TECHNISCHE UNIVERSITÄT DARMSTADT

A Topological Representation Theorem for Tropical Oriented Matroids

FPSAC 2012, Nagoya

Silke Horn

2 August 2012

Discrete Optimization

Oriented Matroids

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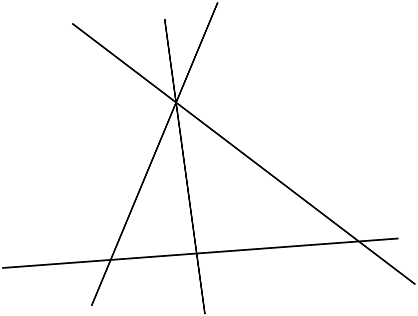

- arrangements of real hyperplanes
- covectors: describe position relative to the hyperplanes
- oriented matroid (OM): combinatorial model for the set of covectors
- non-realisable OMs

Theorem („Topological Representation Theorem“, Folkman & Lawrence, 1978)
Every OM can be realised as an arrangement of pseudohyperplanes.

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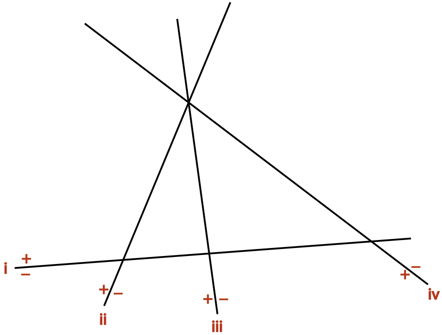

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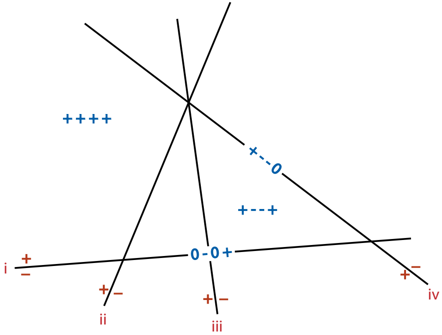

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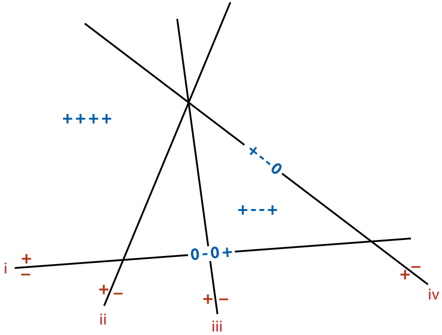

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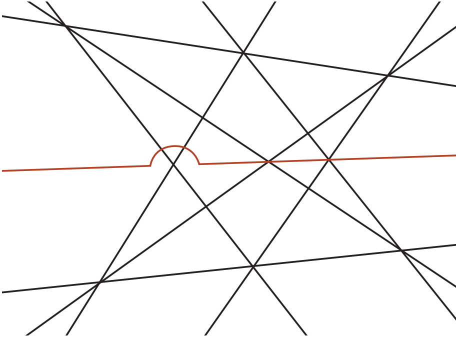

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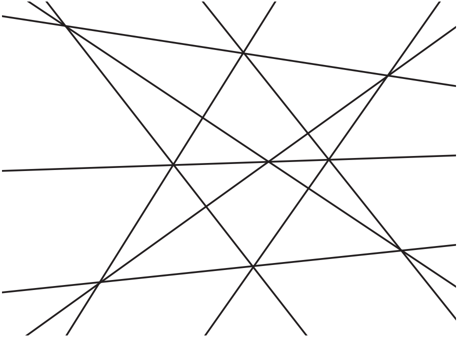

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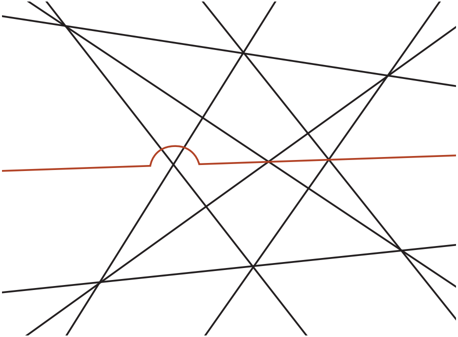

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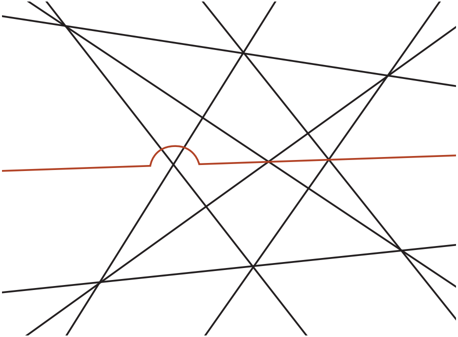

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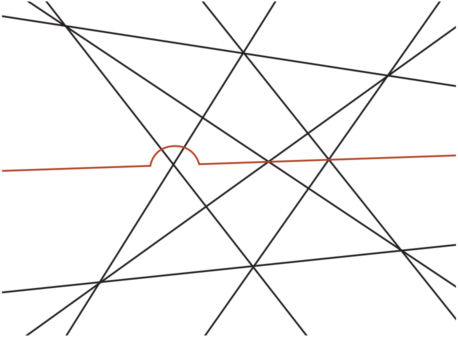

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Every OM can be realised as an arrangement of pseudohyperplanes.

We want a similar theory in the tropical world!

Discrete Optimization

Tropical Geometry in a Nutshell

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- named “tropical” in honour of Brazilian mathematician Imre Simon
- algebraic geometry over the tropical semiring $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$
  $x \oplus y := \min\{x, y\}, x \odot y := x + y$
- linear tropical polynomial: $p(x) = \bigoplus_{i=1}^{d} a_i \odot x_i = \min_{1 \leq i \leq d}\{a_i + x_i\}$
- vanishing locus / tropical hypersurface: minimum attained twice
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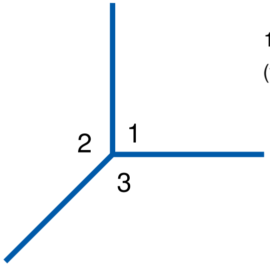

1-dimensional tropical hyperplane
(tropical line)

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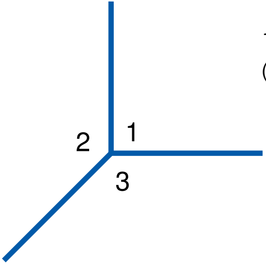

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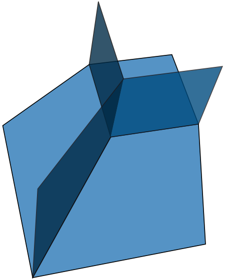
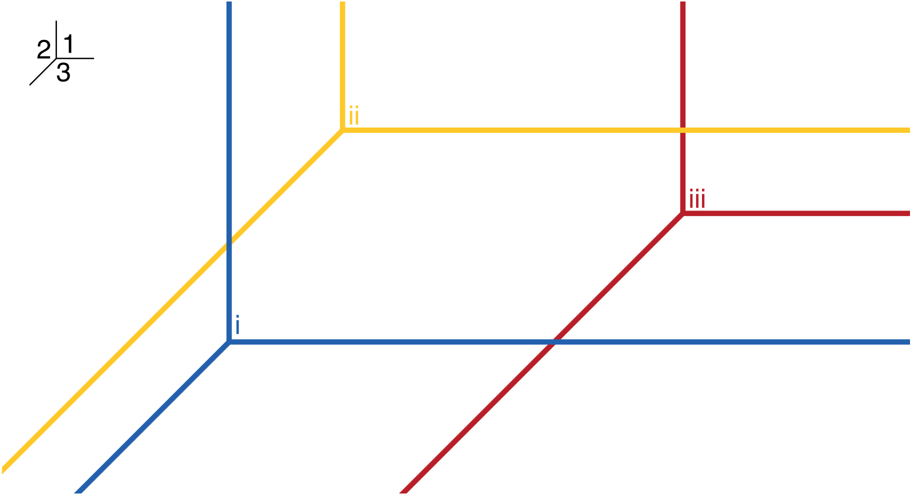
2-dimensional tropical hyperplane



Arrangements of Tropical Hyperplanes

(n, d)-types and tropical covectors



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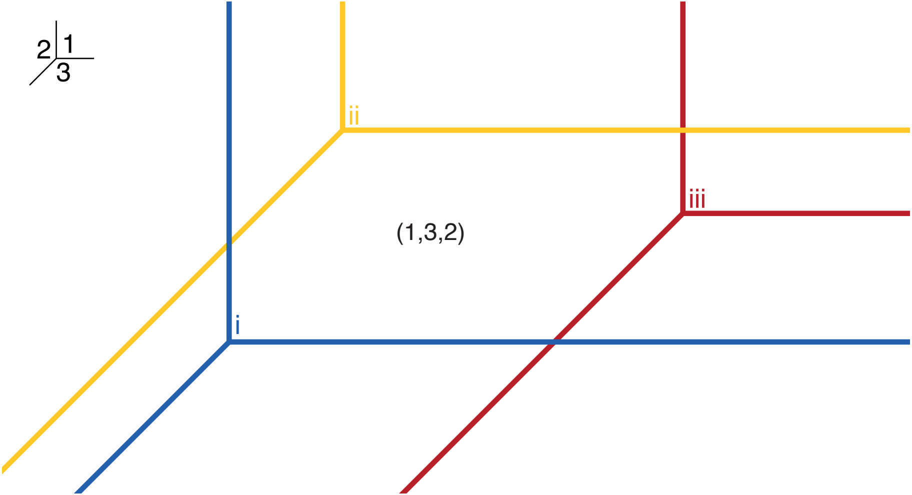

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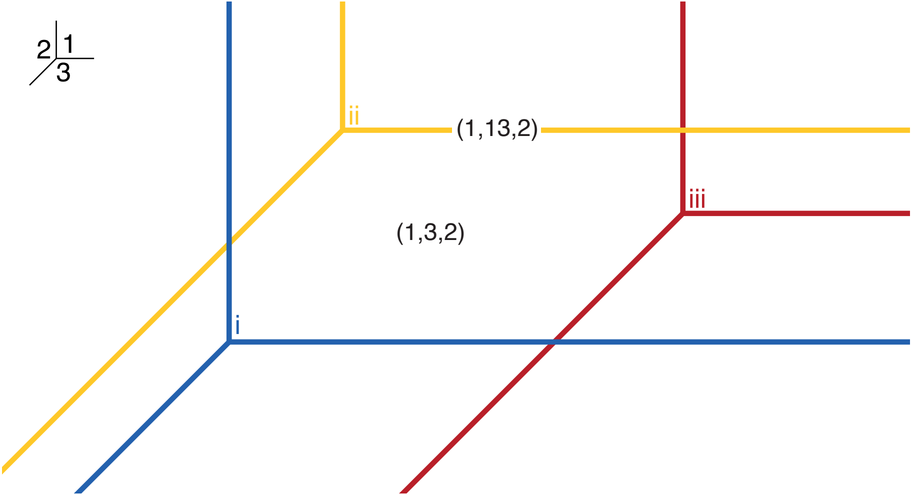

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Tropical Oriented Matroids (TOMs) and Tropical Pseudohyperplanes

- Definition by Ardila and Develin via covector-axioms
- Ardila, Develin: The types in an arrangement of tropical hyperplanes yield a TOM.
- There are non-realisable TOMs.
- Analogue to the Topological Representation Theorem?

Definition

A tropical pseudohyperplane (TROPHY) is the image of a tropical hyperplane under a PL homeomorphism of $\mathbb{TP}^{d-1}$ that fixes the boundary.

Problem: Define tropical pseudohyperplane arrangements!

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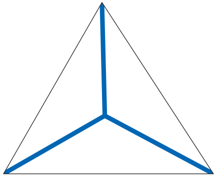

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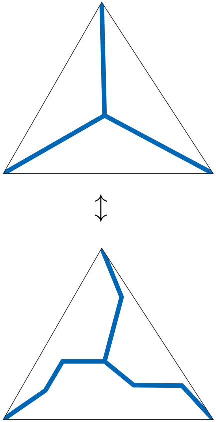

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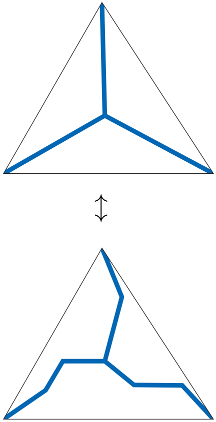

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TOMs and Mixed Subdivisions

The Minkowski sum of two sets X, Y is
$X + Y := \{x + y \mid x \in X, y \in Y\}$.

Definition

A polytopal subdivision of $n\triangle^{d-1}$ is mixed if every face is a Minkowski sum of faces of $\triangle^{d-1}$.

Theorem (Ardila, Develin, 2007)

Every TOM yields a mixed subdivision.

Conjecture (Ardila, Develin, 2007)

The converse also holds.

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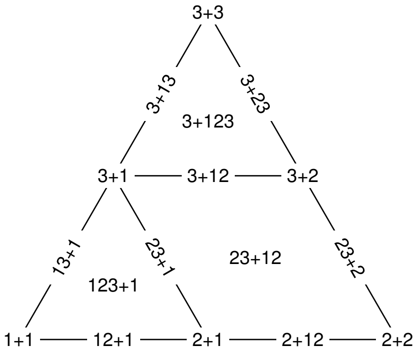

$2\triangle^2$ ($n = 2$, $d = 3$)

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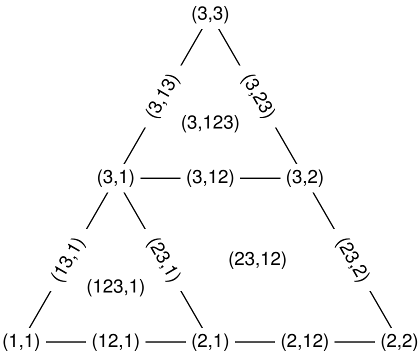

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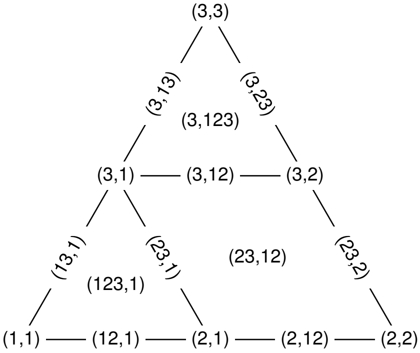

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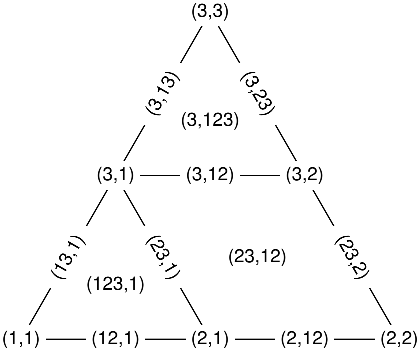

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“The Bigger Picture”

tropical oriented
matroids

mixed subdivisions
of $n\triangle^{d-1}$

tropical hyperplane
arrangements

tropical pseudohyperplane
arrangements ???

“The Bigger Picture”

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“The Bigger Picture”

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tropical oriented matroids → (Ardila/Develin) → mixed subdivisions of $n\triangle^{d-1}$

tropical hyperplane arrangements → (Ardila/Develin) → tropical oriented matroids

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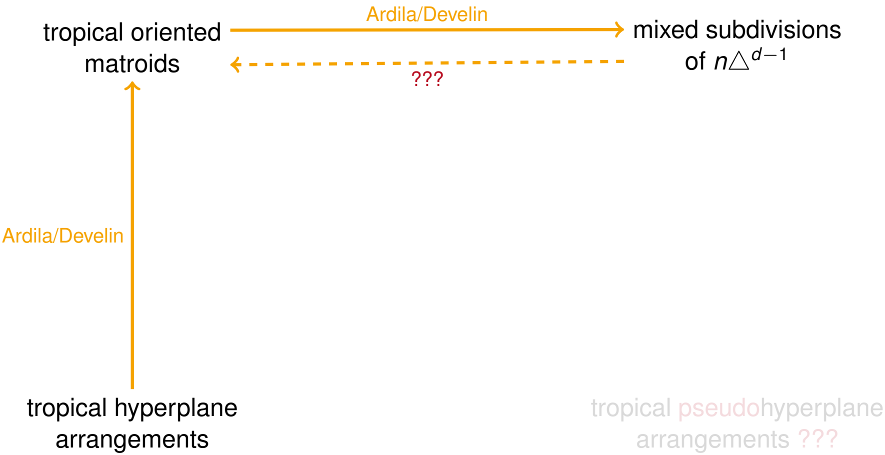

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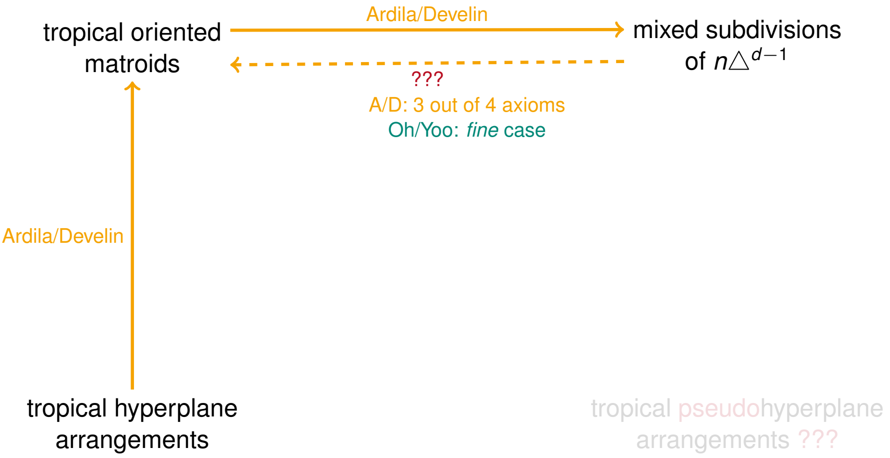

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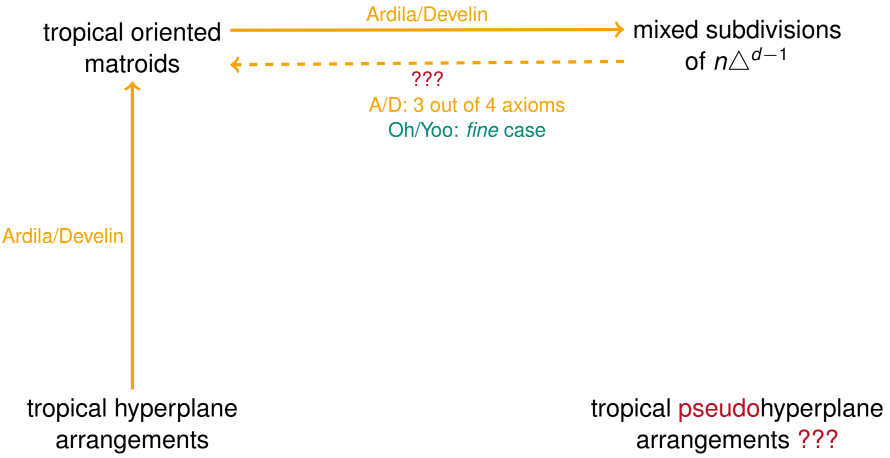

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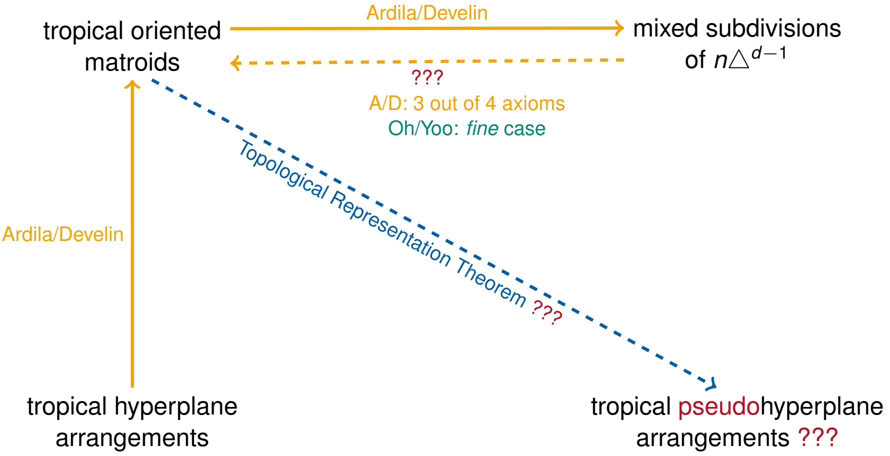

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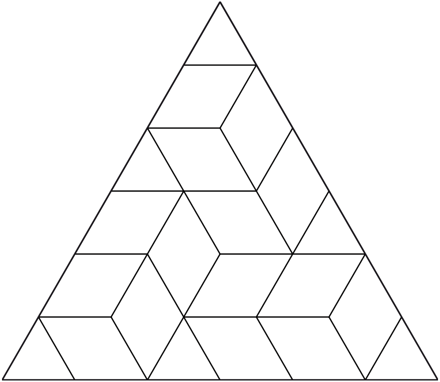
Mixed Subdivisions and TROPHYs

Theorem (H., 2011)

The Poincaré dual of a mixed subdivision of $n\triangle^{d-1}$ yields a family of tropical pseudohyperplanes.



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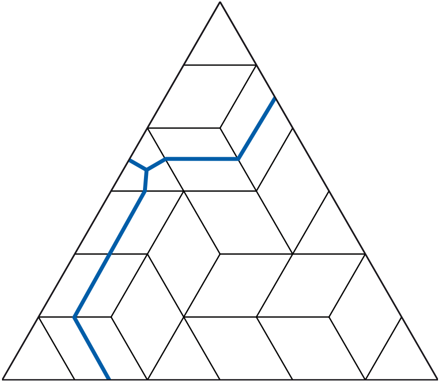

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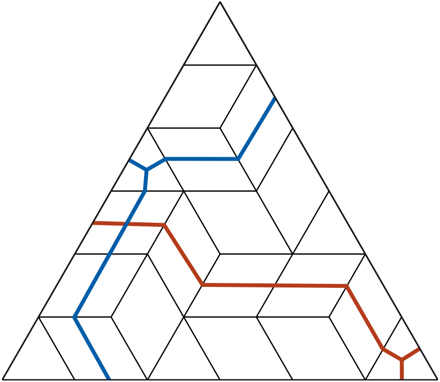

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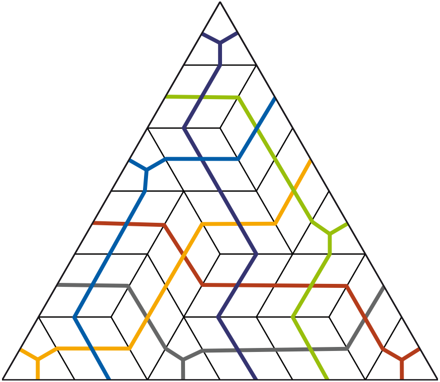

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Elimination and Convexity in TOMs

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Let M be a TOM.

Elimination property: For $A, B \in M$, $k \in [n]$ there is $C \in M$ such that

- $C_k = A_k \cup B_k$,
- $C_i \in \{A_i, B_i, A_i \cup B_i\}$.

convex hull of A and B:
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Discrete Optimization

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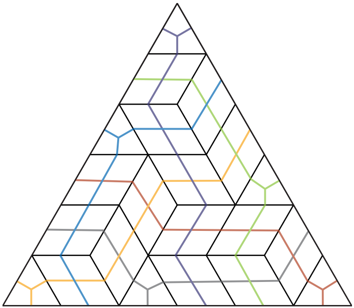

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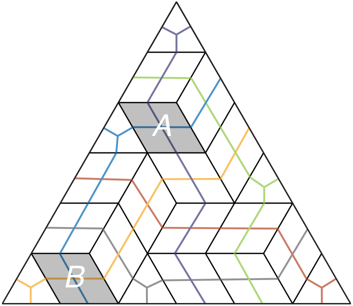

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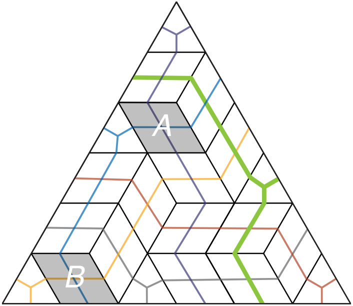

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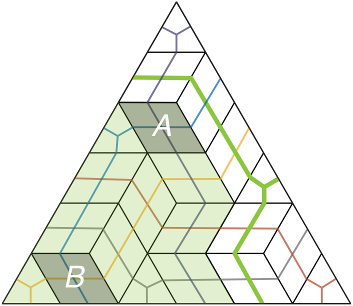

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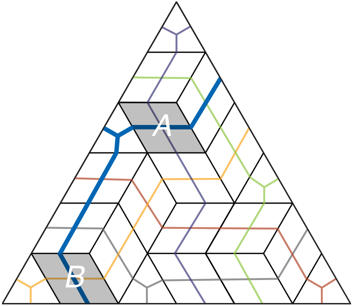

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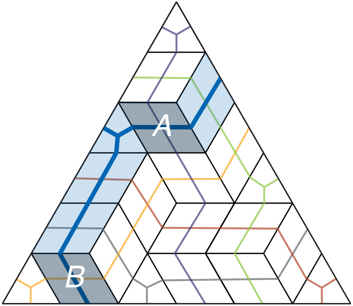

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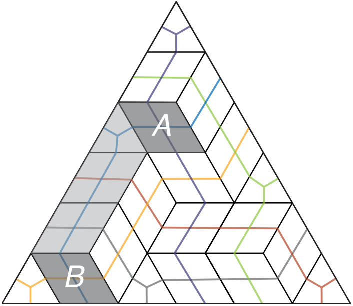

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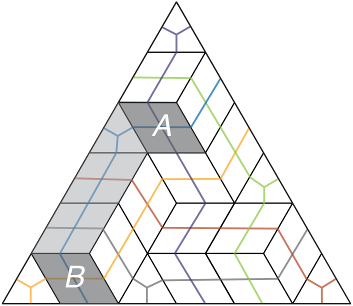

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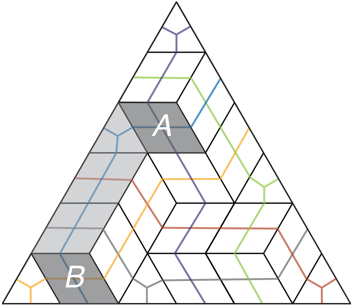

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Arrangements of Tropical Pseudohyperplanes (TROPHYs)

$\mathcal{I} = (2, 23, 1, 12, 3, 3)$

$\mathcal{A}_{\mathcal{I}}$: induced family of (linear) pseudohyperplanes

IDEA: Represent convex hull as intersection of affine pseudohalfspaces.

Definition (H., 2010/2011)

A finite family $\mathcal{A}$ of TROPHYs is an arrangement if for every $\mathcal{A}' \subseteq \mathcal{A}$ and $\mathcal{I}$

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Theorem (Topological Representation Theorem, H., 2011)

A mixed subdivision of $n\triangle^{d-1}$ yields an arrangement of TROPHYs.

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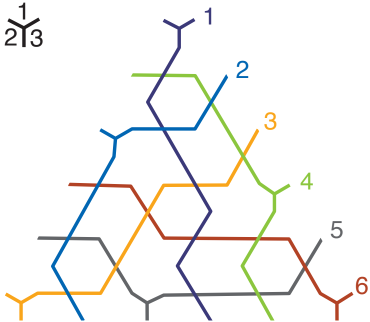

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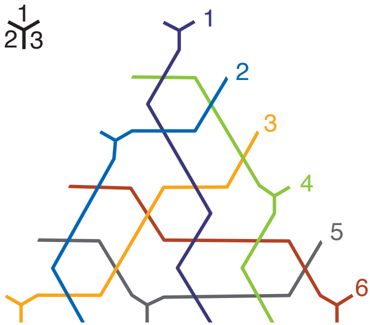

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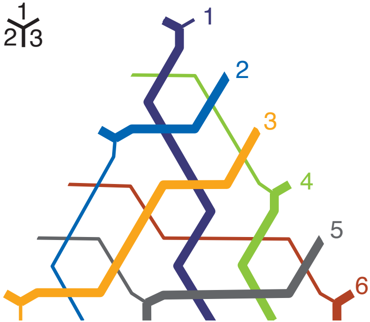

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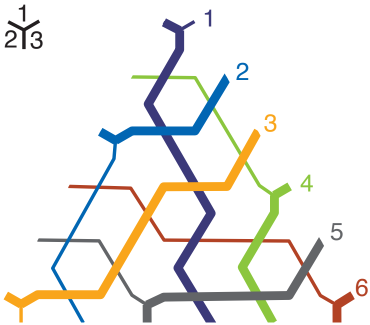

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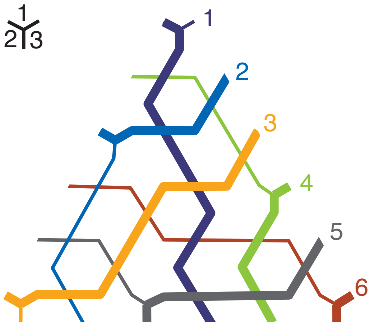

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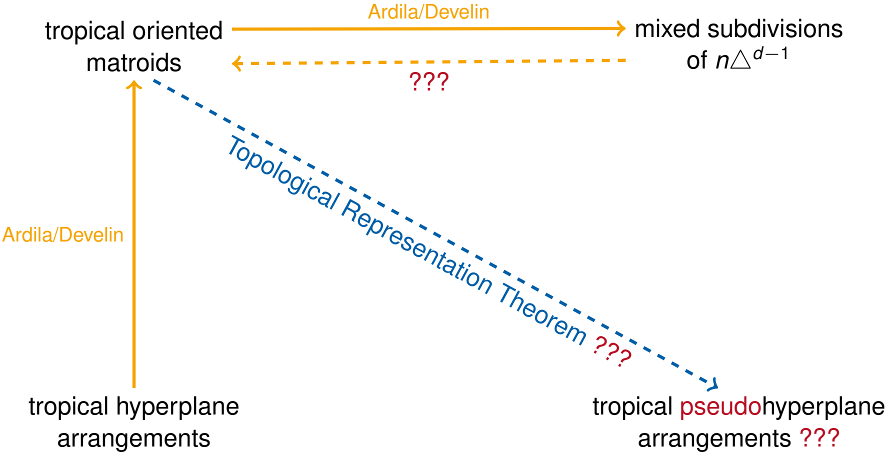

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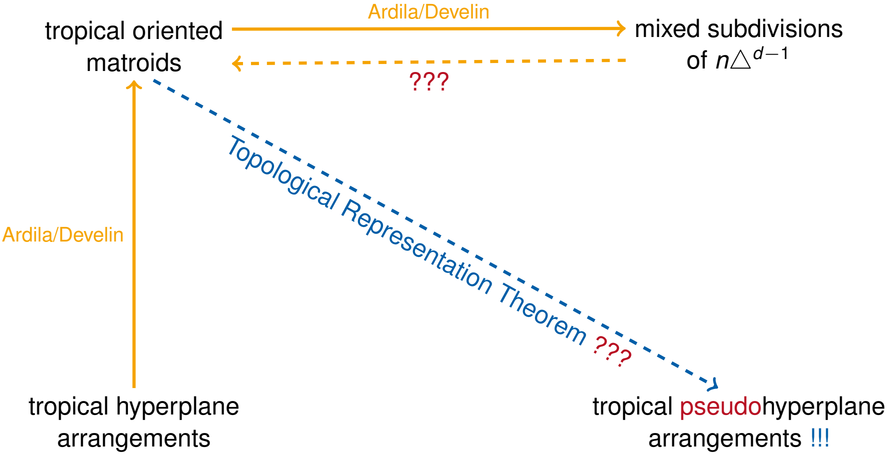

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tropical oriented matroids

Ardila/Develin →

mixed subdivisions of $n\triangle^{d-1}$

← ???

Ardila/Develin ↑

tropical hyperplane arrangements

↓ Topological Representation Theorem !!!

tropical pseudohyperplane arrangements !!!

Discrete Optimization

“The Bigger Picture” revisited

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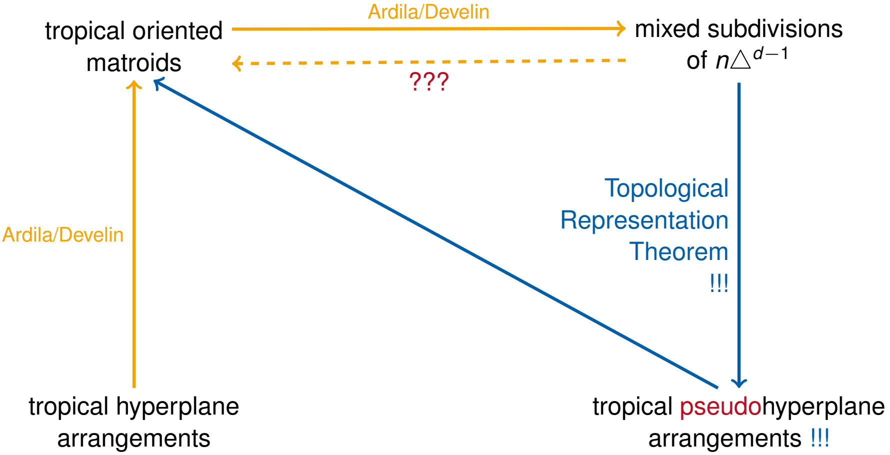

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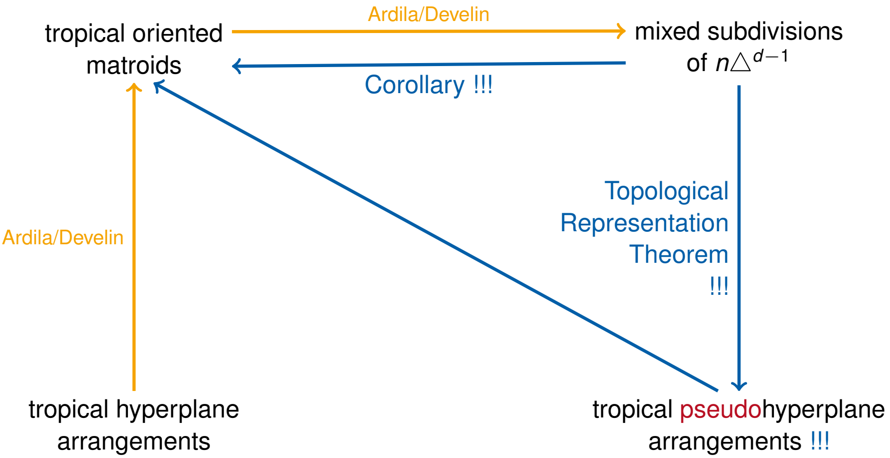

The Missing Arrow
The Elimination Property

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Theorem (H., 2011)

Tropical pseudohyperplane arrangements satisfy the elimination property.

Sketch of proof.

- convex hull of types:
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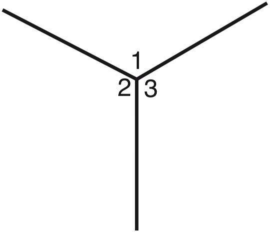

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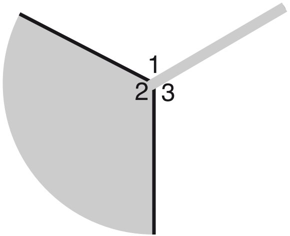

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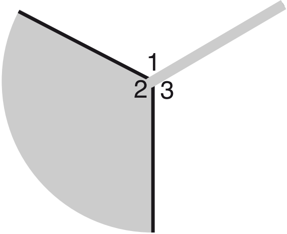

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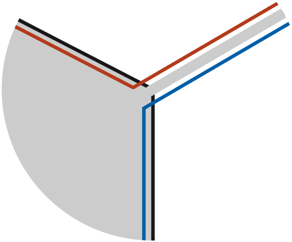

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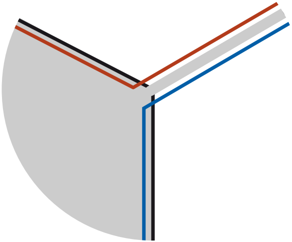

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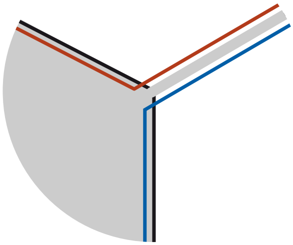

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