Connections Between a Family of Recursive Polynomials and Parking Function Theory

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Definition

Let $T_\mu = t^{\sum (i-1)\mu_i} q^{\sum (i-1)\mu'_i}$. Then ∇ is an important linear operator of the Macdonald polynomials:

$$\nabla \tilde{H}_\mu[X;q,t] = T_\mu \tilde{H}_\mu[X;q,t].$$

Theorem (Haiman)

When applied to the elementary symmetric function e_n, ∇ gives the Frobenius characteristic of the space of diagonal harmonics.

$$\nabla e_n = \sum_{\mu \vdash n} \frac{(1-t)(1-q) T_\mu \tilde{H}_\mu \Pi_\mu B_\mu}{w_\mu}$$

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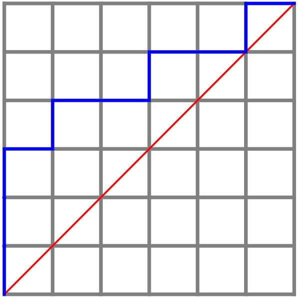
Conjecture (Shuffle Conjecture)

[Haglund, Haiman, Loehr, Remmel, Ulyanov.]

$$\nabla e_n = \sum_{PF \in \mathrm{PF}_n} t^{\mathrm{area}(PF)} q^{\mathrm{dinv}(PF)} Q_{\mathrm{ides}(PF)}$$

Dyck Paths

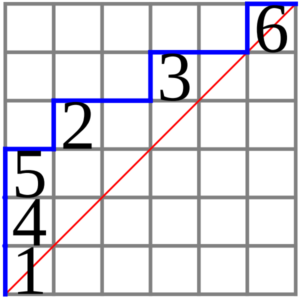


Definition

A dyck path:

1. has only north and east steps,
2. goes from the southwest to the northeast corner, and
3. doesn't cross the main diagonal.

Parking Functions



Definition

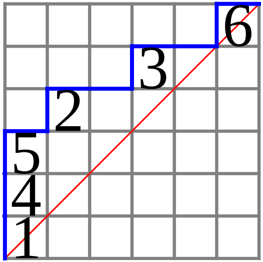
A parking function is a dyck path with:

1. integers 1 to n by the north steps and
2. columns strictly increasing.

Parking Function Statistics

Definition

The *area* of a parking function is the number of complete squares between the dyck path and the main diagonal.



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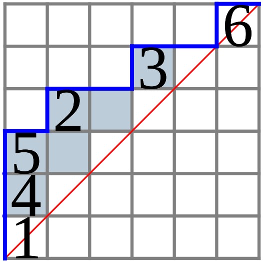
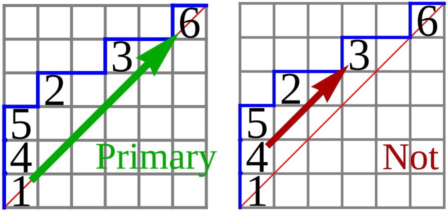


Figure: $\text{area}(PF) = 6$

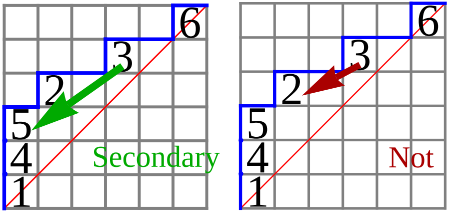
Primary Dinv



Definition

A primary diagonal inversion occurs between a small car and a big car to its right in the same diagonal.

Secondary Dinv



Definition

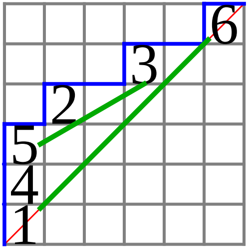
A secondary diagonal inversion occurs between a small car and a big car to its left in the next higher diagonal.

Parking Function Statistics

Definition

The *dinv* of a parking function is the number of primary and secondary diagonal inversions it contains.

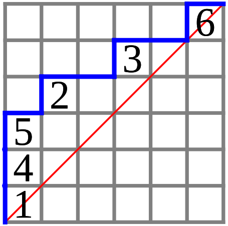
$$\text{dinv}(PF) = 2$$



Reading Word

Definition

The reading word is found by reading the integers along their diagonals.



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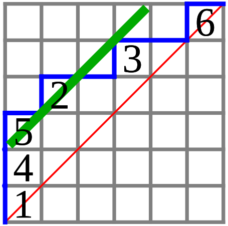

[2,5]

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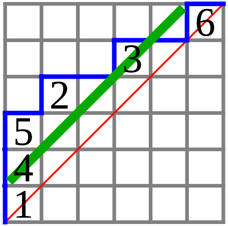

[2,5,3,4]

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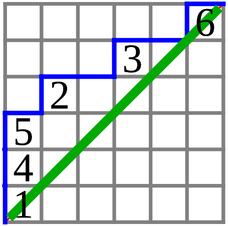


[2,5,3,4,6,1]

I-descents

Definition

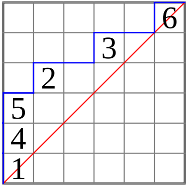
The *i-descent set* of a permutation P, is

$$\mathrm{ides}(P) = \{i : i \text{ occurs after } i+1 \text{ in } P\}.$$

Definition

Let $\mathrm{ides}(PF) = \mathrm{ides}(\mathrm{word}(PF))$.

Example



$$\mathrm{ides}(PF) = \mathrm{ides}([2,5,3,4,6,1]) = \{1,4\}$$

Conjecture (Shuffle Conjecture)

[Haglund, Haiman, Loehr, Remmel, Ulyanov.]

$$\nabla e_n = \sum_{PF \in \mathrm{PF}_n} t^{\mathrm{area}(PF)} q^{\mathrm{dinv}(PF)} Q_{\mathrm{ides}(PF)}$$

Composition

Definition

The composition of a parking function determines where the dyck path touches the main diagonal.

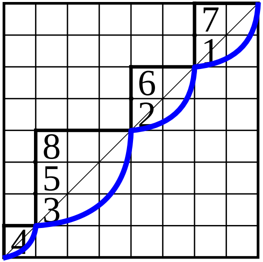


Figure: $\text{comp}(PF) = (1, 3, 2, 2)$

Definition

We are interested in the family of parking functions with a given composition:

$$\mathcal{F}_p = \{PF : \mathrm{comp}(PF) = p\}$$

In particular define the sum:

$$F_p = \sum_{\mathrm{comp}(PF)=p} t^{\mathrm{area}(PF)} q^{\mathrm{dinv}(PF)} Q_{\mathrm{ides}(PF)}.$$

Definition

For a two part composition, if $\mathrm{comp}(PF) = \{n-k, k\}$, let

$$\mathrm{top}(PF) = k.$$

Definition

Let $\mathcal{C}_p 1 = \mathcal{C}_{p_1}\mathcal{C}_{p_2}\ldots\mathcal{C}_{p_k}1$. Then

$$\mathcal{C}_p 1 = \left(-\frac{1}{q}\right)^{\sum p_i - k} H_p[X; 1/q].$$

$C_p 1$ can be generated directly using a particular operator:
For any symmetric function $F[X]$

$$\mathcal{C}_{p_i} F[X] = \left(\frac{-1}{q}\right)^{p_i - 1} \sum_{k \geq 0} F\left[X + \frac{1-q}{q} z\right]\Big|_{z^k} h_{p_i + k}[X].$$

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Conjecture (Haglund, Morse, Zabrocki)

$$\nabla \mathcal{C}_p 1 = F_p = \sum_{\mathrm{comp}(PF)=p} t^{area(PF)} q^{dinv(PF)} Q_{ides(PF)}$$

Theorem (Haglund, Morse, Zabrocki)

1. $e_n = \sum_{p \models n} \mathcal{C}_p 1$.
   - *(Thus the Haglund-Morse-Zabrocki conjecture is a sharpening of the shuffle conjecture.)*
2. $\{C_\mu 1 : \mu \vdash n\}$ *forms a basis for the symmetric functions* Λ^n.
   - *(Since* ∇ *is a linear operator, this gives us that* $\{\nabla \mathcal{C}_\mu 1 : \mu \vdash n\}$ *forms a basis for* Λ^n*.)*
3. *When* $k < n - k$,
   $q(\mathcal{C}_k \mathcal{C}_{n-k} + \mathcal{C}_{n-k-1} \mathcal{C}_{k+1}) = \mathcal{C}_{n-k} \mathcal{C}_k + \mathcal{C}_{k+1} \mathcal{C}_{n-k-1}$
   - *(This is exactly enough information to express any* $C_p 1$ *in terms of* $\{C_\mu 1 : \mu \vdash n\}$*.)*

The Haglund-Morse-Zabrocki Conjecture in Two Parts

1. Is the HMZ conjecture true for p a partition?
2. If the HMZ conjecture is true for every partition p, then is it true for any composition p?

The Partition Case

Theorem

Let V be a vector space with four bases:

$$G = \langle G_1, \ldots, G_n \rangle \text{ and } H = \langle H_1, \ldots, H_n \rangle$$

$$\phi = \langle \phi_1, \ldots, \phi_n \rangle \text{ and } \psi = \langle \psi_1, \ldots, \psi_n \rangle.$$

Say that

$$G_j = \sum_{i \leq j} \phi_i a_{i,j} \text{ and } G_j = \sum_{i \geq j} \psi_i b_{i,j},$$

$$H_j = \sum_{i \leq j} \phi_i c_{i,j} \text{ and } H_j = \sum_{i \geq j} \psi_i d_{i,j},$$

Then there exists constants c_j, such that $G_j = c_j H_j$.

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Then there exists constants c_j, such that $G_j = c_j H_j$.

Is $\nabla C_p 1 = F_p$ for p a partition?

There is an upper triangularity for the two basis:

Theorem (Garsia)

$$\nabla C_p 1 = \sum_{\lambda \leq p} s_\lambda \left[\frac{X}{q-1} \right] \alpha_{\lambda,p}(q,t)$$

$$F_p = \sum_{\lambda \leq p} s_\lambda \left[\frac{X}{q-1} \right] \beta_{\lambda,p}(q,t)$$

If a lower triangularity exists, the two basis are identical:

Theorem (Garsia,H., Xin, Zabrocki)

$$\langle \nabla \mathcal{C}_p 1, e_a h_b \rangle = \langle F_p, e_a h_b \rangle$$

The Compositional Case

Theorem

$$q(\mathcal{C}_k\mathcal{C}_{n-k} + \mathcal{C}_{n-k-1}\mathcal{C}_{k+1}) = \mathcal{C}_{n-k}\mathcal{C}_k + \mathcal{C}_{k+1}\mathcal{C}_{n-k-1}$$

Thus:

$$q(\nabla\mathcal{C}_p\mathcal{C}_k\mathcal{C}_{n-k}\mathcal{C}_{p'}1+\nabla\mathcal{C}_p\mathcal{C}_{n-k-1}\mathcal{C}_{k+1}\mathcal{C}_{p'}1) = \nabla\mathcal{C}_p\mathcal{C}_{n-k}\mathcal{C}_k\mathcal{C}_{p'}1 + \nabla\mathcal{C}_p\mathcal{C}_{k+1}\mathcal{C}_{n-k-1}\mathcal{C}_{p'}1$$

Conjecture (H.)

For $k < n-k$, there exists a bijection f

$$f : \mathcal{F}_{(k,n-k)} \cup \mathcal{F}_{(n-k-1,k+1)} \leftrightarrow \mathcal{F}_{(n-k,k)} \cup \mathcal{F}_{(k+1,n-k-1)}$$

with the following properties:

1. f increases the *dinv* by exactly one
2. f preserves the *area* and the *ides*

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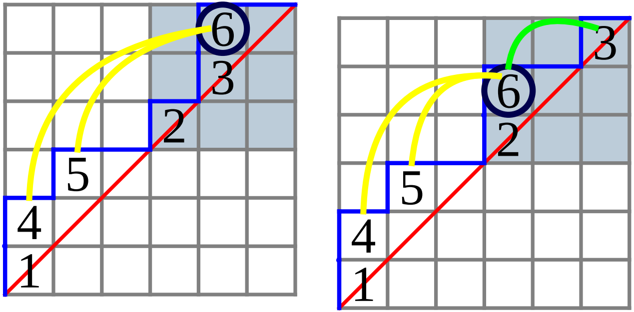
with the following properties:

1. f increases the *dinv* by exactly one
2. f preserves the *area* and the *ides*
3. f keeps the cars in their origional diagonal

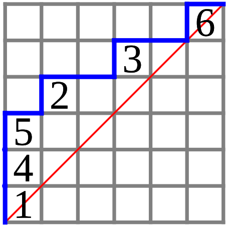
If f keeps cars in their origional diagonal:



1. We can ignore the area.
2. We can just study the two part compositions.

Definition (diagonal word)

The diagonal word lists the cars by diagonal in increasing order.



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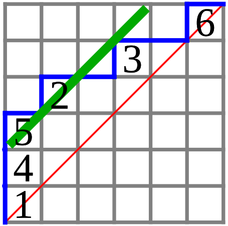

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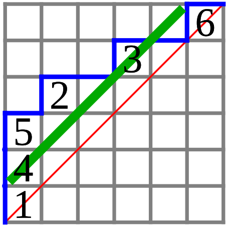

[2,5,3,4]

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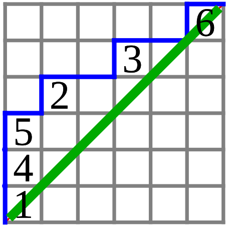
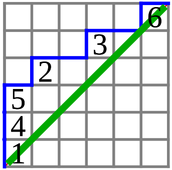


$$[2,5,3,4,1,6]$$

If we split the diagonal word at it's descents, we can reconstruct the diagonal containing any car.



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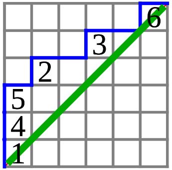

[2,5,3,4,1,6]

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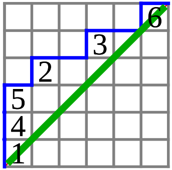


[2,5,3,4,1,6]

2	5	3	4	1	6
2	2	1	1	0	0

Thus parking functions with the same diagonal word are exactly those which have the same set of cars on every diagonal.

Theorem (Haglund and Loehr)

$$\sum_{\mathrm{diag}(PF)=\tau} t^{\mathrm{area}(PF)} q^{\mathrm{dinv}(PF)} = t^{\mathrm{maj}(\tau)} \prod_{i=1}^{n} [w_i^{\tau}]_q$$

We're interested in a different sum:

Definition

$$F^{\tau}(x,t,q) = \sum_{\mathrm{diagword}(PF)=\tau} t^{\mathrm{area}(PF)} q^{\mathrm{dinv}(PF)} x^{\mathrm{top}(PF)} Q_{\mathrm{ides}(PF)}$$

Then we'd like to show that:

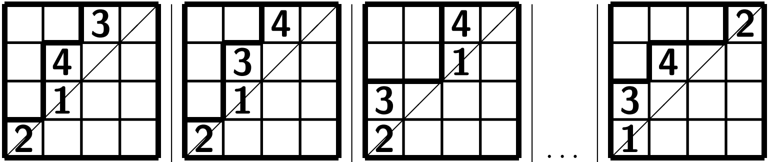
Conjecture (Commutativity)

For $k < n-k$, if $\tau = (\tau_1, \ldots, \tau_n)$ where $\tau_{n-2} > \tau_{n-1} < \tau_n$

$$q \left(F^{\tau}(x,t,q)\big|_{x^{n-k}+x^{k+1}} \right) = F^{\tau}(x,t,q)\Big|_{x^k+x^{n-k-1}}$$

Example

$$\tau = (3,4,1,2)$$



$\downarrow$ $t^2x^3Q_2$ | $\downarrow$ $t^2qx^3Q_{2,3}$ | $\downarrow$ $t^2q^2x^2Q_{2,3}$ | $\ldots$ $\downarrow$ $\ldots$ | $\downarrow$ $t^2q^4xQ_{1,2,3}$

$$\begin{aligned} F^{(3,4,1,2)}(x,q,t) = & \, t^2x^3Q_2 + t^2qx^3Q_{2,3} + t^2q^2x^2Q_{2,3} \\ & + t^2qx^2Q_2 + t^2q^2xQ_2 + t^2q^3xQ_{2,3} \\ & + t^2qx^3Q_{1,2} + t^2q^2x^3Q_{1,2,3} + t^2q^3x^2Q_{1,2,3} \\ & + t^2q^2x^2Q_{1,2} + t^2q^3xQ_{1,2} + t^2q^4xQ_{1,2,3} \end{aligned}$$

Conjecture (Functional Equation)

For any diagonal word τ, there exists $A^\tau(q,t)$ such that

$$(1-q/x)F^\tau(x,q,t)+x^{n-1}(1-qx)F^\tau(1/x,q,t) = (1+x^{n-1})A^\tau(q,t).$$

Theorem

$F^\tau(x,q,t)$ satisfies the functional equation if and only if it satisfies the commutativity conjectures.

Thus if we can show that every $F^\tau(x,q,t)$ satisfies the functional equation, we can reduce the compositional case of the HMZ conjecture to the partitional case!

Example

$$F^{(3,4,1,2)}(x, q, t) = t^2 x(x^2 + xq + q^2)(Q_{1,2,3}q^2 + qQ_{2,3} + qQ_{1,2} + Q_2)$$

Surprise! It factors.

Example

$$\begin{aligned}
&(1 - q/x)F^{(3,4,1,2)}(x, q, t; X_n) + x^{n-1}(1 - qx)F^{(3,4,1,2)}(1/x, q, t; X_n) \\
&\quad = t^2(Q_{1,2,3}q^2 + qQ_{2,3} + qQ_{1,2} + Q_2) \\
&\qquad \left((1 - q/x)x(x^2 + xq + q^2) + x^{n-1}(1 - qx)1/x(x^{-2} + x^{-1}q + q^2)\right) \\
&\quad = (1 + x^{n-1})t^2(1 - q)(q^2 + q + 1)(Q_{1,2,3}q^2 + qQ_{2,3} + qQ_{1,2} + Q_2).
\end{aligned}$$

Theorem

For any diagonal word τ, there exists a polynomial $r_1^\tau(x,q)$ and a quasisymmetric polynomial $r_2^\tau(q;X_n)$ such that

$$F^\tau(x,q,t;X_n) = t^{\mathrm{maj}(\tau)} r_1^\tau(x,q) r_2^\tau(q;X_n)$$

Definition

Let

$$R^\tau(x,q) = \sum_{\mathrm{diagword}(PF)=\tau} q^{\mathrm{dinv}(PF)} x^{\mathrm{top}(PF)}$$

Conjecture (Functional Equation)

For any diagonal word τ, there exists $A^\tau(q)$ such that

$$(1-q/x)R^\tau(x,q) + x^{n-1}(1-qx)R^\tau(1/x,q) = (1+x^{n-1})A^\tau(q).$$

Example

$$R^{(4,3,1,2)}(x,q) = x(q+1)(q+x^2) = R^{(1,4,2,3)}(x,q)$$

In fact, for parking functions of length 5 there are 40 distinct diagonal words, but only 14 distinct $R^{\tau}(x,q)$.

Definition (schedule)

A sequence $W = (w_1, \ldots, w_n)$ is a schedule if:

- $w_1 = 1$ and $w_2 = 2$;
- $w_3 \in \{1, 2\}$; and
- (Slow growth.) $w_i \leq w_{i-1} + 1$.

Definition

$$B_{n,w}P(X_{n-1};q) := \frac{1}{1-q}((x_n - q^w)P(x_1, x_2, \dots, x_{n-1}; q) + (1-x_n)P(x_1, x_2, \dots, x_{n-w-1}, qx_{n-w}, \dots, qx_{n-1}; q))$$

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Base Case

$$P_{(1,2)}(X_2; q) := qx_1 + x_2.$$

$$P_{(w_1,\ldots,w_n)}(X_n; q) := B_{n,w_n}P_{(w_1,\ldots,w_{n-1})}(X_{n-1}; q).$$

Definition

$$Q_W(x; q) := P_W(X_n, q)|_{x_1=\cdots=x_n=x}$$

Theorem

For every τ there exists a schedule W such that

$$R^{\tau}(x,q) = Q_W(x,q).$$

Moreover, the converse is also the case.

Example

$$R^{(3,1,2,4)} = (1+q)^2 x(q^2 + qx + x^2) = Q_{(1,2,2,3)}$$

Theorem (Functional Equation)

If for every schedule $W = (w_1, \ldots, w_n)$,

$$\begin{aligned} (1-q/x)Q_W(x;q) + x^{n-1}(1-qx)Q_W(1/x;q) \quad (1) \\ = (1+x^{n-1})(1-q)\prod_{i=1}^{n}[w_i]_q, \quad (2) \end{aligned}$$

then our desired bijections exist.

Theorem

Let $W = (w_1, \ldots, w_{n-1})$ *and* $W' = (w_1, \ldots, w_{n-2})$ *satisfy the functional equation. Then* $(w_1, \ldots, w_{n-1}, 1)$ *also satisfies the functional equation.*

Definition

If a schedule $W = (w_1, \ldots, w_n)$ can be shown to satisfy the functional equation under the assumption that "smaller" schedules satisfy the functional equation, say that the schedule <u>inductively</u> satisfies the functional equation.

Theorem

Let $W = (w_1, \ldots, w_{n-1})$ *and* $W' = (w_1, \ldots, w_{n-2})$ *satisfy the functional equation. Then* $(w_1, \ldots, w_{n-1}, 1)$ *also satisfies the functional equation.*

Definition

If a schedule $W = (w_1, \ldots, w_n)$ can be shown to satisfy the functional equation under the assumption that "smaller" schedules satisfy the functional equation, say that the schedule inductively satisfies the functional equation.

Then to reduce the compositional case of the HMZ conjecture to the partition case, we can show that every schedule inductively satisfies the functional equation.

Theorem (H.)

Schedules of the form $(1, 2, 2, 3, \ldots, s)$ *satisfy the functional equation.*

Theorem (H.)

Schedules of the form $(1, 2, 2, 3, \ldots, s, w_{s+1}, \ldots, w_n)$ *inductively satisfy the functional equation when* $w_{s+1} < s$*.*

Theorem

The remaining schedules (when $w_{s+1} = s$*) of length less than 15 satisfy the functional equation.*

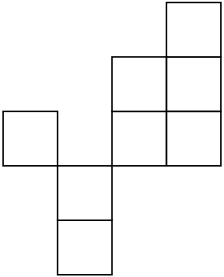
Proof.

by E. Rodemich using exhaustive search (in Fortran!) □

Generating the Polynomials Directly

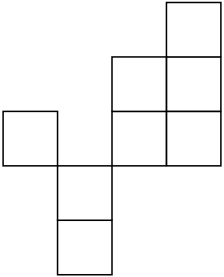
$$\boxed{P_{(1,2,2,3)}\Big|_{x_1x_3x_4} = q(1+q)}$$

1. Use bars of length $(1, 2, 2, 3)$.
2.
3.
4.
5.



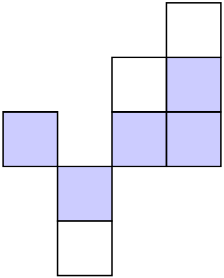
$$\boxed{P_{(1,2,2,3)}\Big|_{x_1 x_3 x_4} = q(1+q)}$$

1. Use bars of length $(1, 2, 2, 3)$.
2. Place the 1st, 3rd, and 4th pointing upward.
3.
4.
5.



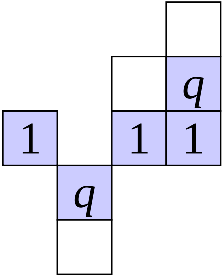
$$\boxed{P_{(1,2,2,3)}\Big|_{x_1x_3x_4} = q(1+q)}$$

1. Use bars of length $(1, 2, 2, 3)$.
2. Place the 1st, 3rd, and 4th pointing upward.
3. Fill in a single square in each of the first two columns.
4. Look back w_i and see how many are pointed the same direction. Fill in that many squares.
5.

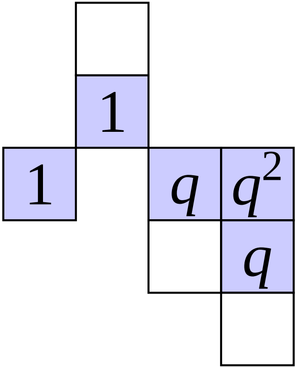


$$P_{(1,2,2,3)}\Big|_{x_1x_3x_4} = q(1+q)$$

1. Use bars of length $(1, 2, 2, 3)$.
2. Place the 1st, 3rd, and 4th pointing upward.
3. Fill in a single square in each of the first two columns.
4. Look back w_i and see how many are pointed the same direction. Fill in that many squares.
5. Count colored squares (from the bottom) to get powers of q.



$$P_{(1,2,2,3)}\big|_{x_2} = q(q + q^2)$$



$$P_{(1,2,2,3)}(X_4; q) = (qx_1 + x_2)(q^2 + q^3 + q^2x_3 + qx_4 + x_3x_4 + qx_3x_4)$$