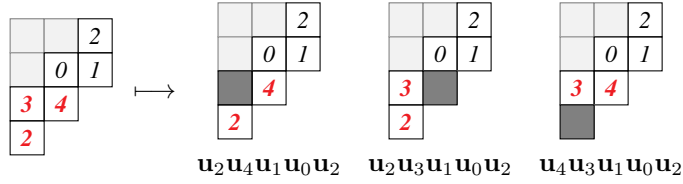
The second diagram gives a term:



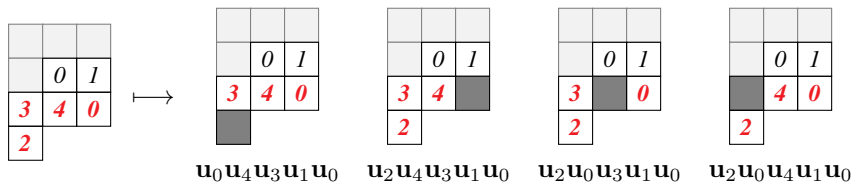
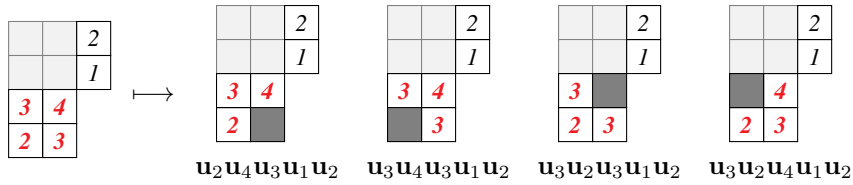
The third and fourth diagrams each give two terms:



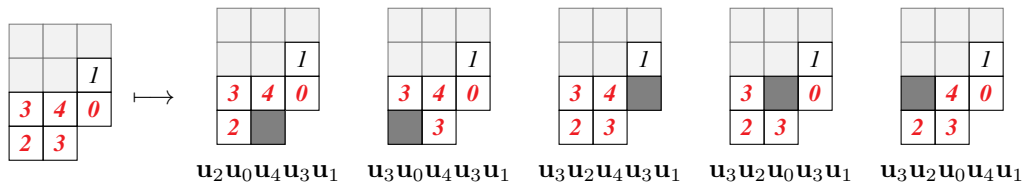
The fifth and sixth diagrams gives 3 terms each:



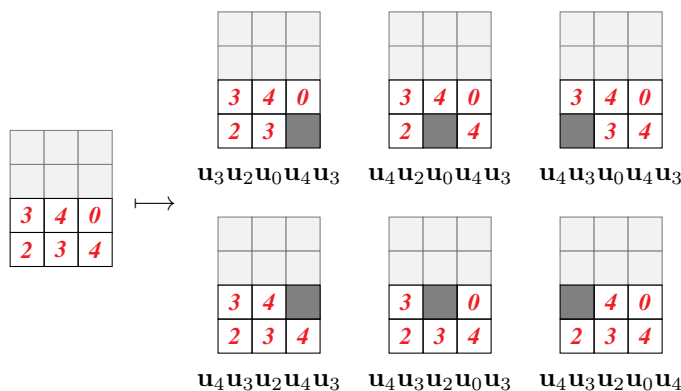
The seventh and eighth diagrams give 4 terms each:



The ninth diagram gives 5 terms:



The tenth and final diagram gives six terms:



Then  $s_{3,2}^{(4)}$  is a sum of the 30 words above.

**Corollary 4.8** Let  $S = (c^{r-1}, c - 1)$  with  $c + r = k + 1$ . Then the coefficient  $c_{\lambda,S}^{\nu,(k)}$  is either 0 or 1.

**Example 4.9** Continuing the example above, we compute  $c_{(2,1),(3,2)}^{(3,3,1,1),3}$ . The element  $\mathbf{u} = u_2 u_3 u_1 u_0 u_2$  satisfies  $\mathbf{u}(2, 1) = (3, 3, 1, 1)$ . Therefore the coefficient  $c_{(2,1),(3,2)}^{(3,3,1,1),3}$  is the coefficient of  $\mathbf{u}$  in the expansion of  $s_{3,2}^{(4)}$ , which is 1.

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We have benefitted from many conversations with Nantel Bergeron, Steven Pon and Mike Zabrocki, as well as email correspondence with Thomas Lam, Jennifer Morse and Mark Shimozono.

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