Constructing combinatorial operads from monoids

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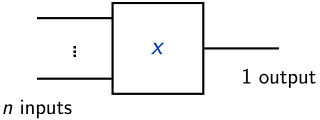
Trees and elements of operads

Element of $\mathcal{P}(n) \rightsquigarrow$ operator of arity n:



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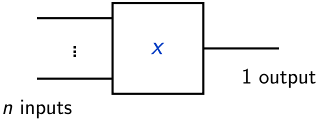

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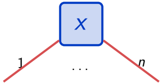

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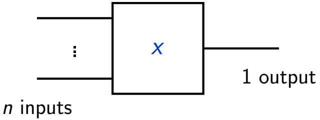

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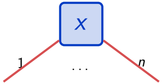

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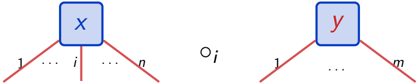

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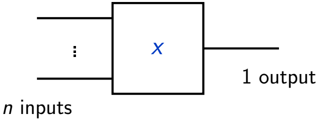

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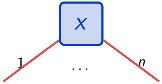

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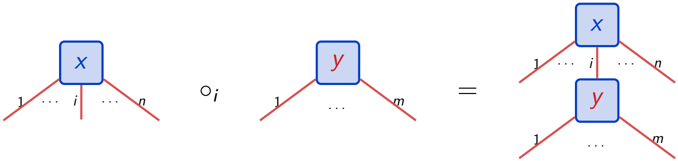

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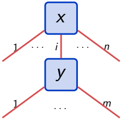
$(x \circ_i y)$



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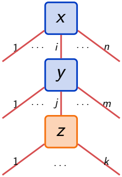

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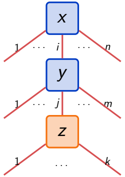

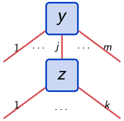

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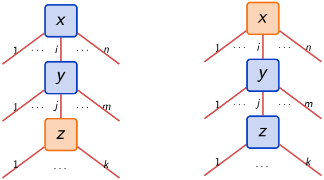

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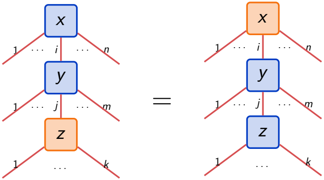

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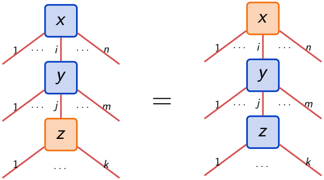

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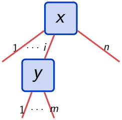

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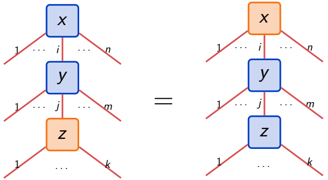

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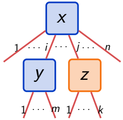

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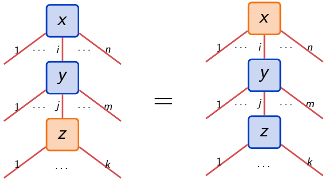

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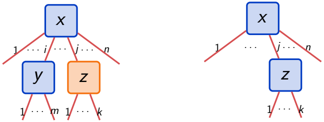

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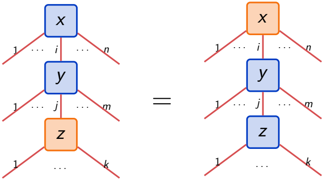

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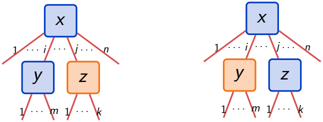

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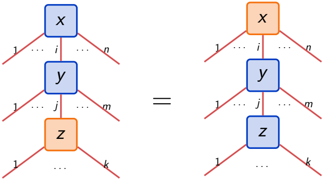

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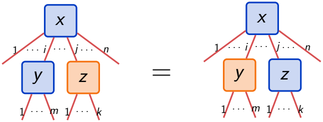

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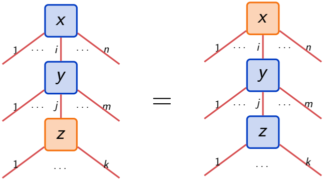

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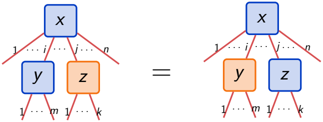

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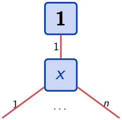

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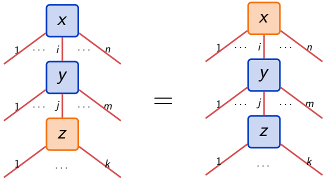

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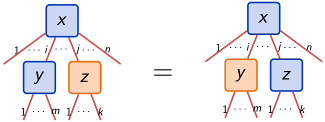

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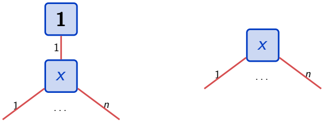

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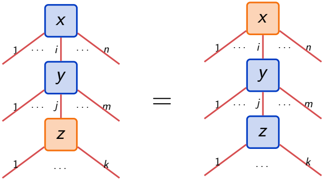

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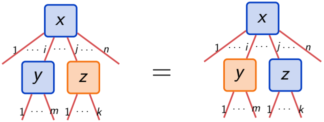

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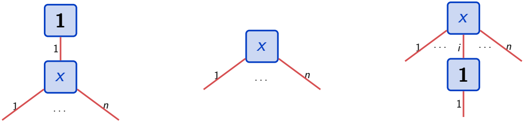

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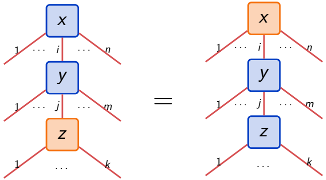

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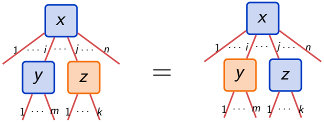

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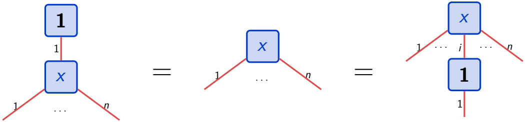

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3. giving a presentation of $\mathcal{P}$ by generators and relations.

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$$\mathsf{Assoc}(n) := \{\mathtt{a}_n\},$$

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The generator $\mathtt{a}_2$ is subject to the relation

$$\mathtt{a}_2 \circ_1 \mathtt{a}_2 = \mathtt{a}_2 \circ_2 \mathtt{a}_2.$$

The associative operad

Let $(\mathrm{Assoc}, \circ_i, \mathtt{a}_1)$ be the operad defined for all $n \geqslant 1$ by

$$\mathrm{Assoc}(n) := \{\mathtt{a}_n\},$$

and for all $n, m \geqslant 1$ and $i \in [n]$ by

$$\mathtt{a}_n \circ_i \mathtt{a}_m := \mathtt{a}_{n+m-1}.$$

Dimensions: 1, 1, 1, 1, 1, 1, ...

Assoc is generated by $\mathtt{a}_2$:

$$\mathtt{a}_1, \quad \mathtt{a}_2, \quad \mathtt{a}_3 = \mathtt{a}_2 \circ_1 \mathtt{a}_2, \quad \mathtt{a}_4 = (\mathtt{a}_2 \circ_1 \mathtt{a}_2) \circ_1 \mathtt{a}_2, \quad \dots$$

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$$\mathtt{a}_2 \circ_1 \mathtt{a}_2 = \mathtt{a}_2 \circ_2 \mathtt{a}_2.$$

Presentation by generators and relations:

$$\mathrm{Assoc} = \langle \mathtt{a}_2 \quad | \quad \mathtt{a}_2 \circ_1 \mathtt{a}_2 = \mathtt{a}_2 \circ_2 \mathtt{a}_2 \rangle .$$

The magmatic operad

Let $(\mathrm{Mag}, \circ_i,$ $)$ be the operad defined for all $n \geqslant 1$ by

$$\mathrm{Mag}(n) := \{T : T \text{ binary tree with } n \text{ leaves}\},$$

and for all $n, m \geqslant 1$ and $i \in [n]$ by

$$S \circ_i T := \text{tree obtained by grafting } T \text{ on the } i\text{th leaf of } S.$$

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Let $(\mathrm{Mag}, \circ_i, \square)$ be the operad defined for all $n \geqslant 1$ by

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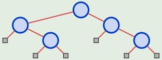
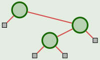


$\circ_4$



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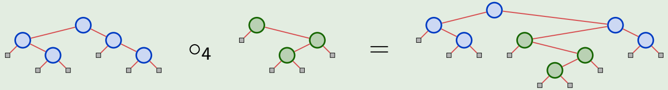

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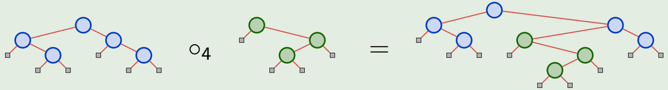

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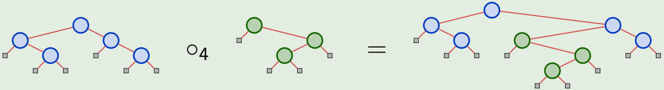

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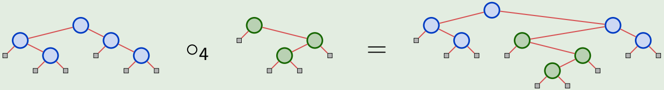

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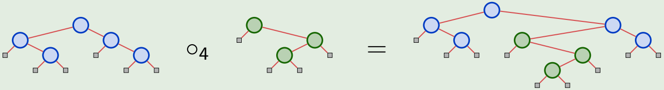

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Let us start with a monoid $(M, \bullet, 1)$.

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Let M and N be two monoids and $\theta : M \to N$ be a monoid morphism.

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$$\mathsf{T}\theta : \mathsf{T}M \to \mathsf{T}N,$$

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$$\mathsf{T}\theta\,(x_1 \dots x_n) := \theta(x_1) \dots \theta(x_n).$$

Some examples of the T construction

$M := (\mathbb{N}, +)$. Elements of $\mathsf{T}M$: words over the alphabet $\mathbb{N}$.

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Properties of the T construction

Theorem

If M is a monoid, $\mathsf{T}M$ is an operad.

If $\theta : M \to N$ is a monoid morphism, $\mathsf{T}\theta$ is an operad morphism.

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Theorem

If M is a monoid, $\mathsf{T}M$ is an operad.

If $\theta : M \to N$ is a monoid morphism, $\mathsf{T}\theta$ is an operad morphism.

Moreover, T preserves injections and surjections.

Hence, T is an exact functor from the category of monoids with monoid morphisms to the category of operads with operad morphisms.

Properties of the T construction

The sets $\mathsf{T}M(n)$ are finite iff M is finite. In this case, the dimensions of $\mathsf{T}M$ are

$$m, m^2, m^3, m^4, \ldots$$

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Example

$\{1\} \uplus \{00\}$ is a generating set of $\mathsf{T}(\mathbb{N}, +)$. For instance,

$$02001 = (((((00 \circ_1 00) \circ_1 00) \circ_1 00) \circ_2 1) \circ_2 1) \circ_5 1.$$

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Main motivations for introducing the T construction:

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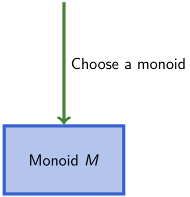

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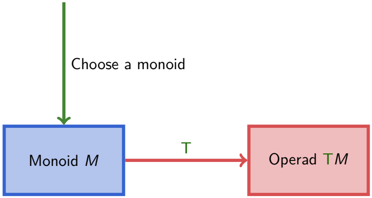

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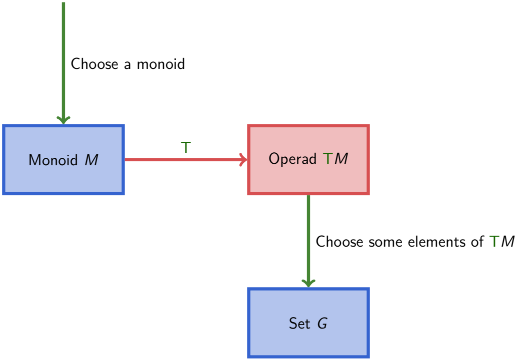

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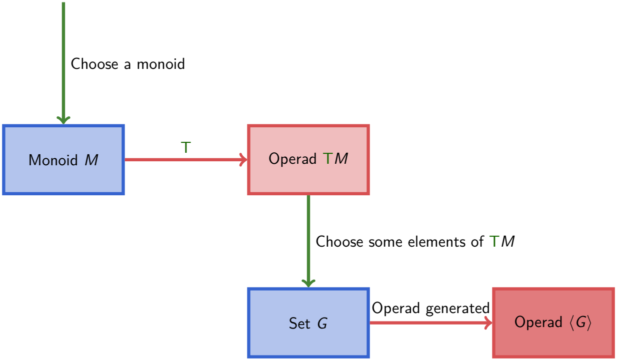

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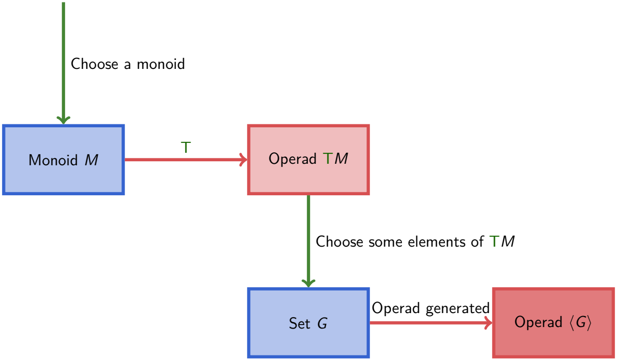

We then ask usual questions about operads on $\langle G \rangle$.

Contents

Survey of some obtained operads

Here are the operads obtained using the T construction with some usual monoids:

Monoid	Operad	Generators	First dimensions	Combinatorial objects
$(\mathbb{N}, +)$	End	—	$1, 4, 27, 256, 3125$	*Endofunctions*
	PF	—	$1, 3, 16, 125, 1296$	*Parking functions*
	PW	—	$1, 3, 13, 75, 541$	*Packed words*
	Per	—	$1, 2, 6, 24, 120$	*Permutations*
	PRT	01	$1, 1, 2, 5, 14, 42$	*Planar rooted trees*
	$\mathsf{FCat}^{(k)}$	00, 01, . . . , 0k	Fuß-Catalan num.	*Trees of arity* $k+1$
	Schr	00, 01, 10	$1, 3, 11, 45, 197$	*Schröder trees*
	Motz	00, 010	$1, 1, 2, 4, 9, 21, 51$	*Motzkin paths*
$(\mathbb{Z}/_{2\mathbb{Z}}, +)$	Comp	00, 01	$1, 2, 4, 8, 16, 32$	*Int. compo.*
$(\mathbb{Z}/_{3\mathbb{Z}}, +)$	DA	00, 01	$1, 2, 5, 13, 35, 96$	*Directed animals*
	SComp	00, 01, 02	$1, 3, 27, 81, 243$	*Segmented int. compo.*
$(\{0, 1\}, \times)$	Dias	01, 10	$1, 2, 3, 4, 5, 6$	*Words with exactly one* 1
	Trias	01, 10, 11	$1, 3, 7, 15, 31, 63$	*Words with at least one* 1

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[0]
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sage:  print Motz.elements(2)
[00]
```

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[00]

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[000, 010]
```

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[0]

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[00]

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[000, 010]

sage:  print Motz.elements(6)
[000000, 000010, 000100, 000110, 001000, 001010, 001100,
001110, 001210, 010000, 010010, 010100, 010110, 011000,
011010, 011100, 011110, 011210, 012100, 012110, 012210]
```

Elements of Motz

Proposition

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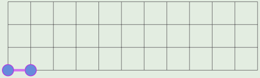
001123221010 $\mapsto$ 

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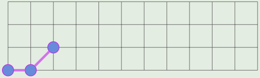
$001123221010 \quad \mapsto$ 

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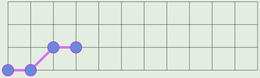
001123221010 $\mapsto$ 

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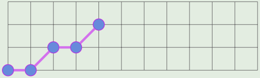
001123221010 $\mapsto$ 

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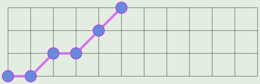
001123221010 $\mapsto$ 

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Bijection between elements of Motz and Motzkin paths:

$$\mathtt{ab} \mapsto \begin{cases} \text{(down step)} & \text{if } \mathtt{b} - \mathtt{a} = -1, \\ \text{(horizontal step)} & \text{if } \mathtt{b} - \mathtt{a} = 0, \\ \text{(up step)} & \text{if } \mathtt{b} - \mathtt{a} = 1. \end{cases}$$

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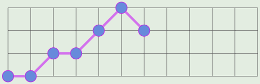

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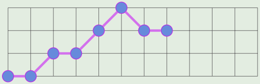
001123221010 $\mapsto$ 

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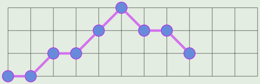

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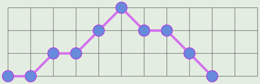
001123221010 $\mapsto$ 

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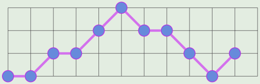

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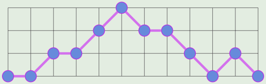
001123221010 $\mapsto$ 

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Let u and v be two Motzkin paths. The grafting $u \circ_i v$ in Motz *returns to replace the ith point of u by v.*

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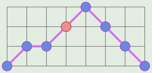

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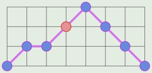

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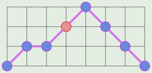
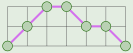
 $\circ_4$ 

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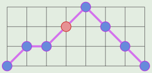
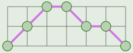
 $\circ_4$ 

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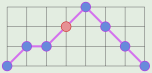
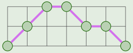
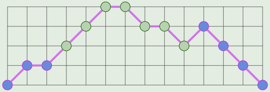
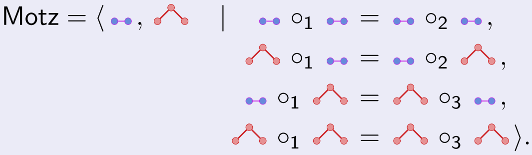
 $\circ_4$  $=$ 

Presentation of Motz

Theorem

The operad Motz *admits the following presentation*



Contents

The diassociative and triassociative operads

The diassociative operad Dias [Loday, 2001] is the operad admitting the following presentation:

$$\mathsf{Dias} := \langle \dashv, \vdash \quad | \quad \dashv \circ_1 \vdash \; = \; \vdash \circ_2 \dashv, \\ \dashv \circ_1 \dashv \; = \; \dashv \circ_2 \dashv \; = \; \dashv \circ_2 \vdash, \\ \vdash \circ_2 \vdash \; = \; \vdash \circ_1 \vdash \; = \; \vdash \circ_1 \dashv \rangle.$$

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sage:  print Tr.elements(3)
[001, 010, 011, 100, 101, 110, 111]
```

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Survey of some obtained operads

These operads fit into following diagram.

$\rightarrowtail$ (resp. $\twoheadrightarrow$) stands for an injective (resp. surjective) operad morphism.