Fermionic formulas for graded tensor product multiplicities from quantum cluster algebras

Rinat Kedem

Quantum cluster algebras were introduced by Berenstein and Zelevinsky [BZ] as the natural deformation of the compatible Poisson structure on cluster algebras [FZ]. The functional equations ("Q-systems") satisfied [KR, KNS, N] by the characters of Kirillov-Reshetikhin modules of (quantum-) affine algebras can be understood combinatorially in the framework of cluster algebras [K08]. The quantum cluster algebra associated with this construction was given in [DFK10] and may be interpreted as a "twisted" tensor product of representations.

The Q-systems are intimately related to the fermionic formulas for the decomposition coefficients in the tensor product of KR-modules, originally conjectured from the Bethe ansatz solution of the generalized Heisenberg spin chain [KR]. These coefficients have a graded version, which has a natural physical interpretation (in terms of the energy of Bethe ansatz states), as well as several representation-theoretical ones (in terms of Crystal theory [OSS], fusion products in conformal field theory [FL], or Betti numbers [L]).

Using the formulation as cluster variables, this grading can now be understood [DFK11] combinatorially in terms of the quantum cluster algebra of the Q-system.

References

[BZ] Arkady Berenstein and Andrei Zelevinsky. Quantum cluster algebras. *Adv. Math.*, 195(2):405–455, 2005.

[DFK10] Philippe Di Francesco and Rinat Kedem. Non-commutative integrability, paths and quasi-determinants. *Adv. Math.*, 228:97–152, 2011. `http://xxx.lanl.gov/abs/1008.0980`.

[DFK11] Philippe Di Francesco and Rinat Kedem, Quantum cluter algebras and fusion products. `http://xxx.lanl.gov/abs/1109.6261v1`.

[FL] B. Feigin and S. Loktev. On generalized Kostka polynomials and the quantum Verlinde rule. In *Differential topology, infinite-dimensional Lie algebras, and applications*, volume 194 of *Amer. Math. Soc. Transl. Ser. 2*, pages 61–79. Amer. Math. Soc., Providence, RI, 1999.

[FZ] Sergey Fomin and Andrei Zelevinsky. Cluster algebras. I. Foundations. *J. Amer. Math. Soc.*, 15(2):497–529 (electronic), 2002.

[K08] Rinat Kedem. Q-systems as cluster algebras. *J. Phys. A*, 41(19):194011, 14, 2008.

[KR] A. N. Kirillov and N. Yu. Reshetikhin. Representations of Yangians and multiplicities of the inclusion of the irreducible components of the tensor product of representations of simple Lie algebras. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 160(Anal. Teor. Chisel i Teor. Funktsii. 8):211–221, 301, 1987.

[KNS] Atsuo Kuniba, Tomoki Nakanishi, and Junji Suzuki. Functional relations in solvable lattice models. I. Functional relations and representation theory. *Internat. J. Modern Phys. A*, 9(30):5215–5266, 1994.

[L] George Lusztig. Fermionic formula and betti numbers (preprint). `math.QA/0005010`.

[N] Hiraku Nakajima. t-analogs of q-characters of Kirillov-Reshetikhin modules of quantum affine algebras. *Represent. Theory*, 7:259–274 (electronic), 2003.

[OSS] Masato Okado, Anne Schilling, and Mark Shimozono. Virtual crystals and fermionic formulas of type $D^{(2)}_{n+1}, A^{(2)}_{2n}$, and $C^{(1)}_n$. *Represent. Theory*, 7:101–163 (electronic), 2003.