What's new in Symbolic Summation

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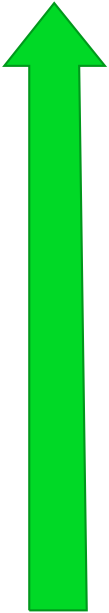
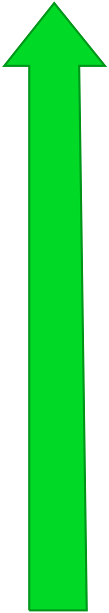


- **prehistory**
  Gosper's algorithm, Sister Celine's algorithm, Karr's algorithm, hypergeometric transformations (nonalgorithmic), table lookup.



- **The 1990s: The stormy decade**
  Z's theory, Z's algorithm, Almkvist-Zeilberger algorithm, Petkovšek's algorithm, WZ-pairs, $A = B$, GFF, q-generalizations, Wegschaider, Paule-Schorn package, gfun, Yen's bound, ...
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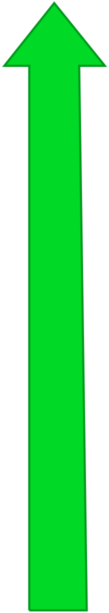

- **The 2000s: Extensions and generalizations**
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- **The 2010s: Efficiency and complexity**
  applications with large input, rational integration exploiting fast arithmetic, worst case bounds on the run time complexity, sharp estimates on the output size, parallel algorithms, ...
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Plan of this talk:

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- Address some developments which are now ready to use.

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Plan of this talk:

- Address some developments which are now ready to use.
- Address some of the hot topics in the area.

A What's old?

- Hypergeometric creative telescoping

B What's new "on the market"?

- Techniques for nested sums and products
- Techniques for multivariate D-finite objects

C What's new "in the labs"?

- Speedup by trading order against degree

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OUTPUT: something like

$$\begin{multline*}(n+1)^3 f(n,k) \\ -(2n+3)(17n^2+51n+39)f(n+1,k) \\ +(n+3)^3 f(n+2,k) = g(n,k+1) - g(n,k)\end{multline*}$$

where $g(n,k) = \frac{4k^4(2n+3)(4n^2+12n-2k^2+3k+8)}{(n-k+1)^2(n-k+2)^2} f(n,k)$.

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polynomials in n only

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rational function in n and k

INPUT: a hypergeometric term $f(n, k)$

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i.e., $\frac{f(n+1,k)}{f(n,k)} \in \mathbb{K}(n,k)$ and $\frac{f(n,k+1)}{f(n,k)} \in \mathbb{K}(n,k)$

INPUT: a hypergeometric term $f(n,k)$

OUTPUT: $T \in \mathbb{K}[n, S_n] \setminus \{0\}$ and $Q \in \mathbb{K}(n,k)$ such that

$$T \cdot f(n,k) = (S_k - 1) \cdot Q\, f(n,k)$$

"telescoper"

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“telescoper” “certificate”

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A telescoper for $f(n,k)$ is **(usually)** an annihilator for $\sum_k f(n,k)$.

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We have

$$p_0(n)f(n,k)$$

$$-\,p_1(n)f(n+1,k)$$

$$+\,p_2(n)f(n+2,k)$$

$$= g(n,k+1) - g(n,k)$$

with $g(n,k) = \frac{4k^4(2n+3)(4n^2+12n-2k^2+3k+8)}{(n-k+1)^2(n-k+2)^2} f(n,k)$.

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$$\Downarrow$$

$$p_0(n)F(n) + p_1(n)F(n+1) + p_2(n)F(n+2) = 0.$$

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van der Poorten on his struggles to check Apéry's argument:

> *"We were quite unable to prove that the sequence $F(n)$ defined above did satisfy the recurrence (Apéry rather tartly pointed out to me in Helsinki that he regarded this more a compliment than a criticism of his method). But empirically (numerically) the evidence in favour was utterly compelling."*

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But Apéry needs a second sum:

$$H(n) = \sum_k \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{i=1}^{n} \frac{1}{i^3} + \sum_{i=1}^{k} \frac{(-1)^{i-1}}{2i^3 \binom{n}{i} \binom{n+i}{i}} \right)$$

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Key step of his proof: $H(n)$ and $F(n)$ satisfy **the same** recurrence.

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We need appropriate generalizations.

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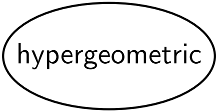
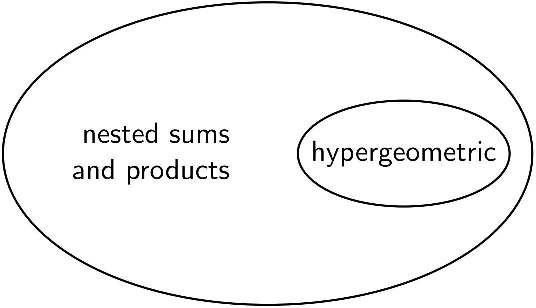

hypergeometric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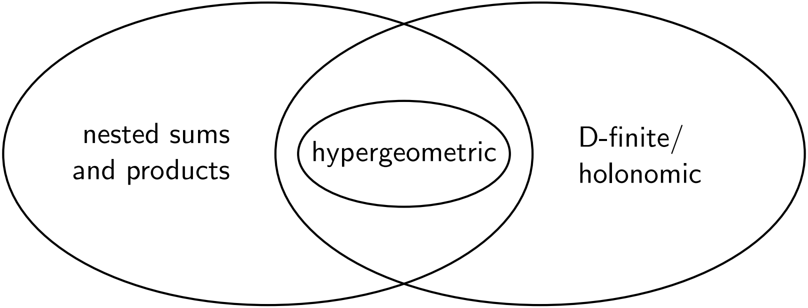
nested sums
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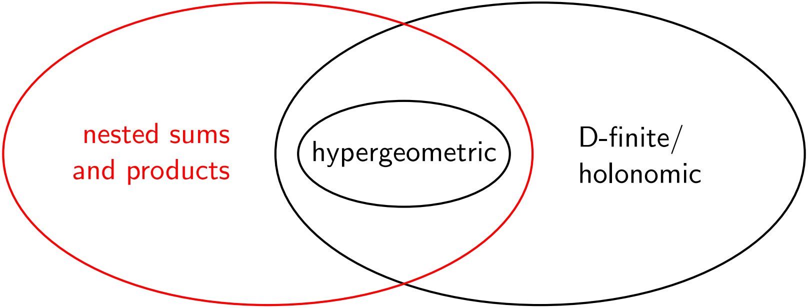
hypergeometric
D-finite/
holonomic


nested sums
and products
hypergeometric
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  OK if either of them is regarded as constant.

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Note: $\Pi\Sigma$-expressions can be easily shifted ($n \rightsquigarrow n+1$) using

$$\sum_{k=1}^{n+1} a_k = \sum_{k=1}^{n} a_k \; + a_{n+1} \qquad \prod_{k=1}^{n+1} a_k = a_{n+1} \prod_{k=1}^{n} a_k$$

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Observation: The field generated by a $\Pi\Sigma$-expression and all its subexpressions is closed under shift.

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  for some $\alpha, \beta \in \mathbb{K}(t_1, t_2, \ldots, t_{i-1})$ (plus some technical restrictions omitted here).
- t_i represents a **product** if $\beta = 0$
  t_i represents a **sum** if $\alpha = 1$

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\sigma(t_4) &= (t_1 + 1)t_4 & t_4 &\sim n! \\
\sigma(t_5) &= t_5 + \tfrac{1+(t_1+1)t_3+(t_1+1)^2 t_4}{(t_1+1)(t_1+1+2t_2)} & t_5 &\sim \textstyle\sum_{k=1}^{n} \tfrac{H_k+k!}{2^k+k}
\end{aligned}$$

Karr's algorithm (1982): **Given** a $\Pi\Sigma$-field $\mathbb{F}$ and an element $f \in \mathbb{F}$, **find** $g \in \mathbb{F}$ with $\sigma(g) - g = f$, or prove that no such element g exists in $\mathbb{F}$.

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This new single sum is not a subexpression of the left hand side

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For a given $\Pi\Sigma$-expression, find an equivalent $\Pi\Sigma$-expression in which the **nesting depth** is as small as can be.

Examples: 

$$\begin{aligned}\cdots = \tfrac{1}{4}\Big(&\tfrac{1}{3}\Big(\sum_{k=1}^{n} \frac{1}{k^2}\Big)^3 + \Big(\sum_{k=1}^{n} \frac{1}{k^4} + \sum_{k=1}^{n} \frac{\big(\sum_{i=1}^{k} \frac{1}{i}\big)^2}{k^2}\Big) \sum_{k=1}^{n} \frac{1}{k^2} + \frac{2}{3} \sum_{k=1}^{n} \frac{1}{k^6} - \\
&\sum_{k=1}^{n} \frac{\big(\sum_{i=1}^{k} \frac{1}{i^4}\big) \sum_{i=1}^{k} \frac{1}{i}}{k} - \sum_{k=1}^{n} \frac{\big(\sum_{i=1}^{k} \frac{1}{i^2}\big)^2 \sum_{i=1}^{k} \frac{1}{i}}{k} + 2 \sum_{k=1}^{n} \frac{\big(\sum_{i=1}^{k} \frac{1}{i}\big)^2}{k^4} + \sum_{k=1}^{n} \frac{\big(\sum_{i=1}^{k} \frac{1}{i}\big)^4}{k^2} + \\
&\Big(\sum_{k=1}^{n} \frac{1}{k}\Big)^2 \sum_{k=1}^{n} \frac{\big(\sum_{i=1}^{k} \frac{1}{i}\big)^2}{k^2} - \sum_{k=1}^{n} \frac{\big(\sum_{i=1}^{k} \frac{1}{i^2}\big)\big(\sum_{i=1}^{k} \frac{1}{i}\big)^2}{k^2} - 2 \sum_{k=1}^{n} \frac{\big(\sum_{i=1}^{k} \frac{1}{i}\big)^3}{k^3} + \\
&\Big(\sum_{k=1}^{n} \frac{1}{k}\Big)\Big(\sum_{k=1}^{n} \frac{\sum_{i=1}^{k} \frac{1}{i^4}}{k} + \sum_{k=1}^{n} \frac{\big(\sum_{i=1}^{k} \frac{1}{i^2}\big)^2}{k} + 2 \sum_{k=1}^{n} \frac{\big(\sum_{i=1}^{k} \frac{1}{i}\big)^2}{k^3} - 2 \sum_{k=1}^{n} \frac{\big(\sum_{i=1}^{k} \frac{1}{i}\big)^3}{k^2}\Big)\Big)\end{aligned}$$

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This requires that the summand $f(n,k)$ is such that $f(n,k)$, $f(n+1,k)$, $f(n+2,k)$, $\dots$ all are $\Pi\Sigma$-expressions with respect to k when n is viewed as a (symbolic) constant.

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- $f(n,k) = \binom{n}{k}^2 \binom{n+k}{k}^2$
- $f(n,k) = \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{i=1}^{n} \frac{1}{i^3} + \sum_{i=1}^{k} \frac{(-1)^{i+1}}{2i^3 \binom{n}{i} \binom{n+i}{i}} \right)$

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- $2(2n+5)(3n+5)F(n) - (6n^3+49n^2+124n+98)F(n+1)$
  $+ (n+2)(2n+3)(3n+8)F(n+2) = 0$
  $\rightsquigarrow$ solutions 1 and $8\sum_{k=1}^{n}\prod_{i=1}^{k}\frac{2}{i} - \sum_{k=0}^{n}\frac{\prod_{i=1}^{k}\frac{2}{i}}{3k+2}$

Solve a given linear recurrence equation in terms of $\Pi\Sigma$-expressions.

Example.

- $$(n^2 H_n + 3nH_n + 2H_n + 2n + 3)F(n) \\ - (n^3 H_n + 6n^2 H_n + 11nH_n + 6H_n + n^2 + 6n + 7)F(n+1) \\ + (n+2)^2 (nH_n + H_n + 1)F(n+2) = 0$$

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Suggested workflow for iterated definite sums:

$$\sum_{k_1} \sum_{k_2} \sum_{k_3} \boxed{\begin{array}{c}\Pi\Sigma\text{-expression in } k_3 \\ \text{with parameters } n, k_1, k_2\end{array}}$$

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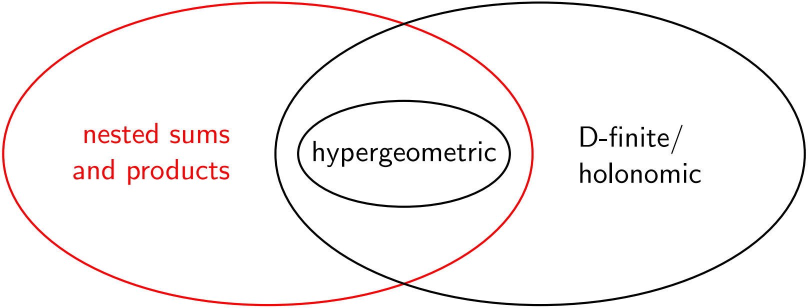
$\Pi\Sigma$-expression in n


nested sums
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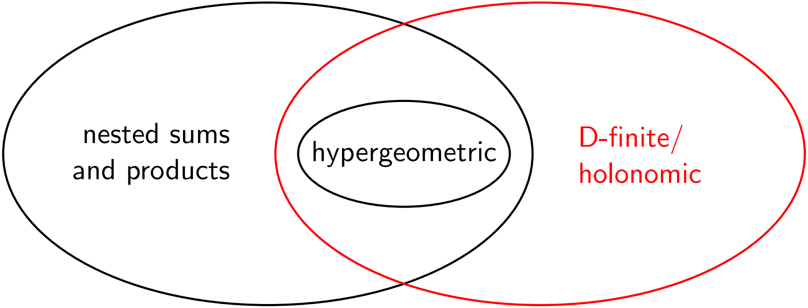
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It is a vector space of dimension 1.

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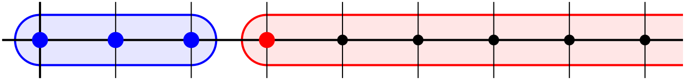
Then $a_n, \ldots, a_{n+r-1}$ generate the vector space. (Possibly fewer.)



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Expert answer:
$$\begin{aligned}&\mathrm{RootOf}(\_Z^5 - 3\_Z + 1, \mathrm{index} = 1),\\ &\mathrm{RootOf}(\_Z^5 - 3\_Z + 1, \mathrm{index} = 2),\\ &\mathrm{RootOf}(\_Z^5 - 3\_Z + 1, \mathrm{index} = 3),\\ &\mathrm{RootOf}(\_Z^5 - 3\_Z + 1, \mathrm{index} = 4),\\ &\mathrm{RootOf}(\_Z^5 - 3\_Z + 1, \mathrm{index} = 5).\end{aligned}$$

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Expert answer: The solutions form a $\mathbb{K}$-vector space V of dimension two. Each solution is uniquely determined by its first two terms, and each choice of two initial terms gives rise to a solution.

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Several variables: An object $a_{n_1,n_2,\ldots,n_p}$ in p variables is **D-finite** if it lives in some **finite-dimensional** $\mathbb{K}(n_1,\ldots,n_p)$-vector space which is closed under shift **for each variable.**

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- $a_{n,k} = n^k$ is D-finite in n for every fixed choice $k \in \mathbb{Z}$, but it is **not D-finite** in n and k.

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$$\begin{array}{ccccc}
a_{n,k+4} & a_{n+1,k+4} & a_{n+2,k+4} & a_{n+3,k+4} & a_{n+4,k+4} \\
a_{n,k+3} & a_{n+1,k+3} & a_{n+2,k+3} & a_{n+3,k+3} & a_{n+4,k+3} \\
a_{n,k+2} & a_{n+1,k+2} & a_{n+2,k+2} & a_{n+3,k+2} & a_{n+4,k+2} \\
a_{n,k+1} & a_{n+1,k+1} & a_{n+2,k+1} & a_{n+3,k+1} & a_{n+4,k+1} \\
a_{n,k} & a_{n+1,k} & a_{n+2,k} & a_{n+3,k} & a_{n+4,k}
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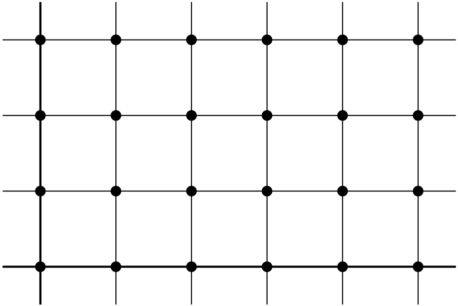

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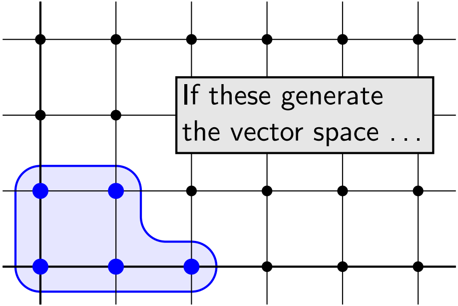

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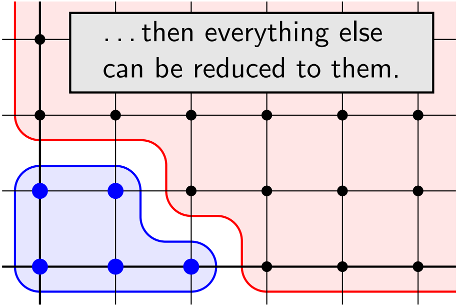

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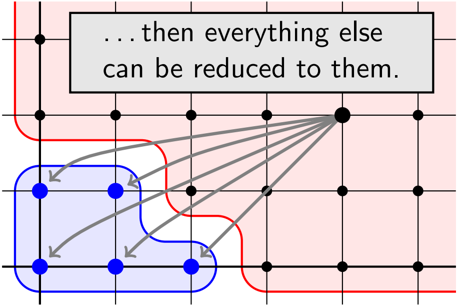

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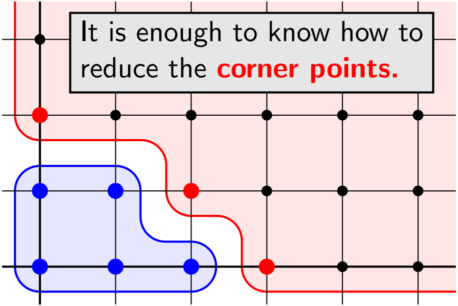

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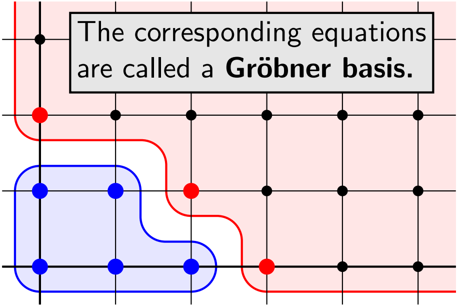

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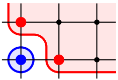
Examples: 

- A Gröbner basis for $a_{n,k} = \binom{n}{k}^2 \binom{n+k}{k}^2$:

$$\left\{ a_{n+1,k} = \frac{(k+n+1)^2}{(n-k+1)^2} a_{n,k}, \quad a_{n,k+1} = \frac{(n-k)^2(k+n+1)^2}{(k+1)^4} a_{n,k} \right\}$$



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- A Gröbner basis for $a_{n,k} = 2^k H_{n+2k}$:

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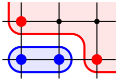

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More generally: An object $a(n_1,n_2,\ldots,n_p,x_1,x_2,\ldots,x_r)$ in p discrete (or q-discrete) variables $n_1,\ldots,n_p$ and r continuous (or q-continuous) variables $x_1,\ldots,x_r$ is called **D-finite** if all the infinitely many mixed (q-)shifts and (q-)derivatives

$$S_{n_1}^{e_1}S_{n_2}^{e_2}\cdots S_{n_p}^{e_p}D_{x_1}^{f_1}D_{x_2}^{f_2}\cdots D_{x_r}^{f_r}\cdot a(n_1,\ldots,n_p,x_1,x_2,\ldots,x_r)$$

$(e_1,\ldots,e_p,f_1,\ldots,f_r\in\mathbb{N})$ generate only a **finite dimensional** vector space over $\mathbb{K}(n_1,\ldots,n_p,x_1,\ldots,x_r)$.

Closure properties: If $a(n_1, \dots, n_p, x_1, \dots, x_r)$ and $b(n_1, \dots, n_p, x_1, \dots, x_r)$ are D-finite, then so are

- their sum $a + b$ and product $a \cdot b$,
- their shifts $a(n_1 + 1, n_2, \dots, n_p, x_1, \dots, x_r)$,
- their derivatives $D_{x_1} \cdot a(n_1, \dots, n_p, x_1, \dots, x_r)$,
- translates $a(u_1 n_1 + u_2 n_2 + \cdots + u_p n_p, n_2, \dots, n_p, x_1, \dots, x_r)$ for any fixed integers $u_1, u_2, \dots, u_p \in \mathbb{Z}$, $u_1 \neq 0$.
- compositions $a(n_1, \dots, n_r, u(x_1, \dots, x_r), x_2, \dots, x_r)$ with algebraic functions u free of $n_1, \dots, n_r$, not free of x_1.

Creative telescoping (Zeilberger's algorithm):

INPUT: a hypergeometric term $f(n,k)$

OUTPUT: $T \in \mathbb{K}[n, S_n] \setminus \{0\}$ and $Q \in \mathbb{K}(n,k)$ such that

$$T \cdot f(n,k) = (S_k - 1)Q \cdot f(n,k)$$

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- Existence of telescopers is guaranteed whenever input is not only D-finite but also "holonomic". This is usually the case.

Example:

$$f(n,k)=\binom{n}{k}^2\binom{n+k}{k}^2\left(\sum_{i=1}^{n}\frac{1}{i^3}+\sum_{i=1}^{k}\frac{(-1)^{i+1}}{2i^3\binom{n}{i}\binom{n+i}{i}}\right)$$

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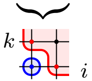

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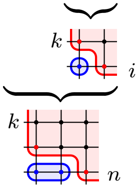

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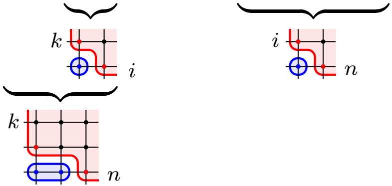

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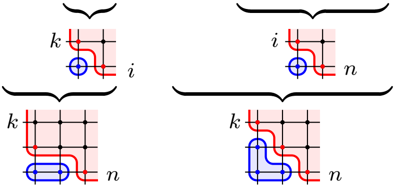

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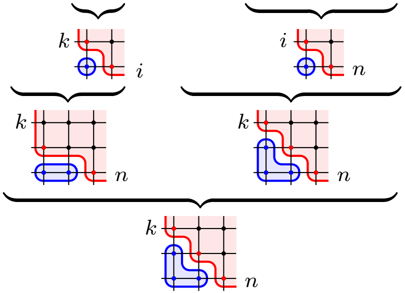

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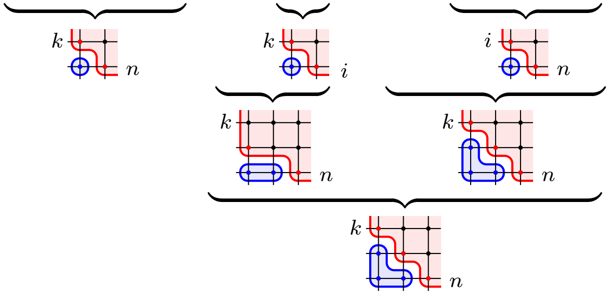

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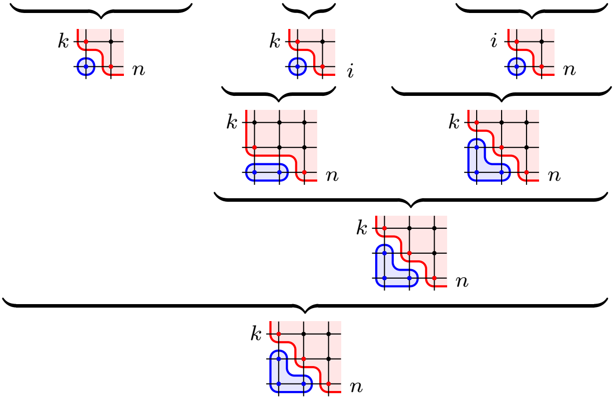

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$$-(\text{messy}) f(n,k)$$
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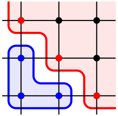

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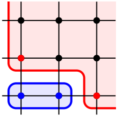

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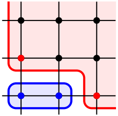
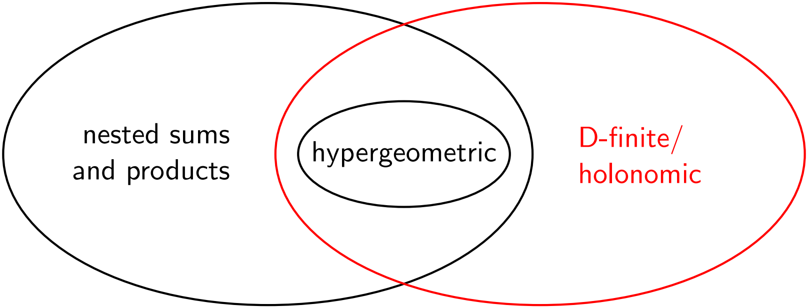


Such extra knowledge can make calculations much faster.

Example:

$$f(n,k) = \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{i=1}^{n} \frac{1}{i^3} + \sum_{i=1}^{k} \frac{(-1)^{i+1}}{2i^3 \binom{n}{i}\binom{n+i}{i}} \right)$$

- Computing a recurrence for $\sum_k f(n,k)$ not using the additional relation takes **40sec** and yields a recurrence of **order 4.**
- Computing a recurrence for $\sum_k f(n,k)$ using the additional relation takes **0.2sec** and yields a recurrence of **order 2.**


nested sums
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A What's old?

- Hypergeometric creative telescoping

B What's new "on the market"?

- Techniques for nested sums and products
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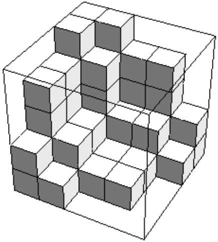
$$\forall\ n \in \mathbb{N} : \sum_{m=0}^{\infty} R_{n,m} q^m = \prod_{k=1}^{n} b_k$$



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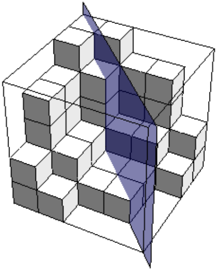

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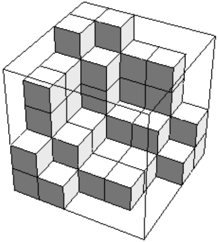

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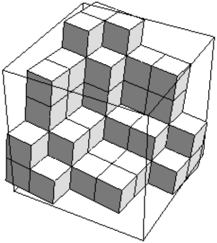

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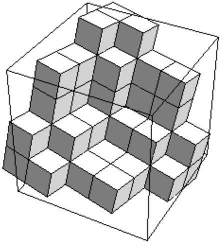

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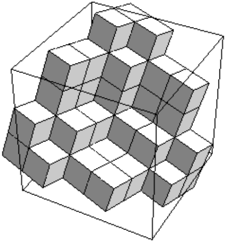

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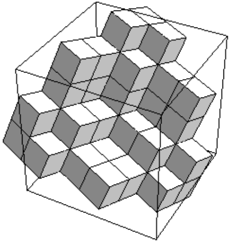

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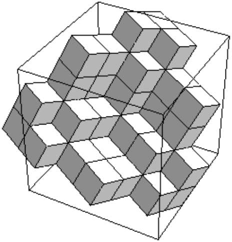

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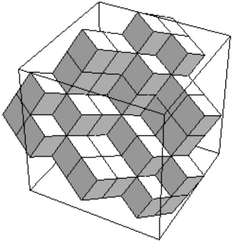

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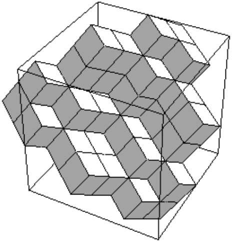

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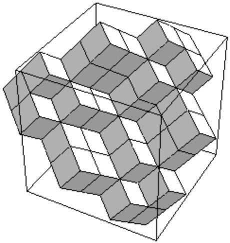

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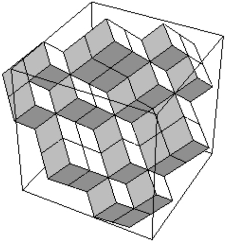

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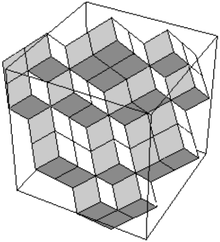

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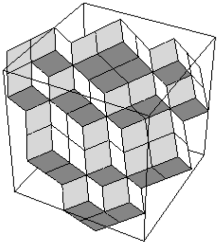

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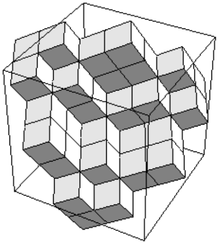

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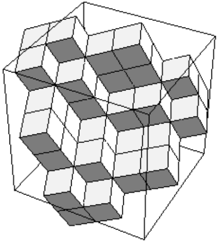

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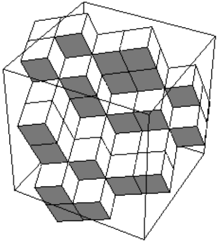

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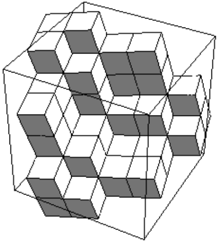

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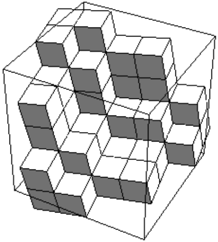

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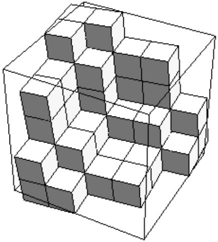

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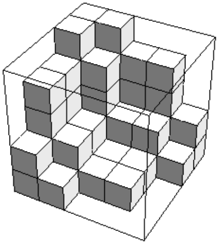

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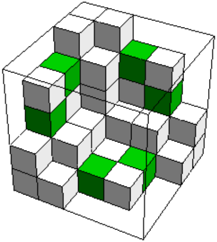

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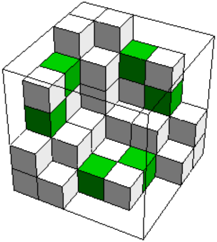

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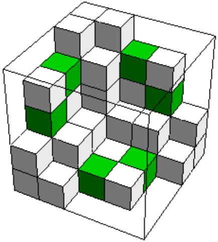

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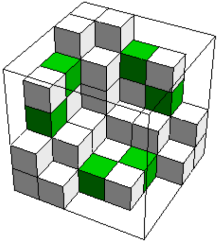

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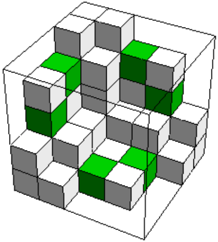

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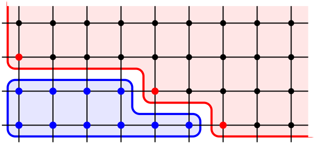

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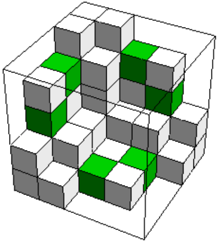

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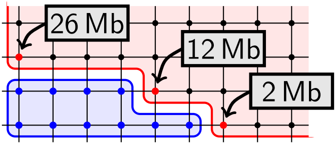

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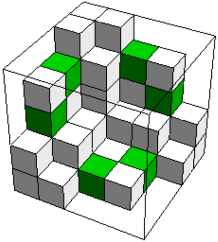

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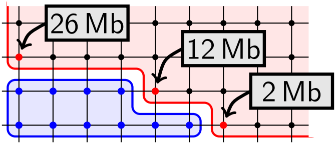

$\Leftarrow$ a **creative telescoping** relation with a certificate Q of size **7Gb.** (Koutschan, MK, Zeilberger, PNAS 2011)

Why are these expressions so big?

How big are they actually?

Can we calculate them more efficiently?

Creative telescoping (Zeilberger's algorithm):

INPUT: a hypergeometric term $f(n,k)$

OUTPUT: $T \in \mathbb{K}[n, S_n] \setminus \{0\}$ and $Q \in \mathbb{K}(n,k)$ such that

$$T \cdot f(n,k) = (S_k - 1)Q \cdot f(n,k)$$

Focus on the Telescoper:

$$
\begin{aligned}
T = &\bigl(a_{0,0} + a_{0,1}n + a_{0,2}n^2 + \cdots + a_{0,d}n^d\bigr) \\
&+\bigl(a_{1,0} + a_{1,1}n + a_{1,2}n^2 + \cdots + a_{1,d}n^d\bigr)S_n \\
&+\bigl(a_{2,0} + a_{2,1}n + a_{2,2}n^2 + \cdots + a_{2,d}n^d\bigr)S_n^2 \\
&+\dots \\
&+\bigl(a_{r,0} + a_{r,1}n + a_{r,2}n^2 + \cdots + a_{r,d}n^d\bigr)S_n^r
\end{aligned}
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$$\left.\begin{aligned} T = \overbrace{\left(a_{0,0} + a_{0,1}n + a_{0,2}n^2 + \cdots + a_{0,d}n^d\right)}^{\text{degree } d} \\ +\left(a_{1,0} + a_{1,1}n + a_{1,2}n^2 + \cdots + a_{1,d}n^d\right)S_n \\ +\left(a_{2,0} + a_{2,1}n + a_{2,2}n^2 + \cdots + a_{2,d}n^d\right)S_n^2 \\ +\ldots \\ +\left(a_{r,0} + a_{r,1}n + a_{r,2}n^2 + \cdots + a_{r,d}n^d\right)S_n^r \end{aligned}\right\}\text{order } r$$

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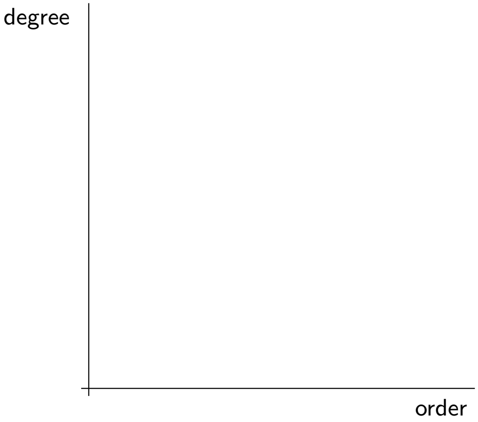
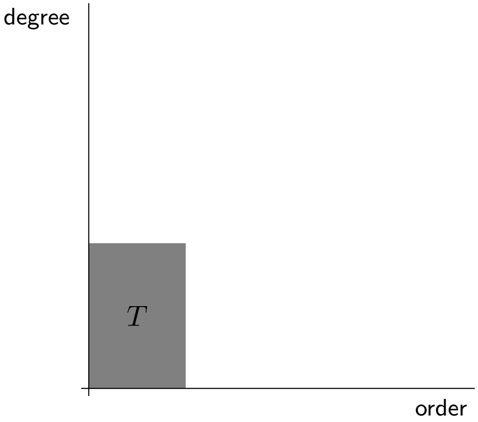
Instead, the set of all telescopers for a fixed term $f(n,k)$ forms a **left ideal** in the operator algebra $\mathbb{K}[n,S_n]$.


degree
order

A telescoper of order r and degree d can be depicted like this.



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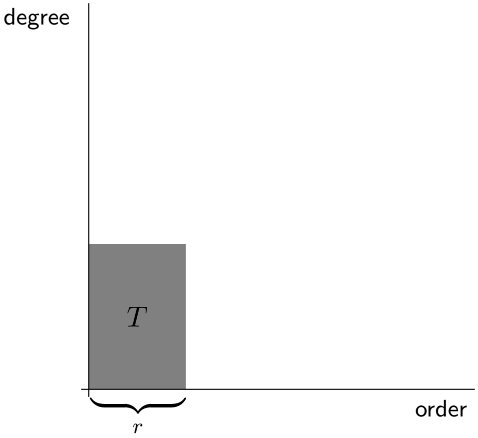

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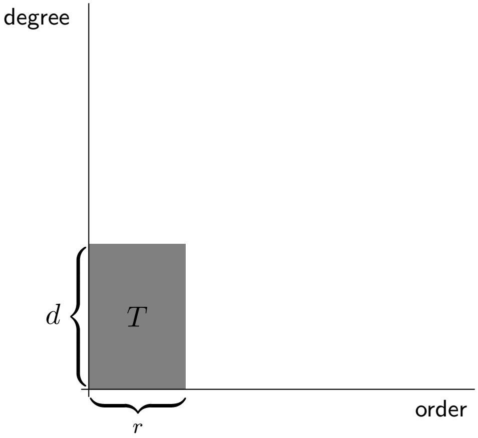
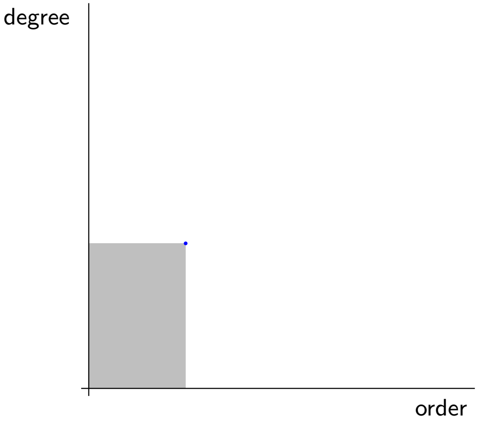


We will however depict it just by its upper right corner (r, d).



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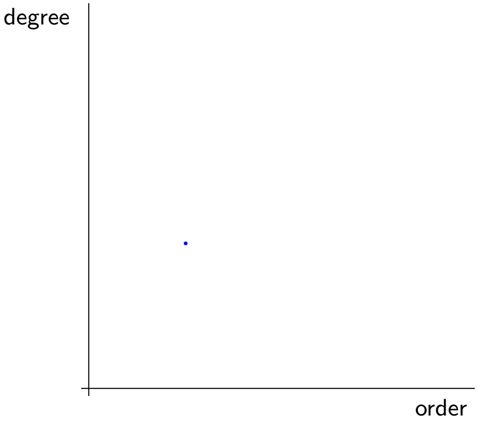
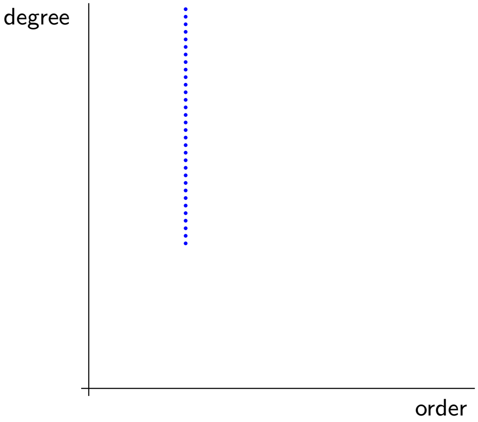
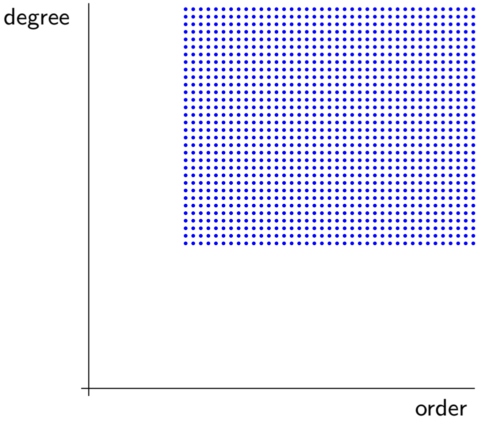


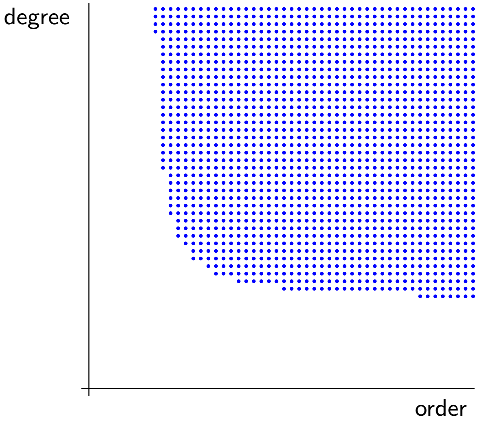
Multiplication by powers of n gives further telescopers.



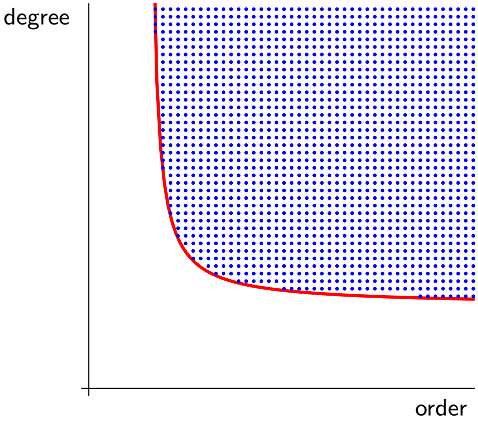
Multiplication by powers of S_n gives even more telescopers.



The set of all telescopers is still bigger.



Want: A curve describing the shape of the blue region.



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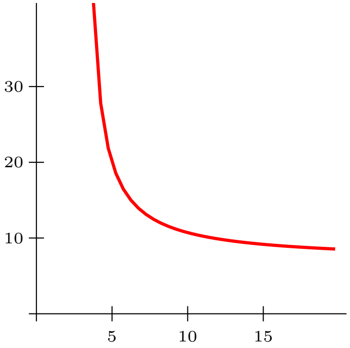

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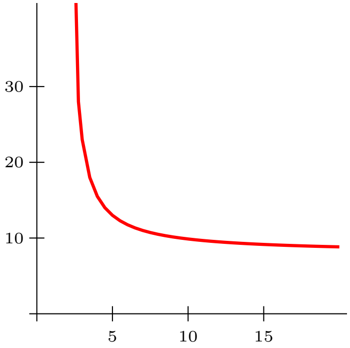

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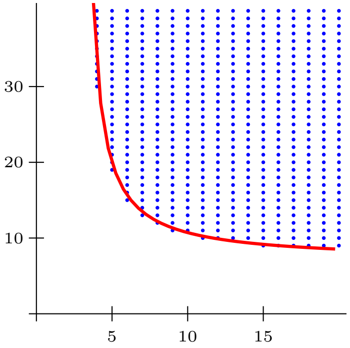

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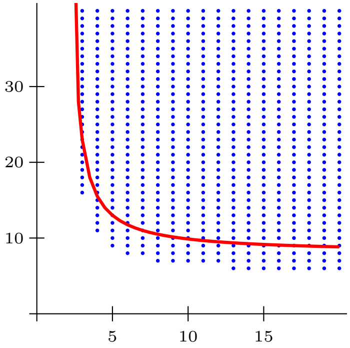

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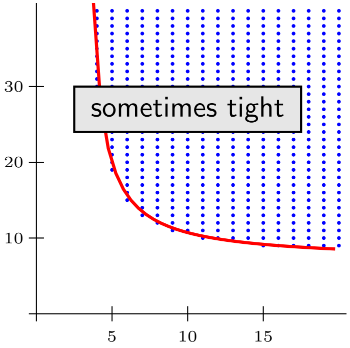

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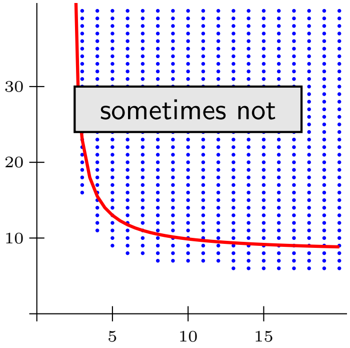

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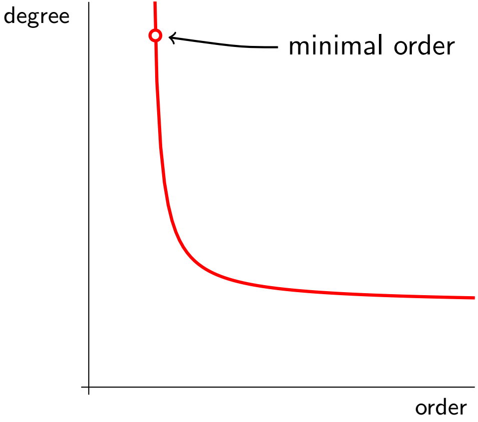

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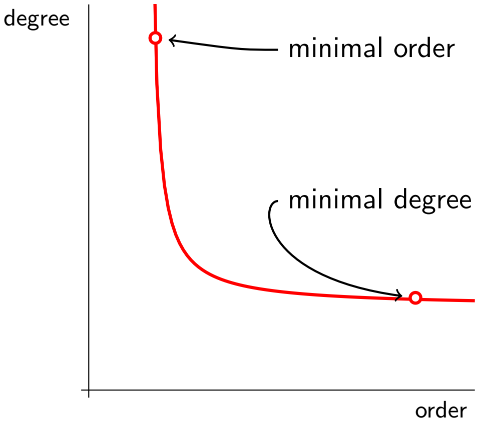

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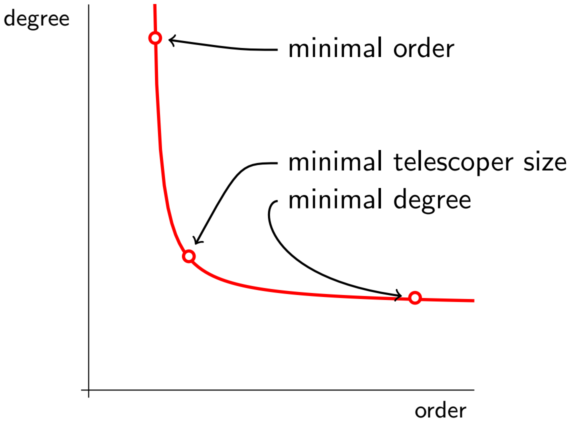

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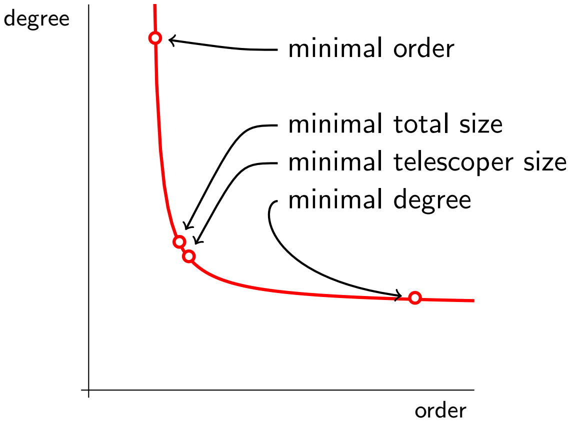

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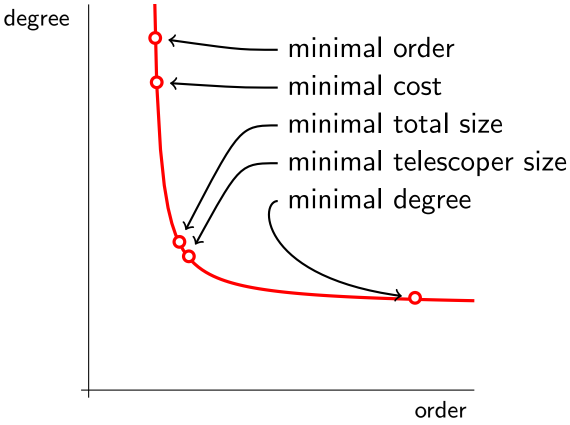

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- Similar effects have already been reported in other circumstances.

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- What is the right question to be asked in the case of several variables?

A What's old?

- Hypergeometric creative telescoping

B What's new "on the market"?

- Techniques for nested sums and products
- Techniques for multivariate D-finite objects

C What's new "in the labs"?

- Speedup by trading order against degree

- **The 2010s: Efficiency and complexity**
  applications with large input, rational integration exploiting fast arithmetic, worst case bounds on the run time complexity, sharp estimates on the output size, parallel algorithms, ...

- **The 2000s: Extensions and generalizations**
  Refined $\Pi\Sigma$-theory, Takayama, Ore algebras and Gröbner bases, Chyzak's algorithm, algorithms for identities involving Abel-type terms or Bernoulli numbers or Stirling numbers, ...

- **The 1990s: The stormy decade**
  Z's theory, Z's algorithm, Almkvist-Zeilberger algorithm, Petkovšek's algorithm, WZ-pairs, $A = B$, GFF, q-generalizations, Wegschaider, Paule-Schorn package, gfun, Yen's bound, ...

- **prehistory**
  Gosper's algorithm, Sister Celine's algorithm, Karr's algorithm, hypergeometric transformations (nonalgorithmic), table lookup.