Combinatorial Reciprocity for Monotone Triangles

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joint work with Ilse Fischer

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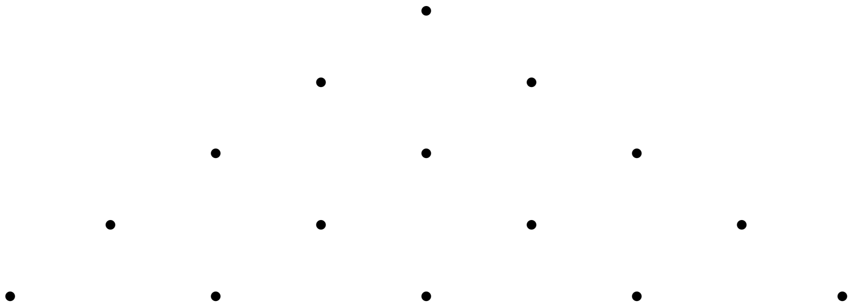
Definition (Monotone Triangle)

Triangular array of integers with

- weak increase along North-East diagonals and South-East diagonals
- strict increase along rows



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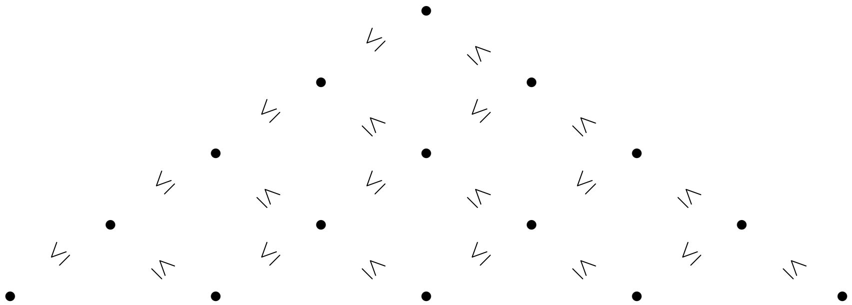

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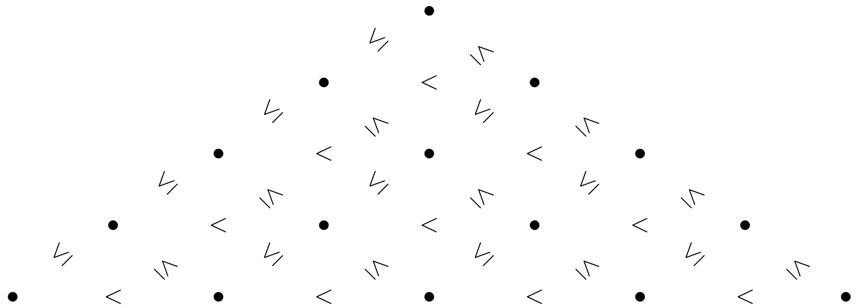

Example (The seven MTs with bottom row $(1, 2, 3)$)

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 2 & \\ 1 & & 2 & & 3 \end{array} \qquad \begin{array}{ccccc} & & 2 & & \\ & 1 & & 2 & \\ 1 & & 2 & & 3 \end{array}$$

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How many MTs with bottom row $k_1 < k_2 < \ldots < k_n$ are there?

Example

$n = 2$: # Monotone Triangles with bottom row (k_1, k_2)

$$\begin{array}{ccccc} & & ? & & \\ & \leq & & \leq & \\ k_1 & & < & & k_2 \end{array} \quad \Rightarrow \quad k_2 - k_1 + 1 \text{ MTs}$$

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Theorem (I. Fischer (2005))

For each $n \geq 1$, there exists a polynomial $\alpha(n; k_1, k_2, \ldots, k_n)$ of degree $n-1$ in each of the n variables satisfying

$$\alpha(n; k_1, k_2, \ldots, k_n) = \#\text{MTs with bottom row } (k_1, k_2, \ldots, k_n),$$

whenever $k_1 < k_2 < \cdots < k_n$.

Example ($\alpha(3; 1, 2, 3) = 7$)

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 2 & \\ 1 & & 2 & & 3 \end{array} \qquad \begin{array}{ccccc} & & 2 & & \\ & 1 & & 2 & \\ 1 & & 2 & & 3 \end{array}$$

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```
    1            2
  1   2        1   2
1   2   3    1   2   3

    1            2            3
  1   3        1   3        1   3
1   2   3    1   2   3    1   2   3

    2            3
  2   3        2   3
1   2   3    1   2   3
```

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Decreasing Monotone Triangles

What does $\alpha(n; k_1, k_2, \ldots, k_n)$ count for $k_1 \geq k_2 \geq \cdots \geq k_n$?

Definition (Decreasing Monotone Triangle)

Triangular array of integers with

- weak decrease along NE- and SE-diagonals
- each row contains an entry at most twice
- two consecutive rows do not contain the same entry exactly once

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Example (The five DMTs with bottom row $(6,3,3,2,1)$)

				2				
			2		2			
		3		2		2		
	3		3		2		2	
6		3		3		2		1

				3				
			3		3			
		3		3		2		
	3		3		2		2	
6		3		3		2		1

				3				
			3		3			
		3		3		2		
	4		3		2		2	
6		3		3		2		1

				2				
			2		2			
		4		2		2		
	5		3		2		2	
6		3		3		2		1

				3				
			3		3			
		3		3		2		
	5		3		2		2	
6		3		3		2		1

Definition (Duplicate-Descendant)

A duplicate-descendant is a pair (x, x), which is either

- in the bottom row, or
- the row below contains the same pair (x, x).

Example (The five DMTs with bottom row $(6, 3, 3, 2, 1)$)

```
        2
      2   2
    3   2   2
  3   3   2   2
6   3   3   2   1
```

```
        3
      3   3
    3   3   2
  3   3   2   2
6   3   3   2   1
```

```
        3
      3   3
    3   3   2
  4   3   2   2
6   3   3   2   1
```

```
        2
      2   2
    4   2   2
  5   3   2   2
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2
2 **2**
3 **2** **2**
3 **3** 2 2
6 **3** **3** 2 1

3
3 **3**
3 **3** 2
3 **3** 2 2
6 **3** **3** 2 1

3
3 **3**
3 3 2
4 3 2 2
6 **3** **3** 2 1

2
2 **2**
4 **2** **2**
5 3 2 2
6 **3** **3** 2 1

3
3 **3**
3 3 2
5 3 2 2
6 **3** **3** 2 1

Theorem 1 (I. Fischer, R. (2011))

Let $k_1 \geq k_2 \geq \cdots \geq k_n$ and $\mathcal{D}_n(k_1, \ldots, k_n)$ denote the set of DMTs with bottom row $(k_1, \ldots, k_n)$.

Then

$$\alpha(n; k_1, \ldots, k_n) = (-1)^{\binom{n}{2}} \sum_{A \in \mathcal{D}_n(k_1, \ldots, k_n)} (-1)^{\mathrm{dd}(A)},$$

where $\mathrm{dd}(A)$ *is the number of duplicate-descendants in A.*

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$$\alpha(n; k_1, \ldots, k_n) = (-1)^{\binom{n}{2}} \sum_{A \in \mathcal{D}_n(k_1,\ldots,k_n)} (-1)^{\mathrm{dd}(A)}$$

Example ($\mathcal{D}_5(6,3,3,2,1)$)

$$\begin{array}{ccccccccc} &&&& 2 &&&& \\ &&& \mathbf{2} && \mathbf{2} &&& \\ && 3 && \mathbf{2} && \mathbf{2} && \\ & \mathbf{3} && \mathbf{3} && 2 && 2 & \\ 6 && \mathbf{3} && \mathbf{3} && 2 && 1 \end{array} \qquad \begin{array}{ccccccccc} &&&& 3 &&&& \\ &&& \mathbf{3} && \mathbf{3} &&& \\ && \mathbf{3} && \mathbf{3} && 2 && \\ & \mathbf{3} && \mathbf{3} && 2 && 2 & \\ 6 && \mathbf{3} && \mathbf{3} && 2 && 1 \end{array} \qquad \begin{array}{ccccccccc} &&&& 3 &&&& \\ &&& \mathbf{3} && \mathbf{3} &&& \\ && 3 && 3 && 2 && \\ & 4 && 3 && 2 && 2 & \\ 6 && \mathbf{3} && \mathbf{3} && 2 && 1 \end{array}$$

$$\begin{array}{ccccccccc} &&&& 2 &&&& \\ &&& \mathbf{2} && \mathbf{2} &&& \\ && 4 && \mathbf{2} && \mathbf{2} && \\ & 5 && 3 && 2 && 2 & \\ 6 && \mathbf{3} && \mathbf{3} && 2 && 1 \end{array} \qquad \begin{array}{ccccccccc} &&&& 3 &&&& \\ &&& \mathbf{3} && \mathbf{3} &&& \\ && 3 && 3 && 2 && \\ & 5 && 3 && 2 && 2 & \\ 6 && \mathbf{3} && \mathbf{3} && 2 && 1 \end{array}$$

$$\alpha(5; 6,3,3,2,1) = (-1)^{\binom{5}{2}} \sum_{A \in \mathcal{D}_5(6,3,3,2,1)} (-1)^{\mathrm{dd}(A)} = 3$$

Some motivation...

$$\alpha(2n; n, n, n-1, n-1, \ldots, 1, 1) = ?$$

$n = 1:$	1
$n = 2:$	2
$n = 3:$	7
$n = 4:$	42
$n = 5:$	429

Number of Alternating Sign Matrices of size n

Some motivation...

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$$
\begin{aligned}
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n = 2 &: \quad 2 \\
n = 3 &: \quad 7 \\
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\end{aligned}
$$

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Definition (Alternating Sign Matrix of size n)

- $(n \times n)$-matrix
- entries in $\{0, 1, -1\}$
- in each row/column: non-zero entries alternate in sign and sum up to 1

Example (The seven ASMs of size 3)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

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Connection between ASMs and MTs

Correspondence (Mills, Robbins, Rumsey, 1983):

$$\text{ASMs of size } n \;\Leftrightarrow\; \text{MTs with bottom row } (1, 2, \ldots, n)$$

Example

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \Longleftrightarrow \begin{array}{ccccccccc} & & & & 2 & & & & \\ & & & 1 & & 4 & & & \\ & & 1 & & 3 & & 5 & & \\ & 1 & & 2 & & 4 & & 5 & \\ 1 & & 2 & & 3 & & 4 & & 5 \end{array}$$

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Some motivation...

$$\alpha(2n; n, n, n-1, n-1, \dots, 1, 1) = ?$$

$$\begin{aligned} n = 1: &\quad 1 \\ n = 2: &\quad 2 \\ n = 3: &\quad 7 \\ n = 4: &\quad 42 \\ n = 5: &\quad 429 \end{aligned}$$

$$\boxed{\alpha(2n; n, n, n-1, n-1, \dots, 1, 1) = \alpha(n; 1, 2, \dots, n)}$$

Towards a combinatorial proof

$$\boxed{\alpha(2n; n, n, n-1, n-1, \dots, 1, 1) \stackrel{!}{=} \alpha(n; 1, 2, \dots, n)}$$

$$(-1)^{\binom{2n}{2}} \sum_{A \in \mathcal{D}_{2n}(n,n,n-1,n-1,\dots,1,1)} (-1)^{\mathrm{dd}(A)} \stackrel{!}{=} \#\ \text{MTs with bottom row } (1, 2, \dots, n)$$

$\rightarrow$ find suitable partition of $\mathcal{D}_{2n}(n, n, n-1, n-1, \dots, 1, 1)$

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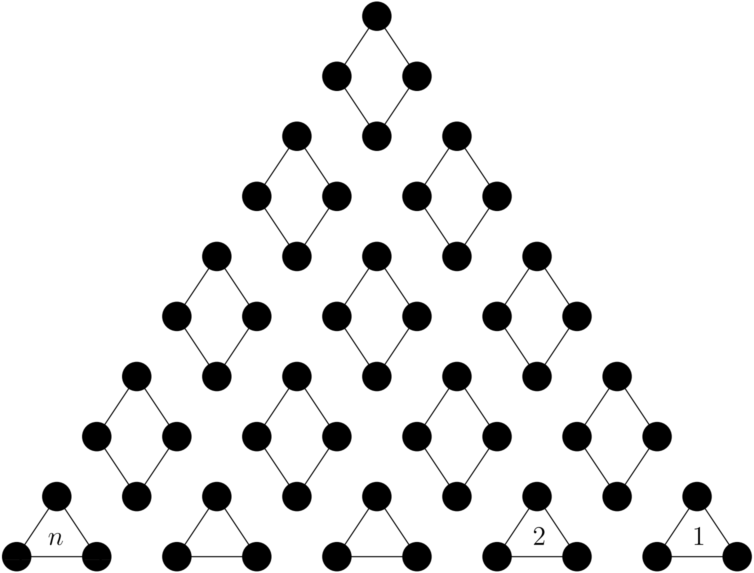
n
2
1

Example

$$
\begin{array}{ccccccccccc}
 & & & & & 2 & & & & & \\
 & & & & 2 & & 2 & & & & \\
 & & & 3 & & 2 & & 1 & & & \\
 & & 3 & & 3 & & 1 & & 1 & & \\
 & 3 & & 3 & & 2 & & 1 & & 1 & \\
3 & & 3 & & 2 & & 2 & & 1 & & 1
\end{array}
\quad \Leftrightarrow \quad
\begin{array}{ccccc}
 & & 2 & & \\
 & 1 & & 3 & \\
1 & & 2 & & 3
\end{array}
$$

Open problem:

Sign-reversing involution on the remaining set of DMTs?

Example

$$
\begin{array}{ccccccccccc}
 & & & & & 2 & & & & & \\
 & & & & 2 & & 2 & & & & \\
 & & & 3 & & 2 & & 1 & & & \\
 & & 3 & & 3 & & 1 & & 1 & & \\
 & 3 & & 3 & & 2 & & 1 & & 1 & \\
3 & & 3 & & 2 & & 2 & & 1 & & 1
\end{array}
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\begin{array}{ccccc}
 & & 2 & & \\
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1 & & 2 & & 3
\end{array}
$$

Open problem:

Sign-reversing involution on the remaining set of DMTs?

Overview of involved combinatorial objects

Monotone Triangles with bottom row $(1, 2, \ldots, n)$

$\Updownarrow$

$(n \times n)$-ASMs

DMTs with bottom row $(n, n, n-1, n-1, \ldots, 1, 1)$

$\Updownarrow$

?

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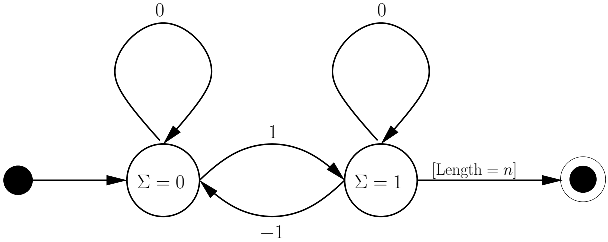

Figure: Machine generating rows and columns of ASMs

2-ASMs

Definition (2-ASM of size n)

- $(2n) \times n$-matrix
- rows generated by ASM-machine
- columns generated by

2-ASMs

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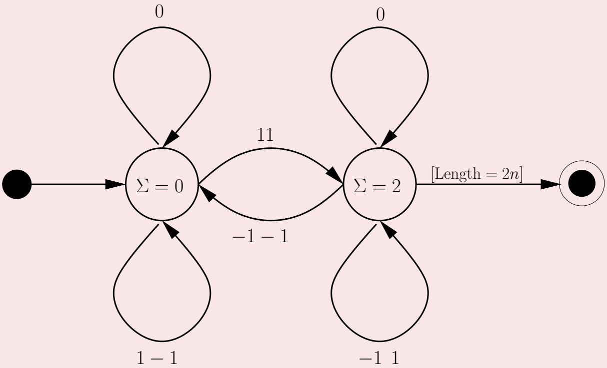
- rows generated by ASM-machine
- columns generated by



Example (DMT $\Leftrightarrow$ 2-ASM)

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 2 & & & & & & & \\
 & & & & & & 2 & & 2 & & & & & & \\
 & & & & & 3 & & 2 & & 2 & & & & & \\
 & & & & 4 & & 2 & & 2 & & 1 & & & & \\
 & & & 4 & & 4 & & 2 & & 1 & & 1 & & & \\
 & & 4 & & 4 & & 2 & & 2 & & 1 & & 1 & & \\
 & 4 & & 4 & & 3 & & 2 & & 2 & & 1 & & 1 & \\
4 & & 4 & & 3 & & 3 & & 2 & & 2 & & 1 & & 1
\end{array}
\Leftrightarrow
$$

$$
\Leftrightarrow
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & -1 & 1 \\
1 & -1 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
$$

Theorem

The set $\mathcal{D}_{2n}(n, n, n-1, n-1, \dots, 1, 1)$ *is in bijection with the set of 2-ASMs of size* n.

Monotone Triangles with bottom row $(1, 2, \dots, n)$

$\Updownarrow$

$(n \times n)$-ASMs

DMTs with bottom row $(n, n, n-1, n-1, \dots, 1, 1)$

$\Updownarrow$

2-ASMs of size n

Theorem 2 (I. Fischer, R. (2011))

Let $A_{n,i}$ denote the number of ASMs with the first row's unique 1 in column i. Then

$$\alpha(2n-1; n-1+i, n-1, n-1, \ldots, 1, 1) = (-1)^{n-1} A_{n,i}$$

holds for $i = 1, \ldots, 2n-1$, $n \geq 1$.

Corollary

$$\alpha(2n; n, n, n-1, n-1, \ldots, 1, 1) = \alpha(n; 1, 2, \ldots, n)$$

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Proof.

$$\alpha(n; 1, 2, \ldots, n) = A_{n+1,1}$$

$$\stackrel{\text{Th.2}}{=} (-1)^n \alpha(2n+1; n+1, n, n, n-1, n-1, \ldots, 1, 1)$$

$$\stackrel{\text{Th.1}}{=} \sum_{A \in \mathcal{D}_{2n+1}(n+1,n,n,\ldots,1,1)} (-1)^{\mathrm{dd}(A)}$$

$$= \sum_{A \in \mathcal{D}_{2n}(n,n,\ldots,1,1)} (-1)^{\mathrm{dd}(A)+n}$$

$$\stackrel{\text{Th.1}}{=} \alpha(2n; n, n, n-1, n-1, \ldots, 1, 1).$$

Corollary

$$\alpha(2n; n, n, n-1, n-1, \ldots, 1, 1) = \alpha(n; 1, 2, \ldots, n)$$

Proof.

$$\begin{aligned}
\alpha(n; 1, 2, \ldots, n) &= A_{n+1,1} \\
&\overset{\text{Th.2}}{=} (-1)^n \alpha(2n+1; n+1, n, n, n-1, n-1, \ldots, 1, 1) \\
&\overset{\text{Th.1}}{=} \sum_{A \in \mathcal{D}_{2n+1}(n+1, n, n, \ldots, 1, 1)} (-1)^{\text{dd}(A)} \\
&= \sum_{A \in \mathcal{D}_{2n}(n, n, \ldots, 1, 1)} (-1)^{\text{dd}(A)+n} \\
&\overset{\text{Th.1}}{=} \alpha(2n; n, n, n-1, n-1, \ldots, 1, 1).
\end{aligned}$$

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$$\alpha(2n; n, n, n-1, n-1, \dots, 1, 1) = \alpha(n; 1, 2, \dots, n)$$

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\end{aligned}$$

By Theorem 1, if $n = 2m$, then

$$\alpha(n; n, n-1, \ldots, 1) = 0 :$$

$$\begin{array}{ccccccccccccccccccc}
 & ↯ & & 2m-2 & & 2m-2 & & \cdots & & & 4 & & 4 & & 2 & & 2 & \\
2m & & 2m-1 & & 2m-2 & & 2m-3 & & \cdots & 5 & & 4 & & 3 & & 2 & & 1
\end{array}$$

Conjecture

For $n = 2m+1$, $m \geq 1$, the equation

$$\begin{aligned}\alpha(n; n, n-1, \ldots, 1) &= (-1)^m \alpha(m; 2, 4, \ldots, 2m) \\ &= (-1)^m \text{ \# vertically symmetric ASMs of size } 2m+1\end{aligned}$$

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