lifted generalized permutohedra and composition polynomials

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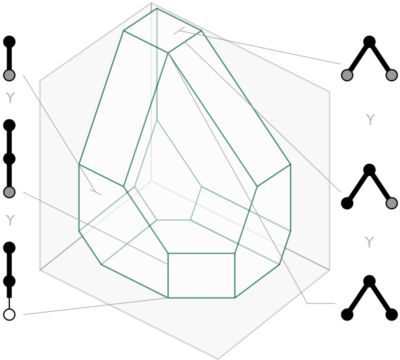
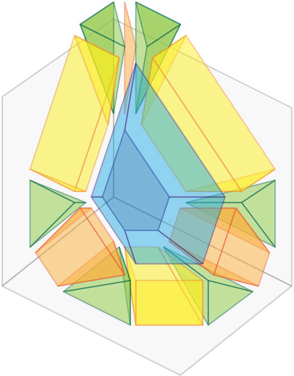




Joint work with: Jeff Doker (UC Berkeley)

Outline

1. Generalized permutahedra and trees
2. Lifted generalized permutahedra
3. The subdivision by compositions
4. Volume and composition polynomials

Generalized permutahedra and trees

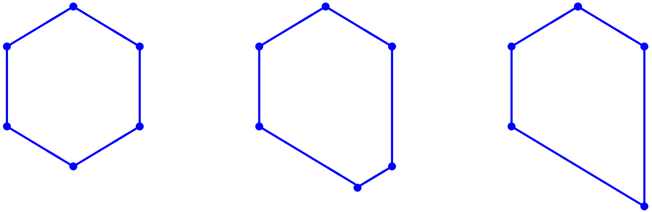
The permutahedron P_n is:

$$P_n = \text{conv}\left\{(\pi_1, \ldots, \pi_n) \ : \ \pi \text{ a permutation of } [n]\right\}$$

Inequality description:

$$\sum_{i=1}^{n} t_i = \binom{n+1}{2}, \qquad \sum_{i \in I} t_i \geq \binom{|I|+1}{2} \text{ for all } I \subseteq [n]$$

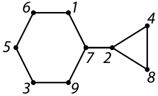
A generalized permutahedron is obtained from P_n by changing the edge lengths while preserving their directions.

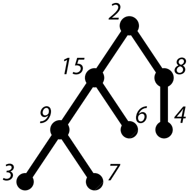


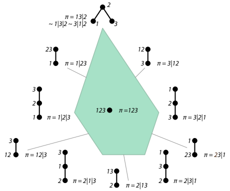
Important examples:

- polytope from empirical distributions (Pitman-Stanley)
- matroid polytope (Edmonds)
- associahedron $\mathcal{K}_n$ (Stasheff, Haiman)
- graph associahedron $\mathcal{K}G$ (Carr-Devadoss, A.-Reiner-Williams)
- nestohedron $\mathcal{K}B$ (Postnikov, Feichtner-Sturmfels)

graph associahedron $\mathcal{K}G$:







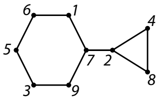
(G-trees) $\longleftrightarrow$ (faces of $\mathcal{K}G$)

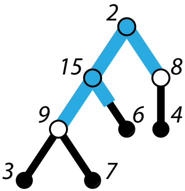
(For $G =$ path, G-trees $\leftrightarrow$ rooted planar trees on n leaves.)

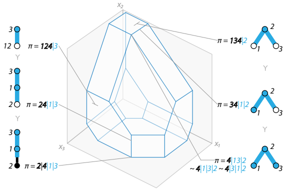
Some new examples:

- multiplihedron $\mathcal{J}_n$ (Stasheff, Forcey, A.-Doker)
- graph multiplihedron $\mathcal{J}G$ (Devadoss-Forcey, A.-Doker)

graph multiplihedron $\mathcal{J}_n$:







(painted G-trees) $\leftrightarrow$ (faces of $\mathcal{J}G$)

Question:

- Is the nestomultiplihedron $\mathcal{J}B$ a polytope? (Devadoss-Forcey)
  (B = building set $\rightarrow$ $\mathcal{J}B$ = polytope(?) of painted B-trees)

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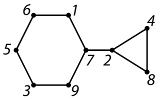

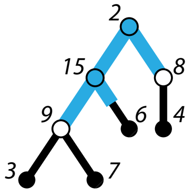

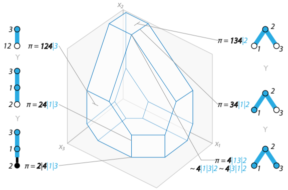

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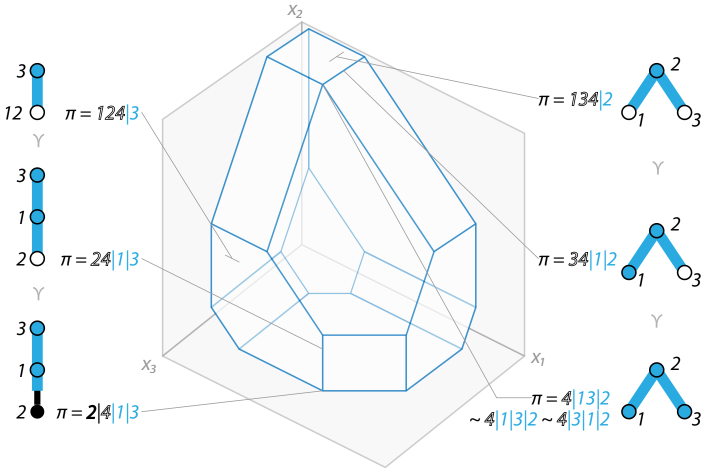
Theorem. (A. - Doker)
There is a generalized permutahedron (the nestomultiplihedron) whose face poset is isomorphic to the poset of painted B-trees.



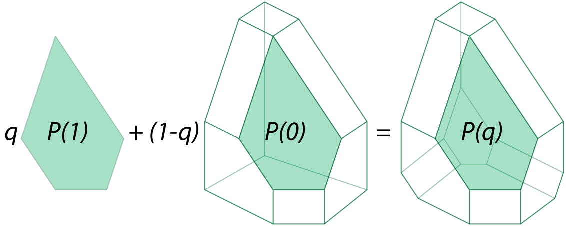
Constructing it: Lifting

Sketch of proof.
The q-lifting $P \mapsto P(q)$ (where $0 \leq q \leq 1$) takes a generalized permutahedron in $\mathbb{R}^n$ to a generalized permutahedron in $\mathbb{R}^{n+1}$.
We define:

$$P(q) := qP(1) + (1-q)P(0)$$

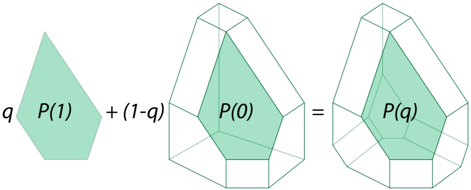
where

$$P(1) = P, \qquad P(0) = \{\mathbf{t} \in \mathbb{R}^n \mid \mathbf{0} \leq \mathbf{t} \leq \mathbf{x} \text{ for some } \mathbf{x} \in P\}$$



We show:

gen. perm. P	lifting $P(q)$
permutahedron P_n	permutahedron P_{n+1}
matroid polytope P_M	independent set polytope I_M $(q = 0)$
associahedron $\mathcal{K}_n$	multiplihedron $\mathcal{J}_n$
graph associahedron $\mathcal{K}G$	graph multiplihedron $\mathcal{J}G$
nestohedron $\mathcal{KB}$	nestomultiplihedron $\mathcal{JB}$



Theorem. (A. - Doker)
There is a generalized permutahedron (the nestomultiplihedron) whose face poset is isomorphic to the poset of painted B-trees.

A subdivision.

$\mathrm{vol}(P(q))$ is polynomial in q. To get a handle on it, subdivide.

For each ordered partition $\pi = B_1|\cdots|B_k$ of $[n]$, let

- P_π be the π-minimal face of P

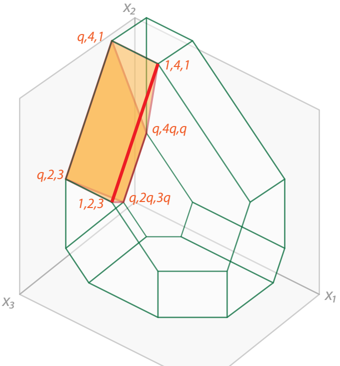
  (minl in direction $w \in \mathbb{R}^n$ where $w_{B_1} < \ldots < w_{B_k}$.) ($P =$ "front facet")
- P_π^i be obtained from P_π by scaling coords. $B_1 \cup \cdots \cup B_i$ by q.
- $P^\pi(q) = \mathrm{conv}(P_\pi^0, P_\pi^1, \ldots, P_\pi^k)$.

Example.

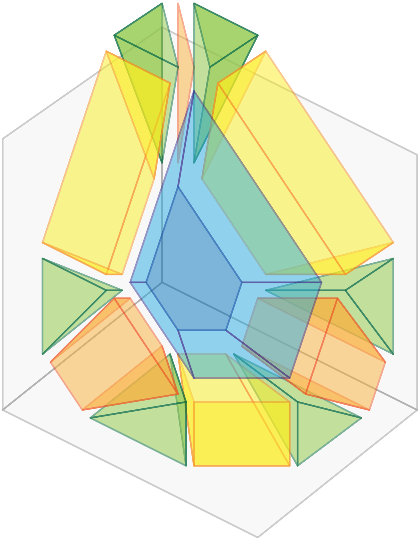
$P = \mathcal{K}(4)$ (associahedron)

$\pi = 1|23$

- $P_\pi = \mathrm{conv}\{(1,2,3),(1,4,1)\}$
- $P_\pi^1 = \mathrm{conv}\{(q,2,3),(q,4,1)\}$
- $P_\pi^2 = \mathrm{conv}\{(q,2q,3q),(q,4q,q)\}$



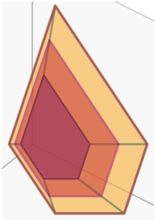
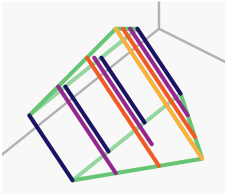
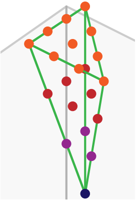
Theorem. (A. - Doker)
The polytopes $P^{\pi}(q)$ form a subdivision of $P(q)$ as π ranges over the ordered partitions of $[n]$.



Volumes and composition polynomials

We get $\mathrm{vol}(P(q)) = \sum_\pi \mathrm{vol}(P^\pi(q))$. What is $\mathrm{vol}(P^\pi(q))$?

- Combinatorially, $P^\pi(q) \cong \Delta_k \times P_\pi$. ($\Delta_k$ = simplex)

- There is a projection $f : P^\pi(q) \to \Delta_k$ whose fibers $f^{-1}(p)$ are predictable modifications of P_π. Integrating over Δ_k,

$$\mathrm{vol}(P^\pi(q)) = z_\pi \mathrm{vol}_{n-k}(P_\pi) \int_q^1 \int_q^{t_k} \cdots \int_q^{t_2} t_1^{|B_1|-1} \cdots t_k^{|B_k|-1} dt_1 \cdots dt_k$$

where $\pi = B_1 | \cdots | B_k$.

Composition polynomials.

For a composition $c = (c_1, \ldots, c_k)$, write $\mathbf{t}^{\mathbf{c}-\mathbf{1}} := t_1^{c_1-1} \cdots t_k^{c_k-1}$.
The **composition polynomial** $g_c(q)$ is

$$g_c(q) := \int_q^1 \int_q^{t_k} \cdots \int_q^{t_2} \mathbf{t}^{\mathbf{c}-\mathbf{1}} dt_1 \cdots dt_k.$$

- $g_{(1,1,1,1)}(q) = \frac{1}{24}(1-q)^4$.
- $g_{(2,2,2,2)}(q) = \frac{1}{384}(1-q)^4(1+q)^4$.
- $g_{(1,2,2)}(q) = \frac{1}{120}(1-q)^3(8+9q+3q^2)$.
- $g_{(2,2,1)}(q) = \frac{1}{120}(1-q)^3(3+9q+8q^2)$.
- $g_{(5,3)}(q) = \frac{1}{120}(1-q)^2(5+10q+15q^2+12q^3+9q^4+6q^5+3q^6)$.

Proposition. $g_c(q)$ is a polynomial of degree n satisfying:

1. $g_{\mathrm{reverse}(c)}(q) = q^n g_c(1/q)$.
2. $g_{mc}(q) = \frac{1}{m^k} g_c(q^m)$ for $m \in \mathbb{N}$.

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Theorem. (A. - Doker) Let $c = (c_1, \ldots, c_k)$ be a composition.

1. $g_c(q) = (1-q)^k f_c(q)$ for a poly. $f_c(q)$ with $f_c(1) = 1/k!$
2. The coefficients of $f_c(q)$ are positive.

Proof: The "easy" recurrences don't suffice. With some work,

$$g_{c^m}(q) = \left(\frac{c_1 + \cdots + c_m}{c_1 + \cdots + c_k}\right) g_{c^R}(q) + \left(\frac{c_{m+1} + \cdots + c_k}{c_1 + \cdots + c_k}\right) q^{c_1} g_{c^L}(q).$$

$c^m := (c_1, \ldots, c_{m-1}, c_m + c_{m+1}, c_{m+2}, \ldots, c_k)$, $\quad c^L := (c_2, \ldots, c_k)$, $\quad c^R := (c_1, \ldots, c_{k-1})$

Question. Are the coefficients of $f_c(q)$ unimodal? Log-concave?

(True for all 335,922 compositions of ≤ 7 parts which are ≤ 6.)

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Polynomial interpolation of exponential functions.

Theorem. (A. - Doker)
Let $c = (c_1, \ldots, c_k)$ be a composition.
Let $\beta_i = c_1 + \cdots + c_i$ be the partial sums. $(i = 0, 1, \ldots, k)$

Let $h(x) = a_0 + a_1 x + \cdots + a_k x^k$ be the polynomial of smallest degree that passes through the $k+1$ points (β_i, q^{β_i}). (Here the coefficients a_i are functions of q.) Then

$$a_k = (-1)^k g_c(q).$$

Proof.

- The recursion gives an explicit formula for $g_c(q)$.
- Lagrange interpolation gives a formula for a_k.
- They match.

Question. What's the real reason for this?

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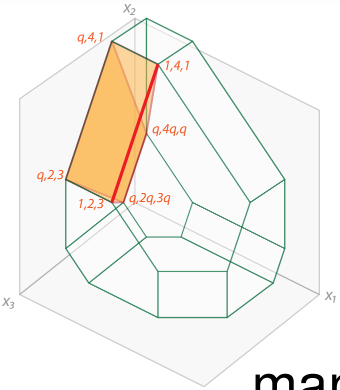


many thanks.

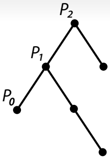
The paper is at:

`math.sfsu.edu/federico`

`arxiv.org`

Advances in Applied Math., to appear.

Linear extensions in posets. (Stanley)



Poset P_c:

- a chain $p_0 < p_1 < \cdots < p_k$
- a chain of size $c_i - 1$ below p_i for $1 \leq i \leq k$

Order polytope $\mathcal{O}(P_c)$: $0 \leq x_i \leq x_j \leq 1$ for $i \leq j \in P$

Then we have:

$$\mathrm{vol}(\mathcal{O}(P_c) \cap (x_{p_0} = q)) = \frac{g_c(q)}{(c_1 - 1)! \cdots (c_k - 1)!}$$

which implies:

$$g_c(q) = \frac{(c_1 - 1)! \cdots (c_k - 1)!}{n!} \sum_{i=0}^{n} N_{i+1} \binom{n}{i} q^i (1-q)^{n-i}$$

where N_j = number of linear extensions f of P_c with $f(x_0) = j$.

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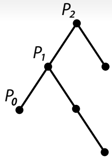

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