The representation of the symmetric group on *m*-Tamari intervals

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The representation on *m*-Tamari intervals

Outline

Motivations and results: Toward a combinatorial description of diagonal coinvariant spaces of the symmetric group

Proof: solving a differential-catalytic equation

Open combinatorial questions

$$X := \begin{pmatrix} x_1 & x_2 & x_3 & \dots & x_n \\ y_1 & y_2 & y_3 & \dots & y_n \\ \dots & & & \dots & \dots \\ z_1 & z_2 & z_3 & \dots & z_n \end{pmatrix}$$

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The symmetric group \mathfrak{S}_n acts on X by permuting the columns:

$$\sigma(X) := \begin{pmatrix} x_{\sigma(1)} & x_{\sigma(2)} & \cdots & x_{\sigma(n)} \\ y_{\sigma(1)} & y_{\sigma(2)} & \cdots & y_{\sigma(n)} \\ \vdots & & & \vdots \\ z_{\sigma(1)} & z_{\sigma(2)} & \cdots & z_{\sigma(n)} \end{pmatrix}$$

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 \mathcal{J} is the ideal generated by constant free polynomials such that $\sigma \cdot f(X) := f(\sigma(X)), \quad \forall \sigma \in \mathfrak{S}_n.$

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The diagonal coinvariant space of the symmetric group: $\mathcal{DR}_{k,n} := \mathbb{C}[X] / \mathcal{J}.$

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 $\mathcal{DR}_{k,n}^{\varepsilon}$ is the sign component (the alternants).

'Higher' diagonal coinvariant spaces

A polynomial f(X) is alternant if $\sigma \cdot f(X) = (-1)^{\operatorname{sign}(\sigma)} f(X).$

 $\ensuremath{\mathcal{A}}$ is the ideal generated by alternants.

The 'higher' diagonal coinvariant spaces of the symmetric group: $\mathcal{DR}_{k,n}^m := \varepsilon^{m-1} \otimes \Big(\mathcal{A}^{m-1} \Big/ \mathcal{JA}^{m-1} \Big).$

 $\mathcal{DR}_{k,n}^{m\ \varepsilon}$ is the sign component.

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Coinvariant spaces (k = 1)

Theorem (Artin ~1950's)

 $\dim(\mathcal{DR}^{m\ \varepsilon}_{1,n})=1$

Theorem (Artin ~1950's)

 $\dim(\mathcal{DR}^m_{1,n}) = n!$

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Bivariate diagonal coinvariant spaces (k = 2)

Theorem (Haiman 2002)
$$\dim(\mathcal{DR}_{2,n}^{m \varepsilon}) = \frac{1}{(m+1)n+1} \binom{(m+1)n+1}{n}.$$

Theorem (Haiman 2002)

$$\dim(\mathcal{DR}^m_{2,n}) = (mn+1)^{n-1}.$$

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Combinatorial interpretations of these numbers (k = 2)

Theorem (classic)

The number of m-Dyck paths of height n is

$$\frac{1}{(m+1)n+1}\binom{(m+1)n+1}{n}$$



Theorem (classic)

The number of m-parking functions of height n is

$$(mn+1)^{n-1}$$
.



m-Dyck paths (m = 2)



- paths consisting of north and east steps.
- starting at (0,0) and finishing at (mn, n).
- never going below the line of equation my = x.

m-parking functions (m = 2)



- *m*-Dyck paths.
- *n* north steps are labelled with the values $\{1, 2, \dots, n\}$.
- labels increase along consecutive north steps.

Trivariate diagonal coinvariant spaces (k = 3)

Conjecture (m=1: Haiman 1994; m > 1: F. Bergeron 2009)

$$\dim(\mathcal{DR}^{m\ \varepsilon}_{3,n}) = \frac{m+1}{n(mn+1)}\binom{(m+1)^2n+m}{n-1}.$$

Conjecture (m = 1: Haiman 1994; m > 1: Bergeron 2009)

$$\dim(\mathcal{DR}_{3,n}^m) = (m+1)^n (mn+1)^{n-2}.$$

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Combinatorial questions: what do these numbers count?

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Combinatorial interpretations of these numbers (k = 3)

Theorem (m = 1: Chapoton 2006; m > 1: MBM, Fusy, LFPR 2011) (Conj. m > 1: Bergeron 2009)

The number of intervals in the m-Tamari lattice defined on m-Dyck paths of height n is

 $\frac{m+1}{n(mn+1)}\binom{(m+1)^2n+m}{n-1}.$



Theorem (MBM, GC, LFPR 2011) (Conj. Bergeron 2008-2009)

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Proof.

Ideas of the proofs will be given in the next section.

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The representation on *m*-Tamari intervals

The cover relation in the *m*-Tamari lattice (m=2)



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The representation on *m*-Tamari intervals

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Tamari lattice on Dyck paths of height 4 (m = 1)



2-Tamari lattice on 2-Dyck paths of height 3



The labelled intervals in the *m*-Tamari lattice (m = 2)



- an interval in the *m*-Tamari lattice
- the top path is an *m*-parking function

A representation on labelled intervals in the *m*-Tamari lattice



We denote this combinatorial representation by $Tam_m(n)$.

A representation on labelled intervals in the *m*-Tamari lattice



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Conjecture (F. Bergeron, LFPR 2010) $\mathcal{DR}_{3,n}^m \cong \varepsilon \otimes \operatorname{Tam}_m(n).$ MBM, GC, LFPR (2012)The representation on *m*-Tamari intervalsFPSAC 201217 / 29

The character of the *m*-Tamari representation

Theorem (MBM, GC, LFPR 2012) (Conj. Bergeron, LFPR 2010)

In the representation $\operatorname{Tam}_m(n)$, the number of labelled intervals in the *m*-Tamari lattice fixed under a permutation of cycle type $\lambda = (\lambda_1, ..., \lambda_\ell)$ is given by

$$(mn+1)^{\ell-2}\prod_{1\leq i\leq \ell}\binom{(m+1)\lambda_i}{\lambda_i}.$$

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Parameters for *m*-Tamari intervals



- t: height of the paths
- x : contacts of the lower path
- y : first rise of the top path

$$F^{(m)}(t; x, y) := \sum_{n \ge 0} \sum_{\substack{\alpha \le \beta \\ \alpha, \beta \in \text{Dyck}_m(n)}} t^n x^{\text{contacts}(\alpha)} y^{\text{rise}(\beta)}$$

Parameters for labelled *m*-Tamari intervals



- t: height of the paths
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- y : first rise of the top path

$$G^{(m)}(t;x,y) := \sum_{n \ge 0} \sum_{\substack{\alpha \le \beta \\ \alpha, \beta \in \operatorname{Dyck}_{m}(n)}} \sum_{P \in \operatorname{Park}_{m}(\beta)} \frac{t^{n}}{n!} x^{\operatorname{contacts}(\alpha)} y^{\operatorname{rise}(\beta)}$$

Recurrence on unlabelled *m*-Tamari intervals (m = 1)



Recurrence on unlabelled *m*-Tamari intervals (m = 1)



With
$$\Delta(R(t;x)) := \frac{R(t;x)-R(t;1)}{x-1}$$
, this reads:

$$F(t;x,y) = x + txy(F(t;x,1)\Delta) (F(t;x,y)).$$

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For general *m*:

$$F^{(m)}(t;x,y) = x + txy(F^{(m)}(t;x,y)\Delta)^m (F^{(m)}(t;x,y)).$$

Polynomial equations with a catalytic variable (y = 1)

$$F(t;x) = x + tx \frac{F(t;x) - F(t;1)}{x-1} F(t;x).$$

Divided difference: x is a catalytic variable (Zeilberger??).

Such equations are ubiquitous in map enumeration.

The solution is always an algebraic series (Brown-Tutte 1960's, Bousquet-Mélou-Jehanne 2006).

i.e. there exist a polynomial Q such that:

Q(F(t;x),t,x)=0.

Recurrence on labelled *m*-Tamari intervals (m = 1)



With $\Delta(R(t;x)) := \frac{R(t;x) - R(t;1)}{x-1}$, for general *m*: $G^{(m)}(t;x,y) = x + tx \int (G^{(m)}(t;x,1)\Delta)^m (G^{(m)}(t;x,y)) dy.$ Alternative forms of the functional equations

$$F^{(m)}(t;x,y) = x + txy(F^{(m)}(t;x,1)\Delta)^m (F^{(m)}(t;x,y))$$

is equivalent to

$$F^{(m)}(t;x,y) = \frac{1}{1 - t \, x \, y (F^{(m)}(t;x,1)\Delta)^m} \, (x).$$

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is equivalent to

$$G^{(m)}(t;x,y) = e^{txy(G^{(m)}(t;x,1)\Delta)^{m}}(x).$$

$$F^{(m)}(t;x,y) = \frac{1}{1 - t \, x \, y (F^{(m)}(t;x,1)\Delta)^m} \, (x)$$

algebraic \Rightarrow use Gfun (Maple)

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$$F^{(m)}(t;x,y) = \frac{1}{1 - t \times y(F^{(m)}(t;x,1)\Delta)^m} (x)$$

$$t = z(1-z)^{m^2+2m}$$

$$x=\frac{1+u}{(1+zu)^{m+1}}$$

$$F^{(m)}(t;x,1) = \frac{1+u}{(1-z)^{m+2}} \left(\frac{1+u-(1+zu)^{m+1}}{(1+zu)^{m+1}u} \right)$$

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$$F^{(m)}(t;x,y) = \frac{1}{1 - t x y (F^{(m)}(t;x,1)\Delta)^m} (x) \qquad G^{(m)}(t;x,y) = e^{t x y (G^{(m)}(t;x,1)\Delta)^m} (x)$$

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$$t = z(1-z)^{m^2+2m}$$
 $t = ze^{-m(m+1)z}$

$$x = \frac{1+u}{(1+zu)^{m+1}} \qquad \qquad x = \frac{1+u}{e^{mzu}}$$

$$F^{(m)}(t;x,1) = \frac{1+u}{(1-z)^{m+2}} \left(\frac{1+u-(1+zu)^{m+1}}{(1+zu)^{m+1}u} \right) \qquad G^{(m)}(t;x,1) = \frac{1+u}{e^{-(m+1)z}} \left(\frac{1+u-e^{mzu}}{e^{(m-1)zu}u} \right)$$

algebraic \Rightarrow use Gfun (Maple)

????

 $\sum_{i=1}^{m} \frac{d(x_i,y)}{d(x_i,y)} < \omega^{(d'(x))},$ By $\prod_{i=1}^{n}(A_i - A_i)$ we near $\prod_{i=1}^{n}(A_i - A_i)$ but we profer the short we needed when the baselines a_i are done. Observe that the A_i saw distinct since the i_i -transmission theorem is d^{-1} in A(i) is $a_i(1+i_i)$. Note since that a_i is an distinct since the a_i -transmission A(i) is $a_i(1+i_i)$. Note since that the same i_i is the same i_i -transmission of A(i) is $a_i(1)$ and A(i) is marked by the same i_i -transmission i_i -transmission A(i) is a A(i) -transmission of A(i) in A(i) -transmission of A(i) -transmission A(i) -trans $\sum_{i=1}^{n} \frac{h_i(\alpha)}{\Pi_{i+1}(\alpha) + i(\alpha)} = (-1)^{\frac{1}{n}} \prod_{i=1}^{n} \frac{1}{\Omega_i}$ $\sum_{i=1}^{n} \frac{Q(\alpha_i)}{|I_{i+1}(Q_i)|^2} = 0.$

Remark. One conceptuate for a 's an Poincar aution in a just [40, Ch. Q. Int this will not be much it have and up will think of them as abstract dimension of an abstract extension of this.

 $\frac{|u_{i}^{(1)}|}{2u}(x,y;u,y) = \sum_{k=0}^{\infty} \frac{2u}{2} \left((x,k)u_{i}^{(0)}u_{i}^{(0)} \right)^{(k)} \hat{u}(x,y;u,y),$

 $= \sum_{n=0}^{\infty} \left(z_{1}^{(1+n)} e^{-z(1+n)z_{1}} \left((1+n) e^{2z(z+1)} A_{z} \right)^{(n)} \right)^{p_{1}} \hat{d}(z, p, n, p_{1}, (n))$

 $\frac{2 h_{1}^{2}}{2 h_{1}^{2} + \mu (n,q)} = \sum_{n \leq n} \frac{h_{1}}{2} \left(e^{(1+n)(n-2) + h^{2}(n))} \left(\frac{2 h_{1}^{2}(n)}{(1+n)(n-2) + h^{2}(n-1)} h_{n}^{2} \right)^{2 n} \right)^{2 n} \hat{\mathcal{O}}(1+\mu (n,q))$

 $G(n) = (0, n) e^{-i(n-n)^2}, \qquad (30)$ Theorem that the first point of equations of the $(0, n) = nO_{11} + nO_{12} + nO_{12}$

with A 1814 + No. 1914 and the initial condition

 $\frac{\partial C}{\partial x}(x,y) = \sum \frac{\partial c}{\partial x} \left(c(1+x)) - c(0^{-1}(x+x)) - \frac{\partial C(x,y)}{\partial x} \left(\frac{\partial C(x,y)}{\partial x} - \frac{\partial C(x,y)}{\partial x} \right)^{2/2} \right)^{2/2} C(x,y), \quad (46)$

A Party and an end states, and the case of 1

 $= \prod_{i=1}^{n} \exp \left(\chi \frac{2\pi}{n} \left(h c(F(x, i); \Delta_i)^{-1} \right)^{(n)} \right) (x)$ $= -\exp \left(y \sum_{i=1}^{m} \left[h c(F(x, 1)h)^{(m)} \right]^{(n)} \right) (x).$

 $F(x,y) = \sum_{i=1}^{n} \prod_{j=1}^{n} \left(\frac{1}{2\gamma_{j}} \left(\frac{1}{2\gamma_{j}} \right)^{n} \left((x_{j})^{j} (x_{j}) (x_{j})^{n-1} \right)^{n-1} \right) (x)$

 $A^{(e)}\delta(u) = \frac{1}{A^{(e)}} \left(\delta(u) - \sum_{i=1}^{n} g_{i}^{e} A(u)^{i}\right).$ (40) $-K^{(n-1)} \bar{u}(u) = \frac{K^{(n)} \bar{u}(u) - \mu_{1}^{'}}{\bar{u}(u)} = \frac{1}{\bar{u}(u)^{n-1}} \left(\bar{u}(u) - \sum_{i=1}^{n} \mu_{1}^{'} \bar{u}(u) r \right)$

 $A(X) = \sum_{n=0}^{\infty} A(n) \prod_{n=0}^{N} \frac{X - x_n}{x_n - x_n}$

 $\frac{\partial}{\partial u} \Big(\hat{u}(u,y) - \hat{u}(u,y) \Big) = \sum_{i=1}^{m} v(i+u)^{ij} (1+i)^{ij} \left\{ \hat{u}(u,y) - \hat{u}(u,y) \right\} = V(i) \left\{ \hat{u}(u,y) - \hat{u}(u,y) \right\}.$

 $\frac{|a|^2}{2}(n, g) = \sum \frac{\mu}{2} n^2 \left[(1 + n)(1 + i) d \right]^{2/2} \tilde{G}(n, g), \quad (26)$ $\frac{2n^2}{2n}(0, y) = \sum \frac{2n}{2} e^{i \phi} \left((1 + n)(1 + n) \theta \right)^{2n} \hat{G}(0, y),$

August - 100 - 100 A(z)and $z = (1 + u)^{-\alpha/2} u^{-\alpha}$ as before. Apping it is to put that for any that ((4)) and the initial con-lister ((4)) define a unique notice in z_1 denoted of $z_1 = z_2 = z_1$. The coefficients of this ratio for its X(z, z). The principle of our proof can be described as before.

 $A(u) = \frac{u}{1+u} e^{-i u (u)},$

 $\frac{\partial \hat{\omega}}{\partial x}|_{X=|g|} = \sum \frac{n_{e}}{2} \left(|z|^2 \hat{u}|_{X=|g|}^2 - P_{\theta}(A, z)\right).$ $B(n, y) = \sum_{i=1}^{m} \frac{\hat{G}(n, y)}{m - 1}$

 $\frac{\langle n\rangle}{2\pi}(n,y) = \sum_{i} \frac{\partial}{\partial t} \Big((\alpha)^{i} \hat{u}_{i}^{i}(n,y) - P_{i}(A(n),z) \Big),$

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Proof of checking: le début ...

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Open combinatorial questions

- Simplify the checking (m > 1).
- Prove the formulas without guessing.
- Bijective proofs? Connections with certain maps? Why is F(t;x,y) symmetric in x and y?
- Combinatorial problems involving statistics on these objects (the statistics correspond to the grading of these spaces).