What's New in Symbolic Summation

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Combinatorics is the primary application area of symbolic summation algorithms. The classical hypergeometric summation machinery consisting of Gosper's algorithm [7], Zeilberger's algorithm [23, 25], and Petkovšek's algorithm [16] (see [17, 10, 9] for introductory textbooks covering these algorithms) is nowadays widely known and widely used. Also q-analogs of these algorithms [15, 18, 1, 19], multivariate hypergeometric summation [22, 21], and algorithms for computing with univariate D-finite sequences (i.e., sequences defined through linear recurrence equations with polynomial coefficients) [20, 14, 9] belong to the well-established computational standard.

These algorithms formed the state of the art already at the end of the 20th century, after a decade of stormy developments which had been initiated by Zeilberger's influential papers from 1990 [24, 23]. In the computer algebra community, the development of symbolic summation has advanced far beyond this point since then. However, we have the feeling that some of the more modern algorithms are not yet as widely known as they deserve, and therefore the main goal in this talk will be to bring some of these algorithms to the attention to the combinatorics community, hoping that in the near future they will also be used as routinely as hypergeometric summation machinery is used today. In particular, we plan to explain how to use algorithms for multivariate D-finite functions. These algorithms were pioneered by Zeilberger [24], Chyzak and Salvy [6, 5], and they are implemented for instance in a Mathematica package by Koutschan [11, 12]. We will mention two recent success stories in which this package plays a key role: our proof of the Gessel conjecture [8, 2] and our proof of the qTSPP conjecture [13].

If time permits, we will finally comment on some topics of ongoing research in the area of symbolic summation. These concern an estimate of the computational cost of symbolic summation when the input is very big. It turns out that the classical algorithms are not optimal and can be improved considerably, at least in principle [4, 3]. Without going into any technical details, we will try to explain the underlying algebraic phenomenon on which this improvement is based.

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