Enumeration of graded $(3+1)$-avoiding posets

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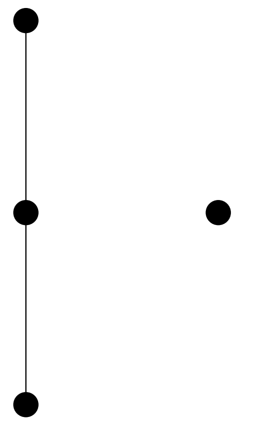
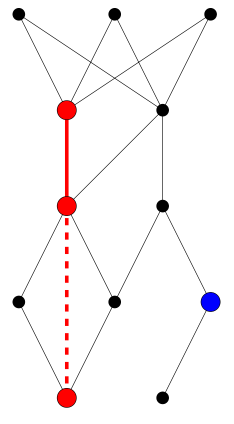
joint with Yan Zhang (MIT)

July 31, 2012

Background

- All partially ordered sets (posets) in this talk are finite, labeled by $[n] = \{1, 2, \ldots, n\}$
- A poset is $(\mathbf{3}+\mathbf{1})$-*free* if has no four elements that induce a copy of $\mathbf{3}+\mathbf{1}$





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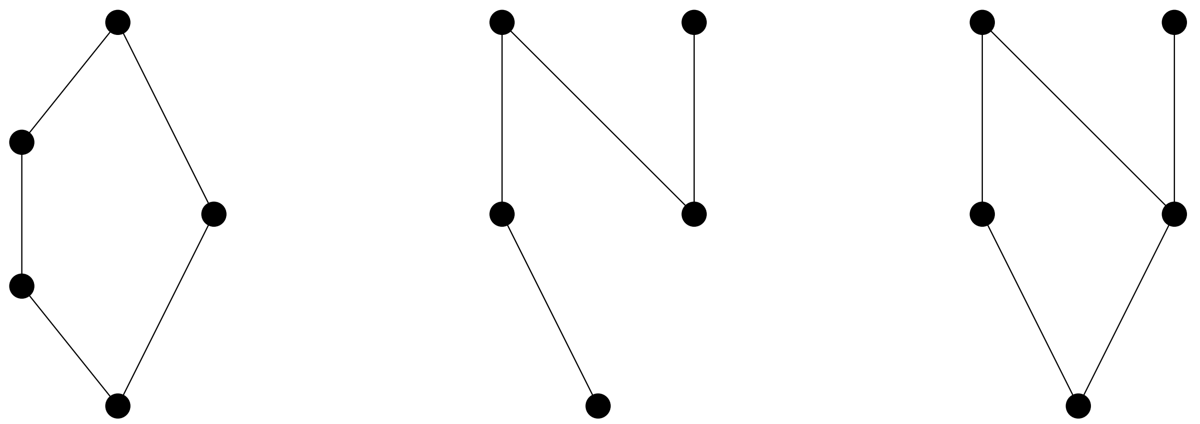
Motivation:

- Stanley-Stembridge conjecture
- Semiorders ($(\mathbf{3}+\mathbf{1})$- and $(\mathbf{2}+\mathbf{2})$-free)
- Recent work on $(\mathbf{2}+\mathbf{2})$-free posets (Bousquet-Mélou, Claesson, Dukes, Kitaev, and subsequent work)
- Other related work (Skandera; Atkinson, Sagan and Vatter)

Graded posets

- A poset P is **weakly graded** if there is a rank function $r : P \to \mathbb{N}$ such that if x covers y then $r(x) - r(y) = 1$.
- **Strongly graded**: also, all minimal vertices have rank 0 and all maximal vertices have the same rank. (Equivalently, all maximal chains have same length.)
- **In this talk, “graded” = “strongly graded”**



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Longstanding open question. Enumerate $(\mathbf{3}+\mathbf{1})$-free posets.

Question whose solution I'll present. Enumerate graded $(\mathbf{3}+\mathbf{1})$-free posets.

Main result

Main Theorem. *We have*

$$\sum_{n\geq 0}(\#\textit{strongly graded } \mathbf{3}+\mathbf{1}\textit{-free posets on } [n])\cdot x^n/n! =$$

$$e^x - 1 + \frac{2e^x + (e^x-2)\Psi(x)}{2e^{2x}+e^x+(e^{2x}-2e^x-1)\Psi(x)}$$

where

$$\Psi(x) = \sum_{n\geq 0}\sum_{0\leq m\leq n}\binom{n}{m}2^{m(n-m)}\cdot x^n/n!.$$

(Something similar is true for weakly graded $(\mathbf{3}+\mathbf{1})$-free posets.)

Proof idea

1. Give a local condition for $(\mathbf{3}+\mathbf{1})$-avoidance in graded posets
2. Decompose graded $(\mathbf{3}+\mathbf{1})$-free posets into simpler objects
3. Count these simpler objects
4. Use generating function magic to count ways to combine the simpler objects to get graded $(\mathbf{3}+\mathbf{1})$-free posets

Avoidance is local

Proposition 1. *If a graded poset* P *does not have three consecutive ranks that induce a copy of* $\mathbf{3} + \mathbf{1}$ *then* P *is* $(\mathbf{3} + \mathbf{1})$*-free.*

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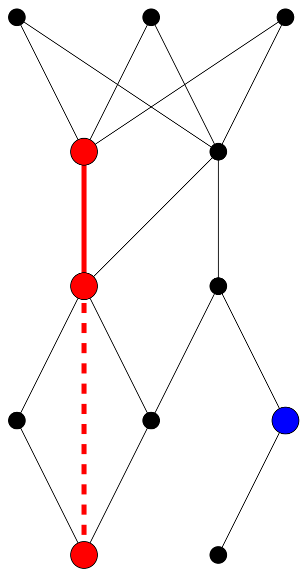
Proposition 1. *If a graded poset P does not have three consecutive ranks that induce a copy of $\mathbf{3}+\mathbf{1}$ then P is $(\mathbf{3}+\mathbf{1})$-free.*

Proof idea. If P contains $\mathbf{3}+\mathbf{1}$,



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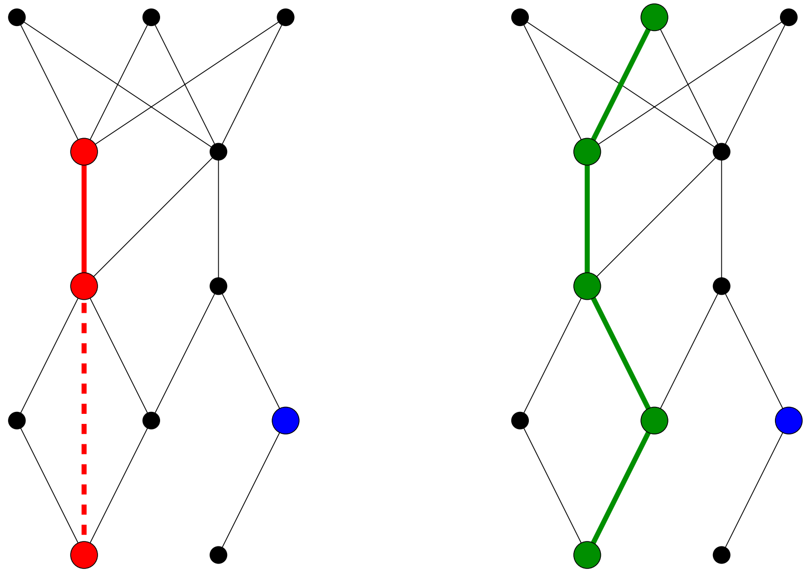

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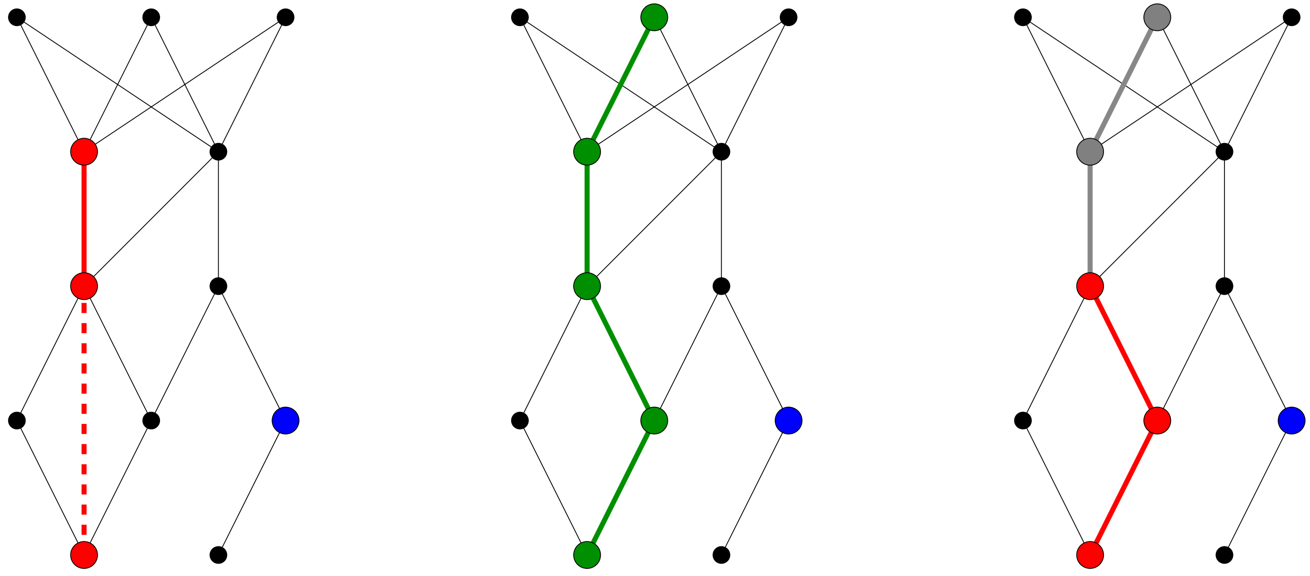
Proof idea. If P contains $\mathbf{3}+\mathbf{1}$, extend the $\mathbf{3}$ to a maximal chain and choose a new $\mathbf{3}$:



□

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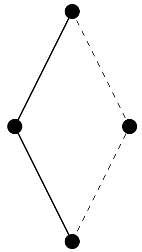
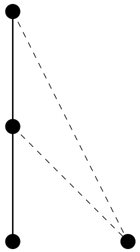
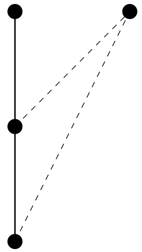
Obstacles:  ,  , and 

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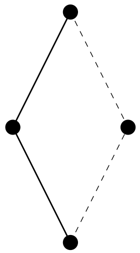
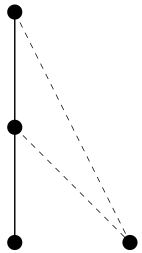
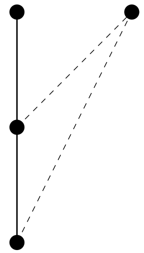
Obstacles:  ,  , and 

Proposition 2. *A graded poset is $(\mathbf{3}+\mathbf{1})$-free if and only if*

- *every vertex is covered by everything on the rank above or covers everything on the rank below, and*
- *every vertex is comparable to all vertices two or more ranks away.*

Decomposing into graphs

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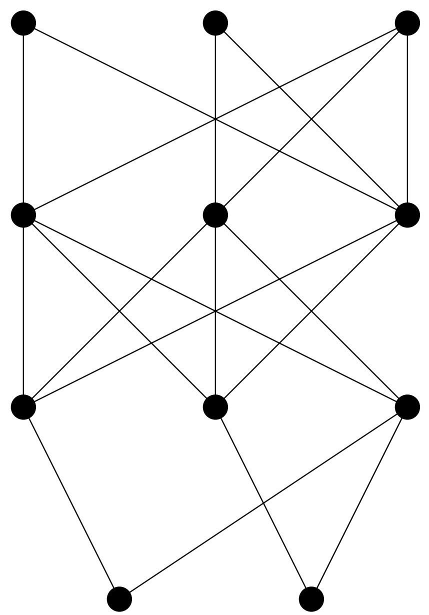
In a graded $(\mathbf{3}+\mathbf{1})$-free poset, focus on the vertices that don't have all cover relations with neighboring ranks.



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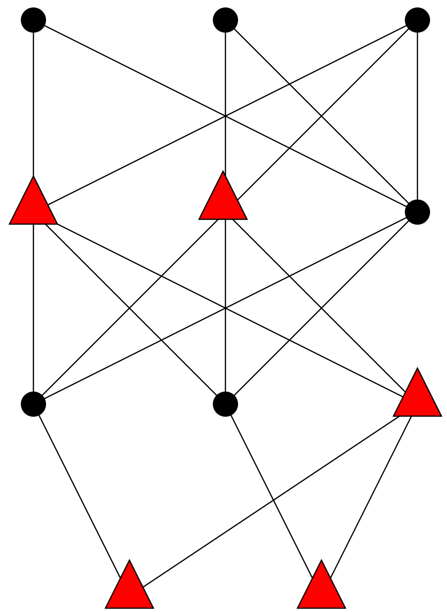

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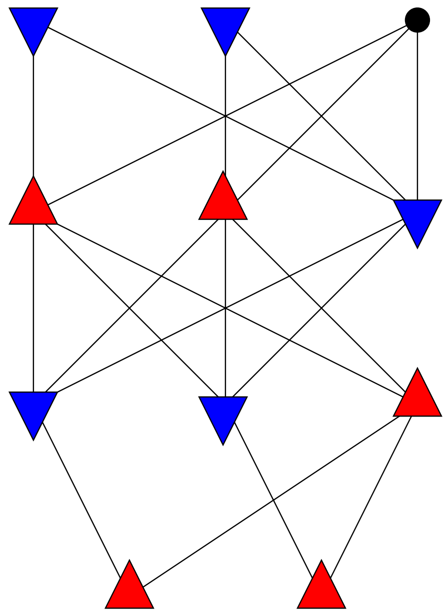

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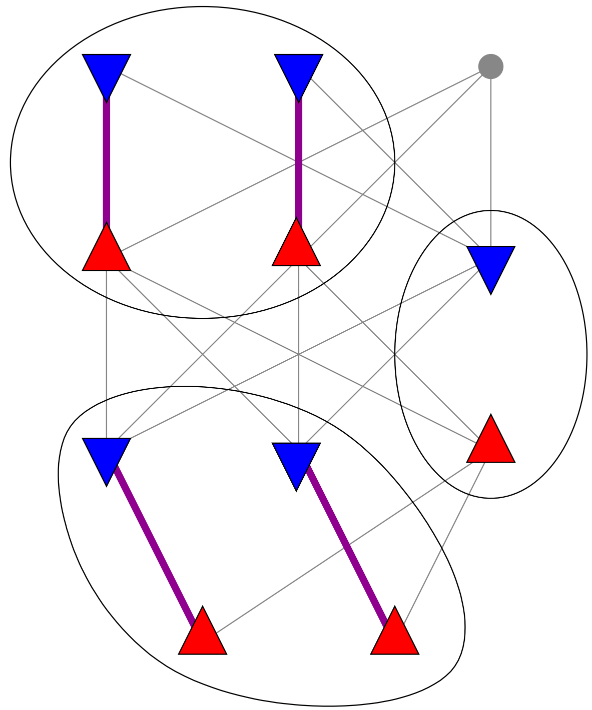
▲: not complete up
▼: not complete down
Group them in adjacent rows



Counting

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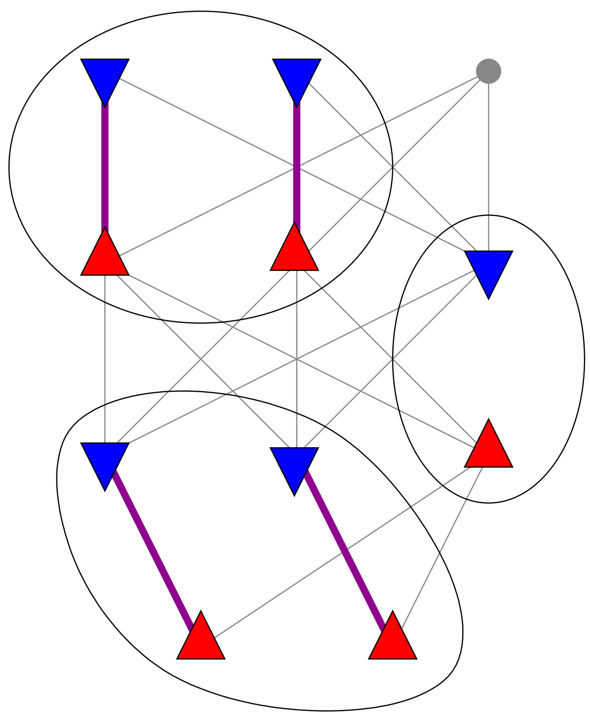

- Outside the circled regions is easy
- Each piece is a bipartite graph with no complete vertices
- Compatibility condition for isolated vertices in consecutive pieces
- Counting the pieces with generating functions is not hard (it involves $\sum_{n\geq 0}\sum_{0\leq m\leq n}\binom{n}{m}2^{m(n-m)}\cdot x^n/n!$)
- Use transfer-matrix method to complete the enumeration

Déjà vu

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where

$$\Psi(x) = \sum_{n\geq 0} \sum_{0\leq m\leq n} \binom{n}{m} 2^{m(n-m)} \cdot x^n/n!.$$

Loose ends

Lingering questions:

- Other classes of posets for which this sort of local approach works?
- Is any of this useful for counting all $(\mathbf{3}+\mathbf{1})$-free posets?

Thanks for listening!

J. Lewis and Y. Zhang, `arXiv:1106.5480`