Primitive derivations, Shi arrangements, Bernoulli polynomials and the height-free conjecture

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What are primitive derivations?

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Theorem

(Chevalley 1955) The exist algebraic independent homogeneous polynomials $P_1, P_2, \ldots, P_\ell$ (basic invariants) such that

$$2 = \deg P_1 < \deg P_2 \leq \cdots \leq \deg P_{\ell-1} < \deg P_\ell = h$$

and $R = \mathbb{R}[P_1, \ldots, P_\ell]$. Let $d_i := \deg P_i - 1$ (exponents) and $h := \deg P_\ell = d_\ell + 1$ (Coxeter number).

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(chain rule) $\partial_{x_i} = \sum_{k=1}^{\ell} (\partial P_k/\partial x_i)\, \partial_{P_k} \Rightarrow Der_R \subset (1/Q)Der_S$, where
$Q := \det\left[\partial P_j/\partial x_i\right]$: (defining polynomial)

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Definition

(K. Saito 1977) Define $D := \partial_{P_\ell}$, which is independent of choice of basic invariants $P_1, P_2, \ldots, P_\ell$. The derivation D is unique up to a constant multiple and is called a primitive derivation.

Logarithmic derivation module $D(\mathcal{A}, m)$

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Definition

(K. Saito ($m = 1$) 1977, G. Ziegler ($m \geq 2$) 1988) Define, for $m \geq 0$,

$$D(\mathcal{A}, m) := \{\theta \in Der_S \mid \theta(\alpha_H) \in \alpha_H^m S \text{ for all } H \in \mathcal{A}\}.$$

$$D(\mathcal{A}) := D(\mathcal{A}, 1) = \{\theta \in Der_S \mid \theta(Q) \in QS\}.$$

Primitive covariant derivative ∇_D

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The covariant derivative of a primitive derivation D gives a shifting T-isomorphism

$$\nabla_D : D(\mathcal{A}(W), 2k+1)^W \xrightarrow{\sim} D(\mathcal{A}(W), 2k-1)^W,$$

where $T := \{f \in S \mid D(f) = 0\} = \mathbb{R}[P_1, P_2, \ldots, P_{\ell-1}]$.

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We may define $D(\mathcal{A}(W), m)$ for $m \in \mathbb{Z}$. For example,

$$\nabla_D : (Der_S)^W = D(\mathcal{A}(W), 1)^W \xrightarrow{\sim} D(\mathcal{A}(W), -1)^W = Der_R.$$

Primitive covariant derivative ∇_D

Define

$$I^* : \Omega_R \tilde{\to} D(\mathcal{A}(W), 1)^W \text{ by } I^*(dP_i)(f) := I^*(dP_i, df),$$

where I^* is the W-invariant inner product on Ω_S^1.

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The derivations $I^*(dP_i)$ $(1 \le i \le \ell)$ form a W-invariant basis for $D(\mathcal{A}(W), 1)$.

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The derivations $I^*(dP_i)$ $(1 \le i \le \ell)$ form a W-invariant basis for $D(\mathcal{A}(W), 1)$.
The the contact order filtration

$$\cdots \subset D(\mathcal{A}(W), 2k+1)^W \subset D(\mathcal{A}(W), 2k-1)^W \subset \ldots$$

coincides with the Hodge filtration in the sense of K. Saito (flat structure $\approx$ Frobenius manifold structure). Recall that

$$\nabla_D : D(\mathcal{A}(W), 2k+1)^W \tilde{\longrightarrow} D(\mathcal{A}(W), 2k-1)^W$$

shifts up the filtration.

Free arrangements

Definition

*For a central arrangement $\mathcal{A}$ and a positive integer m, we say that $(\mathcal{A}, m)$ is **free**, if $D(\mathcal{A}, m)$ is a free S-module. When $(\mathcal{A}, m)$ is free,*

$$D(\mathcal{A}, m) \simeq S(-d_1) \oplus S(-d_2) \oplus \cdots \oplus S(-d_\ell)$$

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(isomorphic as graded S-modules). The nonnegative integers $(d_1, d_2, \ldots, d_\ell)$ are called the ***exponents*** *of $(\mathcal{A}, m)$.*

Theorem

(K. Saito ($m = 1$), L. Solomon-H.T. ($m = 2$), H. T. ($m \geq 3$))
$D(\mathcal{A}(W), m)$ is a free S-module with exponents

$$\begin{aligned} (kh, kh, \ldots, kh) &\text{ if } m = 2k, \\ (kh + d_1, \ldots, kh + d_\ell) &\text{ if } m = 2k + 1. \end{aligned}$$

Primitive covariant derivative ∇_D

The simplest example is as follows: Let $W = A_1$, $\mathcal{A}(W) = \{\text{one point}\} = \{x_1 = 0\}$, $P_1 = x_1^2$. Then

$$D = \frac{1}{2x_1}\partial_{x_1},$$

$$\nabla_D : D(\mathcal{A}(W), 3)^W = R(x_1^3\partial_{x_1}) \tilde{\to} D(\mathcal{A}(W), 1)^W = R(x_1\partial_{x_1}).$$

Let $E := x_1\partial_{x_1}$ be the Euler derivation. Then

$$\nabla_D^{-1}E = \frac{2}{3}x_1^3\partial_{x_1} \in D(\mathcal{A}(W), 3)^W.$$

Note that

$$\nabla_{\partial_{x_1}}\nabla_D^{-1}E = 2x_1^2\partial_{x_1}$$

forms a basis for $D(\mathcal{A}(W), 2)$. In general, we have …

Primitive covariant derivative ∇_D

Theorem

(M. Yoshinaga 2002) The W-isomorphism

$$\Xi_k : (Der_S)_0 = V \xrightarrow{\sim} D(\mathcal{A}(W), 2k)_{kh}$$

can be described as $\Xi_k(\theta) = \nabla_\theta \nabla_D^{-k} E$ *for a primitive derivation* D*. Thus* $\nabla_{\partial_{x_i}} \nabla_D^{-k} E \ (1 \le i \le \ell)$ *form a basis for* $D(\mathcal{A}(W), 2k)$.

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Now we will go to the world with an extra coordinate z.

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Now we will go to the world with an extra coordinate z.

From now on, assume that W is an irreducible Weyl group arising from an irreducible root system Φ.

What are the Shi arrangements?

Φ: an irreducible root system in V, W: the corresponding Weyl group
Φ_+: a set of positive roots
H_α : the hyperplane orthogonal to a positive root $\alpha \in \Phi_+$
$H_{\alpha,j}$: the affine hyperplane defined by the equation $\alpha = j$ for $\alpha \in \Phi_+$ and $j \in \mathbb{Z}$

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The (generalized) Shi arrangement is defined by

$$Shi^k := \{H_{\alpha,j} \mid 1-k \le j \le k, \alpha \in \Phi_+\} \quad (k \ge 1).$$

(J.-H. Shi defined Shi^1 for the type A_ℓ (the braid arrangement case) in 1986. Studied by R. Stanley, Ch. Athanasiadis et al,)

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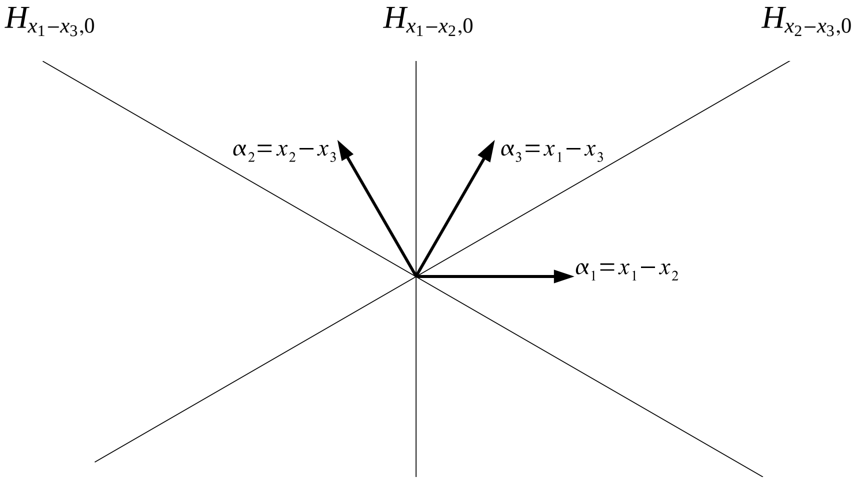
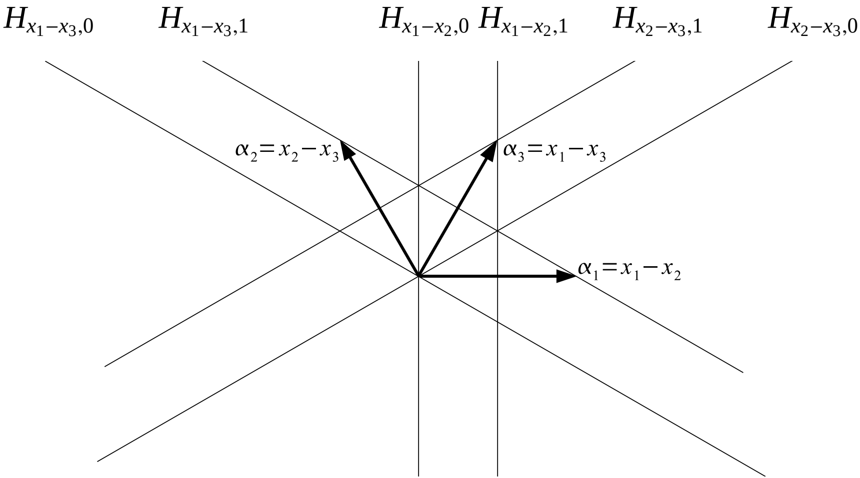
(J.-H. Shi defined Shi^1 for the type A_ℓ (the braid arrangement case) in 1986. Studied by R. Stanley, Ch. Athanasiadis et al,)
Define a central arrangement
$\mathcal{S}^k := \mathcal{S}^k(\Phi)$:= the cone of Shi^k in $\mathbb{R}^{\ell+1}$ (by homogenizing using an extra coordinate z: $\alpha = j \longrightarrow \alpha = jz$)

the braid arrangement $\mathcal{A}(A_2)$



2-dim braid arrangement $\mathcal{A}(A_2)$

the Shi arrangement Shi^1 of the type A_2



2-dim Shi arrangement $Shi^1(A_2)$

Free arrangements

Theorem

(**Factorization Theorem** H.T. (1981)) *When $\mathcal{A}$ is a free arrangement with exponents $d_1, d_2, \dots, d_\ell$, its Poincaré polynomial factors as:*

$$\pi(\mathcal{A}, t) = \prod_{i=1}^{\ell} (1 + d_i t).$$

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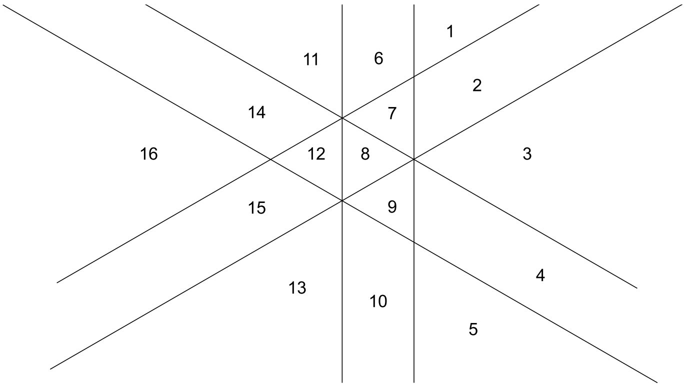
Thanks to Zaslavsky's chamber-counting formula, the number of chambers of $\mathcal{A}$ is equal to $\prod_{i=1}^{\ell}(1 + d_i)$.

Theorem

(Yoshinaga (2004) (conjectured by Edelman-Reiner(1996)) *The cone of every Shi arrangement $\mathcal{S}^k(\Phi)$ is a free arrangement with exponents* $(1, kh, kh, \ldots, kh) = (1, (kh)^\ell)$.

The number of chambers of $\mathcal{S}^1(A_2)$



(with the hyperplane defined by $z = 0$ at infinity)

free arrangement with exponents $(1, 3, 3)$

$2 \times (3 + 1)^2 = 2 \times 16 = 32$(chambers)

Proof of Yoshinaga's theorem

Yoshinaga proved the theorem by proving the surjectivity of the restriction map (setting $z = 0$):

$$\rho : D_0(\mathcal{S}^k)_{kh} \longrightarrow D(\mathcal{A}(W), 2k)_{kh}$$

by showing a sheaf cohomology vanishing. Here,

$$D_0(\mathcal{S}^k)_{kh} := \{\theta \in D(\mathcal{S}^k) \mid \deg\theta = kh, \theta(z) = 0\}.$$

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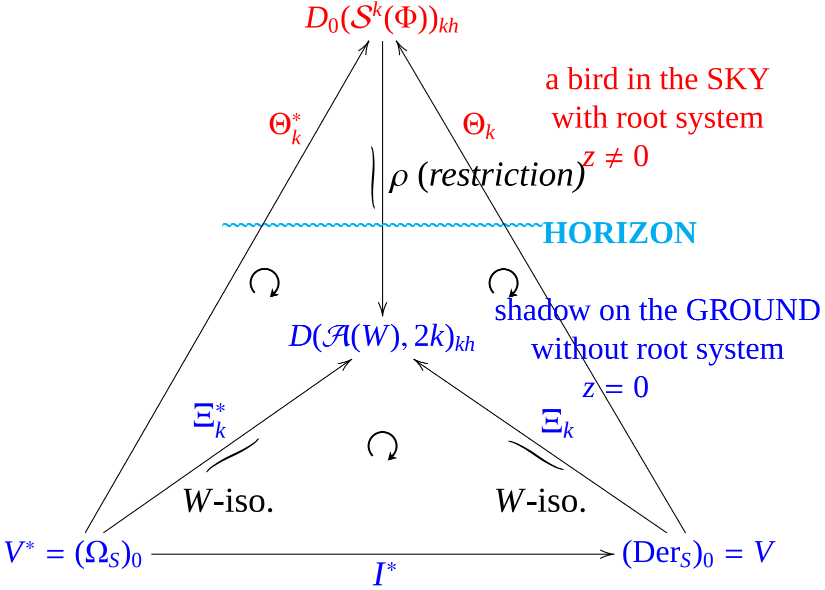
Let us see the big picture:

the big picture

a bird and its shadow

the big picture



Θ_k is the lifting of $\Xi_k(\theta) := \nabla_\theta \nabla_D^{-k} E$ (unique by Schur's lemma)

Two bases: SRB and dSRB

Problem

Find an **explicit description of Θ_k and/or Θ_k^*.**

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Fix a primitive derivation D and a simple system $\{\alpha_1, \ldots, \alpha_\ell\}$.

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Definition

The basis $\varphi_i^* := \Theta_k^*(d\alpha_i)$, $1 \le i \le \ell$, is called the simple root basis =SRB for $D_0(\mathcal{S}^k)$ and another basis $\varphi_i := \Theta_k(\partial_{\alpha_i})$, $1 \le i \le \ell$, is called the dual simple root basis =dSRB for $D_0(\mathcal{S}^k)$.

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They have the following nice characterization:

Two bases: SRB and dSRB

Proposition

(T. Abe-H.T. arXiv: 1111.3510)
(1) The φ_i^* *(SRB) is divisible by* $\alpha_i - kz$ *for each i,*
(2) If $\theta_i \in D_0(\mathcal{S}^k)$ *satisfy* $(\alpha_i - kz) \mid \theta_i$ *for* $1 \le i \le \ell$*, then* $\theta_i = c_i\varphi_i^*$ *for suitable nonzero constant* c_i *for* $1 \le i \le \ell$*.*

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(1)' The $\varphi_i(\alpha_j)$ *(dSRB) is divisible by* $\alpha_j + kz$ *if* $j \ne i$*,*

(2)' If $\theta_i \in D_0(\mathcal{S}^k)$ *satisfy* $(\alpha_j + kz) \mid \theta_i(\alpha_j)$ *for* $1 \le i \le \ell$*,* $1 \le j \le \ell$*,* $i \ne j$*, then* $\theta_i = d_i\varphi_i$ *for suitable nonzero constant* d_i *for* $1 \le i \le \ell$*.*

Two bases: SRB and dSRB (the type A_ℓ)

Example

(root system of the type A_ℓ) Suppose that
$V := \{(x_1, \dots, x_{\ell+1}) \in \mathbb{R}^{\ell+1} \mid x_1 + \cdots + x_{\ell+1} = 0\}$,
$\Phi := \{x_i - x_j \mid 1 \le i \le \ell + 1, 1 \le j \le \ell + 1, i \ne j\}$ *and*
$\Phi_+ := \{x_i - x_j \mid 1 \le i < j \le \ell + 1\}$. *Then*

$$\{\alpha_i := x_i - x_{i+1} \mid 1 \le i \le \ell\}$$

is a set of simple roots. In this case, there exists an explicit formula for SRB and dSRB (D. Suyama-H.T. 2012).

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The SRB for the type A_ℓ looks like this:

An example : the SRB in the case of A_3

$$\varphi_1^* = (x_1 - x_2 - z)\left\{x_3x_4x_1 - \frac{1}{2}(x_3 + x_4)(x_1^2 - x_1 z) + \frac{1}{3}\left(x_1^3 - \frac{3}{2}x_1^2 z + \frac{1}{2}x_1 z^2\right)\right\}\partial_1$$
$$+ (x_1 - x_2 - z)\left\{x_3x_4x_2 - \frac{1}{2}(x_3 + x_4)(x_2^2 - x_2 z) + \frac{1}{3}\left(x_2^3 - \frac{3}{2}x_2^2 z + \frac{1}{2}x_2 z^2\right)\right\}\partial_2$$
$$- \frac{1}{6}x_3(x_1 - x_2 - z)(x_3 + z)(x_3 - 3x_4 - z)\partial_3$$
$$- \frac{1}{6}x_4(x_1 - x_2 - z)(x_4 + z)(x_4 - 3x_3 - z)\partial_4,$$
$$\varphi_2^* = -\frac{1}{6}x_1(x_2 - x_3 - z)(x_1 - z)(x_1 - 3x_4 - 2z)\partial_1$$
$$+ (x_2 - x_3 - z)\left\{x_1x_4x_2 - \frac{1}{2}x_1(x_2^2 - x_2 z) - \frac{1}{2}x_4(x_2^2 + x_2 z) + \frac{1}{3}(x_2^3 - x_2 z^2)\right\}\partial_2$$
$$+ (x_2 - x_3 - z)\left\{x_1x_4x_3 - \frac{1}{2}x_1(x_3^2 - x_3 z) - \frac{1}{2}x_4(x_3^2 + x_3 z) + \frac{1}{3}(x_3^3 - x_3 z^2)\right\}\partial_3$$
$$+ \frac{1}{6}x_4(x_2 - x_3 - z)(x_4 + z)(3x_1 - x_4 - 2z)\partial_4,$$
$$\varphi_3^* = -\frac{1}{6}x_1(x_3 - x_4 - z)(x_1 - z)(x_1 - 3x_2 + z)\partial_1$$
$$- \frac{1}{6}x_2(x_3 - x_4 - z)(x_2 - z)(x_2 - 3x_1 + z)\partial_2$$
$$+ (x_3 - x_4 - z)\left\{x_1x_2x_3 - \frac{1}{2}(x_1 + x_2)(x_3^2 + x_3 z) + \frac{1}{3}\left(x_3^3 + \frac{3}{2}x_3^2 z + \frac{1}{2}x_3 z^2\right)\right\}\partial_3$$
$$+ (x_3 - x_4 - z)\left\{x_1x_2x_4 - \frac{1}{2}(x_1 + x_2)(x_4^2 + x_4 z) + \frac{1}{3}\left(x_4^3 + \frac{3}{2}x_4^2 z + \frac{1}{2}x_4 z^2\right)\right\}\partial_4$$

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$$+ (x_1 - x_2 - z)\left\{x_3x_4x_2 - \frac{1}{2}(x_3+x_4)(x_2^2 - x_2 z) + \frac{1}{3}\left(x_2^3 - \frac{3}{2}x_2^2 z + \frac{1}{2}x_2 z^2\right)\right\}\partial_2 \quad \longleftarrow \sigma_2^{(1,2)}\overline{B}_{0,0}(x_2,z) - \sigma_1^{(1,2)}\overline{B}_{0,1}(x_2,z) + \sigma_0^{(1,2)}\overline{B}_{0,2}(x_2,z)$$

$$- \frac{1}{6}x_3(x_1 - x_2 - z)(x_3 + z)(x_3 - 3x_4 - z)\partial_3$$

$$- \frac{1}{6}x_4(x_1 - x_2 - z)(x_4 + z)(x_4 - 3x_3 - z)\partial_4,$$

$$\varphi_2^* = -\frac{1}{6}x_1(x_2 - x_3 - z)(x_1 - z)(x_1 - 3x_4 - 2z)\partial_1$$

$$+ (x_2 - x_3 - z)\left\{x_1x_4x_2 - \frac{1}{2}x_1(x_2^2 - x_2 z) - \frac{1}{2}x_4(x_2^2 + x_2 z) + \frac{1}{3}(x_2^3 - x_2 z^2)\right\}\partial_2$$

$$+ (x_2 - x_3 - z)\left\{x_1x_4x_3 - \frac{1}{2}x_1(x_3^2 - x_3 z) - \frac{1}{2}x_4(x_3^2 + x_3 z) + \frac{1}{3}(x_3^3 - x_3 z^2)\right\}\partial_3$$

Bernoulli polynomials are the main ingredients!

$$+ \frac{1}{6}x_4(x_2 - x_3 - z)(x_4 + z)(3x_1 - x_4 - 2z)\partial_4,$$

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Explicit formulas for the SRB/dSRB

So far, explicit formulas (in terms of Bernoulli-like polynomilas) have been obtained in the cases of the types A, B, C, D, F and G (by R. Gao, D. Pei, D. Suyama, H.T. 2012).

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However, not for the types E_6, E_7 or E_8.

A characterization for simple roots

Fix a set of simple roots. Let β be a positive root. Then we have

Proposition

(T. Abe-H.T. arXiv: 1111.3510) The following three condtions are equivalent:
(1) β *is a simple root (=of height one) ,*
(2) $\mathcal{S}^k \setminus \{\beta = kz\}$ *is a free arrangement,*
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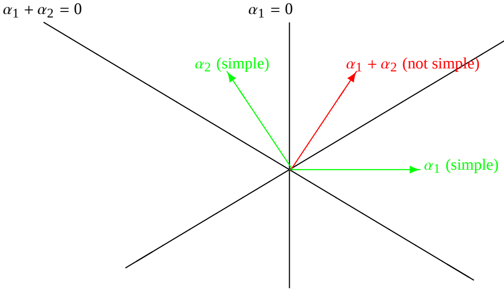

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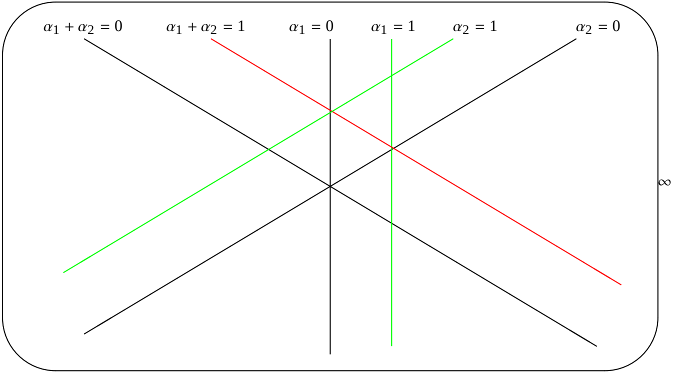

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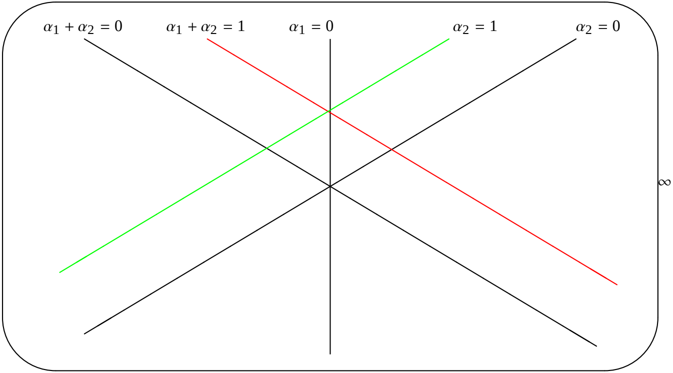

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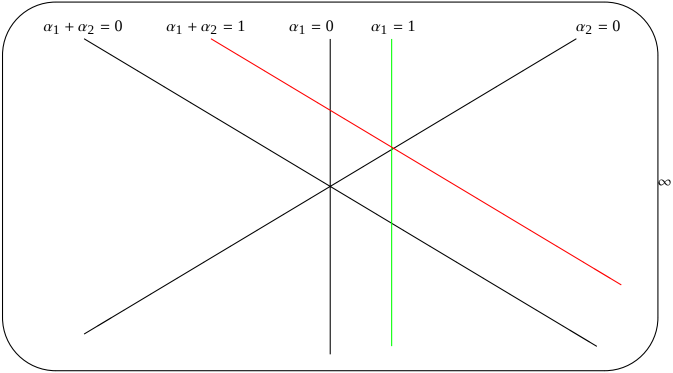

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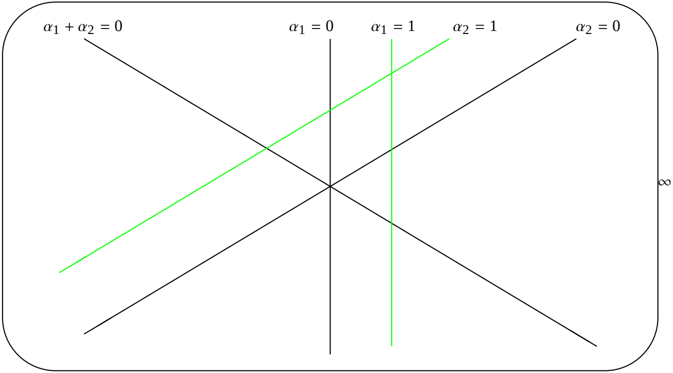

$\mathcal{S}^1 \setminus \{\alpha_1 + \alpha_2 = 1\}$ of the type A_2 is NOT free

Height distribution and exponents

This characterization reminds me of the dual partition theorem due to A. Shapiro, R. Steinberg, B. Kostant, I.G. Macdonald et al. (I learned this from L. Solomon back in 1990.)

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height distribution					
1	●				
1	●				
2	●	●			
3	●	●	●		
4	●	●	●	●	
4	●	●	●	●	
5	●	●	●	●	●
exponents	7	5	4	3	1

Height-free conjecture

Problem

Verify (or disprove) that every height subarrangement is a free arrangement with the exponents determined by the height distribution? (the height-free conjecture)

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Definition

Let $\{\alpha_1, \ldots, \alpha_\ell\}$ be a set of simple roots. For a positive root β, express $\beta = \sum_{i=1}^{\ell} c_i \alpha_i$ with nonnegative integers c_i $(1 \leq i \leq \ell)$. Recall the height of β, denote by $ht(\beta)$: $ht(\beta) = \sum_{i=1}^{\ell} c_i$. Define

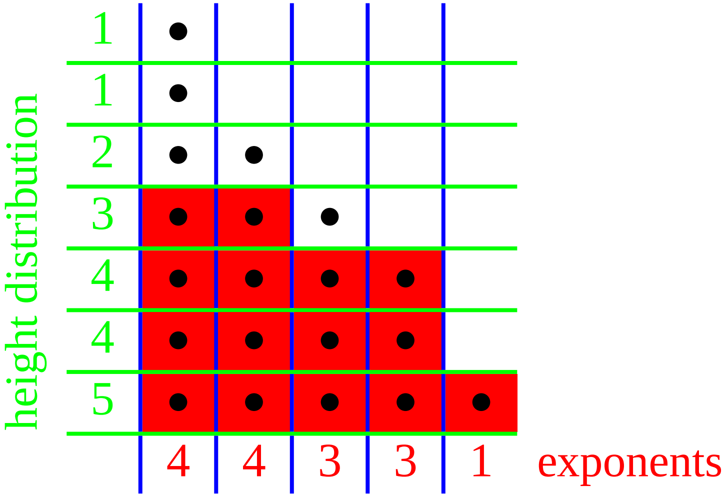
$$\mathcal{A}_{\leq m} := \{\ker(\alpha) \mid \alpha \in \Phi_+, ht(\alpha) \leq m\}.$$

A subarrangement $\mathcal{B}$ of $\mathcal{A}$ is called a height subarrangement if

$$\mathcal{A}_{\leq m} \subseteq \mathcal{B} \subseteq \mathcal{A}_{\leq m+1}.$$

an example of height subarrangement

$\mathcal{A}_{\leq 3} \subseteq \mathcal{B} \subseteq \mathcal{A}_{\leq 4}$. The red squares indicate $\mathcal{B}$.



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Proposition

(T. Abe-H.T. 2012) Suppose the height-free conjecture is true. Let $\mathcal{B}$ be a height subarrangement of $\mathcal{A}(W)$ in $\mathbb{R}^\ell$. Then

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$$\mathcal{S}^k \cup \{H_{\alpha,-k} \mid H_\alpha \in \mathcal{B}\} \quad (= \textit{Catalan if } \mathcal{B} = \mathcal{A}(W))$$

in $\mathbb{R}^{\ell+1}$ is a free arrangement with exponents

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So far, the height-free conjecture has been verified by T. Abe (-H.T.) in the cases of the types A, B, C, D, E_6, F and G.

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However, not for the types E_7 or E_8.

I stop here.

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Thank you!

Bernoulli numbers and Bernoulli polynomials

Definition

The Bernoulli numbers B_n are characterized by

$$B_0 = 1, \ \sum_{k=0}^{n} \binom{n+1}{k} B_k = 0 \ \ (n \geq 1).$$

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$$B_0 = 1,\ B_1 = -1/2,\ B_2 = 1/6,\ B_3 = 0,$$
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$$B_0(x) = 1,\ B_1(x) = x - (1/2),\ B_2(x) = x^2 - x + (1/6),$$
$$B_3(x) = x^3 - (3/2)x^2 + (1/2)x,\ B_4(x) = x^4 - 2x^3 + x^2 - (1/30),$$
$$B_5(x) = x^5 - (5/2)x^4 + (5/3)x^3 - (1/6)x,$$
$$B_6(x) = x^6 - 3x^5 + (5/2)x^4 - (1/2)x^2 + (1/42),$$
$$B_7(x) = x^7 - (7/2)x^6 + (7/2)x^5 - (7/6)x^3 + (1/6)x,$$
$$B_8(x) = x^8 - 4x^7 + (14/3)x^6 - (7/3)x^4 + (2/3)x^2 - (1/30),$$

$$\ldots$$

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Let $1 \le j \le \ell$. *Define* $I_1^{(j)} = \{x_1, x_2, \ldots, x_{j-1}\}$, $I_2^{(j)} = \{x_{j+2}, x_{j+3}, \ldots, x_{\ell+1}\}$. *Let* $\sigma_k^{(j,s)}$ *denote the* ***elementary symmetric function*** *in the variables in* $I_s^{(j)}$ *of degree* k $(s = 1, 2\ ,\ k \in \mathbb{Z}_{\ge 0})$. *Let* ∂_i $(1 \le i \le \ell + 1)$ *and* ∂_z *denote* $\partial/\partial x_i$ *and* $\partial/\partial z$ *respectively. Define homogeneous derivations*
$\eta_1 := \sum_{i=1}^{\ell+1} \partial_i \in D(\mathcal{S}_\ell)$, $\eta_2 := z\partial_z + \sum_{i=1}^{\ell+1} x_i\partial_i \in D(\mathcal{S}_\ell)$, *and*

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$$\varphi_j^* := (x_j - x_{j+1} - z) \sum_{i=1}^{\ell+1} \sum_{\substack{0 \le k_1 \le j-1 \\ 0 \le k_2 \le \ell - j}} (-1)^{k_1+k_2} \sigma_{j-1-k_1}^{(j,1)} \sigma_{\ell-j-k_2}^{(j,2)} \overline{B}_{k_1,k_2}(x_i, z) \partial_i$$

for $1 \le j \le \ell$.