Triangulations of Cayley and Tutte polytopes

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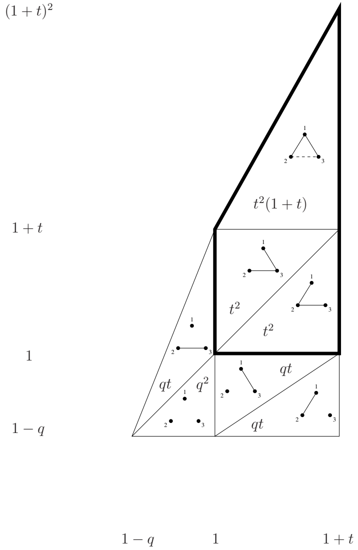
Preview of coming attractions



Cayley's theorem and Braun's conjecture

Theorem (Cayley, 1857)

The number of integer sequences $(a_1, \ldots, a_n)$ such that $1 \leq a_1 \leq 2$ and $1 \leq a_i \leq 2a_{i-1}$ for $i = 2, \ldots, n$, is equal to the total number of partitions of integers $N \in \{0, 1, \ldots, 2^n - 1\}$ into parts $1, 2, 4, \ldots, 2^{n-1}$.

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Conjecture (Braun, 2011)

Define the Cayley polytope $\mathbf{C}_n \subseteq \mathbb{R}^n$ by inequalities

$$1 \leq x_1 \leq 2, \quad \text{and} \quad 1 \leq x_i \leq 2x_{i-1} \quad \text{for} \quad i = 2, \ldots, n.$$

Then $n! \operatorname{vol} \mathbf{C}_n$ is equal to the number of connected graphs on $n + 1$ nodes.

Main result

Theorem (K-Pak)

Define the Tutte polytope $\mathbf{T}_n(q,t) \subseteq \mathbb{R}^n$ (by inequalities or by vertices), $\mathbf{T}_n(0,1) = \mathbf{C}_n$. Then

$$n!\,\mathrm{vol}\,\mathbf{T}_n(q,t) = \sum q^{k(G)-1} t^{|E(G)|},$$

where the sum is over all graphs on $n+1$ nodes, and $k(G)$ is the number of connected components of G.

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In other words, $n! \operatorname{vol} \mathbf{T}_n(q,t) = t^n T_{K_{n+1}}(1+q/t, 1+t)$, where $T_H(x,y)$ denotes the Tutte polynomial of the graph H.

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We call $n! \operatorname{vol} \mathbf{P}$ the normalized volume of $\mathbf{P} \subseteq \mathbb{R}^n$.

Triangulation of Cayley polytope

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We will define:

- a map from connected graphs to (labeled) trees
- a map from trees to simplices

so that:

- the simplices triangulate $\mathbf{C}_n$
- the normalized volume of each simplex is equal to the number of graphs that map into the corresponding tree

Connected graphs to trees: neighbors first search

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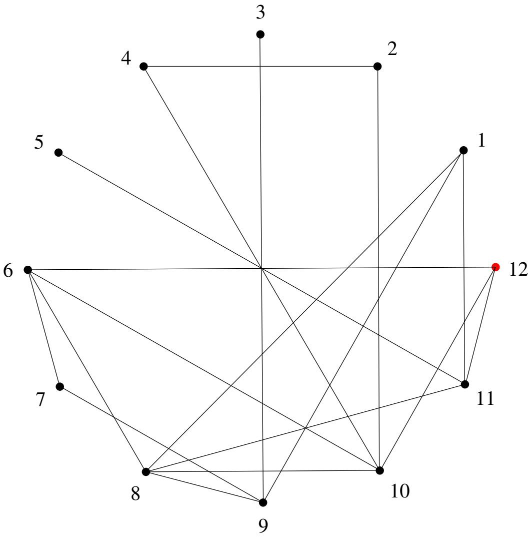
- the result is an ordering of the nodes and a search tree
- this is a variant of the neighbors first search introduced by Gessel and Sagan (1996)

Example



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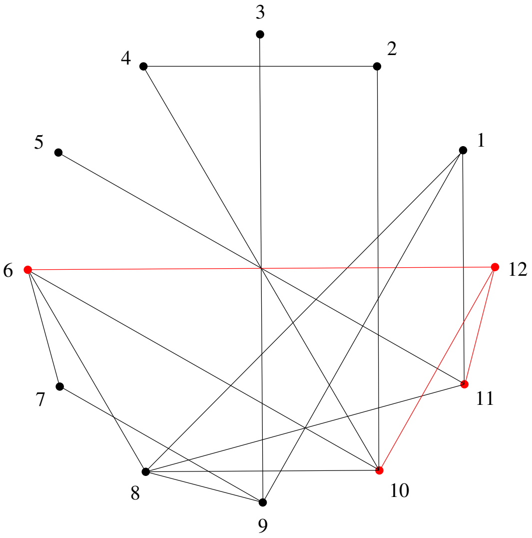

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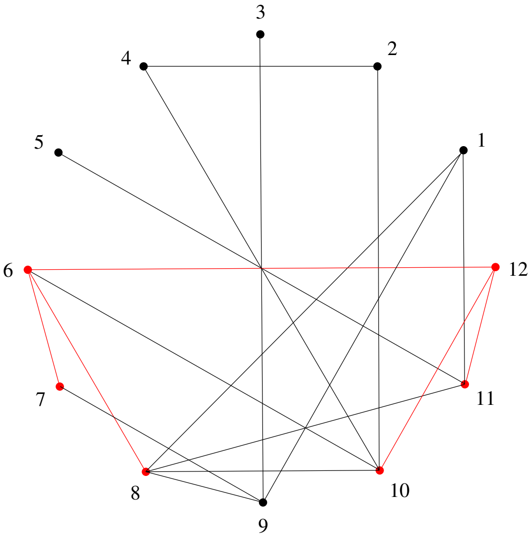

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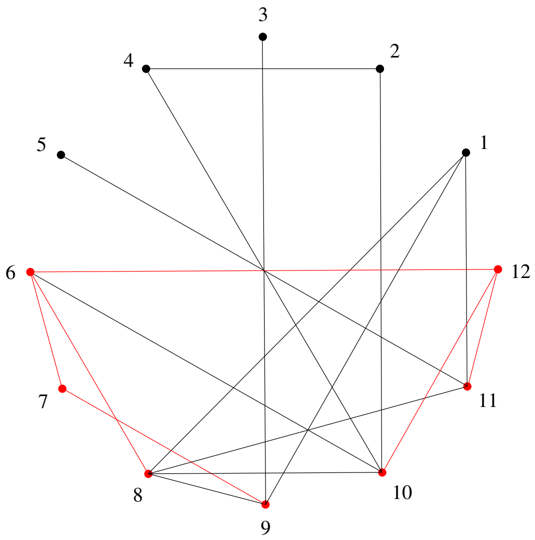

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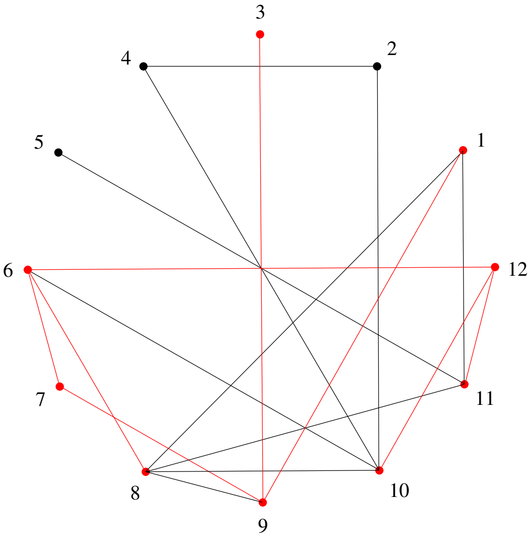

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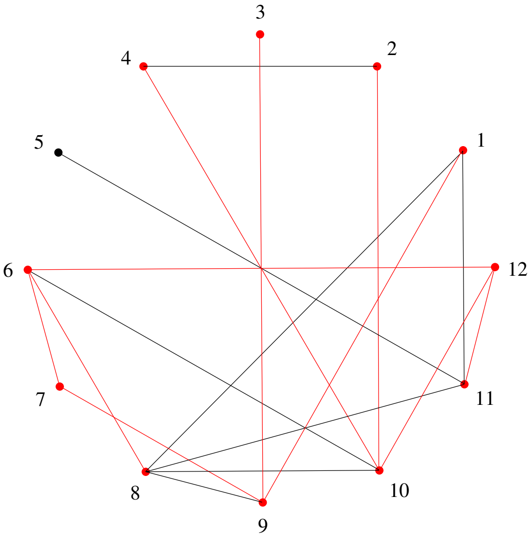

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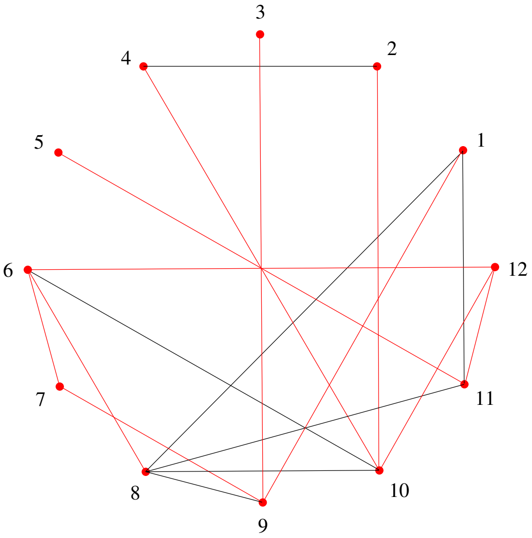
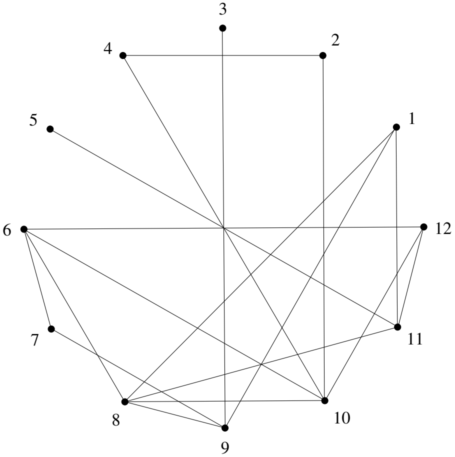
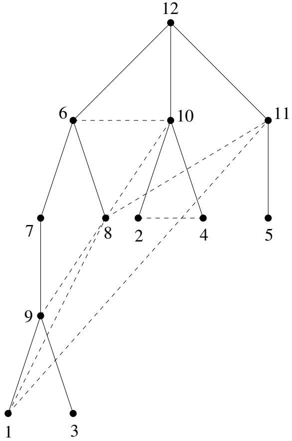


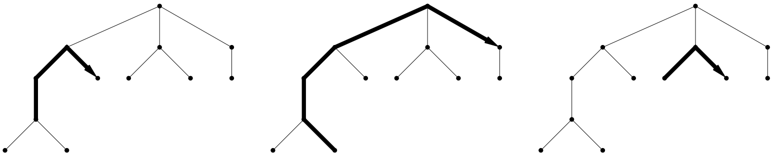
Resulting tree





Cane paths

A cane path is an up-up-. . .-up-down right path.



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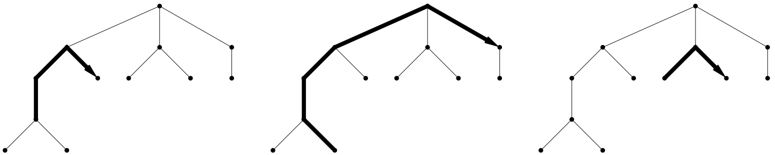
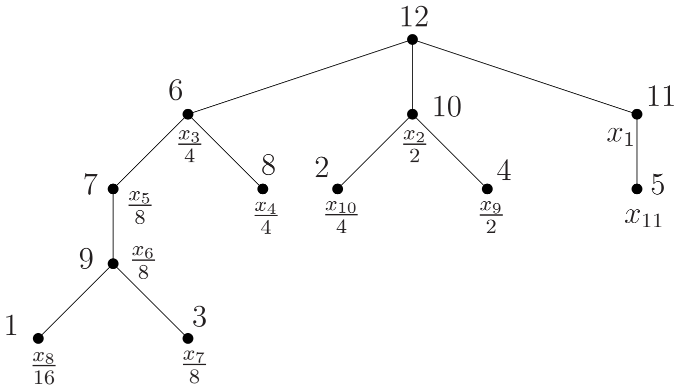


Fact

Number of graphs with search tree T is $2^{\alpha(T)}$, where $\alpha(T)$ is the number of cane paths in T.

Coordinates of nodes in a tree



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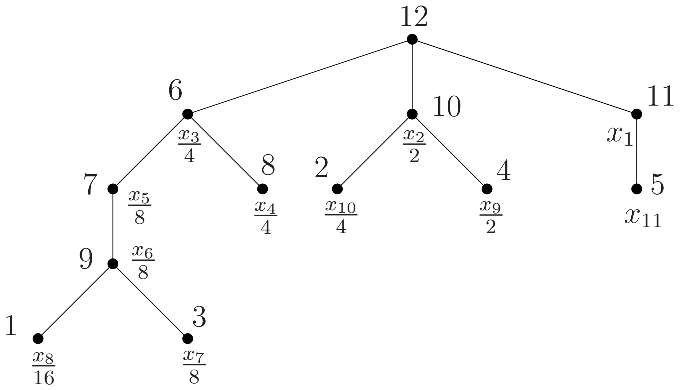
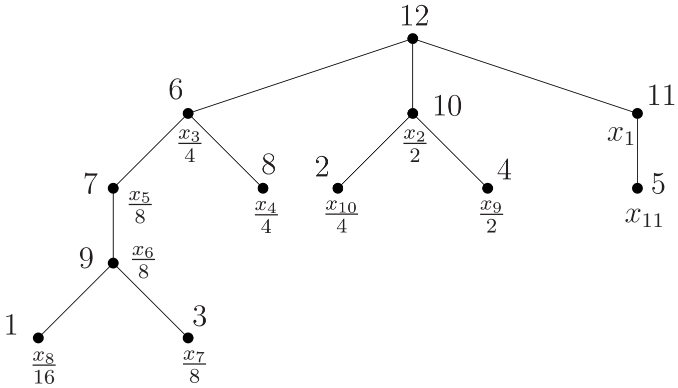


Fact

If the node v is visited i-th in the neighbors first search and j is the number of cane paths starting in v, then the coordinate of v is $x_i/2^j$.

Trees to simplices



$$1 \le \frac{x_8}{16} \le \frac{x_{10}}{4} \le \frac{x_7}{8} \le \frac{x_9}{2} \le x_{11} \le \frac{x_3}{4} \le \frac{x_5}{8} \le \frac{x_4}{4} \le \frac{x_6}{8} \le \frac{x_2}{2} \le x_1 \le 2.$$

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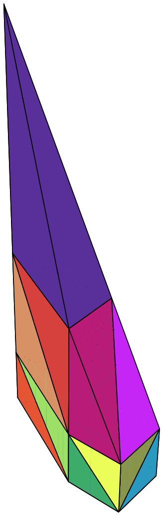
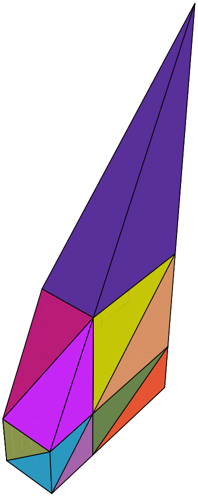
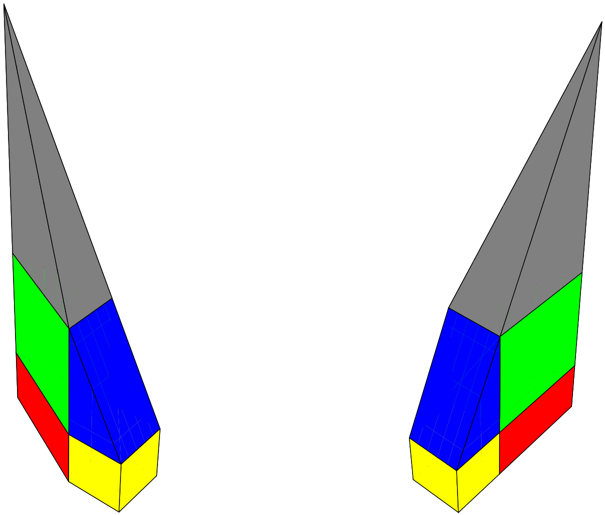
The resulting simplices triangulate Cayley's polytope. So this proves Braun's conjecture.

Triangulation of $\mathbf{C}_3$





Another subdivision of $\mathbf{C}_3$



Gayley polytope

Cayley polytope $\mathbf{C}_n$:

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Gayley polytope $\mathbf{G}_n$:

$$0 \le x_1 \le 2, \quad \text{and} \quad 0 \le x_i \le 2x_{i-1} \quad \text{for} \quad i = 2, \ldots, n$$

It is an orthoscheme with sides $2, 4, \ldots, 2^n$, so its normalized volume is $2^{\binom{n+1}{2}}$, i.e. the number of all graphs on $n+1$ nodes.

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Charles Mills Gayley (1858 – 1932), professor of English and Classics at UC Berkeley

Triangulation of Gayley polytope

Neighbors first search on a general graph: arrange connected components so that their maximal labels are decreasing from left to right, perform neighbors first search on each tree from left to right.

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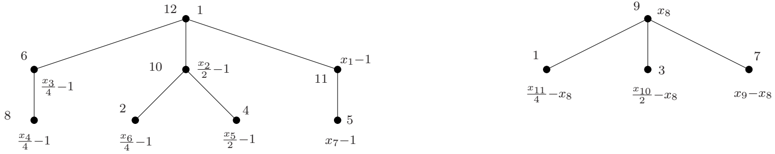

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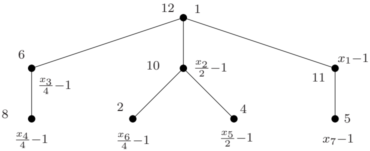

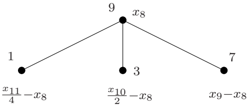

$$0 \leq \frac{x_{11}}{4} - x_8 \leq \frac{x_6}{4} - 1 \leq \frac{x_{10}}{2} - x_8 \leq \frac{x_5}{2} - 1 \leq x_7 - 1 \leq$$

$$\leq \frac{x_3}{4} - 1 \leq x_9 - x_8 \leq \frac{x_4}{4} - 1 \leq x_8 \leq \frac{x_2}{2} - 1 \leq x_1 - 1 \leq 1.$$

t-Cayley and t-Gayley polytope

Replace powers of 2 by powers of $1+t$, $t>0$:

- t-Cayley polytope $\mathbf{C}_n(t)$:

$$1 \leq x_1 \leq 1+t, \quad \text{and} \quad 1 \leq x_i \leq (1+t)x_{i-1} \quad \text{for} \quad i = 2, \ldots, n$$

- t-Gayley polytope $\mathbf{G}_n(t)$:

$$0 \leq x_1 \leq 1+t, \quad \text{and} \quad 0 \leq x_i \leq (1+t)x_{i-1} \quad \text{for} \quad i = 2, \ldots, n$$

- coordinates of the form $x_i/2^j - x_l$ become $x_i/(1+t)^j - x_l$
- coordinates of the form x_l (for roots) become tx_l

Normalized volumes

Theorem

The normalized volume of $\mathbf{C}_n(t)$ is

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where the sum is over all connected graphs G on $n+1$ nodes.
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Tutte polytope: hyperplanes

Take $0 < q \leq 1$ and $t > 0$. Define Tutte polytope $\mathbf{T}_n(q, t)$ by

$$x_n \geq 1 - q,$$

$$qx_i \leq q(1+t)x_{i-1} - t(1-q)(1 - x_{j-1}),$$

where $1 \leq j \leq i \leq n$ and $x_0 = 1$.

Theorem

The normalized volume of Tutte polytope is

$$\sum q^{k(G)-1} t^{|E(G)|},$$

where the sum is over all graphs on $n+1$ nodes.

t-Cayley polytope: vertices

Define $V_n(t)$ as the set of points with properties $x_1 \in \{1, 1+t\}$, $x_i \in \{1, (1+t)x_{i-1}\}$ for $i = 2, \ldots, n$.

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\begin{array}{ccc}
1+t & (1+t)^2 & (1+t)^3 \\
1+t & (1+t)^2 & 1 \\
1+t & 1 & 1+t \\
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\end{array}$$

It is easy to see that $V_n(t)$ is the set of vertices of $\mathbf{C}_n(t)$.

Tutte polytope: vertices

Replace the trailing 1's of each point in $V_n(t)$ by $1 - q$, denote the resulting set $V_n(q,t)$.

$$
\begin{array}{ccc}
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1+t & 1 & 1+t \\
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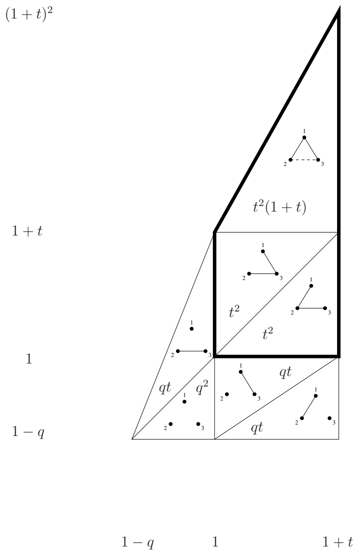
1 & 1+t & 1-q \\
1 & 1 & 1+t \\
1-q & 1-q & 1-q
\end{array}
$$

Then $V_n(q,t)$ is the set of vertices of $\mathbf{T}_n(q,t)$.

Triangulation of $\mathbf{T}_2(q,t)$



Application: recursive formula

Theorem

Define polynomials $r_n(t)$, $n \geq 0$, by

$$r_0(t) = 1, \qquad r_n(t) = -\sum_{j=1}^{n} \binom{n}{j} (1+t)^{\binom{j}{2}} r_{n-j}(t).$$

Then

$$\sum t^{|E(G)|} = \sum_{j=0}^{n} \binom{n}{j} (1+t)^{\binom{j+1}{2}} r_{n-j}(t),$$

where the sum is over connected graphs on $n+1$ nodes.