# A simple model of trees for unicellular maps

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joint work with
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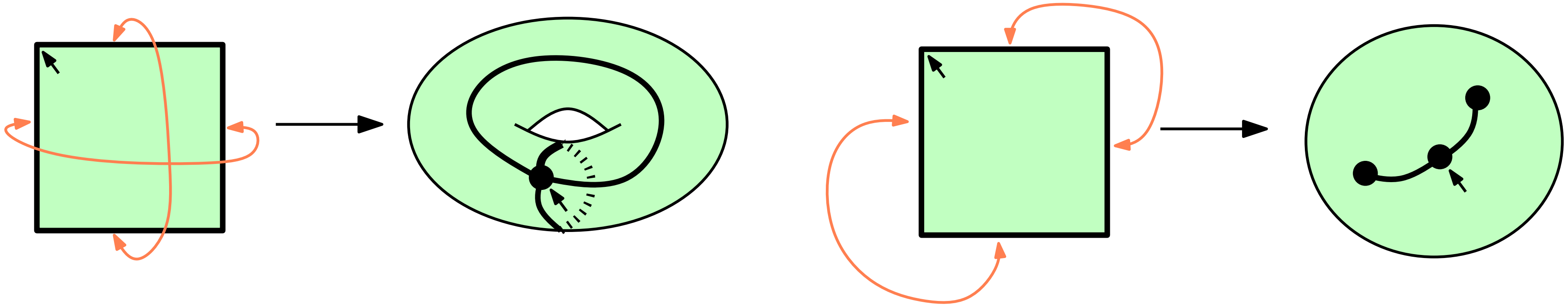
# Unicellular maps (a.k.a. “one-face” maps)

- Start with a (rooted) $2n$-gon, and paste the edges pairwise in order to form an orientable surface.



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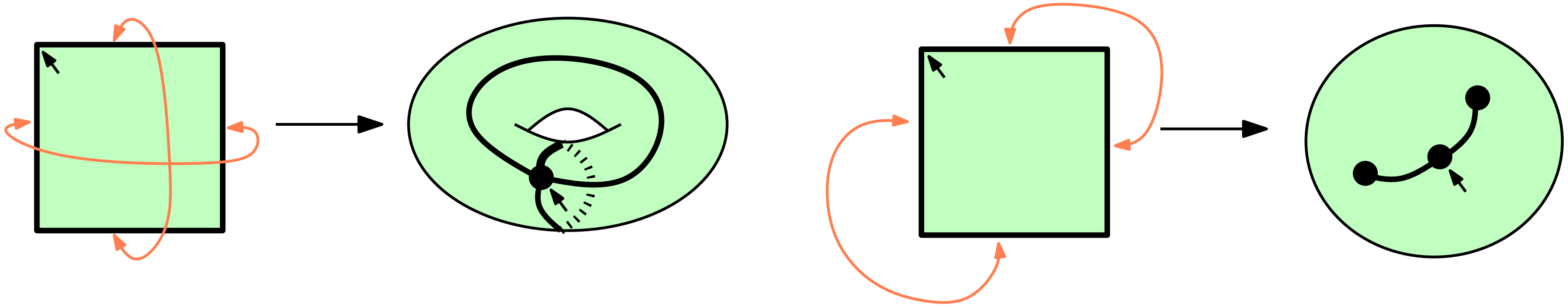

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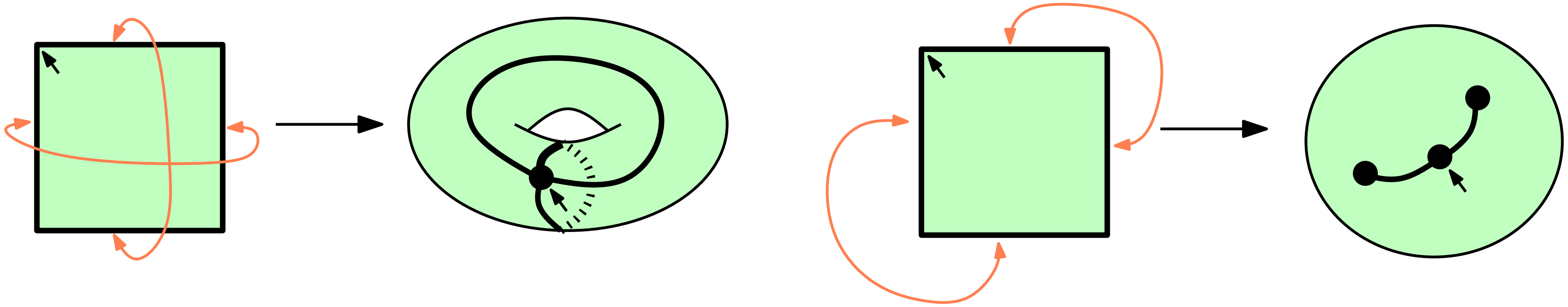

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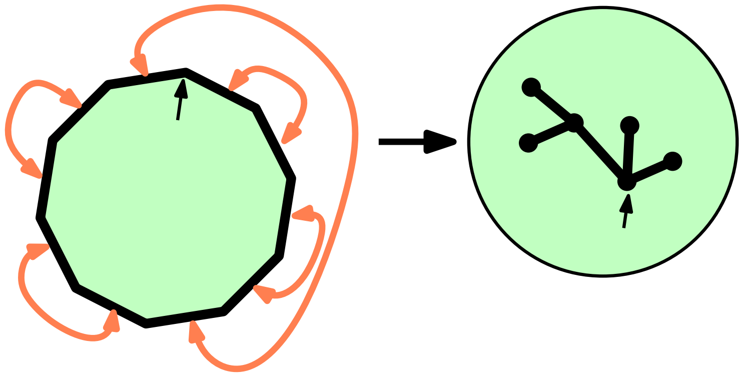
- The number of unicellular maps of size $n$ is $(2n-1)!!$

- What if we fix the genus ? For example, on the sphere (genus $0$), unicellular maps = plane trees... so there are $\mathrm{Cat}(n)$ of them.



# Unicellular maps: counting!

- Let $\epsilon_g(n)$ be the number of unicellular maps with $n$ edges and genus $g$.

- Are these numbers interesting ? Yes!

$$\begin{aligned}
\epsilon_0(n) &= \mathrm{Cat}(n)\\
\epsilon_1(n) &= \frac{(n+1)n(n-1)}{12}\mathrm{Cat}(n)\\
\epsilon_2(n) &= \frac{(n+1)n(n-1)(n-2)(n-3)(5n-2)}{1440}\mathrm{Cat}(n)\\
\dots
\end{aligned}$$

- These numbers are connection coefficients in $\mathcal{Z}(\mathbb{C}[\mathfrak{S}_n])$ (all map numbers are, more or less - but this is not really the point of this talk).

# Unicellular maps: some chosen formulas

[Lehman-Walsh 72]

$$\epsilon_g(n) = \left( \sum_{\gamma \vdash g} \frac{(n+1-2g)_{2\ell(\gamma)+1}}{2^{2g} \prod_i m_i!(2i+1)^{m_i}} \right) \mathrm{Cat}(n)$$

no bijective proof!

[Harer-Zagier 86]
(summation form)

$$\sum_{g \geq 0} \epsilon_g(n) y^{n+1-2g} = (2n-1)!! \sum_{i \geq 1} 2^{i-1} \binom{n}{i-1} \binom{y}{i}$$

nice bijective proof [Bernardi10] building on [Lass 01, Goulden Nica 05]

[Harer-Zagier 86] (recurrence form)

$$(n+1)\epsilon_g(n) = 2(2n-1)\epsilon_g(n-1) + (n-1)(2n-1)(2n-3)\epsilon_{g-1}(n-2)$$

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[Goupil-Schaeffer 98] for $\lambda \vdash 2n$, $\lambda = 1^{m_1} 2^{m_2} \ldots$ :

$$\epsilon_g(n;\lambda) = \frac{(l+2g-1)!}{2^{2g-1} \prod_i m_i!} \sum_{\gamma_1+\gamma_2+\dots+\gamma_l=g} \prod_i \frac{1}{2\gamma_i+1} \binom{\lambda_i - 1}{2\gamma_i}$$

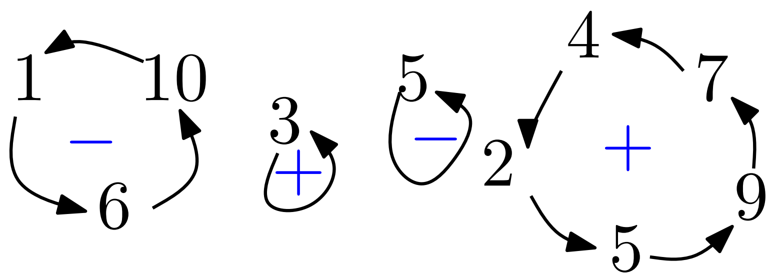
vertex degrees

no bijective proof!

# Our result, in short

- A C-permutation of a set $S$:
  - all cycles have odd length
  - each cycle carries a sign in $\{+,-\}$
  - its genus is $g := \sum_i k_i$ where $(2k_i+1)$ are the cycle-lengths

$$\#\text{cycles} = |S| - 2g$$



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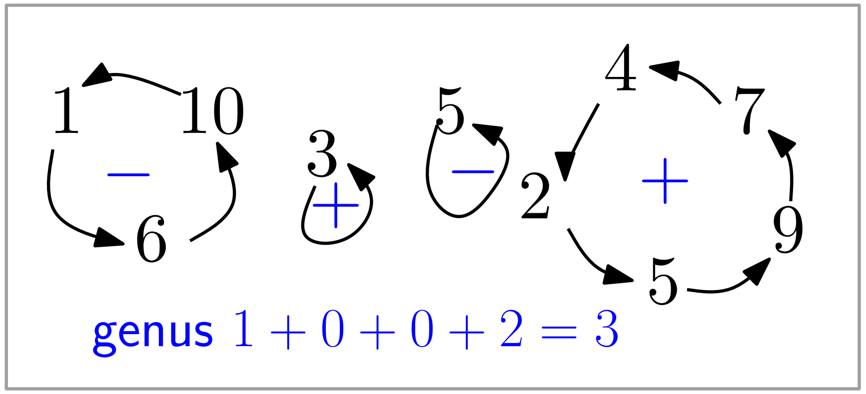

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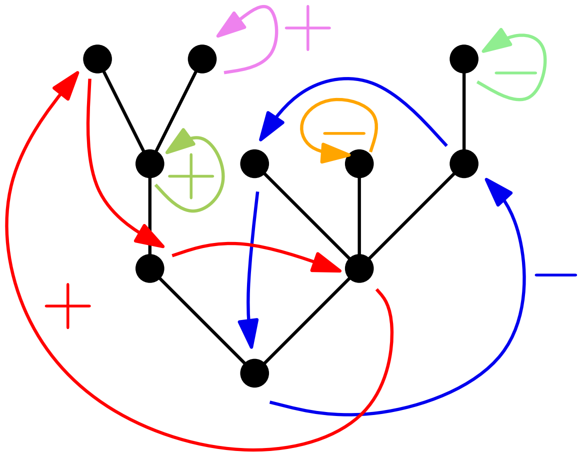
- C-decorated tree = a rooted plane tree equipped with a C-permutation of its vertices:



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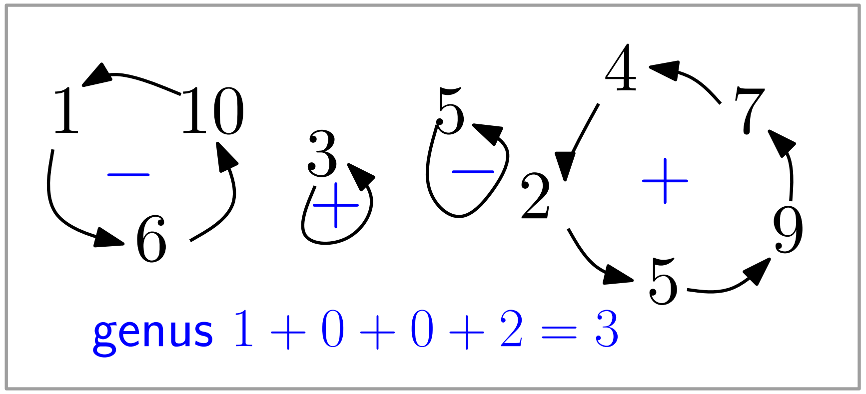

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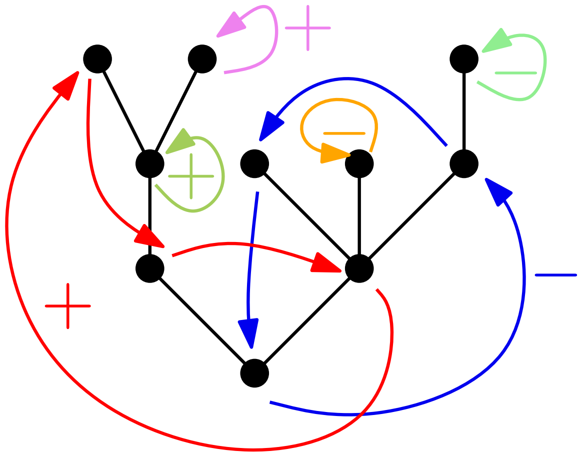

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- **Theorem** [C., Féray, Fusy] (our main result!)

There is a $2^{n+1}$-to-1-jection between unicellular maps of size $n$ and C-decorated trees with $n$ edges. It preserves the genus.

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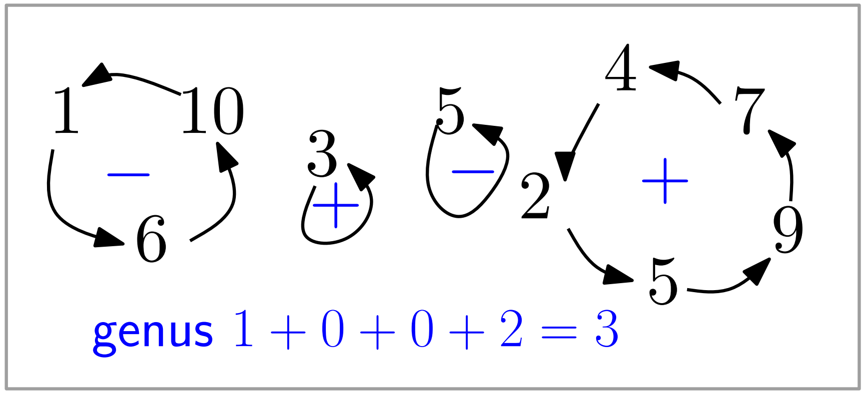

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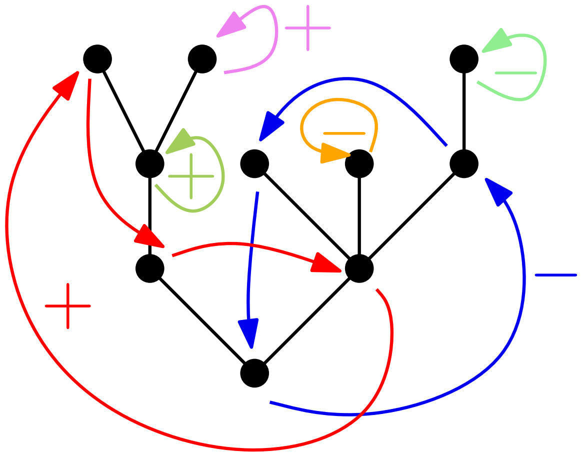

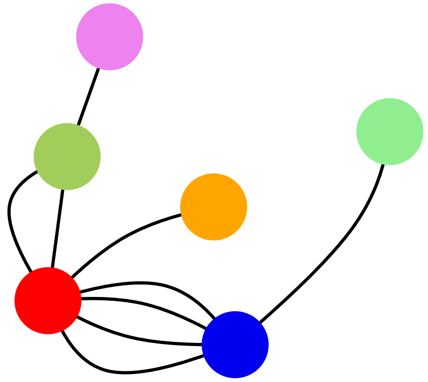

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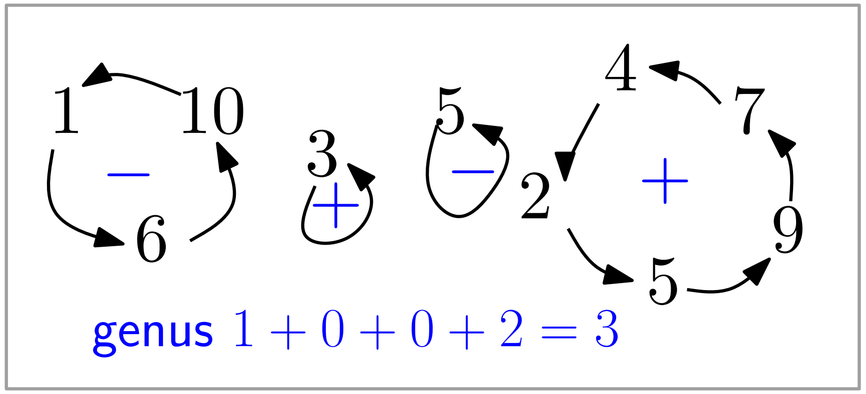

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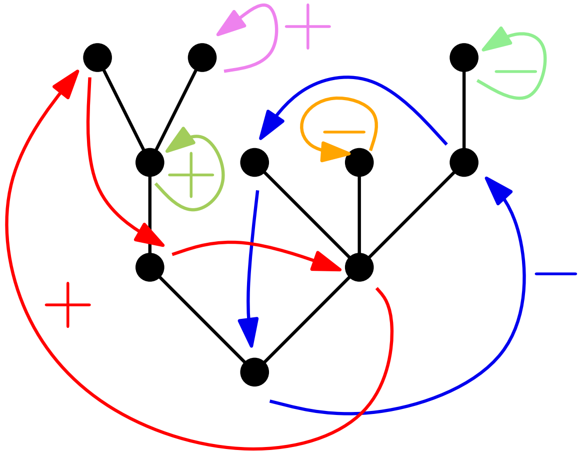

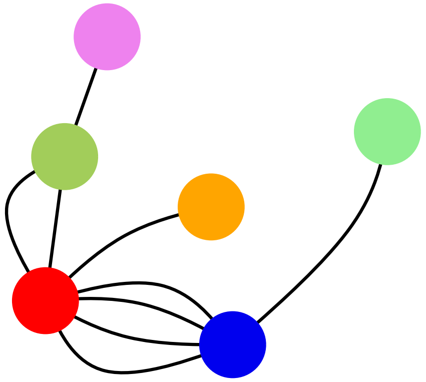

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FROM THERE ALL KNOWN FORMULAS FOLLOW, BIJECTIVELY!

# First ingredient: a bijection from FPSAC'09 (1)

Let $\mathcal{E}_g^{(k)}(n)$ = unicellular maps, genus $g$, $n$ edges, $k$ marked vertices.

- **Theorem** [Ch.09] There is an explicit $2g$-to-$1$-jection that realizes:
$$2g \cdot \mathcal{E}_g(n) = \mathcal{E}_{g-1}^{(3)}(n) + \mathcal{E}_{g-2}^{(5)}(n) + \cdots + \mathcal{E}_0^{(2g+1)}(n)$$

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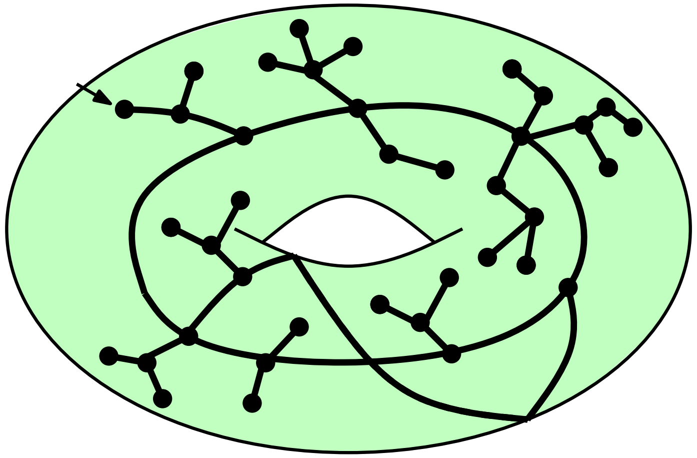

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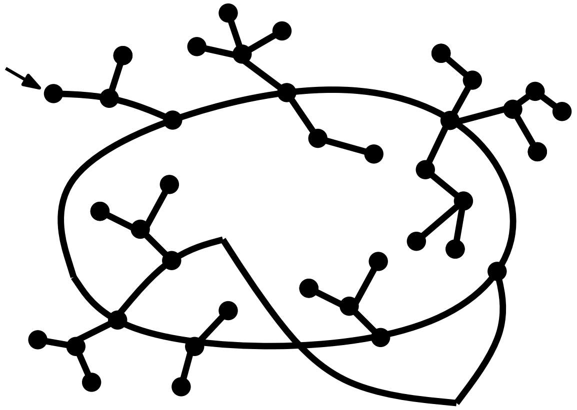

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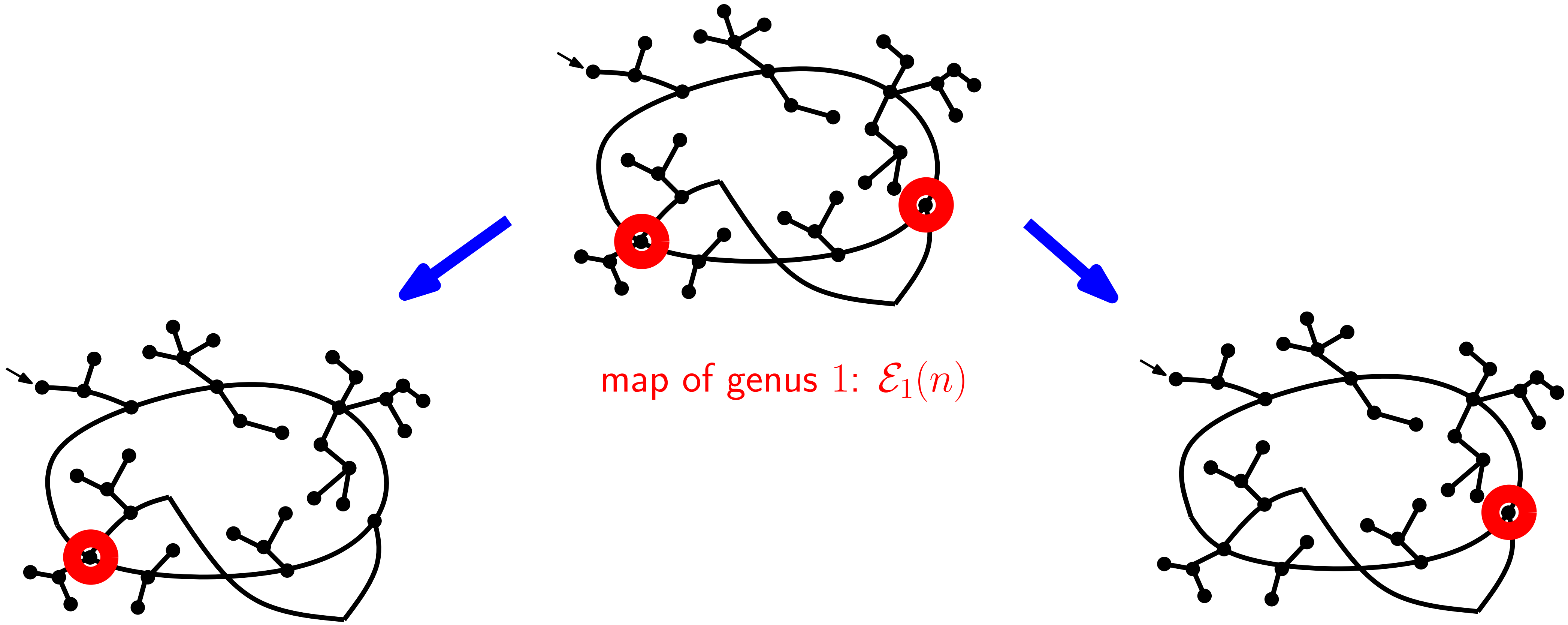

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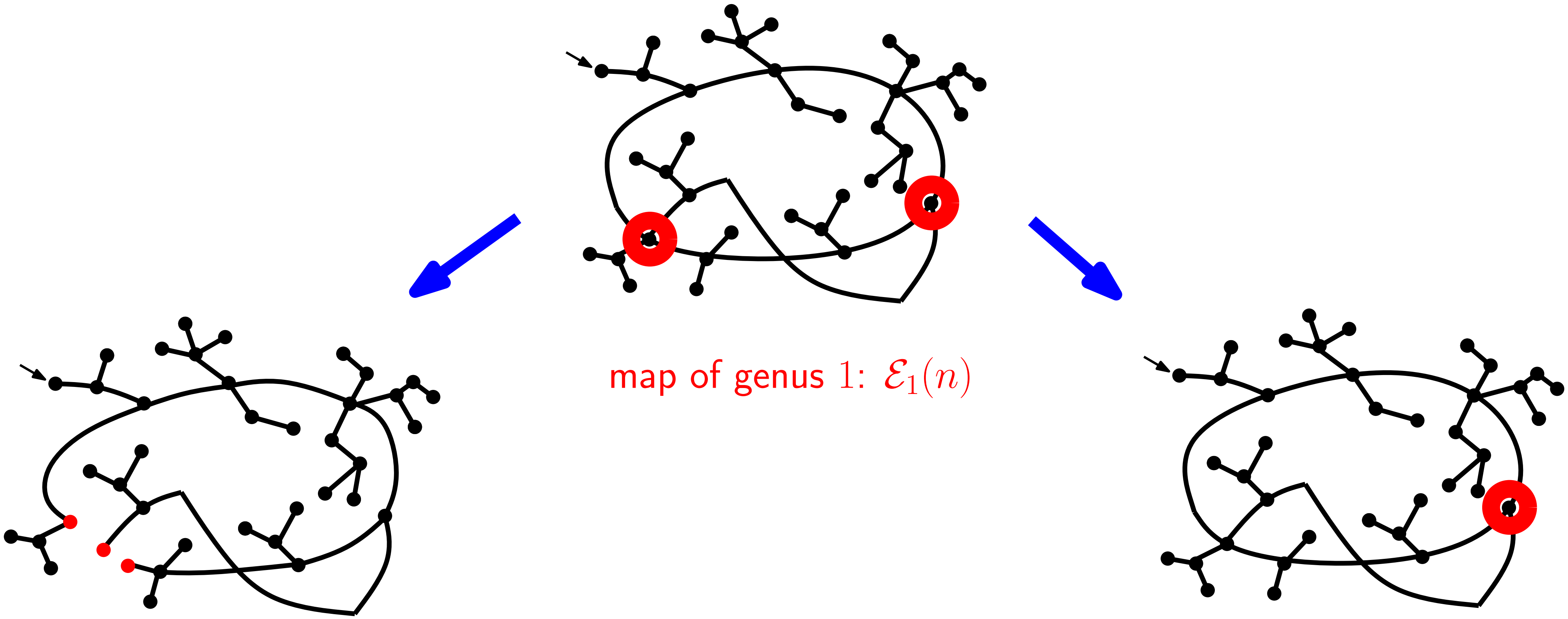

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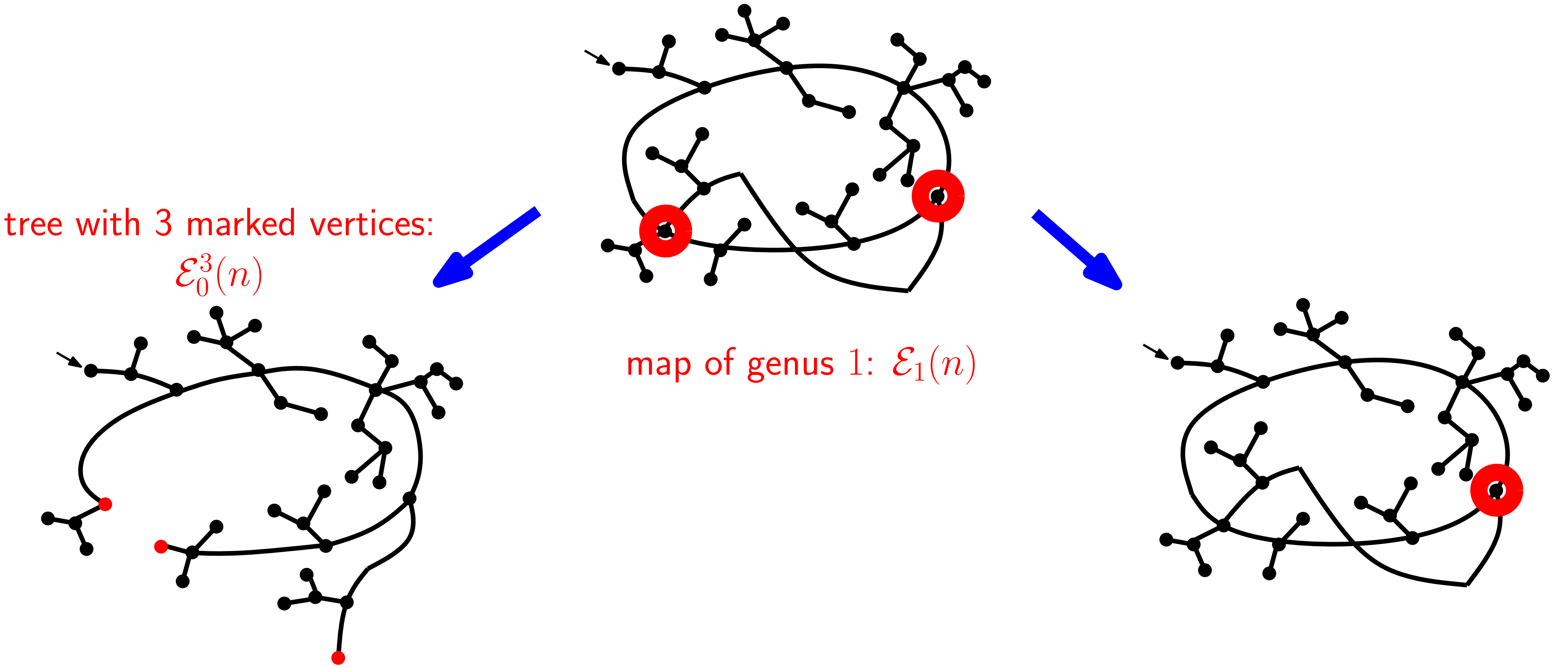

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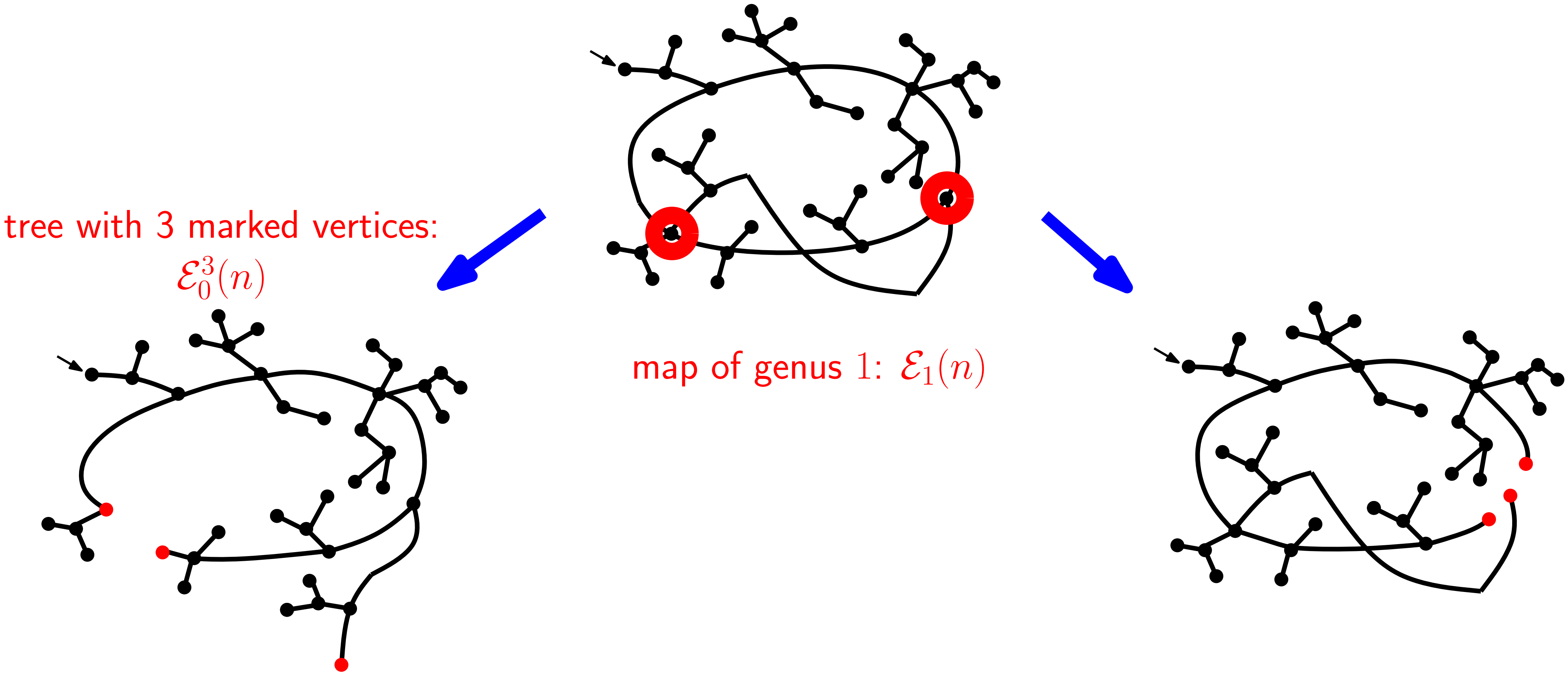

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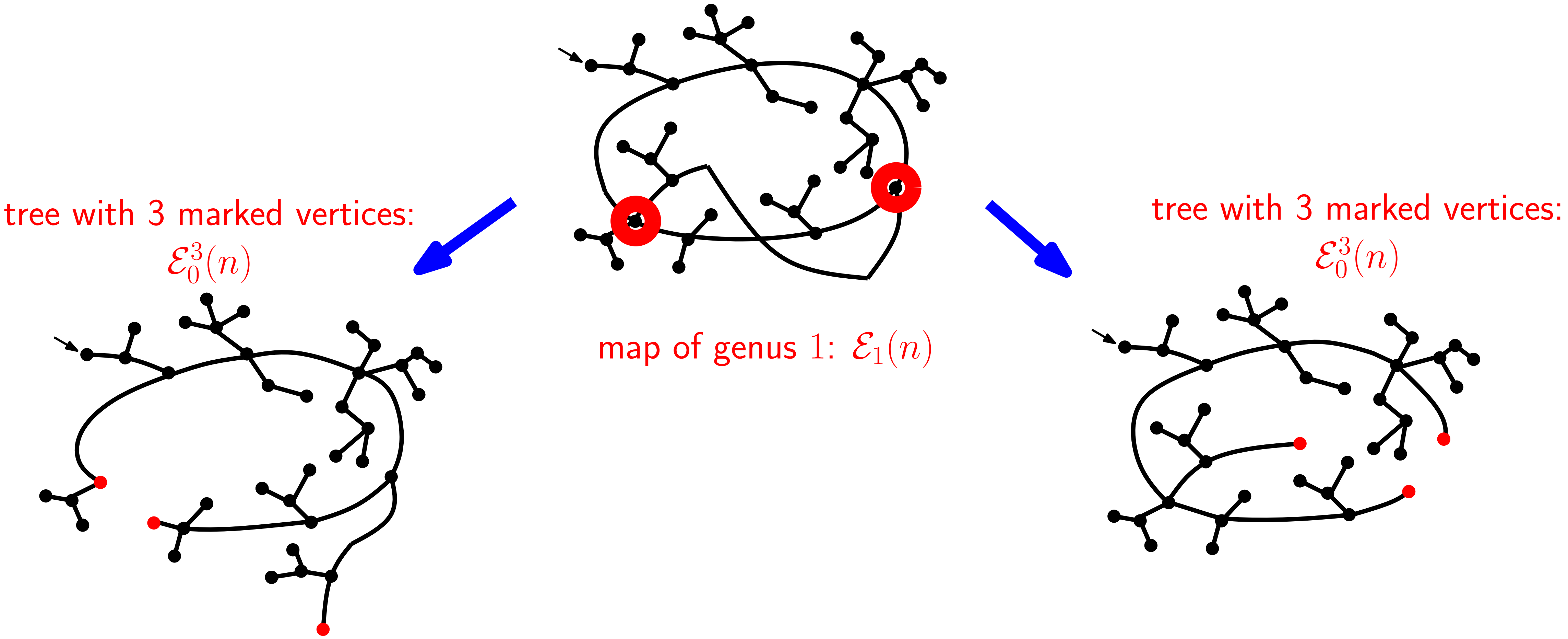

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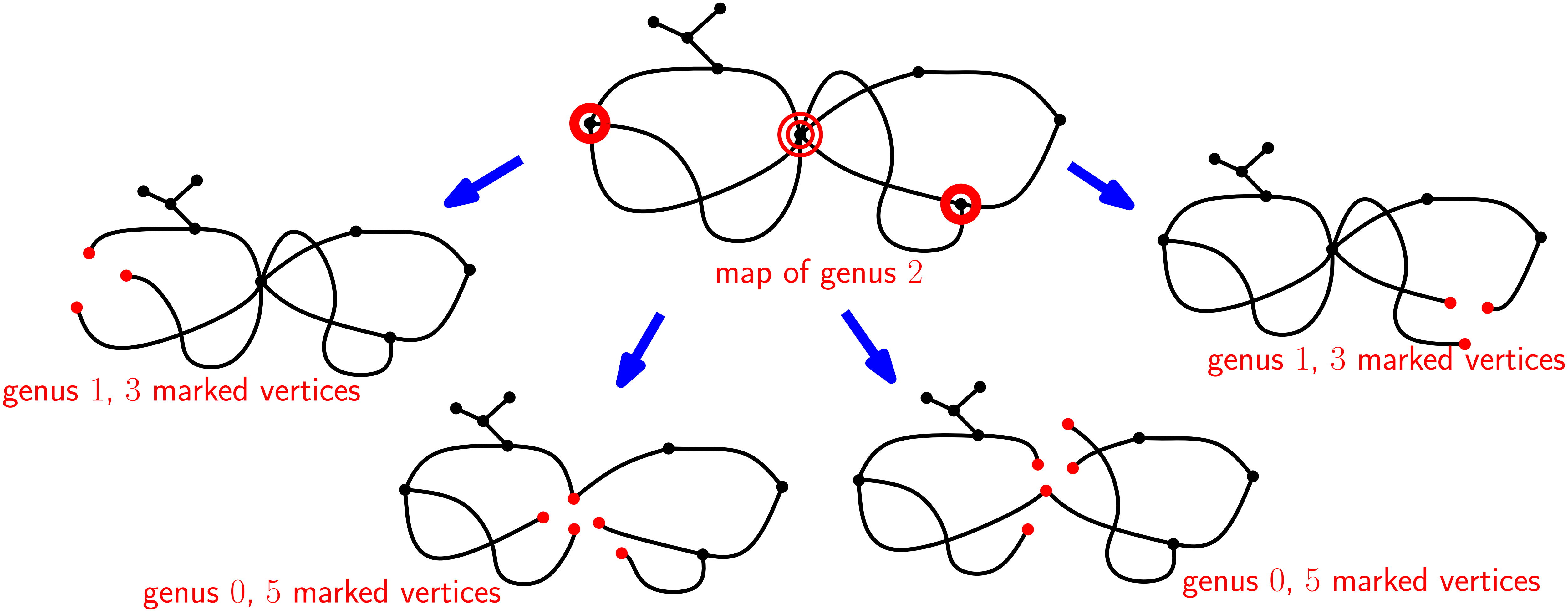

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- **Corollary:** $\epsilon_g(n) = P_g(n) \times \mathrm{Cat}(n)$ where the polynomial $P_g$ is defined recursively:

$$2g \cdot P_g(n) = \binom{n+3-2g}{3} P_{g-1}(n) + \binom{n+5-2g}{5} P_{g-2}(n) + \cdots + \binom{n+1}{2g+1} P_0(n)$$

...but now we can say more !

# $C$-permutations solve the recurrence combinatorially

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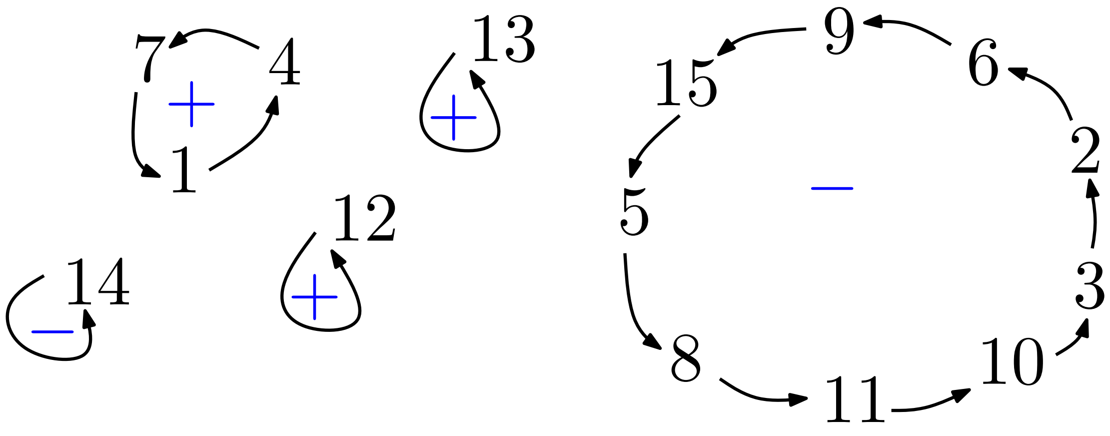
- **Sketch (left-to-right):** Start with a $C$-permutation of genus $g$.
  $2g = (\#\text{elements}) - \#(\text{cycles})$ is the number of non-minimal elements



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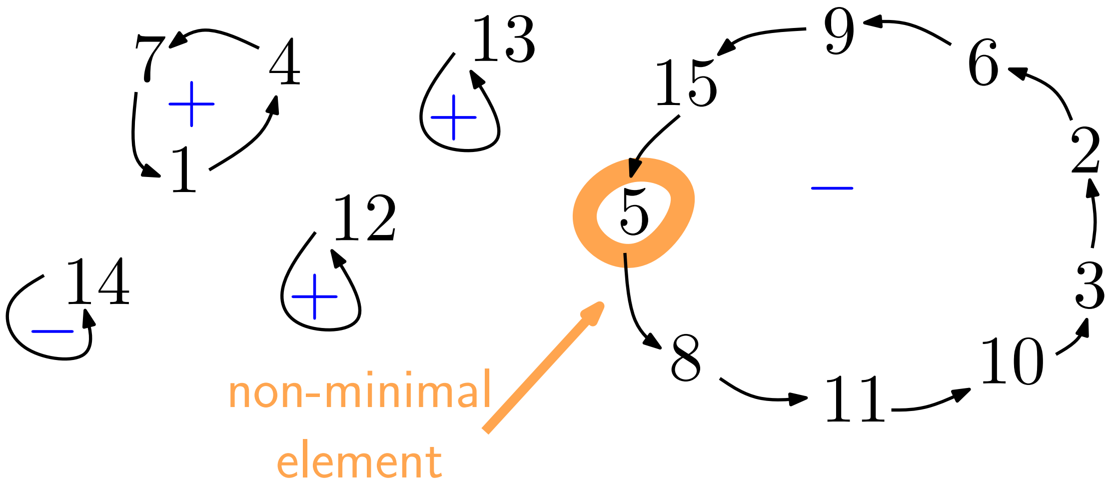

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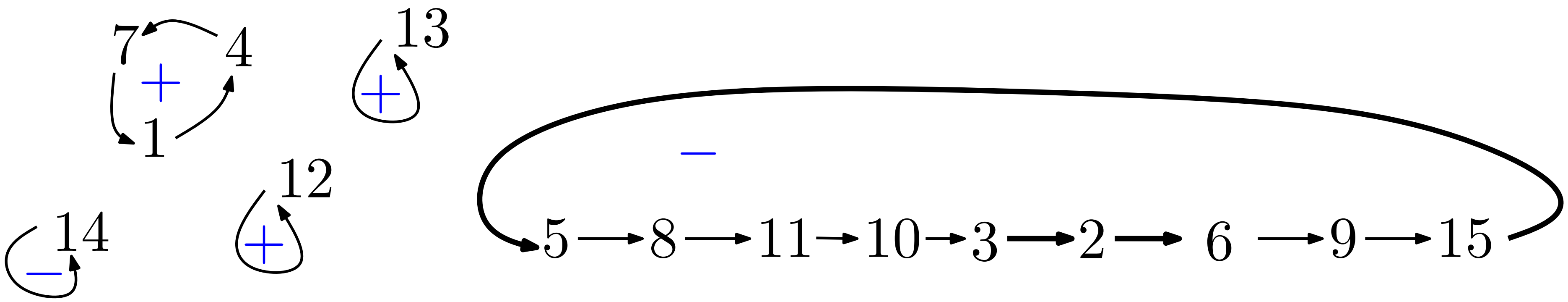

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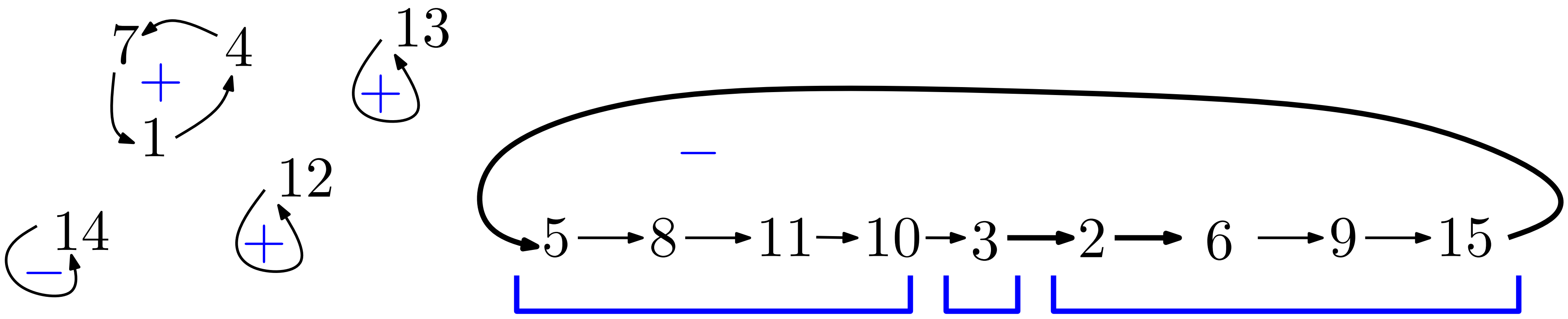

1. look at runs separating lower records
2. some may have even length...

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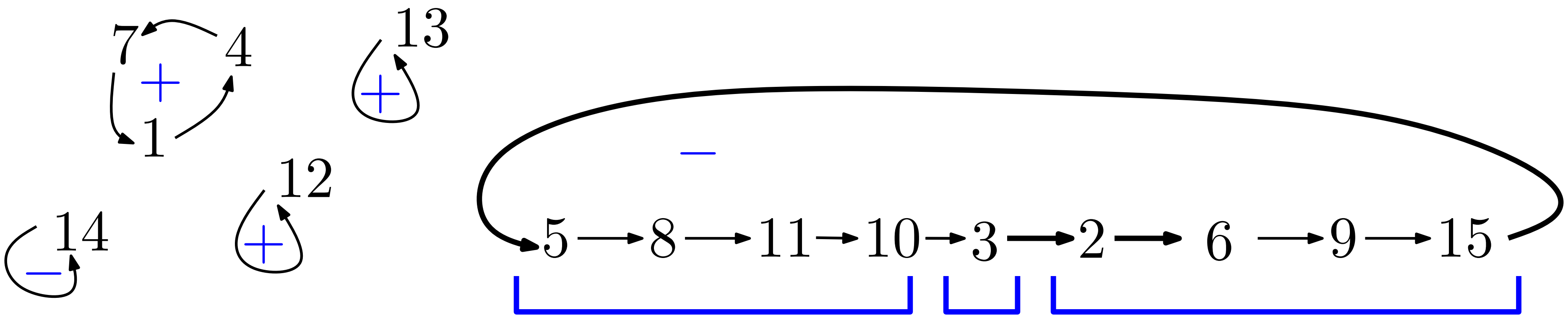

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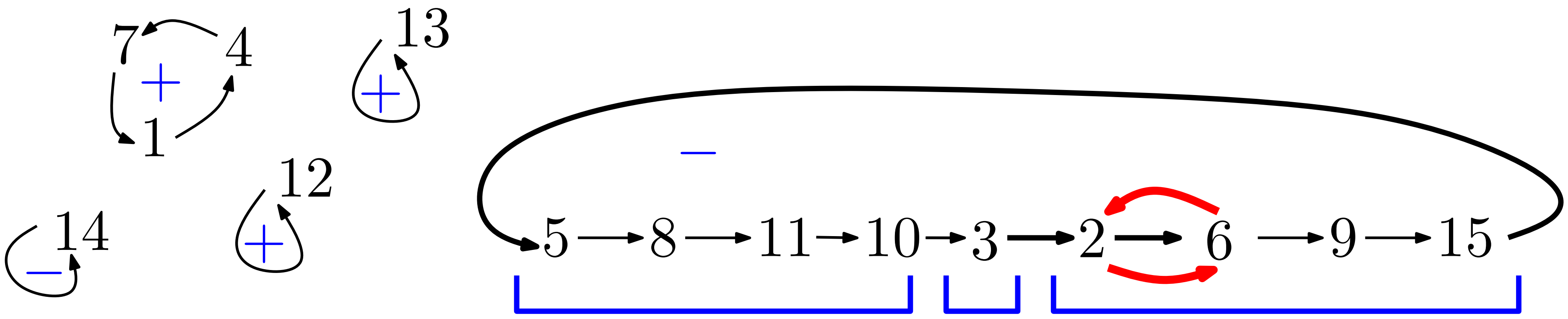

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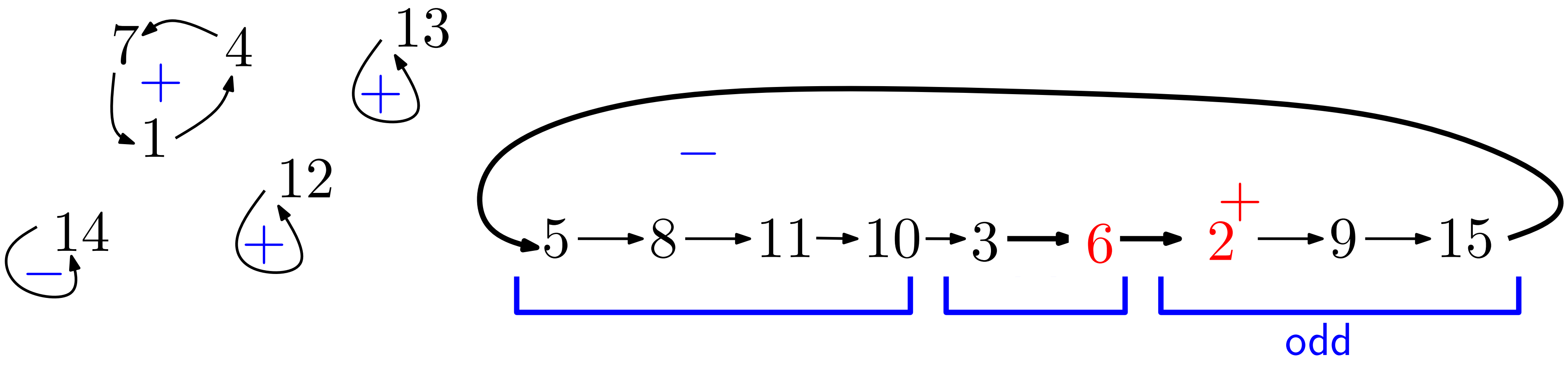

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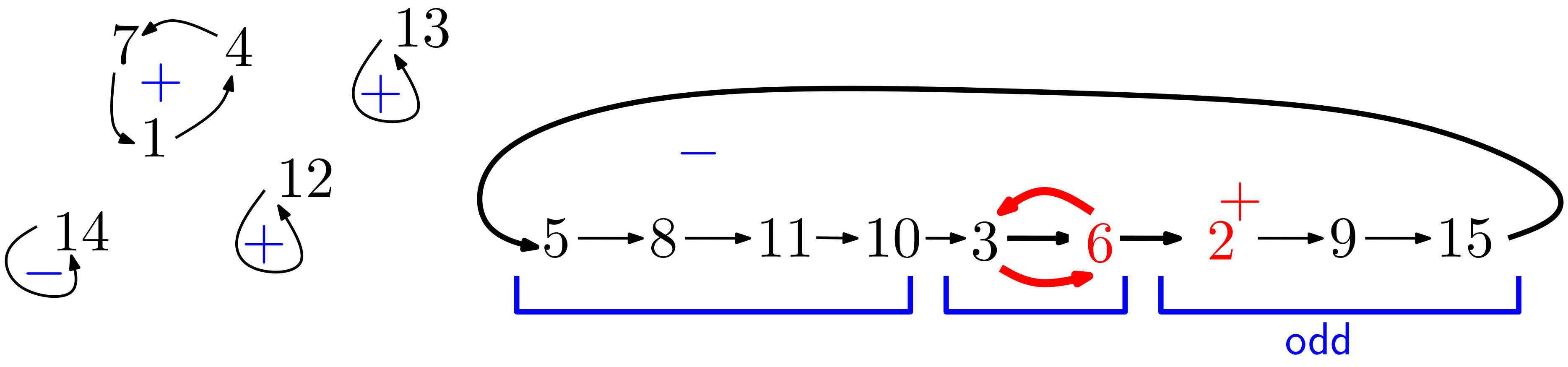

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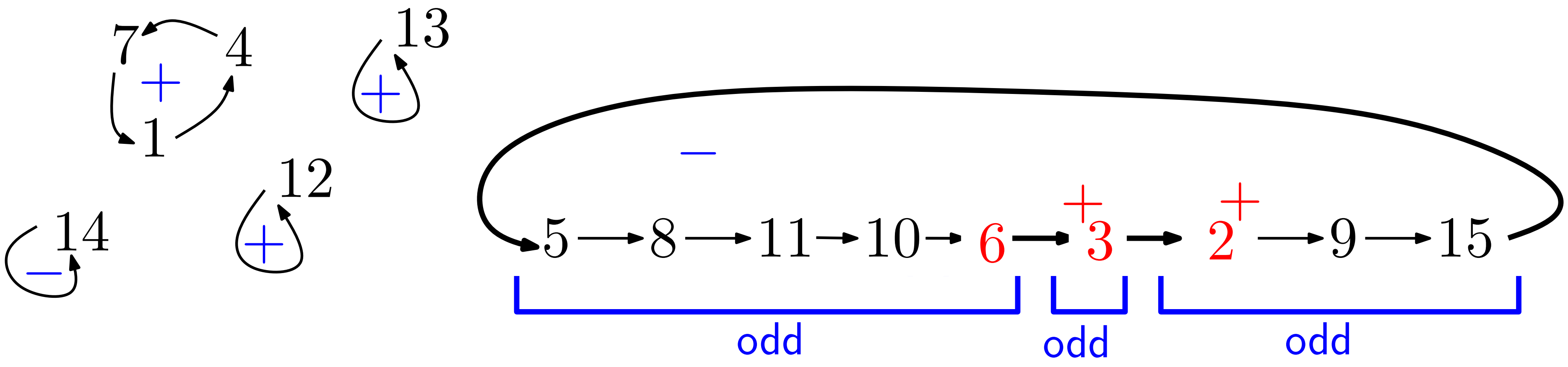

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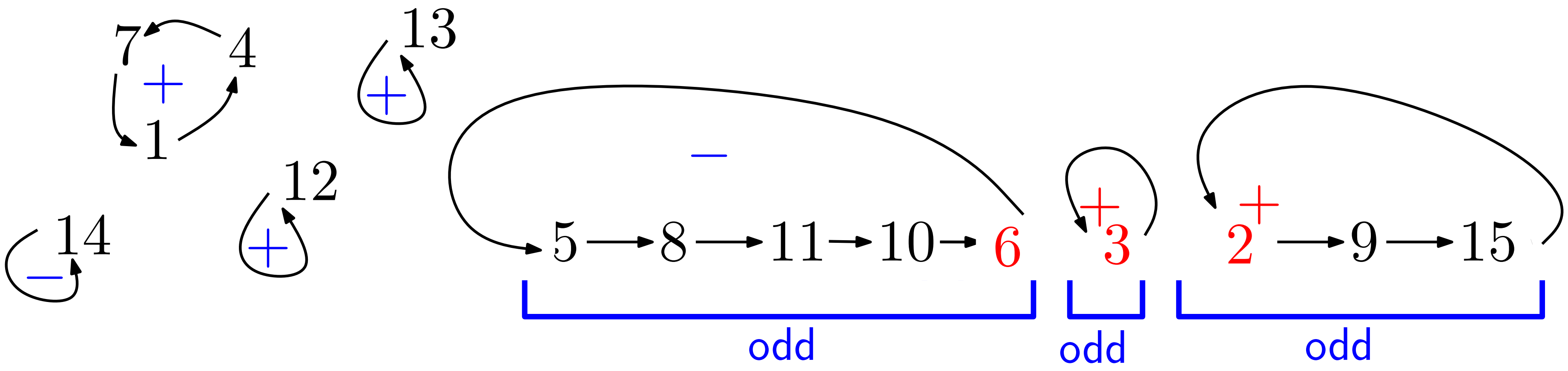

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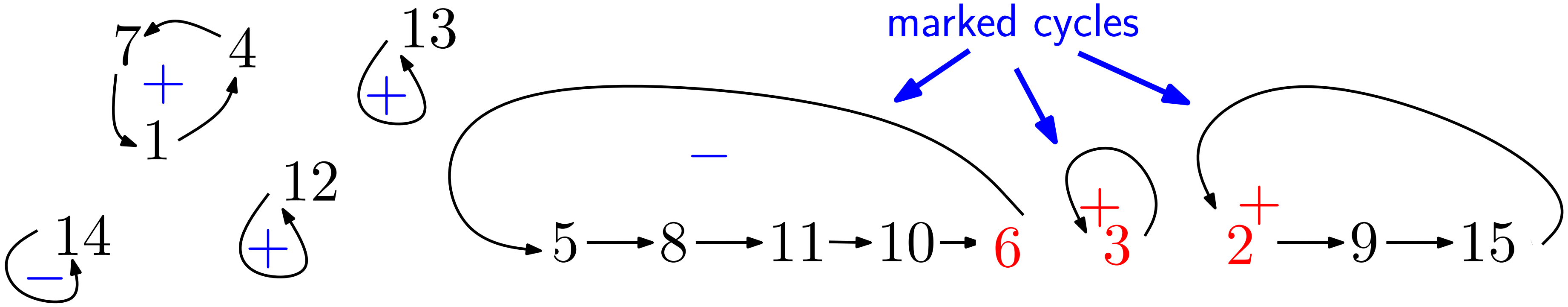

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# Counting $C$-decorated trees is straightforward

- **Theorem** [C., Féray, Fusy]

The number of unicellular maps of genus $g$ with $n$ edges satisfies:

$$2^{n+1}\epsilon_g(n) = C_g(n+1)\mathrm{Cat}(n)$$

where $C_g(n+1)$ is the number of $C$-perm. of genus $g$ on $n+1$ elements.

- but $C_g(n+1)$ = easy numbers!
  - $C_0(n+1) = 2^{n+1}$

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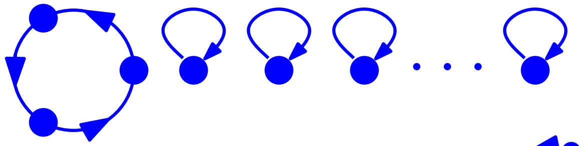
  - $C_1(n+1) = \frac{(n+1)n(n-1)}{3}2^{n-1}$



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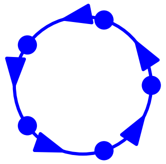

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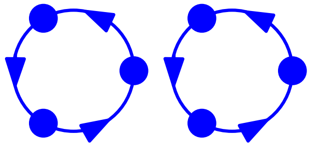

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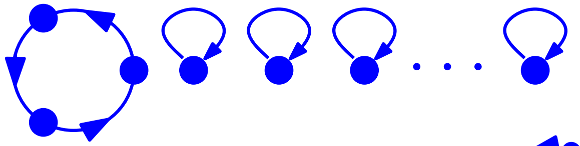

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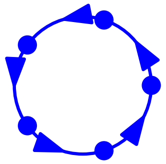

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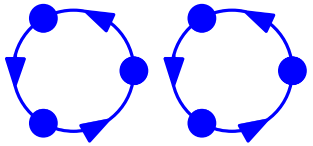

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- In general: $$C_g(n+1) = \left(\sum_{\gamma \vdash g} \frac{(n+1-2g)_{2\ell(\gamma)+1}}{\prod_i m_i!(2i+1)^{m_i}}\right) 2^{n+1-2g}$$

sum is over the cyle type of the $C$-permutation: $(2\gamma_i + 1) =$ cycle lengths.

$\surd$ Lehman-Walsh formula !

# Conclusion

- It is a series of exercises to recover ALL the known formulas concerning unicellular maps, bijectively with C-decorated trees. You just need to know your classics (count permutations, count trees...). Take a look at the paper!

- For example the beautiful Harer-Zagier recurrence formula has been waiting for a combinatorial interpretation since 1986...

# Harer-Zagier recurrence formula (1986)

- Classic: for $g = 0$, Rémy's bijection [Rémy 85]

$$(n+1)\mathrm{Cat}(n) \quad = \quad 2 \times (2n-1)\mathrm{Cat}(n-1)$$

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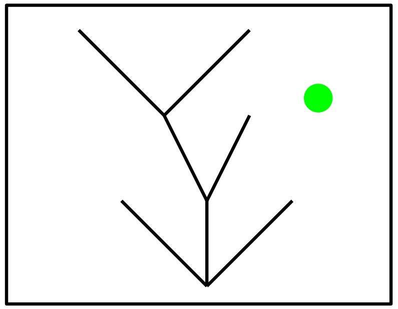
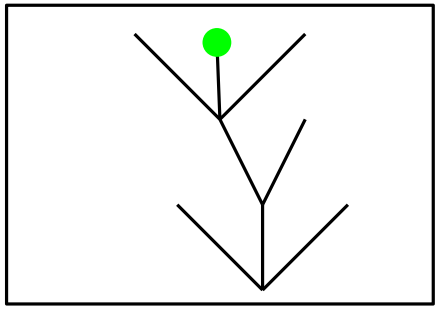


rooted tree, $n$ edges, one marked vertex



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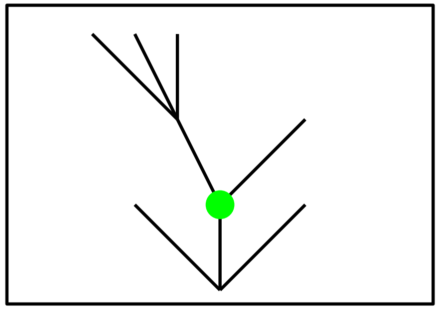

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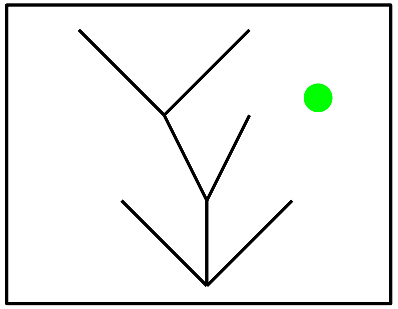

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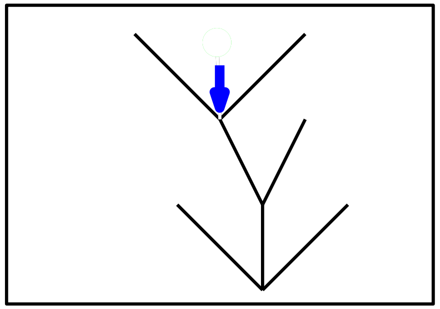

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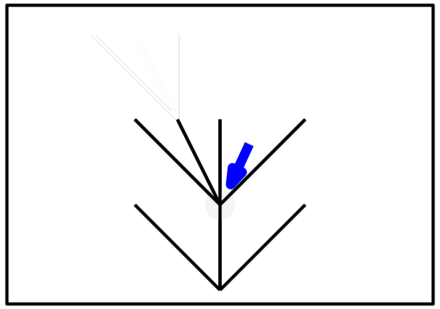

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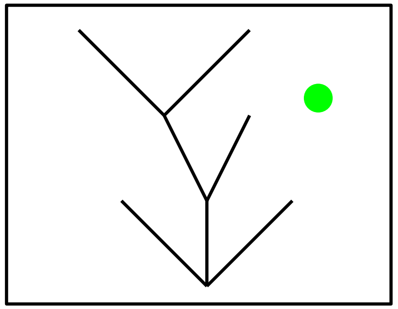

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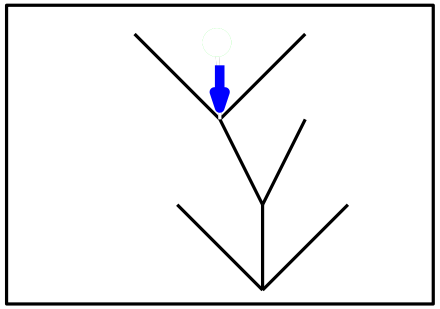

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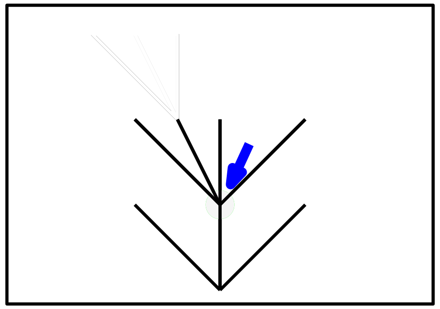

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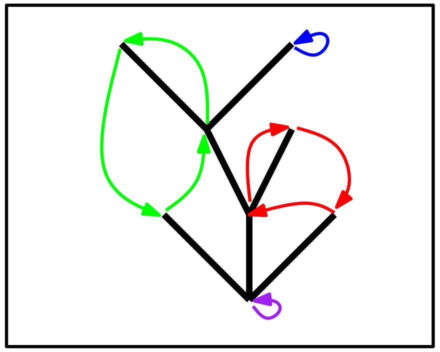

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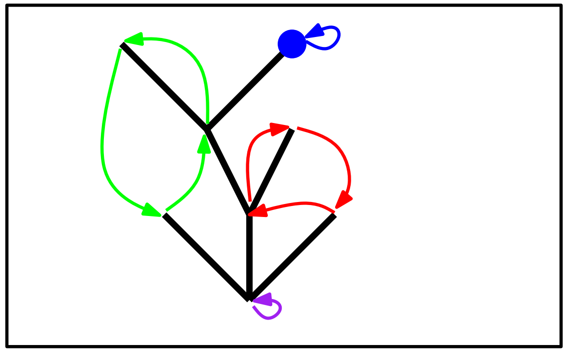

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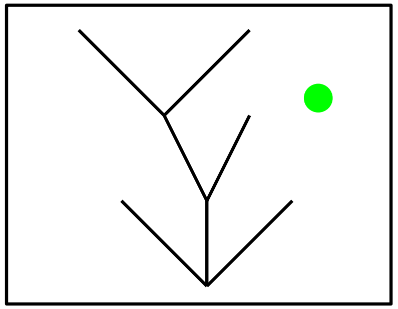

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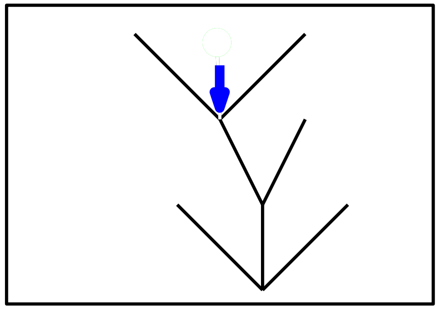

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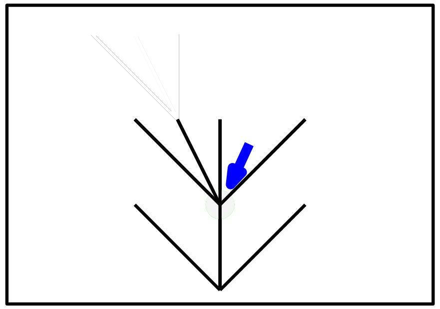

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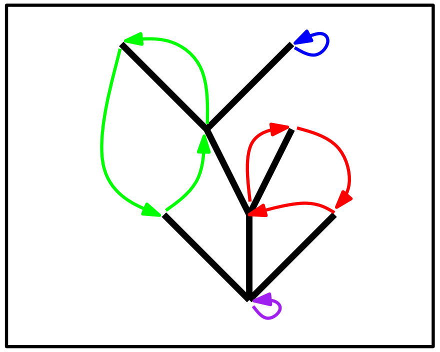

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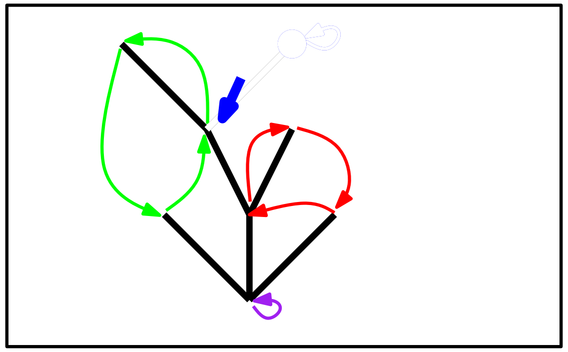

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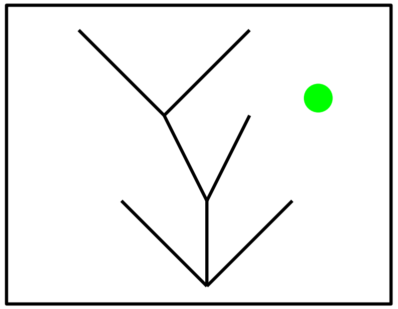

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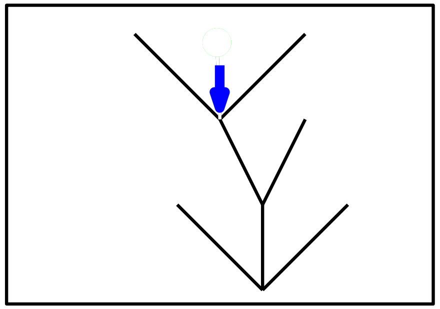

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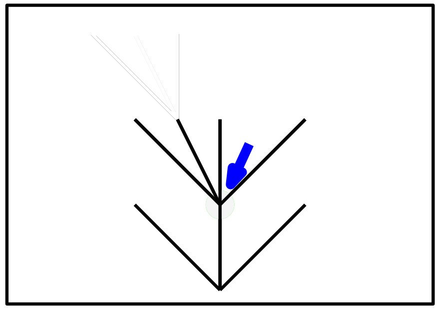

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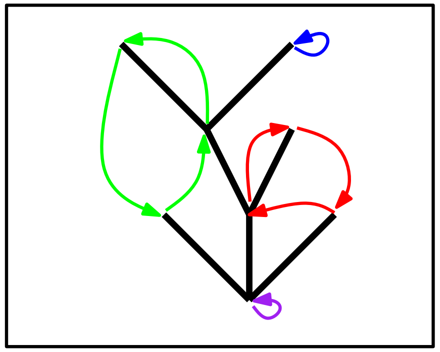

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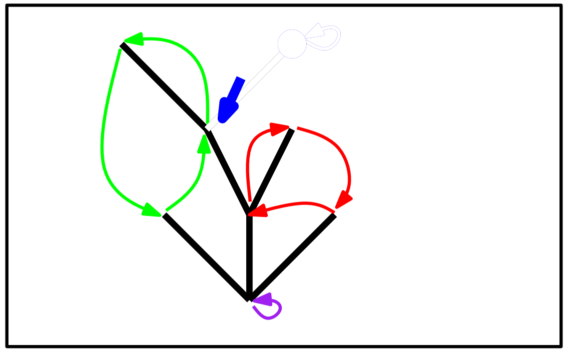

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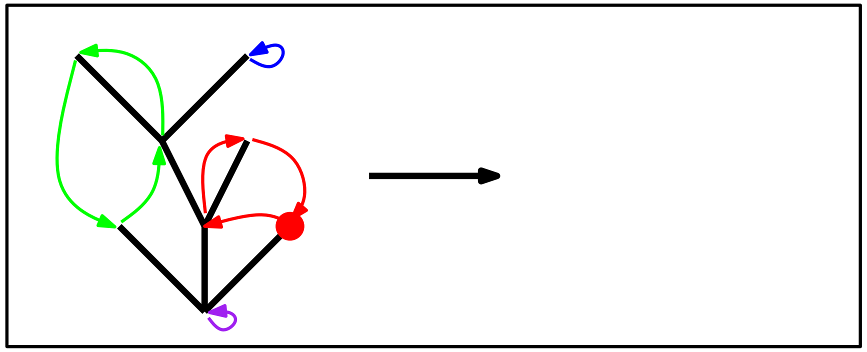

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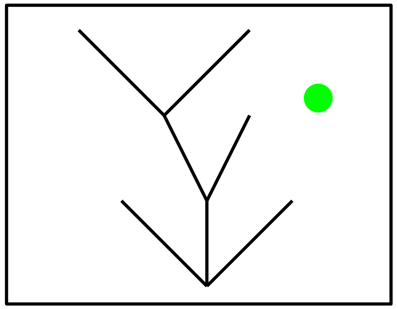

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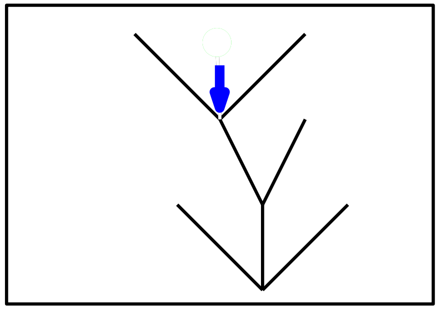

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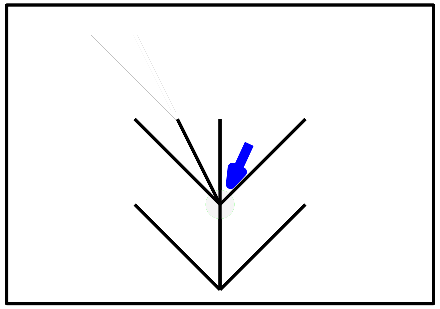

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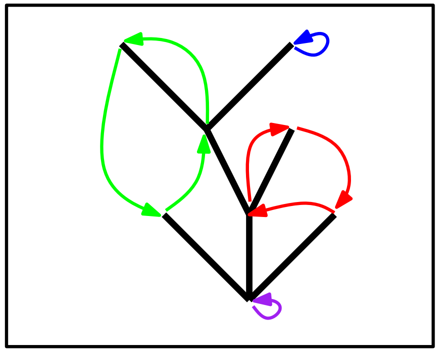

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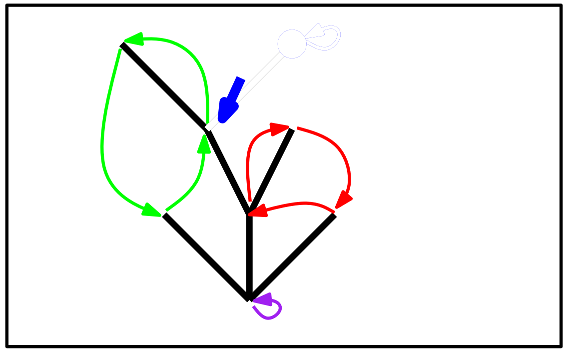

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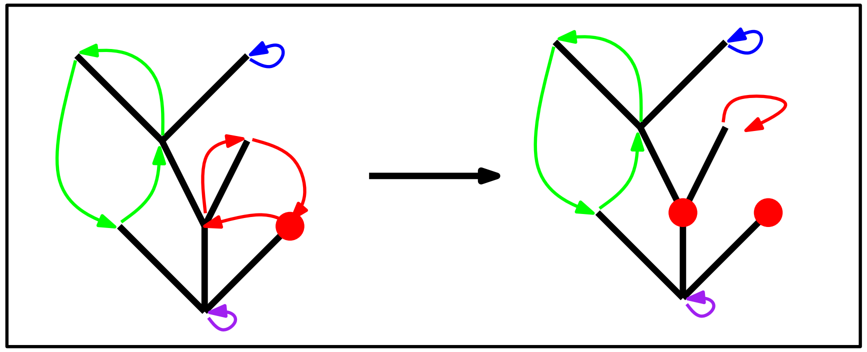

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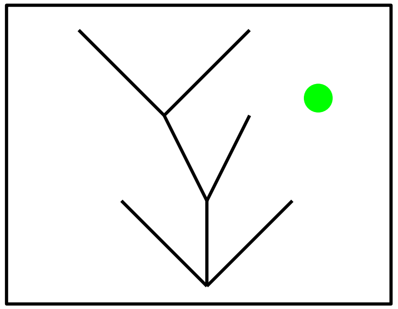
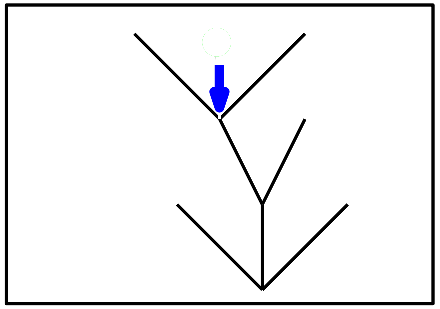
- Classic: for $g = 0$, Rémy's bijection [Rémy 85]

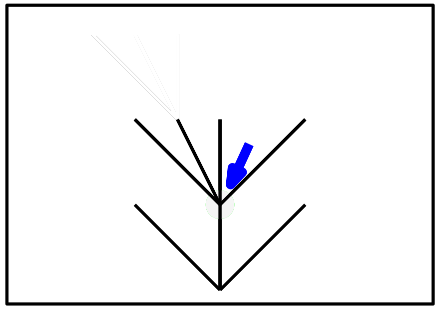
$$(n+1)\mathrm{Cat}(n) \quad = \quad 2 \times (2n-1)\mathrm{Cat}(n-1)$$



rooted tree, $n$ edges, one marked vertex



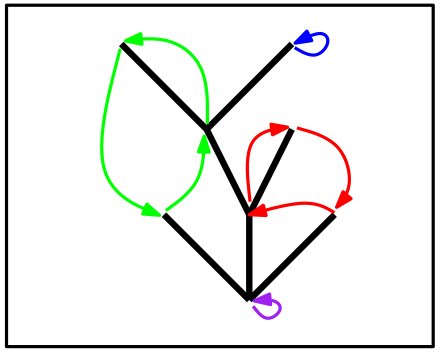
case a: vertex is a leaf. Delete it.



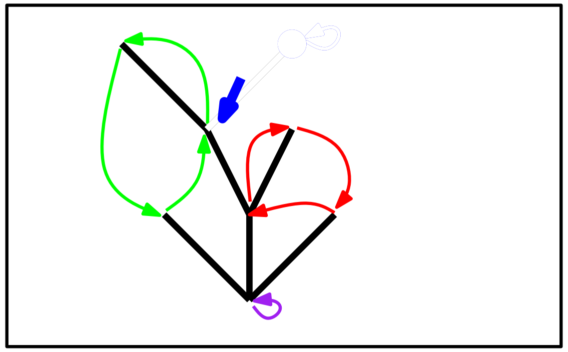
case b: vertex is not a leaf. Contract the leftmost outgoing edge.

- Then for general $g$:

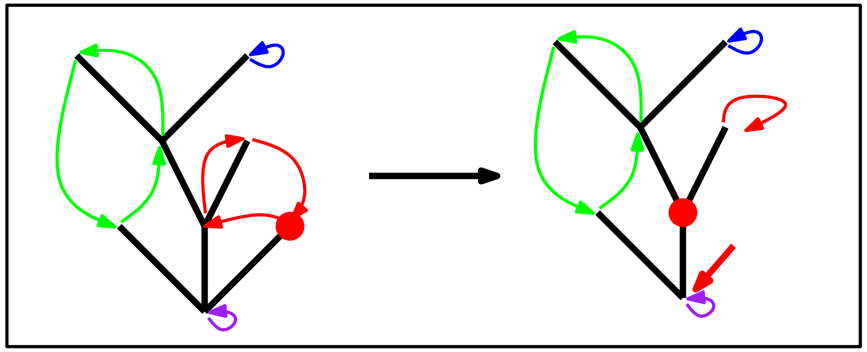
$$(n+1)\epsilon_g(n) \quad = \quad 2(2n-1)\epsilon_g(n-1) + (n-1)(2n-1)(2n-3)\epsilon_{g-1}(n-2)$$



$C$-decorated tree, $n$ edges, genus $g$, one marked vertex



case 1: vertex is a fixed point: apply Rémy's bijection (one vertex disappears)



case 2: vertex is in a $(2k+1)-$cycle. Apply Rémy's bijection twice (two vertices disappear, cycle length decreases by $2$)

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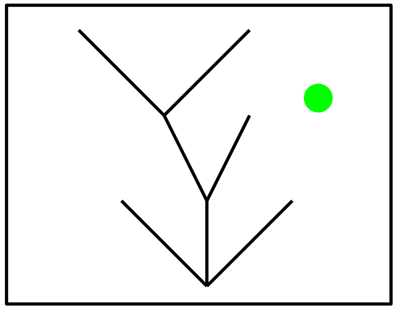

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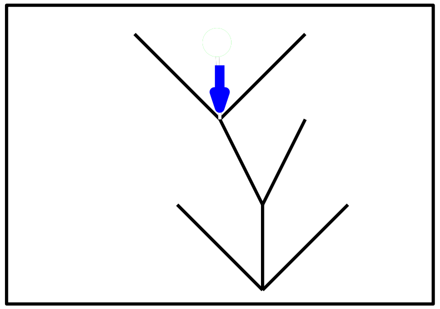

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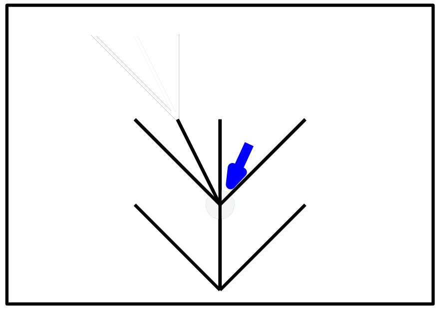

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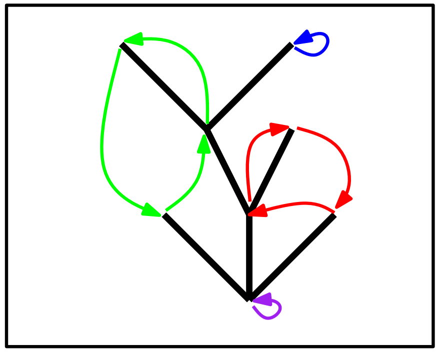

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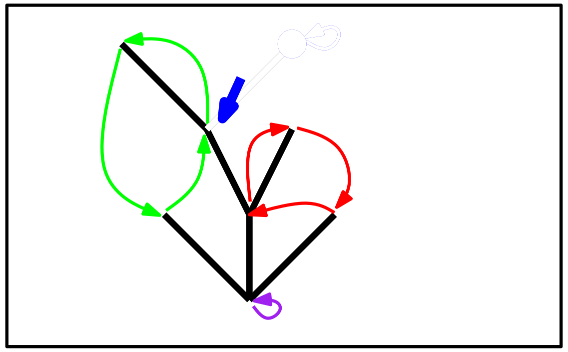

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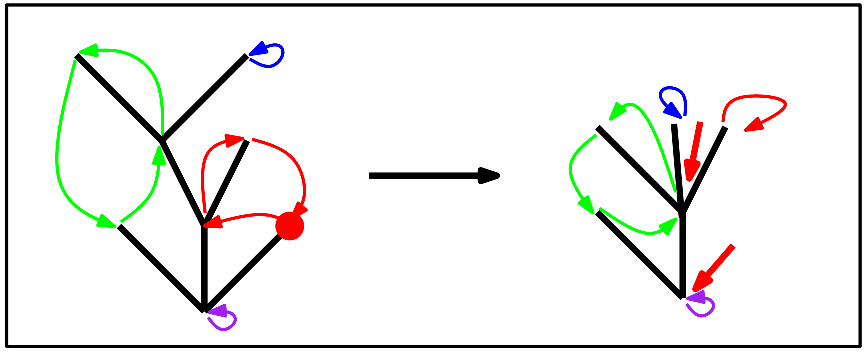

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# Conclusion

- It is a series of exercises to recover ALL the known formulas concerning unicellular maps, bijectively with C-decorated trees. You just need to know your classics (count permutations, count trees...). Take a look at the paper!

- The bijection also applies to Féray's expression of Stanley character polynomials in terms of unicellular maps (we obtain a new expression - is it useful?)

- Next ?

  → unicellular constellations? ([Poulalhon-Schaeffer 02, Bernardi-Morales 11])

  (problem: FPSAC'09 bijection does not work well)

  (very partial results in the full version - take a look!)

  → many-face maps? (KP hierarchy?)

  (problem: seems much harder!)

  → non-orientable surfaces?

  (problem: FPSAC'09 bijection only exists in asymptotic version - [Bernardi-Ch., FPSAC'10])