

Correlations for the Novak process

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FPSAC '12, Nagoya, Japan

The Aztec Diamond

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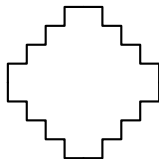
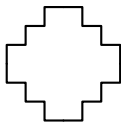
Nordenstam
Young

Novak
half-hexagon

Limit shape

Correlation
kernel

Aztec diamonds of orders 1, 2, 3 and 4.



The Aztec Diamond

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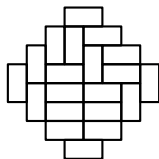
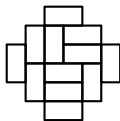
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Aztec diamonds of orders 1, 2, 3 and 4.



The diamond of order n can be tiled in $2^{n(n+1)/2}$ ways.
Elkies, Kuperberg, Larsen & Propp 1992

The Aztec Diamond

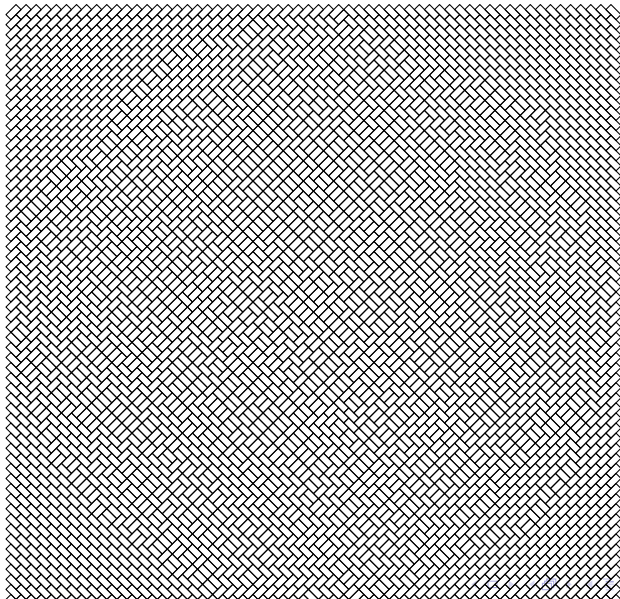
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The number of tilings of an order n Aztec diamond is $2^{\binom{n+1}{2}}$.
Jonathan Novak observed that

$$\det \left[\binom{2i}{j} \right]_{i,j=1}^n = 2^{\binom{n+1}{2}}.$$

Novak half-hexagon

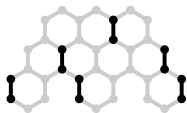
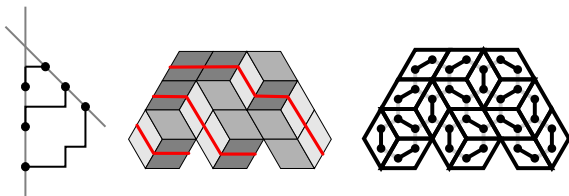
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3			
2	5		
1	3	6	
1	3	5	7

Limit shape

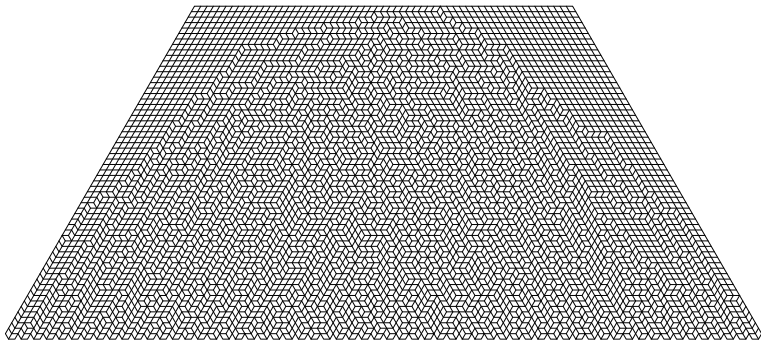
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- The shuffling algorithm
- The Arctic Parabola Theorem.
- Correlation kernel

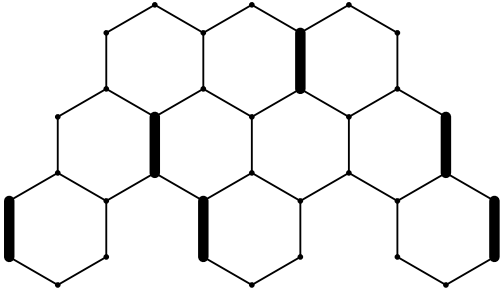
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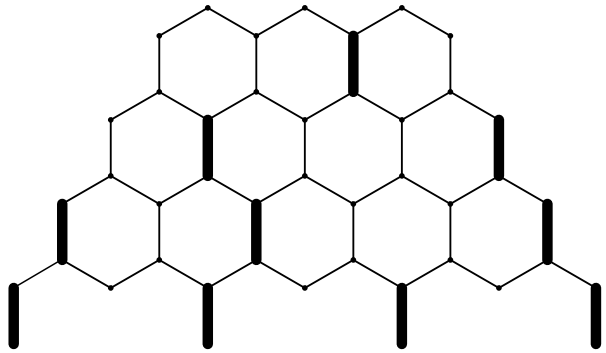
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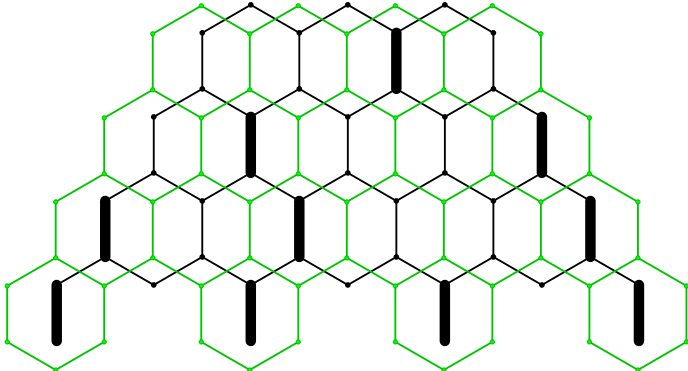
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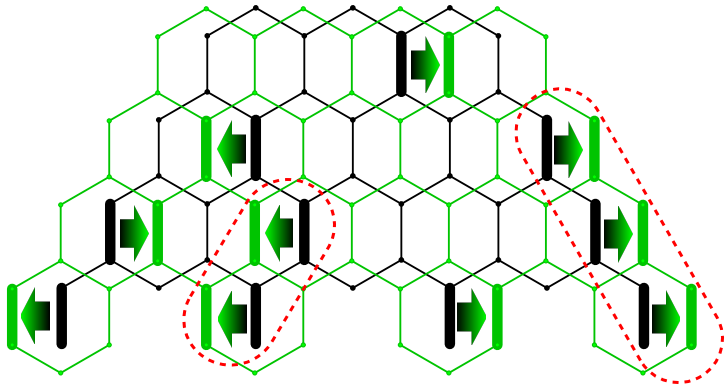
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Introduce a coordinate system: $X_j^i(t)$ is the position of the j th particle on level i at time t .



3
2 5
1 3 6
1 3 5 7

Note that

$$X_j^i(t) \leq X_j^{i-1}(t) < X_{j+1}^i(t)$$

Recursion equations

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$$X_1^1(t) = X_1^1(t-1) + \beta_1^1(t)$$

$$X_1^j(t) = X_1^j(t-1) + \beta_1^j(t)$$

$$- \mathbf{1}\{X_1^j(t-1) + \beta_1^j(t) = X_1^{j-1}(t) + 1\} \quad \text{for } j \geq 2$$

$$X_j^j(t) = X_j^j(t-1) + \beta_j^j(t)$$

$$+ \mathbf{1}\{X_j^j(t-1) + \beta_j^j(t) = X_{j-1}^{j-1}(t)\} \quad \text{for } j \geq 2$$

$$X_i^j(t) = X_i^j(t-1) + \beta_i^j(t)$$

$$+ \mathbf{1}\{X_i^j(t-1) + \beta_i^j(t) = X_{i-1}^{j-1}(t)\}$$

$$- \mathbf{1}\{X_i^j(t-1) + \beta_i^j(t) = X_i^{j-1}(t) + 1\} \quad \text{for } j > i > 1$$

where all $\beta_j^i(t)$ are independent coin flips.

Particle dynamics

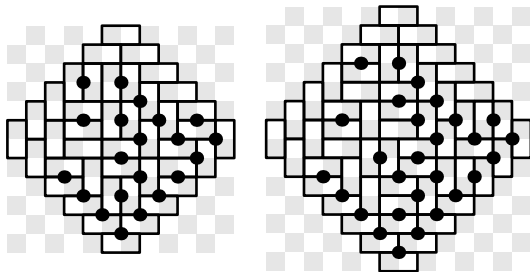
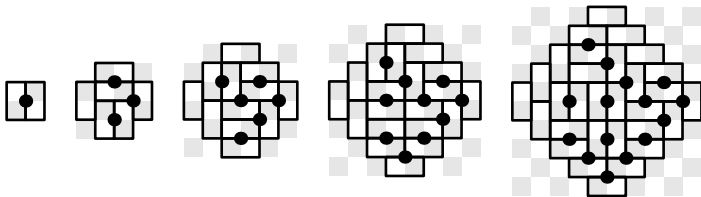
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Time shift: $x_i^j(t) = X_i^j(t - j)$

Half-Aztec diamond

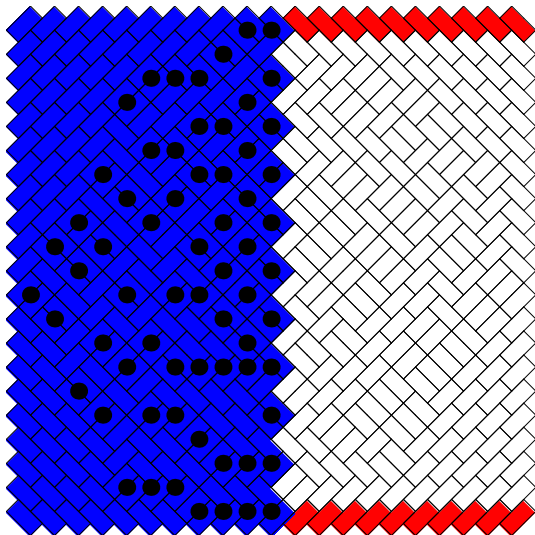
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The Arctic Parabola Theorem

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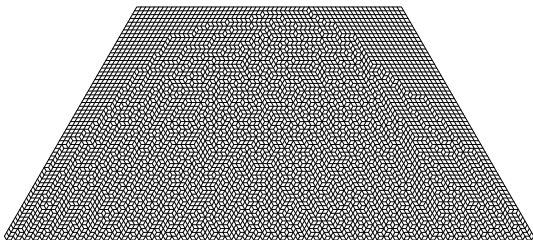
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Theorem

Consider uniform measure on tilings of the Novak half-hexagon. The region in which the density of particles (i.e. vertical lozenges) is asymptotically non-zero is bounded by a parabola.



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Proposition (N & Y 2011)

The limit shape in the Half-Aztec diamond is the semi-circle.

Jockusch, Propp & Shor (1998)

Cohn, Kenyon & Okounkov (2001)

The Arctic Parabola Theorem

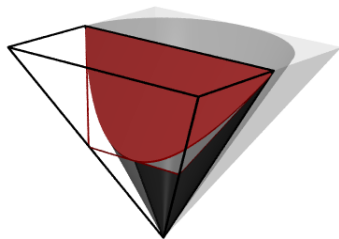
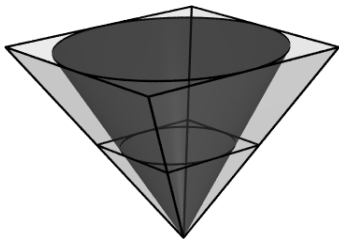
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Correlations

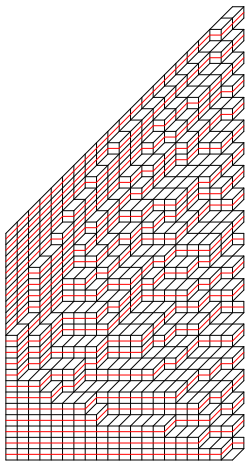
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Consider n Bernoulli walkers started at $1, 2, \dots, n$, and conditioned to end up at positions y_1, \dots, y_n , at time N conditioned never to intersect.

The number of such configurations is given by the Lindström-Gessel-Viennot Theorem as the determinant of

$$M = \left[\binom{N}{y_i - j} \right]_{i,j=1}^n$$

Eynard-Mehta Theorem

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Eynard & Mehta (1998)

Borodin & Rains (2005)

The probability that there is a walker at each of $(x_1, t_1), \dots, (x_k, t_k)$ is

$$\det[K(t_i, x_i; t_j, x_j)]_{i,j=1}^k$$

where

$$K(r, x; s, y) = -\mathbf{1}\{s > r\} \binom{s-r}{y-x} + \sum_{i,j=1}^n \binom{N-r}{y_i-x} [M^{-1}]_{i,j} \binom{s}{y-j}$$

$$\det \left[\binom{N}{y_j - i - \mathbf{1}\{i \geq s\}} \right]_{i,j=1}^{n-1} = \left(\prod_{i=1}^{n-1} \frac{N!}{(y_i - 1)!(N - y_i + n)!} \right) \\ \times \det \left[\frac{(y_i - j + 1) \cdots (y_i - 1)(N - y_i + j + 1) \cdots (N - y_i + n)}{(y_i - j) \cdots (y_i - 1)(N - y_i + j + 2) \cdots (N - y_i + n)} \right]_{i,j=1}^{n-1}$$

Let Δ mean taking the Vandermonde determinant in the variables. For $s = 1$, sage gave us

$$P_{n,1}(N, \bar{y}) = \Delta(\bar{y}) \left(\prod_{i=1}^{n-2} (N + i)^{n-1-i} \right) \left(\prod_{j=1}^{n-1} (\bar{y}_j - 1) \right)$$

Let Δ mean taking the Vandermonde determinant in the variables. For $s = 2$, sage gave us

$$P_{3,2}(N, y) = (N + 1)\Delta(y)(-2e_2(y) + (N + 4)e_1(y) - (3N + 8))$$

$$P_{4,2}(N, y) = (N + 1)^2(N + 2)\Delta(y)(-3e_3(y) + (N + 6)e_2(y) - (3N + 12)e_1(y) + (7N + 24))$$

$$P_{5,2}(N, y) = (N + 1)^3(N + 2)^2(N + 3)\Delta(y)(-4e_4(y) + (N + 8)e_3(y) - (3N + 16)e_2(y) + (7N + 32)e_1(y) - (15N + 64))$$

$$P_{6,2}(N, y) = (N + 1)^4(N + 2)^3(N + 3)^2(N + 4)\Delta(y)(-5e_5(y) + (N + 10)e_4(y) - (3N + 20)e_3(y) + (7N + 40)e_2(y) - (15N + 80)e_1(y) + (31N + 160))$$

$$\begin{aligned}
 P_{n,s}(N, y) &= \Delta(y) \prod_{r=1}^{n-2} (N+r)^{n-1-r} \times \\
 &\times \sum_{l=0}^{n-1} \sum_{k=0}^{s-1} \sum_{j=1}^s \sum_{i=0}^j (-1)^{n+s+l+j} \frac{N^k j^l e_{n-1-l}(y)}{i!(s-1)!} s(s-1-j, k-i) \times \\
 &\quad \times \left(\left(\frac{d}{dn} \right)^i (n-1) \cdots (n-j) \right) \binom{s-1}{j} \quad (1)
 \end{aligned}$$

where $s(n, k)$ are the Stirling numbers of the first kind.

Eynard-Mehta Theorem

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Theorem

Let

$$M = \left[\binom{N}{y_i - j} \right]_{i,j=1}^n.$$

Then

$$[M^{-1}]_{i,j} = \sum_{k=1}^j \frac{\binom{N+n-1}{k-1} \binom{N-1+j-k}{j-k}}{\binom{N+n-1}{y_i-1}} (-1)^{k+j} \prod_{l=1, l \neq i}^n \frac{k - y_l}{y_i - y_l}.$$

Proof

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$$\begin{aligned} [MM^{-1}]_{\alpha,\gamma} &= \sum_{\beta=1}^n [M]_{\alpha,\beta} [M^{-1}]_{\beta,\gamma} \\ &= \sum_{\beta=1}^n \sum_{k=1}^{\gamma} (-1)^{k+\gamma} \binom{N+n-1}{y_{\beta}-1}^{-1} \binom{N+n-1}{k-1} \times \\ &\quad \binom{N-1+\gamma-k}{\gamma-k} \binom{N}{y_{\beta}-\alpha} \prod_{i=1, i \neq \beta}^n \frac{k-y_i}{y_{\beta}-y_i}. \quad (2) \end{aligned}$$

Lagrange interpolation

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Let $(x_1, y_1), \dots, (x_n, y_n) \in \mathbb{R}^2$ and let

$$p_k(x) = \prod_{i=1, i \neq k}^n \frac{x - x_i}{x_k - x_i}.$$

Then

$$f(x) = \sum_{k=1}^n y_k p_k(x)$$

has the property that $f(x_i) = y_i$ for $i = 1, \dots, n$.

Proof

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$$\begin{aligned} [MM^{-1}]_{\alpha,\gamma} &= \sum_{k=1}^{\gamma} (-1)^{\beta+j} \binom{N-1+\beta-k}{\beta-k} \binom{N}{k-\gamma} \\ &= \binom{0}{\alpha-\gamma} = \delta_{\alpha,\gamma} \quad (3) \end{aligned}$$

Corollary

The correlation functions for the Novak half-hexagon are determinantal, with kernel given by

$$\begin{aligned}
 K(r, x; s, y) = & -\phi_{r,s}(x, y) \\
 & + \sum_{i,j=1}^n \frac{\binom{n+1-r}{2i-x} \binom{s}{y-j}}{\binom{2n}{2i-1}} \sum_{k=1}^j \binom{2n}{k-1} \binom{n+j-k}{j-k} \times \\
 & \times \frac{(-1)^{k+j+i+n}}{(i-1)!(n-i)!} \prod_{l=1, l \neq i}^n (k-2l)
 \end{aligned}$$

where for $r \geq s$, $\phi \equiv 0$ and for $r < s$,

$$\phi_{r,s}(x, y) = \binom{s-r}{y-x}.$$

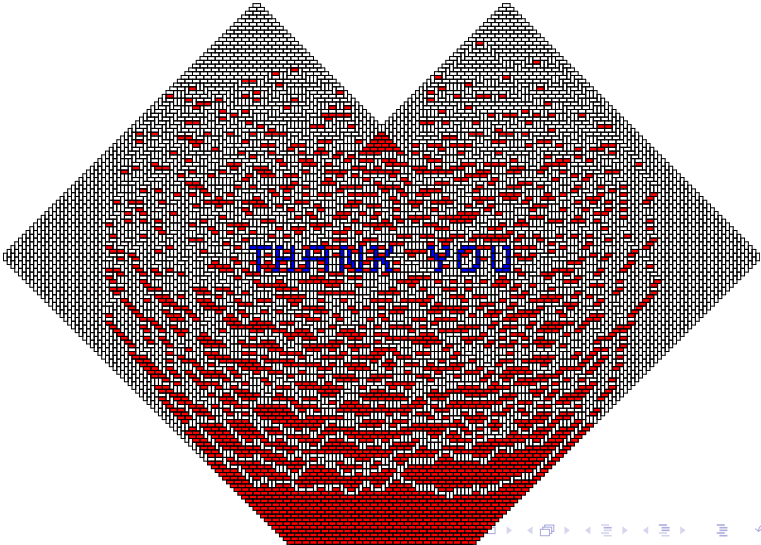
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Thank you four your attention

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Nordenstam, *On the Shuffling Algorithm for Domino Tilings*, arXiv:0802.2592, Electron. J. of Probab. 15 (2010), no. 3, pp. 75-95



Nordenstam, Young, *Domino shuffling on Novak half-hexagons and Aztec half-diamonds*, Electron. J. of Combin. 18 (2011), no. 1.



Nordenstam, Young, *Correlations for the Novak Process*, FPSAC 2012 proceedings, arXiv:1201.4138.

Theorem

$$N = \left[\left[\begin{matrix} A \\ B_j - i \end{matrix} \right]_q q^{\binom{B_j - i}{2}} \right]_{i,j=1}^n,$$

has inverse

$$[N^{-1}]_{i,j} = \frac{q^{nB_i - \binom{B_i}{2}}}{\left[\begin{matrix} A+n-1 \\ B_i-1 \end{matrix} \right]_q} \left(\prod_{k=1, k \neq i}^n \frac{1}{q^{B_i} - q^{B_k}} \right) \times$$

$$\sum_{a=0}^{j-1} \sum_{b=0}^{n-1} \left[\begin{matrix} b \\ j-1-a \end{matrix} \right]_q \left[\begin{matrix} n-b-1 \\ a \end{matrix} \right]_q q^{\binom{j-1}{2} + (a+b)(a-j-1) - b - 1 + aA} \times$$

$$(-1)^b e_b(q^{B_1}, \dots, \widehat{q^{B_i}}, \dots, q^{B_n}).$$