Maximal Newton Polygons via the Quantum Bruhat Graph

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Abstract

This paper discusses a surprising relationship between the quantum cohomology of the variety of complete flags and the partially ordered set of Newton polygons associated to an element in the affine Weyl group. One primary key to establishing this connection is the fact that paths in the quantum Bruhat graph, which is a weighted directed graph with vertices indexed by elements in the finite Weyl group, encode saturated chains in the strong Bruhat order on the affine Weyl group. This correspondence is also fundamental in the work of Lam and Shimozono [LS10] establishing Peterson's isomorphism between the quantum cohomology of partial flag varieties and the homology of the affine Grassmannian [Pet96].

Newton Polygons

Let $F = \mathbb{F}_q((t))$, and let $G(F)$ be a split connected reductive group. Given $A \in G(F)$, such as

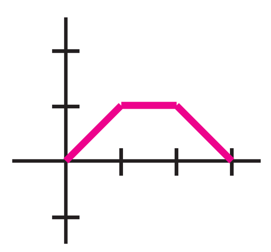
$$A = \begin{bmatrix} -t^2 & t^2 & t^2 \\ 1 & 0 & t \\ 0 & t^{-2} & t^{-1} \end{bmatrix} \in SL_3(F),$$

the *Newton polygon* $\nu(A)$ is constructed as follows.

- **Step 1:** Compute the characteristic polynomial $\det(A - \lambda I)$ for the matrix A:

$$p_\lambda(A) = \lambda^3 - (t^{-1} - t^2)\lambda^2 - (t^{-1} + t + t^2)\lambda - 1.$$

- **Step 2:** Plot $-\mathrm{val}(a_i)$ for each of the coefficients a_i in $p_\lambda(A)$, where $\mathrm{val}(a_i)$ equals the lowest power of t appearing in a_i.
- **Step 3:** Take the upper convex hull of this set of points.



A Newton polygon is also uniquely determined by its *slope sequence*, in which we record successive slopes of the edges left to right, repeated with multiplicity. For example, we also say $\nu(A) = (1, 0, -1)$.

The Poset and Maximal Elements

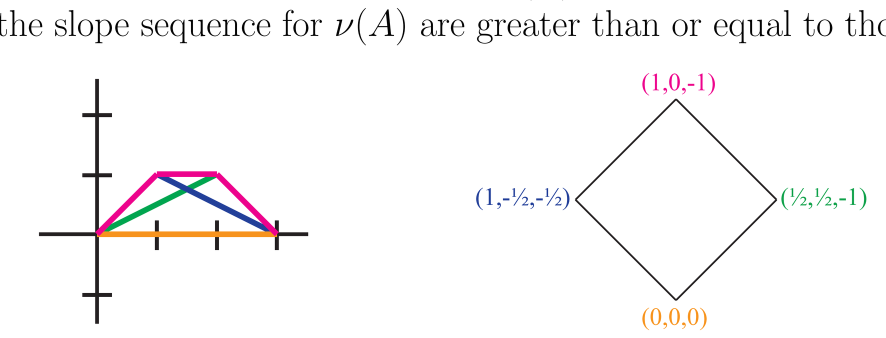
Let I be the Iwahori subgroup, and let $x \in \widetilde{W}$ be an element of the affine Weyl group. If $\mathcal{O} = \mathbb{F}[[t]]$,

$$I = \begin{bmatrix} \mathcal{O}^\times & t\mathcal{O} & t\mathcal{O} \\ \mathcal{O} & \mathcal{O}^\times & t\mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O}^\times \end{bmatrix} \leq SL_3(F) \quad \text{and} \quad \begin{bmatrix} 0 & 0 & t^2 \\ 1 & 0 & 0 \\ 0 & t^{-2} & 0 \end{bmatrix} = t^{(2,0,-2)} s_1 s_2 \in \widetilde{S_3}.$$

The affine Bruhat decomposition says that $G(F) = \coprod_{x \in \widetilde{W}} IxI$. We may then study

$$\mathcal{N}(G)_x = \{\nu(g) \mid g \in IxI\}.$$

$\mathcal{N}(G)_x$ is a partially ordered set containing a *unique maximal element* ν_x. We say $\nu(A) \geq \nu(B)$ if they share a left and rightmost vertex and all edges of $\nu(A)$ lie either on or above those of $\nu(B)$, or if all partial sums for the slope sequence for $\nu(A)$ are greater than or equal to those for $\nu(B)$.



The poset $\mathcal{N}(G)_x$ for $x = t^{(2,0,-2)} s_1 s_2$, for which $\nu_x = (1, 0, -1)$.

The Quantum Bruhat Graph

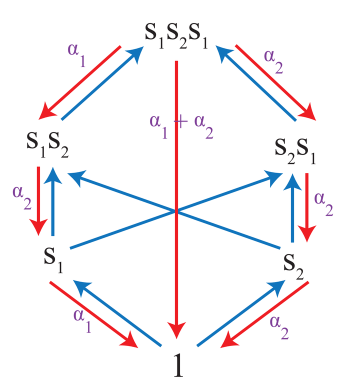
The *quantum Bruhat graph* is a weighted directed graph with vertices indexed by the Weyl group and weights given by the reflection used to get from one element to the other.

- **Vertices:** The elements $w \in W$ of the finite Weyl group.
- **Edges:** Draw an edge if the elements are related by a reflection satisfying one of two conditions:

$$w \longrightarrow ws_\alpha \ \text{ if } \ \ell(ws_\alpha) = \ell(w) + 1, \ \text{ or}$$
$$w \longrightarrow ws_\alpha \ \text{ if } \ \ell(ws_\alpha) = \ell(w) - \langle \alpha^\vee, 2\rho \rangle + 1.$$

Note that if G is of type ADE, then we have $\ell(w) - \langle \alpha^\vee, 2\rho \rangle + 1 = \ell(w) - \ell(s_\alpha)$.

- **Weights:** Label any downward edge from $w \longrightarrow ws_\alpha$ by the coroot $\alpha^\vee$ corresponding to s_α.



The quantum Bruhat graph for S_3.

The *weight of a path* in the quantum Bruhat graph is then defined to be the sum of the weights of these downward edges in the path.

Main Theorem

Theorem 1. *Fix M sufficiently large, and let $x = t^{v\lambda^+} w \in \widetilde{W}$, where λ^+ is dominant and $|\langle \lambda, \alpha \rangle| \geq M$ for all $\alpha \in R^+$. Then*

$$\nu_x = \lambda^+ + w_0(\alpha_d^\vee), \tag{1}$$

where $\alpha_d^\vee$ is the weight of any path of minimal length in the quantum Bruhat graph from vw_0 to $w^{-1}vw_0$, and w_0 denotes the longest element in the finite Weyl group W.

Example 2. Consider $x = t^{(2,0,-2)} s_1 s_2$ in the affine symmetric group $\widetilde{S_3}$. Here, $v = 1$ and $w = s_1 s_2$ so that $vw_0 = s_1 s_2 s_1$ and $w^{-1}vw_0 = (s_2 s_1)(s_1 s_2 s_1) = s_1$. In the quantum Bruhat graph for $W = S_3$, the weight of both paths of minimal length from $s_1 s_2 s_1$ to s_1, each of which has length 2, equals $\alpha_1^\vee + \alpha_2^\vee = (1, 0, -1)$. Therefore, by Theorem 1

$$\nu_x = (2, 0, -2) + w_0(1, 0, -1) = (2, 0, -2) + (-1, 0, 1) = (1, 0, -1).$$

Key Idea: Finding Pure Translations

Theorem 3 (Viehmann [Vie09])**.** *Given $x \in \widetilde{W}$, then*

$$\nu_x = \max\{\nu(y) \mid y \in \widetilde{W} \text{ with } y \leq x\}, \tag{2}$$

where the maximum is taken with respect to dominance order and $y \leq x$ in Bruhat order.

Proposition 4. *Let $y = t^\lambda w \in \widetilde{W}$, and suppose that the order of w in W equals k. Then*

$$\nu(y) = \left(\frac{1}{k} \sum_{i=1}^{k} w^i(\lambda) \right)^+. \tag{3}$$

In general, $\nu(t^\lambda w) < \nu(t^\lambda)$ if w is non-trivial, and so it suffices to find pure translations $t^\mu \leq x$.

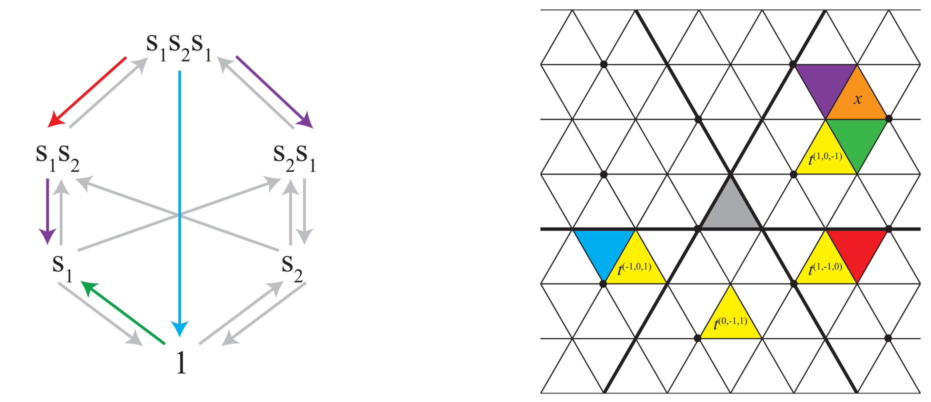
Proposition 5. *Chose M sufficiently large, and suppose that $|\langle \lambda, \alpha \rangle| \geq M$ for all $\alpha \in R^+$. Given any saturated chain of minimal length in strong Bruhat order from $x = t^\lambda w \in \widetilde{W}$ to a pure translation, say $x \gtrdot x_1 \gtrdot x_2 \gtrdot \cdots \gtrdot x_k = t^\mu$, then $\nu_x = \mu^+$.*

Quantum Bruhat Graph and Affine Bruhat Order

Using work of Lam and Shimozono [LS10], we can related cocovers of $x = t^{v\lambda^+} w \in \widetilde{W}$ in affine Bruhat order to edges in the quantum Bruhat graph:

- an edge $vw_0 \longrightarrow vw_0 r_\alpha$ means $x \gtrdot t^{vr_{w_0\alpha}(\lambda)} r_{vw_0\alpha} w$
- an edge $vw_0 \longrightarrow vw_0 r_\alpha$ means $x \gtrdot t^{vr_{w_0\alpha}(\lambda + w_0\alpha^\vee)} r_{vw_0\alpha} w$
- an edge $w^{-1}vw_0 r_\alpha \longrightarrow w^{-1}vw_0$ means $x \gtrdot t^{v(\lambda)} r_{vw_0\alpha} w$
- an edge $w^{-1}vw_0 r_\alpha \longrightarrow w^{-1}vw_0$ means $x \gtrdot t^{v(\lambda + w_0\alpha^\vee)} r_{vw_0\alpha} w$

Any path of minimal length k in the quantum Bruhat graph from vw_0 to $w^{-1}vw_0$ then corresponds to exactly 2^k saturated chains of minimal length from x to a pure translation.



Saturated chains from $x = t^{(2,0,-2)} s_1 s_2$ to translations and corresponding paths in the QBG for S_3.

Connection to Quantum Schubert Calculus

The quantum cohomology ring of the complex complete flag variety G/B equals

$$QH^*(G/B) = H^*(G/B, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[q_1, \ldots, q_r],$$

where r is the rank of G. As a $\mathbb{Z}[q_1, \ldots, q_r]$-module, $QH^*(G/B)$ has a basis of Schubert classes σ_w where $w \in W$. The main problem in modern quantum Schubert calculus is to explicitly compute

$$\sigma_u * \sigma_v = \sum_{w,d} c_{u,v}^{w,d} q^d \sigma_w,$$

by finding non-recursive, positive combinatorial formulas for the coefficients $c_{u,v}^{w,d}$ and monomials q^d.

Theorem 6 (Postnikov [Pos05])**.** *The unique minimal monomial occurring in the quantum product $\sigma_u * \sigma_v$ equals $q^d = q_1^{d_1} \cdots q_r^{d_r}$, where $\alpha_d^\vee = d_1\alpha_1^\vee + \cdots + d_r\alpha_r^\vee$ is the weight of any shortest path from u to $w_0 v$ in the quantum Bruhat graph.*

Theorem 7. *Fix M sufficiently large, and suppose that $|\langle \lambda, \alpha \rangle| \geq M$ for all $\alpha \in R^+$. Then q^d is the minimal monomial in the quantum product $\sigma_u * \sigma_v$ if and only if $\lambda^+ + w_0(\alpha_d^\vee)$ is the maximal Newton polygon in $\mathcal{N}(G)_x$, where $x = t^{uw_0(\lambda^+)} uv^{-1} w_0$.*

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