Arithmetic matroids and Tutte polynomials

(joint work with Luca Moci)

Michele D'Adderio

Georg-August Universität Göttingen

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Definition of Matroid

We use the word *list* for *multiset* (repetitions allowed).

A *matroid* $\mathfrak{M} = \mathfrak{M}_X = (X, rk)$ is a list of *vectors* X with a *rank function* $rk : \mathbb{P}(X) \to \mathbb{N} \cup \{0\}$ such that:

1. if $A \subseteq X$, then $rk(A) \leq |A|$;
2. if $A, B \subseteq X$ and $A \subseteq B$, then $rk(A) \leq rk(B)$;
3. if $A, B \subseteq X$, then $rk(A \cup B) + rk(A \cap B) \leq rk(A) + rk(B)$.

In particular $rk(\emptyset) = 0$.

We say that a sublist A is *independent* $\Leftrightarrow rk(A) = |A|$.

An independent sublist of maximal rank $rk(X)$ is called a *basis*.

$rk(X)$ is called the *rank* of the matroid.

The independent sublists determine the matroid structure:
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Examples

1. X is a finite list of vectors of a vector space (e.g. $\mathbb{R}^n$);
   $rk(A) = dim(span(A))$;
   independent = linearly independent;
2. X a finite list of edges of a graph $\mathcal{G}$;
   $rk(A) = |\text{maximal subforest of } A|$;
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Dual Matroid

The *dual* of the matroid $\mathfrak{M} = (X, rk)$ is defined as the matroid with the same set X of vectors, and with bases the complements of the bases of $\mathfrak{M}$.
We will denote it by $\mathfrak{M}^*$. The rank function of $\mathfrak{M}^*$ is given by

$$rk^*(A) := |A| - rk(X) + rk(X \setminus A).$$

In particular the rank of $\mathfrak{M}^*$ is $|X| - rk(X)$.

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Tutte Polynomial

The *Tutte polynomial* of the matroid $\mathfrak{M} = (X, rk)$ is defined as

$$T_X(x,y) := \sum_{A \subseteq X} (x-1)^{rk(X)-rk(A)}(y-1)^{|A|-rk(A)}.$$

From the definition it is clear that $T_X(1,1)$ is equal to the number of bases of the matroid.
The coefficients of the Tutte polynomial are positive, and they have a nice combinatorial interpretation in terms of *internal* and *external activity*.
A vector $v \in X$ is *dependent* on $A \subseteq X$ if $rk(A \cup \{v\}) = rk(A)$.
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Crapo's Theorem

We fix a total order on X, and let B be a basis extracted from X.
We say that $v \in X \setminus B$ is *externally active* on B if v is dependent on the list of elements of B following it.
We say that $v \in B$ is *internally active* on B if v is externally active on the complement $B^c := X \setminus B$ in the dual matroid.
The number $e(B)$ of externally active vectors is called the *external activity* of B, while the number $i(B) = e^*(B^c)$ of internally active vectors is called the *internal activity* of B.

Theorem (Crapo)

$$T_X(x, y) = \sum_{\substack{B \subseteq X \\ B\, basis}} x^{e^*(B^c)} y^{e(B)}.$$

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Definition of Arithmetic Matroid

An *arithmetic matroid* is a pair $(\mathfrak{M}_X, m)$, where $\mathfrak{M}_X$ is a matroid on a list of vectors X, and m is a *multiplicity function*, i.e. $m : \mathbb{P}(X) \to \mathbb{N} \setminus \{0\}$ has the following properties:

1. if $A \subseteq X$ and $v \in X$ is dependent on A, then $m(A \cup \{v\})$ divides $m(A)$;
2. if $A \subseteq X$ and $v \in X$ is independent on A, then $m(A)$ divides $m(A \cup \{v\})$;
3. if $A \subseteq B \subseteq X$ and B is a disjoint union $B = A \cup F \cup T$ such that for all $A \subseteq C \subseteq B$ we have $rk(C) = rk(A) + |C \cap F|$, then $m(A) \cdot m(B) = m(A \cup F) \cdot m(A \cup T)$.
4. if $A \subseteq B \subseteq X$ and $rk(A) = rk(B)$, then $\mu_B(A) := \sum_{A \subseteq T \subseteq B} (-1)^{|T|-|A|} m(T) \geq 0$;
5. if $A \subseteq B \subseteq X$ and $rk^*(A) = rk^*(B)$, then $\mu^*_B(A) := \sum_{A \subseteq T \subseteq B} (-1)^{|T|-|A|} m(X \setminus T) \geq 0$.

Definition of Arithmetic Matroid

An *arithmetic matroid* is a pair $(\mathfrak{M}_X, m)$, where $\mathfrak{M}_X$ is a matroid on a list of vectors X, and m is a *multiplicity function*, i.e. $m : \mathbb{P}(X) \to \mathbb{N} \setminus \{0\}$ has the following properties:

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Definition of Arithmetic Matroid

If $A \subseteq B = X$, then we denote $\mu_X(A)$ simply by $\mu(A)$. Similarly for $\mu^*(A)$.

Setting $m(A) = 1$ for all $A \subseteq X$ we get a *trivial* multiplicity function, which essentially does not add anything to the matroid structure.

So any matroid is trivially an arithmetic matroid.

In this sense the notion of an arithmetic matroid is a generalization of the one of a matroid.

But of course there are more interesting examples.

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The main example

Let X be a finite list of elements of a finitely generated abelian group $G \cong \mathbb{Z}^r \oplus \mathbb{Z}/d_1\mathbb{Z} \oplus \mathbb{Z}/d_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_s\mathbb{Z}$.
For $A \subseteq X$ we set

$$rk(A) := \text{maximal rank of a free abelian subgroup of } \langle A \rangle;$$

$$m(A) := |G_A : \langle A \rangle|, \text{ where } G_A \text{ is the maximal subgroup of } G \text{ such that } \langle A \rangle \leq G_A \text{ and } |G_A : \langle A \rangle| < \infty.$$

Theorem (D.-Moci)

If we set $\mathfrak{M}_X := (X, rk)$, then $(\mathfrak{M}_X, m)$ is an arithmetic matroid.

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Consider the matrix $\begin{pmatrix} 3 & 2 & -3 \\ 0 & -2 & 3 \end{pmatrix}$ whose columns are v_1, v_2, v_3.

Remark

The multiplicity of $A \subseteq X$ is the *GCD* of the minors of maximal rank in the submatrix corresponding to A.

So $m(\emptyset) = 1$, $m(\{v_2\}) = 2$, $m(\{v_1, v_2, v_3\}) = 3$, $m(\{v_2, v_3\}) = 1$, $m(\{v_1, v_2\}) = 6$, $m(\{v_1\}) = m(\{v_3\}) = 3$, $m(\{v_1, v_3\}) = 9$.

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Dual arithmetic matroid

Given an arithmetic matroid $(\mathfrak{M}_X, m)$, its *dual* is $(\mathfrak{M}^*_X, m^*)$, where $\mathfrak{M}^*_X$ is the dual matroid of $\mathfrak{M}_X$, and for all $A \subseteq X$ we set $m^*(A) := m(X \setminus A)$.

Lemma (D.-Moci)

The dual of an arithmetic matroid is an arithmetic matroid.

Theorem (D.-Moci)

The dual of a representable arithmetic matroid is representable.

In fact we give an explicit construction.

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$$M_X(x,y) := \sum_{A \subseteq X} m(A)(x-1)^{rk(X)-rk(A)}(y-1)^{|A|-rk(A)}.$$

From the definition it is clear that $M_X(1,1)$ is equal to the sum of the multiplicities of the bases of the matroid.
For the trivial multiplicity function $m(A) = 1$ for all $A \subseteq X$ we get the Tutte polynomial $T_X(x,y)$ of $\mathfrak{M}_X$.

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Positive coefficients!

The combinatorial problem

Let $(\mathfrak{M}_X, m)$ be an arithmetic matroid, and $M_X(x, y)$ its arithmetic Tutte polynomial.

Question

Does $M_X(x, y)$ have positive coefficients for any arithmetic matroid?

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Is there a combinatorial interpretation of $M_X(x, y)$?

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What is the problem?

Remember that $M_X(1,1)$ is the sum of the multiplicities of the bases extracted from X.
$X_1 := \{v_1 := (3,0), v_2 := (2,-2)\} \subseteq G := \mathbb{Z}^2$.
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Same bases give different statistics!

The construction I

Consider an arithmetic matroid $(\mathfrak{M}_X, m)$. Let $S \subseteq X$ be of maximal rank, i.e. $rk(S) = rk(X)$.
Then $\mu(S) = \sum_{X \supseteq T \supseteq S} (-1)^{|T|-|S|} m(T) \geq 0$.
We call L_X the list in which every maximal rank sublist S appears $\mu(S)$ many times.
We construct dually L_X^* from $(\mathfrak{M}_X^*, m^*)$ using $\mu^*(S)$.
We define the lists $\mathcal{B} := \{(B, T) \mid B \text{ basis}, B \subseteq T, T \in L_X\}$ and its dual $\mathcal{B}^* := \{(B^c, \widetilde{T}) \mid B \text{ basis}, B^c \subseteq \widetilde{T}, \widetilde{T} \in L_X^*\}$.
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We fix a total order on X. For every $(B, T) \in \mathcal{B}$ we define its *local external activity* $e(B, T)$ to be the number of elements of $T \setminus B$ that are externally active on B. We define $e^*(B^c, \widetilde{T})$ dually (using the same order).

Are we done?

Not quite: we need to decide how to match the pairs from $\mathcal{B}$ with the pairs from $\mathcal{B}^*$.

Clearly $(B, T) \in \mathcal{B}$ goes to some $(B^c, \widetilde{T}) \in \mathcal{B}^*$, but how do we choose $\widetilde{T}$?

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We define a matching $\psi : \mathcal{B} \to \mathcal{B}^*$: given a basis $B \subseteq X$, we identify the pairs $(B, T) \in \mathcal{B}$ having the same elements in T active on B, ignoring the non-active elements.
We do the same with the pairs $(B^c, \widetilde{T}) \in \mathcal{B}^*$. Then we *equidistribute* these pairs among each others.

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$$M_X(x,y) = \sum_{(B,T)\in\mathcal{B}} x^{e^*(\psi(B,T))} y^{e(B,T)}.$$

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An example

$X = \{v_1 := (3,0) < v_2 := (2,-2) < v_3 := (-3,3)\} \subseteq G := \mathbb{Z}^2$.

Consider the matrix $\begin{pmatrix} 3 & 2 & -3 \\ 0 & -2 & 3 \end{pmatrix}$ whose columns are v_1, v_2, v_3.

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$x^2 + 3x + 2 + xy + 2y$

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$x^2 + 3x + 2 + xy + 2y + 2x + 4$

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$x^2 + 3x + 2 + xy + 2y + 2x + 4 =$

$= x^2 + 3x + 2 + (y+2)(x+2)$

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References

1. M. D'Adderio, L. Moci, *Arithmetic matroids, Tutte Polynomial and toric arrangements*, arXiv:1105.3220.
2. C. De Concini, C. Procesi, *Topics in hyperplane arrangements, polytopes and box-splines*, Springer 2010.

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THANKS!

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