Flow polytopes of signed graphs and the Kostant partition function

Alejandro Morales (MIT $\rightarrow$ LaCIM)

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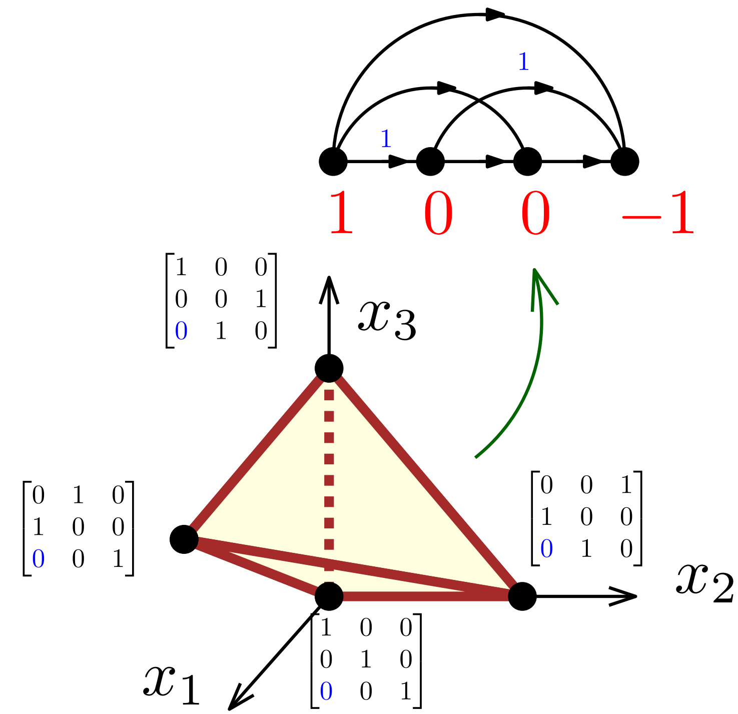
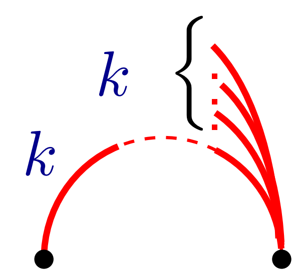
joint work with Karola Mészáros (Cornell)





Example of a type A flow polytope ($\mathcal{CRY}(n)$)

$$\mathcal{CRY}(n) := \left\{ (b_{ij}) \in \mathbb{R}^{n^2} \mid \text{ doubly-stochastic matrix, } b_{ij} = 0, i - j \geq 2 \right\}$$

Example:

b_{11}	b_{12}	b_{13}	b_{14}
b_{21}	b_{22}	b_{23}	b_{24}
0	b_{32}	b_{33}	b_{34}
0	0	b_{43}	b_{44}

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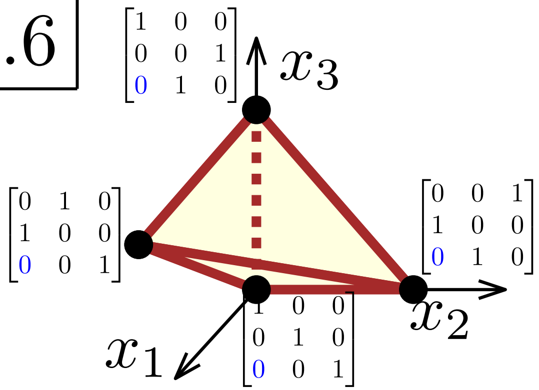
- $\mathcal{CRY}(n)$ is the **Chan-Robbins-Yuen polytope**
- has 2^{n-1} vertices and $\dim(\mathcal{CRY}(n)) = \binom{n}{2}$



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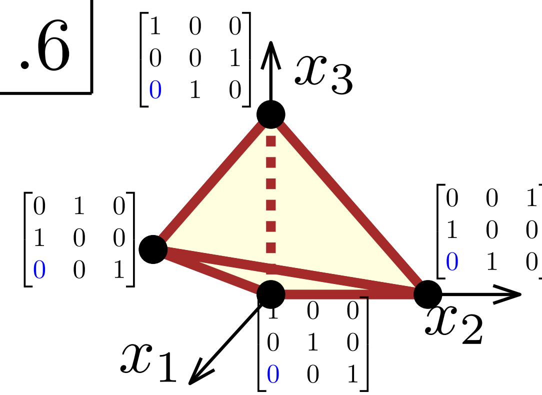

Data: $v_n = \binom{n}{2}! \cdot \text{vol}(\mathcal{CRY}(n))$

n	2	3	4	5	6	7
v_n	1	1	2	10	140	5880
$\frac{v_n}{v_{n-1}}$		**1**	**2**	**5**	**14**	**42**

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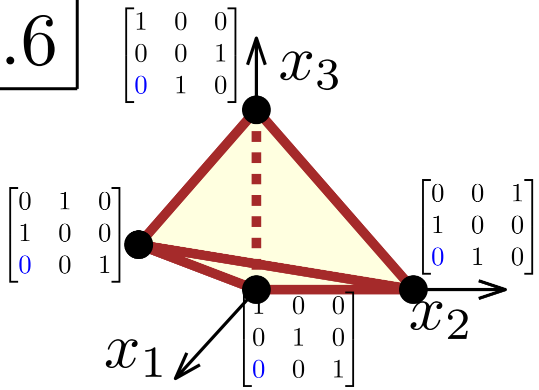

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Theorem [Zeilberger 99]:

$$\binom{n}{2}! \cdot \mathrm{vol}(\mathcal{CRY}(n)) = Cat(0)Cat(1)Cat(2)\cdots Cat(n-2).$$

Example of a Kostant partition function

$$f_n := \# \left\{ \begin{array}{l} \text{ways of writing } (1,2,\ldots,n-1,-\binom{n}{2}) \text{ as} \\ \mathbb{N}\text{-combination of } e_i - e_j \end{array} \right\}$$

Example:

$$\begin{aligned}
n = 2: \quad & (1,-1) && = 1(1,-1) && f_2 = 1 \\
n = 3: \quad & (1,2,-3) && = 1(1,-1,0) + 3(0,1,-1) && \\
& && = 1(1,0,-1) + 2(0,1,-1) && f_3 = 2 \\
n = 4: \quad & (1,2,3,-6) && = 1(1,-1,0,0) + 3(0,1,-1,0) && \\
& && \quad + 6(0,0,1,-1) && \\
& && = \cdots && f_4 = 10
\end{aligned}$$

Example of a Kostant partition function

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$$= 1(1,0,-1) + 2(0,1,-1) \qquad f_3 = 2$$

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Theorem [Zeilberger 99]:

$$f_{n-1} = Cat(0)Cat(1)Cat(2)\cdots Cat(n-2).$$

PROOF OF A CONJECTURE OF CHAN, ROBBINS, AND YUEN

Doron ZEILBERGER [1]

Abstract: Using the celebrated Morris Constant Term Identity, we deduce a recent conjecture of Chan, Robbins, and Yuen (math.CO/9810154), that asserts that the volume of a certain $n(n-1)/2$-dimensional polytope is given in terms of the product of the first $n-1$ Catalan numbers.

Chan, Robbins, and Yuen[CRY] conjectured that the cardinality of a certain set of triangular arrays $\mathcal{A}_n$ defined in pp. 6-7 of [CRY] equals the product of the first $n-1$ Catalan numbers. It is easy to see that their conjecture is equivalent to the following *constant term identity* (for any rational function $f(z)$ of a variable z, $CT_z f(z)$ is the coeff. of z^0 in the formal Laurent expansion of $f(z)$ (that always exists)):

$$CT_{x_n} \dots CT_{x_1} \prod_{i=1}^{n} (1-x_i)^{-2} \prod_{1 \le i < j \le n} (x_j - x_i)^{-1} = \prod_{i=1}^{n} \frac{1}{i+1} \binom{2i}{i} \quad . \qquad (CRY)$$

But this is just the special case $a=2$, $b=0$, $c=1/2$, of the *Morris Identity*[M] (where we made some trivial changes of discrete variables, and 'shadowed' it)

$$CT_{x_n} \dots CT_{x_1} \prod_{i=1}^{n} (1-x_i)^{-a} \prod_{i=1}^{n} x_i^{-b} \prod_{1 \le i < j \le n} (x_j - x_i)^{-2c} = \frac{1}{n!} \prod_{j=0}^{n-1} \frac{\Gamma(a+b+(n-1+j)c)\Gamma(c)}{\Gamma(a+jc)\Gamma(c+jc)\Gamma(b+jc+1)} \, . \qquad (Chip)$$

To show that the right side of $(Chip)$ reduces to the right side of (CRY) upon the specialization $a=2, b=0, c=1/2$, do the plugging in the former and call it M_n. Then manipulate the products to simplify M_n/M_{n-1}, and *then* use *Legendre's duplication formula* $\Gamma(z)\Gamma(z+1/2) = \Gamma(2z)\Gamma(1/2)/2^{2z-1}$ three times, and *voilà*, up pops the Catalan number $\binom{2n}{n}/(n+1)$. □

Remarks: 1. By converting the left side of $(Chip)$ into a contour integral, we get the same integrand as in the Selberg integral (with $a \to -a$, $b \to -b-1$, $c \to -c$). Aomoto's proof of the Selberg integral (SIAM J. Math. Anal. **18**(1987), 545-549) goes verbatim. **2.** Conjecture 2 in [CRY] follows in the same way, from (the obvious contour-integral analog of) Aomoto's extension of Selberg's integral. Introduce a new variable t, stick $CT_t t^{-k}$ in front of (CRY), and replace $(1-x_i)^{-2}$ by $(1-x_i)^{-1}(t + x_i/(1-x_i))$. **3.** Conjecture 3 follows in the same way from another specialization of $(Chip)$.

References

[CRY] Clara S. Chan, David P. Robbins, and David S. Yuen, *On the volume of a certain polytope*, `math.CO/9810154`.

[M] Walter Morris, *"Constant term identities for finite and affine root systems, conjectures and theorems"*, Ph.D. thesis, University of Wisconsin, Madison, Wisconsin, 1982.

[1] Department of Mathematics, Temple University, Philadelphia, PA 19122, USA. `zeilberg@math.temple.edu` `http://www.math.temple.edu/~zeilberg/` . Nov. 17, 1998. Supported in part by the NSF.

Outline

1. What are type A flow polytopes?

2. What are type D flow polytopes?

3. How do we calculate volumes of flow polytopes?

4. Connection between type A flow polytopes and Kostant partition function?

5. Is there such a connection for type D flow polytopes?

From $\mathcal{CRY}(n)$ to flow polytopes $\mathcal{F}_G(\mathbf{a})$

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a	b	c	d
$\bullet$	e	f	g
0	$\bullet$	h	i
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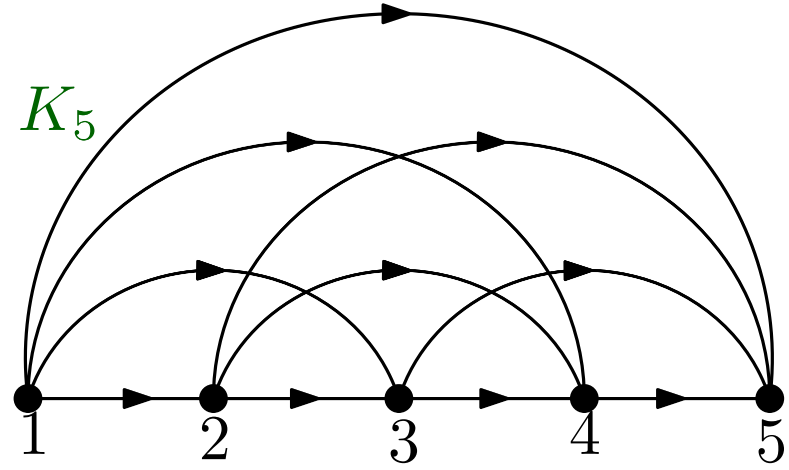

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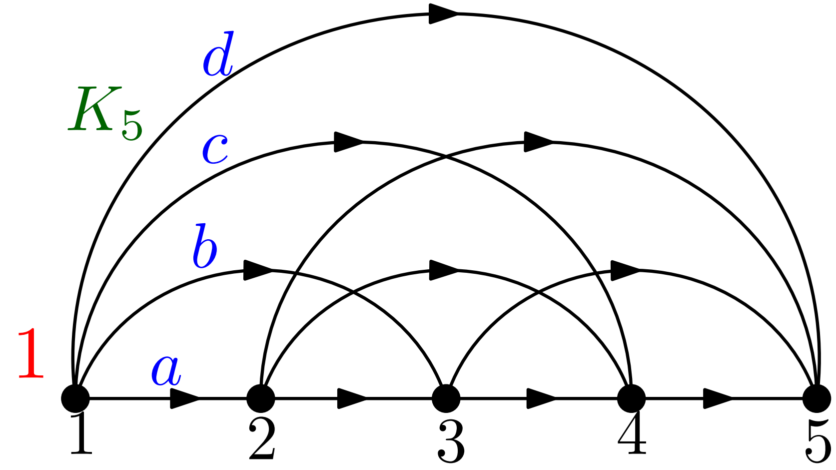
$1 = a + b + c + d$



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$$1 = a + b + c + d$$
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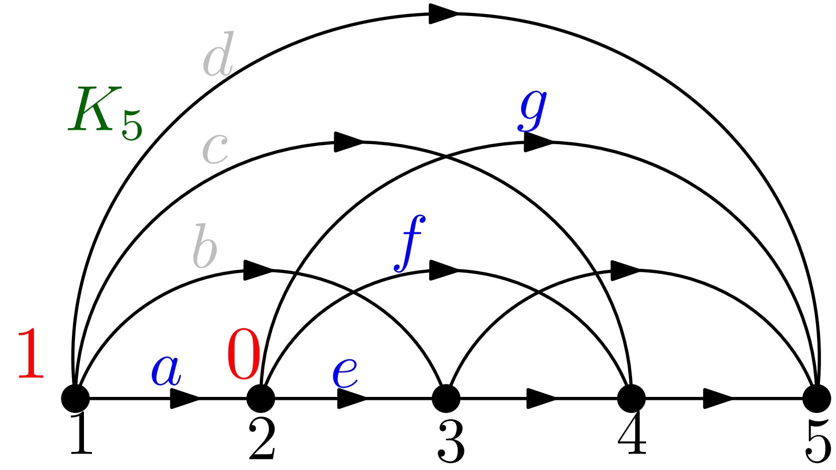

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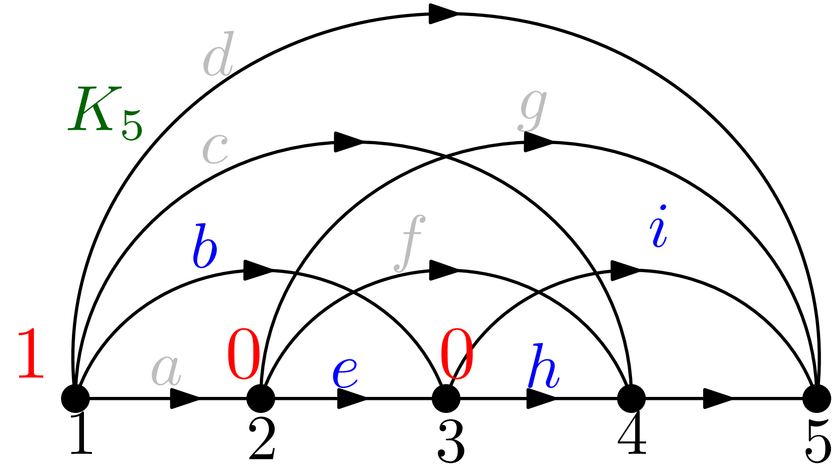

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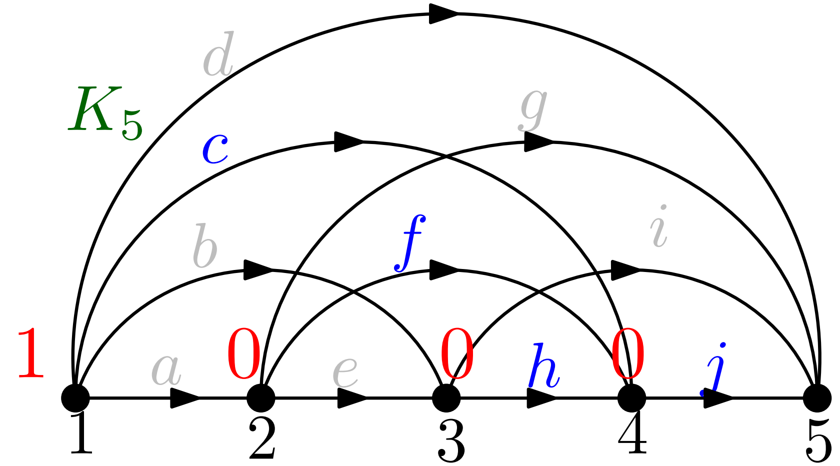

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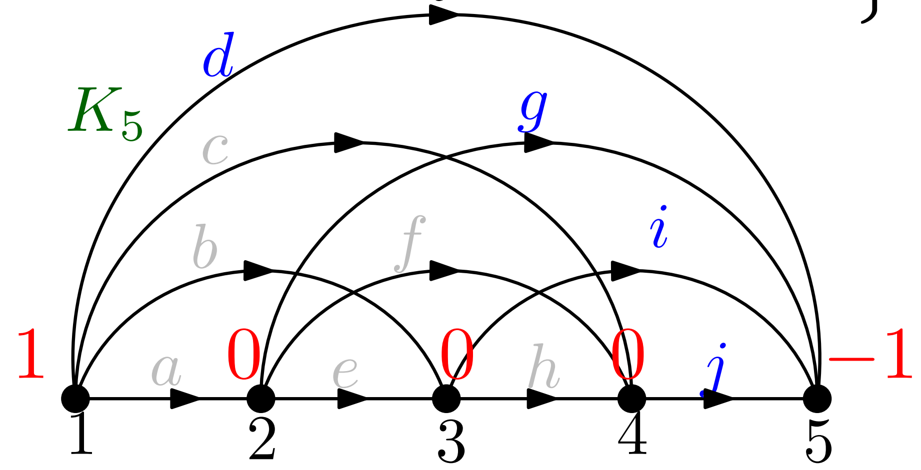

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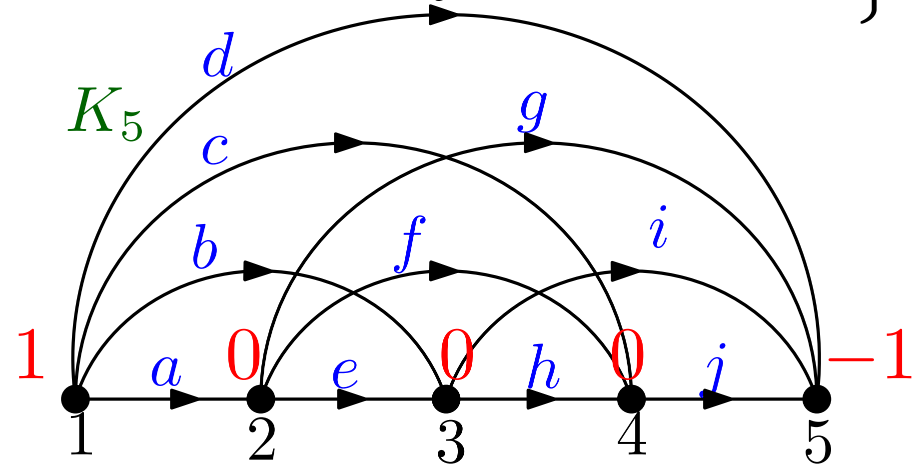

Correspondence between $\mathcal{CRY}(n)$ and **flows** in K_{n+1} with netflow: 1 first vertex, -1 last vertex, 0 other vertices.

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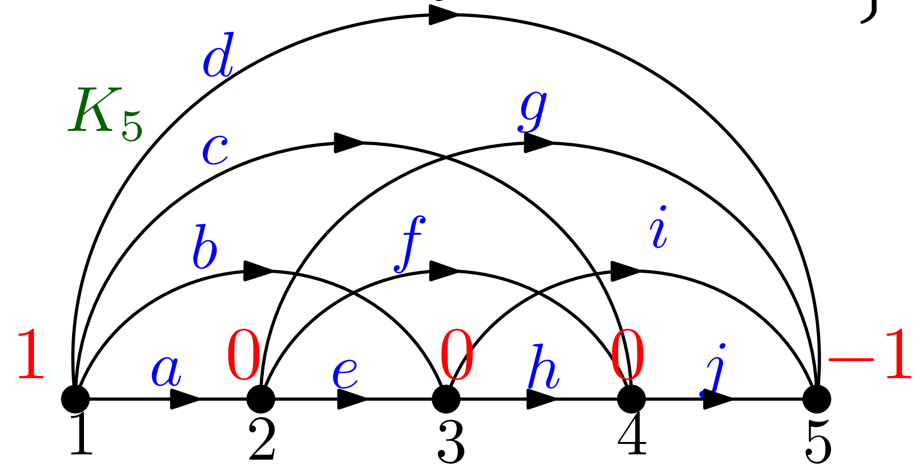

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Example: (other graphs and netflow)

$\mathcal{F}_G((3, 4, -7))$

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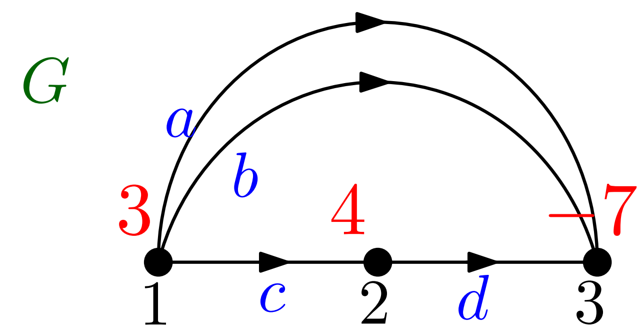
$$4 = d - c$$
$$-7 = -a - b - d$$



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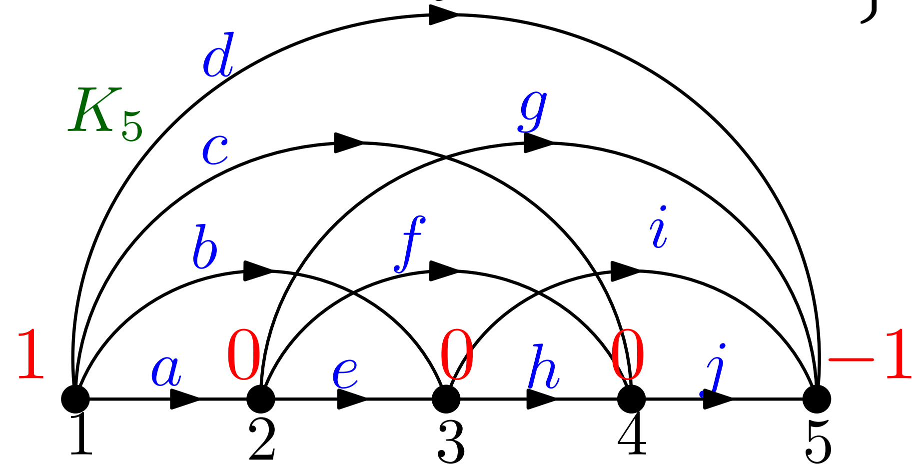

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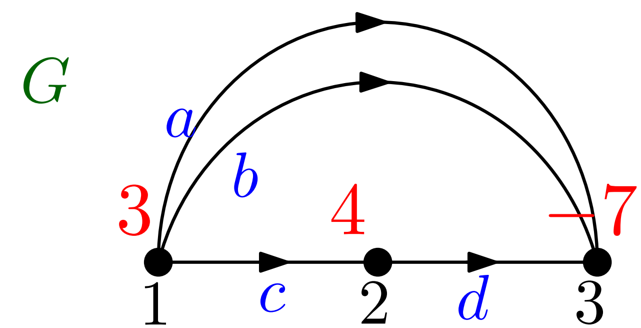

For graph G, vertices $\{1, 2, \ldots, n\}$, $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, the **flow polytope** of G is (Postnikov-Stanley 05, Baldoni-Vergne 08)

$$\mathcal{F}_G(\mathbf{a}) := \{\text{flows } b(\epsilon) \in \mathbb{R}_{\geq 0},\ \epsilon \in E(G) \mid \text{ netflow vertex } i = a_i\}$$

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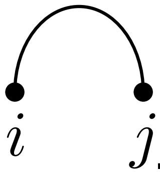
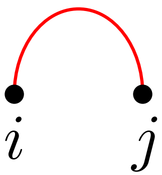
Flow polytopes for signed graphs

edges 

$(i < j)$ correspond to $e_i - e_j$ (roots in A_{n-1}^+)

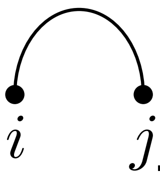
we also consider:

edges 

and 

correspond to $e_i + e_j$ and $2e_i$ (roots in C_n^+, D_n^+)

Flow polytopes for signed graphs

edges 

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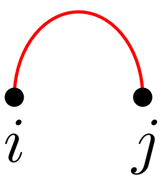
edges 

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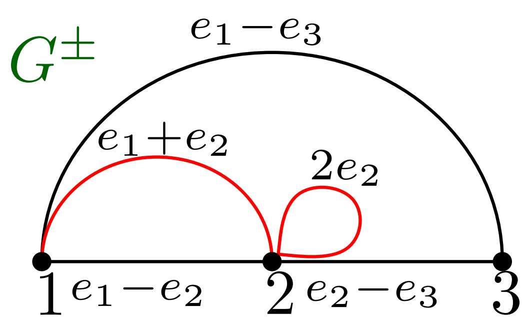

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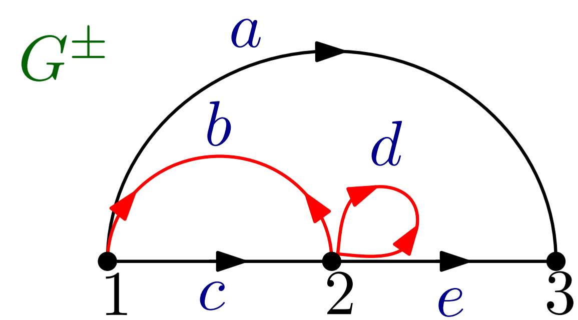

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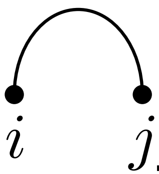
edges  ($i<j$) correspond to $e_i - e_j$ (roots in A^+_{n-1})

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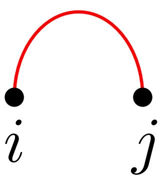
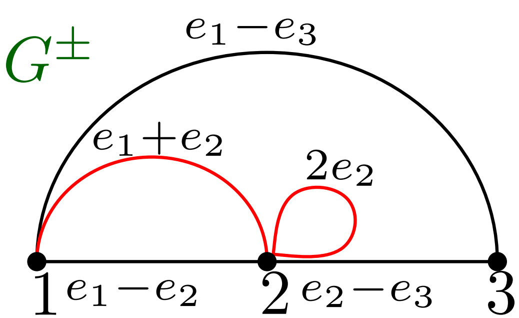
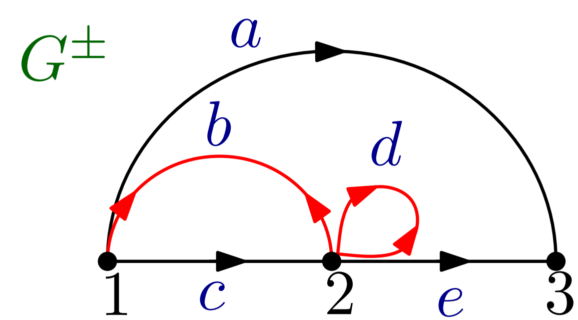
edges  and  correspond to $e_i + e_j$ and $2e_i$ (roots in C^+_n, D^+_n)

Example: (signed graphs)

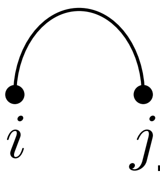


$$\mathbf{a} = (1,3,-2)$$
$$1 = a + b + c$$
$$3 = b + 2d + e - c$$
$$-2 = -a - e$$



i.e. $(1,3,-2) = a\cdot(e_1 - e_3) + b\cdot(e_1+e_2) + c\cdot(e_1-e_2) + \cdots$

Flow polytopes for signed graphs

edges 

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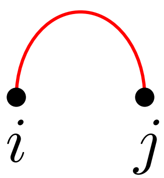
edges 

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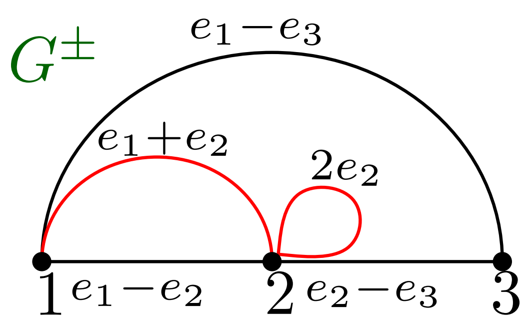
Example: (signed graphs)



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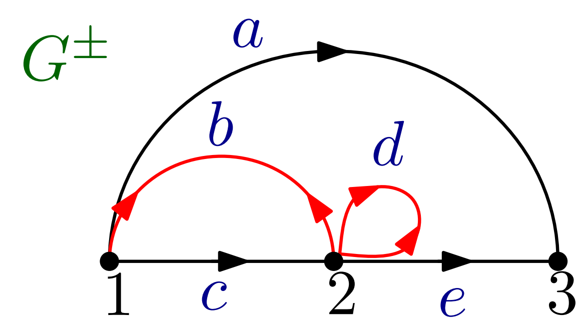

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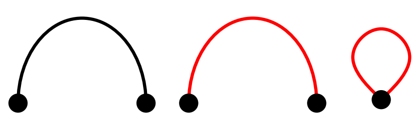
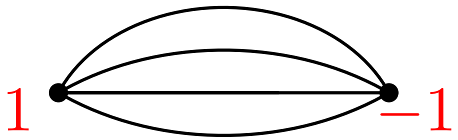
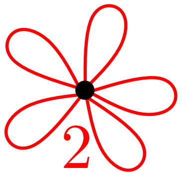
$G^{\pm}$ graph with edges  vertices $\{1, 2, \ldots, n\}$,
$\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, the **signed flow polytope** of $G^{\pm}$ is

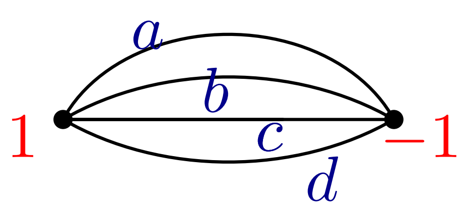
$$\mathcal{F}_{G^{\pm}}(\mathbf{a}) := \{\text{flows } b(\epsilon) \in \mathbb{R}_{\geq 0},\ \epsilon \in E(G^{\pm}) \mid \text{ netflow vertex } i = a_i\}$$

Examples flow polytopes





Examples flow polytopes



$\mathbf{a} = (1, -1)$

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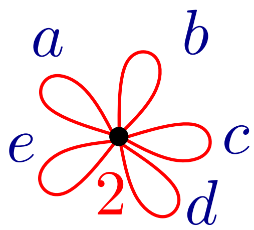
$a, b, c, d \geq 0$

simplex



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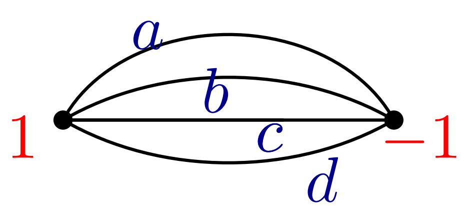

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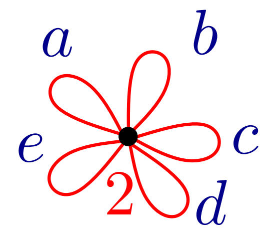

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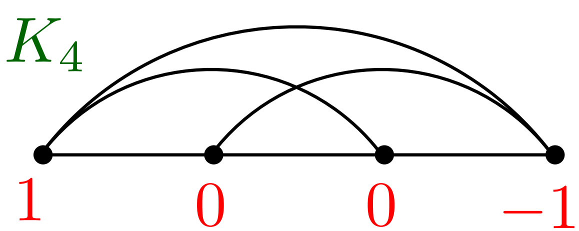
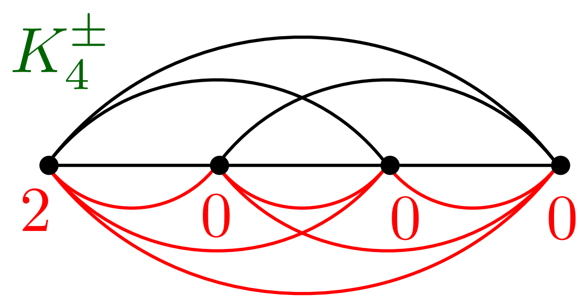


$\mathcal{CRY}$



type D $\mathcal{CRY}$

Outline

1. What are type A flow polytopes? ✓

2. What are type D flow polytopes? ✓

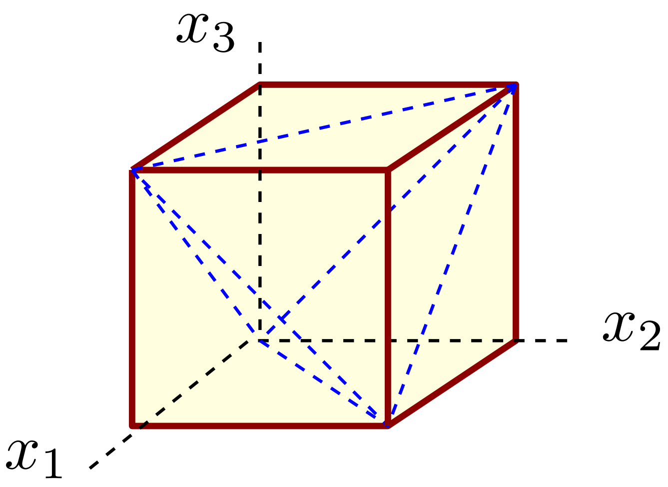
3. How do we calculate volumes of flow polytopes?

4. Connection between type A flow polytopes and Kostant partition function?

5. Is there such a connection for type D flow polytopes?

Volumes and triangulations

- $\mathcal{P} \subset \mathbb{R}^n$ convex polytope, $\dim(\mathcal{P}) = n$,
- A **triangulation** T is collection of n-simplices:
  (i) $\mathcal{P} = \bigcup_{\Delta \in T} \Delta$,
  (ii) for $\Delta, \Delta' \in T$, $\Delta \cap \Delta'$ is face common to Δ, Δ'.



Triangulation into 6 Δs.

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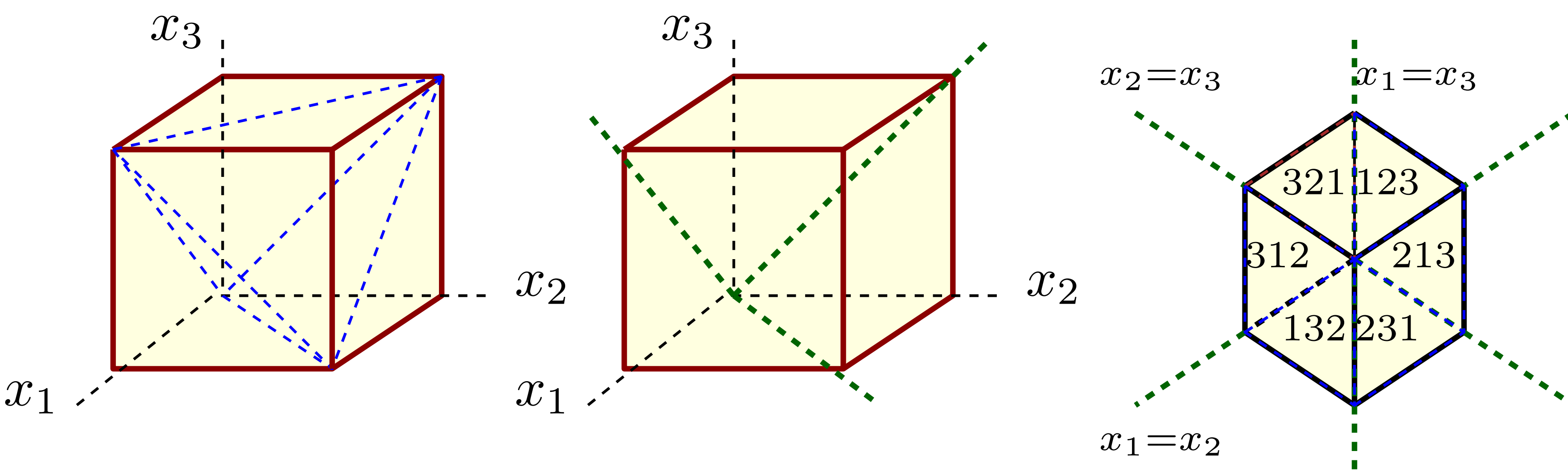

Triangulation into 6 Δs. Δs parametrized by S_3.

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  $\Rightarrow$ normalized volume of $\mathcal{P} = \#T = \#$ **objects**.

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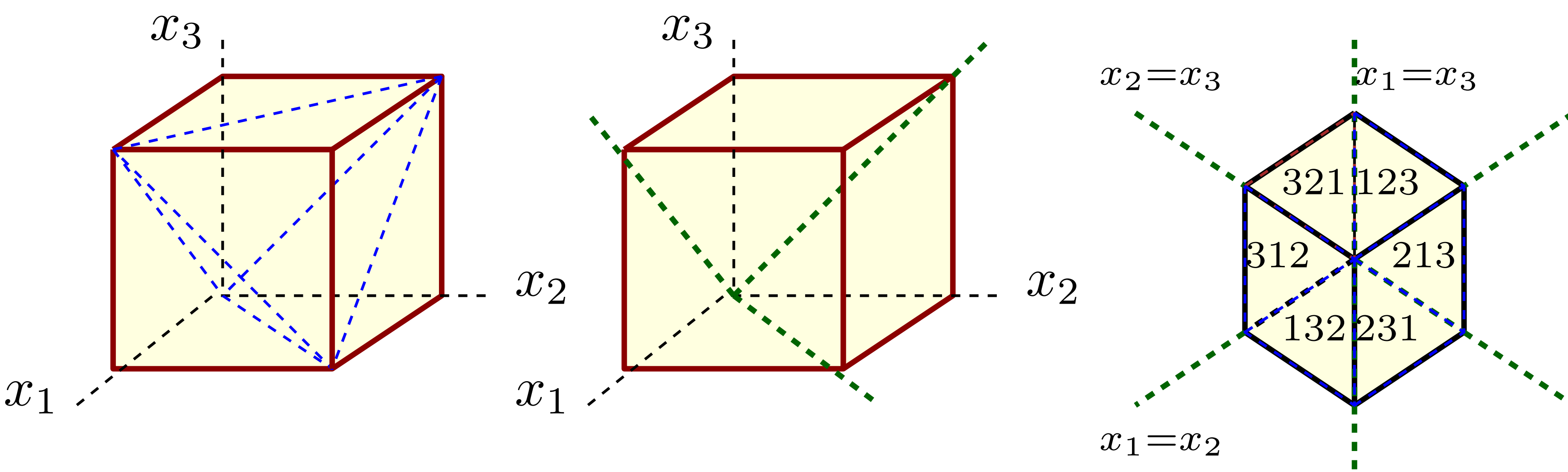

Triangulation into 6 Δs. Δs parametrized by S_3.

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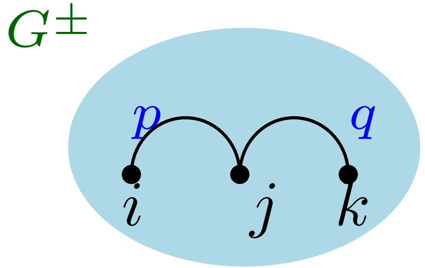
- we triangulate $\mathcal{F}_{G^\pm}$, triangulation indexed by certain **integral flows** on $G^\pm$

Triangulating flow polytopes



underlying relation:
$(e_i - e_j) + (e_j - e_k) = e_i - e_k$.

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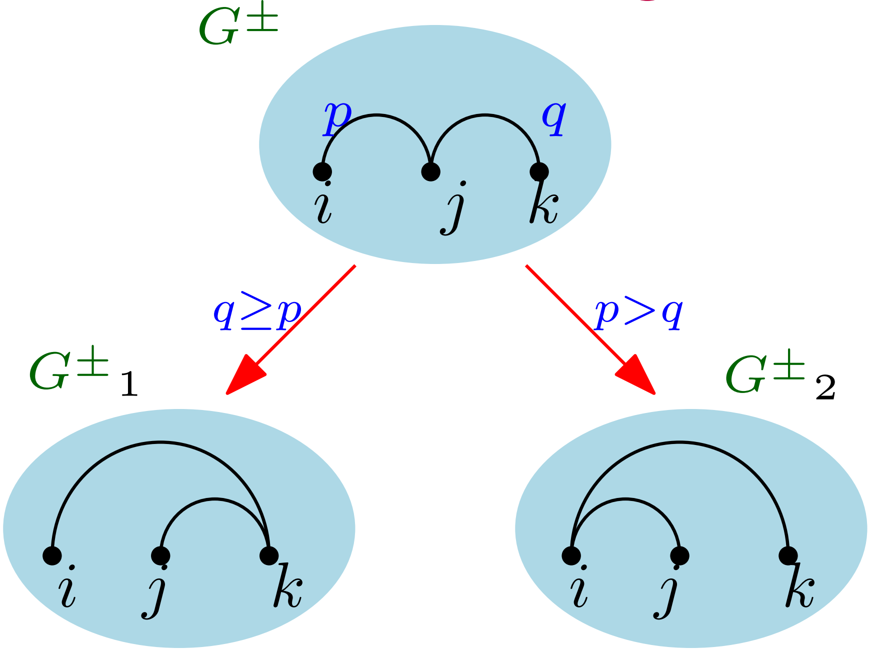

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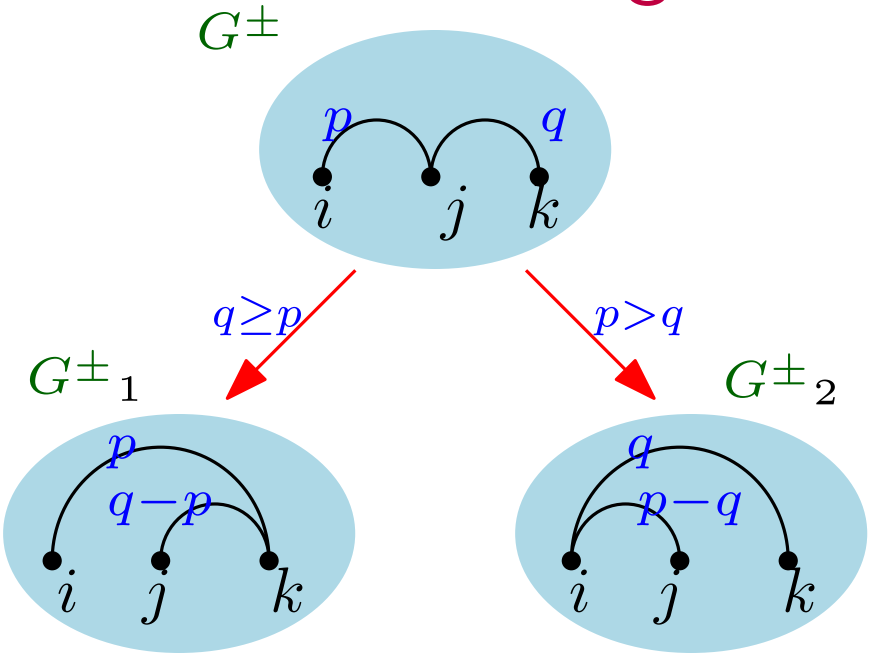

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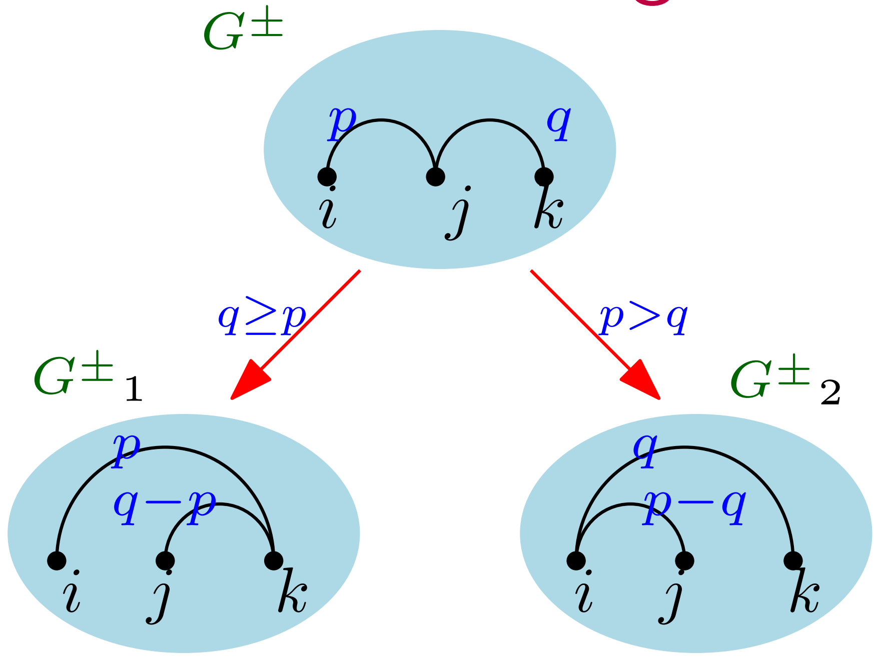

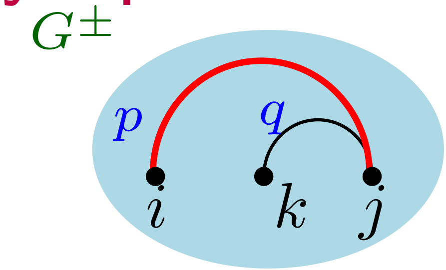

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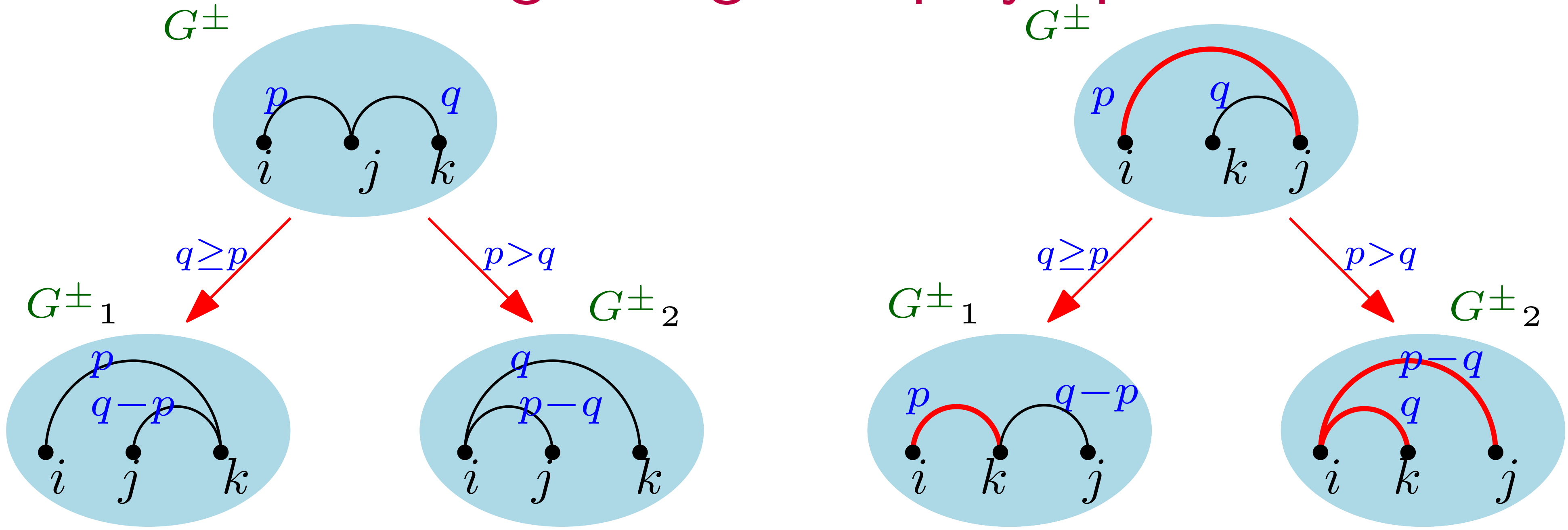

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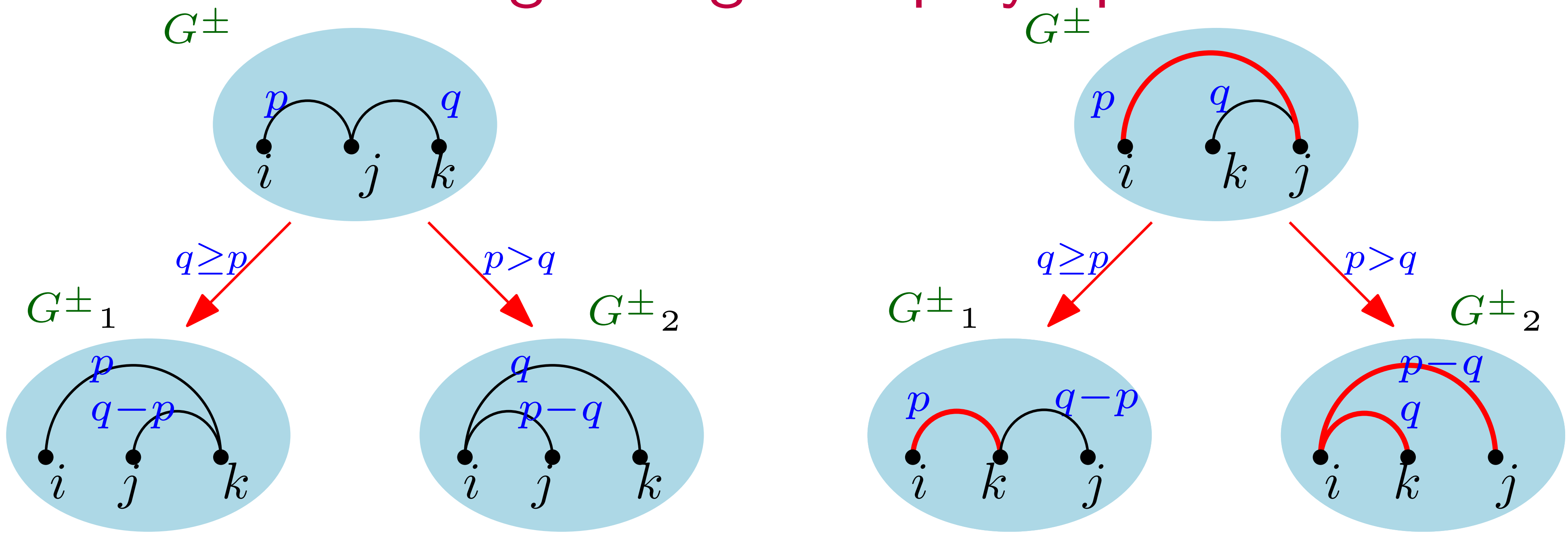

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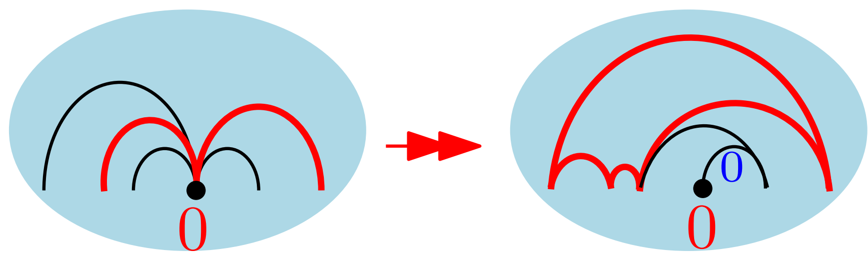

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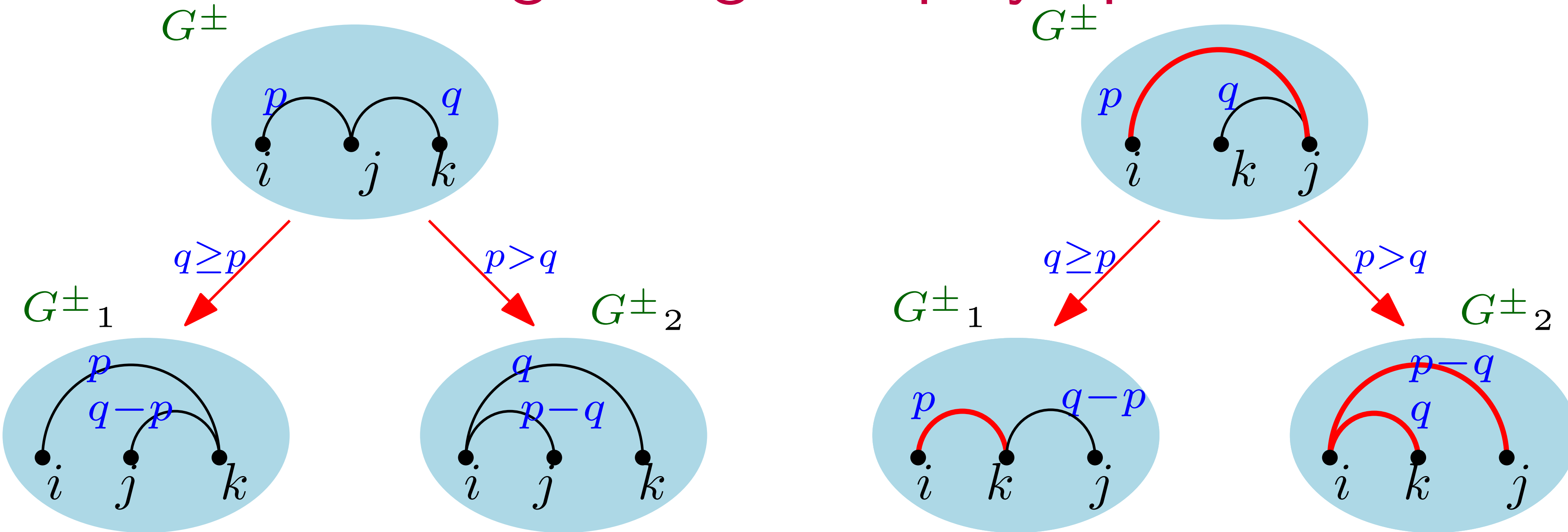

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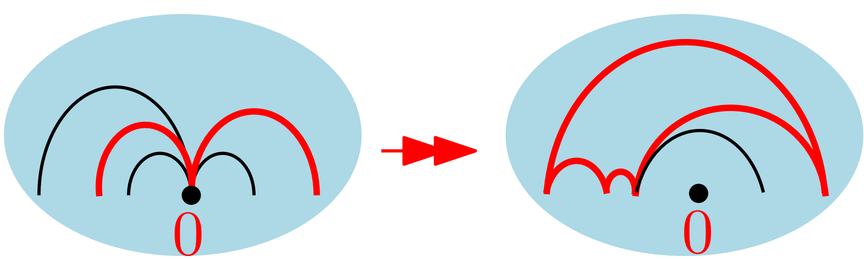

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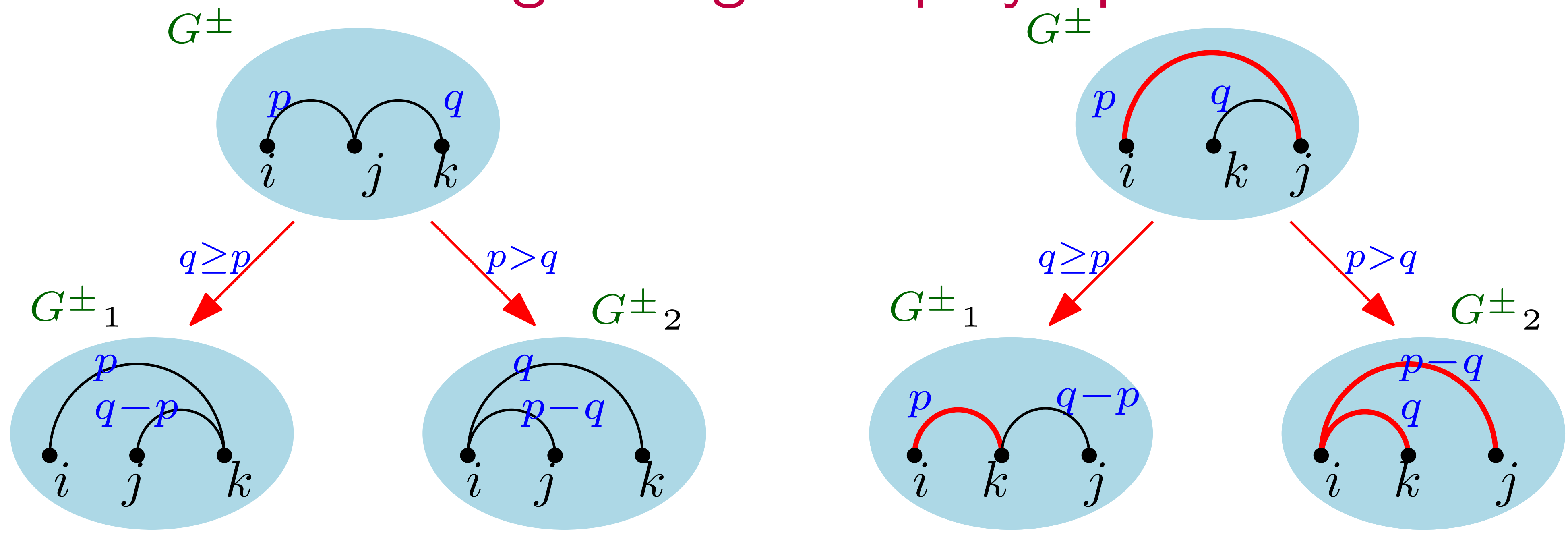

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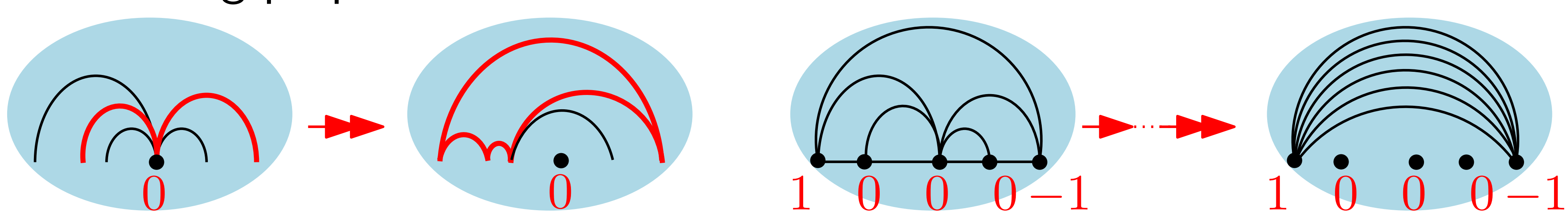

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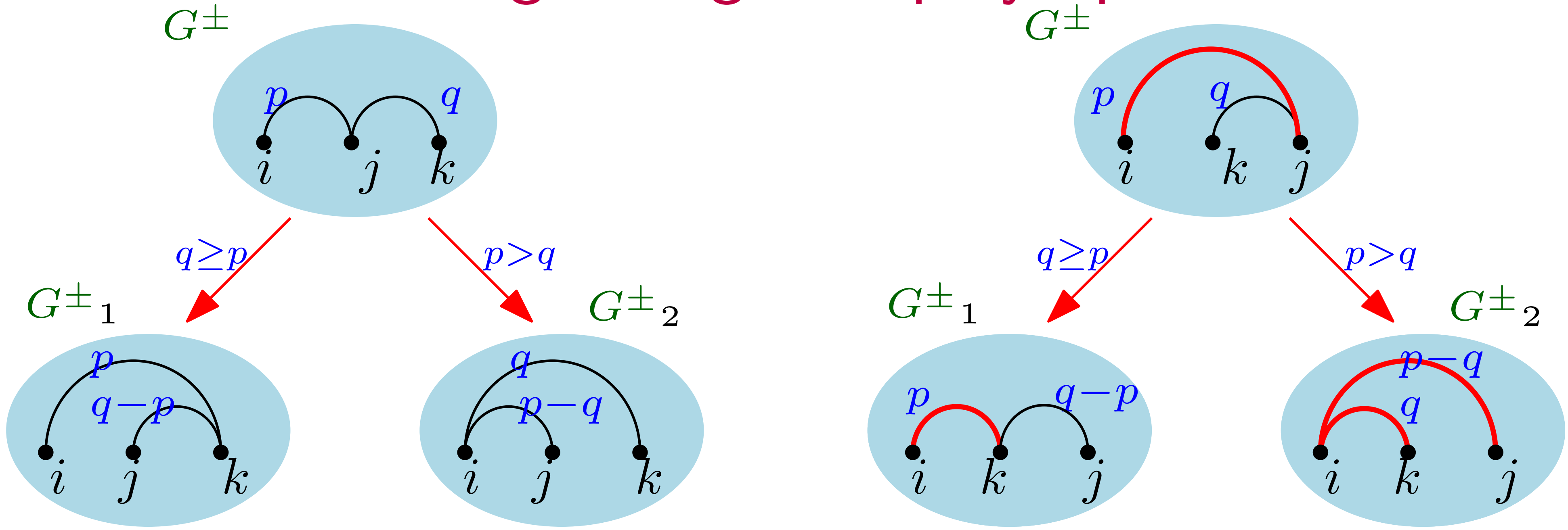

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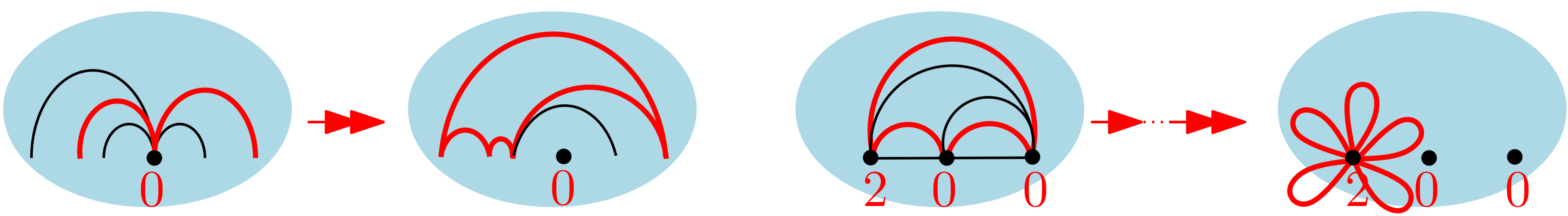

Integral flows / lattice points: Kostant partition function

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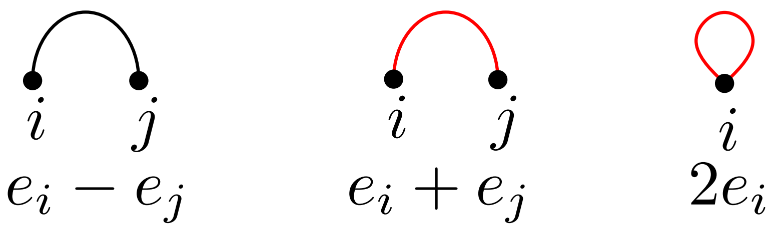
Interpret $E(G^\pm)$ as multiset of roots:



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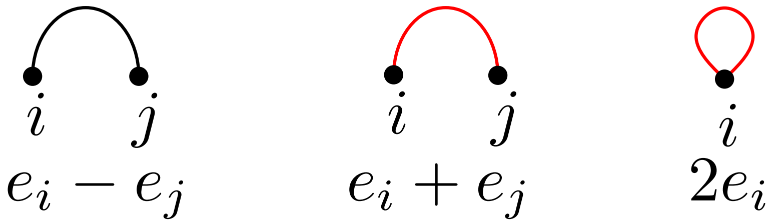

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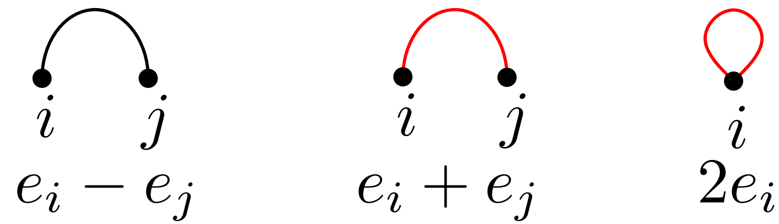

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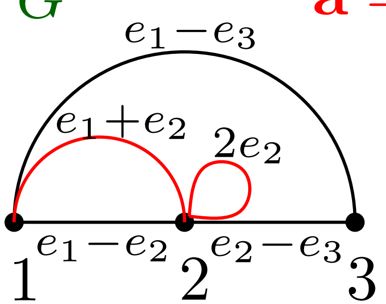

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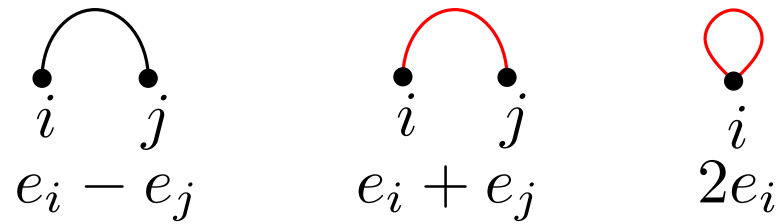

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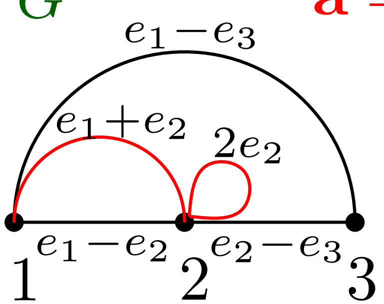

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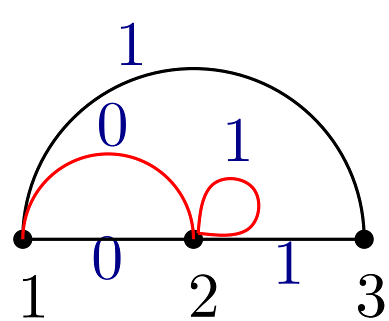

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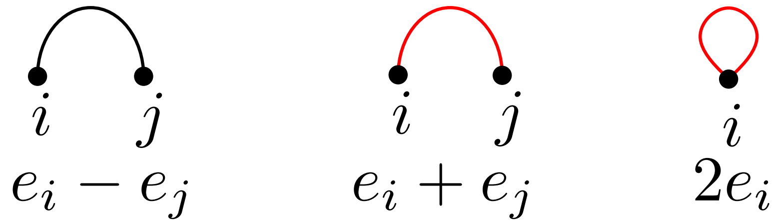

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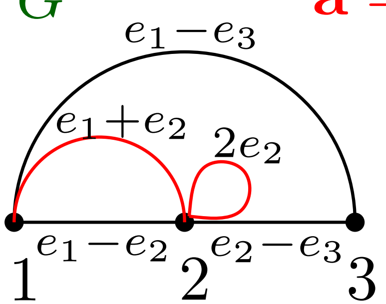

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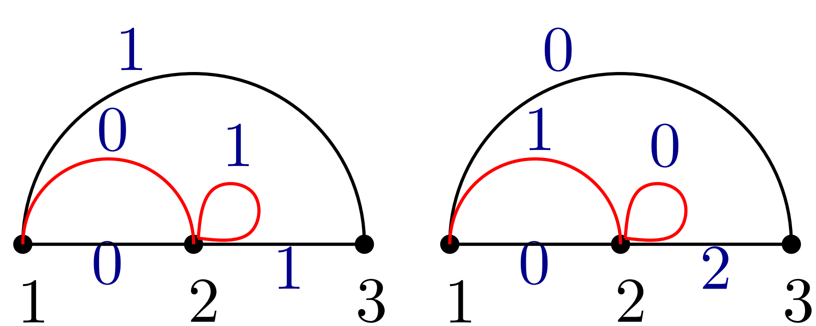
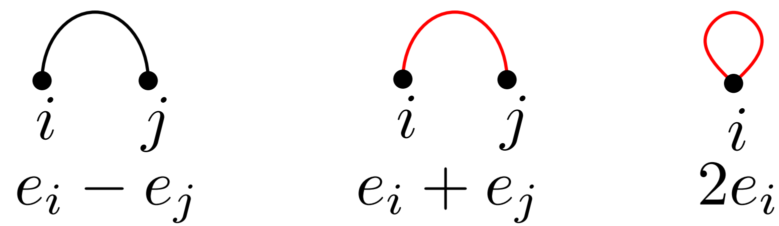


Integral flows / lattice points: Kostant partition function

$$\mathcal{F}_{G^\pm}(\mathbf{a}) := \{\text{flows } b(\epsilon) \in \mathbb{R}_{\geq 0},\ \epsilon \in E(G^\pm) \mid \text{ netflow vertex } i = a_i\}$$

Interpret $E(G^\pm)$ as multiset of roots:

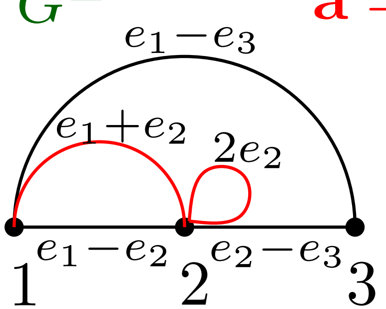


$$\underbrace{\{\textbf{lattice points of} \mathcal{F}_{G^\pm}(\mathbf{a})\}}_{\text{integral flows netflow } \mathbf{a}} \equiv \# \left\{ \begin{matrix} \text{ways of expressing } \mathbf{a} \text{ as an} \\ \mathbb{N}\text{-combination of roots of } G^\pm \end{matrix} \right\}$$

We call this number: $K_{G^\pm}(\mathbf{a})$, the **Kostant partition function**.

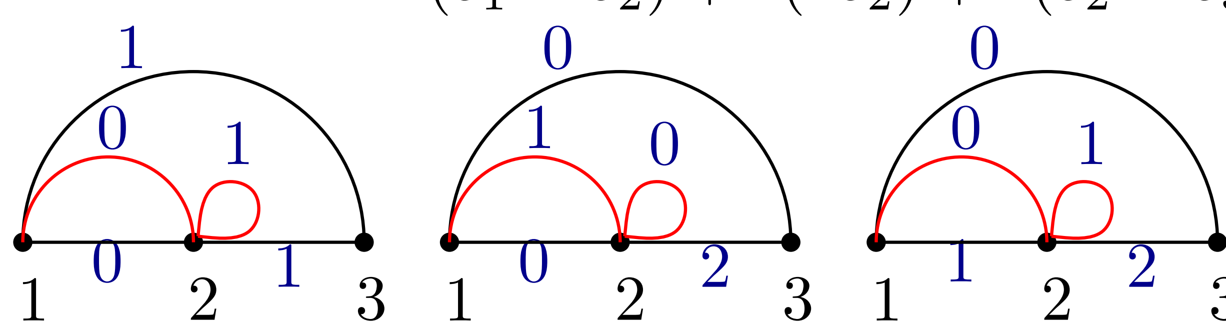
Example:

$G^\pm$ $\quad \mathbf{a} = (1, 3, -2)$



$K_{G^\pm}((1, 3, -2)) = 3$, since:

$$\begin{aligned}(1, 3, -2) &= 1(e_1 - e_3) + 1(2e_2) + 1(e_2 - e_3) \\ &= 1(e_1 + e_2) + 2(e_2 - e_3) \\ &= 1(e_1 - e_2) + 1(2e_2) + 2(e_2 - e_3)\end{aligned}$$



Outline

1. What are type A flow polytopes? ✓

2. What are type D flow polytopes? ✓

3. How do we calculate volumes of flow polytopes? ✗

4. Connection between type A flow polytopes and Kostant partition function?

5. Is there such a connection for type D flow polytopes?

Volume of $\mathcal{F}_G(\mathbf{1}, \mathbf{0}, \ldots, \mathbf{0}, \mathbf{-1})$

Theorem [Postnikov-Stanley 00]:

For a graph G, vertices $\{1, 2 \ldots, n\}$, only negative edges

$$\boxed{\dim(\mathcal{F}_G)! \cdot \mathrm{vol}\,(\mathcal{F}_G(1, 0, \ldots, 0, -1)) = K_G(0, d_2, \ldots, d_{n-1}, -\sum_{i=2}^{n-1} d_i),}$$

where $d_i = (\text{indegree of } i) - 1$.

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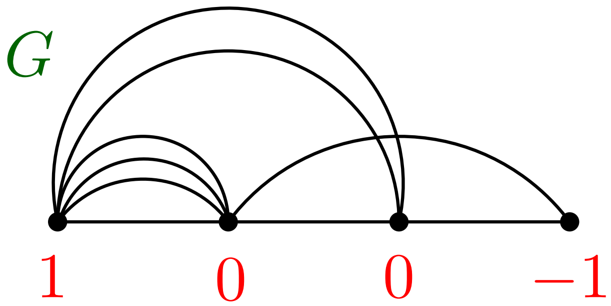

Volume of flow polytope $\mathcal{F}_G(1, 0, 0, -1)$ for

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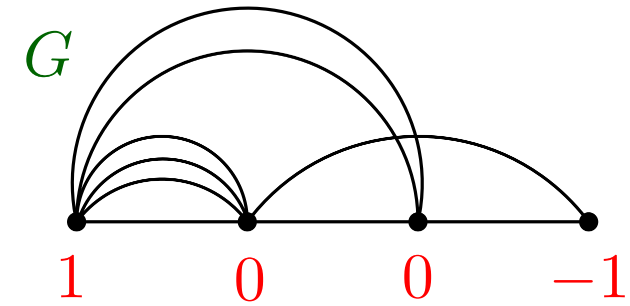

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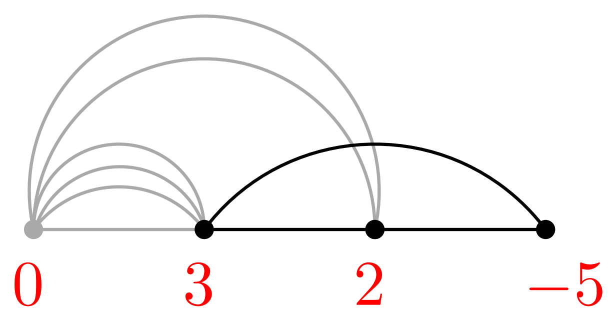

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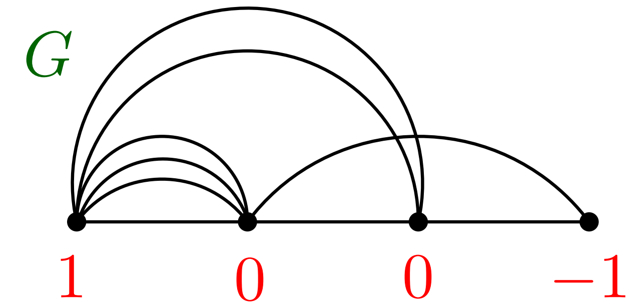

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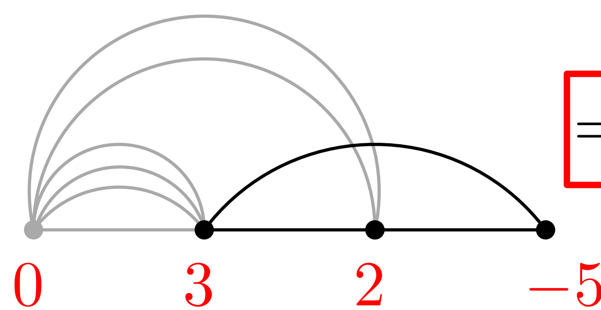

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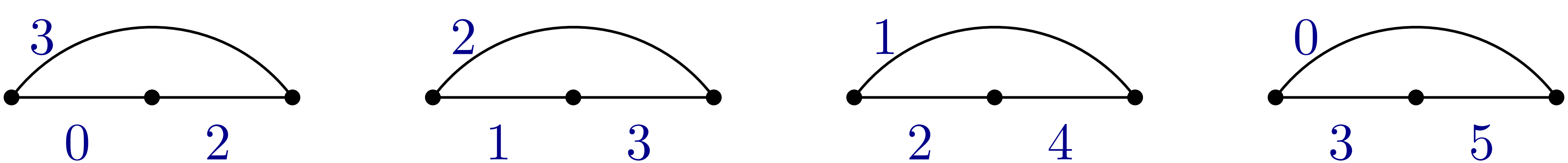

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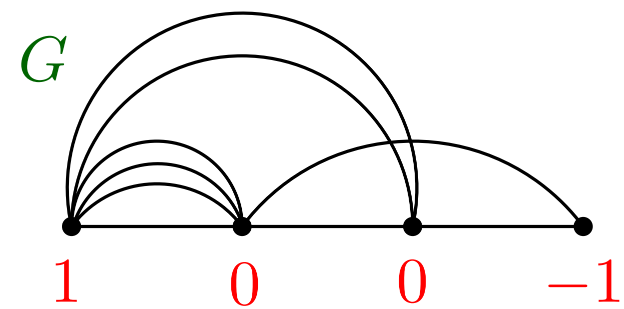

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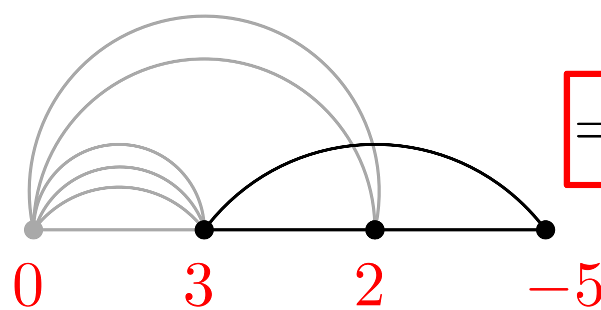

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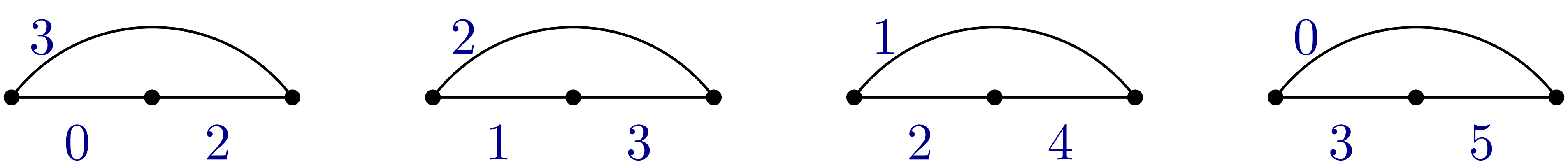
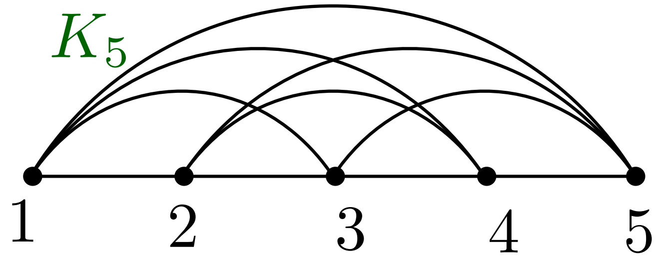


Note:

$\mathrm{vol}(\mathcal{F}_G(1, 0, \ldots, 0, -1))$ given by $\#$ lattice points of $\mathcal{F}_G(0, d_2, d_3, \ldots)$.

Application to $\mathcal{CRY}(n)$

Since $\mathcal{CRY}(n) = \mathcal{F}_{K_{n+1}}(1, 0, \ldots, 0, -1)$, then



Corollary

$$\binom{n}{2}! \cdot \mathrm{vol}(\mathcal{CRY}(n)) = K_{K_{n+1}}\left(0, 0, 1, 2, \ldots, n-2, -\binom{n-1}{2}\right)$$
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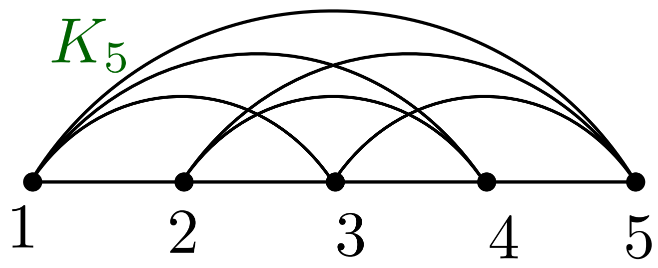

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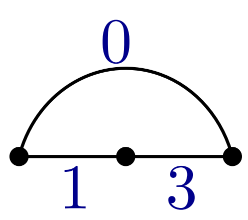
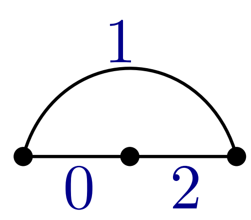
Example:

$6! \cdot \mathrm{vol}(\mathcal{CRY}(4)) = K_{K_3}(1, 2, -3) = 2$:

$(1, 2, -3) = 1(e_1 - e_2) + 3(e_2 - e_3) = 1(e_1 - e_3) + 2(e_2 - e_3)$





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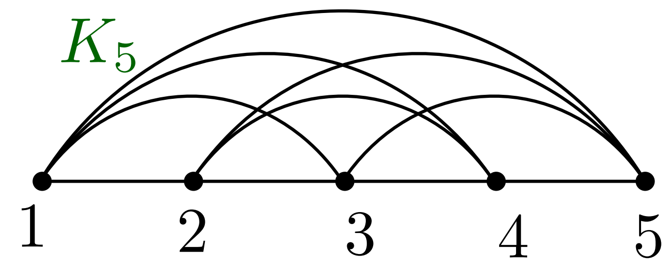

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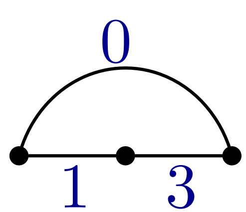

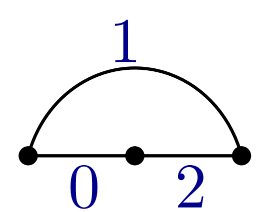

Remarks:

- Zeilberger used $(\dagger)$, the generating series of $K_G(\mathbf{a})$, and the *Morris Identity* to calculate $\binom{n}{2}! \cdot \mathrm{vol}(\mathcal{CRY}(n)) = \prod_{i=0}^{n-2} Cat(i)$,

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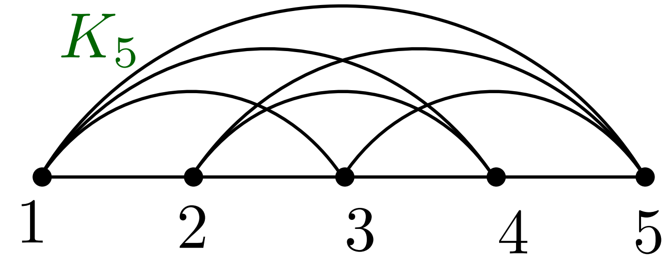

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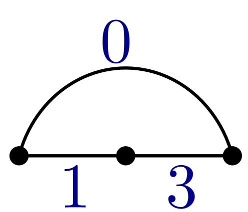

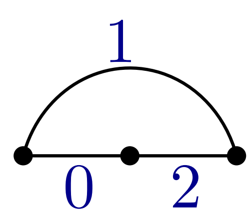

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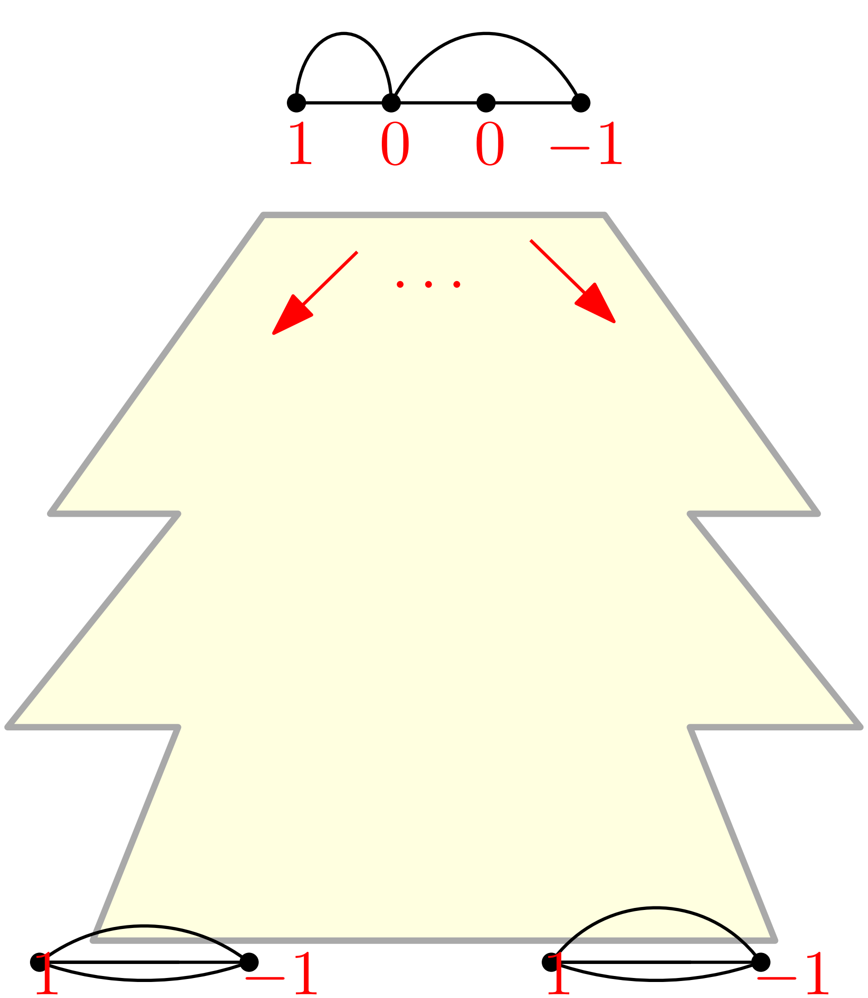
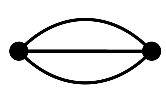
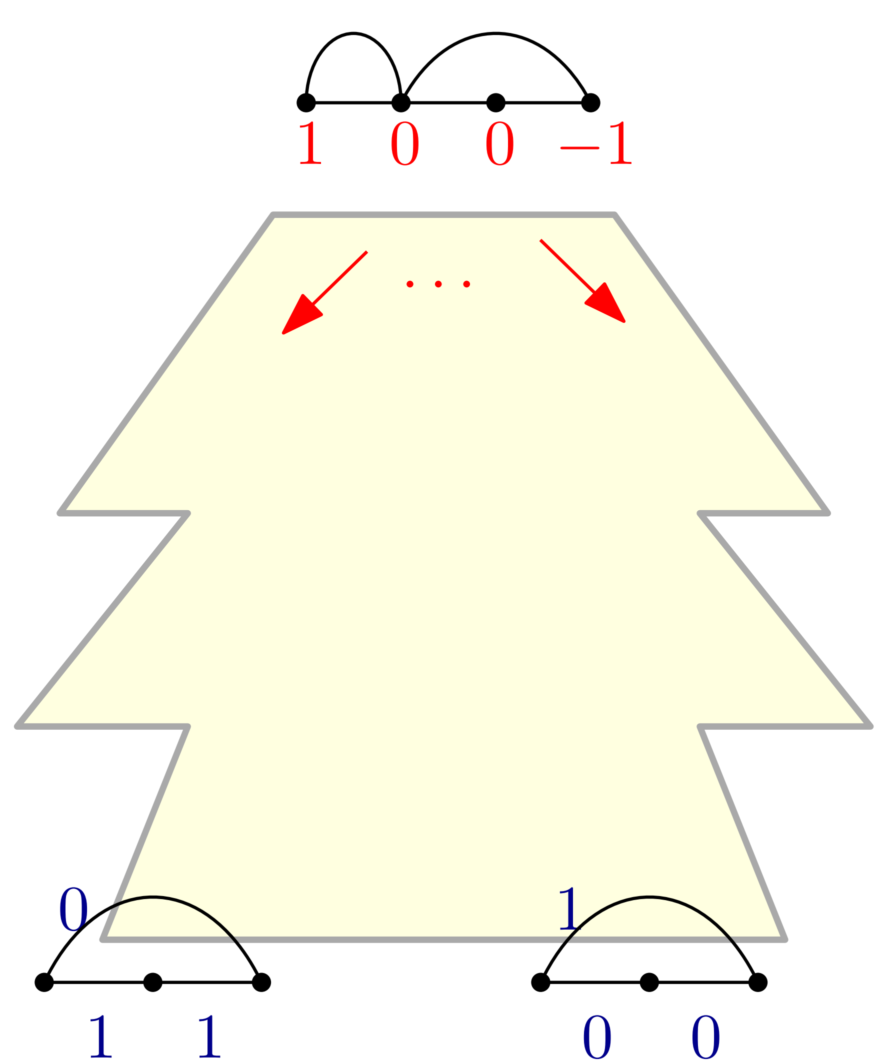
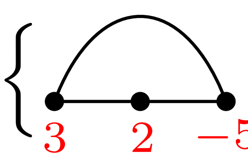
- No combinatorial proof for this formula of $\mathrm{vol}(\mathcal{CRY}(n))$.

Idea proof of Theorem on $\mathrm{vol}\mathcal{F}_G(e_1 - e_n)$



$\mathrm{vol}(\mathcal{F}_G(\mathbf{a})) = \frac{1}{\dim(\mathcal{F}_G)!}\#\{$  $\}$



$\mathrm{vol}(\mathcal{F}_G(\mathbf{a})) = \frac{1}{\dim(\mathcal{F}_G)!}\#\{$  $\}$

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Dynamic Flow

For signed graphs: $\mathrm{vol}(\mathcal{F}_{G^\pm}(2e_1)) \neq \#\{\text{integral flows on } G^\pm\} = K_{G^\pm}(\cdot)$

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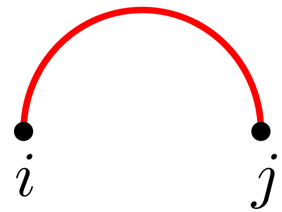
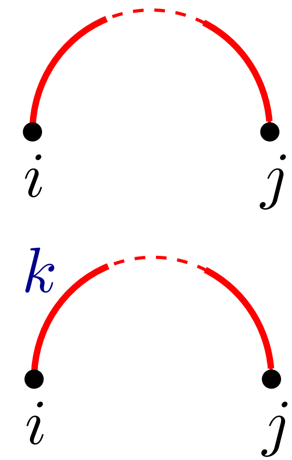
- split edges 

into two half-edges 

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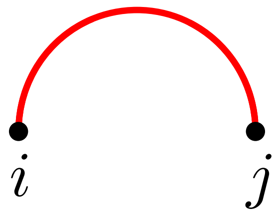
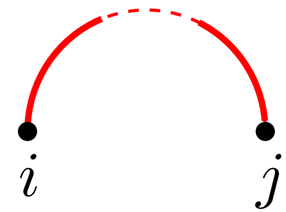
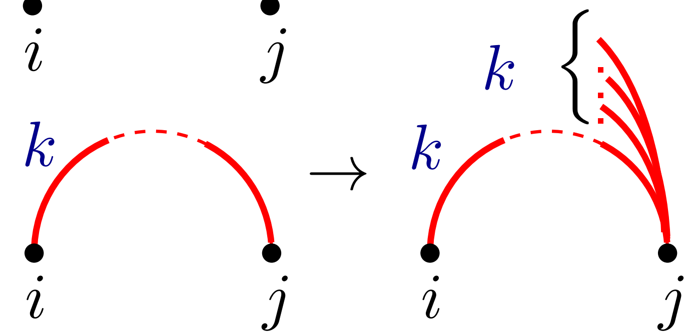
- split edges 

into two half-edges 

- if left half-edge has flow k
  $\rightarrow$ **add** k new right half-edges



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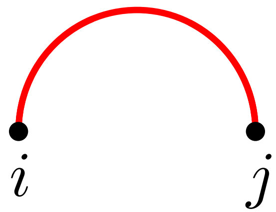
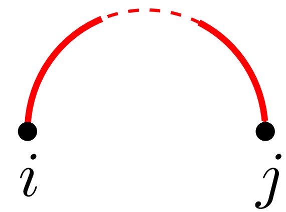
- split edges 

into two half-edges 

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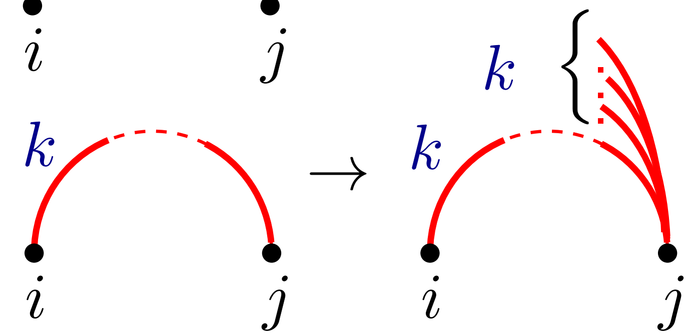

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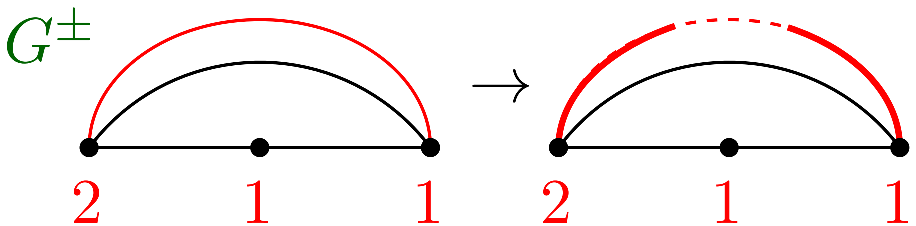

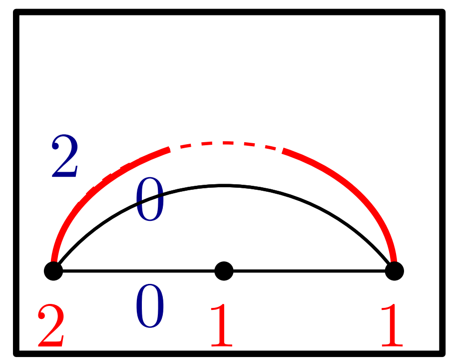

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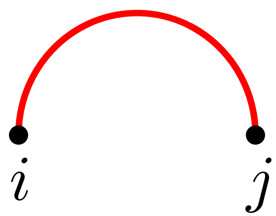
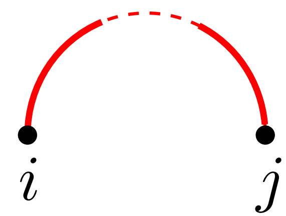
- split edges 

into two half-edges 

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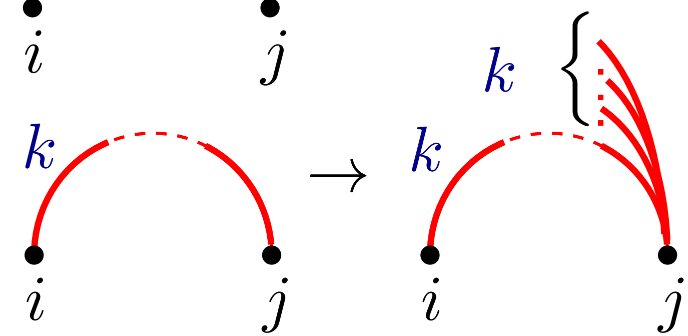

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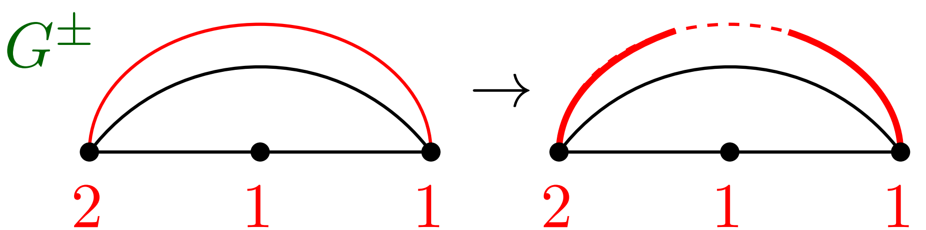

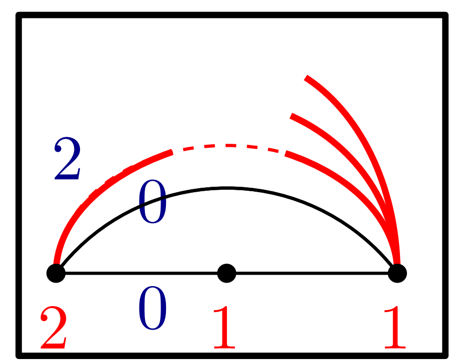

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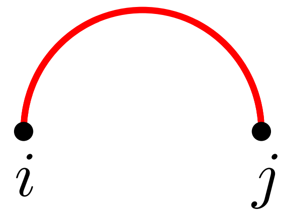
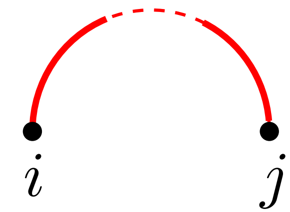
- split edges 

into two half-edges 

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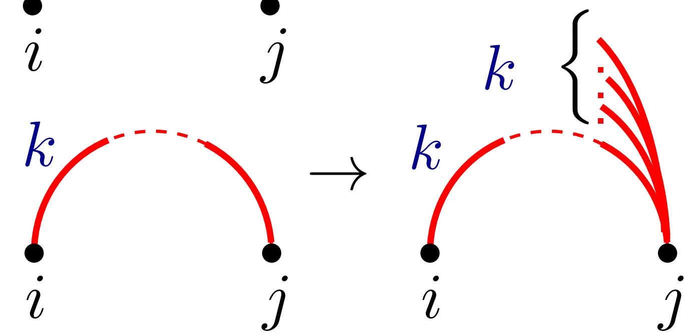

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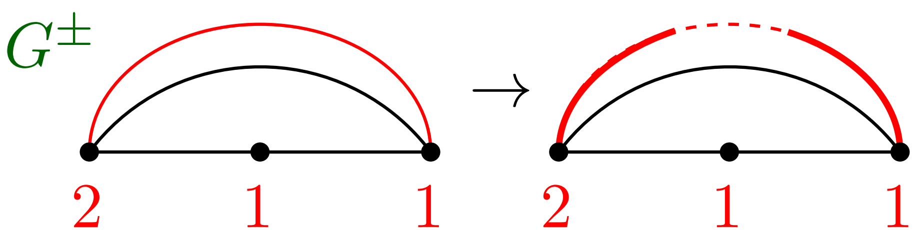
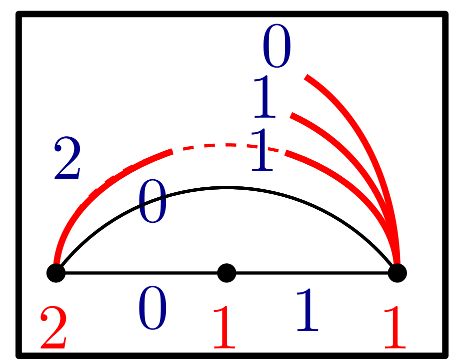
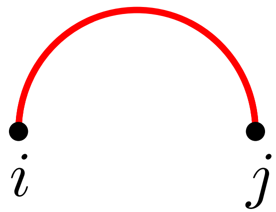
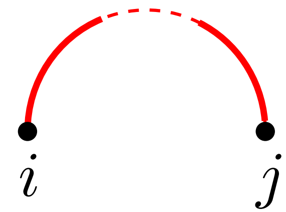




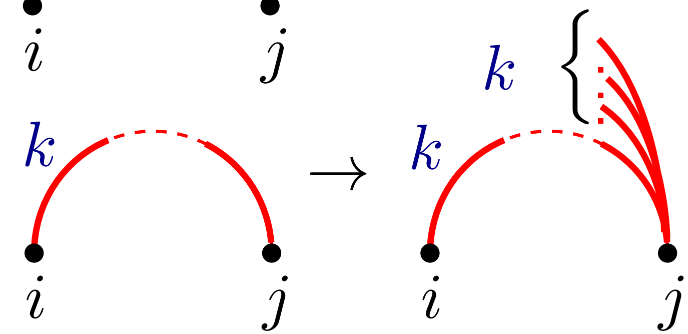
Dynamic Flow

For signed graphs: $\mathrm{vol}(\mathcal{F}_{G^\pm}(2e_1)) \neq \#\{\text{integral flows on } G^\pm\} = K_{G^\pm}(\cdot)$
$= \#\{\text{integral } \textbf{dynamic} \text{ flows on } G^\pm\}$

- split edges 

into two half-edges 

- if left half-edge has flow k
  → **add** k new right half-edges



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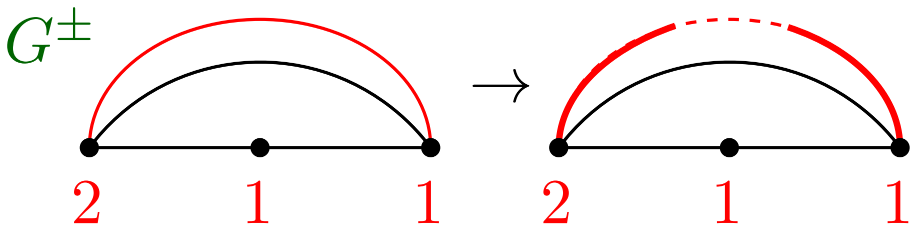

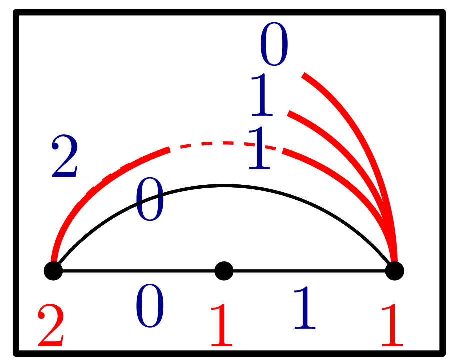

we define:

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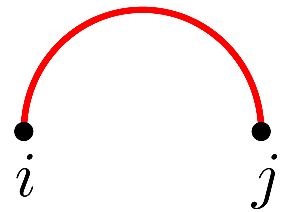
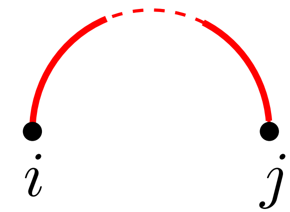
- split edges  into two half-edges 
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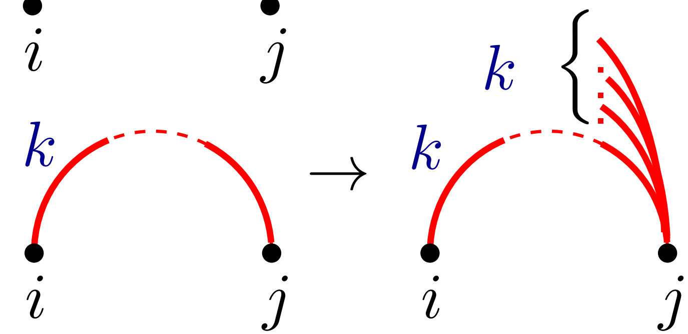

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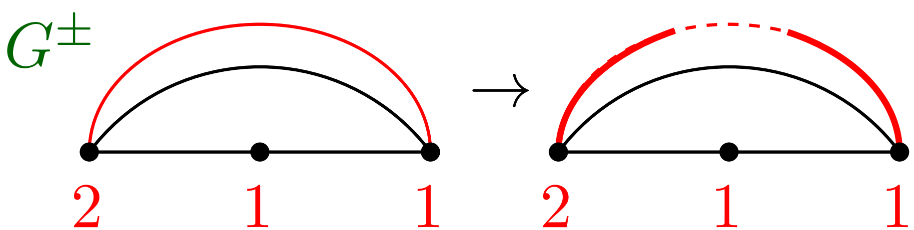

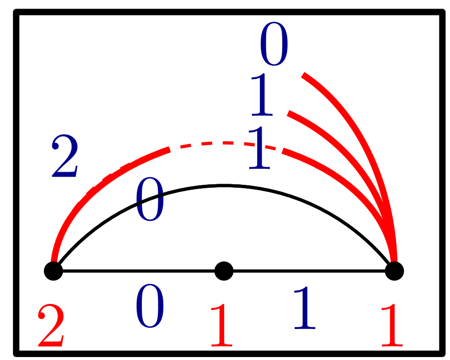

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Generating function:

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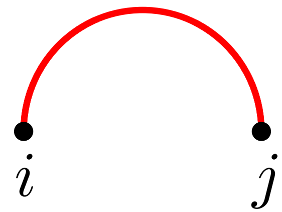
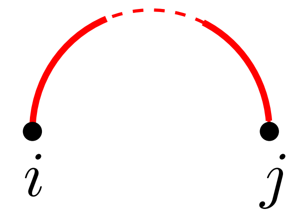
- split edges 

into two half-edges 

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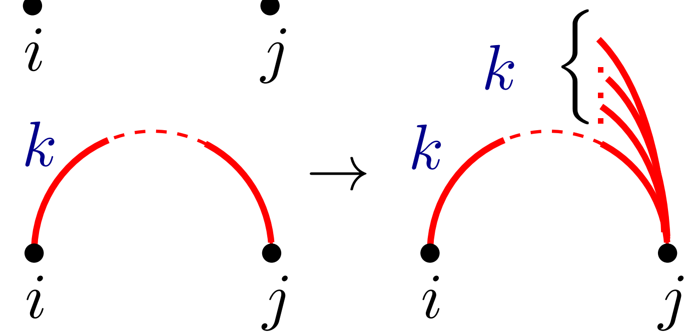

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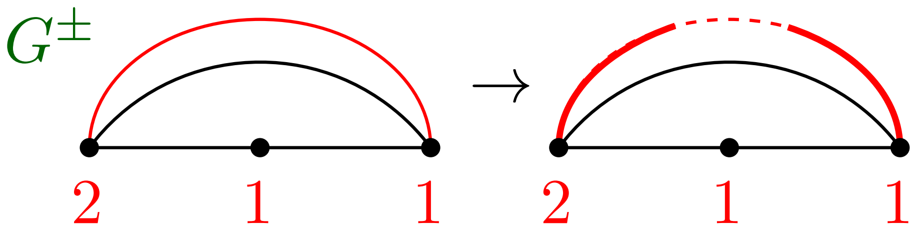

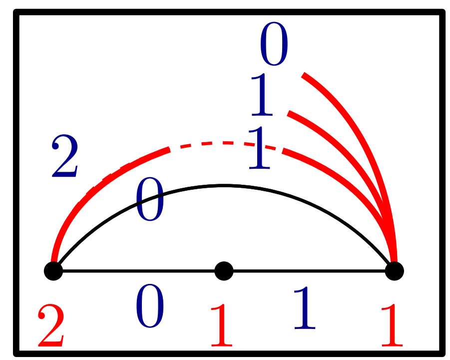

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Volume of $\mathcal{F}_{G^\pm}(\mathbf{2}, \mathbf{0}, \ldots, \mathbf{0})$

Theorem [Mészáros-M 11]:

For a signed graph $G^\pm$, vertices $\{1, 2 \ldots, n\}$

$$\boxed{\dim(\mathcal{F}_{G^\pm})! \cdot \mathrm{vol}\,(\mathcal{F}_{G^\pm}(2, 0, \ldots, 0)) = K_{G^\pm}^{\textbf{dyn.}}(0, d_2, \ldots, d_{n-1}, d_n),}$$

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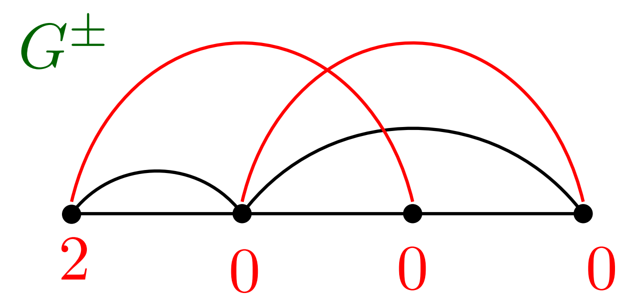

Volume of flow polytope $\mathcal{F}_{G^\pm}(2,0,0,0)$ for

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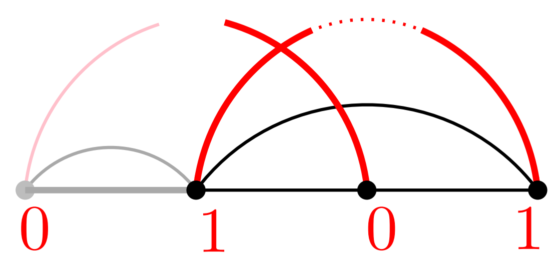
$\rightarrow$ #integral **dynamic** flows in



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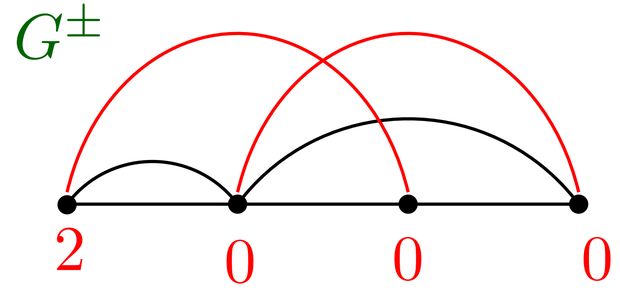

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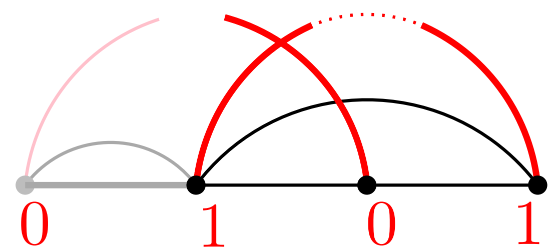

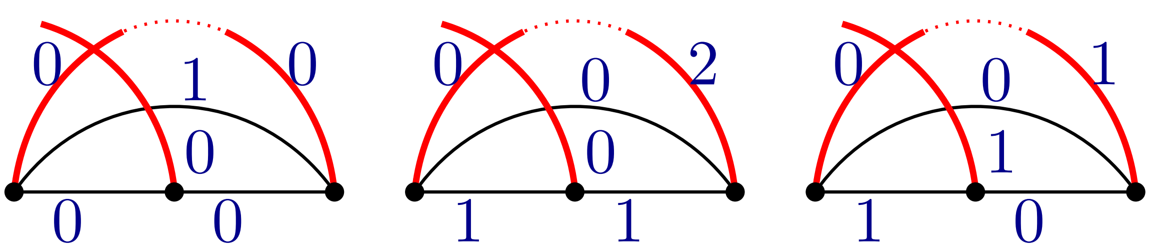

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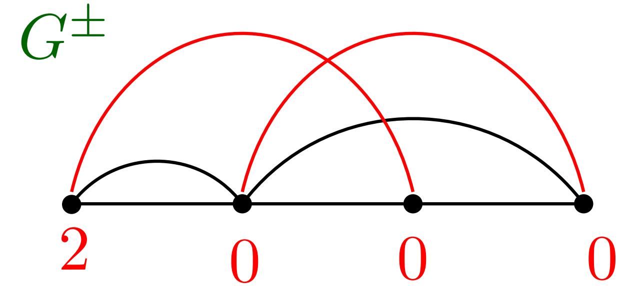

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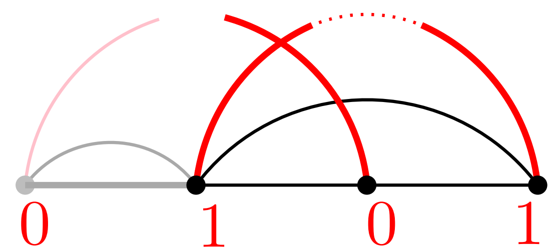

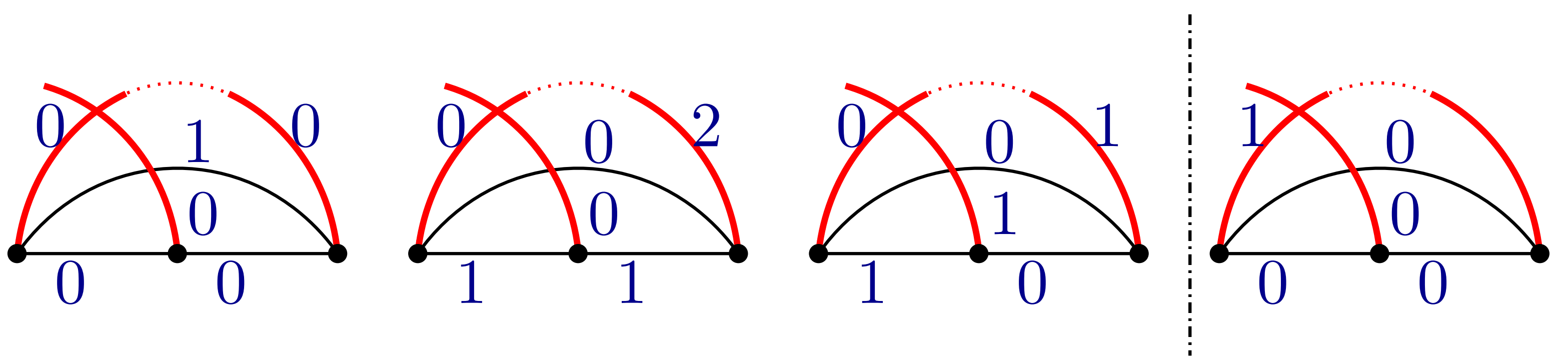

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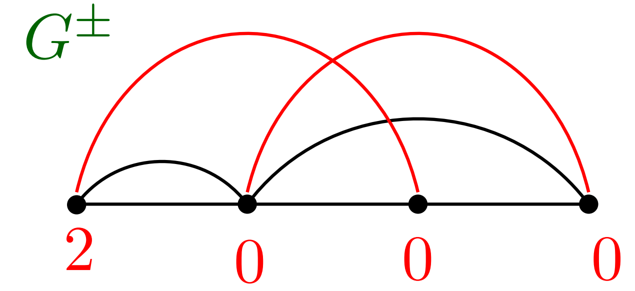

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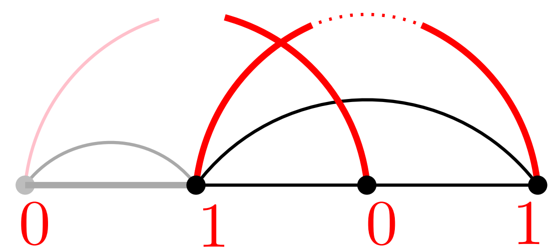

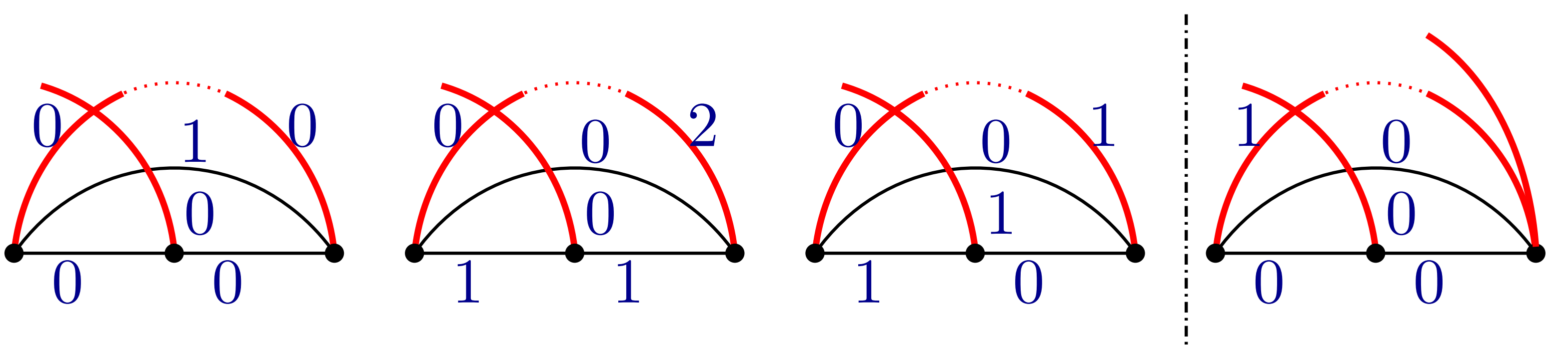

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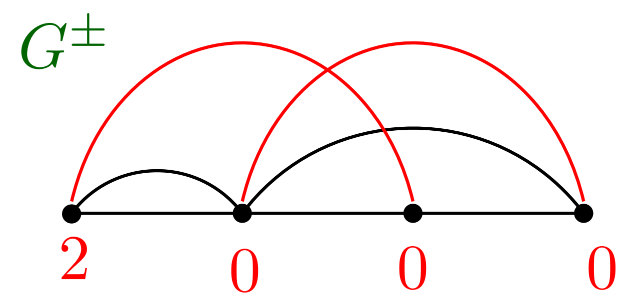

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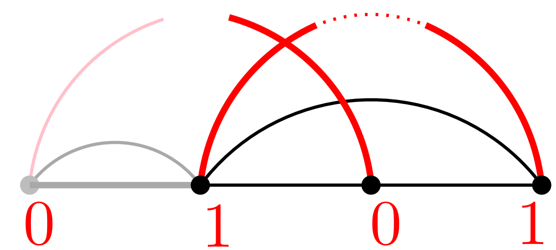

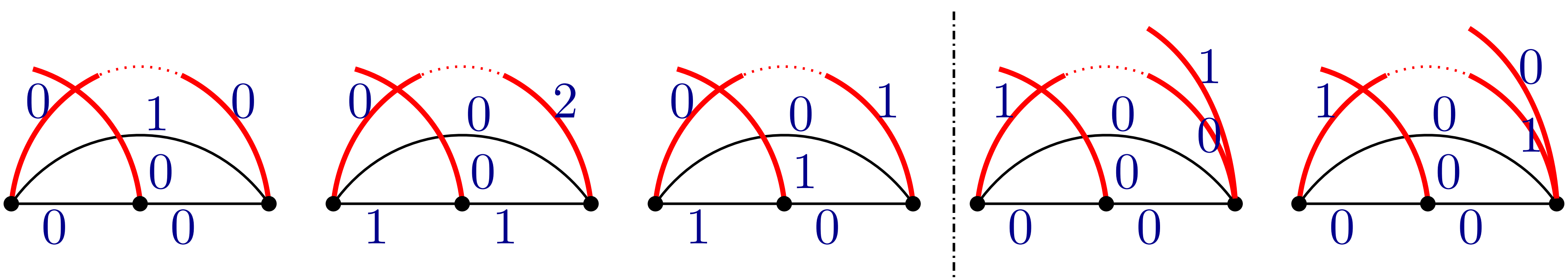

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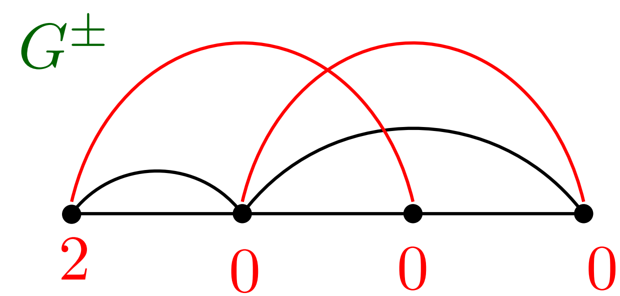

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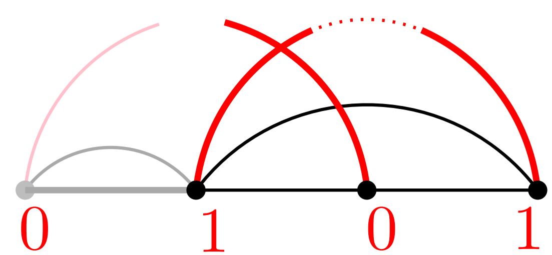
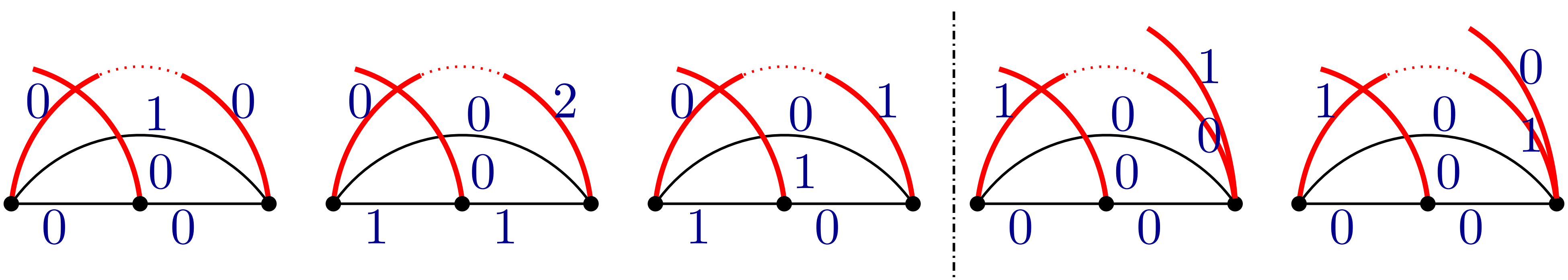
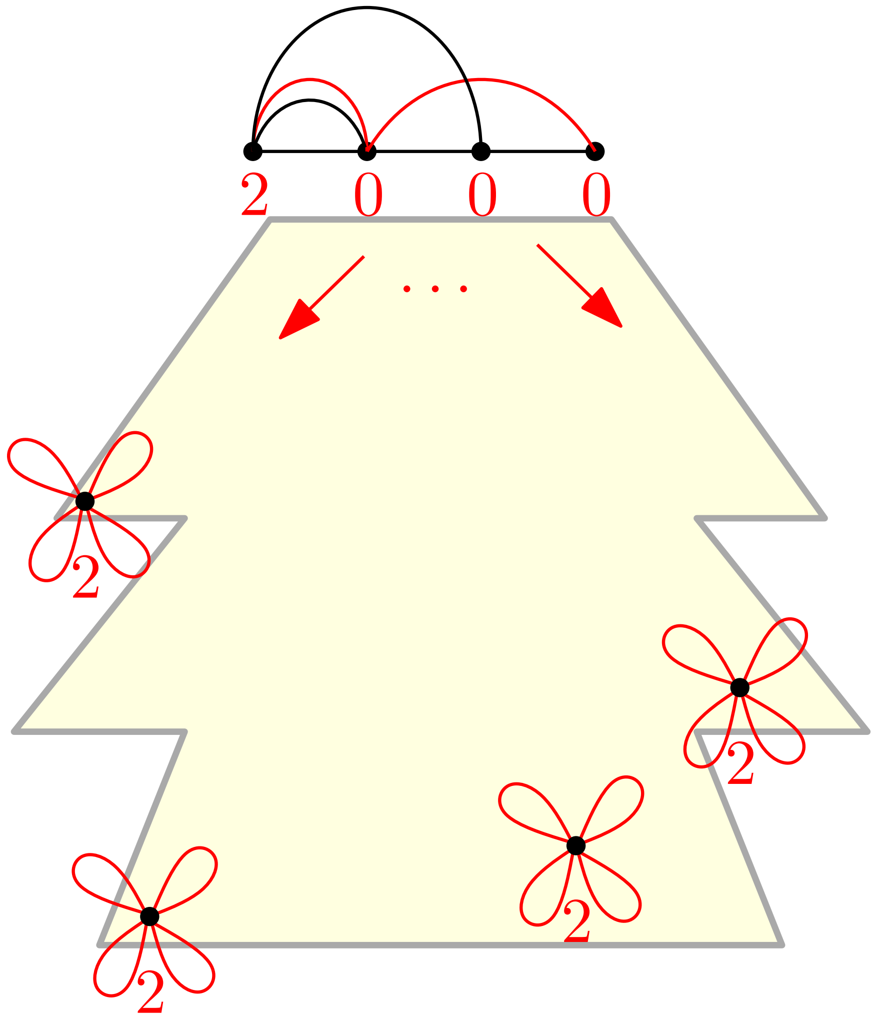
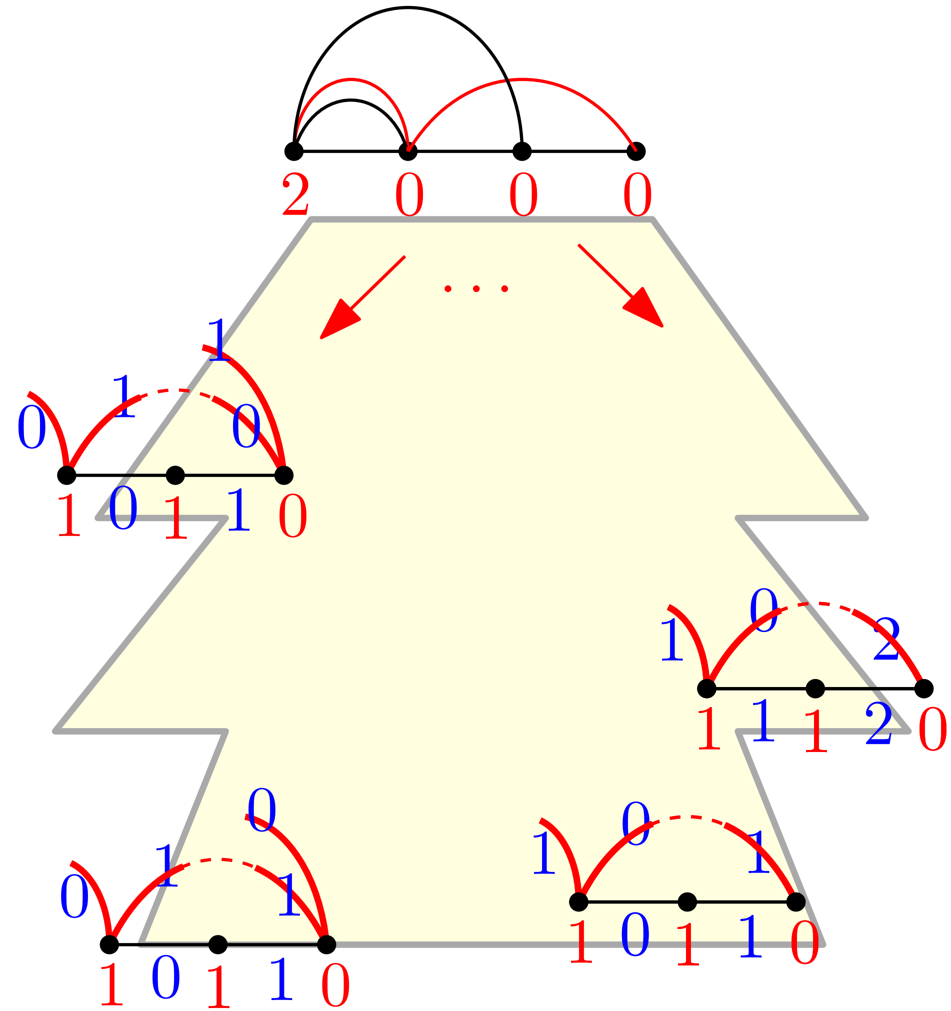


$= 5 :$

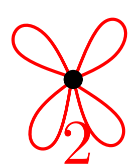


Idea proof of Theorem on $\mathrm{vol}\mathcal{F}_{G^\pm}(2e_1)$





$$\mathrm{vol}(\mathcal{F}_G(\mathbf{a})) = \frac{1}{\dim(\mathcal{F}_G)!} \# \left\{ \quad \right\}$$



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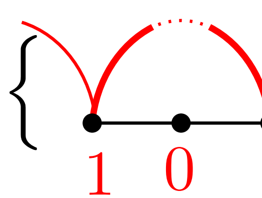
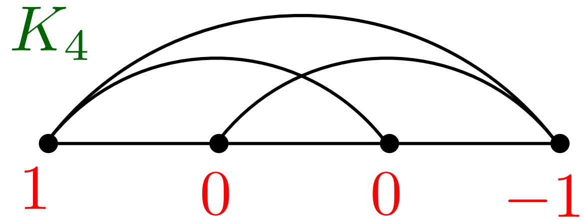


Application to type D analogue of $\mathcal{CRY}(n)$



Recall $\mathcal{CRY}(n) = \mathcal{F}_{K_{n+1}}(e_1 - e_{n+1})$:

- dimension $\binom{n}{2}$, 2^{n-1} vertices, volume $\prod_{i=0}^{n-2} Cat(i)$.

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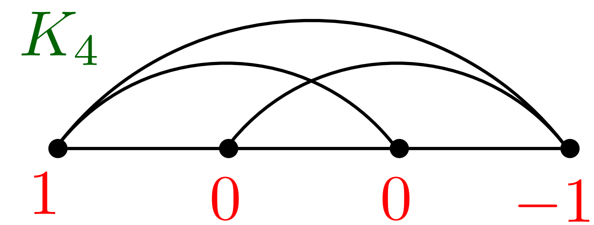

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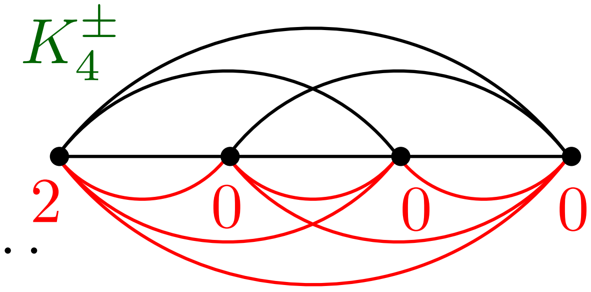
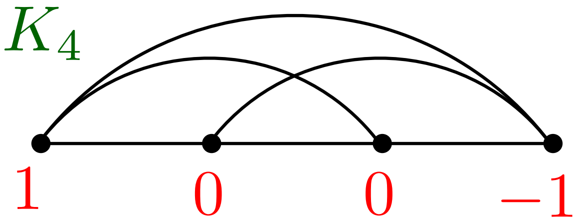


- dimension $n(n-2)$, $3^{n-1} - 2^{n-1}$ vertices, volume $\cdots$

Application to type D analogue of $\mathcal{CRY}(n)$



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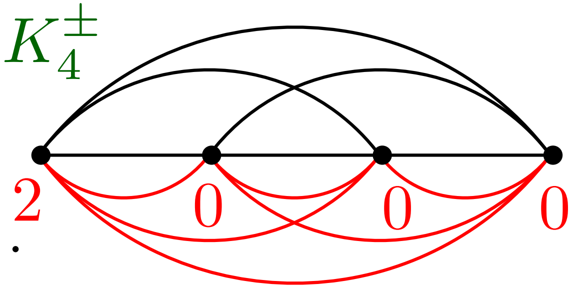

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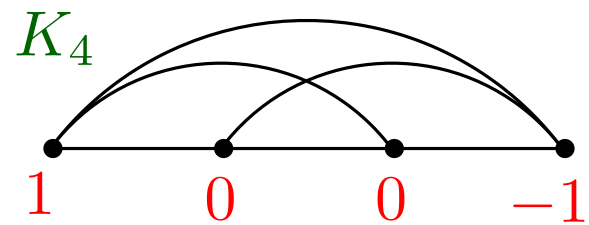
Corollary

$$(n(n-2))! \cdot \mathrm{vol}(\mathcal{CRY}^{\pm}(n)) = K^{\textbf{dyn.}}_{K_n^{\pm}}(0, 0, 1, 2, \ldots, n-3, n-2).$$

Application to type D analogue of $\mathcal{CRY}(n)$



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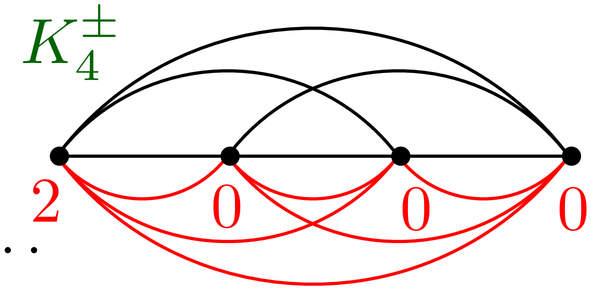

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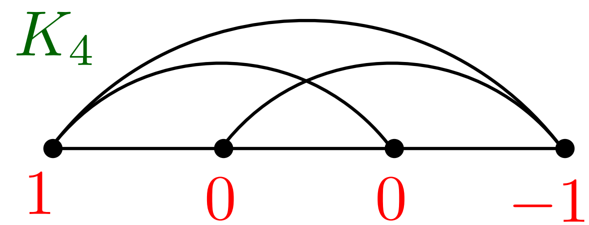
Data: $v_n = \dim(\mathcal{CRY}^{\pm}(n))! \cdot \mathrm{vol}(\mathcal{CRY}^{\pm}(n))$

n	2	3	4	5	6	7
v_n	1	2	32	5120	9175040	197300060160
$\frac{v_n}{v_{n-1}}$		2	$2^3 \cdot \mathbf{2}$	$2^5 \cdot \mathbf{5}$	$2^7 \cdot \mathbf{14}$	$2^9 \cdot \mathbf{42}$

Application to type D analogue of $\mathcal{CRY}(n)$



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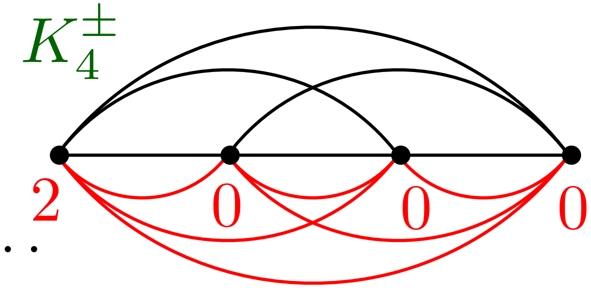

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- dimension $n(n-2)$, $3^{n-1} - 2^{n-1}$ vertices, volume $\cdots$

Corollary

$$(n(n-2))! \cdot \mathrm{vol}(\mathcal{CRY}^{\pm}(n)) = K^{\textbf{dyn.}}_{K_n^{\pm}}(0,0,1,2,\ldots,n-3,n-2).$$

Data: $v_n = \dim(\mathcal{CRY}^{\pm}(n))! \cdot \mathrm{vol}(\mathcal{CRY}^{\pm}(n))$

n	2	3	4	5	6	7
v_n	1	2	32	5120	9175040	197300060160
$\frac{v_n}{v_{n-1}}$		2	$2^3 \cdot \mathbf{2}$	$2^5 \cdot \mathbf{5}$	$2^7 \cdot \mathbf{14}$	$2^9 \cdot \mathbf{42}$

Conjecture $v_n = 2^{(n-2)^2} \cdot Cat(0)Cat(1)Cat(2)\cdots Cat(n-2)$.

Outline

1. What are type A flow polytopes? ✓

2. What are type D flow polytopes? ✓

3. How do we calculate volumes of flow polytopes? ✓

4. Connection between type A flow polytopes and Kostant partition function? ✓

5. Is there such a connection for type D flow polytopes? ✓

References:

- W. Baldoni, M. Vergne, **Kostant partition functions and flow polytopes**, Transform. Groups, **13**, 3, 2008, 447-469.
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- with K. Mészáros, **Flow polytopes of signed graphs and the Kostant partition function**, **arXiv:1208.0140**, code at sites.google.com/site/flowpolytopes/