Primitive derivations, Shi arrangements and Bernoulli polynomials

Hiroaki Terao

Let W be a finite irreducible real reflection group, which is a Coxeter group. A **primitive derivation** D, introduced and studied by K. Saito (e.g., [4]), plays a crucial role in the theory of differential forms with logarithmic poles along the Coxeter arrangement. For example, we may describe the contact order filtration of the logarithmic derivation module using the primitive derivations ([10, 1]). The contact order filtration is closely related to the Frobenius manifold structure of the orbit space.

In particular, let W be a finite Weyl group. We have the central (extended) **Shi arrangements** $\mathrm{Shi}^k(W)$ in V by "deforming" the associated Weyl arrangement $\mathcal{A}(W)$ using the root system as in [5]. The arrangement $\mathrm{Shi}^k(W)$ was proved to be free by M. Yoshinaga in [12]. Let $(\mathcal{A}(W), 2k)$ be the multiarrangements with constant multiplicity $2k$. Since the derivation modules $D(\mathcal{A}(W), 2k)$ is W-isomorphic to the W-module V by [6, 9], we have the following key commutative diagram

$$
\begin{array}{ccc}
V & \overset{\Theta_k}{\dashrightarrow} & D_0(\mathrm{Shi}^k(W))_{kh} \\
 & \searrow^{\text{W-iso. } \Xi_k} \ \circlearrowright & \downarrow \rho \\
 & & D(\mathcal{A}(W), 2k)_{kh}.
\end{array}
$$

Here, the map ρ is the Ziegler restriction map in [13], which is a linear isomorphism. Note that, by Schur's lemma, the W-isomorphism Ξ_k is unique up to a nonzero constant multiple and thus the map Ξ_k is proportional to the covariant derivative $\Xi_k(v) = \nabla_{\partial_v} \nabla_D^{-k} E$, where E is the Euler derivation, D is a primitive derivation and ∂_v is the derivation uniquely determined by $v \in V$ as in [11]. This diagram shows us that there exists a **unique lifting** Θ_k which is a linear isomorphism. We should, however, note that the vector spaces $D_0(\mathrm{Shi}^k(W))_{kh}$ do not admit a W-action. The set of spaces $\{D_0(\mathrm{Shi}^k(W))\}_k$ can be regarded as a one-parameter deformation of the contact order filtration $\{D(\mathcal{A}(W), 2k)\}_k$.

Fix a set Δ of simple roots in V^*. Then the set $\Theta_k(\Delta^*)$ gives a basis for the module $D_0(\mathrm{Shi}^k(W))$ over the symmetric algebra $S := Sym(V^*)$, where the set Δ^* is the dual basis of Δ for V. This basis is called a **dual simple root basis (dSRB)**. The dual object, called a **simple root basis (SRB)**, also exists. Each basis is uniquely determined by Δ up to a nonzero constant as seen in [2].

The SRB of the types A, B, C and D are explicitly expressed by D. Suyama et al. in [8, 7, 3]. These are the first explicitly-described basis for the derivation module of the corresponding Shi arrangements. Those descriptions are in terms of the **Bernoulli polynomials**. We don't know why.

References

[1] T. Abe and H. Terao, A primitive derivation and logarithmic differential forms of Coxeter arrangements. *Math. Z.* **264** (2010), 813–828.

[2] T. Abe and H. Terao, Simple-root bases for Shi arrangements. (arXiv:1111.3510) preprint.

[3] R. Gao, D. Pei and H. Terao, The Shi arrangement of the type D_ℓ. *Proc. Japan Acad. Ser. A Math. Sci.,* **88** (2012), 41-45.

[4] K. Saito, On a linear structure of the quotient variety by a finite reflection group. *Publ. RIMS, Kyoto Univ.* **29** (1993), 535–579.

[5] J.-Y. Shi, The Kazhdan-Lusztig cells in certain affine Weyl groups. *Lecture Notes in Math.*, **1179**, Springer-Verlag, 1986.

[6] L. Solomon and H. Terao, The double Coxeter arrangements. *Comment. Math. Helv.* **73** (1998), 237–258.

[7] D. Suyama, A basis construction for the Shi arrangement of the type B_ℓ or C_ℓ. (arXiv:1205.6294) preprint.

[8] D. Suyama and H. Terao, The Shi arrangements and the Bernoulli polynomials. *Bull. London Math. Soc.* **44** (2012), 563-570.

[9] H. Terao, Multiderivations of Coxeter arrangements. *Invent. Math.* **148** (2002), 659–674.

[10] H. Terao, The Hodge filtration and the contact-order filtration of derivations of Coxeter arrangements. *Manuscripta Math.* **118** (2005), 1–9.

[11] M. Yoshinaga, The primitive derivation and freeness of multi-Coxeter arrangements. *Proc. Japan Acad. Ser. A* **78** (2002), no. 7, 116–119.

[12] M. Yoshinaga, Characterization of a free arrangement and conjecture of Edelman and Reiner. *Invent. Math.* **157** (2004), no. 2, 449–454.

[13] G. M. Ziegler, Multiarrangements of hyperplanes and their freeness. in *Singularities* (Iowa City, IA, 1986), 345–359, Contemp. Math., **90**, Amer. Math. Soc., Providence, RI, 1989.