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Bienvenue à Paris!

Nous sommes ravis de vous accueillir à Paris pour la **25^e** édition de la conférence Séries Formelles et Combinatoire Algébrique (SFCA). Cette conférence a été initiée a Lille en 1988 puis a eu lieu a Paris en 1990. Elle s'est ensuite baladée partout dans le monde mais son attache française reste importante. Ainsi elle a été organisée à Bordeaux en 1992 puis à Marne la Vallée en 1995.

Le succès de SFCA est maintenant garanti. Chaque année, nous avons de plus en plus de soumissions et de participants!

Longue vie à SFCA!

Sylvie Corteel
pour le comité d'organisation

L'édition 2013 de SFCA à Paris a fait l'objet d'un nombre record de 176 soumissions. Le comité de programme et les rapporteurs auxiliaires ont souligné dans leur ensemble la qualité exceptionnelle des contributions proposées. Au vu du format de la conférence, 27 exposés et 75 posters ont été acceptés soit un taux de 57% d'acceptation. La conférence SFCA a comme tradition d'accueillir et d'encourager particulièrement les contributions d'étudiants, ainsi que d'offrir un programme varié, intéressant et solide scientifiquement autour de la combinatoire et de ses applications.

This year, a record number of 176 contributions were submitted to the scientific committee of FPSAC 2013. According to our panel of evaluators, the quality of the submitted papers was exceptionally high. In view of temporal and physical constraints, 27 talks and 75 posters were accepted for a ratio of acceptance of 57%. FPSAC has a tradition of welcoming and encouraging student submissions. We are also careful in selecting a broad, interesting and solid scientific program.

Bonne lecture et bonne conférence !

Alain Goupil et Gilles Schaeffer
Program committee co-chairs

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A unified bijective framework for planar maps

Olivier Bernardi

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Abstract. Planar maps are connected planar graphs embedded in the plane. In the last fifteen years, a bijective approach for studying planar maps has been developped; and there are now dozens of bijections between classes of maps and classes of trees. We present a unified way to think about these bijections. Roughly speaking, we show that all the bijective results for maps can be recovered by specializing a unique "master bijection". The subtlety is that the master bijection acts on *oriented maps*. Thus, the known bijections are recovered by choosing a suitable orientation for the maps in the class considered, and then applying the master bijection. The suitable orientations implicitly used in the known bijections are in fact part of an infinite family $(\Omega_d)_{d\geq 0}$ of orientations that we characterize. The parameter d is related to the girth of the maps.

Keywords: Bijection, Planar maps, Trees, Girth

Planar maps are connected planar graphs embedded in the plane, considered up continuous deformations. Planar maps have been actively studied in combinatorics ever since the seminal work of William Tutte in the sixties. Along the years, deep connections have been fruitfully exploited between planar maps and subjects as diverse as the combinatorics of the symmetric group, graph drawing algorithms, random matrix theory, statistical mechanics, and 2D quantum gravity.

In the last decade, following the seminal work of Cori and Vauquelin Cori and Vauquelin (1981), Arquès Arquès (1986) and Schaeffer Schaeffer (1998), many bijections have been discovered between classes of maps (e.g. triangulations, bipartite maps) and classes of trees Schaeffer (1998). These bijections provide the "proofs from the Book" for the many simple-looking counting formulas discovered by Tutte and his followers. Moreover they proved to be invaluable tools in order to study the metric properties of maps, finding algorithms for maps, and solving statistical mechanics models on maps.

There are now dozens of bijections between classes of planar maps and classes of trees; see for instance Schaeffer (1998, 1997); Bouttier et al. (2002); Poulalhon and Schaeffer (2003); Bouttier et al. (2004); Fusy et al. (2008); Fusy (2009); Bousquet-Mélou and Schaeffer (2000, 2002); Bouttier et al. (2007); Bernardi (2007); Fusy et al. (2009). We will present a bijective framework, developed jointly with Eric Fusy, which unifies and extend these bijections Bernardi and Fusy (2012a,b, 2013b,a). There are two ingredients to our approach:

- A master bijection between a class of oriented maps and a class of trees. The master bijection Φ is illustrated in Figures 1 and 2.
- The existence of certain canonical orientations for planar maps of given girth.

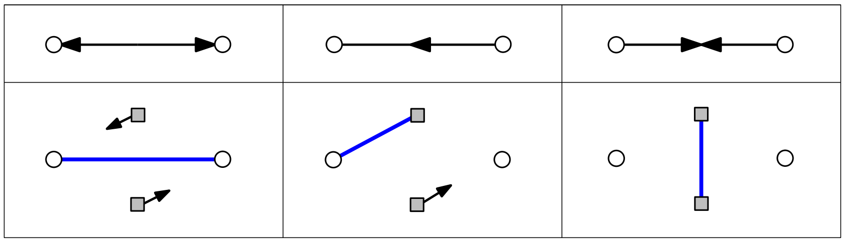

Fig. 1: The three types of edges in a bi-oriented map and the local rule of the master bijection Φ.

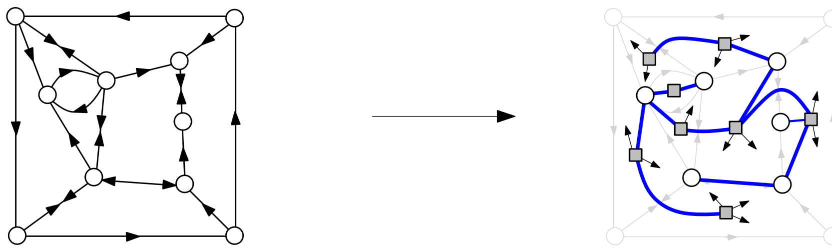

Fig. 2: The master bijection Φ applied to a bi-oriented map: the image is a tree with black and white vertices with some decorations on black vertices.

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Self-avoiding walks on the honeycomb lattice

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In 2010, Duminil-Copin and Smirnov [2] proved a long standing conjecture [3], according to which the number of n-step self-avoiding walks (SAWs) on the honeycomb lattice grows like μ^n, up to sub-exponential factors, where $\mu = \sqrt{2+\sqrt{2}}$.

Their proof is in fact rather simple, but also very original, at least to a combinatorialist's eyes. At the heart of the proof is a remarkable identity, that relates several generating functions of SAWs *evaluated at the critical point* $1/\mu$. We will discuss this identity and some of its extensions, with applications to SAWs interacting with a surface [1], and to the $O(n)$ loop model.

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Topological combinatorics of Bruhat order and total positivity

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This talk will focus on the rich interplay of combinatorics, topology, and representation theory arising in the theory of total positivity and in particular in the study of the totally nonnegative part of a matrix Schubert variety. Along the way, we will survey what combinatorics of a closure poset can and what it cannot tell us about the topology of a stratified space. Braid moves on reduced and nonreduced words in the associated 0-Hecke algebra are interpreted topologically, yielding information about the possible relations among (exponentiated) Chevalley generators of a Lie group. The subword complexes introduced by Allen Knutson and Ezra Miller also play a role in this story, giving the face poset structure for the fibers of a map $f_{(i_1,\ldots,i_d)}$ suggested in work of Lusztig where $f_{(i_1,\ldots,i_d)}$ is given by a product of exponentiated Chevalley generators. Sergey Fomin and Michael Shapiro conjectured that totally nonnegative spaces arising as images of these maps, or equivalently as the Bruhat decompositions of the totally nonnegative part of matrix Schubert varieties, together with the links of their cells, are regular CW complexes homeomorphic to balls having the intervals of Bruhat order as their closure posets. We will discuss the new combinatorics and topology which the proof of this conjecture revealed.

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Quiver mutation and combinatorial DT-invariants

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A quiver is an oriented graph. Quiver mutation is an elementary operation on quivers. It appeared in physics in Seiberg duality in the nineties and in mathematics in the definition of cluster algebras by Fomin-Zelevinsky in 2002. We show how, for large classes of quivers Q, using quiver mutation and quantum dilogarithms, one can construct the combinatorial DT-invariant, a formal power series intrinsically associated with Q. When defined, it coincides with the 'total' Donaldson-Thomas invariant of Q (with a generic potential) provided by algebraic geometry (work of Joyce, Kontsevich-Soibelman, Szendroi and many others). We illustrate combinatorial DT-invariants on many examples and point out their links to quantum cluster algebras and to (infinite) generalized associahedra.

Un carquois est un graphe orienté. La mutation des carquois est une opération élémentaire sur les carquois. Elle est apparue en physique dans la dualité de Seiberg dans les années 90 et en mathématiques dans la définition des algèbres amassées par Fomin–Zelevinsky en 2002. Nous montrons comment, pour de grandes classes de carquois Q, à l'aide de la mutation des carquois et des dilogarithmes quantiques, on peut construire l'invariant DT combinatoire, une série formelle associée intrinsèquement à Q. Quand cet invariant est défini, il est égal à l'invariant de Donaldson–Thomas 'total' associé à Q (avec un potentiel générique) qui est fourni par la géométrie algébrique (travaux de Joyce, Kontsevich–Soibelman, Szendroi et beaucoup d'autres). Nous illustrons les invariants DT combinatoires sur beaucoup d'exemple et évoquons leurs liens avec les algèbres amassées quantiques et des associaèdres généralisés (infinis).

Keywords: Cluster algebra, quiver mutation, Donaldson-Thomas invariants

1 Introduction

A quiver is an oriented graph. Quiver mutation is an elementary operation on quivers. It appeared in physics already in the nineties in Seiberg duality, cf. Seiberg (1995). In mathematics, quiver mutation was introduced by Fomin and Zelevinsky (2002) as the basic combinatorial ingredient of their definition of cluster algebras. Thus, quiver mutation is linked to the large array of subjects where cluster algebras have subsequently turned out to be relevant, cf. for example the cluster algebras portal maintained by

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Fomin (2002) and the survey articles by Fomin (2010), Leclerc (2010), Reiten (2010), Williams (2012). Among these links, the one to representation theory and algebraic geometry has been particularly fruitful. It has allowed to 'categorify' cluster algebras and thereby to prove conjectures about them which seem beyond the scope of the purely combinatorial methods, cf. for example the articles of Derksen et al. (2010), Geißet al. (2011), Plamondon (2011), Cerulli Irelli et al. (2012), ...

The constructions and results we present in this talk are another manifestation of this fruitful interaction. They are inspired by the theory of Donaldson–Thomas invariants as it has been developped by Bridgeland, Joyce and Song (2009), Kontsevich and Soibelman (2008, 2010), Nagao (2010), Reineke (2011), Szendrői (2008) and many others. In this theory, one assigns Donaldson–Thomas invariants to three-dimensional, possibly non commutative, Calabi–Yau varieties. These invariants exist in many different versions. Here we use the 'total' Donaldson–Thomas invariant, which is a certain power series in (slighly) non commutative variables. One important construction of non commutative 3-Calabi–Yau varieties takes as its input a quiver (with a generic potential). Thus, there is a Donaldson–Thomas invariant associated with 'each' quiver (some technical problems remain to be solved for a completely general definition). It turns out that for a suprisingly large class of quivers, it is possible to construct this invariant in a combinatorial way using products of quantum dilogarithm series associated with so-called reddening sequences of quiver mutations. This construction yields the definition of the combinatorial DT-invariant, which is the main point of this talk (section 4). It is an important fact that a given quiver may admit many distinct reddening sequences. Each of them yields a product decomposition for the combinatorial DT-invariant and in this way, one obtains many interesting quantum dilogarithm identities (section 5).

Let us emphasize that the use of (products of) quantum dilogarithm series in the study of (quantum) cluster algebras goes back to the insight of Fock and Goncharov (2009a,b). They also pioneered their application in the study of (quantum) dilogarithm identities, which was subsequently developped by many authors. We refer to Nakanishi (2012) for a survey. Geometric as well as combinatorial constructions of DT-invariants also appear in physics, cf. for example Cecotti et al. (2010), Alim et al. (2011a,b), Cecotti et al. (2011), Cecotti and Vafa (2011), Gaiotto et al. (2009, 2010a,b), Xie (2012), ...

2 Quiver mutation

A *quiver* is an oriented graph, i.e. a quadruple $Q = (Q_0, Q_1, s, t)$ formed by a set of vertices Q_0, a set of arrows Q_1 and two maps s and t from Q_1 to Q_0 which send an arrow α respectively to its source $s(\alpha)$ and its target $t(\alpha)$. In practice, a quiver is given by a picture as in the following example

$$\begin{tikzcd}
Q: & & 3 \arrow[dl, "\lambda"'] & & 5 \arrow[loop left, "\alpha"] \arrow[r, shift left] \arrow[r] \arrow[r, shift right] & 6 \\
& 1 \arrow[rr, "\nu"'] & & 2 \arrow[ul, "\mu"'] \arrow[r, shift left, "\beta"] & 4. \arrow[l, shift left, "\gamma"]
\end{tikzcd}$$

An arrow α whose source and target coincide is a *loop*; a 2-*cycle* is a pair of distinct arrows β and γ such that $s(\beta) = t(\gamma)$ and $t(\beta) = s(\gamma)$. Similarly, one defines n-*cycles* for any positive integer n. A vertex i of a quiver is a *source* (respectively a *sink*) if there is no arrow with target i (respectively with source i). A *Dynkin quiver* is a quiver whose underlying graph is a Dynkin diagram of type A_n, $n \geq 1$, D_n, $n \geq 4$, or E_6, E_7, E_8.

By convention, in the sequel, by a quiver we always mean a finite quiver without loops nor 2-cycles whose set of vertices is the set of integers from 1 to n for some $n \geq 1$. Up to an isomorphism fixing the vertices, such a quiver Q is given by the *skew-symmetric matrix* $B = B_Q$ whose coefficient b_{ij} is the difference between the number of arrows from i to j and the number of arrows from j to i for all $1 \leq i, j \leq n$. Conversely, each skew-symmetric matrix B with integer coefficients comes from a quiver.

Let Q be a quiver and k a vertex of Q. The *mutation* $\mu_k(Q)$ is the quiver obtained from Q as follows:

1) for each subquiver $i \xrightarrow{\beta} k \xrightarrow{\alpha} j$, we add a new arrow $[\alpha\beta] : i \to j$;
2) we reverse all arrows with source or target k;
3) we remove the arrows in a maximal set of pairwise disjoint 2-cycles.

For example, if k is a source or a sink of Q, then the mutation at k simply reverses all the arrows incident with k. In general, if B is the skew-symmetric matrix associated with Q and B' the one associated with $\mu_k(Q)$, we have

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \text{ ;} \\ b_{ij} + \operatorname{sign}(b_{ik}) \max(0, b_{ik} b_{kj}) & \text{else.} \end{cases} \tag{2.1}$$

This is the *matrix mutation rule* for skew-symmetric (more generally: skew-symmetrizable) matrices introduced by Fomin and Zelevinsky (2002), cf. also Fomin and Zelevinsky (2007).

One checks easily that μ_k is an involution. For example, the quivers

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 2 \arrow[ur] & & 3 \arrow[ll] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & 1 \arrow[dl] & \\ 2 & & 3 \arrow[ul] \end{tikzcd} \tag{2.2}$$

are linked by a mutation at the vertex 1. Notice that these quivers are drastically different: The first one is a cycle, the second one the Hasse diagram of a linearly ordered set.

Two quivers are *mutation equivalent* if they are linked by a finite sequence of mutations. For example, it is an easy exercise to check that any two orientations of a tree are mutation equivalent. Using the quiver mutation applet by Keller (2006) or the Sage package by Musiker and Stump (2011) one can check that the following three quivers are mutation equivalent

$$\begin{tikzcd}[column sep=small] & & & 1 \arrow[dr] & & & \\ & & 2 \arrow[ur] \arrow[dr] & & 3 \arrow[ll] \arrow[dr] & & \\ & 4 \arrow[ur] \arrow[dr] & & 5 \arrow[ll] \arrow[ur] \arrow[dr] & & 6 \arrow[ll] \arrow[dr] & \\ 7 \arrow[ur] & & 8 \arrow[ll] \arrow[ur] & & 9 \arrow[ll] \arrow[ur] & & 10 \arrow[ll] \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & 10 \arrow[r] \arrow[dl] & 5 \arrow[dd] & & \\ 1 & & & & \\ & & 7 \arrow[dl] & 6 \arrow[l] & \\ & 8 \arrow[dr] & & & 4 \arrow[ul] \\ & & 9 \arrow[urr] \arrow[d] & & 3 \arrow[u] \\ & & 2 & & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] 4 \arrow[dr] & & 5 \arrow[dl] & 3 \arrow[l] \\ & 6 \arrow[dl] & & \\ 7 \arrow[d] & & & 2 \arrow[uu] \\ 8 \arrow[r] & 9 \arrow[r] & 10 \arrow[ur] \arrow[r] & 1 \end{tikzcd} . \tag{2.3}$$

The common *mutation class* of these quivers contains 5739 quivers (up to isomorphism). The mutation class of ‘most’ quivers is infinite. The classification of the quivers having a finite mutation class was achieved by by Felikson et al. (2012a) (and by Felikson et al. (2012b) in the skew-symmetric case): in addition to the quivers associated with triangulations of surfaces (with boundary and marked points, cf. Fomin et al. (2008)), the list contains 11 exceptional quivers, the largest of which is in the mutation class of the quivers (2.3).

3 Green quiver mutation

Let Q be a quiver without loops nor 2-cycles. The *framed quiver* $\tilde{Q}$ is obtained from Q by adding, for each vertex i, a new vertex i' and a new arrow $i \to i'$. Here is an example:

$$Q: \; 1 \to 2 \qquad \tilde{Q}: \; \begin{array}{ccc} 1 & \to & 2 \\ \downarrow & & \downarrow \\ 1' & & 2' \end{array} \quad .$$

The vertices i' are called *frozen vertices* because we never mutate at them. Now suppose that we have transformed $\tilde{Q}$ into $\tilde{Q}\,'$ by a finite sequence of mutations (at non frozen vertices). A vertex i of Q is *green* in $\tilde{Q}\,'$ if there are no arrows $j\,' \to i$ in $\tilde{Q}\,'$. Otherwise, it is *red*. A sequence $\mathbf{i} = (i_1, \ldots, i_N)$ is *green* if for each $1 \leq t \leq N$, the vertex i_t is green in the partially mutated quiver $\tilde{Q}(\mathbf{i}, t)$ defined by

$$\tilde{Q}(\mathbf{i}, t) = \mu_{i_{t-1}} \ldots \mu_{i_2} \mu_{i_1}(\tilde{Q}),$$

where for $t = 1$, we have the empty mutation sequence and obtain the initial quiver $\tilde{Q}$. It is *maximal green* if it is green and all the vertices of the final quiver $\mu_{\mathbf{i}}(\tilde{Q})$ are red (so that indeed, the sequence $\mathbf{i}$ cannot be extended to any strictly longer green sequence).

In Figure 1, we have encircled the green vertices. We see that in this example, we have two maximal green sequences 12 and 212 and that the final quivers associated with these two sequences are isomorphic by an isomorphism which fixes the frozen vertices. We call such an isomorphism a *frozen isomorphism*. A sequence $\mathbf{i} = (i_1, \ldots, i_N)$ is *reddening* if all vertices of the final quiver

$$\mu_{\mathbf{i}}(\tilde{Q}) = \tilde{Q}(\mathbf{i}, N)$$

are red. Of course, maximal green sequences are reddening. In the example of Figure 1, the sequence $1, 2, 1, 2, 1, 2, 1$ is reddening but not green.

Theorem 3.1 *If* $\mathbf{i}$ *and* $\mathbf{i}'$ *are reddening sequences, there is a frozen isomorphism between the final quivers*

$$\mu_{\mathbf{i}}(\tilde{Q}) \xrightarrow{\sim} \mu_{\mathbf{i}'}(\tilde{Q}).$$

The statement of the theorem is purely combinatorial but the known proofs (cf. section 7 of Keller (2012) and the references given there) are based on representation theory and geometry. For an *arbitrary* sequence $\mathbf{i}$ of non frozen vertices, we define the c-matrix $C(\mathbf{i})$ as the $n \times n$-matrix occuring in the right upper corner of the skew-symmetric matrix associated with the final quiver $\mu_{\mathbf{i}}(\tilde{Q})$, so that we have

$$B_{\mu_{\mathbf{i}}(\tilde{Q})} = \begin{bmatrix} * & C(\mathbf{i}) \\ * & * \end{bmatrix}.$$

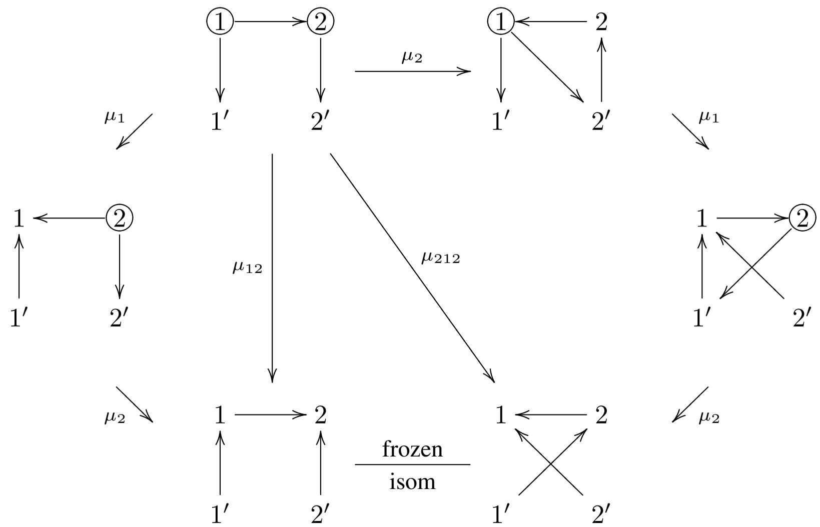


Fig. 1: The two maximal green sequences for A_2

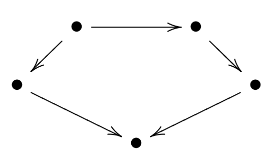
Thus, the (i, j)-coefficient of the matrix $C(\mathbf{i})$ is the difference between the number of arrows $i \to j'$ and $j' \to i$. The *c-vectors* associated with the sequence $\mathbf{i}$ are by definition the columns of the matrix $C(\mathbf{i})$. The following statement is known as the *sign coherence of c-vectors*.

Theorem 3.2 (Derksen et al. (2010)) *Each c-vector lies in* $\mathbb{N}^n$ *or* $(-\mathbb{N})^n$.

Again, the known proof of this combinatorial statement uses representation theory and geometry. The *oriented exchange graph* of the quiver Q is defined to be the quiver $\mathcal{E}_Q$ whose vertices are the frozen isomorphism classes of the quivers $\mu_{\mathbf{i}}(\tilde{Q})$, where $\mathbf{i}$ is an arbitrary sequence of vertices of Q, and where we have an arrow

$$\tilde{Q}' \to \mu_j(\tilde{Q}')$$

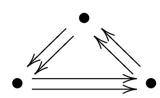
whenever j is a green vertex of $\tilde{Q}'$. For example, if Q is the quiver $1 \to 2$, then we see from Figure 1 that the oriented exchange graph is the oriented pentagon



By Theorem 3.1, the quiver $\mathcal{E}_Q$ has at most one sink. One can also show that it always has a unique source. An arbitrary sequence $\mathbf{i}$ of non frozen vertices corresponds to a *walk* in $\mathcal{E}_Q$ (a sequence of arrows and formal inverses of arrows) and a reddening sequence to a *path* (a formal composition of arrows) from the source to the sink. If Q is mutation equivalent to a quiver whose underlying graph is a Dynkin diagram of type A_n (resp. Δ), then $\mathcal{E}_Q$ is an orientation of the 1-skeleton of the nth Stasheff associahedron (resp. of the generalized associahedron of type Δ, cf. Chapoton et al. (2002)).

One can show that $\mathcal{E}_Q$ is the Hasse graph of a poset (a subposet of the set of torsion subcategories, cf. section 7.7 of Keller (2012)). If Q is the linear orientation of A_n, this poset the Tamari lattice. For certain classes of quivers, this poset is studied from the viewpoint of representation theory for example by Adachi et al. (2012), Brüstle et al. (2012), King and Qiu (2011), Koenig and Yang (2012) and Ladkani (2007).

Not all quivers admit reddening sequences. For example the quiver



does not admit a reddening sequence. On the other hand, reddening sequences do exist for large classes of quivers, cf. section 5 below. In particular, each acyclic quiver (=quiver without oriented cycles) admits a maximal green sequence corresponding to an increasing enumeration of the vertices for the order defined by the existence of a path.

4 Combinatorial DT-invariants

Our aim is to associate an intrinsic formal power series $\mathbb{E}_Q$ with each quiver Q admitting a reddening sequence. For quiver with a unique vertex and no arrows, this series will be the *quantum dilogarithm series*

$$\begin{aligned}\mathbb{E}(y) &= 1 + \frac{q^{1/2}}{q-1} \cdot y + \cdots + \frac{q^{n^2/2} y^n}{(q^n-1)(q^n-q)\cdots(q^n-q^{n-1})} + \cdots \\ &\in \mathbb{Q}(q^{1/2})[[y]],\end{aligned}$$

where $q^{1/2}$ is an indeterminate whose square is denoted by q and y is an indeterminate. This series is a classical object with many remarkable properties, cf. for example Zagier (1991). We will focus on one of them, namely the *pentagon identity*: If y_1 and y_2 are two indeterminates which q-commute, i.e. $y_1 y_2 = q y_2 y_1$, then we have

$$\mathbb{E}(y_1)\mathbb{E}(y_2) = \mathbb{E}(y_2)\mathbb{E}(q^{-1/2} y_1 y_2)\mathbb{E}(y_1). \tag{4.1}$$

It is due to Faddeev and Volkov (1993) and Faddeev and Kashaev (1994); a recent account can be found in Volkov (2012). Notice a striking structural similarity between this identity and the diagram in Figure 1: The two factors $\mathbb{E}(y_1)\mathbb{E}(y_2)$ on the left correspond to the two mutations in the path on the left, the three factors $\mathbb{E}(y_2)\mathbb{E}(q^{-1/2}y_1y_2)\mathbb{E}(y_1)$ on the right correspond to the three mutations in the path on the right and the equality corresponds to the frozen isomorphism between the final quivers. The common value of the two products will be defined as the combinatorial DT-invariant $\mathbb{E}_Q$ associated with the quiver $Q : 1 \to 2$.

Let Q be a quiver. For *any* sequence $\mathbf{i} = (i_1, \ldots, i_N)$ of vertices of Q, we will define a *product* $\mathbb{E}_{Q,\mathbf{i}}$ of quantum dilogarithm series. For this, let $\lambda_Q : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ be the bilinear antisymmetric form associated with the matrix B_Q. Define the *complete quantum affine space* as the algebra

$$\hat{\mathbb{A}}_Q = \mathbb{Q}(q^{1/2})\langle\langle y^\alpha, \alpha \in \mathbb{N}^n \mid y^\alpha y^\beta = q^{1/2\,\lambda(\alpha,\beta)} y^{\alpha+\beta}\rangle\rangle.$$

This is a slightly non commutative deformation of an ordinary commutative power series algebra in n indeterminates. We define the *product*

$$\mathbb{E}_{Q,\mathbf{i}} = \mathbb{E}(y^{\varepsilon_1\beta_1})^{\varepsilon_1} \cdots \mathbb{E}(y^{\varepsilon_N\beta_N})^{\varepsilon_N},$$

where the product is taken in $\hat{\mathbb{A}}_Q$, the vector β_t is the t-th column of the c-matrix $C(i_1, \ldots, i_{t-1})$ and ε_t is the common sign of the entries of this column (Theorem 3.2), $1 \leq t \leq N$.

Theorem 4.1 *If* $\mathbf{i}$ *and* $\mathbf{i}'$ *are two sequences of vertices of Q such that there is a frozen isomorphism between $\mu_{\mathbf{i}}(\tilde{Q})$ and $\mu_{\mathbf{i}'}(\tilde{Q})$, then we have the equality*

$$\mathbb{E}_{Q,\mathbf{i}} = \mathbb{E}_{Q,\mathbf{i}'}.$$

The theorem is proved in section 7.11 of Keller (2012) and independently in Nagao (2011). In particular, if $\mathbf{i}$ and $\mathbf{i}'$ are two reddening sequences, then by Theorem 4.1, the above equality holds. More generally, the theorem shows that to each vertex of the oriented exchange graph $\mathcal{E}_Q$, a canonical power series in $\hat{\mathbb{A}}_Q$ is associated. The one associated to the unique sink (if it exists) is $\mathbb{E}_Q$ and any reddening sequence gives a product expansion for $\mathbb{E}_Q$.

Definition 4.2 *If Q admits a reddening sequence* $\mathbf{i}$*, the* combinatorial DT-invariant of *Q is defined as*

$$\mathbb{E}_Q = \mathbb{E}_{Q,\mathbf{i}} \in \hat{\mathbb{A}}_Q.$$

The adjoint combinatorial DT-invariant of *Q is $DT_Q = \Sigma \circ Ad(\mathbb{E}_Q) : Frac(\hat{\mathbb{A}}_Q) \to Frac(\hat{\mathbb{A}}_Q)$, where $Frac(\hat{\mathbb{A}}_Q)$ is the non commutative field of fractions of $\hat{\mathbb{A}}_Q$ (cf. the Appendix of Berenstein and Zelevinsky (2005)) and Σ its automorphism determined by $\Sigma(y^\alpha) = y^{-\alpha}$ for all $\alpha \in \mathbb{N}^n$.*

For the agreement with the geometrically defined DT-invariant, we refer to section 7 of Keller (2012). In physics, an equivalent procedure has been discovered independently, cf. Xie (2012) and the references given there. It is easy to check that for $Q : 1 \to 2$, the above definition yields the left and right hand sides of the pentagon identity (4.1) associated with the two maximal green sequences of Figure 1 so that indeed, the combinatorial DT-invariant $\mathbb{E}_Q$ equals these two products. One can show that in this case, the adjoint combinatorial DT-invariant satisfies $(DT_Q)^5 = \mathrm{Id}$. Below, we will explore some remarkable generalizations of this example.

5 Examples

5.1 Dynkin quivers

Let Q be an *alternating* Dynkin quiver, i.e. a simply laced Dynkin diagram endowed with an orientation such that each vertex is a source or a sink, for example

$$Q = \vec{A}_5 : \bullet \longleftarrow \circ \longrightarrow \bullet \longleftarrow \circ \longrightarrow \bullet\ .$$

Let i_+ be the sequence of all sources $\circ$ and i_- the sequence of all sinks $\bullet$ (in any order). One can show that in this case, the sequence $\mathbf{i} = i_+ i_-$ is maximal green and so is

$$\mathbf{i}' = \underbrace{i_- i_+ i_- \ldots}_{h \text{ factors}}\ ,$$

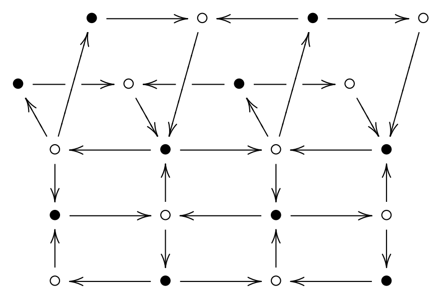


Fig. 2: The quiver $\vec{A}_4 \square \vec{D}_5$

where h is the Coxeter number of the underlying graph of Q. Thus, we have $\mathbb{E}(\mathbf{i}) = \mathbb{E}(\mathbf{i}')$ and $\mathbb{E}_Q$ is the common value. The identity $\mathbb{E}(\mathbf{i}) = \mathbb{E}(\mathbf{i}')$ is due to Reineke (2010). Using the geometry of the generalized associahedra of Chapoton et al. (2002), one can show that it is a consequence of the pentagon identity, cf. Qiu (2011) for another approach. For the adjoint combinatorial DT-invariant, we have

$$DT_Q^{h+2} = \text{Id}.$$

This is closely related to the original form of the periodicity conjecture of Zamolodchikov (1991) proved by Fomin and Zelevinsky (2003). We refer to Brüstle et al. (2012) for the study of maximal green sequences for more general acyclic quivers.

5.2 Square products of Dynkin quivers

Let Q_1 and Q_2 be alternating Dynkin quivers and $Q = Q_1 \square Q_2$ their square product as defined in section 8 of Keller (2010). For example, for suitable orientations of A_4 and D_5, the square product is depicted in Figure 2. There are no longer sources or sinks in the square product. However, we can consider the sequence i_+ of all even vertices $\circ$ (corresponding to a pair of sources or a pair of sinks) and the sequence i_- of all odd vertices $\bullet$ (corresponding to a mixed pair). Let

$$\begin{aligned} \mathbf{i} \; &= i_+ i_- i_+ \dots \quad \text{with } h \text{ factors,} \\ \mathbf{i}' &= i_- i_+ i_- \dots \quad \text{with } h' \text{ factors,} \end{aligned}$$

where h and h' are the Coxeter numbers of the Dynkin diagrams underlying the two quivers. One can check that both of these sequences are maximal green. In particular, the combinatorial DT-invariant is well-defined and we have $\mathbb{E}_Q = \mathbb{E}(\mathbf{i}) = \mathbb{E}(\mathbf{i}')$. It is an open question whether these identities are consequences of the pentagon identity. One can show (cf. section 5.7 of Keller (2011) or section 8.3.2 of Cecotti et al. (2010)) that in this case, the adjoint combinatorial DT-invariant satisfies

$$(DT_Q)^m = \text{Id}, \quad \text{where} \quad m = \frac{2(h+h')}{\gcd(h,h')}.$$

5.3 Quivers from reduced expressions in Coxeter groups

If R is an acyclic quiver and $\tilde{w}$ a reduced expression for an element of the Coxeter group associated with the underlying graph of R, there is a canonical quiver Q associated with the pair $(R, \tilde{w})$, cf. Berenstein et al. (2005). For example, if Q is A_4 with the linear orientation and $\tilde{w}$ a suitable expression for the longest element, one obtains the first quiver in (2.3). As shown by Geißet al. (2011), such quivers always admit maximal green sequences. In the A_4-example, such a sequence is given by

$$7, 8, 9, 10, 4, 5, 6, 2, 3, 1, 7, 8, 9, 2, 3, 1, 7, 8, 4, 7.$$

Thus, the combinatorial DT-invariant is well-defined. In the above example (and for all members of the 'triangular' family it belongs to), the adjoint combinatorial DT-invariant satisfies $(DT_Q)^6 = \mathrm{Id}$. It is an open question for which pairs $(R, \tilde{w})$ the invariant DT_Q is of finite order.

5.4 Another product construction

The following quiver is obtained as the triangle product (cf. section 8 of Keller (2010)) of a quiver of type A_3 with the quiver appearing in the first column (the arrows marked by 2 are double arrows).

$$\begin{tikzcd}
7 \arrow[r] & 8 \arrow[r] \arrow[dl] & 9 \arrow[dl] \\
4 \arrow[u] \arrow[r] & 5 \arrow[u] \arrow[r] \arrow[dl, "2"] & 6 \arrow[u] \arrow[dl, "2"] \\
1 \arrow[u, "2"] \arrow[r] & 2 \arrow[u, "2"] \arrow[r] & 3 \arrow[u, "2"]
\end{tikzcd}$$

It admits the maximal green sequence $3, 6, 9, 2, 5, 8, 1, 4, 7, 3, 6, 9, 2, 5, 8, 3, 6, 9$. In this case, the invariant DT_Q is of infinite order.

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Particles jumping on a cycle: a process on permutations and words

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Abstract. I will describe recent research regarding the so called TASEP on a cycle. It describes permutations (or more generally words) on a cycle, where a small number may jump over a larger number. This process has been studied for reasons coming both from algebraic combinatorics and probability. It exhibits a number of very nice structural, probabilistic and enumerative properties, several of which are still unproved.

Keywords: TASEP, exclusion process, cyclic permutations, cyclic words

Assume we have n particles labeled $\{1, 2, \ldots, n\}$ on a cycle. A particle i can jump to the left if the particle to the left is labelled j, where $j > i$. A jump means that i and j switch places. This Markov chain is an example of a TASEP (Totally asymmetric simple exclusion process). See Figure 1 for the full Markov chain when $n = 3$ and all possible jumps occur with the same rate.

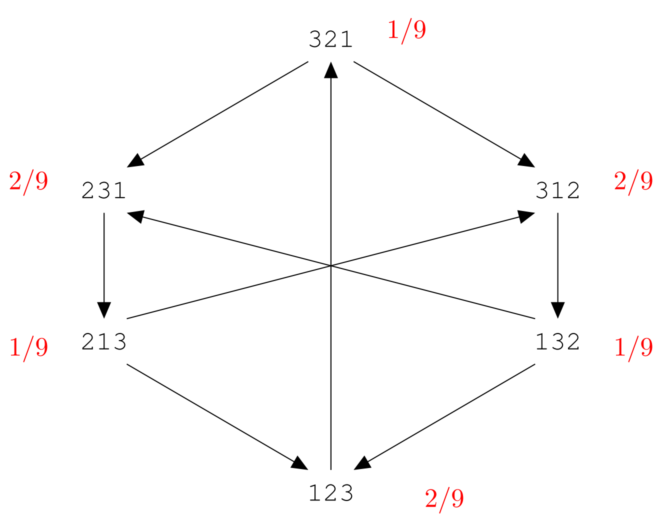

Fig. 1: The cyclic-TASEP Markov chain for $n = 3$. The stationary probabilities are given in red.

TASEPs (and other exclusion processes) on a line have been studied intensively in combinatorics in recent years, see e.g. Duchi and Scheaffer [DS], and Corteel and Williams [CW].

The cyclic TASEP described above when all particles are equally likely to try to jump (i.e. having the same rate) exhibits many interesting properties. For example it was proved by Ferrari and Martin [FM] that the stationary probability for the (cyclic) reverse permutation $w_0 = n...21$ is exactly

$$\frac{1}{\prod_{i=1}^{n} \binom{n}{i}}.$$

It was conjectured by Lam [L] that the probability for the (cyclic) identity permutation is

$$\frac{\prod_{i=1}^{n-1} \binom{n-1}{i}}{\prod_{i=1}^{n} \binom{n}{i}},$$

which was later proved by Aas [A].

One key ingredient to understanding these probabilities are the multiline queues (MLQ) introduced by Ferrari and Martin. These are intricate combinatorial objects such that each of them map to a cyclic permutation (or more generally a cyclic word). Let $q(\pi)$ be the number of MLQs that map to the permutation π. Then Ferrari and Martin proved that the stationary probability for π is

$$\frac{q(\pi)}{\prod_{i=1}^{n} \binom{n}{i}}.$$

Hence one way to understand the cyclic TASEP is to study the combinatorics of the MLQs.

I will describe the MLQs in the talk and discuss what is known and present combinatorial conjectures. I will also discuss the more general case when different particles have different jump rates, see e.g.[AL]. Very interesting positivity properties were conjectured in [LW], some of which now have been proved and some remain open.

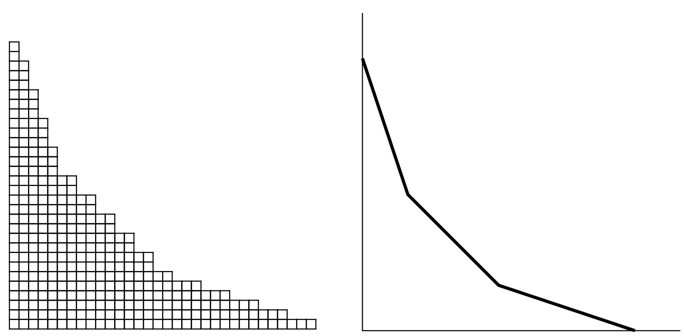

Fig. 2: A large random 4-core, and the limiting piecewise-linear curve.

I will also give some motivation to why this cyclic TASEP is particularly interesting. As Lam [L] showed, it is connected to the shape of both infinite reduced words in the affine Weyl group $\tilde{A}_n$ and the shape of a random n-core partition, i.e. partitions where no hook has length n. Recent unpublished work by Ayyer and Linusson proves that the latter turns to a specific piecewise linear form as conjectured by Lam, see Figure 2.

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Boundaries of branching graphs

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A branching graph is an infinite graded graph, sometimes with an additional structure. The boundary of such a graph describes all possible ways of escaping to infinity along ”regular” monotone paths. This notion emerged about 30 years ago in the work of Vershik and Kerov on characters of the infinite symmetric group. I will survey old and new results related to boundaries of concrete graphs, and state open questions. The problems here originate from representation theory and probability theory, while the methods are mainly of combinatorial nature and rely on the theory of symmetric functions and their analogs, such as supersymmetric and quasisymmetric functions.

Frontières des graphes de branchement

Un graphe de branchement est un graphe gradué infini, muni parfois d'une structure supplémentaire. La frontière d'un tel graphe décrit toutes les manières possibles d'échapper à l'infini le long des chemins monotones “ réguliers”. Cette notion a émergé il y a 30 années d'environ dans le travail de Vershik et Kerov sur les caractères du groupe symétrique infini. Je vais présenter un synthèse des résultats anciens et nouvaux liés aux frontières des graphes concrets et formuler des questions ouvertes. Les problèmes ici proviennent de la théorie des représentations et des probabilités, tandis que les méthodes sont principalement de nature combinatoire et sont fondées sur la théorie des fonctions symétriques et leur analogues, telles que les fonctions supersymétriques et quasisymmétriques.

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Beyond q: special functions on elliptic curves

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An important thread in modern representation theory (and combinatorics) is that many important objects have so- called q-analogues, generalizations depending on a parameter q which reduce to more familiar objects when $q = 1$. For instance, the Schur functions (irreducible characters of the unitary group) have q,t-analogues, namely the famous Macdonald polynomials, and similarly the Koornwinder polynomials are six-parameter q-analogues of the characters of other classical groups. It turns out that many q-analogues extend further to elliptic analogues, in which q is replaced by a point on an elliptic curve. The Macdonald/Koornwinder polynomials are no exception; I will describe a relatively elementary approach to those polynomials and how to modify the approach to obtain an elliptic analogue.

Recent Progress on the Diameter of Polyhedra and Simplicial Complexes

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We review several recent results on the diameter of polytopes, polyhedra and simplicial complexes, motivated by the (now disproved, but not quite solved) Hirsch Conjecture.

Keywords: Polyhedra, simplicial complexes, diameter, simplex method

Introduction

The Hirsch Conjecture, understood in a broad sense, asked what is the maximum possible combinatorial diameter of a convex polyhedron of dimension d and with n facets. Let us denote this number $H(n, d)$. Although the original conjecture $H(n, d) \leq n - d$ has been disproved [7, 9], the underlying problem is still wide open:

- The known counter-examples violate the conjecture only by a constant and *small* factor (25% in the case of unbounded polyhedra, 5% for bounded polytopes).
- No polynomial upper bound is known for $H(n, d)$. All we know is $H(n, d) \leq n^{\log d+2}$ (*quasi-polynomial* bound of Kalai and Kleitman [6]) and $H(n, d) \leq 2^{d-3}n$ (linear bound *in fixed dimension* by Larman [8]).

Some recent attempts of settling this question go by looking at the problem in the more general context of *pure simplicial complexes*: *What is the maximum diameter of the dual graph of a simplicial* $(d-1)$*-sphere or* $(d-1)$*-ball with* n *vertices?*

Here a simplicial $(d-1)$-ball or sphere is a simplicial complex homeomorphic to the $(d-1)$-ball or sphere. These complexes are necessarily *pure* (all the maximal simplices have the same dimension). The dual graph of a pure simplicial complex is the graph whose vertices are the maximal simplices (a. k. a. *facets*) and whose edges correspond to adjacent facets. We can also remove the sphere/ball condition and ask the same for all pure simplicial complexes. Some recent results in this direction are:

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- For arbitrary pure simplicial complexes the diameter can be exponential, in the order of $n^{2d/3}$ [5].
- For complexes in which every star is strongly connected (that is, the dual graph of every star is connected) the Kalai-Kleitman and the Larman bounds stated above hold, essentially with the same proofs. These complexes have been called *normal* or *locally strongly connected* in the literature.
- For complexes which are not only normal but also *flag* (meaning that the complex is the *clique complex* of its 1-skeleton), the original Hirsch bound holds [1].

Going back to polytopes, there is also a recent bound in terms of n, d and the maximum determinant of the system defining the polytope [2] and a recent construction of polytopes which fail to have the k-decomposability property, for arbitrarily large k [4].

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Razumov-Stroganov–type Correspondences in the 6-Vertex and $\mathrm{O}(1)$ Dense Loop Model

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Razumov and Stroganov conjectured in 2001 a correspondence between the enumerations of Fully-Packed Loops (FPL) on a square domain (a version of the 6-Vertex Model), refined according to the link pattern, and the ground-state components of the Hamiltonian in the periodic XXZ Quantum Spin Chain at $\Delta = -1/2$, a realisation of the $\mathrm{O}(1)$ Dense Loop Model (DLM) on a cylinder.

Extensions have been considered later on. In particular, Di Francesco in 2004 suggested a one-parameter generalization: on the 'DLM side', the ground state of the Hamiltonian H is replaced by the one of the Scattering Matrix, $S(t)$; on the 'FPL side', one also considers the refinement on the last row.

Similar conjectures existed for two large families of domains: those with a 'hidden dihedral symmetry', or with 'vertical symmetry', respectively. Both the basic and extended conjectures have been proven, in the dihedral case, by L. Cantini and the speaker, while the vertical cases are open.

We present the subject, its implications on Algebraic Combinatorics and Statistical Mechanics, and how the forementioned conjectures have been proven.

A combinatorial method to find sharp lower bounds on flip distances

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Abstract. Consider the triangulations of a convex polygon with n vertices. In 1988, Daniel Sleator, Robert Tarjan, and William Thurston have shown that the flip distance of two such triangulations is at most $2n-10$ when n is greater than 12 and that this bound is sharp when n is large enough. They also conjecture that “large enough” means greater than 12. A proof of this conjecture was recently announced by the author. A sketch of this proof is given here, with emphasis on the intuitions underlying the construction of lower bounds on the flip distance of two triangulations.

Résumé. En 1988, Daniel Sleator, Robert Tarjan et William Thurston ont montré que la distance, en nombre de flips, de deux triangulations d'un polygone convexe de n sommets est au plus $2n-10$ quand n est supérieur à 12. Ils ont également montré que cette borne est atteinte si n est suffisamment grand et ils conjecturent qu'il existe deux triangulations à distance $2n-10$ dès que n est supérieur à 12. Un preuve de cette conjecture a récemment été proposée par l'auteur. Une ébauche de cette preuve est présentée ici qui explique de manière intuitive les méthodes permettant d'obtenir des bornes inférieures sur la distance, en nombre de flips, de deux triangulations.

Keywords: Triangulations, Flip-graph, Rotation distance, Binary tree, Associahedra

1 Introduction

Consider a convex polygon π with n vertices. For convenience, π will be identified with the set of its vertices. An edge of π is a subset of π with two elements. A triangulation of π is a maximal set of pairwise non-crossing edges of π. The elements of a triangulation will be referred to as *its* edges.

According to this definition, a triangulation T of π contains every boundary edge of π. All the other edges of T will be called its *interior edges*. Note that an interior edge ε of T is the diagonal of a quadrilateral q whose four boundary edges belong to T. Replacing ε within T by the other diagonal, say ς, of quadrilateral q results in another triangulation T/ε of π:

$$T/\varepsilon = [T\backslash\{\varepsilon\}] \cup \{\varsigma\}.$$

This operation is called a *flip*. While working on the dynamic optimality conjecture [7], Daniel Sleator, Robert Tarjan, and William Thurston have shown that, when n is greater than 12, one can transform any triangulation of π into any other triangulation of π by performing at most $2n-10$ flips [8]. In other words, the flip distance of two triangulations of π is at most $2n-10$ when n is greater than 12. It is also proven in [8] that when n is "large enough", there exist pairs of triangulations of π whose flip distance is precisely $2n-10$. Unfortunately their proof does not give an hint on how large n should be, and they propose the following conjecture: *there exist pairs of triangulations of π whose flip distance is precisely $2n-10$ whenever n is greater than* 12.

While the upper bound obtained by Daniel Sleator, Robert Tarjan, and William Thurston follows from an easy combinatorial argument, they use constructions in hyperbolic space to prove the existence of pairs of triangulations at maximal flip distance. They comment that the role played by hyperbolic geometry in this problem, whose statement is purely combinatorial, may seem mysterious and they ask for a combinatorial proof of their existence result. Three years ago, Patrick Dehornoy made progress toward such a proof by finding a lower bound of the form $2d-O(\sqrt{d})$ on the diameter of the d-dimensional associahedron, using combinatorial arguments [1].

A solution to these two open problems has been announced recently [6]. The purpose of this extended abstract is to sketch this proof with emphasis on the underlying intuitions.

Two triangulations at maximal flip distance are described in Section 3. The proof that these triangulations are indeed maximally distant is sketched in Section 4, and the combinatorial methods used to do so (i.e. general equalities and inequalities on flip distances) are presented in Section 2. In some places, proofs are omitted. In these cases, the interested reader is referred to [6] for the complete argument.

2 Equalities and inequalities on flip distances

In this section, several types of equalities and inequalities on flip-distance are obtained. Consider two triangulations U and V of a same polygon. A *path* of length k between U and V is a sequence of k flips that transforms U into V. A such path is called *minimal* when its length is minimal among all the path between U and V. Hereafter, the length of any minimal path between U and V is denoted by $\delta(\{U,V\})$. Part (a) of Lemma 3 from [8] states that, if an edge of V can be introduced in U by some flip, then there is a minimal path from U to V that begins with this flip. This result is generalized by the following theorem, whose straightforward proof can be found in [6]:

Theorem 1 *Let T, U, and V be three triangulations of a convex polygon. If:*

i. T is found along a minimal path between U and V,

ii. An edge of T can be introduced in U by some flip,

then there is a minimal path from U to V that begin with this flip.

This theorem makes it possible to prescribe the first flip along some minimal path between two triangulations. It can also be thought of as an equality on flip distances. Indeed, if ε is the edge removed by the flip mentioned in the second statement of Theorem 1, then the conclusion of this theorem is:

$$\delta(\{U,V\}) = \delta(\{U/\varepsilon, V\}) + 1.$$

Theorem 1 can further be generalized to sequences flips [6]. This generalization will be replaced here by less complete as in the proof of Theorem 6 thereafter. The proof sketched in Section 4 will also require inequalities that compare flip distances in the case of two polygons with different numbers of vertices. In order to obtain such inequalities, one can use the contraction operation that is now introduced.

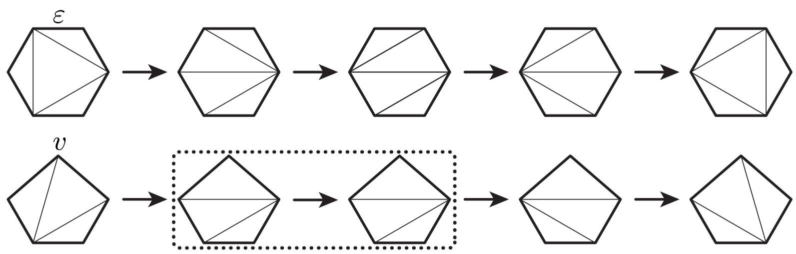

Fig. 1: A (minimal) path between the two triangulations of the hexagon whose interior edges form a triangle (top row), and the sequence of triangulations resulting from the contraction of edge ε to vertex v (bottom row). The framed triangulations in the bottom row are identical, and one of them can be removed from the sequence.

Let ε be some boundary edge of a given polygon π. Contracting ε in a triangulation T of π consists in replacing the two vertices of ε by a single point v within every edge of $T\backslash\{\varepsilon\}$. If v belongs to $\mathrm{conv}(\varepsilon)$, then this operation results in a triangulation of $(\pi\backslash\varepsilon)\cup\{v\}$ (a proof of this is given in [6]). It is assumed in the following that v inherits the labels of the two vertices of ε, and can be referred to indifferently using these two labels. This convention on the labeling of the contracted polygon is slightly different from that in [6]. Denote by $T\curlyvee\varepsilon$ the triangulation obtained by contracting edge ε in a triangulation T of π.

The effect of contractions on a path between two triangulations is now described using an example. Denote by U and V the two triangulations of the hexagon whose interior edges form a triangle. A path between U and V is shown in the top of Figure 1. Contracting the boundary edge ε at the top of the hexagon (as shown in the figure) in every triangulation along this path results in the sequence of triangulations of the pentagon depicted in the bottom of the figure. It can be seen that two consecutive triangulations in this sequence are either identical or connected by a flip. Now observe that the second and the third triangulation in the second row are identical precisely because the two corresponding triangulations in the top row are connected by a flip that modifies the triangle containing edge ε. In other words, the length of the path between $U\curlyvee\varepsilon$ and $V\curlyvee\varepsilon$ resulting from the contraction is less than the length of the path between U and V by the number of flips along this path that modify the triangle containing ε. According to the following theorem, proven in [6], this property holds in general:

Theorem 2 *Let U and V be two triangulations of a same polygon and ε a boundary edge of this polygon. If ψ is a path of length k between U and V, then there exists a path of length $k-j$ between $U\curlyvee\varepsilon$ and $V\curlyvee\varepsilon$, where j is the number of flips along path ψ that modify the triangle containing edge ε.*

Consider a pair P of triangulations of a polygon π and a boundary edge ε of π. Theorem 2 provides a convenient way to obtain inequalities between the flip distances of P and of $P\curlyvee\varepsilon$. Call $\vartheta(P,\varepsilon)$ the maximal number of flips that modify the triangle containing ε along any minimal path between the two elements of P. The following corollary is a direct consequence of Theorem 2:

Corollary 1 *Let P be a pair of triangulations of a polygon π. If ε is a boundary edge of π then:*

$$\delta(P)\geq\delta(P\curlyvee\varepsilon)+\vartheta(P,\varepsilon).$$

This corollary provides lower bounds on $\delta(P)$ that depend on $\vartheta(P,\varepsilon)$. The remainder of the section is dedicated to finding a lower bound on $\vartheta(P,\varepsilon)$.

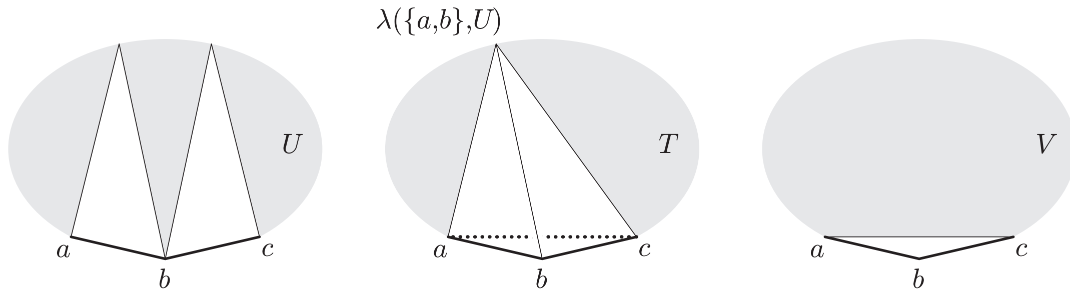

Fig. 2: Triangulations U (left) and V (right) depicted according to the requirements of Theorem 3, and triangulation T (center) used in the proof of this theorem. The dotted line shows that edge $\{b\} \cup \lambda(\{a,b\},U)$ is replaced by edge $\{a,c\}$ in triangulation T by some flip along a minimal path from U to V.

Let T be a triangulation of a polygon π and ε a boundary edge of π. Call a and b the two vertices of ε. Denote by c the vertex of π so that $\{a,c\}$ and $\{b,c\}$ are two edges of T. In other words, a, b, and c are the three vertices of the triangle in T that contains edge ε. The *link* of ε in T, denoted by $\lambda(\varepsilon,T)$ hereafter, is the set $\{c\}$. The following theorem is borrowed from [6]:

Theorem 3 *Let U and V be two triangulations of a polygon π. If a, b, and c three vertices of π so that:*

i. $\{a,b\}$ *and* $\{b,c\}$ *are boundary edges of* π *and* $\{a,c\}$ *belongs to* V*,*

ii. $\lambda(\{a,b\},U)$ *and* $\lambda(\{b,c\},U)$ *are distinct subsets of* $\pi\backslash\{a,c\}$*,*

then $\vartheta(\{U,V\},\{a,b\})$ *and* $\vartheta(\{U,V\},\{b,c\})$ *are not both less than* 2*.*

Proof: Consider three vertices a, b, and c of π. Assume that $\{a,b\}$ and $\{b,c\}$ are boundary edges of π and that $\{a,c\}$ is an edge of V. As a consequence, triangle $\{a,b,c\}$ is found in triangulation V. Further assume that $\lambda(\{a,b\},U)$ and $\lambda(\{b,c\},U)$ are distinct subsets of $\pi\backslash\{a,c\}$. It follows that the two edges $\{a,b\}$ and $\{b,c\}$ are contained in two distinct triangles of U whose unique common vertex is b. Triangulations U and V are depicted in the left and in the right of Figure 2 in this case.

Now assume that $\vartheta(\{U,V\},\{a,b\}) \leq 1$. Since the triangles in U containing $\{a,b\}$ and $\{b,c\}$ are distinct, edge $\{a,b\}$ cannot be contained in the same triangle in U and in V. Hence, there is exactly one flip that modifies the triangle containing $\{a,b\}$ along any minimal path from U to V. Consider such a path ψ, and call T the triangulation in which this flip is performed along path ψ. Since there is only one such flip along path ψ, this flip necessarily replaces edge $\{b\} \cup \lambda(\{a,b\},U)$ by edge $\{a,c\}$ as shown in the center of figure 2, where the latter edge is depicted as a dotted line. As $\lambda(\{a,b\},U)$ and $\lambda(\{b,c\},U)$ are distinct, then the triangle containing edge $\{b,c\}$ must have been modified at least once along path ψ before T is reached. Moreover, the flip performed in T also modifies the triangle containing edge $\{b,c\}$. As a consequence, the triangle containing edge $\{b,c\}$ is modified at least twice along path ψ, which shows that $\vartheta(\{U,V\},\{b,c\})$ is not less than 2. $\square$

Theorem 1, Corollary 1 and Theorem 3 are the basic building blocks of the combinatorial method used in Section 4 to obtain sharp lower bounds of flip distances. While they need to be generalized to allow for a rigorous proof of the desired result (see [6]), such generalizations will not be given here. Intuition on how the proof works will be provided instead using the results presented above.

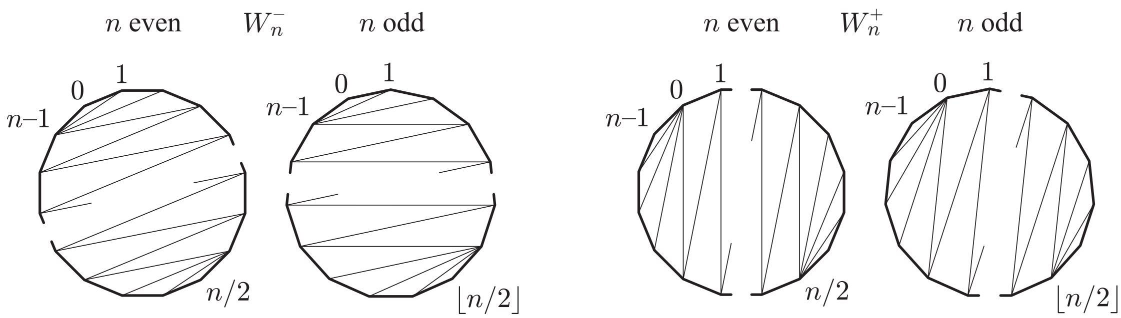

Fig. 3: Triangulations W_n^- (left) and W_n^+ (right) depicted when n is greater than 8.

3 A pair of triangulations at maximal distance

Let π be a convex polygon with n vertices labeled clockwise from 0 to $n-1$. Consider the triangulations W_n^- and W_n^+ of π depicted in Figure 3 depending on the parity of n. It is shown in [6] that these two triangulations have flip distance $2n-10$ when n is greater than 12. As can be seen in the figure, W_n^- contains three interior edges incident to vertex $n-1$. A such set of edges will be referred to as a *comb with three teeth at vertex* $n-1$. Triangulation W_n^- has another comb at vertex $\lfloor n/2 \rfloor - 1$ with three or four teeth depending on the parity of n. Triangulation W_n^+ contains a comb with four teeth at vertex 0, and another comb at vertex $\lfloor n/2 \rfloor$ with three or four teeth depending on the parity of n. The remaining interior edges of W_n^- and W_n^+ form a zigzag (i.e. a simple path whose vertices of edges belong to "opposite" sides of the polygon) that connects the two combs contained in these triangulations. One can see in Figure 3 that the size of these zigzags depends on n. When n is equal to 9, the polygon is precisely large enough to contain the two combs. In this case, the combs have a common tooth, and the zigzag disappears. For this reason, the above description of W_n^- and W_n^+, and their representation in Figure 3 are only valid when n is greater than 8.

The definition of W_n^- and W_n^+ given in [6] is more general. In particular, it is valid whenever n is greater than 2. Obviously, when $3 \leq n \leq 8$, the combs found in these triangulations lose teeth, or even disappear. For these small values of n, though, W_n^- and W_n^+ keep a number of important properties that are needed in the proof. Triangulations W_n^- and W_n^+ are sketched in Figure 4 when $4 \leq n \leq 8$. If n is equal to 3, both triangulations shrink to a single triangle, and for this reason, they are omitted in the figure. For any n greater than 2, denote:

$$A_n = \{W_n^-, W_n^+\}.$$

When $3 \leq n \leq 12$, the flip distance of pair A_n can be easily found using a computer program or a mathematical proof. In particular, the flip distances reported in the following table are obtained in [6] using the methods presented in Section 2, and their generalizations:

n	3	4	5	6	7	8	9	10	11	12
$\delta(A_n)$	0	1	2	4	5	7	8	10	12	14

One can see in this table that $\delta(A_n)$ is greater than $2n-10$ when $3 \leq n \leq 8$ and equal to this value when $9 \leq n \leq 12$. Using the results of the previous section, a recursive lower bound on $\delta(A_n)$ will be

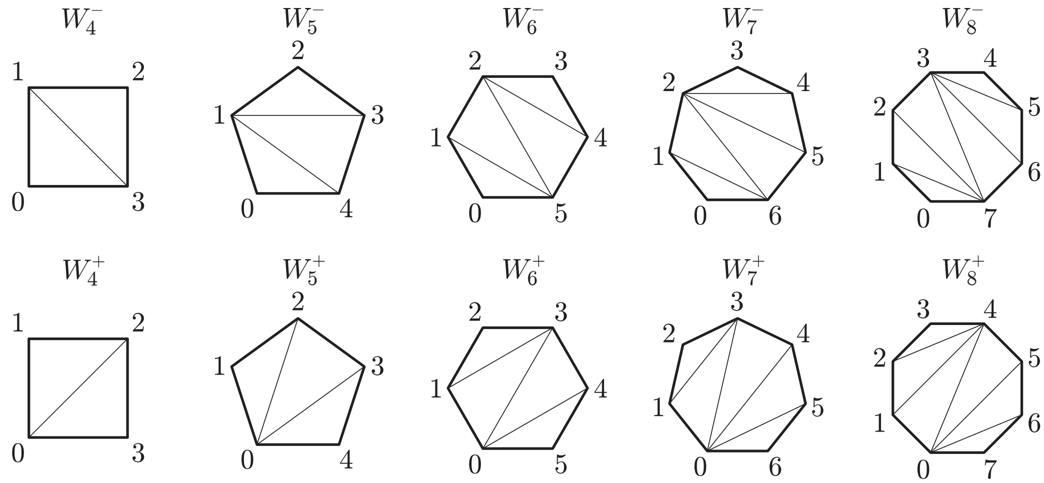

Fig. 4: Triangulations W_n^- (left) and W_n^+ (right) depicted when $4 \leq n \leq 8$.

obtained in the next section for any $n > 12$. This lower bound together with the values of $\delta(A_n)$ reported in the above table will produce the desired result.

4 Sketch of the proof

It can be seen in Figures 3 and 4 that contracting edge $\{n-2, n-1\}$ in triangulations W_n^- and W_n^+ results in pair A_{n-1}, up to a relabeling of the vertices. Note that this observation holds whenever n is greater than 3. Hence, one obtains the following result as a consequence of Corollary 1:

Theorem 4 *Let n be an integer greater than 3, if $\vartheta(A_n, \{n-2, n-1\}) \geq 2$,*

$$\delta(A_n) \geq \delta(A_{n-1}) + 2.$$

Assuming that the inequality $\delta(A_n) \geq \delta(A_{n-1}) + 2$ holds in general for any n greater than 3, the desired result immediately follows. Unfortunately, this inequality is only obtained under a rather strong condition, that is the existence of a minimal path between W_n^- and W_n^+ that modifies at least twice the triangle containing edge $\{n-2, n-1\}$. In fact, such a minimal path does not necessarily exist. In order to overcome this difficulty, other contractions will be considered.

Observe, for instance, that the contraction considered in Theorem 4 can be iterated. If n is greater than 4, contracting edge $\{n-2, n-1\}$ in A_n and then edge $\{0, 1\}$ will result in pair A_{n-2}, up to a relabeling of the vertices. This observation remains true if one exchanges the order of the two contractions: first contracting edge $\{0, 1\}$ and then edge $\{n-2, n-1\}$ in A_n still results in a pair of triangulations whose flip-distance is $\delta(A_{n-2})$. It can be seen in Figures 3 and 4 that the triangle containing edge $\{n-2, n-1\}$ is not the same in triangulations $W_n^- \curlyvee \{0, 1\}$ and $W_n^+ \curlyvee \{0, 1\}$. Note that this observation holds whenever n is greater than 5. Hence, at least one flip modifies this triangle in any minimal path between these two triangulations and, as a consequence, Corollary 1 yields:

$$\delta(A_n \curlyvee \{0, 1\}) \geq \delta(A_{n-2}) + 1.$$

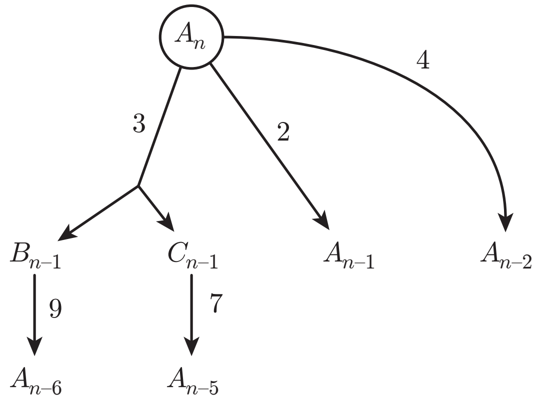

Fig. 5: Sketch of the proof of Theorem 9. An arc of weight w from a pair P to a pair Q represents the inequality $\delta(P) \geq \delta(Q) + w$ obtained under some condition. These conditions are omitted in this sketch but can be found in the statements of the corresponding theorems.

Combining this inequality with the inequality obtained invoking Corollary 1 for the contraction of $\{0, 1\}$ in pair A_n results in the following theorem:

Theorem 5 *Let n be an integer greater than 5, if $\vartheta(A_n, \{0, 1\}) \geq 3$,*

$$\delta(A_n) \geq \delta(A_{n-2}) + 4.$$

As was the case with Theorem 4, the inequality provided by Theorem 5 is subject to an (even stronger) condition involving the existence of a particular path between W_n^- and W_n^+. Again, nothing tells us that such a path exists. For this reason, the condition complementary to the requirements of Theorems 4 and 5 will have to be investigated. Before going any further, consider Figure 5, where the main proof is sketched using a tree. In this figure, each arc corresponds to an inequality obtained under some condition. The inequality corresponding to an arc of weight w from a pair P to a pair Q is:

$$\delta(P) \geq \delta(Q) + w.$$

Hence, the arcs from A_n to A_{n-1} and from A_n to A_{n-2} respectively correspond to Theorems 4 and 5. Since these two inequalities are obtained under the condition that particular paths exist between W_n^- and W_n^+, the third arc originating at pair A_n in Figure 4 will be considered under the complementary condition. Observe that this arc has two ends. This is due to the fact that the inequality corresponding to this arc relates $\delta(A_n)$ with either $\delta(B_{n-1})$ or $\delta(C_{n-1})$. The two pairs of triangulations B_n and C_n can be defined for any integer n greater than 8 as:

$$B_n = \{Y_n^-, Y_n^+\} \text{ and } C_n = \{X_n^-, Y_n^+\},$$

where Y_n^-, Y_n^+, and X_n^- are depicted in Figure 6. Triangulation Y_n^- is depicted using solid lines in the left of the figure. It contains two combs at vertices 3 and $\lceil n/2 \rceil - 1$. The comb at vertex 3 always has three

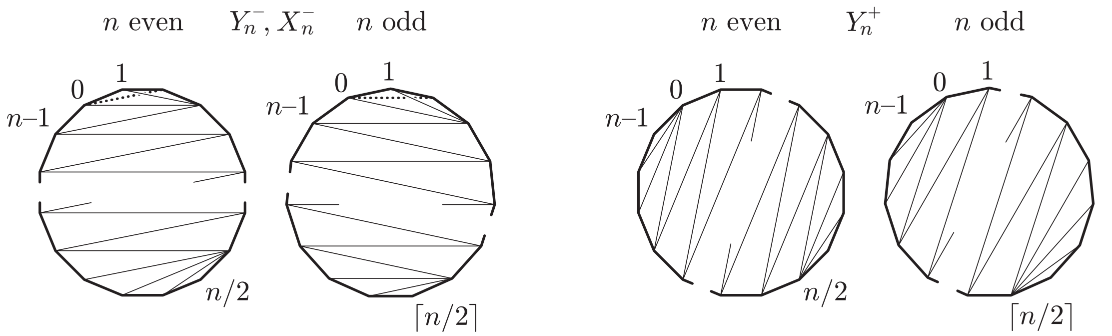

Fig. 6: Triangulation Y_n^- (left) and Y_n^+ (right) shown in solid lines when n is greater than 8. Triangulation X_n^-, obtained by flipping edge $\{1,3\}$ in Y_n^- is depicted using dotted lines (left).

teeth, and the other comb has four teeth if n is even, and only three is n is odd. The remaining interior edges of the triangulation form a zigzag that connects the two combs. Note that when n is equal to 9 or to 10, the zigzag disappears as the two combs become adjacent. Triangulation X_n^- is obtained by flipping edge $\{1,3\}$ in triangulation Y_n^-, which is sketched using dotted lines in the left of Figure 6. Observe that this flip removes the comb at vertex 3. Triangulation Y_n^+ is shown in the right side of the figure. As can be seen, Y_n^+ can be built by appropriately relabeling the vertices of W_n^-. In particular Y_n^+ contains a comb with three teeth at vertex 0 and another comb at vertex $\lceil n/2 \rceil$ whose number of teeth (three or four) depends on the parity of n.

Observe that pairs B_n and C_n are also defined when $7 \leq n \leq 8$ in [6]. This is needed for the computer-free proof that $\delta(A_n) \geq 2n - 10$ when $3 \leq n \leq 12$. This result has already been given above without a proof (see [6] for one). It is therefore not necessary to define B_n and C_n when $7 \leq n \leq 8$ here. For the same reason, the following theorem is stated here when n is greater than 9 rather than when n is greater than 7. It gives the inequality corresponding to the leftmost arc originating at pair A_n in Figure 5:

Theorem 6 *Let n be an integer greater than 9. If $\vartheta(A_n, \{n-2, n-1\}) \leq 1$ and $\vartheta(A_n, \{0,1\}) \leq 2$, then there exists $P \in \{B_{n-1}, C_{n-1}\}$ so that $\delta(A_n) = \delta(P) + 3$ and $\vartheta(P, \{0,1\}) \leq 1$.*

Proof (sketch): Assume that $\vartheta(A_n, \{n-2, n-1\}) \leq 1$ and that $\vartheta(A_n, \{0,1\}) \leq 2$. Observe that $\{n-2, n-1\}$ is not contained in the same triangle in W_n^- and W_n^+. Hence, there is exactly one flip that modifies the triangle containing this edge along any minimal path from W_n^- to W_n^+. Consider such a path ψ, and call T the triangulation resulting from the flip along this path that modifies the triangle containing $\{n-2, n-1\}$. Since there is only one such flip along path ψ, this flip necessarily replaces edge $\{3, n-1\}$ by edge $\{0, n-2\}$. In particular, T contains edges $\{0, n-2\}$ and $\{0,3\}$. As a first consequence, all the boundary edges of quadrilateral $\{0,1,2,3\}$ are contained in T and one diagonal of this quadrilateral (i.e. $\{0,2\}$ or $\{1,3\}$) must also be contained in T. Denote by ε this diagonal.

It can be seen in Figure 3 that edges $\{0,2\}$, $\{0,3\}$, and $\{0, n-2\}$ can be introduced in W_n^- by successively flipping $\{1, n-1\}$, $\{2, n-1\}$, and $\{3, n-1\}$ in this order. Reversing the order of the first two flips, this sequence will introduce edges $\{1,3\}$, $\{0,3\}$, and $\{0, n-2\}$ instead. This proves that edges ε, $\{0,3\}$, and $\{0, n-2\}$ can be introduced in W_n^- by performing three consecutive flips. As shown above, these three edges are contained in T. Hence, it follows from Theorem 1 that this sequence of three flips is

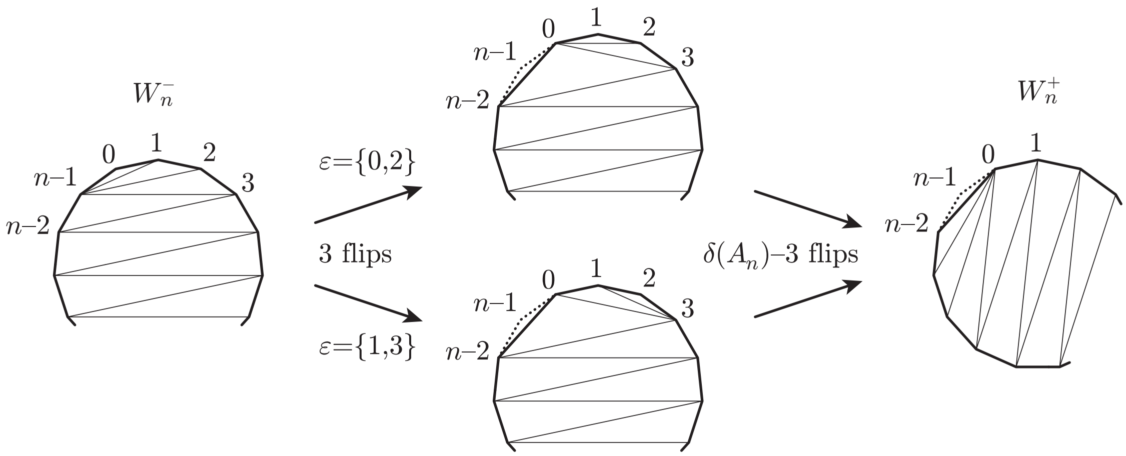

Fig. 7: Triangulation U (center) used in the proof of Theorem 6, and shown here depending on ε. In this proof, triangulation U is reached after three flips along a path ϕ from W_n^- to W_n^+.

the beginning of some minimal path ϕ from W_n^- to W_n^+.

The triangulation U obtained after the three first flips along path ϕ is shown in Figure 7 depending on whether ε is equal to $\{0,2\}$ or to $\{1,3\}$. One can see that U and W_n^+ both contain the triangle with vertices 0, $n-1$, and $n-2$. Hence, it follows from part (b) of Lemma 3 from [8] that this triangle will be contained in every triangulation found along any minimal path between U and W_n^+. As a consequence, one can remove the two boundary edges of π incident to vertex $n-1$ from U and from W_n^+ without changing the flip distance. The pair of triangulations thus obtained is either equal to B_{n-1} (if $\varepsilon=\{1,3\}$) or to C_{n-1} (if $\varepsilon=\{0,2\}$). Observe in particular that relabeling the vertices of π is unnecessary here.

Denote $P=B_{n-1}$ if $\varepsilon=\{1,3\}$ and $P=C_{n-1}$ if $\varepsilon=\{0,2\}$. It has been proven that:

$$\delta(A_n)=\delta(\{U,W_n^+\})+3 \text{ and } \delta(\{U,W_n^+\})=\delta(P).$$

Combining these two equalities yields $\delta(A_n)=\delta(P)+3$. Now observe that exactly one of the first three flips along path ϕ modifies the triangle that contains in $\{0,1\}$. Since a minimal path between W_n^- and W_n^+ can be built from the three first flips along path ϕ and from any minimal path between the two elements of P, one obtains:

$$\vartheta(A_n,\{0,1\})\geq\vartheta(P,\{0,1\})+1.$$

As $\vartheta(A_n,\{0,1\})\leq 2$, this proves that $\vartheta(P,\{0,1\})$ is not greater than 1. □

Two more inequalities have to be obtained, corresponding to the arcs originating at pairs B_{n-1} and C_{n-1} in Figure 5. According to the statement of Theorem 6, these two inequalities can be proven under the assumption that the image by $\vartheta(P,\{0,1\})$ is at most 1, where P is the pair at the origin of the arc. The proofs of these inequalities rely on Theorem 3 and its generalizations. Here, only the simplest of these inequalities will be proven. Moreover, instead of using the generalization of Theorem 3 found in [6], the proof will be sketched in an alternative, more intuitive way.

Theorem 7 *Let n be an integer greater than 11. If $\vartheta(C_n,\{0,1\})\leq 1$, then $\delta(C_n)\geq\delta(A_{n-4})+7$.*

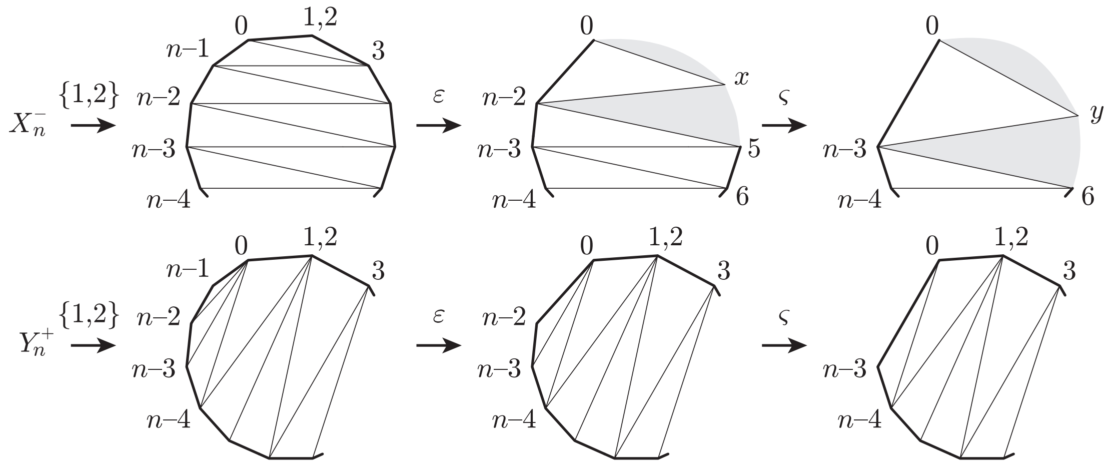

Fig. 8: The first three contractions used in the proof of Theorem 7. The arrows indicate the successive contractions of edges $\{1, 2\}$, ε, and ς in triangulations X_n^- and Y_n^+. The vertex resulting from the contraction of edge $\{1, 2\}$ is labeled by 1 and by 2 following the convention introduced in Section 2. The vertex labeled x is either equal to 3 or to 4, and the vertex labeled y is either equal to 3, to 4, or to 5 depending on ε and ς.

Proof (sketch): Assume that $\vartheta(C_n, \{0, 1\}) \leq 1$. First observe that successively contracting edges $\{1, 2\}$, $\{n-3, n-2\}$, $\{n-2, n-1\}$, and $\{0, n-1\}$ in pair C_n results in pair A_{n-4} up to a renumbering of the vertices. It can be seen in Figure 6 that, in pair C_n, vertices 0, 1, and 2 satisfy the conditions on a, b, and c required by Theorem 3. As $\vartheta(C_n, \{0, 1\}) \leq 1$, then according to this theorem,

$$\vartheta(C_n, \{1, 2\}) \geq 2. \tag{1}$$

The two elements of pair $C_n \curlyvee \{1, 2\}$ are depicted in the left of Figure 8. It can be seen in this figure that the links of edges $\{n-3, n-2\}$, $\{n-2, n-1\}$, and $\{0, n-1\}$ in triangulation $X_n^- \curlyvee \{1, 2\}$ are respectively $\{5\}$, $\{4\}$, and $\{3\}$. This is a consequence of n being greater than 11. Indeed, recall that X_n^- has a comb at vertex $\lceil n/2 \rceil - 1$ (see Figure 6). Since n is greater than 11 then $5 \leq \lceil n/2 \rceil - 1$ and, among the three edges $\{n-3, n-2\}$, $\{n-2, n-1\}$, and $\{0, n-1\}$, only the first one is possibly placed between two teeth of the comb at vertex $\lceil n/2 \rceil - 1$. Hence, the three links are necessarily distinct.

Now observe that contracting edges $\{n-3, n-2\}$, $\{n-2, n-1\}$, and $\{0, n-1\}$ in any order in this pair always result in A_{n-4} (recall that when an edge is contracted to a vertex, this vertex inherits the labels of the two vertices of the contracted edge). It can be proven, using Theorem 3, that one of these orders provides the desired inequality. Indeed, it can be seen in the left of Figure 8 that, in pair $C_n \curlyvee \{1, 2\}$, vertices 0, $n-1$, and $n-2$ satisfy the conditions on a, b, and c in the statement of this theorem. Hence, the following inequality holds, either with $\varepsilon = \{0, n-1\}$ or with $\varepsilon = \{n-2, n-1\}$:

$$\vartheta(C_n \curlyvee \{1, 2\}, \varepsilon) \geq 2. \tag{2}$$

The two elements of pair $C_n \curlyvee \{1, 2\} \curlyvee \varepsilon$ are depicted in the center of Figure 8. In this figure, x is either equal to 3 (if $\varepsilon = \{n-2, n-1\}$) or to 4 (if $\varepsilon = \{0, n-1\}$). It can be seen that, in pair $C_n \curlyvee \{1, 2\} \curlyvee \varepsilon$, vertices 0, $n-2$, and $n-3$ satisfy the conditions on a, b, and c in the statement of Theorem 3. Hence, the following inequality holds, with $\varsigma = \{0, n-2\}$ or with $\varsigma = \{n-3, n-2\}$:

$$\vartheta(C_n \curlyvee \{1,2\} \curlyvee \varepsilon, \varsigma) \geq 2. \tag{3}$$

The two elements of pair $C_n \curlyvee \{1,2\} \curlyvee \varepsilon \curlyvee \varsigma$ are depicted in the right of Figure 8. Here, y is equal to 3, to 4, or to 5 depending on the values of ε and ς. Now recall that n is greater than 11. As a consequence, $y < n-4$. It can be seen in Figure 8 that, in this case, the triangle containing edge $\{0, n-3\}$ is not the same in the two elements of pair $C_n \curlyvee \{1,2\} \curlyvee \varepsilon \curlyvee \varsigma$. Hence, every minimal path between these two triangulations modifies at least once the triangle containing edge $\{0, n-3\}$, and therefore:

$$\vartheta(C_n \curlyvee \{1,2\} \curlyvee \varepsilon \curlyvee \varsigma, \{0, n-3\}) \geq 1. \tag{4}$$

Invoking four times Corollary 1 and using (1), (2), (3), and (4) yields the desired inequality. □

Observe that Theorem 7 can be proven using a generalization of Corollary 1, which avoids invoking this corollary four times (see [6]). The above alternative proof explains how this generalization works, and may help the interested reader in understanding the argument used in [6]. Note in particular, that the sequence of contractions used in the above proof is built so that every contraction “eats up” at least two flips, except for the last one.

Using a similar argument, one obtains the following result, that corresponds to the arc originating at pair B_{n-1} in Figure 5. The reader will find a rigorous proof of this result in [6]. Note that this proof requires five successive contractions instead of just four.

Theorem 8 *Let n be an integer greater than 11. If $\vartheta(B_n, \{0,1\}) \leq 1$ then $\delta(B_n) \geq \delta(A_{n-5}) + 9$.*

This theorem provides the last of the inequalities corresponding to the arcs shown in Figure 5. Recall that every inequality is obtained under some condition. The conditions associated to the different arcs originating at a given pair in Figure 5 exhaust all possibilites, though, and one can read this figure as the following recursive lower bound on $\delta(A_n)$:

Theorem 9 *For any integer n greater than 12,*

$$\delta(A_n) \geq \min(\delta(A_{n-1}) + 2, \delta(A_{n-2}) + 4, \delta(A_{n-5}) + 10, \delta(A_{n-6}) + 12).$$

This result can be used to prove inductively that $\delta(A_n) \geq 2n + O(1)$ for every integer n greater than 2. Now recall that, as mentioned in Section 3, the flip-distance of W_n^- and W_n^+ is at least $2n - 10$ when $3 \leq n \leq 12$. Hence, one obtains that $\delta(A_n) \geq 2n + 10$ for every integer n greater than 2. Further recall that, as shown in [8] using a combinatorial argument, the flip distance of two such triangulations is at most $2n - 10$ when n is larger than 12. One therefore obtains the following result:

Theorem 10 *For any integer n greater than 12,*

i. The flip distance of triangulations W_n^- and W_n^+ is exactly $2n - 10$,

ii. The flip graph of a polygon with n vertices has diameter $2n - 10$.

5 Conclusion

The method given in [6] to obtain sharp lower bounds on flip distances has been presented here with emphasis on the underlying intuitions, sometimes at the expense of completeness. The reader is referred to [6] for a complete and rigorous argument. This method provides a way to solve the two open problems formulated in [8]. In other words, it can be used to obtain a combinatorial proof that the flip-graph (i.e. the graph whose vertices are the triangulations, and whose edges are the flips) of a polygon with n vertices has diameter $2n-10$ when n is greater than 12. A direct consequence of this result is that the diameter of the d-dimensional associahedron [4] is $2d-4$ for all integers d greater than 9.

It is natural to ask whether this method, or a variation of it, could be applied to the several generalizations of triangulations and flips that can be found in the literature. One of these generalizations consists in considering the regular triangulations of a finite, but otherwise arbitrary d-dimensional set of points (see for instance [2] and [3]). In this case, though, the operation of edge contraction may not be possible [6]. Another way to generalize the problem is to consider multitriangulations and their flips [5]. It has been shown that edge contraction is possible in this case. Unfortunately, Lemma 3 from [8] may not carry over to multitriangulations, and one may need alternative arguments.

Finally, it was suggested by a referee of this paper that the methods presented above could also be used to obtain the maximal distance of centrally symmetric triangulations, under the modified flip operation that either exchanges two diameters of the polygon or simultaneously exchanges the diagonals of two centrally symmetric quadrilaterals. In this case, contractions should affect two opposite edges of the polygon. Solving this problem would, in addition, provide the diameters of cyclohedra.

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A generalization of the quadrangulation relation to constellations and hypermaps

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Abstract. Constellations and hypermaps generalize combinatorial maps, *i.e.* embedding of graphs in a surface, in terms of factorization of permutations. In this paper, we extend a result of Jackson and Visentin (1990) on an enumerative relation between quadrangulations and bipartite quadrangulations. We show a similar relation between hypermaps and constellations by generalizing a result in the original paper on factorization of characters. Using this enumerative relation, we recover a result on the asymptotic behavior of hypermaps of Chapuy (2009).

Résumé. Les constellations et les hypercartes généralisent les cartes combinatoires, *i.e.* les plongements de graphe dans une surface, en terme de factorisation de permutations. Dans cet article, nous généralisons un résultat de Jackson et Visentin (1990) sur une relation énumérative entre les quadrangulations ordinaires et biparties. Nous montrons une relation similaire entre les constellations et les hypercartes en généralisant un résultat de factorisation de caractère. Avec cette relation, on retrouve un résultat sur le comportement asymptotique des hypercartes dans Chapuy (2009).

Keywords: combinatorial maps, constellations, enumeration, character factorization

1 Introduction

Maps are combinatorial structures describing an embedding of a graph in a surface. They can be encoded as factorizations of identity in the symmetric group. Enormous efforts have been devoted to the enumeration of these combinatorial objects and their variants, see *e.g.* [LZ04] and references therein. In [JV99], the following strikingly simple enumerative relation was established:

$$E_{n,D}^{(g)} = \sum_{i=0}^{g} 4^{g-i} B_{n,D}^{(g-i,2i)} = 4^g B_{n,D}^{(g,0)} + 4^{g-1} B_{n,D}^{(g-1,2)} + \dots .$$

Here, for $D \subseteq \mathbb{N}^+$, we define $B_{n,D}^{(g,k)}$ as the number of rooted bipartite maps with every face degree of the form $2d$ and $d \in D$, whose vertices are colored black and white, of genus g with n edges such that k black vertices are marked. The number $E_{n,D}^{(g)}$ is the counterpart for rooted (non necessarily bipartite) maps with the same restriction on face degrees, without the marking part. In the planar case, we have $E_{n,D}^{(0)} = B_{n,D}^{(0,0)}$, meaning that every planar map of all faces with even degree is always bipartite. This is

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not true for higher genera. Consider a rectangular grid of size $m \times n$ on a torus. It is a map of all faces with even degree, but it is bipartite if and only if m, n are both even, which is not always true.

The special case on $D = \{2\}$ had been proved in [JV90a], and the maps in concern are quadrangulations, which gives this special case the name *quadrangulation relation*. It had been then extended to $D = \{p\}$ in [JV90b]. Despite its nice form, the combinatorial meaning of the quadrangulation relation remains unclear, though some effort is done in [JV99] to explore properties of the possible hinted bijection.

In enumeration of maps, there is a recurrent phenomenon: results on bipartite maps can often be generalized to constellations (see *e.g.* [BMS00, BDFG04, PS02]). In the same spirit, we will generalize the quadrangulation relation to m-constellations and m-hypermaps. As an example, our result in the case $m = 3$ gives rise to the following relation (*c.f.* Corollary 4.3):

$$H^{(g)}_{n,3,D} = \sum_{i=0}^{g} 3^{2g-2i} \sum_{l=0}^{2i} \frac{2^{l+1} - (-1)^{l+1}}{3} C^{(g-i,l,2i-l)}_{n,3,D}.$$

Here, $C^{(g,a,b)}_{n,3,D}$ is the number of rooted 3-constellations with n hyperedges, and hyperface degree restricted by the set D, with a vertices of color 1 and b vertices of color 2 marked. The number $H^{(g)}_{n,3,D}$ is the counterpart for rooted 3-hypermaps without markings. See Section 2.1 for the definitions of these notions. This simple relation suggests a more general bijection for constellations and hypermaps than the one implied by the quadrangulation relation. Finally, we recover a relation between the asymptotic behavior of m-constellations and m-hypermaps in [Cha09], which can be seen as an asymptotic version of our relation.

Given a partition $\mu \vdash n$, we note $m\mu$ the partition obtained by multiplying every part in μ by m. In [JV90a], the quadrangulation relation was obtained using a factorization of irreducible characters of the symmetric group on partitions of the form $[(mk)^n]$ using a notion called m-balanced partition. A generalization to partitions of the form 2λ is stated in [JV99]. In this paper, we will present a generalization of this character factorization to partitions of the form $m\lambda$ (Theorem 3.1). This result can be derived from two different perspectives, algebraic or combinatorial. We then give our generalization of the quadrangulation relation in Corollaries 4.2, 4.3 and 4.4 using our generalized character factorization.

2 Preliminaries

2.1 Constellations and hypermaps

A *map* M is an embedding of a connected graph G, with possibly multi-edges or loops, into a closed, connected and orientable surface S such that all *faces*, *i.e.* components of $S \setminus M$, are topological disks. Maps are defined up to orientation-preserving homeomorphisms. We define the genus g of a map to be that of the surface it is embedded into. We thus have the Euler relation $|V| - |E| + |F| = 2 - 2g$.

We now define two special kinds of maps following [LZ04, Cha09]. An *m-hypermap* is a map with two types of faces, *hyperedges* with degree m and *hyperfaces* with degree divisible by m, such that every edge is located between a hyperedge and a hyperface. Each edge then is naturally oriented with the hyperedge on its right. Conventionally hyperedges are colored black and hyperfaces white. An *m-constellation* is an m-hypermap with additional condition that all vertices are colored with an integer between 1 and m in a fashion that every hyperedge has its vertices colored by $1, 2, \ldots, m$ in clockwise order. A map with faces of even degree can be considered as a 2-hypermap by replacing every edge with a 2-hyperedge, and a bipartite map can be considered as a 2-constellation in the same way. A *rooted* m-hypermap is an m-hypermap with a distinguished edge. Rooted m-constellations are similarly defined, with the convention that the starting vertex of the root in natural orientation has color 1. We consider only rooted m-hypermaps and rooted m-constellations hereinafter.

Figure 1 provides an example of planar 3-hypermap. It can also be considered as a planar 3-constellation. More generally, every planar m-hypermap can have its vertices colored to meet the additional condition to be an m-constellation, that is to say, every planar m-hypermaps can be considered as an m-constellation. However, this is not necessarily true for higher genera, in which m-hypermaps do not necessarily have a coloring that conforms with the additional condition to be an m-constellations.

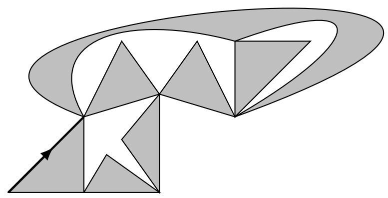

Fig. 1: Example of planar 3-hypermap.

We use $\underline{\mathbf{x}}$ to denote a sequence of variables $x_1, \ldots, x_m$, and $[x_i \leftarrow f(i)]$ to denote the substitution of $\underline{\mathbf{x}}$ by $x_i = f(i)$. We define $H(x, y, z, u)$ to be the ordinary generating series of rooted m-hypermaps, with x marking the number of vertices, y the number of hyperfaces, z the number of hyperedges and u twice the genus. Similarly, we define $C(\underline{\mathbf{x}}, y, z, u)$ to be the ordinary generating series of rooted m-constellations, except that with x_i we mark the number of vertices with color i.

A *k-factorization of identity* (or simply *k-factorization*) in S_n is a family of k permutations $(\sigma_1, \ldots, \sigma_k)$ in S_n such that $\sigma_1 \cdots \sigma_k = id$. Such a factorization is *transitive* if the family acts transitively on $\{1, \ldots, n\}$. There is a 1-to-$(n-1)!m^{n-1}$ correspondence between rooted m-hypermaps with n hyperedges and transitive 3-factorizations in S_{mn} with cycle lengths in σ_1 all divisible by m. Similarly, rooted m-constellations with n hyperedges are in 1-to-$(n-1)!$ correspondence with transitive $(m+1)$-factorizations in S_n (*c.f.* [LZ04]). By noting C_λ the set of permutations with cycle type λ and $l(\pi)$ the number of cycles in a permutation π, we define the following generating series:

$$R_H(x, y, z) = \sum_{n \geq 1} \frac{z^n}{n!} \sum_{\substack{\sigma\tau\pi = id_{mn} \\ \pi \in C_{m\mu}}} x^{l(\sigma)} y^{l(\pi)}, \quad R_C(\underline{\mathbf{x}}, y, z) = \sum_{n \geq 1} \frac{z^n}{n!} \sum_{\sigma_1 \ldots \sigma_m \pi = id_n} y^{l(\pi)} \prod_{i=1}^{m} x_i^{l(\sigma_i)}.$$

By taking the logarithm of the corresponding generating series, we can pass from general k-factorizations to transitive ones. We can now easily verify the following relations concerning generating series H, C of m-hypermaps and m-constellations, and R_H, R_C defined above:

$$H(x, y, z, u) = mu^2 \left(z \frac{\partial}{\partial z} (\log R_H) \right) (xu^{-1}, yu^{-1}, \frac{1}{m} zu^{m-1}), \tag{1}$$

$$C(\underline{\mathbf{x}}, y, z, u) = u^2 \left(z \frac{\partial}{\partial z} (\log R_C) \right) ([x_i \leftarrow x_i u^{-1}], yu^{-1}, zu^{m-1}). \tag{2}$$

In an algebraic point of view, the series R_H and R_C are much easier to manipulate than H and C. To investigate the link between m-hypermaps and m-constellations, we will start by analyzing R_H and R_C using the group algebra of the symmetric group.

2.2 Characters and group algebra of the symmetric group

The group algebra $\mathbb{C}S_n$ of the symmetric group S_n is a complex vector space with a canonical basis indexed by elements of S_n and a multiplication of elements extending distributively the group law of S_n. For θ a partition of n (noted as $\theta \vdash n$), we define K_θ to be the formal sum of elements in S_n with cycle type θ. The elements $(K_\theta)_{\theta \vdash n}$ form a basis of the center of $\mathbb{C}S_n$. According to the classic representation theory (*c.f.* [Ser77]), the center of $\mathbb{C}S_n$ has another basis $(F_\theta)_{\theta \vdash n}$ formed by orthogonal idempotents.

For a partition $\lambda = [1^{m_1} 2^{m_2} \ldots] \vdash n$ in which i appears m_i times, we note $z_\lambda = \prod_{i>0} i^{m_i} m_i!$, and we know that $n! z_\lambda^{-1}$ is the number of permutations of cycle type λ. The change of basis between $(K_\theta)_{\theta \vdash n}$ and $(F_\theta)_{\theta \vdash n}$ is thus given by $F_\lambda = f^\lambda (n!)^{-1} \sum_{\theta \vdash n} \chi_\theta^\lambda K_\theta$ and $K_\lambda = n! z_\lambda^{-1} \sum_{\theta \vdash n} \chi_\lambda^\theta (f^\theta)^{-1} F_\theta$, where χ_θ^λ is the irreducible character indexed by λ evaluated on the conjugacy class of cycle type θ, and f^λ the dimension of the irreducible representation indexed by λ (*c.f.* [Sta99]).

We now consider the coefficient of K_θ in $K_\alpha K_{\beta^{(1)}} \cdots K_{\beta^{(k)}}$ for arbitrary partitions $\theta, \alpha, \beta^{(1)}, \ldots, \beta^{(k)}$ of n. This coefficient, noted as $[K_\theta] K_\alpha K_{\beta^{(1)}} \cdots K_{\beta^{(k)}}$, can be interpreted as the number of factorizations $\pi \tau_1 \cdots \tau_k \sigma = id$ with π and τ_i of cycle type α and $\beta^{(i)}$ respectively, and σ a fixed permutation of cycle type θ. With this interpretation, using the change of basis between $(K_\theta)_{\theta \vdash n}$ and $(F_\theta)_{\theta \vdash n}$ given above and the fact that $(F_\theta)_{\theta \vdash n}$ are orthogonal idempotents, we can rewrite R_H and R_C as following.

$$R_H(x, y, z) = \sum_{n \geq 1} \frac{z^n}{n!} \sum_{\lambda \vdash mn, \mu \vdash n} n! z_\lambda^{-1} z_{m\mu}^{-1} x^{l(\lambda)} y^{l(m\mu)} \sum_{\theta \vdash mn} \frac{1}{f^\theta} \chi_\lambda^\theta \chi_{[m^n]}^\theta \chi_{m\mu}^\theta \tag{3}$$

$$R_C(\underline{\mathbf{x}}, y, z) = \sum_{n \geq 1} \frac{z^n}{n!} \sum_{\lambda^{(1)}, \ldots, \lambda^{(m)}, \mu \vdash n} \left(\prod_{i=1}^{m} x_i^{l(\lambda_{(i)})} \right) y^{l(\mu)} \sum_{\theta \vdash n} (f^\theta)^{(1-m)} z_\mu^{-1} \chi_\mu^\theta \prod_{i=1}^{k} n! z_{\lambda^{(i)}}^{-1} \chi_{\lambda^{(i)}}^\theta \tag{4}$$

To further simplify the expressions above, we define the *rising factorial function* $x^{(n)} = x(x+1) \cdots (x+n-1)$ for $n \in \mathbb{N}$. For a partition θ, we define the polynomial $H_\theta(x)$ as $\prod_{i=1}^{l(\theta)} (x-i+1)^{(\theta_i)}$. Using this notation, we give the following expressions of R_H and R_C.

Proposition 2.1 *We can rewrite R_H and R_C as follows:*

$$R_H(x, y, z) = \sum_{n \geq 1} \frac{z^n}{n!} \sum_{\mu \vdash n} m^{-l(\mu)} z_\mu^{-1} y^{l(\mu)} \sum_{\theta \vdash mn} \chi_{[m^n]}^\theta \chi_{m\mu}^\theta H_\theta(x),$$

$$R_C(\underline{\mathbf{x}}, y, z) = \sum_{n \geq 1} \frac{z^n}{n!} \sum_{\mu \vdash n} y^{l(\mu)} z_\mu^{-1} \sum_{\theta \vdash n} f^\theta \chi_\mu^\theta \left(\prod_{i=1}^{m} H_\theta(x_i) \right).$$

This proposition comes from direct application of the following lemma (Lemma 3.4 in [JV90a]) to (3) and (4). We omit the proof of this lemma here. Readers can refer to the original paper for a proof.

Lemma 2.1 *We have the following equality:*

$$n! \sum_{\alpha \vdash n} z_\alpha^{-1} \chi_\alpha^\theta x^{l(\alpha)} = f^\theta H_\theta(x).$$

We can see that the characters in R_H in Proposition 2.1 are all evaluated at the partition $[m^n]$ and partitions of the form $m\mu$ with $\mu \vdash n$. In [JV90a], $\chi^\theta_{[m^n]}$ is proved to have an expression as a product of smaller characters, which is a crucial step towards the quadrangulation relation. This factorization is also presented in [JK81] (Section 2.7) under the framework of p-core and abacus display of a partition. By extending this approach in the next section to all partitions of the form $m\mu$, we will give a similar relation between m-hypermaps and m-constellations in Section 4.

3 Factorization of characters evaluated at $m\lambda$

In this section we give a result on factorizing $\chi^\theta_{[m\lambda]}$ into smaller characters. Our result can be viewed in both algebraic and combinatorial perspectives.

- **Algebraic approach**

  This approach exploits algebraic relations between symmetric functions and characters χ^θ_λ (*c.f.* [Sta99]). More specifically, to evaluate $\chi^\theta_{[m\lambda]}$, we want to use the Jacobi-Trudi identity to express Schur function s_θ as a determinant D in homogeneous functions, then extract the coefficient of power sum function $p_{m\lambda}$ from D. Due to properties of symmetric functions, many terms in the determinant are irrelevant in extraction of coefficient, and thus we only need to evaluate a simpler determinant D' that has a block structure which can be revealed by the m-decomposition of θ introduced in [JV90a]. Since each block has a similar structure to D itself, we obtain the factorization desired. Though the basic ingredients already exist in [JV90a, JV99], our presentation here stresses more on the block structure of the reduced Jacobi-Trudi determinant, rendering our proof much more conceptual and accessible.

- **Combinatorial approach**

  This approach gives a purely combinatorial interpretation of our factorization of characters using ribbon tableaux and the boson-fermion correspondence. With the Murnaghan-Nakayama rule (*c.f.* [Sta99]), character evaluation can be expressed using enumeration of signed ribbon tableaux. By coding partitions as infinite lattice paths, we define a notion of m-split for partitions θ with an empty m-core, and we establish a bijection between ribbon tableaux of shape θ and of content $m\lambda$ to tuples of smaller ribbon tableaux whose shapes are exactly components in the m-split of θ. With the boson-fermion correspondence and the infinite wedge space (*c.f.* [Oko01]), we also verify that the sign incorporates well in our bijection for the desired factorization. This approach is also related to the notion of p-core and p-quotient (c.f. [JK81]).

These two approaches are essentially two sides of the same coin, since we can show that the notions of m-decomposition and m-split coincide. For consistency of style, we only present the algebraic approach here. For details on the combinatorial approach, readers can refer to a future full version of this paper.

Before proceeding to our algebraic approach, we try to convey our idea about character factorization via an example. Consider the partition $\theta = (6,6,4,4,4,3,3)$. To evaluate $\chi^{\theta}_{3\lambda}$ for arbitrary λ, we want to extract the coefficient of the power sum function $p_{3\lambda}$ in $\det(D)$ for D below provided by the Jacobi-Trudi identity for the Schur function s_θ. Terms in D are all homogeneous symmetric functions $(h_k)_{k\in\mathbb{N}}$.

$$D = \begin{pmatrix} h_6 & h_7 & h_8 & h_9 & h_{10} & h_{11} & h_{12} \\ h_5 & h_6 & h_7 & h_8 & h_9 & h_{10} & h_{11} \\ h_2 & h_3 & h_4 & h_5 & h_6 & h_7 & h_8 \\ h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 \\ h_0 & h_1 & h_2 & h_3 & h_4 & h_5 & h_6 \\ 0 & 0 & h_0 & h_1 & h_2 & h_3 & h_4 \\ 0 & 0 & 0 & h_0 & h_1 & h_2 & h_3 \end{pmatrix}$$

Since the power sum functions $(p_k)_{k>0}$ are algebraically independent, when extracting the coefficient of $p_{3\lambda}$ in D, we only need to consider h_k with k divisible by 3. We can thus ignore a lot of terms in D, leading to evaluating $\det(D_1)$ on a simpler matrix D_1, which can be arranged with permutation of rows and columns to be a block matrix D_2 as below.

$$D_1 = \begin{pmatrix} h_6 & 0 & 0 & h_9 & 0 & 0 & h_{12} \\ 0 & h_6 & 0 & 0 & h_9 & 0 & 0 \\ 0 & h_3 & 0 & 0 & h_6 & 0 & 0 \\ 0 & 0 & h_3 & 0 & 0 & h_6 & 0 \\ h_0 & 0 & 0 & h_3 & 0 & 0 & h_6 \\ 0 & 0 & h_0 & 0 & 0 & h_3 & 0 \\ 0 & 0 & 0 & h_0 & 0 & 0 & h_3 \end{pmatrix}, D_2 = \begin{pmatrix} h_6 & h_9 & h_{12} & 0 & 0 & 0 & 0 \\ h_0 & h_3 & h_6 & 0 & 0 & 0 & 0 \\ 0 & h_0 & h_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & h_6 & h_9 & 0 & 0 \\ 0 & 0 & 0 & h_3 & h_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & h_3 & h_6 \\ 0 & 0 & 0 & 0 & 0 & h_0 & h_3 \end{pmatrix}$$

We can see that each block of D_2 has a similar form to the matrix in Jacobi-Trudi identity, up to some variable substitutions. From the block structure we can clearly observe a factorization of $\chi^{\theta}_{3\lambda}$ in 3 parts.

To achieve a rigorous description of the phenomenon presented in the example above, we start with the following definition of the m-decomposition of a partition and of m-balanced partitions, first introduced in [JV90a]. Though apparently artificial and technical at first sight, the notion of m-balanced partition can arise from the boson-fermion correspondence in a very natural way in our combinatorial perspective, which will be discussed in a future full version of this paper.

Definition 3.1 (m-decomposition of a partition, m-balanced partition) *Given $m, n \in \mathbb{N}$, let α be a partition of mn. For j from 1 to m, we define the following objects:*

- *the set $P_j = \{i \mid \alpha_i - i + j \equiv 0 \mod m\}$,*
- *m_j the cardinality of P_j,*
- *$p_{j,1}, \dots, p_{j,m_j}$ as a list of elements of P_j with increasing order,*
- *$\alpha^{(j)} = (\alpha^{(j)}_1, \dots, \alpha^{(j)}_{m_j})$ for all j, where $\alpha^{(j)}_i = i - 1 + (\alpha_{p_{j,i}} - p_{j,i} + j)/m$ for all i from 1 to m_j.*

*We can see that the sets P_j form a set partition of $\{1, 2, \ldots, l(\alpha)\}$.We note $\underline{\alpha}_m = (\alpha^{(1)}, \ldots, \alpha^{(m)})$ the m-*decomposition *of α, and we define two permutations π_α, π'_α defined explicitly as follows:*

$$\pi_\alpha = \begin{pmatrix} 1 & 2 & \ldots & m_1 & m_1+1 & \ldots & l(\alpha) \\ p_{1,1} & p_{1,2} & \ldots & p_{1,m_1} & p_{2,1} & \ldots & p_{k,m_k} \end{pmatrix},$$

$$\pi'_\alpha = \begin{pmatrix} 1 & 2 & 3 & \ldots & \ldots & \ldots & l(\alpha) \\ 1 & k+1 & 2k+1 & \ldots & 2 & k+2 & \ldots \end{pmatrix}.$$

*If for all j we have $m_j = \lceil (l(\alpha) - j + 1)/m \rceil$, we say that α is m-*balanced*.*

In our running example, we verify that the partition $\theta = (6, 6, 4, 4, 4, 3, 3)$ is 3-balanced, and its 3-decomposition is $((2, 1, 1), (2, 2), (1, 1))$. The permutations π_θ and π'_θ are respectively row and column permutation used to go from matrix D_1 to D_2.

Given a partition $\lambda \vdash n$, we note $|\lambda| = n$ the *weight* of λ. For an m-balanced partition $\alpha \vdash mn$ with $\underline{\alpha}_m = (\alpha^{(1)}, \ldots, \alpha^{(m)})$, we have $\sum_{i=1}^m |\alpha^{(i)}| = n$. Conversely, given a list of partitions $(\alpha^{(1)}, \ldots, \alpha^{(m)})$ with $\sum_{i=1}^m |\alpha^{(i)}| = n$, we can recover an m-balanced partition $\alpha \vdash mn$. There is a bijection between m-balanced partitions of mn and lists of m partitions that sums in total to n (*c.f.* Proposition 4.2, 4.3, 4.5 and Lemma 4.6 in [JV90a]).

The m-decomposition of a partition introduces the following factorization of the polynomial $H_\theta(x)$.

Lemma 3.1 *For $\theta \vdash n$, we have*

$$H_\theta(x) = m^{mn} \prod_{i=1}^{m} \prod_{j=0}^{m-1} H_{\theta^{(i)}} \left(\frac{x - i + j + 1}{m} \right).$$

Proof: We define $H_m^\theta(x) = \prod_{i=1}^{l(\theta)} (x - mi + m)^{(m\theta_i)}$. By direct computation, we easily verify that $H_m^\theta(x) = m^{mn} \prod_{j=0}^{m-1} H_\theta(\frac{x+j}{m})$. We then obtain our result by applying Lemma 4.11 of [JV90a], which states that $H_\theta(x) = \prod_{i=1}^m H_m^{\theta^{(i)}}(x - i + 1)$, to the equality we just verified. □

For an alternative proof that is purely combinatorial, readers can refer to the combinatorial perspective of character factorization in a future full version of this paper.

In [JV90a], the notion of m-decomposition is used to give a factorization of irreducible characters of the symmetric group evaluated at "semiregular" partitions, *i.e.* partitions of the form $[(mk)^n]$, into characters in smaller symmetric groups. We now extend this result to partitions of the form $m\lambda$. The theorem and its proof we are giving below stress more on the block structure of the reduced Jacobi-Trudi determinant. As a result, not only is our result more general, but our proof is also more conceptual.

Theorem 3.1 (Main result on character factorization) *Let m, n be two natural numbers, and $\lambda \vdash n$, $\theta \vdash mn$ be two partitions. We consider partitions as multisets and we denote multiset sum by $\uplus$. If θ is m-balanced, we have*

$$\chi^\theta_{m\lambda} = z_\lambda \operatorname{sgn}(\pi_\theta \pi'_\theta) \sum_{\lambda^{(1)} \uplus \cdots \uplus \lambda^{(m)} = \lambda} \prod_{i=1}^{m} \chi^{\theta^{(i)}}_{\lambda^{(i)}} z^{-1}_{\lambda^{(i)}}.$$

If θ is not m-balanced, $\chi^\theta_{m\lambda} = 0$.

Proof: According to the Jacobi-Trudi identity, we can express χ_λ^θ as a determinant.

$$\begin{aligned}\chi_\lambda^\theta &= z_\lambda [p_\lambda] s_\theta = z_\lambda [p_\lambda] \det(h_{\theta_i - i + j})_{l(\theta)\times l(\theta)} \\ &= z_\lambda \sum_{\sigma \in S_{l(\theta)}} \operatorname{sgn}(\sigma) [p_\lambda] \prod_{i=1}^{l(\theta)} h_{\theta_i - i + \sigma(i)} \\ &= z_\lambda \sum_{\lambda^{(1)} \uplus \cdots \uplus \lambda^{(l(\theta))} = \lambda} \sum_{\sigma \in S_{l(\theta)}} \operatorname{sgn}(\sigma) \prod_{i=1}^{l(\theta)} [p_{\lambda^{(i)}}] h_{\theta_i - i + \sigma(i)}\end{aligned}$$

In the case of $\chi_{m\lambda}^\theta$, we have

$$\chi_{m\lambda}^\theta = z_{m\lambda} \sum_{\lambda^{(1)} \uplus \cdots \uplus \lambda^{(l(\theta))} = \lambda} \sum_{\sigma \in S_{l(\theta)}} \operatorname{sgn}(\sigma) \prod_{i=1}^{l(\theta)} [p_{m\lambda^{(i)}}] h_{\theta_i - i + \sigma(i)}.$$

We notice that $[p_\lambda] h_k \neq 0$ implies $\lambda \vdash k$. Consider the matrix $M = (h_{\theta_i - i + j})_{i,j}$ in the Jacobi-Trudi identity. Since we always take the coefficient $[p_{m\lambda^{(i)}}] h_{\theta_i - i + \sigma(i)}$ when computing the determinant of M, we can ignore entries h_k in M when k is not a multiple of m. As the entry at (i,j) is $h_{\theta_i - i + j}$ in M, we consider only the entries $h_{\theta_i - i + j}$ where $\theta_i - i + j$ is a multiple of m, which is equivalent to $i \in P_j$ in the m-decomposition of θ. If we permute the columns of M with π_θ and the rows of M with π'_θ, we obtain a block diagonal matrix M'. For M' to have full rank, we need θ to be m-balanced. As a result, when θ is not m-balanced, M' does not have full rank, and we have $\chi_{m\lambda}^\theta = 0$ since the determinant is zero.

We now analyze the case where θ is m-balanced. We can see that, in the sum over σ, only those σ with $i \in P_{\sigma(i) \bmod m}$ for every i have a non-zero contribution. In the perspective of the block diagonal matrix M', they are exactly permutations compatible with its blocks. It is now natural to see σ as a list of permutations over each P_i, and to decompose the sum as a product of sums over permutations of each P_i, which is equivalent to evaluating $\det(M')$ as the product of determinants of blocks in M'.

We notice that, for $\lambda = [1^{t_1} 2^{t_2} \ldots] \vdash n$, we have $[p_\lambda] h_n = z_\lambda^{-1}$, and $z_{m\lambda} = \prod_{i \geq 1} (mi)^{t_i} t_i! = m^{l(\lambda)} \prod_{i \geq 1} (i)^{t_i} t_i! = m^{l(\lambda)} z_\lambda$. With the equality above, we now conclude our proof by the promised decomposition.

$$\begin{aligned}\chi_{m\lambda}^\theta &= z_{m\lambda} \operatorname{sgn}(\pi_\theta \pi'_\theta) \sum_{\substack{\lambda^{(1)} \uplus \cdots \uplus \lambda^{(l(\theta))} = \lambda \\ \sigma_1 \in S_{m_1}, \ldots, \sigma_m \in S_{m_m}}} \prod_{j=1}^{m} \left(\operatorname{sgn}(\sigma_j) \prod_{i=1}^{m_j} [p_{m\lambda^{(i)}}] h_{m(\theta_i^{(j)} - i + \sigma_j(i))} \right) \\ &= z_{m\lambda} \operatorname{sgn}(\pi_\theta \pi'_\theta) \sum_{\substack{\lambda^{(1,1)} \uplus \cdots \uplus \lambda^{(m, m_m)} = \lambda \\ \sigma_1 \in S_{m_1}, \ldots, \sigma_m \in S_{m_m}}} \prod_{j=1}^{m} \left(\operatorname{sgn}(\sigma_j) \prod_{i=1}^{m_j} m^{-m_j} [p_{\lambda^{(i,j)}}] h_{\theta_i^{(j)} - i + \sigma_j(i)} \right) \\ &= z_\lambda \operatorname{sgn}(\pi_\theta \pi'_\theta) \sum_{\lambda^{(1)} \uplus \cdots \uplus \lambda^{(m)} = \lambda} \prod_{i=1}^{m} \chi_{\lambda^{(i)}}^{\theta^{(i)}} z_{\lambda^{(i)}}^{-1}\end{aligned}$$

$\square$

4 Generalization of the quadrangulation relation

In this section, using Theorem 3.1, we establish a relation between m-hypermaps and m-constellations in arbitrary genus that generalizes the quadrangulation relation. We then recover a result in [Cha09] on asymptotic behavior of m-hypermaps related to that of m-constellations. We start by showing a link between the series R_H and R_C, using Theorem 3.1.

Proposition 4.1 *The generating series R_H and R_C are related by the following equation.*

$$R_H(x,y,z) = \prod_{j=1}^{m} R_C\left(\left[x_i \leftarrow \frac{x-j+i}{m}\right], \frac{y}{m}, m^m z\right)$$

Proof: We take the expressions of R_H and R_C from Proposition 2.1. We observe that, in the expression of R_H, we only need to consider those θ which are m-balanced. For θ m-balanced, let $(\theta^{(1)}, \ldots, \theta^{(m)})$ be the m-decomposition of θ, and we have the following equality derived from Theorem 3.1.

$$\chi^{\theta}_{[m^n]}\chi^{\theta}_{m\mu} = n! z_\mu \sum_{\mu^{(1)} \uplus \cdots \uplus \mu^{(m)} = \mu} \prod_{i=1}^{m} \frac{f^{\theta^{(i)}} \chi^{\theta^{(i)}}_{\mu^{(i)}}}{(|\theta^{(i)}|)! z_{\mu^{(i)}}}$$

We then substitute the equality above and Lemma 3.1 into the expression of R_H in Corollary 2.1 to factorize R_H into a product of R_C evaluated on different points as follows:

$$\begin{aligned}
&R_H(x,y,z) \\
&= \sum_{n\geq 1} (m^m z)^n \sum_{\mu \vdash n} \left(\frac{y}{m}\right)^{l(\mu)} \sum_{\theta \vdash mn} \sum_{\mu^{(1)} \uplus \cdots \uplus \mu^{(m)} = \mu} \prod_{i=1}^{m} \left(\frac{f^{\theta^{(i)}} \chi^{\theta^{(i)}}_{\mu^{(i)}}}{(|\theta^{(i)}|)! z_{\mu^{(i)}}} \prod_{j=0}^{m-1} H_{\theta^{(i)}}\left(\frac{x-i+j+1}{m}\right)\right) \\
&= \prod_{j=1}^{m} R_C\left(\left[x_i \leftarrow \frac{x-j+i}{m}\right], \frac{y}{m}, m^m z\right).
\end{aligned}$$

□

This link between R_H and R_C can be translated directly into a link between the series $H(x,y,z,u)$ of m-hypermaps and the series $C(\underline{\mathbf{x}}, y, z, u)$ of m-constellations, resulting in our main result as follows.

Theorem 4.1 *The generating series of m-constellations and m-hypermaps are related by the following formula:*

$$H(x,y,z,u) = m \sum_{j=1}^{m} C\left(\left[x_i \leftarrow \frac{x+(i-j)u}{m}\right], \frac{y}{m}, m^{m-1}z, u\right).$$

Proof: This comes directly from a substitution of the equality in Proposition 4.1 into (1) and (2). □

We note $H^{(g)}(x,y,z) = [u^{2g}]H(x,y,z,u)$ and $C^{(g)}(x,y,z) = [u^{2g}]C(x,y,z,u)$ the generating functions of m-hypermaps and m-constellations of genus g respectively. We can now express the following corollary concerning the link between m-hypermaps and m-constellations with respect to the genus.

Corollary 4.1 *We have the following relation between the generating series $H^{(g)}$ and $C^{(g)}$:*

$$H^{(g)}(x,y,z) = \sum_{k=0}^{g} \frac{m^{2g-2k}}{m(2k)!} \left(\sum_{j=1}^{m} \left(\sum_{i=1}^{m} (i-j) \frac{\partial}{\partial x_i} \right)^{2k} C^{(g-k)} \right) ([x_i \leftarrow x], y, z).$$

Proof: We want to compute $H^{(g)}(x,y,z) = [u^{2g}]H(x,y,z,u)$.

$$\begin{aligned}
[u^{2g}]H(x,y,z,u) &= m \sum_{j=1}^{m} \sum_{k=0}^{g} [u^{2k}] C^{(g-k)} \left(\left[x_i \leftarrow \frac{x+(i-j)u}{m} \right], \frac{y}{m}, m^{m-1} z \right) \\
&= m \sum_{j=1}^{m} \sum_{k=0}^{g} \frac{1}{(2k)!} \left(\frac{\partial}{\partial u} \right)^{2k} C^{(g-k)} \left(\left[x_i \leftarrow \frac{x+(i-j)u}{m} \right], \frac{y}{m}, m^{m-1} z \right) \Big|_{u=0} \\
&= m \sum_{j=1}^{m} \sum_{k=0}^{g} \frac{1}{(2k)!} \left(\left(\sum_{i=1}^{m} \frac{i-j}{m} \frac{\partial}{\partial x_i} \right)^{2k} C^{(g-k)} \right) \left(\left[x_i \leftarrow \frac{x}{m} \right], \frac{y}{m}, m^{m-1} z \right)
\end{aligned}$$

To obtain the final result, we then simplify the formula above with the fact that each term in $C^{(g)}$ has the form $x_1^{v_1} \cdots x_m^{v_m} y^{f_1} z^{f_2}$ with $v_1 + \cdots + v_m - m f_2 + f_1 + f_2 = 2 - 2g$, according to the Euler relation. □

We can further generalize these results. Let D be a subset of $\mathbb{N}^*$. We define (m,D)-hypermaps and (m,D)-constellations as m-hypermaps and m-constellations with the restriction that every hyperface has its degree in mD. We note respectively $H_D(x,y,z,u)$ and $C_D(x,y,z,u)$ their generating functions. We have the following corollary. We omit its proof here, but we can see that the whole proof mechanism for Theorem 4.1 transfers directly onto $H_D(x,y,z,u)$ and $C_D(x,y,z,u)$ with the observation that, in the proof of Theorem 3.1, if all parts of λ are in mD, then all parts of every $\lambda^{(i)}$ are in D.

Corollary 4.2 (Main result in the form of series) *We have the following equations:*

$$H_D(x,y,z,u) = m \sum_{j=1}^{m} C_D \left(\left[x_i \leftarrow \frac{x+(i-j)u}{m} \right], \frac{y}{m}, m^{m-1} z, u \right)$$

$$H_D^{(g)}(x,y,z) = \sum_{k=0}^{g} \frac{m^{2g-2k}}{m(2k)!} \left(\sum_{j=1}^{m} \left(\sum_{i=1}^{m} (i-j) \frac{\partial}{\partial x_i} \right)^{2k} C_D^{(g-k)} \right) ([x_i \leftarrow x], y, z).$$

By taking $m = 2$ and $D = \{2\}, \{p\}$ or D arbitrary, we recover the quadrangulation relation and its extensions in [JV90b] and [JV99] respectively. Though the relation seems to be a bit monstrous at first glance, it actually has an elegant combinatorial interpretation. We define $C_{n,m,D}^{(g,a_1,\ldots,a_{m-1})}$ to be the number of rooted m-constellations with n hyperedges, and hyperface degree restricted by the set D, with a_i marked vertices of color i for i from 1 to $m-1$. The number $H_{n,m,D}^{(g)}$ is the counterpart for rooted m-hypermaps without the marking part. For $m = 3$ and $m = 4$, we have the following elegant relations.

Corollary 4.3 (Generalization of the quadrangulation relation, special case $m = 3, 4$) *For $m = 3, 4$, we have*

$$H_{n,3,D}^{(g)} = \sum_{i=0}^{g} 3^{2g-2i} \sum_{l=0}^{2i} \frac{2 \cdot 2^l + (-1)^l}{3} C_{n,3,D}^{(g-i,l,2i-l)},$$

$$H_{n,4,D}^{(g)} = \sum_{i=0}^{g} 4^{2g-2i} \sum_{l_1,l_2 \geq 0, l_1+l_2 \leq 2i} \frac{2(3^{l_1} 2^{l_2} + 2^{l_2}(-1)^{l_1})}{4} C_{n,4,D}^{(g-i,l_1,l_2,2i-l_1-l_2)}.$$

We notice that the coefficients are always positive integers. This is not a coincidence. In fact, by carefully rearranging terms, we can obtain the following more general relation.

Corollary 4.4 (Generalization of the quadrangulation relation, for arbitrary m) *With certain coefficients $c_{k_1,\dots,k_{m-1}}^{(m)}$ all integral and positive, we have*

$$H_{n,m,D}^{(g)} = \sum_{i=0}^{g} m^{2g-2i} \sum_{\substack{k_1,\dots,k_{m-1} \geq 0 \\ k_1+\cdots+k_{m-1}=2i}} c_{k_1,\dots,k_{m-1}}^{(m)} C_{n,m,D}^{(g-i,k_1,\dots,k_{m-1})}.$$

A detailed proof can be found in a future full version of this paper. This relation might hint an unknown combinatorial bijection between m-hypermaps with given genus and some families of decorated m-constellations with lower genus. It is thus interesting to try to understand the combinatorial meaning of these coefficients.

According to Theorem 3.1 in [Cha09], the number $C_{n,m,D}^{(g)} = C_{n,m,D}^{(g,0,\dots,0)}$ of (m, D)-constellations with n hyperedges without marking grows asymptotically in $\Theta(n^{\frac{5}{2}(g-1)} \rho_{m,D}^n)$ when n tends to infinity in multiples of $\gcd(D)$. Using Corollary 4.2, we now give a new proof of Theorem 3.2 of [Cha09] about the asymptotic behavior of the number of (m, D)-hypermaps.

Corollary 4.5 (Asymptotic behavior of (m, D)-hypermaps) *We have the following asymptotic behavior of (m, D)-hypermaps when n tends to infinity in multiples of $\gcd(D)$:*

$$H_{n,m,D}^{(g)} \sim m^{2g} C_{n,m,D}^{(g)}.$$

Proof: We observe that, in the second part of Corollary 4.2, for a fixed k, the number of differential operators applied to $C_D^{(g-k)}$ does not depend on n, and they are all of order $2k$. Since in an m-hypermap, the number of vertices with a fixed color i is bounded by the number of hyperedges n, the contribution of the term with $k = t$ is $O(n^{\frac{5}{2}(g-t-1)+2t} \rho_{m,D}^n) = O(n^{\frac{5}{2}(g-1)-\frac{1}{2}t} \rho_{m,D}^n)$. The dominant term is therefore given by the case $k = 0$, with $C_{n,m,D}^{(g)} = \Theta(n^{\frac{5}{2}(g-1)} \rho_{m,D}^n)$, and we can easily verify the multiplicative constant. □

This corollary, alongside with its proof, is a refinement of the asymptotic enumerative results established in [Cha09] on the link between m-hypermaps and m-constellations.

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A uniform model for Kirillov–Reshetikhin crystals

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Abstract. We present a uniform construction of tensor products of one-column Kirillov–Reshetikhin (KR) crystals in all untwisted affine types, which uses a generalization of the Lakshmibai–Seshadri paths (in the theory of the Littelmann path model). This generalization is based on the graph on parabolic cosets of a Weyl group known as the parabolic quantum Bruhat graph. A related model is the so-called quantum alcove model. The proof is based on two lifts of the parabolic quantum Bruhat graph: to the Bruhat order on the affine Weyl group and to Littelmann's poset on level-zero weights. Our construction leads to a simple calculation of the energy function. It also implies the equality between a Macdonald polynomial specialized at $t = 0$ and the graded character of a tensor product of KR modules.

Résumé. Nous présentons une construction uniforme pour les produits tensoriels des cristaux de Kirillov–Reshetikhin de type colonne, pour tous les types affines symétriques, qui utilise une généralisation des chemins de Lakshmibai–Seshadri (dans la théorie des chemins de Littelmann). Cette généralisation est basée sur un graphe sur les classes parabolique d'un groupe de Weyl appelé le graphe de Bruhat parabolique quantique. Un modèle lié est le modèle des alcôves quantique. La preuve est basée sur deux relèvements du graphe de Bruhat parabolique quantique: dans l'ordre de Bruhat affine et dans un ensemble ordonné des poids de niveau zero. Notre construction donne une formule simple pour la fonction d'énergie. Elle donne aussi l'égalité d'un polynôme de Macdonald spécialisé à $t = 0$ avec le caractère gradué d'un produit tensoriel des modules de Kirillov–Reshetikhin.

Keywords: Parabolic quantum Bruhat graph, Kirillov–Reshetikhin crystals, energy function, Lakshmibai–Seshadri paths, Macdonald polynomials.

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1 Introduction

Our goal in this series of papers (see [LNSSS1, LNSSS2]) is to obtain a uniform construction of tensor products of one-column Kirillov–Reshetikhin (KR) crystals. As a consequence we shall prove the equality $P_\lambda(q) = X_\lambda(q)$, where $P_\lambda(q)$ is the Macdonald polynomial $P_\lambda(q,t)$ specialized at $t = 0$ and $X_\lambda(q)$ is the graded character of a simple Lie algebra coming from tensor products of KR modules. Both the Macdonald polynomials and KR modules are of arbitrary untwisted affine type. The index λ is a dominant weight for the simple Lie subalgebra obtained by removing the affine node. Macdonald polynomials and characters of KR modules have been studied extensively in connection with various fields such as statistical mechanics and integrable systems, representation theory of Coxeter groups and Lie algebras (and their quantized analogues given by Hecke algebras and quantized universal enveloping algebras), geometry of singularities of Schubert varieties, and combinatorics.

Our point of departure is a theorem of Ion [Ion], which asserts that the nonsymmetric Macdonald polynomials at $t = 0$ are characters of Demazure submodules of highest weight modules over affine algebras. This holds for the Langlands duals of untwisted affine root systems (and type $A_{2n}^{(2)}$ in the case of nonsymmetric Koornwinder polynomials). Our results apply to the untwisted affine root systems. The overlapping cases are the simply-laced affine root systems $A_n^{(1)}$, $D_n^{(1)}$ and $E_{6,7,8}^{(1)}$.

It is known [FL, FSS, KMOU, KMOTU, ST, Na] that certain affine Demazure characters (including those for the simply-laced affine root systems) can be expressed in terms of KR crystals, which motivates the relation between P and X. For types $A_n^{(1)}$ and $C_n^{(1)}$, the equality $P = X$ was achieved in [Le2, LeS] by establishing a combinatorial formula for the Macdonald polynomials at $t = 0$ from the Ram–Yip formula [RY], and by using explicit models for the one-column KR crystals [FOS]. It should be noted that, in types $A_n^{(1)}$ and $C_n^{(1)}$, the one-column KR modules are irreducible when restricted to the canonical simple Lie subalgebra, while in general this is not the case. For the cases considered by Ion [Ion], the corresponding KR crystals are perfect. This is not necessarily true for the untwisted affine root systems considered in this work, especially for the untwisted non-simply-laced affine root systems.

In this work we provide a type-free approach to the equality $P = X$ for untwisted affine root systems. Lenart's specialization [Le2] of the Ram–Yip formula for Macdonald polynomials uses the quantum alcove model [LeL1], whose objects are paths in the quantum Bruhat graph (QBG), which was defined and studied in [BFP] in relation to the quantum cohomology of the flag variety. On the other hand, Naito and Sagaki [NS1, NS2, NS4, NS5] gave models for tensor products of KR crystals of one-column type in terms of projections of level-zero Lakshmibai–Seshadri (LS) paths to the classical weight lattice. Hence we need to establish a bijection between the quantum alcove model and projected level-zero LS paths.

In analogy with [BFP] and inspired by the quantum Schubert calculus of homogeneous spaces [Mi, P] we define the parabolic quantum Bruhat graph (PQBG), which is a directed graph structure on parabolic quotients of the Weyl group with respect to a parabolic subgroup. We construct two lifts of the PQBG. The *first lift* is from the PQBG to the Bruhat order of the affine Weyl group. This is a parabolic analogue of the lift of the QBG to the affine Bruhat order [LS], which is the combinatorial structure underlying Peterson's theorem [P]; the latter equates the Gromov-Witten invariants of finite-dimensional homogeneous spaces with the Pontryagin homology structure constants of Schubert varieties in the affine Grassmannian. We obtain Diamond Lemmas for the PQBG via projection of the standard Diamond Lemmas for the affine

Weyl group. We find a *second lift* of the PQBG into a poset of Littelmann [Li] for level-zero weights and characterize its local structure (such as cover relations) in terms of the PQBG. Littelmann's poset was defined in connection with LS paths for arbitrary (not necessarily dominant) weights, but the local structure was not previously known. The weight poset precisely controls the combinatorics of the level-zero LS paths and therefore their classical projections, which we formulate directly as quantum LS paths. Finally, we describe a bijection between the quantum alcove model and the quantum LS paths.

The paper is organized as follows. In Section 2 we prepare the background and define the PQBG. Section 3 is reserved for the two lifts of the PQBG and the statement of the Diamond Lemmas. In Section 4 we describe KR crystals in terms of the quantum LS paths and the quantum alcove model in [LeL1]; we also give simple combinatorial formulas for the energy function. Finally, we conclude in Section 5 with the results on (nonsymmetric) Macdonald polynomials at $t = 0$.

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2 Background

2.1 Untwisted affine root datum

Let $I_{\mathrm{af}} = I \sqcup \{0\}$ (resp. I) be the Dynkin node set of an untwisted affine algebra $\mathfrak{g}_{\mathrm{af}}$ (resp. its canonical subalgebra $\mathfrak{g}$), W_{af} (resp. W) the affine (resp. finite) Weyl group with simple reflections r_i for $i \in I_{\mathrm{af}}$ (resp. $i \in I$), and $X_{\mathrm{af}} = \mathbb{Z}\delta \oplus \bigoplus_{i \in I_{\mathrm{af}}} \mathbb{Z}\Lambda_i$ (resp. $X = \bigoplus_{i\in I} \mathbb{Z}\omega_i$) the affine (resp. finite) weight lattice. Let $\{\alpha_i \mid i \in I_{\mathrm{af}}\}$ be the simple roots, $\Phi^{\mathrm{af}} = W_{\mathrm{af}}\{\alpha_i \mid i \in I_{\mathrm{af}}\}$ (resp. $\Phi = W\{\alpha_i \mid i \in I\}$) the set of affine real roots (resp. roots), and $\Phi^{\mathrm{af}+} = \Phi^{\mathrm{af}} \cap \bigoplus_{i\in I_{\mathrm{af}}} \mathbb{Z}_{\geq 0}\alpha_i$ (resp. $\Phi^+ = \Phi \cap \bigoplus_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$) the set of positive affine real (resp. positive) roots. Furthermore, $\Phi^{\mathrm{af}-} = -\Phi^{\mathrm{af}+}$ (resp. $\Phi^- = -\Phi^+$) are the negative affine real (resp. negative) roots. Let $X_{\mathrm{af}}^\vee = \mathrm{Hom}_{\mathbb{Z}}(X_{\mathrm{af}}, \mathbb{Z})$ be the dual lattice, $\langle \cdot\,, \cdot\rangle : X_{\mathrm{af}}^\vee \times X_{\mathrm{af}} \to \mathbb{Z}$ the evaluation pairing, and $\{d\} \cup \{\alpha_i^\vee \mid i \in I_{\mathrm{af}}\}$ the dual basis of $X_{\mathrm{af}}^\vee$. The natural projection $\mathrm{cl} : X_{\mathrm{af}} \to X$ has kernel $\mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\delta$ and sends $\Lambda_i \mapsto \omega_i$ for $i \in I$.

The affine Weyl group W_{af} acts on X_{af} and $X_{\mathrm{af}}^\vee$ by

$$r_i\lambda = \lambda - \langle \alpha_i^\vee\,, \lambda\rangle\alpha_i \qquad \text{and} \qquad r_i\mu = \mu - \langle \mu\,, \alpha_i\rangle\alpha_i^\vee\,,$$

for $i \in I_{\mathrm{af}}$, $\lambda \in X_{\mathrm{af}}$, and $\mu \in X_{\mathrm{af}}^\vee$. For $\beta \in \Phi^{\mathrm{af}}$, let $w \in W_{\mathrm{af}}$ and $i \in I_{\mathrm{af}}$ be such that $\beta = w\alpha_i$. Define the associated reflection $r_\beta \in W_{\mathrm{af}}$ and associated coroot $\beta^\vee \in X_{\mathrm{af}}^\vee$ by $r_\beta = wr_iw^{-1}$ and $\beta^\vee = w\alpha_i^\vee$.

The null root is the unique element $\delta \in \bigoplus_{i \in I_{\mathrm{af}}} \mathbb{Z}_{>0}\alpha_i$ which generates the rank 1 sublattice $\{\lambda \in X_{\mathrm{af}} \mid \langle \alpha_i^\vee\,, \lambda\rangle = 0 \text{ for all } i \in I_{\mathrm{af}}\}$. We have $\delta = \alpha_0 + \theta$, where θ is the highest root for $\mathfrak{g}$. The canonical central element is the unique element $c \in \bigoplus_{i\in I_{\mathrm{af}}} \mathbb{Z}_{>0}\alpha_i^\vee$ which generates the rank 1 sublattice $\{\mu \in X_{\mathrm{af}}^\vee \mid \langle \mu\,, \alpha_i\rangle = 0 \text{ for all } i \in I_{\mathrm{af}}\}$. The level of a weight $\lambda \in X_{\mathrm{af}}$ is defined by $\mathrm{level}(\lambda) = \langle c\,, \lambda\rangle$. Let $X_{\mathrm{af}}^0 \subset X_{\mathrm{af}}$ be the sublattice of level-zero elements.

We denote by $\ell(w)$ for $w \in W_{\mathrm{af}}$ (resp. W) the length of w and by $\lessdot$ the Bruhat cover. The element $t_\mu \in W_{\mathrm{af}}$ is the translation by the element μ in the coroot lattice $Q^\vee = \bigoplus_{i\in I} \mathbb{Z}\alpha_i^\vee$.

2.2 Affinization of a weight stabilizer

Let $\lambda \in X$ be a dominant weight, which is fixed throughout the remainder of Sections 2 and 3. Let W_J be the stabilizer of λ in W. It is a parabolic subgroup, being generated by r_i for $i \in J$, where $J = \{i \in I \mid \langle \alpha_i^\vee , \lambda \rangle = 0\}$. Let $Q_J^\vee = \bigoplus_{i \in J} \mathbb{Z}\alpha_i^\vee$ be the associated coroot lattice, W^J the set of minimum-length coset representatives in W/W_J, $\Phi_J \supset \Phi_J^+$ the set of roots and positive roots respectively, and $\rho_J = (1/2)\sum_{\alpha \in \Phi_J^+} \alpha$ (if $J = \emptyset$, then ρ_J is denoted by ρ). Define

$$(W_J)_{\mathrm{af}} = W_J \ltimes Q_J^\vee = \{wt_\mu \in W_{\mathrm{af}} \mid w \in W_J , \mu \in Q_J^\vee\}, \tag{1}$$

$$\Phi_J^{\mathrm{af}+} = \{\beta \in \Phi^{\mathrm{af}+} \mid \mathrm{cl}(\beta) \in \Phi_J\} = \Phi_J^+ \cup (\mathbb{Z}_{>0}\delta + \Phi_J), \tag{2}$$

$$(W^J)_{\mathrm{af}} = \{x \in W_{\mathrm{af}} \mid x \cdot \beta > 0 \text{ for all } \beta \in \Phi_J^{\mathrm{af}+}\}. \tag{3}$$

By [LS, Lemma 10.5] [P], any $w \in W_{\mathrm{af}}$ factors uniquely as $w = w_1 w_2$, where $w_1 \in (W^J)_{\mathrm{af}}$ and $w_2 \in (W_J)_{\mathrm{af}}$. Therefore, we can define $\pi_J : W_{\mathrm{af}} \to (W^J)_{\mathrm{af}}$ by $w \mapsto w_1$. We say that $\mu \in Q^\vee$ is *J-adjusted* if $\pi_J(t_\mu) = z_\mu t_\mu$ with $z_\mu \in W$. Say that $\mu \in Q^\vee$ is *J-superantidominant* if μ is antidominant (i.e., $\langle \mu , \alpha \rangle \le 0$ for all $\alpha \in \Phi^+$), and $\langle \mu , \alpha \rangle \ll 0$ for $\alpha \in \Phi^+ \setminus \Phi_J^+$.

2.3 The parabolic quantum Bruhat graph

The *parabolic quantum Bruhat graph* $\mathrm{QB}(W^J)$ is a directed graph with vertex set W^J, whose directed edges have the form $w \xrightarrow{\alpha} \lfloor wr_\alpha \rfloor$ for $w \in W^J$ and $\alpha \in \Phi^+ \setminus \Phi_J^+$. Here we denote by $\lfloor v \rfloor$ the minimum-length representative in the coset vW_J for $v \in W$. There are two kinds of edges:

1. (Bruhat edge) $w \lessdot wr_\alpha$. (One may deduce that $wr_\alpha \in W^J$.)
2. (Quantum edge) $\ell(\lfloor wr_\alpha \rfloor) = \ell(w) + 1 - \langle \alpha^\vee , 2\rho - 2\rho_J \rangle$.

If $J = \emptyset$, then we recover the quantum Bruhat graph $\mathrm{QB}(W)$ defined in [BFP].

3 Two lifts of the parabolic quantum Bruhat graph

3.1 Lifting $\mathrm{QB}(W^J)$ to W_{af}

We construct a parabolic analogue of the lift of $\mathrm{QB}(W)$ to W_{af} given in [LS].

Let $\Omega_J^\infty \subset W_{\mathrm{af}}$ be the subset of elements of the form $w\pi_J(t_\mu)$ with $w \in W^J$ and $\mu \in Q^\vee$ J-superantidominant and J-adjusted. We have $\Omega_J^\infty \subset (W^J)_{\mathrm{af}} \cap W_{\mathrm{af}}^-$, where W_{af}^- is the set of minimum-length coset representatives in W_{af}/W. Impose the Bruhat covers in Ω_J^∞ whenever the connecting root has classical part in $\Phi \setminus \Phi_J$. Then Ω_J^∞ is a subposet of the Bruhat poset W_{af}.

Proposition 3.1 *Every edge in $\mathrm{QB}(W^J)$ lifts to a downward Bruhat cover in Ω_J^∞, and every cover in Ω_J^∞ projects to an edge in $\mathrm{QB}(W^J)$. More precisely:*

1. *For any edge $\lfloor wr_\alpha \rfloor \xleftarrow{\alpha} w$ in $\mathrm{QB}(W^J)$, and $\mu \in Q^\vee$ that is J-superantidominant and J-adjusted with $\pi_J(t_\mu) = zt_\mu$, there is a covering relation $y \lessdot x$ in Ω_J^∞ where*

   $$x = wzt_\mu, \quad y = xr_{\widetilde{\alpha}} = wr_\alpha t_{\chi\alpha^\vee} zt_\mu, \quad \widetilde{\alpha} = z^{-1}\alpha + (\chi + \langle \mu , z^{-1}\alpha \rangle)\delta \in \Phi^{\mathrm{af}-},$$

   and χ is 0 or 1 according as the arrow in $\mathrm{QB}(W^J)$ is of Bruhat or quantum type respectively.

2. *Suppose $y \lessdot x$ is an arbitrary covering relation in Ω_J^∞. Then we can write $x = wzt_\mu$ with $w \in W^J$, $z = z_\mu \in W_J$, and $\mu \in Q^\vee$ J-superantidominant and J-adjusted, as well as $y = xr_\gamma$ with $\gamma = z^{-1}\alpha + n\delta \in \Phi^{\mathrm{af}}$, $\alpha \in \Phi^+ \setminus \Phi_J^+$, and $n \in \mathbb{Z}$. With the notation $\chi := n - \langle \mu\,,\, z^{-1}\alpha\rangle$, we have*

$$\chi \in \{0,1\}\,, \quad \gamma = z^{-1}\alpha + (\chi + \langle \mu\,,\, z^{-1}\alpha\rangle)\delta \in \Phi^{\mathrm{af}-}\,;$$

furthermore, there is an edge $wr_\alpha z \xleftarrow{z^{-1}\alpha} wz$ in $\mathrm{QB}(W)$ and an edge $\lfloor wr_\alpha \rfloor \xleftarrow{\alpha} w$ in $\mathrm{QB}(W^J)$, where both edges are of Bruhat type if $\chi = 0$ and of quantum type if $\chi = 1$.

3.2 The Diamond Lemmas

In the following, a dotted (resp. plain) edge represents a quantum (resp. Bruhat) edge in $\mathrm{QB}(W^J)$, whereas a dashed edge can be of both types. Given $w \in W^J$ and $\gamma \in \Phi^+$, define $z \in W_J$ by $r_\theta w = \lfloor r_\theta w \rfloor z$. We now state the Diamond Lemmas for $\mathrm{QB}(W^J)$. They are proved based on the lift of $\mathrm{QB}(W^J)$ to W_{af} in Proposition 3.1 and the fact that such a lemma holds for any Coxeter group [BB].

Lemma 3.2 *Let $\alpha \in \Phi$ be a simple root, $\gamma \in \Phi^+ \setminus \Phi_J^+$, and $w \in W^J$. Then we have the following cases, in each of which the bottom two edges imply the top two edges, and vice versa.*

1. *In the left diagram we assume $\gamma \neq w^{-1}(\alpha)$. In both cases we have $r_\alpha\lfloor wr_\gamma \rfloor = \lfloor r_\alpha wr_\gamma \rfloor$.*

$$\begin{tikzcd}
 & r_\alpha\lfloor wr_\gamma\rfloor & & & & r_\alpha\lfloor wr_\gamma\rfloor & \\
r_\alpha w \arrow[ur, "\gamma"] & & \lfloor wr_\gamma\rfloor \arrow[ul, "\lfloor wr_\gamma\rfloor^{-1}(\alpha)"'] & & r_\alpha w \arrow[ur, dotted, "\gamma"] & & \lfloor wr_\gamma\rfloor \arrow[ul, "\lfloor wr_\gamma\rfloor^{-1}(\alpha)"'] \\
 & w \arrow[ul, "w^{-1}(\alpha)"] \arrow[ur, "\gamma"'] & & & & w \arrow[ul, "w^{-1}(\alpha)"] \arrow[ur, dotted, "\gamma"'] &
\end{tikzcd} \tag{4}$$

2. *Here z is defined as above. We assume $\gamma \neq -w^{-1}(\theta)$ whenever both of the hypothesized edges are quantum ones. In the left diagram, the dashed edge is a quantum (resp. a Bruhat) edge depending on $\langle \gamma^\vee, w^{-1}(\theta)\rangle$ being nonzero (resp. zero). In the right diagram, the dashed edge is a Bruhat (resp. a quantum) edge depending on $\langle \gamma^\vee, w^{-1}(\theta)\rangle$ being nonzero (resp. zero).*

$$\begin{tikzcd}
 & \lfloor r_\theta wr_\gamma\rfloor & & & & \lfloor r_\theta wr_\gamma\rfloor & \\
\lfloor r_\theta w\rfloor \arrow[ur, dashed, "z(\gamma)"] & & \lfloor wr_\gamma\rfloor \arrow[ul, dotted, "-\lfloor wr_\gamma\rfloor^{-1}(\theta)"'] & & \lfloor r_\theta w\rfloor \arrow[ur, dashed, "z(\gamma)"] & & \lfloor wr_\gamma\rfloor \arrow[ul, dotted, "-\lfloor wr_\gamma\rfloor^{-1}(\theta)"'] \\
 & w \arrow[ul, dotted, "-w^{-1}(\theta)"] \arrow[ur, "\gamma"'] & & & & w \arrow[ul, dotted, "-w^{-1}(\theta)"] \arrow[ur, dotted, "\gamma"'] &
\end{tikzcd} \tag{5}$$

3.3 Lifting $\mathrm{QB}(W^J)$ to the level-zero weight poset

In [Li], Littelmann introduced a poset related to LS paths for arbitrary (not necessarily dominant) integral weights. Littelmann did not give a precise local description of it. We consider this poset for level-zero weights and characterize its cover relations in terms of the PQBG.

Let $\lambda \in X$ be a fixed dominant weight (cf. Section 2.2 and the notation thereof, e.g., W_J is the stabilizer of λ). We view X as a sublattice of X^0_{af}. Let $X^0_{\mathrm{af}}(\lambda)$ be the orbit $W_{\mathrm{af}}\lambda$.

Definition 3.3 (Level-zero weight poset [Li]) *A poset structure is defined on $X^0_{\mathrm{af}}(\lambda)$ as the transitive closure of the relation*

$$\mu < r_\beta(\mu) \quad \Leftrightarrow \quad \langle \beta^\vee, \mu\rangle > 0\,,$$

where $\beta \in \Phi^{\mathrm{af}+}$. This poset is called the level-zero weight poset for λ.

The cover $\mu \lessdot \nu = r_\beta(\mu)$ of $X^0_{\mathrm{af}}(\lambda)$ is labeled by the root $\beta \in \Phi^{\mathrm{af}+}$. The projection map cl restricts to the map $\mathrm{cl} : X^0_{\mathrm{af}}(\lambda) \to W\lambda$. We identify $W\lambda \simeq W/W_J \simeq W^J$, and consider $\mathrm{QB}(W^J)$. Our main result is the construction of a lift of $\mathrm{QB}(W^J)$ to $X^0_{\mathrm{af}}(\lambda)$. The proof is based on Lemma 3.2.

Theorem 3.4 *Let $\mu \in X^0_{\mathrm{af}}(\lambda)$ and $w := \mathrm{cl}(\mu) \in W^J$. If $\mu \lessdot \nu$ is a cover in $X^0_{\mathrm{af}}(\lambda)$, then its label β is in Φ^+ or $\delta - \Phi^+$. Moreover, $w \to \mathrm{cl}(\nu)$ is an up (respectively down) edge in $\mathrm{QB}(W^J)$ labeled by $w^{-1}(\beta) \in \Phi^+ \setminus \Phi^+_J$ (respectively $w^{-1}(\beta - \delta)$), depending on $\beta \in \Phi^+$ (respectively $\beta \in \delta - \Phi^+$). Conversely, if $w \xrightarrow{\gamma} wr_\gamma = w'$ (respectively $w \overset{\gamma}{\dashrightarrow} \lfloor wr_\gamma \rfloor = w'$) in $\mathrm{QB}(W^J)$ for $\gamma \in \Phi^+ \setminus \Phi^+_J$, then there exists a cover $\mu \lessdot \nu$ in $X^0_{\mathrm{af}}(\lambda)$ labeled by $w(\gamma)$ (respectively $\delta + w(\gamma)$) with $\mathrm{cl}(\nu) = w'$.*

4 Models for KR crystals and the energy function

4.1 Quantum LS paths

Throughout this section, we fix a dominant integral weight $\lambda \in X$, and set $J := \{i \in I \mid \langle \alpha_i^\vee\,,\, \lambda\rangle = 0\}$.

Definition 4.1 *Let $x,\, y \in W^J$, and let $\sigma \in \mathbb{Q}$ be such that $0 < \sigma < 1$. A directed σ-path from y to x is, by definition, a directed path*

$$x = w_0 \xleftarrow{\gamma_1} w_1 \xleftarrow{\gamma_2} w_2 \xleftarrow{\gamma_3} \cdots \xleftarrow{\gamma_n} w_n = y$$

from y to x in $\mathrm{QB}(W^J)$ such that $\sigma\langle \gamma_k^\vee\,,\, \lambda\rangle \in \mathbb{Z}$ for all $1 \le k \le n$.

A quantum LS path of shape λ is a pair $\eta = (\underline{x}\,;\, \underline{\sigma})$ of a sequence $\underline{x}\ :\ x_1,\, x_2,\, \ldots,\, x_s$ of elements in W^J with $x_u \ne x_{u+1}$ for $1 \le u \le s-1$ and a sequence $\underline{\sigma}\ :\ 0 = \sigma_0 < \sigma_1 < \cdots < \sigma_s = 1$ of rational numbers such that there exists a directed σ_u-path from x_{u+1} to x_u for each $1 \le u \le s-1$. Denote by $\mathrm{QLS}(\lambda)$ the set of quantum LS paths of shape λ. We identify an element $\eta = (x_1,\, x_2,\, \ldots,\, x_s\,;\, \sigma_0,\, \sigma_1,\, \ldots,\, \sigma_s) \in \mathrm{QLS}(\lambda)$ with the following piecewise linear, continuous map $\eta : [0,1] \to \mathbb{R} \otimes_{\mathbb{Z}} X$:

$$\eta(t) = \sum_{u'=1}^{u-1} (\sigma_{u'} - \sigma_{u'-1}) x_{u'} \cdot \lambda + (t - \sigma_{u-1}) x_u \cdot \lambda \quad \text{for } \ \sigma_{u-1} \le t \le \sigma_u,\ 1 \le u \le s,$$

and set $\mathrm{wt}(\eta) =: \eta(1)$. Following [Li], we define the root operators e_i and f_i for $i \in I_{\mathrm{af}} = I \sqcup \{0\}$ as follows. For $\eta \in \mathrm{QLS}(\lambda)$ and $i \in I_{\mathrm{af}}$, we set

$$H(t) = H_i^\eta(t) := \langle \alpha_i^\vee\,,\, \eta(t)\rangle \quad \text{for } \ t \in [0,1],$$

$$m = m_i^\eta := \min\{H_i^\eta(t) \mid t \in [0,1]\};$$

in fact, $m \in \mathbb{Z}_{\le 0}$. If $m = 0$, then $e_i\eta := \mathbf{0}$. If $m \le -1$, then we define $e_i\eta$ by:

$$(e_i\eta)(t) = \begin{cases} \eta(t) & \text{if } 0 \le t \le t_0, \\ r_{i,\,m+1}(\eta(t)) = \eta(t_0) + s_i(\eta(t) - \eta(t_0)) & \text{if } t_0 \le t \le t_1, \\ r_{i,\,m+1}r_{i,\,m}(\eta(t)) = \eta(t) + \widetilde{\alpha}_i & \text{if } t_1 \le t \le 1, \end{cases}$$

where

$$t_1 := \min\bigl\{t \in [0,1] \mid H(t) = m\bigr\},$$
$$t_0 := \max\bigl\{t \in [0,t_1] \mid H(t) = m+1\bigr\},$$

$r_{i,\,n}$ is the reflection with respect to the hyperplane $H_{i,\,n} := \bigl\{\mu \in \mathbb{R} \otimes_{\mathbb{Z}} X \mid \langle \alpha_i^\vee\,,\, \mu\rangle = n\bigr\}$ for each $n \in \mathbb{Z}$, and

$$\widetilde{\alpha}_i := \begin{cases} \alpha_i & \text{if } i \ne 0, \\ -\theta & \text{if } i = 0, \end{cases} \qquad s_i := \begin{cases} r_i & \text{if } i \ne 0, \\ r_\theta & \text{if } i = 0. \end{cases}$$

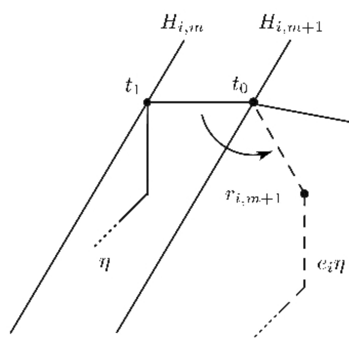

Fig. 1: Root operator e_i.

The definition of $f_i\eta$ is similar. The following theorem is one of our main results.

Theorem 4.2 *(1) The set* QLS(λ) *together with crystal operators e_i, f_i for $i \in I_{\mathrm{af}}$ and weight function* wt*, becomes a regular crystal with weight lattice X.*

(2) For each $i \in I$, the crystal QLS(ω_i) *is isomorphic to the crystal basis of $W(\omega_i)$, the fundamental representation of level zero, introduced by Kashiwara* [Ka].

(3) Let $\mathbf{i} = (i_1, i_2, \dots, i_p)$ be an arbitrary sequence of elements of I, and set $\lambda_{\mathbf{i}} := \omega_{i_1} + \omega_{i_2} + \cdots + \omega_{i_p}$. There exists a crystal isomorphism $\Psi_{\mathbf{i}} : \mathrm{QLS}(\lambda_{\mathbf{i}}) \xrightarrow{\sim} \mathrm{QLS}(\omega_{i_1}) \otimes \mathrm{QLS}(\omega_{i_2}) \otimes \cdots \otimes \mathrm{QLS}(\omega_{i_p})$.

Remark 4.3 It is known that the fundamental representation $W(\omega_i)$ (of level zero) is isomorphic to the KR module $W_1^{(i)}$ in the sense of [HKOTT, §2.3] (but for the explicit form of the Drinfeld polynomials of $W(\omega_i)$, see [N, Remark 3.3]), and that it has a global crystal basis (see [Ka, Theorem 5.17]). Furthermore, the crystal basis of $W(\omega_i) \cong W_1^{(i)}$ is unique, up to a nonzero constant multiple (see also [NS3, Lemma 1.5.3]); we call it a (one-column) KR crystal. By the theorem above, the crystal $\mathrm{QLS}(\lambda)$ is a model for the corresponding tensor product of KR crystals.

4.2 Sketch of the proof of Theorem 4.2

First, let us recall the definition of LS paths of shape λ from [Li].

Definition 4.4 *For $\mu,\, \nu \in X_{\mathrm{af}}^0(\lambda)$ with $\mu > \nu$ (see Definition 3.3) and a rational number $0 < \sigma < 1$, a σ-chain for (μ, ν) is, by definition, a sequence $\mu = \xi_0 \gtrdot \xi_1 \gtrdot \cdots \gtrdot \xi_n = \nu$ of covers in $X_{\mathrm{af}}^0(\lambda)$ such that $\sigma\langle \gamma_k^\vee\,,\, \xi_{k-1}\rangle \in \mathbb{Z}$ for all $k = 1,\, 2,\, \dots,\, n$, where γ_k is the label for $\xi_{k-1} \gtrdot \xi_k$.*

Definition 4.5 *An LS path of shape λ is, by definition, a pair $(\underline{\nu}\,;\, \underline{\sigma})$ of a sequence $\underline{\nu} : \nu_1 > \nu_2 > \cdots > \nu_s$ of elements in $X_{\mathrm{af}}^0(\lambda)$ and a sequence $\underline{\sigma} : 0 = \sigma_0 < \sigma_1 < \cdots < \sigma_s = 1$ of rational numbers such that there exists a σ_u-chain for $(\nu_u,\, \nu_{u+1})$ for each $u = 1,\, 2,\, \dots,\, s-1$.*

We denote by $\mathbb{B}(\lambda)$ the set of all LS paths of shape λ. We identify an element

$$\pi = (\nu_1,\, \nu_2,\, \dots,\, \nu_s\,;\, \sigma_0,\, \sigma_1,\, \dots,\, \sigma_s) \in \mathbb{B}(\lambda)$$

with the following piecewise linear, continuous map $\eta : [0,1] \to \mathbb{R} \otimes_{\mathbb{Z}} X_{\mathrm{af}}$:

$$\pi(t) = \sum_{u'=1}^{u-1} (\sigma_{u'} - \sigma_{u'-1})\nu_{u'} + (t - \sigma_{u-1})\nu_u \quad \text{for } \sigma_{u-1} \le t \le \sigma_u,\ 1 \le u \le s.$$

Let $\mathbb{B}(\lambda)_{\mathrm{cl}} := \bigl\{\mathrm{cl}(\pi) \mid \pi \in \mathbb{B}(\lambda)\bigr\}$, where $\mathrm{cl}(\pi)$ is the piecewise linear, continuous map $[0,1] \to \mathbb{R} \otimes_{\mathbb{Z}} X$ defined by $(\mathrm{cl}(\pi))(t) := \mathrm{cl}(\pi(t))$ for $t \in [0,1]$. For $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$ and $i \in I_{\mathrm{af}}$, we define $e_i\eta$ and $f_i\eta$ in exactly the same way as above. Then it is known from [NS1, NS2] that the same statement as in Theorem 4.2 holds for $\mathbb{B}(\lambda)_{\mathrm{cl}}$. Thus, Theorem 4.2 follows immediately from the following proposition.

Proposition 4.6 *The affine crystals $\mathrm{QLS}(\lambda)$ and $\mathbb{B}(\lambda)_{\mathrm{cl}}$ are isomorphic.*

This proposition is a consequence of Theorem 3.4. Let us show that if $\eta \in \mathrm{QLS}(\lambda)$, then $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$. Here, for simplicity, we assume that $\eta = (x,\, y\,;\, 0,\, \sigma,\, 1) \in \mathrm{QLS}(\lambda)$, and $x = w_0 \xleftarrow{\gamma_1} w_1 \xleftarrow{\gamma_2} w_2 = y$ is a directed σ-path from y to x. Take an arbitrary $\mu \in X_{\mathrm{af}}^0(\lambda)$ such that $\mathrm{cl}(\mu) = y$. By applying Theorem 3.4 to $w_1 \xleftarrow{\gamma_2} w_2 = y$, we obtain a cover $\nu_1 \gtrdot \mu$ for some $\nu_1 \in X_{\mathrm{af}}^0(\lambda)$ with $\mathrm{cl}(\nu_1) = w_1$. Then, by applying Theorem 3.4 to $x = w_0 \xleftarrow{\gamma_1} w_1$, we obtain a cover $\nu \gtrdot \nu_1$ for some $\nu \in X_{\mathrm{af}}^0(\lambda)$ with $\mathrm{cl}(\nu) = x$. Thus we get a sequence $\nu \gtrdot \nu_1 \gtrdot \mu$ of covers in $X_{\mathrm{af}}^0(\lambda)$. It can be easily seen that this is a σ-chain for $(\nu,\, \mu)$, which implies that $\pi = (\nu,\, \mu\,;\, 0,\, \sigma,\, 1) \in \mathbb{B}(\lambda)$. Therefore, $\eta = \mathrm{cl}(\pi) \in \mathbb{B}(\lambda)_{\mathrm{cl}}$. The reverse inclusion can be shown similarly.

4.3 Description of the energy function in terms of quantum LS paths

Recall the notation in Theorem 4.2 (2); for simplicity, we set $\lambda := \lambda_{\mathbf{i}}$. In [NS5], Naito and Sagaki introduced a degree function $\mathrm{Deg}_\lambda : \mathbb{B}(\lambda)_{\mathrm{cl}} = \mathrm{QLS}(\lambda) \to \mathbb{Z}_{\le 0}$, and proved that Deg_λ is identical to the *energy function* [HKOTT, HKOTY] on the tensor product $\mathbb{B}(\omega_{i_1})_{\mathrm{cl}} \otimes \mathbb{B}(\omega_{i_2})_{\mathrm{cl}} \otimes \cdots \otimes \mathbb{B}(\omega_{i_p})_{\mathrm{cl}} = \mathrm{QLS}(\omega_{i_1}) \otimes \mathrm{QLS}(\omega_{i_2}) \otimes \cdots \otimes \mathrm{QLS}(\omega_{i_p})$ (which is isomorphic to the corresponding tensor product of

KR crystals; see Remark 4.3) via the isomorphism $\Psi_{\mathbf{i}}$. The function $\mathrm{Deg}_\lambda : \mathbb{B}(\lambda)_{\mathrm{cl}} \to \mathbb{Z}_{\leq 0}$ is described in terms of $\mathrm{QB}(W^J)$ as follows. For $x,\, y \in W^J$ let

$$\mathbf{d} : x = w_0 \stackrel{\beta_1}{\leftarrow} w_1 \stackrel{\beta_2}{\leftarrow} \cdots \stackrel{\beta_n}{\leftarrow} w_n = y$$

be a shortest directed path from y to x, and define

$$\mathrm{wt}(\mathbf{d}) := \sum_{\substack{1 \leq k \leq n \text{ such that} \\ w_{k-1} \stackrel{\beta_k}{\leftarrow} w_k \text{ is a down arrow}}} \beta_k^\vee .$$

The value $\langle \mathrm{wt}(\mathbf{d})\, ,\, \lambda \rangle$ does not depend on the choice of a shortest directed path $\mathbf{d}$ from y to x, and

Theorem 4.7 *Let $\eta = (x_1,\, x_2,\, \ldots,\, x_s\, ;\, \sigma_0,\, \sigma_1,\, \ldots,\, \sigma_s) \in \mathrm{QLS}(\lambda) = \mathbb{B}(\lambda)_{\mathrm{cl}}$. Then,*

$$\mathrm{Deg}(\eta) = -\sum_{u=1}^{s-1} (1 - \sigma_u) \big\langle \lambda,\, \mathrm{wt}(\mathbf{d}_u) \big\rangle, \tag{6}$$

where $\mathbf{d}_u$ is a shortest directed path from x_{u+1} to x_u.

4.4 The quantum alcove model

The quantum alcove model is a generalization of the alcove model of the first author and Postnikov [LP1, LP2], which, in turn, is a discrete counterpart of the Littelmann path model [Li]. For the affine Weyl group terminology below, we refer to [H]. Fix a dominant weight λ. We say that two alcoves are adjacent if they are distinct and have a common wall. Given a pair of adjacent alcoves A and B, we write $A \xrightarrow{\beta} B$ for $\beta \in \Phi$ if the common wall is orthogonal to β and β points in the direction from A to B.

Definition 4.8 [LP1] *The sequence of roots $(\beta_1, \beta_2, \ldots, \beta_m)$ is called a* λ-chain *if*

$$A_0 = A_\circ \xrightarrow{-\beta_1} A_1 \xrightarrow{-\beta_2} \cdots \xrightarrow{-\beta_m} A_m = A_\circ - \lambda$$

is a shortest sequence of alcoves from the fundamental alcove $A_\circ$ to its translation by $-\lambda$.

The *lex λ-chain* Γ_{lex} is a particular λ-chain defined in [LP2, Section 4]. Given an arbitrary λ-chain $\Gamma = (\beta_1, \beta_2, \ldots, \beta_m)$, let $r_i = r_{\beta_i}$ and define the *level sequence* $(l_1, \ldots, l_m)$ of Γ by $l_i = |\,\{j \geq i \mid \beta_j = \beta_i\}\,|$.

Definition 4.9 [LeL1] *A (possibly empty) finite subset $J = \{j_1 < j_2 < \cdots < j_s\}$ of $\{1, \ldots, m\}$ is a* Γ-admissible subset *if we have the following path in $\mathrm{QB}(W)$:*

$$1 \xrightarrow{\beta_{j_1}} r_{j_1} \xrightarrow{\beta_{j_2}} r_{j_1} r_{j_2} \xrightarrow{\beta_{j_3}} \cdots \xrightarrow{\beta_{j_s}} r_{j_1} r_{j_2} \cdots r_{j_s} = \kappa(J)\,. \tag{7}$$

Let $\mathcal{A}^\Gamma(\lambda)$ be the collection of Γ-admissible subsets. Define the level of $J \in \mathcal{A}^\Gamma(\lambda)$ by $\mathrm{level}(J) = \sum_{j \in J^-} l_j$*, where $J^- \subseteq J$ corresponds to the down steps in Bruhat order in* (7)*.*

Theorem 4.10 *There is an isomorphism of graded classical crystals $\Xi : \mathcal{A}^{\Gamma_{\mathrm{lex}}}(\lambda) \to \mathrm{QLS}(\lambda)$.*

The map Ξ is the following forgetful map. Given the path (7), based on the structure of Γ_{lex} we select a subpath, and project its elements under $W \to W/W_\lambda$ where W_λ is the stabilizer of λ in W, thereby obtaining a quantum LS path. The inverse map is more subtle, and is based on the so-called *tilted Bruhat theorem* of [LNSSS1]; this is a QBG analogue of the minimum-length Deodhar lift [De] $W/W_\lambda \to W$.

An affine crystal structure was defined on $\mathcal{A}^{\Gamma_{\text{lex}}}(\lambda)$ in [LeL1]. We show that Ξ is an affine crystal isomorphism, up to removing some f_0-arrows from QLS(λ). The remaining arrows are called *Demazure arrows* in [ST], as they correspond to the arrows of a certain affine Demazure crystal, cf. [FSS]. Moreover the above bijection sends the level statistic to the Deg statistic of (6). Thus, we have proved that $\mathcal{A}^{\Gamma_{\text{lex}}}(\lambda)$ is also a model for KR crystals.

In [LeL2], we show that all the affine crystals $\mathcal{A}^{\Gamma}(\lambda)$, for various Γ, are isomorphic. Making particular choices in classical types, we can translate the level statistic into a so-called *charge statistic* on sequences of the corresponding *Kashiwara–Nakashima columns* [KN]. In type A, we recover the classical Lascoux–Schützenberger charge [LSc]. Type C was worked out in [Le2, LeS], while type B is considered in [BL].

5 Macdonald polynomials

The symmetric Macdonald polynomials $P_\lambda(x; q, t)$ [M] are a remarkable family of orthogonal polynomials associated to any affine root system (where λ is a dominant weight for the canonical finite root system), which depend on parameters q, t. They generalize the corresponding irreducible characters, which are recovered upon setting $q = t = 0$. For untwisted affine root systems the Ram–Yip formula [RY] expresses $P_\lambda(x; q, t)$ in terms of all subsequences of any λ-chain Γ (cf. Definition 4.8). In [Le2], it was shown that the Ram–Yip formula takes the following simple form for $t = 0$:

$$P_\lambda(x; q, 0) = \sum_{J \in \mathcal{A}^{\Gamma}(\lambda)} q^{\text{level}(J)}\, x^{\text{wt}(J)}\,, \tag{8}$$

where wt(J) is a weight associated with J. By using the results in Section 4.4, we can write the right-hand side of (8) as a sum over the corresponding tensor product of KR crystals. It follows that $P_\lambda(x; q, 0) = X_\lambda(q)$. Furthermore, we can express the nonsymmetric Macdonald polynomial $E_{w\lambda}(x; q, 0)$, for $w \in W$, in a similar way to (8), by summing over those $J \in \mathcal{A}^{\Gamma}(\lambda)$ with $\kappa(J) \leq w$ (in Bruhat order), where $\kappa(J)$ was defined in (7). This formula can be derived both by induction, based on Demazure operators, and from the Ram–Yip formula (in this case, for nonsymmetric Macdonald polynomials); however, the latter derivation is more involved than the one of (8), as it uses the transformation on admissible subsets defined in [Le1, Section 5.1].

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Asymptotics of symmetric polynomials with applications to statistical mechanics and representation theory

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Abstract. We develop a new method for studying the asymptotics of symmetric polynomials of representation–theoretic origin as the number of variables tends to infinity. Several applications of our method are presented: We prove a number of theorems concerning characters of infinite–dimensional unitary group and their q–deformations. We study the behavior of uniformly random lozenge tilings of large polygonal domains and find the GUE–eigenvalues distribution in the limit. We also investigate similar behavior for Alternating Sign Matrices (equivalently, six–vertex model with domain wall boundary conditions). Finally, we compute the asymptotic expansion of certain observables in the $O(n = 1)$ dense loop model.

Résumé. Nous développons une nouvelle méthode pour étudier l'asymptotique des polynômes symétriques d'origine représentation théorique quand le nombre de variables tend vers l'infini. Plusieurs applications de notre méthode seront présentées: Nous démontrons un certain nombre de théorèmes concernant les caractères du groupe unitaire de dimension infinie et leurs q–déformations. Nous étudions le comportement des pavages en losange a distribution uniforme et aléatoire de grands domaines polygonaux et nous trouvons la distribution des valeurs propres des GUE à la limite. Nous étudions également le comportement similaire des ASM. Enfin, nous calculons l'expansion asymptotique de certains paramètres observables en $O(n = 1)$ modèle de la boucle dense.

Keywords: Schur functions, asymptotics, GUE, ASM, infinite-dimensional unitary group, dense loop model, lozenge tilings

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1 Introduction

In this article we study the asymptotic behavior of symmetric functions as the number of variables tends to infinity. The functions of interest originate in representation theory but have interpretations in combinatorics and statistical mechanics. In this extended abstract we focus on the Schur functions, but most of

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our results hold in a greater generality, e.g. for the characters of the symplectic and orthogonal groups, BC_n characters.

The [rational] Schur function $s_\lambda(x_1,\dots,x_n)$ is a symmetric Laurent polynomial in variables $x_1,\dots,$ x_n parameterized by N–tuple of integers $\lambda=(\lambda_1\ge\lambda_2\ge\dots\ge\lambda_N)$ (we call such N–tuples *signatures*, they from the set $\mathbb{GT}_N$) and given by Weyl's character formula as

$$s_\lambda(x_1,\dots,x_N)=\frac{\det\left[x_i^{\lambda_j+N-j}\right]_{i,j=1}^N}{\prod_{i<j}(x_i-x_j)}.$$

Here we study the asymptotic behavior of the following *normalized* symmetric polynomials

$$S_\lambda(x_1,\dots,x_k;N,1)=\frac{s_\lambda(x_1,\dots,x_k,\overbrace{1,\dots,1}^{N-k})}{s_\lambda(\underbrace{1,\dots,1}_{N})} \tag{1}$$

$$S_\lambda(x_1,\dots,x_k;N,q)=\frac{s_\lambda(x_1,\dots,x_k,1,\dots,q^{N-k-1})}{s_\lambda(1,\dots,q^{N-1})} \tag{2}$$

for some $q>0$. Here $\lambda=\lambda(N)$ is allowed to vary with N, k is any fixed number and $x_1,\dots,x_k$ are complex numbers, which may or may not vary together with N, depending on the context.

The asymptotic analysis of expressions (1), (2) is important because of the various applications in representation theory, statistical mechanics and probability, including:

- Convergence of (1) for any k and any *fixed* $x_1,\dots,x_k$ with $|x_i|=1$ to some limit and identification of this limit can be put in representation–theoretic framework as the approximation of indecomposable characters of infinite–dimensional unitary group $U(\infty)$ by normalized characters of unitary groups $U(N)$, the latter problem was first studied by Vershik and Kerov [VK].

- Convergence of (2) for any k and any *fixed* $x_1,\dots,x_k$ is similarly related to the *quantization* of characters of $U(\infty)$, see [G].

- Asymptotic behavior of (1) can be put in the context of Random Matrix Theory as the study of Harish–Chandra-Itzykson–Zuber integral

$$\int_{U(N)}\exp(\mathrm{Trace}(AUBU^{-1}))dU, \tag{3}$$

  where A is a fixed Hermitian matrix of finite rank and $B=B(N)$ is an $N\times N$ matrix changing in a regular way as $N\to\infty$. This problem was thoroughly studied by Guionnet and Maïda [GM].

- (1) can be interpreted as the expectation of a certain observable in the probabilistic model of uniformly random lozenge tilings of planar domains. The asymptotical analysis of (1) as $N\to\infty$ with $x_i=\exp(y_i/\sqrt{N})$ and fixed y_i gives a direct way to prove the local convergence of random tilings to a distribution of random matrix origin — GUE–corners process. Informal argument explaining that such convergence should hold was suggested earlier by Okounkov and Reshetikhin [OR1].

- When λ is the *double staircase Young diagram* with $2N$ rows $\lambda(2N) = (N-1, N-1, N-2, N-2, \dots, 1, 1, 0, 0)$, then (1) gives the expectation of a certain observable for the uniformly random configurations of the six–vertex model with domain wall boundary conditions, equivalently, Alternating Sign Matrices. Asymptotic behavior $N \to \infty$ with $x_i = \exp(y_i/\sqrt{N})$ and fixed y_i gives a way to study the local limit of this model near the boundary, equivalently, the positions of 1s and -1s in ASMs near the edges.

- For the same staircase λ the expression involving (1) with $k = 4$ and Schur polynomials replaced by the characters of symplectic group is related to the boundary-to-boundary current for the completely packed $O(n = 1)$ model, see [GNP]. The asymptotic (now with fixed x_i not depending on N) gives the limit behavior of this current, significant for the understanding of this critical model. The problem of finding the asymptotic behavior was presented by Jan de Gier during the MSRI program on Random Spatial Processes and it is solved in the present paper.

We develop a new unified approach to the study of the asymptotics of Schur functions (1), (2) (and also for more general symmetric functions like symplectic characters and polynomials corresponding to the root system BC_n), which gives a way to answer all of the above limit questions. There are 3 main ingredients of our method:

1. We find a simple contour integral representations for the normalized Schur polynomials (1), (2) with $k = 1$, i.e. for
$$\frac{s_\lambda(x, 1, \dots, 1)}{s_\lambda(1, \dots, 1)} \quad \text{and} \quad \frac{s_\lambda(x, 1, \dots, q^{N-2})}{s_\lambda(1, \dots, q^{N-1})}, \tag{4}$$
and also for more general symmetric functions of representation–theoretic origin.

2. We study the asymptotics of the above contour integrals using the *steepest descent* method.

3. We find the formulas expressing (1), (2) as $k \times k$ determinant of expressions involving (4), and combining these formulas with the asymptotics of (4) compute the limits of (1), (2).

In the rest of this abstract we will state our asymptotic results and then explain in more detail how they are applied to solve the limit problems listed above. Full details, background and a complete list of references can be found in our 67-page long paper by the same name.

2 Method and asymptotic results

The main ingredient of our approach to the asymptotic analysis of symmetric functions is the following integral formula.

Theorem 2.1 . *Let* $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N)$*, and let* x *be a complex number other than* 0 *and* 1 *, then*
$$S_\lambda(x; N, 1) = \frac{s_\lambda(x, 1^{N-1})}{s_\lambda(1^N)} = \frac{(N-1)!}{(x-1)^{N-1}} \frac{1}{2\pi\sqrt{-1}} \oint_C \frac{x^z}{\prod_{i=1}^N (z - (\lambda_i + N - i))} dz, \tag{5}$$
where the contour C *encloses all the singularities of the integrand.*

We also prove various generalizations of formula (5), omitted here for brevity: one can replace 1s by geometric series $1, q, q^2, \dots$, Schur functions can be replaced with characters of symplectic group or, more, generally, with multivariate Jacobi polynomials. In all these cases a normalized symmetric function is expressed as a contour integral with integrand being the product of elementary factors. The only exception is the most general case of Jacobi polynomials, where we have to use certain hypergeometric series. Similar formula appears in [CPZ] and [HJ].

Applying tools from complex analysis to formula (5), mainly the method of steepest descent, we compute the limit behavior of (1) for $k = 1$ under different convergence regimes for λ, as described below.

Suppose that there exists a function $f(t)$ for which the vector $(\lambda_1(N)/N, \dots, \lambda_N(N)/N)$ converges to $(f(1/N), \dots, f(N/N))$ pointwise as $N \to \infty$. Let R_1, R_∞ denote the corresponding norms of the difference of vectors $(\lambda_j(N)/N)$ and $f(j/N))$:

$$R_1(\lambda, f) = \sum_{j=1}^{N} \left| \frac{\lambda_j(N)}{N} - f(j/N) \right|, \quad R_\infty(\lambda, f) = \sup_{j=1\dots,N} \left| \frac{\lambda_j(N)}{N} - f(j/N) \right|.$$

We also introduce the function $\mathcal{F}(w)$ defined as

$$\mathcal{F}(w) = \int_0^1 \ln(w - f(t) - 1 + t)dt. \tag{6}$$

Proposition 2.2 *Suppose that $f(t)$ is piecewise-continuous, $R_\infty(\lambda(N), f)$ is bounded and $R_1(\lambda(N), f)/N$ tends to zero as $N \to \infty$, then we have for any fixed $y \in \mathbb{R} \setminus \{0\}$*

$$\lim_{N\to\infty} \frac{\ln S_{\lambda(N)}(e^y; N, 1)}{N} = yw_0 - F(w_0) - 1 - \ln(e^y - 1),$$

where w_0 is a root of $\mathcal{F}'(w) = y$ (when y is real then w is real also).

Remark 1. This proposition holds for any real y and in many cases when y is complex under some mild technical assumptions on $\mathcal{F}(w)$ which will not be discussed here.

Remark 2. Note that piecewise-continuity of $f(t)$ is a reasonable assumption since f is monotonous.

Remark 3. The solution w to $\partial/\partial w \mathcal{F}(w) = y$ can be interpreted as *inverse Hilbert transform.*

Remark 4. A somewhat similar statement was proven by Guionnet and Maïda, [GM, Theorem 1.2].

Proposition 2.2 can be refined as follows.

Proposition 2.3 *Suppose that the limit shape of λ, given by $f(t)$, is twice-differentiable and*

$$\lim_{N\to\infty} \prod_{j=1}^{N} \left(1 + \frac{f(j/N) - \lambda_j(N)/N}{w - f(j/N) - 1 + j/N}\right) = g(w) \tag{7}$$

uniformly on an open $\mathcal{M}$ set in $\mathbb{C}$, containing w_0. Then as $N \to \infty$ we have for any fixed $y \in \mathbb{R} \setminus \{0\}$

$$S_{\lambda(N)}(e^y; N, 1) = \sqrt{-\frac{w_0 - f(0) - 1}{\mathcal{F}''(w_0)(w_0 - f(1))}} g(w_0) \frac{\exp(N(yw_0 - \mathcal{F}(w_0)))}{e^N(e^y - 1)^{N-1}} \Big(1 + o(1)\Big).$$

where w_0 is a root of $\mathcal{F}'(w) = y$. The remainder $o(1)$ is uniform over y belonging to compact subsets of $\mathbb{R} \setminus \{0\}$ and such that $w_0 = w_0(y) \in \mathcal{M}$.

The last crucial ingredient in the asymptotic analysis is expressing the multivariate normalized symmetric functions through the univariate ones.

Theorem 2.4 *: $S_\lambda(x_1,\dots,x_k;N,1)$ can be expressed in terms of $S_\lambda(x_i;N,1)$ as follows*

$$S_\lambda(x_1,\dots,x_k;N,1)=\frac{1}{\prod_{i<j}(x_i-x_j)}\prod_{i=1}^{k}\frac{(N-i)!}{(x_i-1)^{N-k}} \times\det\left[D_{x_i}^{k-j}\right]_{i,j=1}^{k}\left(\prod_{j=1}^{k}S_\lambda(x_j;N,1)\frac{(x_j-1)^{N-1}}{(N-1)!}\right), \quad (8)$$

where D_x is the differential operator $x\frac{\partial}{\partial x}$.

Formula (8) can again be generalized: the 1s can be replaced with geometric series $1,q,q^2,\dots$ (and D is replaced by $D_q(f)(x)=\frac{f(qx)-f(x)}{q-1}$), Schur functions can be replaced with characters of symplectic group or, more, generally, with multivariate Jacobi polynomials. In principle, the formulas similar to (8) can be found in the literature, see e.g. [GP, Proposition 6.2].

Formula (8) allows us to derive the full asymptotics for the multivariate normalized Schur function, $S_\lambda(x_1,\dots,x_k;N,1)$ from the asymptotics for $S_\lambda(x;N,1)$. As a side remark, since we deal with analytic functions and convergence in our formulas is always (at least locally) uniform, differentiation in formula (8) does not introduce any issues.

3 Applications

3.1 Asymptotic representation theory

The asymptotic analysis developed in this paper can be applied to obtain new simpler proofs of some classical theorems in asymptotic representation theory.

Let $U(N)$ denote the group of all $N\times N$ unitary matrices. The infinite dimensional unitary group is defined as the inductive limit of $U(N)$s, where $U(N)$ embeds in $U(N+1)$ by fixing the $N+1$st vector:

$$U(\infty)=\bigcup_{N=1}^{\infty}U(N).$$

A [normalized] *character* of a group G is a continuous function which is constant on conjugacy classes, positive definite and evaluates to 1 at the unit element. *Extreme character* is an extreme point of the convex set of all characters. If G is a compact group, then extreme characters are normalized matrix traces of irreducible representations. Applying this result to $U(N)$, and using the classical fact that irreducible representations of the unitary group $U(N)$ are parameterized by signatures $\lambda\in\mathbb{GT}_N$, and their characters are the Schur functions $s_\lambda(u_1,\dots,u_N)$ we have that normalized characters of $U(N)$ are the normalized Schur functions $s_\lambda(u)/s_\lambda(1^N)$.

For "big groups" such as $U(\infty)$ the situation is more delicate. The classification theorem for the characters of infinite dimensional unitary group is summarized in the Voiculescu-Edrei ([Vo], [Ed]) theorem, which gives the explicit form of the extreme characters. The following *approximation theorem* explains the connection of characters of $U(\infty)$ with limits of normalized Schur functions.

Proposition 3.1 ([VK],[OO]) *Every extreme normalized character χ of $U(\infty)$ is a uniform limit of extreme characters of $U(N)$: for every χ there exists a sequence $\lambda(N) \in \mathbb{GT}_N$ such that for every k*

$$\chi(u_1, \dots, u_k, 1, \dots) = \lim_{N\to\infty} S_\lambda(u_1, \dots, u_k; N, 1)$$

uniformly on the torus $(S_1)^k$.

The sequences of characters of $U(N)$ which approximate characters of $U(\infty)$ has originally been found in [VK] as follows. Let μ be a Young diagram with row lengths μ_i, column lengths μ'_i and the length of main diagonal d. Introduce *modified Frobenius coordinates*:

$$p_i = \mu_i - i + 1/2, \quad q_i = \mu'_i - i + 1/2, \quad i = 1, \dots, d.$$

Note that $\sum_{i=1}^d p_i + q_i = |\mu|$. Let $\lambda \in \mathbb{GT}_N$ be a signature, we associate two Young diagrams λ^+ and λ^- to it: row lengths of λ^+ are positive of λ_i's, while row lengths of λ^- are minus negative ones. In this way we get two sets of modified Frobenius coordinates: p_i^+, q_i^+, $i = 1, \dots, d^+$ and p_i^-, q_i^-, $i = 1, \dots, d^-$. As a direct corollary of our results on asymptotics of normalized Schur polynomials from Section 2 we can reprove the following Theorem.

Theorem 3.2 ([VK], [OO], [BO],[P2]) *Let $\omega = (\alpha^\pm, \beta^\pm, ; \delta^\pm)$, such that*

$$\alpha^\pm = (\alpha_1^\pm \ge \alpha_2^\pm \ge \dots \ge 0) \in \mathbb{R}^\infty, \quad \beta^\pm = (\beta_1^\pm \ge \beta_2^\pm \ge \dots \ge 0) \in \mathbb{R}^\infty,$$

$$\sum_{i=1}^{\infty}(\alpha_i^\pm + \beta_i^\pm) \le \delta^\pm, \ \beta_1^+ + \beta_1^- \le 1.$$

Suppose that the sequence $\lambda(N) \in \mathbb{GT}_N$ is such that

$$p_i^+(N)/N \to \alpha_i^+, \quad p_i^-(N)/N \to \alpha_i^-, \quad q_i^+(N)/N \to \beta_i^+, \quad q_i^-(N)/N \to \beta_i^+,$$

$$|\lambda^+|/N \to \delta^+, \quad |\lambda^-|/N \to \delta^-.$$

Then for every k the normalized character of $U(\infty)$, parameterized by ω, satisfies

$$\chi^\omega(u_1, \dots, u_k, 1, \dots) = \lim_{N\to\infty} S_{\lambda(N)}(u_1, \dots, u_k; N, 1)$$

uniformly on torus $(S_1)^k$.

Note that every normalized character of $U(\infty)$ is in fact parametrized by such ω.

Voiculescu–Edrei's formula for the characters of $U(\infty)$ exhibits a remarkable multiplicavity where the value of character on the matrix is expressed as a product of values on its eigenvalues of a single function. Formula (8) should be viewed as a manifestation of *approximate multiplicativity* for (normalized) characters of $U(N)$, in particular, formula (8) implies informally that

$$S_\lambda(x_1, \dots, x_k; N, 1) = S_\lambda(x_1; N, 1) \cdots S_\lambda(x_k; N, 1) + O(1/N), \tag{9}$$

so normalized characters of $U(N)$ are approximately multiplicative and exactly multiplicative as $N \to \infty$.

A q–deformation of the notion of character of $U(\infty)$ was suggested in [G] . Let $\lambda(N)$ be such that

$$\frac{s_{\lambda(N)}(x_1,\dots,x_k,q^{-k},q^{-k-1},\dots,q^{1-N})}{s_{\lambda(i)}(1,q^{-1},\dots,q^{1-N})} \tag{10}$$

converges uniformly on $\{(x_1,\dots,x_k)\in\mathbb{C}^k \mid |x_i|=q^{1-i}\}$ for every k. The q–analogues of formulas (5) and (8) give a short proof of the second part of the following quantized version of Theorem 3.2.

Theorem 3.3 ([G]) *Let $0<q<1$. Extreme q–characters of $U(\infty)$ are parameterized by the points of set $\mathcal{N}$ of all non-decreasing sequences of integers: $\mathcal{N}=\{\nu_1\le\nu_2\le\nu_3\le\dots\}\subset\mathbb{Z}^\infty$, Suppose that $\lambda(N)\in\mathbb{GT}_N$ is such that for any $j>0$ $\lim\limits_{i\to\infty}\lambda(N)_{N+1-j}=\nu_j$, then for every k*

$$\frac{s_{\lambda(N)}(x_1,\dots,x_k,q^{-k},q^{-k-1},\dots,q^{1-N})}{s_{\lambda(N)}(1,q^{-1},\dots,q^{1-N})} \tag{11}$$

converges uniformly on $\{(x_1,\dots,x_k)\in\mathbb{C}^k \mid |x_i|=q^{1-i}\}$. These limits define the q–character of $U(\infty)$.

3.2 Random lozenge tilings

Consider a tiling of a domain drawn on the regular triangular lattice of the kind shown at Figure 1 with rhombi of 3 types, called *lozenges*, where each rhombus is a union of 2 elementary triangles. The configuration of the domain is encoded by the number N which is its width and N integers $\mu_1>\mu_2>\dots>\mu_N$ which are the positions of *horizontal lozenges* sticking out of the right boundary. If we write $\mu_i=\lambda_i+N-i$, then λ is a partition of size N and we denote the corresponding domain by Ω_λ. Tilings of such domains are in correspondence with tilings of certain polygonal domains, see Figure 1. It is well–known that each lozenge tiling can be identified with a stepped surface in $\mathbb{R}^3$, see e.g. [Ke].

Let Υ_λ denote a uniformly random lozenge tiling of Ω_λ. Lozenge tilings have remarkable asymptotic behavior. When N is large the rescaled stepped surface corresponding to Υ_λ concentrates near a deterministic limit shape (this holds for more general domains as well, see [CKP]). One feature of the limit shape is the formation of so–called *frozen regions*; these are the regions where asymptotically with high probability only a single type of lozenges is observed. For general polygonal domains the frozen boundary is an inscribed algebraic curve, see [KO] and [P1].

In this article we study the local behavior of lozenge tiling near a *turning point* of the frozen boundary, which is the point where the boundary of the frozen region touches (and is tangent to) the boundary of the domain. Okounkov and Reshetikhin gave in [OR1] a non-rigorous argument explaining that the scaling limit of tilings in such situation should be governed by, what we call *GUE–corners process* (introduced and studied in [Bar] and [JN] as *GUE-minors process*) and defined below. In one model of tilings of infinite polygonal domains presented in [OR1], the proof of the convergence is based on the determinantal structure of the correlation functions of the model and on the double–integral representation for the correlation kernel.

The GUE random matrix ensemble is a probability measure on the set of $k\times k$ Hermitian matrix with density proportional to $\exp(-Trace(X^2)/2)$ and GUE–distribution $\mathbb{GUE}_k$ is the distribution of the k (ordered) eigenvalues of such random matrix. The *GUE–corners process* is the joint distribution of the eigenvalues of the k principal submatrices of a $k\times k$ matrix from a GUE ensemble.

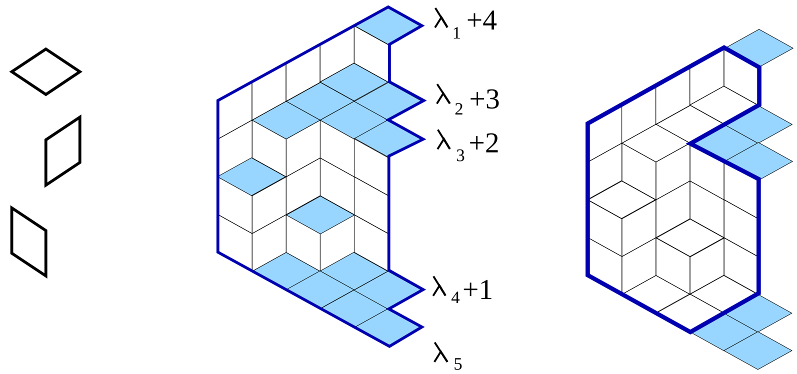

Fig. 1: Left: the 3 types of lozenges, top one is called "horizontal". Middle and right: a lozenge tiling of the domain encoded by a signature λ (here $\lambda = (4,3,3,0,0)$) and of the corresponding polygonal domain (right).

Theorem 3.4 *Let $\lambda(N) \in \mathbb{GT}_N$, $N = 1,2,\dots$ be a sequence of signatures. Suppose that there exist a non-constant piecewise-differentiable weakly decreasing function $f(t)$ such that*

$$\sum_{i=1}^{N} \left| \frac{\lambda_i(N)}{N} - f(i/N) \right| = o(\sqrt{N})$$

as $N \to \infty$ and also $\sup_{i,N} |\lambda_i(N)/N| < \infty$. Then for every k as $N \to \infty$ we have

$$\frac{\Upsilon^k_{\lambda(N)} - NE(f)}{\sqrt{NS(f)}} \to \mathbb{GUE}_k$$

in the sense of weak convergence, where

$$E(f) = \int_0^1 f(t)dt, \quad S(f) = \int_0^1 f(t)^2 dt - E(f)^2 + \int_0^1 f(t)(1-2t)dt.$$

Corollary 3.5 *Under the same assumptions as in Theorem 3.4 the (rescaled) joint distribution of the $k(k+1)/2$ horizontal lozenges (tiles) on the left k lines weakly converges to the GUE–corners process.*

Approach to the proof of Theorem 3.4: The moment generating function for the position of the horizontal lozenges along the kth vertical section of the tiling is essentially given by $S_\lambda(e^{x_1},\dots,e^{x_k};N,1)$. Using the asymptotic results of Section 2 we derive the following asymptotic expansion

$$S_\lambda(e^{\frac{h_1}{\sqrt{N}}},\dots,e^{\frac{h_k}{\sqrt{N}}};N,1) \;=\; \exp\left(\sqrt{N}E(f)(h_1+\dots+h_k) + \frac{1}{2}S(f)(h_1^2+\dots+h_k^2) + o(1)\right),$$

which corresponds to the moment generating function for the GUE–corners process. □

Remark. As $N \to \infty$ our domains may approximate a non–polygonal limit domain, where the results of [KO] describing the limit shape as an algebraic curves do not apply and the exact shape of the frozen boundary is unknown. Even an explicit expression for the position of the point where the frozen boundary touches the left boundary (a side result of Theorem 3.4) seems not to have been present in the literature.

3.3 The six–vertex model and random Alternating Sign Matrices

An *Alternating Sign Matrix* (ASM) of size N is a $N \times N$ matrix filled with 0s 1s and $-1s$ such that the sum along every row and column is 1 and along each row and each column the nonzero entries alternate in sign. ASMs are in bijection with configurations of the six-vertex model with domain-wall boundary conditions as shown at Figure 2 and have been a subject of interest in both combinatorics and statistical mechanics, see [Br] for a review. They are enumerated by a remarkable formula, proven independently by Zeilberger and Kuperberg twenty years ago, but not much more of their refined enumeration or statistics is known, see [BFZ] for some state of the art results.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

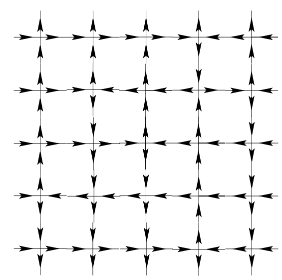

Fig. 2: Alternating sign matrix of size 5 and the corresponding configuration of the 6–vertex model .

We are interested in what the *uniformly random* ASM of size N looks like when N is large. Conjecturally, the features of this model should be similar to those of lozenge tilings: we expect the formation of a limit shape and various connections with random matrices. The existence and properties of the limit shape were studied by Colomo and Pronko [CP2], however their arguments are physical, while a mathematical proof is yet unavailable.

In the present article we prove a partial result toward the following conjecture.

Conjecture 3.6 *Fix any k. As $N \to \infty$ the probability that the number of (-1)s in the first k rows of uniformly random ASM of size N is maximal (i.e. there is one (-1) in second row, two (-1)s in third row, etc) tends to 1, and, thus, the 1s in the first k rows are interlacing. After proper centering and rescaling the distribution of the positions of 1s tends to GUE–corners process as $N \to \infty$.*

Let $\Psi_k(N)$ denote the sum of horizontal coordinates of 1s minus the sum of horizontal coordinates of (-1)s in the kth row of a uniformly random ASM of size N. We prove that

Theorem 3.7 *For any fixed k the random variable $\frac{\Psi_k(n)-N/2}{\sqrt{N}}$ weakly converges to the normal random variable $\mathcal{N}(0, \sqrt{3/8})$. Moreover, the joint distribution of any collection of such variables converges to the distribution of independent normal random variables $\mathcal{N}(0, \sqrt{3/8})$.*

Our proof of Theorem 3.7 is based on the results of Okada [Ok] and Stroganov [St] that sums of certain quantities over all ASMs (i.e. the partition functions of the 6-vertex model) can be expressed through Schur polynomials and on the asymptotic analysis of these polynomials. In fact, we claim that the precise statement Theorem 3.7 together with additional probabilistic argument implies Conjecture 3.6. However, this argument in itself is unrelated to the asymptotics of symmetric polynomials and is postponed to a future publication.

3.4 The $O(n=1)$ dense loop model

Recently found parafermionic observables in the *completely packed* $O(n=1)$ *dense loop model* in a strip are also simply related to symmetric polynomials, see [GNP]. The $O(n=1)$ dense loop model is one of the representations of the percolation model on the square lattice. For the critical percolation models similar observables and their asymptotic behavior were studied (see e.g. [Sm]), however, the methods involved are usually completely different from ours.

A configuration of the $O(n=1)$ loop model in a vertical strip consists of two parts: a tiling of the strip on a square grid of width L and infinite height with squares of two types shown in Figure 3, and a choice of one of the two types of boundary conditions for each 1×2 segment along each of the vertical boundaries of the strip, see3. Let $\urcorner_L$ denote the set of all configurations of the model in the strip of width L. As the arcs drawn on squares and boundary segments form closed loops and paths joining the boundaries, the elements of $\urcorner_L$ are interpreted as collections of non-intersecting paths and closed loops.

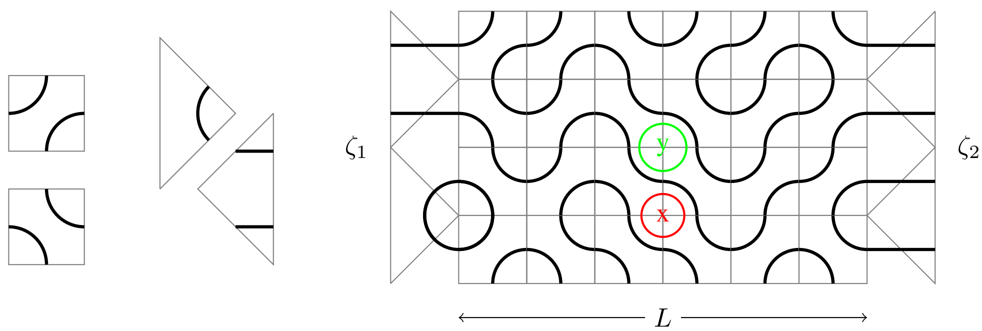

Fig. 3: Left: the two types of squares. Middle: the two types of boundary conditions. Right:A particular configuration of the dense loop model showing a path passing between two vertically adjacent points x and y.

A probability distribution on $\urcorner_L$ is defined by choosing the type of square for each point on the grid according to a weight defined as a certain function of its horizontal coordinate and depending on L parameters $z_1, \ldots, z_L$; two other parameters ζ_1, ζ_2 control the probabilities of the boundary conditions and, using a parameter q, the whole configuration is further weighted by its number of closed loops. Setting $z_i = 1$ and $q = \exp(-\sqrt{-1}\pi/6)$ makes the choice of square type for each position an I.I.D. Bernoulli RV with parameter $1/2$. See [GNP] for exact details.

Fix two points x and y and consider a configuration $\omega \in \urcorner_L$. For each path τ passing between x and y, define the current $c(\tau)$ as 1 if τ joins the two boundaries and x lies above τ; -1 if τ joins the two boundaries and x lies below τ; and 0 otherwise. The total current $C^{x,y}(\omega)$ is the sum of $c(\tau)$ over all [necessarily finitely many] paths passing between x and y. The *mean total current* $F^{x,y}$ is defined as the expectation of $C^{x,y}$. As $F^{x,y}$ is skew-symmetric and additive, it can be expressed as a sum of several instances of the mean total current between two horizontally adjacent points, $F^{(i,j),(i,j+1)}$, and two vertically adjacent points, $F^{(j,i),(j+1,i)}$. The authors of [GNP] present a formula for $F^{(i,j),(i,j+1)}$, and $F^{(j,i),(j+1,i)}$, which, based on certain assumptions, expresses them through the symplectic characters $\chi_{\lambda^L}(z_1^2,\ldots,z_L^2,\zeta_1^2,\zeta_2^2)$ as certain functions denoted Y_L and $X_L^{(j)}$, where $\lambda^L = (\lfloor \frac{L-1}{2} \rfloor, \lfloor \frac{L-2}{2} \rfloor, \ldots, 1, 0, 0)$.

Our approach from Section 2 allows us to compute the asymptotic behavior of the formulas of [GNP] as the lattice width $L \to \infty$ in the homogenous case with $q = \exp(-\sqrt{-1}\pi/6)$ as follows.

Theorem 3.8 *As $L \to \infty$, the formula of [GNP] for the mean total current between two horizontally adjacent points is asymptotically*

$$X_L^{(j)}\Big|_{z_j=z;\, z_i=1,\ i\neq j} = \frac{\sqrt{-3}}{4L}(z^3 - z^{-3}) + o\left(\frac{1}{L}\right).$$

The formula of [GNP] for the mean total current between two vertically adjacent points is asymptotically

$$Y_L\Big|_{z_i=1,\ i=1,\dots,L; z_{L+1}=w, z_{L+2}=q^{-1}w} = \frac{\sqrt{-3}}{4L}(w^3 - w^{-3}) + o\left(\frac{1}{L}\right).$$

Remark 1. When $z = 1$, $X_L^{(j)}$ is identically zero and so is our asymptotics.
Remark 2. The fully homogeneous case corresponds to $w = \exp(-\sqrt{-1}\pi/6)$, then $Y_L = \frac{\sqrt{3}}{2L} + o\left(\frac{1}{L}\right)$.
Remark 3. The leading asymptotics terms do not depend on the boundary parameters ζ_1 and ζ_2.

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Combinatorics of non-ambiguous trees[‡]

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Abstract. This article investigates combinatorial properties of non-ambiguous trees. These objects we define may be seen either as binary trees drawn on a grid with some constraints, or as a subset of the tree-like tableaux previously defined by Aval, Boussicault and Nadeau. The enumeration of non-ambiguous trees satisfying some additional constraints allows us to give elegant combinatorial proofs of identities due to Carlitz, and to Ehrenborg and Steingrímsson. We also provide a hook formula to count the number of non-ambiguous trees with a given underlying tree. Finally, we use non-ambiguous trees to describe a very natural bijection between parallelogram polyominoes and binary trees.

Résumé. Cet article s'intéresse aux propriétés combinatoires des arbres non-ambigus. Ces objets, que nous définissons, peuvent être vus soit comme des arbres dessinés sur une grille sous certaines contraintes, soit comme un sous-ensemble des tableaux boisés précédemment définis par Aval, Boussicault et Nadeau. L'énumération des arbres non-ambigus satisfaisant des contraintes supplémentaires nous permet de donner des preuves combinatoires élégantes d'identités dues à Carlitz, et à Ehrenborg et Steingrímsson. Nous donnons aussi une formule des équerres pour le comptage des arbres non-ambigus dont l'arbre sous-jacent est fixé. Enfin, nous utilisons les arbres non-ambigus pour décrire une bijection très naturelle entre polyominos parallélogrammes et arbres binaires.

Keywords: tree, polyomino, non-ambiguous tree, tree-like tableau, hook formula, Bessel function

1 Introduction

It is well known that Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ enumerate many combinatorial objects, such as binary trees and parallelogram polyominoes. Several bijective proofs in the literature show that parallelogram polyominoes are enumerated by Catalan numbers, the two most classical being Delest-Viennot's bijection with Dyck paths [DV84] and Viennot's bijection with bicolored Motzkin paths [DV84].

In this paper we demonstrate a bijection –which we believe is more natural– between binary trees and parallelogram polyominoes. In some sense, we show that parallelogram polyominoes may be seen as two-dimensional drawings of binary trees. This point of view gives rise to a new family of objects – we call them *non-ambiguous trees* – which are particular compact embeddings of binary trees in a grid.

[†]This short paper is an extended abstract of [ABBS], where details of the proofs are provided.

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The tree structure of these objects leads to a hook formula for the number of non-ambiguous trees with a given underlying tree. Unlike the classical hook formula for trees due to Knuth (see [Knu98], §5.1.4, Exercise 20), this one is defined on the edges of the tree.

Non-ambiguous trees are in bijection with permutations such that all their (strict) excedances stand at the beginning of the permutation word. Ehrenborg and Steingrímsson in [ES00] give a closed formula (involving Stirling numbers of the second kind) for the number of such permutations. We show that this formula can be easily proved using non-ambiguous trees and a variation of the insertion algorithm for tree-like tableaux introduced in [ABN11]. Indeed, non-ambiguous trees can also be seen as a subclass of tree-like tableaux, objects defined in [ABN11], that are in bijection with permutation tableaux [SW07] or alternative tableaux [Nad11, Vie07].

A particular subclass of non-ambiguous trees leads to unexpected combinatorial interpretations. We study complete non-ambiguous trees, defined as non-ambiguous trees such that their underlying binary tree is complete, and show that their enumerating sequence is related to the formal power series of the logarithm of the Bessel function of order 0. This gives rise to new combinatorial interpretations of some identities due to Carlitz [Car63].

The paper is organized as follows: in Section 2 we define non-ambiguous trees. Then, in Section 3 we give the enumeration of non-ambiguous trees satisfying certain constraints: those contained into a given rectangular box, and those with a fixed underlying tree. Section 4 introduces the family of complete non-ambiguous trees, and studies the relations between this family and the Bessel function. Finally, in Section 5 we describe our new bijection between binary trees and parallelogram polyominoes.

2 Definitions and notations

In this paper, trees are embedded in a bidimensional grid $\mathbb{N} \times \mathbb{N}$. The grid is not oriented as usual: the x-axis has south-west orientation, and the y-axis has south-east orientation, as shown on Figure 1.

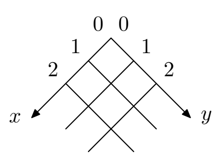


Fig. 1: The underlying grid for non-ambiguous trees

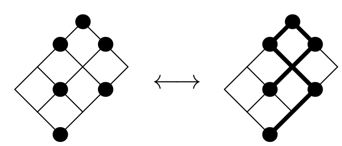
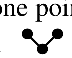


Fig. 2: The edges of a non-ambiguous tree are not necessary

Every x-oriented (resp. y-oriented) line will be called column (resp. row). Each column (resp. row) on this grid is numbered with an integer corresponding to its y (resp. x) coordinate. A vertex v located on the intersection of two lines has the coordinate representation: $(X(v), Y(v))$.

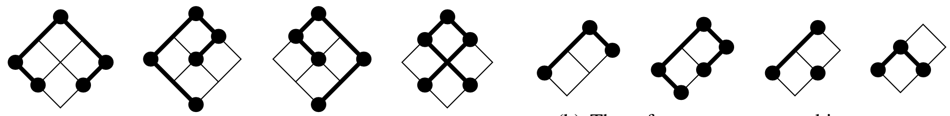
A non-ambiguous tree may be seen as a binary tree embedded in the grid in such a way that the embedding of its vertices in the grid determines the tree completely (i.e. determines its edges – see Figure 2).

Formally, a *non-ambiguous tree* of size n is a set A of n points $(x, y) \in \mathbb{N} \times \mathbb{N}$ such that:

1. $(0, 0) \in A$; we call this point the *root* of A;
2. given a non-root point $p \in A$, there exists one point $q \in A$ such that $Y(q) < Y(p)$ and $X(q) = X(p)$, or one point $r \in A$ such that $X(r) < X(p)$, $Y(r) = Y(p)$, but not both (which means that the pattern  is avoided);

3. there is no empty line between two given points: if there exists a point $p \in A$ such that $X(p) = x$ (resp. $Y(p) = y$), then for every $x' < x$ (resp. $y' < y$) there exists $q \in A$ such that $X(q) = x'$ (resp. $Y(q) = y'$).

Figure 3 shows some examples and counterexamples of non-ambiguous trees.



(a) Four non-ambiguous trees (b) These four are not non-ambiguous trees

Fig. 3: Some examples and counterexamples of non-ambiguous trees

It is straightforward that a non-ambiguous tree A has a tree structure: except for the root, every point $p \in A$ has a unique parent, which is the nearest point q preceding p in the same row (resp. column). In this case, we will say that p is the *right child* (resp. *left child*) of q. In this paper, we orient every edge of a tree from the root to the leaves. We shall denote by $\mathcal{T}(A)$ the underlying binary tree associated to A.

Figure 4 shows all the non-ambiguous trees of size 4, grouping inside a rectangle those having the same underlying binary tree.

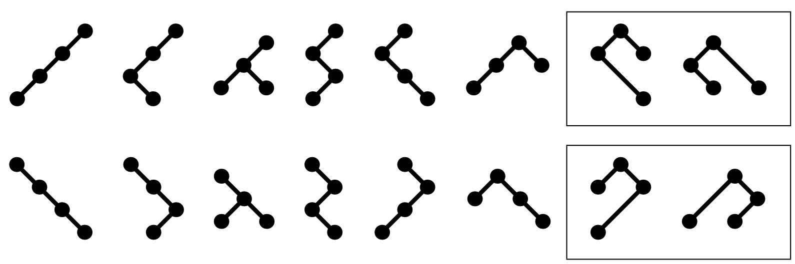


Fig. 4: The 16 non-ambiguous trees of size 4

Remark 1 *A tree-like tableau [ABN11] of size n is a set of n points placed in the boxes of a Ferrers diagram such that conditions 1, 2, 3 defining non-ambiguous trees are satisfied. Figure 5 shows an example of a tree-like tableau of size 7. It should be clear that non-ambiguous trees are in bijection with tree-like tableaux with rectangular shape.*

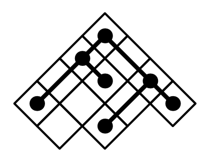


Fig. 5: A tree-like tableau

3 Enumeration of non-ambiguous trees

Non ambiguous trees of size n are in bijection with permutations of size n with all their strict excedances at the beginning. This fact is a consequence of Lemma 5 in [SW07] and of results proved in [ABN12]. The sequence $(a_n)_{n \geq 1}$ counting the number of non-ambiguous tree of size n is referenced in [Slo] as `A136127` $= [1, 2, 5, 16, 63, 294, 1585, 9692, \dots]$, but no simple formula is known.

3.1 Non-ambiguous trees inside a fixed box

Given a non-ambiguous tree, its x-size (resp. y-size) may be defined as the maximum of the x-coordinate (resp. y-coordinate) of its points. The aim of this subsection is to give a formula for the number $A(k, \ell)$

of non-ambiguous trees with x-size equal to k and y-size equal to ℓ. We denote by $c(n, j)$ the unsigned Stirling numbers of the first kind, *i.e.* the number of permutations of size n with exactly j disjoint cycles.

Proposition 2 *For every integers n, ℓ, one has:*

$$\sum_{k=1}^{n} c(n,k)\, A(k,\ell) = n^{\ell-1}\, n!\,. \tag{1}$$

We may inverse (1) to get:

$$A(k,\ell) = \sum_{i=1}^{k} (-1)^{k-i}\, S(k,i)\, i!\, i^{\ell-1}, \tag{2}$$

where $S(k, i)$ denotes the Stirling numbers of the second kind, *i.e.* the number of partitions of a set of k elements into i non-empty parts. Since (from [SW07, ABN12]) $A(k, \ell)$ is equal to the number of permutations of size $k+\ell$ with exactly k strict excedances in position $1, 2, \ldots, k$, Equation (1) is equivalent to Corollary 6.6 in [ES00]. In that paper, (2) is obtained through an inclusion-exclusion argument, and (1) is deduced by inversion.

In our setting, we may interpret $c(n, k)$ through tree-like tableaux. We refer to [ABN11] for definitions, and basic properties. As mentioned in Remark 1, non-ambiguous trees are nothing but tree-like tableaux with a rectangular shape. Permutations of size n with exactly j disjoint cycles are in bijection with tree-like tableaux of size n with exactly j points in their first row (a consequence of Theorem 4.2 in [Bur07] and of the results contained in [ABN12]). We are thus able to interpret (1) with unified objects: tree-like tableaux and non-ambiguous trees.

With these tools, the proof of Proposition 2 is a *simple* use of a variation of the insertion algorithm defined on tree-like tableaux in [ABN11], but we cannot give the details in this extended abstract.

3.2 Non-ambiguous trees with a fixed underlying tree: a new hook formula

Let T be a binary tree. We define $NA(T)$ as the number of non-ambiguous trees A such that their underlying binary tree $\mathcal{T}(A)$ is T. The aim of this section is to get a formula for $NA(T)$: this will be done by Proposition 6, which shows that $NA(T)$ may be expressed by a new and elegant hook formula on the edges of T. To do this, we encode any non-ambiguous tree A by a triple $\Phi(A) = (T, \alpha_L, \alpha_R)$ where T is a binary tree, and α_L (resp. α_R) is a word called the left (resp. right) code of A. To distinguish the vertices of A, we label them by integers from 1 to the size of A, as shown on Figure 6.

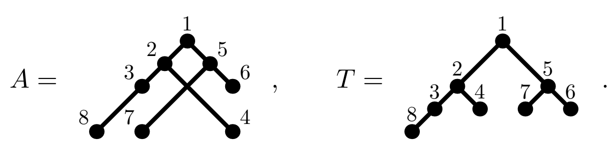
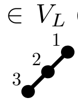


Fig. 6: A non-ambiguous tree A with labeled vertices, and the associated binary tree T

The first entry in $\Phi(A)$ is the underlying binary tree T associated to A. Observe that we keep the labels on vertices when we extract the underlying binary tree. Now we denote by V_L (resp. V_R) the set of the end points of the left (resp. right) edges of A, which gives $V_L = \{2, 3, 8, 7\}$ and $V_R = \{5, 6, 4\}$ on the example in Figure 6. The definition of non-ambiguous trees ensures that the set $\{X(v), v \in V_L\}$ is the interval $\{1, \ldots, |V_L|\}$. Thus for $i = 1, \ldots, |V_L|$, we may set $\alpha_L(i)$ as the unique label $v \in V_L$ such that $X(v) = i$, and we proceed symmetrically for α_R. On the example of Figure 6, we have: $\alpha_L = 2378$ and $\alpha_R = 564$. Our starting point is the following lemma.

Lemma 3 *The application Φ which sends A to the triple (T, α_L, α_R) is injective.*

Proof: The proof shall not be detailed here: it is elementary to check that, given the tree T, the left and right codes uniquely determine the coordinates of every point in A. □

Lemma 3 allows us to encode a non-ambiguous tree A by a triple (T, α_L, α_R), where T is a binary tree, and α_L (resp. α_R) is a word in which every label $v \in V_L$ (resp. V_R) appears exactly once. Of course, Φ is not surjective on such triples. If we take $T =$ 

, it should be clear that α_L is forced to be 23. Consequently, our next task is to characterize the pairs of codes (α_L, α_R) which are compatible with a given binary tree T, *i.e.* such that (T, α_L, α_R) is in the image of Φ. In order to describe this characterization, we need to define partial orders on the sets V_L and V_R. The pairs (α_L, α_R) of compatible codes will be seen to correspond to pairs of linear extensions of the posets V_L and V_R. The posets are defined as follows: given $a, b \in V_L$ (resp. V_R), we say that $a \leq b$ if and only if there exists a path in the oriented tree starting from a and ending at b. Figure 7 and Figure 9 (with minima at the top) illustrate this notion.

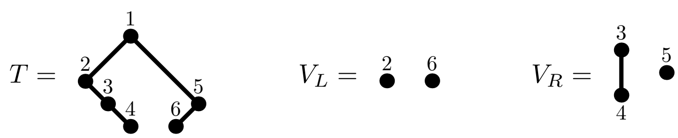


Fig. 7: The posets V_L and V_R of a tree T

The next lemma is the crucial step to prove Proposition 6.

Lemma 4 *Given a binary tree T, the pairs of codes compatible with T are exactly the pairs (α_L, α_R) where α_L is a linear extension of V_L and α_R is a linear extension of V_R.*

Figure 8 gives these compatible codes, together with the corresponding non-ambiguous trees, in the case of the tree T of Figure 7.

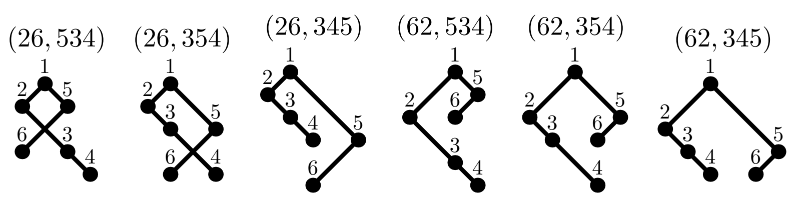


Fig. 8: Non-ambiguous trees of the tree T of Figure 7

Proof: We shall only give the main arguments of the proof.

Given a tree T, consider the map Φ_T defined on the set of non-ambiguous trees with underlying tree T as follows:

$$\Phi_T(A) := (\alpha_L, \alpha_R),$$

where $\Phi(A) = (T, \alpha_L, \alpha_R)$. Since Φ is injective (by Lemma 3), so is Φ_T. It remains to prove that the image of Φ_T is $\mathcal{L}(V_L) \times \mathcal{L}(V_R)$, where we denote by $\mathcal{L}(P)$ the set of linear extensions of a poset P.

First, we prove that $Im\,\Phi_T \subseteq \mathcal{L}(V_L) \times \mathcal{L}(V_R)$. Without loss of generality, we will prove that $\alpha_L \in \mathcal{L}(V_L)$. We need to prove that, if $s <_{V_L} t$, then s precedes t in α_L, which we shall write $s <_{\alpha_L} t$. If $s <_{V_L} t$, there exists a path in T starting from s and ending at t. When we go through the path, the X-coordinates of the vertices remain unchanged along right edges, while they increase along left edges. Since $s \neq t$, we have $X(s) < X(t)$, which is equivalent to $s <_{\alpha_L} t$.

Now the hard part is to prove that $\mathcal{L}(V_L) \times \mathcal{L}(V_R) \subseteq Im\,\Phi_T$. Let $(\alpha_L, \alpha_R) \in \mathcal{L}(V_L) \times \mathcal{L}(V_R)$. It is always possible to use the triple (T, α_L, α_R) to build a set of points in the grid which we may denote by A: we just have to place the root at position $(0,0)$ and every other vertex v of T at the position

$$\begin{cases} X(v) = i \text{ with } \alpha_L(i) = v \quad \text{and} \quad Y(v) = Y(\text{parent}(v)) & \text{if } v \in V_L; \\ X(v) = X(\text{parent}(v)) \quad \text{and} \quad Y(v) = j \text{ with } \alpha_R(j) = v & \text{if } v \in V_R. \end{cases}$$

The goal is to prove that A is a non-ambiguous tree, which is quite technical. The main steps are:

1. check that for every left (resp. right) edge (s,t) of T, we have $X(s) < X(t)$ (resp. $Y(s) < Y(t)$) in A;
2. prove that A avoids the pattern  ;
3. check that two different vertices in T occupy different positions in A. □

Now we come to the final step toward proving Proposition 6.

Lemma 5 *The Hasse diagrams of V_L and V_R are forests.*

Figure 9 shows an example of the forests obtained by computing the Hasse diagrams of V_L and V_R.

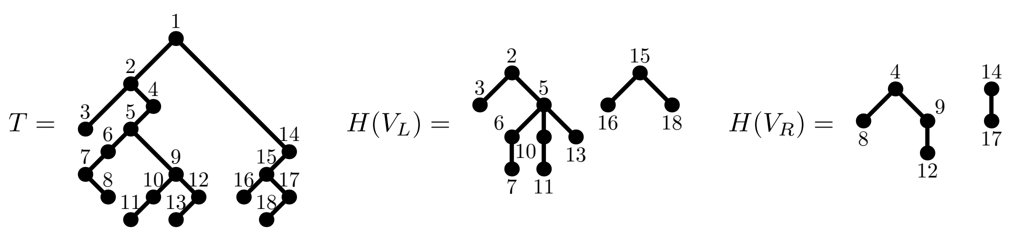


Fig. 9: The Hasse diagrams $H(V_L)$ and $H(V_R)$ of V_L and V_R are forests

Proof: We prove this proposition by contradiction. Suppose that there is a cycle in the Hasse diagram of V_R (the case of V_L is analogous). We can deduce from the poset structure that there are two paths in V_R starting from an element v and ending at w. This would imply that in the tree there are two different paths from v to w, and hence there would be a cycle in the tree. □

As a consequence the number of non-ambiguous trees with underlying tree T is given by the product of the results of Knuth's hook formula [Knu98] applied to the Hasse diagram of V_L and to the Hasse diagram of V_R. We can make this more precise. To do so, we associate to each edge an integer n_e. Given a left edge e (resp. right edge) the integer n_e is the number of left edges (resp. right edges) contained in the subtree whose root is the ending point of e, plus 1.

Proposition 6 *The number of non-ambiguous trees with underlying tree T is given by*

$$NA(T) = \frac{\#\{\textit{left edges}\}!\ \#\{\textit{right edges}\}!}{\prod_{e\in V_L} n_e \prod_{e\in V_R} n_e}. \tag{3}$$

This new hook formula is illustrated by Figure 10.

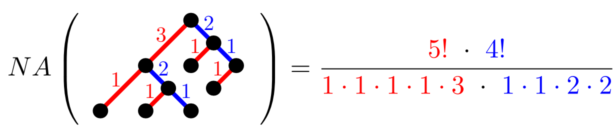


Fig. 10: A hook formula for non-ambiguous trees

Remark 7 *Equation (3) gives a way to compute the number of permutations of size n with all their strict excedances at the beginning, by summing over all binary trees T with n vertices.*

4 Complete non-ambiguous trees and Bessel function

4.1 Definition and enumeration of complete non-ambiguous trees

A non-ambiguous tree is *complete* whenever its vertices have either 0 or 2 children. An example of complete non-ambiguous tree can be found in Figure 11. A complete non-ambiguous tree has always an odd number of vertices. Moreover, as in complete binary trees, a complete non-ambiguous tree with $2k+1$ vertices has exactly k internal vertices, $k+1$ leaves, k right edges and k left edges. Denote by b_k the number of complete non-ambiguous trees with k internal vertices. The sequence $(b_k)_{k\geq 0}$ is known in [Slo] as A002190 $= [1, 1, 4, 33, 456, 9460, \dots]$, and two remarkable identities satisfied by this sequence are given by Carlitz [Car63]. Propositions 8 and 10 give combinatorial interpretations for these identities.

Denote by C_n the number of complete binary trees with n internal vertices. It is well-known that $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number, and that, for every $n \geq 0$, we have the identity:

$$C_{n+1} = \sum_{i+j=n} C_i C_j. \tag{4}$$

Proposition 8 gives a variant of this identity for complete non-ambiguous trees:

Proposition 8 *For every $n \geq 0$, we have:*

$$b_{n+1} = \sum_{i+j=n} \binom{n+1}{i}\binom{n+1}{j} b_i\, b_j. \tag{5}$$

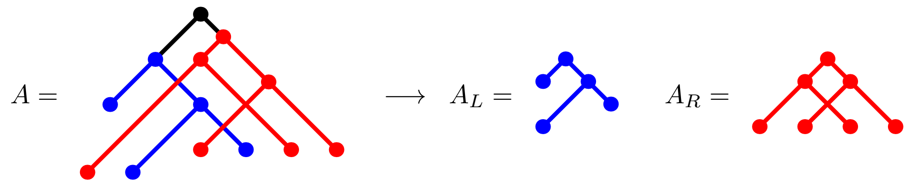


Fig. 11: The root suppression in a complete non-ambiguous tree

Proof: The proof of this proposition is similar to the classical proof of (4): the left (resp. right) subtree A_L (resp. A_R) of a complete non-ambiguous tree A with $n+1$ internal vertices is a complete non-ambiguous tree with i (resp. j) internal vertices, where $i+j=n$.

Figure 11 shows an example of left and right subtree of a complete non-ambiguous tree.

Hence, in order to construct an arbitrary complete non-ambiguous tree A with $n+1$ internal vertices, we need to choose:

- the number i of internal vertices contained in A_L (i may range between 0 and n, the number j is equal to $n-i$);
- the complete non-ambiguous tree structure of A_L (resp. A_R) – we have b_i (resp. b_j) choices;
- the way of interlacing the right (resp. left) edges of A_L and A_R.

We denote by $u_1, u_2, \dots, u_i$ (resp. $v_1, v_2, \dots, v_j$) the end points of the right edges in A_L (resp. A_R) such that if $k < l$, then $Y(u_k) < Y(u_l)$ (resp. $Y(v_k) < Y(v_l)$), and by u_0 and v_0 the roots of A_L and A_R. Now, if we want to interlace the right edges in A_L with those in A_R, we need to decide at what positions we want to insert the vertices $u_1, u_2, \dots, u_i$ with respect to $v_0, v_1, v_2, \dots, v_j$, saving the relative order among $u_0, u_1, u_2, \dots, u_i$ and $v_0, v_1, v_2, \dots, v_j$. A vertex u_k can be placed either to the left of v_0, or between v_t and v_{t+1} ($0 \le t \le j-1$), or to the right of v_j.

Hence, we must choose the i positions of $u_1, u_2, \dots, u_i$ (multiple choices of the same position are allowed) among $j+2$ possible ones. This shows that there are $\left(\binom{j+2}{i}\right) = \binom{i+j+1}{i} = \binom{n+1}{i}$ ways of interlacing the right edges of the subtrees A_L and A_R, where $\left(\binom{a}{b}\right)$ denotes the number of way of choosing b objects within a, with possible repetitions.

Analogous arguments apply to left edges. In this case, we have $\left(\binom{i+2}{j}\right) = \binom{n+1}{j}$ different interlacements. This ends the proof. $\square$

Corollary 9 *The sequence b_k satisfies the following identity*

$$\sum_{k\geq 0} b_k \frac{x^{2(k+1)}}{((k+1)!2^{k+1})^2} = -\ln\left(J_0(x)\right). \tag{6}$$

Proof: It is well known (see, e.g., [AS64]) that the Bessel function $J_0(x) = \sum_{k\geq 0} j_k x^k$ satisfies the differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{d\,y}{dx} + y = 0, \tag{7}$$

The first coefficients in its series expansion are $j_0 = 1$ and $j_1 = 0$.

Consider now the function $B(x) = exp\left(-\sum_{k\geq 0} b_k \frac{x^{2(k+1)}}{((k+1)!2^{k+1})^2}\right) = \sum_{k\geq 0} \beta_k x^k$. Equation (5) ensures that $B(x)$ satisfies Equation (7), *i.e.* the same second order differential equation as $J_0(x)$.

Setting $x = 0$, we have $\beta_0 = B(0) = 1 = j_0$. Moreover, in $Z(x) = -\sum_{k\geq 0} b_k \frac{x^{2(k+1)}}{((k+1)!2^{k+1})^2}$ only the even powers of x have non-zero coefficients. Hence, since $B(x) = exp\,(Z(x)) = \sum_{k\geq 0} \frac{Z(x)^k}{k!}$, we have $\beta_{2i+1} = 0$ for every $i \geq 0$. In particular, $\beta_1 = 0 = j_1$. These arguments imply that $B(x) = J_0(x)$. □

4.2 Proving identities combinatorially

Corollary 9 shows that non-ambiguous trees provide a combinatorial interpretation –and to our knowledge, the first one– of sequence `A002190` [Slo].
In [Car63], the author shows analytically that identities (5) and (8) below are equivalent. We give a combinatorial proof of this fact.

Proposition 10 *For every $n \geq 1$, we have:*

$$\sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1}\binom{n-1}{k} b_k = 1. \tag{8}$$

Proof: We fix an integer n and we take $0 \leq k \leq n-1$. We define a gridded tree of size (k, n) to be a set of $2k+1$ points placed in a $n \times n$ grid, such that Condition 2 defining non-ambiguous trees is satisfied (which means we consider a non-ambiguous tree of size $2k+1$ embedded in a $n \times n$ grid) and such that the underlying tree is complete and that its root belongs to the first column. This implies that there are $n-k-1$ empty columns and $n-k-1$ empty rows, and that the first column is not empty. Figure 12 shows an example of a gridded tree of size $(2, 6)$.

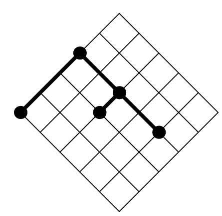


Fig. 12: An example of gridded tree with 2 internal vertices drawn on a 6×6 grid

It is easy to verify that there are $\binom{n}{k+1}\binom{n-1}{k} b_k$ gridded tree of size (k, n). We call trivial gridded tree the tree of size $(0, n)$ consisting of a single vertex in $(0, 0)$. Now, for every integer n, we define an involution on the set of non trivial gridded trees. This involution associates a gridded tree of size (k, n) with a gridded tree either of size $(k-1, n)$ or $(k+1, n)$.
To define this involution, consider a gridded tree of size (k, n) and add a virtual root at position $(-1, 0)$; the previous root becomes the left child of the virtual root. Now consider the path starting from the virtual root, going down through the tree, turning at each internal vertex, and ending at a leaf. This path is unique. There are two cases:

1. the path does not cross an empty row, nor an empty column: we erase the leaf and its parent from the tree, getting a new gridded tree of size $(k-1, n)$. We can always erase the leaf and its parent, except if the parent were the virtual root. This happens only if the tree is the trivial gridded tree. As we restricted to non trivial gridded trees, this case never happens.

2. the path crosses an empty row or an empty column: we choose the first empty row or column met while visiting the path. Without loss of generality, we suppose that it is a column, say c. Then, we add a new vertex v at the position where c crosses the path, and we add in the same column a new leaf (whose parent is v) in the topmost empty row. While visiting the path, we did not meet an empty row. Since there are as many empty rows as empty columns, there is always an empty row below v. This operation gives rise to a new gridded tree of size $(k+1, n)$.

Figure 13 shows how the involution acts on two examples.

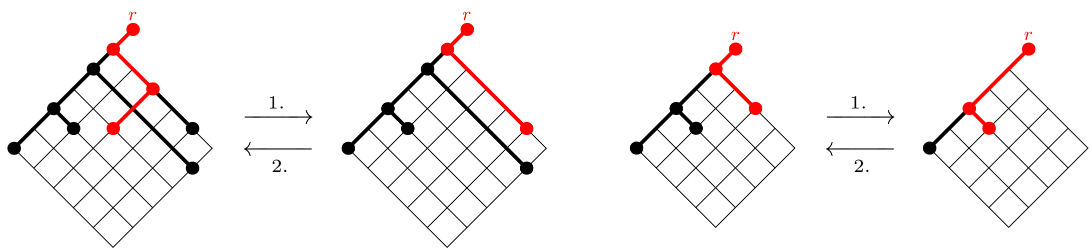


Fig. 13: The involution acting on two examples of non trivial gridded trees

Remark that adding (resp. removing) a leaf and its parent p in (resp. from) a gridded tree following the previous algorithm does not remove (resp. add) any empty row or column that crosses the path from the virtual root to p. For this reason, this operation is an involution. □

In a similar fashion to the proof of Proposition 10, it is possible to prove that Catalan numbers satisfy $\sum_{k=0}^{n}(-1)^{n+k}\binom{n+k}{n-k}C_k = 0$, for any $n \geq 1$. This identity and Proposition 10 allow us to prove a further identity involving the sequence b_n. Our proof uses the methodology described in [PWZ96] and settles a conjecture of P. Hanna (see [Slo] sequence A002190):

Proposition 11 *For every $n \geq 1$, we have* $\displaystyle\sum_{k=0}^{n}(-1)^k b_k C_k \binom{n+k}{n-k}^2 = 0.$

5 A new bijection between trees and parallelogram polyominoes

We recall that a parallelogram polyomino of size n is a pair of lattice paths of length $n+1$ with south-west and south-east steps starting at the same point, ending at the same point, and never meeting each other. Figure 14 shows some examples of parallelogram polyominoes of size 4. The two paths defining a given parallelogram polyomino delimit a connected set of boxes. We will consider the parallelogram polyomino from this point of view.

We now describe a bijection between parallelogram polyominoes of size n and binary trees with n vertices by showing that a parallelogram polyomino hides a non-ambiguous tree.

Given a parallelogram polyomino P, consider the set S_P of dots defined as follows:

- we enlighten P from north-west to south-east and from north-east to south-west;
- we put a dot in the enlightened boxes.

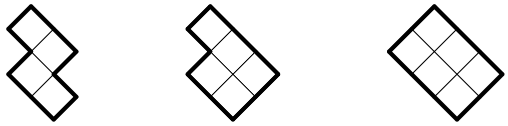


Fig. 14: Example of parallelogram polyominoes of size 4

It is easy to verify that S_P is a non-ambiguous tree. Indeed, it is impossible that all three points in the pattern  are enlightened. Moreover, only the northernmost box in the parallelogram polyomino can be enlightened twice. This implies that every dot (except for the one in the northernmost box) has a parent. Let Ψ be the application that associates to a parallelogram polyomino the underlying binary tree of S_P. An example of this application is shown in Figure 15.

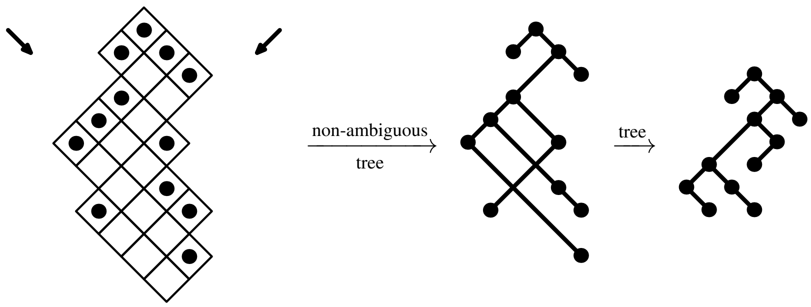


Fig. 15: Parallelogram polyominoes are just a way of drawing a binary tree in the plane

Proposition 12 *The map Ψ is a bijection between the set of parallelogram polyominoes of size n and the set of binary trees with n vertices.*

Proof: We are able in [ABBS] to describe explicitly the inverse of Ψ. But in this extended abstract, we shall only prove that Ψ is injective. Since the two considered sets have the same cardinality, this is enough to prove Proposition 12. In order to do that, we will construct a parallelogram polyomino P and the associated tree $\Psi(P)$ (actually, a non-ambiguous tree of shape $\Psi(P)$) at the same time. More precisely, when creating the parallelogram polyomino, we start from the origin of the two paths, and we add:

- one step to each of the two paths at a time in the parallelogram polyomino;
- the enlightened dot(s) corresponding to the inserted steps, when needed.

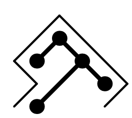


Fig. 16: An example of parallelogram polyomino with its tree under construction

Figure 16 shows an example of this construction.
Consider two different parallelogram polyominoes and construct them simultaneously, together with the associated non-ambiguous trees. While the beginning of the paths are the same, the associated trees are also the same. Consider the first time where one path in the first parallelogram polyomino differs from its homologous in the other parallelogram polyomino. One of the two added steps will be SW-oriented, and the other will be SE-oriented. This means that,

only one of these steps is associated with a new dot, connected to its parent v. The dot v exists in both trees, but it does not have the same number of children in both trees. □

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Network parameterizations for the Grassmannian

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Abstract. Deodhar introduced his decomposition of partial flag varieties as a tool for understanding Kazhdan-Lusztig polynomials. The Deodhar decomposition of the Grassmannian is also useful in the context of soliton solutions to the KP equation, as shown by Kodama and the second author. Deodhar components S_D of the Grassmannian are in bijection with certain tableaux D called *Go-diagrams*, and each component is isomorphic to $(\mathbb{K}^*)^a \times (\mathbb{K})^b$ for some non-negative integers a and b. Our main result is an explicit parameterization of each Deodhar component in the Grassmannian in terms of networks. More specifically, from a Go-diagram D we construct a weighted network N_D and its *weight matrix* W_D, whose entries enumerate directed paths in N_D. By letting the weights in the network vary over $\mathbb{K}$ or $\mathbb{K}^*$ as appropriate, one gets a parameterization of the Deodhar component S_D. One application of such a parameterization is that one may immediately determine which Plücker coordinates are vanishing and nonvanishing, by using the Lindstrom-Gessel-Viennot Lemma. We also give a (minimal) characterization of each Deodhar component in terms of Plücker coordinates.

Résumé. Deodhar a introduit une décomposition des variétés drapeaux pour comprendre les polynômes de Kazhdan-Lusztig. La décomposition de Deodhar des Grassmanniennes est aussi utile dans le contexte des solutions solitons de l'équation KP, ce qui a été établi par Kodama et le deuxième auteur. Les composantes de Deodhar S_D sont en bijection avec certains tableaux D appelés *diagrammes de Go*, et chaque composante est isomorphe à $(\mathbb{K}^*)^a \times (\mathbb{K})^b$ où a et b sont des entiers positifs. Notre résultat principal est une paramétrisation explicite de chaque composante de Deodhar des Grassmanniennes en termes de réseaux. Plus précisément, à partir d'un diagramme de Go D, nous construisons un réseau N_D et sa *matrice de poids* W_D, dont les composantes énumèrent les chemins dirigés dans N_D. En faisant varier les poids dans $\mathbb{K}$ ou $\mathbb{K}^*$, nous obtenons une paramétrisation de la composante de Deodhar S_D. Une application de cette paramétrisation est que nous pouvons déterminer quelles coordonnées de Plücker s'annulent, en utilisant le lemme de Lindstrom-Gessel-Viennot. Nous donnons aussi une caractérisation minimale de chaque composante en termes de coordonnées de Plücker.

Keywords: Grassmannian, Deodhar decomposition, networks

1 Introduction

There is a remarkable subset of the real Grassmannian $Gr_{k,n}(\mathbb{R})$ called its *totally non-negative part* $(Gr_{k,n})_{\geq 0}$ [7, 9], which may be defined as the subset of the real Grassmannian where all Plücker coordinates have the same sign. Postnikov showed that $(Gr_{k,n})_{\geq 0}$ has a decomposition into *positroid cells*,

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which are indexed by certain tableaux called Ⅎ-diagrams. He also gave explicit parameterizations of each cell. In particular, he showed that from each Ⅎ-diagram one can produce a planar *network*, and that one can write down a parameterization of the corresponding cell using the *weight matrix* of that network. This parameterization shows that the cell is isomorphic to $\mathbb{R}^d_{>0}$ for some d. Such a parameterization is convenient, because for example, one may read off formulas for Plücker coordinates from non-intersecting paths in the network, using the Lindstrom-Gessel-Viennot Lemma.

A natural question is whether these network parameterizations for positroid cells can be extended from $(Gr_{k,n})_{\geq 0}$ to the entire real Grassmannian $Gr_{k,n}(\mathbb{R})$. In this paper we give an affirmative answer to this question, by replacing the positroid cell decomposition with the Deodhar decomposition of the Grassmannian $Gr_{k,n}(\mathbb{K})$ (here $\mathbb{K}$ is an arbitrary field).

The components of the Deodhar decomposition are not in general cells, but nevertheless have a simple topology: by [2, 3], each one is isomorphic to $(\mathbb{K}^*)^a \times (\mathbb{K})^b$. The relation of the Deodhar decomposition of $Gr_{k,n}(\mathbb{R})$ to Postnikov's cell decomposition of $(Gr_{k,n})_{\geq 0}$ is as follows: the intersection of a Deodhar component $S_D \cong (\mathbb{R}^*)^a \times (\mathbb{R})^b$ with $(Gr_{k,n})_{\geq 0}$ is precisely one positroid cell isomorphic to $(\mathbb{R}_{>0})^a$ if $b = 0$, and is empty otherwise. In particular, when one intersects the Deodhar decomposition with $(Gr_{k,n})_{\geq 0}$, one obtains the positroid cell decomposition of $(Gr_{k,n})_{\geq 0}$. There is a related *positroid stratification* of the real Grassmannian, and each positroid stratum is a union of Deodhar components.

As for the combinatorics, components of the Deodhar decomposition are indexed by *distinguished subexpressions* [2, 3], or equivalently, by certain tableaux called *Go-diagrams* [6], which generalize Ⅎ-diagrams. In this paper we associate a network to each Go-diagram, and write down a parameterization of the corresponding Deodhar component using the weight matrix of that network. Our construction generalizes Postnikov's, but our networks are no longer planar in general.

Our main results can be summed up as follows. See Theorems 3.15 and 4.3 and the constructions preceding them for complete details.

Theorem. *Let $\mathbb{K}$ be an arbitrary field.*

- *Every point in $Gr_{k,n}(\mathbb{K})$ can be realized as the weight matrix of a unique network associated to a Go-diagram, and we can explicitly construct the corresponding network. The networks corresponding to points in the same Deodhar component have the same underlying graph, but different weights.*

- *Every Deodhar component may be characterized by the vanishing and nonvanishing of certain Plücker coordinates. Using this characterization, we can also explicitly construct the network associated to a point given either by a matrix repsresentative or by a list of Plücker coordinates.*

To illustrate the main results, we provide a small example here. More complicated examples may be seen throughout the rest of the paper.

Example 1.1. *Consider the Grassmannian $Gr_{2,4}$. The large Schubert cell in this Grassmannian can be characterized as*

$$\Omega_\lambda = \{A \in Gr_{2,4} \mid \Delta_{1,2}(A) \neq 0\},$$

where Δ_J denotes the Plücker coordinate corresponding to the column set J in a matrix representative of a point in $Gr_{2,4}$. This Schubert cell contains multiple positroid strata, including $S_{\mathcal{I}}$, where $\mathcal{I}$ is the Grassmann necklace $\mathcal{I} = (12, 23, 34, 14)$. This positroid stratum can also be characterized by the nonvanishing

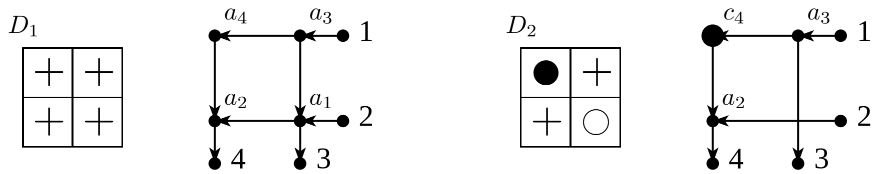

Fig. 1: The diagrams and networks associated to S_{D_1} and S_{D_2} in Example 1.1.

of certain Plücker coordinates:

$$S_{\mathcal{I}} = \{A \in Gr_{2,4} \mid \Delta_{1,2}(A) \neq 0,\ \Delta_{2,3}(A) \neq 0,\ \Delta_{3,4}(A) \neq 0,\ \Delta_{1,4}(A) \neq 0\}.$$

Figure 1 shows two Go-diagrams D_1 and D_2 and their associated networks. Note that the network on the right is not planar. The weight matrices associated to these diagrams are

$$\begin{pmatrix} 1 & 0 & -a_3 & -(a_3a_4 + a_3a_2) \\ 0 & 1 & a_1 & a_1a_2 \end{pmatrix} \textit{ and } \begin{pmatrix} 1 & 0 & -a_3 & -a_3c_4 \\ 0 & 1 & 0 & a_2 \end{pmatrix}.$$

The positroid stratum $S_{\mathcal{I}}$ is the disjoint union of the two corresponding Deodhar components S_{D_1} and S_{D_2}, which can be characterized in terms of vanishing and nonvanishing of minors as:

$$S_{D_1} = \{A \in S_{\mathcal{I}} \mid \Delta_{1,3} \neq 0\} \textit{ and } S_{D_2} = \{A \in S_{\mathcal{I}} \mid \Delta_{1,3} = 0\}.$$

Note that if one lets the a_i's range over $\mathbb{K}^*$ and lets c_4 range over $\mathbb{K}$, then we see that $S_{D_1} \cong (\mathbb{K}^*)^4$ and $S_{D_2} \cong (\mathbb{K}^*)^2 \times \mathbb{K}$.

There are several applications of our construction. First, as a special case of our theorem, one may parameterize all $k \times n$ matrices using networks. Second, by applying the Lindstrom-Gessel-Viennot Lemma to a given network, one may write down explicit formulas for Plücker coordinates in terms of collections of non-intersecting paths in the network. Third, building upon work of [6], we obtain (minimal) descriptions of Deodhar components in the Grassmannian, in terms of vanishing and nonvanishing of Plücker coordinates. It follows that each Deodhar component is a union of matroid strata.

Although less well known than the Schubert decomposition and matroid stratification, the Deodhar decomposition is very interesting in its own right. Deodhar's original motivation for introducing his decomposition was the desire to understand Kazhdan-Lusztig polynomials. In the flag variety, one may intersect two opposite Schubert cells, obtaining a Richardson variety, which Deodhar showed is a union of Deodhar components. Each Richardson variety $\mathcal{R}_{v,w}(q)$ may be defined over a finite field $\mathbb{K} = \mathbb{F}_q$, and in this case, the number of points determines the R-polynomials $R_{v,w}(q) = \#(\mathcal{R}_{v,w}(\mathbb{F}_q))$, introduced by Kazhdan and Lusztig [4] to give a recursive formula for the Kazhdan-Lusztig polynomials. Since each Deodhar component is isomorphic to $(\mathbb{F}_q^*)^a \times (\mathbb{F}_q)^b$ for some a and b, if one understands the decomposition of a Richardson variety into Deodhar components, then in principle one may compute the R-polynonomials and hence Kazhdan-Lusztig polynomials.

Another reason for our interest in the Deodhar decomposition is its relation to soliton solutions of the KP equation. It is well-known that from each point A in the real Grassmannian, one may construct a soliton solution $u_A(x, y, t)$ of the KP equation. It was shown in recent work of Kodama and the second author [6] that when the time variable t tends to $-\infty$, the combinatorics of the solution $u_A(x, y, t)$ depends precisely on which Deodhar component A lies in.

The outline of this paper is as follows. In Section 2, we give some background on the Grassmannian and its decompositions, including the Schubert decomposition, the positroid stratification, and the matroid stratification. In Section 3, we present our main construction: we explain how to construct a network from each diagram, then use that network to write down a parameterization of a subset of the Grassmannian that we call a network component. Our main result is that this network component coincides with the corresponding Deodhar component in the Grassmannian. Finally in Section 4 we give a characterization of Deodhar components in terms of the vanishing and nonvanishing of certain Plücker coordinates.

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2 Background on the Grassmannian

The *Grassmannian* $Gr_{k,n}$ is the space of all k-dimensional subspaces of an n-dimensional vector space $\mathbb{K}^n$. In this paper we will usually let $\mathbb{K}$ be an arbitrary field, though we will often think of it as $\mathbb{R}$ or $\mathbb{C}$. An element of $Gr_{k,n}$ can be viewed as a full-rank $k \times n$ matrix modulo left multiplication by nonsingular $k \times k$ matrices. In other words, two $k \times n$ matrices represent the same point in $Gr_{k,n}$ if and only if they can be obtained from each other by row operations. Let $\binom{[n]}{k}$ be the set of all k-element subsets of $[n] := \{1, \dots, n\}$. For $I \in \binom{[n]}{k}$, let $\Delta_I(A)$ be the *Plücker coordinate*, that is, the maximal minor of the $k \times n$ matrix A located in the column set I. The map $A \mapsto (\Delta_I(A))$, where I ranges over $\binom{[n]}{k}$, induces the *Plücker embedding* $Gr_{k,n} \hookrightarrow \mathbb{KP}^{\binom{n}{k}-1}$ into projective space.

We now describe several useful decompositions of the Grassmannian: the Schubert decomposition, the positroid stratification, and the matroid stratification. Note that the matroid stratification refines the positroid stratification, which refines the Schubert decomposition. The main subject of this paper is the *Deodhar* decomposition of the Grassmannian, which refines the positroid stratification, and is refined by the matroid stratification (as we prove in Corollary 4.4).

2.1 The Schubert decomposition of $Gr_{k,n}$

Throughout this paper, we identify partitions with their Young diagrams. Recall that the partitions λ contained in a $k \times (n-k)$ rectangle are in bijection with k-element subset $I \subset [n]$. The boundary of the Young diagram of such a partition λ forms a lattice path from the upper-right corner to the lower-left corner of the rectangle. Let us label the n steps in this path by the numbers $1, \dots, n$, and define $I = I(\lambda)$ as the set of labels on the k vertical steps in the path. Conversely, we let $\lambda(I)$ denote the partition corresponding to the subset I.

Definition 2.1. *For each partition λ contained in a $k \times (n-k)$ rectangle, we define the* Schubert cell

$$\Omega_\lambda = \{A \in Gr_{k,n} \mid I(\lambda) \textit{ is the lexicographically minimal subset such that } \Delta_{I(\lambda)}(A) \neq 0\}.$$

As λ ranges over the partitions contained in a $k \times (n-k)$ rectangle, this gives the Schubert decomposition *of the Grassmannian $Gr_{k,n}$, i.e.*

$$Gr_{k,n} = \bigsqcup_{\lambda \subset (n-k)^k} \Omega_\lambda.$$

We now define the *shifted linear order* $<_i$ (for $i \in [n]$) to be the total order on $[n]$ defined by

$$i <_i i+1 <_i i+2 <_i \cdots <_i n <_i 1 <_i \cdots <_i i-1.$$

One can then define *cyclically shifted Schubert cells* as follows.

Definition 2.2. *For each partition λ contained in a $k \times (n-k)$ rectangle, and each $i \in [n]$, we define the* cyclically shifted Schubert cell

$\Omega^i_\lambda = \{A \in Gr_{k,n} \mid I(\lambda)$ *is the lexicographically minimal subset with respect to* $<_i$ *such that* $\Delta_{I(\lambda)} \neq 0\}$.

2.2 The positroid stratification of $Gr_{k,n}$

The *positroid stratification* of the Grassmannian $Gr_{k,n}$ is obtained by taking the simultaneous refinement of the n Schubert decompositions with respect to the n shifted linear orders $<_i$. This stratification was first considered by Postnikov [9], who showed that the strata are conveniently described in terms of *Grassmann necklaces*, as well as *decorated permutations* and Ⅎ-*diagrams*. Postnikov coined the terminology *positroid* because the intersection of the positroid stratification of the real Grassmannian with the *totally non-negative part of the Grassmannian* $(Gr_{k,n})_{\geq 0}$ gives a cell decomposition of $(Gr_{k,n})_{\geq 0}$ (whose cells are called *positroid cells*).

Definition 2.3. *[9, Definition 16.1] A* Grassmann necklace *is a sequence $\mathcal{I} = (I_1, \ldots, I_n)$ of subsets $I_r \subset [n]$ such that, for $i \in [n]$, if $i \in I_i$ then $I_{i+1} = (I_i \setminus \{i\}) \cup \{j\}$, for some $j \in [n]$; and if $i \notin I_i$ then $I_{i+1} = I_i$. (Here indices i are taken modulo n.) In particular, we have $|I_1| = \cdots = |I_n|$, which is equal to some $k \in [n]$. We then say that $\mathcal{I}$ is a Grassmann necklace of* type (k, n).

Example 2.4. $\mathcal{I} = (1345, 3456, 3456, 4567, 4567, 1467, 1478, 1348)$ *is an example of a Grassmann necklace of type* $(4, 8)$.

Lemma 2.5. *[9, Lemma 16.3] Given $A \in Gr_{k,n}$, let $\mathcal{I}(A) = (I_1, \ldots, I_n)$ be the sequence of subsets in $[n]$ such that, for $i \in [n]$, I_i is the lexicographically minimal subset of $\binom{[n]}{k}$ with respect to the shifted linear order $<_i$ such that $\Delta_{I_i}(A) \neq 0$. Then $\mathcal{I}(A)$ is a Grassmann necklace of type (k, n).*

The *positroid stratification* of $Gr_{k,n}$ is defined as follows.

Definition 2.6. *Let $\mathcal{I} = (I_1, \ldots, I_n)$ be a Grassmann necklace of type (k, n). The* positroid stratum $S_{\mathcal{I}}$ *is defined to be*

$$S_{\mathcal{I}} = \{A \in Gr_{k,n} \mid \mathcal{I}(A) = \mathcal{I}\}.$$

Equivalently, each positroid stratum is an intersection of n cyclically shifted Schubert cells, that is,

$$S_{\mathcal{I}} = \bigcap_{i=1}^{n} \Omega^i_{\lambda(I_i)}\ .$$

Grassmann necklaces are in bijection with tableaux called Ⅎ-*diagrams*.

Definition 2.7. *[9, Definition 6.1] Fix k, n. A* Ⅎ*-diagram $(\lambda, D)_{k,n}$ of type (k, n) is a partition λ contained in a $k \times (n-k)$ rectangle together with a filling $D : \lambda \to \{0, +\}$ of its boxes which has the* Ⅎ*-property: there is no 0 which has a $+$ above it and a $+$ to its left.*(i) *(Here, "above" means above and in the same column, and "to its left" means to the left and in the same row.)*

In Figure 2 we give an example of a Ⅎ-diagram.

(i) This forbidden pattern is in the shape of a backwards L, and hence is denoted Ⅎ and pronounced "Le."

+	0	+	+
+	0	+	
0	0	+	
+	0		

Fig. 2: A Le-diagram $L = (\lambda, D)_{k,n}$.

2.3 The matroid stratification of $Gr_{k,n}$

Definition 2.8. *A* matroid *of* rank k *on the set* $[n]$ *is a nonempty collection* $\mathcal{M} \subset \binom{[n]}{k}$ *of* k*-element subsets in* $[n]$*, called* bases *of* $\mathcal{M}$*, that satisfies the* exchange axiom*:*
For any $I, J \in \mathcal{M}$ *and* $i \in I$ *there exists* $j \in J$ *such that* $(I \setminus \{i\}) \cup \{j\} \in \mathcal{M}$*.*

Given an element $A \in Gr_{k,n}$, there is an associated matroid $\mathcal{M}_A$ whose bases are the k-subsets $I \subset [n]$ such that $\Delta_I(A) \neq 0$.

Definition 2.9. *Let* $\mathcal{M} \subset \binom{[n]}{k}$ *be a matroid. The* matroid stratum $S_{\mathcal{M}}$ *is defined to be*

$$S_{\mathcal{M}} = \{A \in Gr_{k,n} \mid \Delta_I(A) \neq 0 \textit{ if and only if } I \in \mathcal{M}\}.$$

This gives a stratification of $Gr_{k,n}$ *called the* matroid stratification*, or* Gelfand-Serganova stratification*.*

Remark 2.10. *Clearly the matroid stratification refines the positroid stratification, which in turn refines the Schubert decomposition.*

3 The main result: network parameterizations from Go-diagrams

In this section we define certain tableaux called *Go-diagrams*, then explain how to parameterize the Grassmannian using networks associated to Go-diagrams. First we will define more general tableaux called *diagrams*.

3.1 Diagrams and networks

Definition 3.1. *Let* λ *be a partition contained in a* $k \times (n-k)$ *rectangle. A* diagram *in* λ *is an arbitrary filling of the boxes of* λ *with pluses* $+$*, black stones* ●*, and white stones* ○*.*

To each diagram D we associate a network N_D as follows.

Definition 3.2. *Let* λ *be a partition with* ℓ *boxes contained in a* $k \times (n-k)$ *rectangle, and let* D *be a diagram in* λ*. Label the boxes of* λ *from* 1 *to* ℓ*, starting from the rightmost box in the bottom row, then reading right to left across the bottom row, then right to left across the row above that, etc. The* (weighted) network N_D *associated to* D *is a directed graph obtained as follows:*

- *Associate an* internal vertex *to each* $+$ *and each* ●*;*
- *After labeling the southeast border of the Young diagram with the numbers* $1, 2, \ldots, n$ *(from northeast to southwest), associate a* boundary vertex *to each number;*

- *From each internal vertex, draw an edge right to the nearest $+$-vertex or boundary vertex;*
- *From each internal vertex, draw an edge down to the nearest $+$-vertex or boundary vertex;*
- *Direct all edges left and down. After doing so, k of the boundary vertices become* sources *and the remaining $n-k$ boundary vertices become* sinks*.*
- *If e is a horizontal edge whose left vertex is a $+$-vertex (respectively a ●-vertex) in box b, assign e the weight a_b (respectively c_b). We think of a_b and c_b as indeterminates, but later they will be elements of $\mathbb{K}^*$ and $\mathbb{K}$ respectively.*
- *If e is a vertical edge, assign e the weight 1.*

Note that in general such a directed graph is not planar, as two edges may cross over each other without meeting at a vertex. See Figure 3 for an example of a diagram and its associated network.

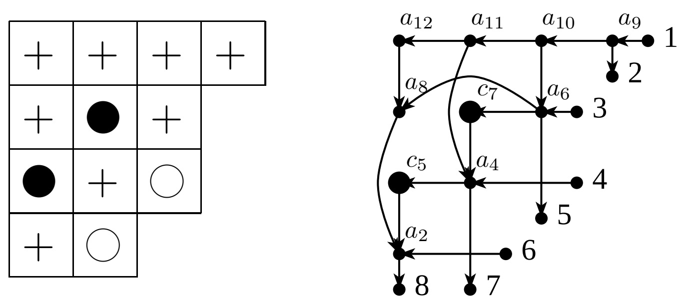

Fig. 3: An example of a diagram and its associated network.

We now explain how to associate a *weight matrix* to such a network.

Definition 3.3. *Let N_D be a network as in Definition 3.2. Let $I=\{i_1<i_2<\cdots<i_k\}\subset[n]$ denote the sources. If P is a directed path in the network, let $w(P)$ denote the product of all weights along P. If P is the empty path which starts and ends at the same boundary vertex, we let $w(P)=1$. If r is a source and s is any boundary vertex, define*

$$W_{rs}=\pm\sum_P w(P),$$

where the sum is over all paths P from r to s. The sign is chosen (uniquely) so that

$$\Delta_{I\setminus\{r\}\cup\{s\}}(W_D)=\sum_P w(P),\ \textit{where}$$

$$W_D=(W_{rs})$$

is the $k\times(n-k)$ weight matrix*. We make the convention that the rows of W_D are indexed by the sources $i_1,\dots,i_k$ from top to bottom, and its columns are indexed by $1,2,\dots,n$ from left to right.*

Example 3.4. *The weight matrix associated to the network in Figure 3 is*

$$\begin{pmatrix} 1 & a_9 & 0 & 0 & a_9a_{10} & 0 & -a_9a_{10}(a_{11}+c_7) & -a_9a_{10}(a_{11}a_{12}+a_{11}c_5+a_8+c_7c_5) \\ 0 & 0 & 1 & 0 & -a_6 & 0 & a_6c_7 & a_6a_8+a_6c_7c_5 \\ 0 & 0 & 0 & 1 & 0 & 0 & a_4 & -a_4c_5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & a_2 \end{pmatrix}$$

3.2 Distinguished expressions

We now review the notion of distinguished subexpressions, as in [2] and [8]. This definition will be essential for defining Go-diagrams. We assume the reader is familiar with the (strong) Bruhat order $<$ on $W = \mathfrak{S}_n$, and the basics of reduced expressions, as in [1].

Let $\mathbf{w} := s_{i_1} \dots s_{i_m}$ be a reduced expression for $w \in W$. A *subexpression* $\mathbf{v}$ of $\mathbf{w}$ is a word obtained from the reduced expression $\mathbf{w}$ by replacing some of the factors with 1. For example, consider a reduced expression in $\mathfrak{S}_4$, say $s_3s_2s_1s_3s_2s_3$. Then $s_3s_2\,1\,s_3s_2\,1$ is a subexpression of $s_3s_2s_1s_3s_2s_3$. Given a subexpression $\mathbf{v}$, we set $v_{(k)}$ to be the product of the leftmost k factors of $\mathbf{v}$, if $k \geq 1$, and $v_{(0)} = 1$.

Definition 3.5. *[8, 2] Given a subexpression $\mathbf{v}$ of a reduced expression $\mathbf{w} = s_{i_1}s_{i_2}\dots s_{i_m}$, we define*

$$\begin{aligned} J^{\circ}_{\mathbf{v}} &:= \{k \in \{1,\dots,m\} \mid v_{(k-1)} < v_{(k)}\}, \\ J^{+}_{\mathbf{v}} &:= \{k \in \{1,\dots,m\} \mid v_{(k-1)} = v_{(k)}\}, \\ J^{\bullet}_{\mathbf{v}} &:= \{k \in \{1,\dots,m\} \mid v_{(k-1)} > v_{(k)}\}. \end{aligned}$$

The expression $\mathbf{v}$ is called non-decreasing if $v_{(j-1)} \leq v_{(j)}$ for all $j = 1,\dots,m$, e.g. $J^{\bullet}_{\mathbf{v}} = \emptyset$.

Definition 3.6 (Distinguished subexpressions)**.** *[2, Definition 2.3] A subexpression $\mathbf{v}$ of $\mathbf{w}$ is called distinguished if we have*

$$v_{(j)} \leq v_{(j-1)}\, s_{i_j} \qquad \text{for all } j \in \{1,\dots,m\}. \tag{1}$$

In other words, if right multiplication by s_{i_j} decreases the length of $v_{(j-1)}$, then in a distinguished subexpression we must have $v_{(j)} = v_{(j-1)}s_{i_j}$.

We write $\mathbf{v} \prec \mathbf{w}$ if $\mathbf{v}$ is a distinguished subexpression of $\mathbf{w}$.

Definition 3.7 (Positive distinguished subexpressions)**.** *We call a subexpression $\mathbf{v}$ of $\mathbf{w}$ a positive distinguished subexpression (or a PDS for short) if*

$$v_{(j-1)} < v_{(j-1)}s_{i_j} \qquad \text{for all } j \in \{1,\dots,m\}. \tag{2}$$

In other words, it is distinguished and non-decreasing.

Lemma 3.8. *[8] Given $v \leq w$ and a reduced expression $\mathbf{w}$ for w, there is a unique PDS $\mathbf{v}_+$ for v in $\mathbf{w}$.*

3.3 Go-diagrams

In this section we explain how to index distinguished subexpressions by certain tableaux called *Go-diagrams*, which were introduced in [6]. Go-diagrams are fillings of Young diagrams by pluses $+$, *black stones* ●, and *white stones* ○.[ii]

[ii] In KW2, we used a slightly different convention and used blank boxes in place of $+$'s.

s_5	s_4	s_3	s_2	s_1
s_6	s_5	s_4	s_3	s_2
s_7	s_6	s_5	s_4	s_3

15	14	13	12	11
10	9	8	7	6
5	4	3	2	1

15	12	9	6	3
14	11	8	5	2
13	10	7	4	1

Fig. 4: The labeling of a the boxes of a partition by simple generators s_i, and two reading orders.

Fix k and n. Let $W_k = \langle s_1, s_2, \dots, \hat{s}_{n-k}, \dots, s_{n-1}\rangle$ be a parabolic subgroup of $W = \mathfrak{S}_n$. Let W^k denote the set of minimal-length coset representatives of W/W_k. Recall that a *descent* of a permutation π is a position j such that $\pi(j) > \pi(j+1)$. Then W^k is the subset of permutations of $\mathfrak{S}_n$ which have at most one descent; and that descent must be in position $n-k$.

It follows from [11] and [10] that elements w of W^k can be identified with partitions λ_w contained in a $k \times (n-k)$ rectangle. More specifically, let Q^k be the poset whose elements are the boxes of a $k \times (n-k)$ rectangle; if b_1 and b_2 are two adjacent boxes such that b_2 is immediately to the left or immediately above b_1, we have a cover relation $b_1 \lessdot b_2$ in Q^k. The partial order on Q^k is the transitive closure of $\lessdot$. Now label the boxes of the rectangle with simple generators s_i as in the figure below. If b is a box of the rectangle, then let s_b denote its label by a simple generator. Let $w_0^k \in W^k$ denote the longest element in W^k. Then the set of reduced expressions of w_0^k can be obtained by choosing a linear extension of Q^k and writing down the corresponding word in the s_i's. We call such a linear extension a *reading order*; two linear extensions are shown in the figure below. Additionally, given a partition λ contained in the $k \times (n-k)$ rectangle (chosen so that the upper-left corner of its Young diagram is aligned with the upper-left corner of the rectangle), and a linear extension of the sub-poset of Q^k comprised of the boxes of λ, the corresponding word in s_i's is a reduced expression of a minimal length coset representatives $w \in W^k$. The element $w \in W^k$ depends only on the partition, not the linear extension, and all reduced expressions of w can be obtained by varying the linear extension. Finally, this correspondence is a bijection between partitions λ_w contained in the $k \times (n-k)$ rectangle and elements $w \in W^k$.

Definition 3.9. *[6, Section 4] Fix k and n. Let $w \in W^k$, let $\mathbf{w}$ be a reduced expression for w, and let $\mathbf{v}$ be a distinguished subexpression of $\mathbf{w}$. Then w and $\mathbf{w}$ determine a partition λ_w contained in a $k \times (n-k)$ rectangle together with a reading order of its boxes. The* Go-diagram *associated to $\mathbf{v}$ and $\mathbf{w}$ is a filling of λ_w with pluses and black and white stones, such that: for each $k \in J^{\circ}_{\mathbf{v}}$ we place a white stone in the corresponding box; for each $k \in J^{\bullet}_{\mathbf{v}}$ we place a black stone in the corresponding box of λ_w; and for each $k \in J^{+}_{\mathbf{v}}$ we place a plus in the corresponding box of λ_w.*

Remark 3.10. *By [6, Section 4], the Go-diagram associated to $\mathbf{v}$ and $\mathbf{w}$ does not depend on $\mathbf{w}$, only on w. Moreover, whether or not such a filling of a partition λ_w is a Go-diagram does not depend on the choice of reading order of the boxes of λ_w.*

Definition 3.11. *We define the* standard reading order *of the boxes of a partition to be the reading order which starts at the rightmost box in the bottom row, then reads right to left across the bottom row, then right to left across the row above that, then right to left across the row above that, etc. This reading order is illustrated at the right of the figure below.*

By default, we will use the standard reading order in this paper.

Example 3.12. *Let $k = 3$ and $n = 7$, and let $\lambda = (4, 3, 1)$. The standard reading order is shown at the right of the figure below.*

s_4	s_3	s_2	s_1
s_5	s_4	s_3	
s_6			

8	7	6	5
4	3	2	
1			

Then the following diagrams are Go-diagrams of shape λ.

○	○	○	○
○	○	○	
○			

+	○	○	+
+	○	+	
○			

●	+	+	○
+	○	○	
+			

They correspond to the expressions $s_6s_3s_4s_5s_1s_2s_3s_4$, $s_61s_411s_2s_31$, and $1s_3s_41s_111s_4$. The first and second are positive distinguished subexpressions (PDS's), and the third one is a distinguished subexpression (but not a PDS).

Remark 3.13. *The Go-diagrams associated to PDS's are in bijection with ⅃-diagrams, see [6, Section 4]. Note that the Go-diagram associated to a PDS contains only pluses and white stones. This is precisely a ⅃-diagram.*

3.4 The main result

To state the main result, we now consider Go-diagrams (not arbitrary diagrams), the corresponding networks (*Go-networks*), and the corresponding weight matrices.

Definition 3.14. *Let D be a Go-diagram contained in a $k \times (n-k)$ rectangle. We define a subset $\mathcal{R}_D$ of the Grassmannian $Gr_{k,n}$ by letting each variable a_i of the weight matrix (Definition 3.3) range over all nonzero elements $\mathbb{K}^*$, and letting each variable c_i of the weight matrix range over all elements $\mathbb{K}$. We call $\mathcal{R}_D$ the* network component associated to *D.*

We will not define the Deodhar decomposition of the Grassmannian, but refer to [2, 3, 8] for details.

Theorem 3.15. *Let D be a Go-diagram contained in a $k \times (n-k)$ rectangle. Suppose that D has t pluses and u black stones. Then $\mathcal{R}_D$ is isomorphic to the corresponding Deodhar component, and in particular is isomorphic to $(\mathbb{K}^*)^t \times \mathbb{K}^u$. Furthermore, $Gr_{k,n}$ is the disjoint union of the network components $\mathcal{R}_D$, as D ranges over all Go-diagrams contained in a $k \times (n-k)$ rectangle. In other words, each point in the Grassmannian $Gr_{k,n}$ can be represented uniquely by a weighted network associated to a Go-diagram.*

Corollary 3.16. *Every matrix can be represented by a unique weighted network associated to a Go-diagram.*

4 A characterization of Deodhar components by minors

In this section we characterize Deodhar components in the Grassmannian by a list of vanishing and nonvanishing Plücker coordinates.

Definition 4.1. *[6, Definition 5.4] Let $W = \mathfrak{S}_n$, let $\mathbf{w} = s_{i_1} \dots s_{i_m}$ be a reduced expression for $w \in W^k$ and choose $\mathbf{v} \prec \mathbf{w}$. This determines a Go-diagram D of shape $\lambda = \lambda_w$. Let $I = I(\lambda)$. It is not hard to check that $I = w\{n, n-1, \dots, n-k+1\}$.*

Let b be any box of D. Note that the set of all boxes of D which are weakly southeast of b forms a Young diagram λ_b^{in}; also the complement of λ_b^{in} in λ is a Young diagram which we call λ_b^{out} (see Example 4.2 below). By looking at the restriction of $\mathbf{w}$ to the positions corresponding to boxes of λ_b^{in}, we obtained a reduced expression $\mathbf{w}_b^{\mathrm{in}}$ for some permutation w_b^{in}, together with a distinguished subexpression $\mathbf{v}_b^{\mathrm{in}}$ for some permutation v_b^{in}. Similarly, by using the positions corresponding to boxes of λ_b^{out}, we obtained $\mathbf{w}_b^{\mathrm{out}}$, w_b^{out}, $\mathbf{v}_b^{\mathrm{out}}$, and v_b^{out}. When the box b is understood, we will often omit the subscript b.

If b contains a $+$, define $I_b = v^{\mathrm{in}}(w^{\mathrm{in}})^{-1}I \in \binom{[n]}{k}$. If b contains a white or black stone, define $I_b = v^{\mathrm{in}} s_b (w^{\mathrm{in}})^{-1} I \in \binom{[n]}{k}$.

Example 4.2. *Let $W = \mathfrak{S}_7$ and $\mathbf{w} = s_4 s_5 s_2 s_3 s_4 s_6 s_5 s_1 s_2 s_3 s_4$ be a reduced expression for $w \in W^3$. Let $\mathbf{v} = s_4 s_5 1 1 s_4 1 s_5 s_1 1 1 s_4$ be a distinguished subexpression. So $w = (3, 5, 6, 7, 1, 2, 4)$ and $v = (2, 1, 3, 4, 6, 5, 7)$. We can represent this data by the poset λ_w and the corresponding Go-diagram:*

s_4	s_3	s_2	s_1
s_5	s_4	s_3	s_2
s_6	s_5	s_4	

●	$+$	$+$	○
●	○	$+$	$+$
$+$	○	○	

Let b be the box of the Young diagram which is in the second row and the second column (counting from left to right). Then the diagram below shows: the boxes of λ^{in} and λ^{out}; a reading order which puts the boxes of λ^{out} after those of λ^{in}; and the corresponding labeled Go-diagram. Using this reading order, $\mathbf{w}^{\mathrm{in}} = s_4 s_5 s_2 s_3 s_4$, $\mathbf{w}^{\mathrm{out}} = s_6 s_5 s_1 s_2 s_3 s_4$, $\mathbf{v}^{\mathrm{in}} = s_4 s_5 1 1 s_4$, and $\mathbf{v}^{\mathrm{out}} = 1 s_5 s_1 1 1 s_4$.

out	*out*	*out*	*out*
out	*in*	*in*	*in*
out	*in*	*in*	

11	10	9	8
7	5	4	3
6	2	1	

Theorem 4.3. *Let D be a Go-diagram of shape λ contained in a $k \times (n-k)$ rectangle. Let $A \in Gr_{k,n}$. Then A lies in the Deodhar component S_D if and only if the following conditions are satisfied:*

1. *$\Delta_{I_b}(A) = 0$ for all boxes in D containing a white stone.*
2. *$\Delta_{I_b}(A) \neq 0$ for all boxes in D containing a $+$.*
3. *$\Delta_{I(\lambda)}(A) \neq 0$.*
4. *$\Delta_J(A) = 0$ for all k-subsets J which are lexicographically smaller than $I(\lambda)$.*

Corollary 4.4. *The Deodhar decomposition of the Grassmannian is a coarsening of the matroid stratification: in other words, each Deodhar component is a union of matroid strata.*

Remark 4.5. *Theorem 4.3 implicitly gives an algorithm for determining the Deodhar component and corresponding network of a point of the Grassmannian, given by a matrix representative or by a list of its Plücker coordinates. The steps are as follows.*

1. *Find the lexicographically minimal nonzero Plücker coordinate Δ_I. Then the Go-diagram has shape $\lambda(I)$. Fix a reading order for this shape.*
2. *We determine how to fill each box, working in the reading order, as follows. First check whether the box b is forced to contain a black stone. If not, b must contain a white stone if $\Delta_{I(b)} = 0$, and b must contain a plus if $\Delta_{I(b)} \neq 0$. This process will completely determine the Go-diagram.*
3. *Given the Go-diagram, we know the underlying graph of the network. To determine the weights on horizontal edges, work in the reading order again. The Plücker coordinate $\Delta_{I(b)}$ will only use edge weights a_b (when b contains a $+$) or c_b (when b contains a black stone) and weights $a_{b'}$ and $c_{b'}$ corresponding to boxes b' which are earlier than b in the reading order. Thus, we may use the Lindström-Gessel-Viennot Lemma recursively to determine each weight a_b or c_b.*

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Cuts and Flows of Cell Complexes

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Abstract. We study the vector spaces and integer lattices of cuts and flows of an arbitrary finite CW complex, and their relationships to its critical group and related invariants. Our results extend the theory of cuts and flows in graphs, in particular the work of Bacher, de la Harpe and Nagnibeda. We construct explicit bases for the cut and flow spaces, interpret their coefficients topologically, and describe sufficient conditions for them to be integral bases of the cut and flow lattices. Second, we determine the precise relationships between the discriminant groups of the cut and flow lattices and the higher critical and cocritical groups; these are expressed as short exact sequences with error terms corresponding to torsion (co)homology. As an application, we generalize a result of Kotani and Sunada to give bounds for the complexity, girth, and connectivity of a complex in terms of Hermite's constant.

Resumé. Nous étudions les espaces vectoriels et les réseaux entiers des coupures et flots d'un CW-complexe arbitraire fini, et leur relations avec son groupe critical et invariants similaires. Nos résultats développent la théorie des coupures et flots dans les graphes, en particulier le travail de Bacher, de la Harpe et Nagnibeda. Nous construisons des bases explicites pour les espaces des coupures et des flots, donnons une description topologique de leurs coefficients, et décrivons conditions suffisants pour qu'ils soient des bases entières des réseaux des coupures et des flots. De plus, nous déterminons les relations précises entre les groupes discriminantes des réseaux, et les groupes critical et cocritical; ces relations prennent la forme des suites exactes courtes, avec termes correspondant à la torsion (co)homologie. Comme application, nous généralisons un résultat de Kotani et Sunada sur bornes pour la complexité, la circonférence, et la connectivité d'un CW-complexe en termes de la constante d'Hermite.

Keywords: cut lattice, flow lattice, critical group, spanning forest, cell complex

1 Introduction

This paper is about vector spaces, integer lattices of cuts and flows, and finite group invariants associated with a finite cell complex.

By way of background, the critical group of a graph is a finite abelian group whose order is the number of spanning forests. The definition was introduced independently in several different settings, including arithmetic geometry Lorenzini [1991], physics Dhar [1990], and algebraic geometry Bacher et al. [1997]

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(where it is also known as the Picard group or Jacobian group). It has received considerable recent attention for its connections to discrete dynamical systems, tropical geometry, and linear systems of curves; see, e.g., Baker and Norine [2007], Biggs [1999], Bond and Levine [2011], Haase et al. [2012].

In previous work, the authors studied cellular generalizations of the graph-theoretic concepts of spanning trees Duval et al. [2009, 2011a] and the critical group Duval et al. [2011b]. To summarize, a cellular spanning tree of a d-dimensional CW-complex Σ is a subcomplex $\Upsilon \subseteq \Sigma$ generated by facets corresponding to a column basis of the cellular boundary matrix $\partial : C_d(\Sigma) \to C_{d-1}(\Sigma)$. The critical group $K(\Sigma)$ is the torsion part of the cokernel of the combinatorial Laplacian $\partial\partial^*$, and its order is a weighted enumeration of the cellular spanning trees of Σ. Moreover, the action of the critical group on cellular $(d-1)$-cochains gives a model of discrete flow on Σ, generalizing the chip-firing and sandpile models; see, e.g., Biggs [1999], Dhar [1990].

The lattices $\mathcal{C}$ and $\mathcal{F}$ of integral cuts and flows of a graph were first defined in Bacher et al. [1997], in which the authors regarded a graph as an analogue of a Riemann surface and interpreted the discriminant groups $\mathcal{C}^\sharp/\mathcal{C}$ and $\mathcal{F}^\sharp/\mathcal{F}$ respectively as the Picard group of divisors and as the Jacobian group of holomorphic forms. In particular, they showed that the critical group $K(G)$ is isomorphic to both $\mathcal{C}^\sharp/\mathcal{C}$ and $\mathcal{F}^\sharp/\mathcal{F}$. Similar definitions and results appear in Biggs [1999].

Here, we define the cut and flow spaces and lattices of a cell complex Σ by

$$\begin{aligned} \operatorname{Cut}(\Sigma) &= \operatorname{im}_{\mathbb{R}} \partial^*, & \operatorname{Flow}(\Sigma) &= \ker_{\mathbb{R}} \partial, \\ \mathcal{C}(\Sigma) &= \operatorname{im}_{\mathbb{Z}} \partial^*, & \mathcal{F}(\Sigma) &= \ker_{\mathbb{Z}} \partial. \end{aligned}$$

In topological terms, cut- and flow-vectors are cellular coboundaries and cycles, respectively. Equivalently, the vectors in $\operatorname{Cut}(\Sigma)$ support sets of facets whose deletion increases the codimension-1 Betti number, and the vectors in $\operatorname{Flow}(\Sigma)$ support nontrivial rational homology classes. In the language of matroid theory, cuts and flows correspond to cocircuits and circuits, respectively, of the *cellular matroid* represented by the columns of ∂. Indeed, every cellular spanning tree $\Upsilon \subseteq \Sigma$ gives rise to a natural basis for each of the cut and flow spaces, whose elements are supported on fundamental cocircuits and circuits of Υ, respectively. In the graph case the coefficients of these basis vectors are ± 1; in the general case, they are (up to sign) the cardinalities of homology groups of cellular spanning trees obtained from Υ by matroid basis exchange. Under certain conditions, these vector space bases are in fact integral bases for the cut and flow lattices (Theorem 4.3).

The idea of studying cuts and flows of matroids goes back to Tutte [1965]. The recent work Su and Wagner [2010] defines cuts and flows of a regular matroid (i.e., one represented by a totally unimodular matrix M); when M is the boundary matrix of a cell complex, this is the case where the torsion coefficients are all trivial. Su and Wagner's definitions coincide with ours; their focus, however, is on recovering the structure of a matroid from the metric data of its flow lattice.

As we will see, the groups $\mathcal{C}^\sharp/\mathcal{C}$ and $\mathcal{F}^\sharp/\mathcal{F}$ are not necessarily isomorphic to each other. Their precise relationship involves several other groups: the critical group $K(\Sigma)$, a dually defined *cocritical group* $K^*(\Sigma)$, and the *cutflow group* $\mathbb{Z}^n/(\mathcal{C} \oplus \mathcal{F})$. We show (Theorem 5.5) that the critical and cocritical groups are respectively isomorphic to the discriminant groups of the cut lattice and flow lattice, and that the cutflow group mediates between them with an "error term" given by homology. The sizes of the critical and cocritical groups are respectively torsion-weighted enumerators for cellular spanning trees and for relatively acyclic subcomplexes. As an application of our theory, we generalize a theorem of Kotani and Sunada [2000] to obtain a geometric bound for the girth and complexity of a cellular matroid (Theorem 6.2).

This is an extended abstract of the full article Duval et al. [2012], to which the reader is referred for the proofs of all results stated herein.

2 Preliminaries

We assume that the reader is familiar with the basic topology of cell complexes. In general, we adopt the notation of Hatcher [2002] for chain groups, (co)homology, etc. Throughout the paper, Σ will denote a finite CW complex of dimension d. We adopt the convention that Σ has a unique cell of dimension -1 (as though it were an abstract simplicial complex); this will allow our results to specialize correctly to the case $d = 1$ (i.e., that Σ is a graph). We write Σ_i for the set of i-dimensional cells in Σ, and $\Sigma_{(i)}$ for the i-dimensional skeleton of Σ, i.e., $\Sigma_{(i)} = \Sigma_i \cup \Sigma_{i-1} \cup \cdots \cup \Sigma_0$. A cell of dimension d is called a *facet*. Unless otherwise stated, every d-dimensional subcomplex $\Gamma \subseteq \Sigma$ is assumed to have a full codimension-1 skeleton, i.e., $\Gamma_{(d-1)} = \Sigma_{(d-1)}$. Accordingly, for simplicity of notation, we will often make no distinction between the subcomplex Γ itself and its set Γ_d of facets. For a coefficient ring R, we say that Σ is *R-acyclic in codimension one* if $\tilde{H}_{d-1}(\Sigma; R) = 0$. For a graph ($d = 1$), both $\mathbb{Q}$- and $\mathbb{Z}$-acyclicity in codimension one are equivalent to connectedness. The i^{th} reduced Betti number is $\tilde{\beta}_i(\Sigma) = \dim \tilde{H}_i(\Sigma; \mathbb{Q})$, and the i^{th} torsion coefficient $\mathbf{t}_i(\Sigma)$ is the cardinality of the torsion subgroup $\mathbf{T}(\tilde{H}_i(\Sigma; \mathbb{Z}))$.

A *cellular spanning forest* (CSF) of Σ is a subcomplex $\Upsilon \subseteq \Sigma$ such that $\Upsilon_{(d-1)} = \Sigma_{(d-1)}$ and

$$\tilde{H}_d(\Upsilon; \mathbb{Z}) = 0, \tag{1a}$$

$$\operatorname{rank} \tilde{H}_{d-1}(\Upsilon; \mathbb{Z}) = \operatorname{rank} \tilde{H}_{d-1}(\Sigma; \mathbb{Z}), \quad \text{and} \tag{1b}$$

$$|\Upsilon_d| = |\Sigma_d| - \tilde{\beta}_d(\Sigma) + \tilde{\beta}_{d-1}(\Sigma). \tag{1c}$$

These conditions generalize the definition of a spanning forest of a graph G: respectively, it is acyclic, connected, and has $n - c$ edges, where n and c are the numbers of vertices and components of G. (By "spanning forest," we mean a maximal acyclic subgraph, not merely an acyclic subgraph containing all vertices.) Just as in the graphic case, any two of the conditions (1a), (1b), (1c) together imply the third. An equivalent definition is that a subcomplex $\Upsilon \subseteq \Sigma$ is a cellular spanning forest if and only if its d-cells correspond to a column basis for the cellular boundary matrix $\partial = \partial_d(\Sigma)$. In the case that Σ is $\mathbb{Q}$-acyclic in codimension one, this definition specializes to our earlier definition of a cellular spanning tree [Duval et al., 2011a, Definition 2.2].

The *complexity* of Σ is

$$\tau(\Sigma) = \tau_d(\Sigma) = \sum_{\text{CSFs } \Upsilon \subseteq \Sigma} |\mathbf{T}(\tilde{H}_{d-1}(\Upsilon; \mathbb{Z}))|^2. \tag{2}$$

When $d = 1$, this is just the number of spanning forests. More generally, the complexity can be calculated using a generalization of the matrix-tree theorem, as we now describe. Define the i^{th} *up-down*, *down-up* and *total* Laplacian operators of Σ by

$$L_i^{\mathrm{ud}} = \partial_{i+1}\partial_{i+1}^* \colon C_i(\Sigma) \to C_i(\Sigma), \qquad L_i^{\mathrm{du}} = \partial_i^*\partial_i \colon C_i(\Sigma) \to C_i(\Sigma), \qquad L_i^{\mathrm{tot}} = L_i^{\mathrm{ud}} + L_i^{\mathrm{du}}.$$

(These are discrete versions of the Laplacian operators on differential forms of a Riemannian manifold. The interested reader is referred to Eckmann [1945] and Dodziuk and Patodi [1976] for their origins in differential geometry and, e.g., Denham [2001], Friedman [1998], Merris [1994] for more recent appearances in combinatorics.) The *cellular matrix-tree theorem* [Duval et al., 2011a, Theorem 2.8] states that if Σ is $\mathbb{Q}$-acyclic in codimension one and $L_{\bar\Upsilon}$ is the submatrix of $L^{\mathrm{ud}}_{d-1}(\Sigma)$ obtained by deleting the rows and columns corresponding to the facets of a $(d-1)$-spanning tree Υ, then

$$\tau(\Sigma) = \frac{|\mathbf{T}(\tilde{H}_{d-2}(\Sigma;\mathbb{Z}))|^2}{|\mathbf{T}(\tilde{H}_{d-2}(\Upsilon;\mathbb{Z}))|^2} \det L_{\bar\Upsilon}.$$

One of our first results is that the condition $\tilde{H}_{d-1}(\Sigma;\mathbb{Q}) = 0$ can be dropped; see equation (3.3) below. Closely related results, also applicable to all cell complexes, appear in Catanzaro et al. [2012] and Lyons [2009].

A *lattice* $\mathcal{L}$ is a discrete subgroup of a finite-dimensional vector space V; that is, it is the set of integer linear combinations of some basis of V. (For background, see, e.g., [Artin, 1991, Chapter 12], [Godsil and Royle, 2001, Chapter 14], [Hungerford, 1980, Chapter IV].) Every lattice $\mathcal{L} \subseteq \mathbb{R}^n$ is isomorphic to $\mathbb{Z}^r$ for some integer $r \leq n$, called the *rank* of $\mathcal{L}$. The elements of $\mathcal{L}$ span a vector space denoted by $\mathcal{L}\otimes\mathbb{R}$. For $\mathcal{L} \subseteq \mathbb{Z}^n$, the *saturation* of $\mathcal{L}$ is defined as $\hat{\mathcal{L}} = (\mathcal{L}\otimes\mathbb{R})\cap\mathbb{Z}^n$. An *integral basis* of $\mathcal{L}$ is a set of linearly independent vectors $v_1,\dots,v_r \in \mathcal{L}$ such that $\mathcal{L} = \{c_1v_1+\cdots+c_rv_r\colon c_i \in \mathbb{Z}\}$. Fixing the standard inner product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^n$, we define the *dual lattice* of $\mathcal{L}$ by

$$\mathcal{L}^\sharp = \{v \in \mathcal{L}\otimes\mathbb{R}\colon\ \langle v,w\rangle \in \mathbb{Z}\ \ \forall w \in \mathcal{L}\}.$$

Note that $\mathcal{L}^\sharp$ can be identified with the dual $\mathbb{Z}$-module $\mathcal{L}^* = \mathrm{Hom}(\mathcal{L},\mathbb{Z})$, and that $(\mathcal{L}^\sharp)^\sharp = \mathcal{L}$. A lattice is called *integral* if it is contained in its dual; for instance, any subgroup of $\mathbb{Z}^n$ is an integral lattice. The *discriminant group* (or *determinantal group*) of an integral lattice $\mathcal{L}$ is $\mathcal{L}^\sharp/\mathcal{L}$; its cardinality can be calculated as $\det M^TM$, for any matrix M whose columns form an integral basis of $\mathcal{L}$.

Many of our results of the paper may be expressed in the language of matroids, with which we assume the reader is familiar (for a general reference on matroids, see, e.g., Oxley [1992]). We adopt the following notation. If $\mathcal{M}$ is a matroid on ground set E and B is a basis of $\mathcal{M}$, then for every $e \in B$the *fundamental bond* (or *fundamental cocircuit*) of e with respect to B is

$$\mathsf{bo}(B,e) = \{f \in E\colon B\setminus\{e\}\cup\{f\} \text{ is a basis of } \mathcal{M}$$

and for every $e \notin B$ the *fundamental circuit* is

$$\mathsf{ci}(B,e) = \{f \in E\colon B\cup\{e\}\setminus\{f\} \text{ is a basis of } \mathcal{M}.$$

The *cellular matroid* of Σ is the matroid $\mathcal{M}(\Sigma)$ represented over $\mathbb{R}$ by the columns of the boundary matrix ∂. Thus the ground set of $\mathcal{M}(\Sigma)$ naturally corresponds to the d dimensional cells Σ_d, and $\mathcal{M}(\Sigma)$ records which sets of columns of ∂ are linearly independent. If Σ is a graph, then $\mathcal{M}(\Sigma)$ is its usual graphic matroid, while if Σ is a simplicial complex then $\mathcal{M}(\Sigma)$ is its simplicial matroid (see Cordovil and Lindström [1987]). The bases of $\mathcal{M}(\Sigma)$ are the collections of facets of cellular spanning forests of Σ. If r is the rank function of the matroid $\mathcal{M}(\Sigma)$, then for each set of facets $B \subseteq \Sigma_d$, we have $r(B) = \operatorname{rank}\partial_B$, where ∂_B is the submatrix consisting of the columns indexed by the facets in B. Moreover, we have

$$r(\Sigma) := r(\Sigma_d) = \operatorname{rank}\mathcal{M}(\Sigma) = \operatorname{rank}\partial = |\Sigma_d| - \tilde\beta_d(\Sigma)$$

by the definition of Betti number.

A set of facets $B \subsetneq \Sigma_d$ is called a *cut* if deleting B from Σ increases its codimension-one homology, i.e., $\tilde{\beta}_{d-1}(\Sigma \setminus B) > \tilde{\beta}_{d-1}(\Sigma)$. A cut B is a *bond* if $r(\Sigma \setminus B) = r(\Sigma) - 1$, but $r((\Sigma \setminus B) \cup \sigma) = r(\Sigma)$ for every $\sigma \in B$. That is, a bond is a minimal cut. In matroid terminology, a bond of Σ is precisely a cocircuit of $\mathcal{M}(\Sigma)$, i.e., a minimal set that meets every basis of $\mathcal{M}(\Sigma)$. Equivalently, a bond is the complement of a flat of rank $r(\Sigma) - 1$.

It is important to point out that the cut and flow spaces and lattices of a complex Σ are not matroid invariants, i.e., they are not determined by the cellular matroid $\mathcal{M}(\Sigma)$. (See Su and Wagner [2010] for more on this subject.) Below is a table collecting some of the standard terminology from linear algebra, graph theory, and matroid theory, along with the analogous concepts that we will be using for cell complexes.

Linear algebra	**Graph**	**Matroid**	**Cell complex**
Column vectors	Edges	Ground set	Facets
Independent set	Acyclic subgraph	Independent set	Acyclic subcomplex
Min linear dependence	Cycle	Circuit	Circuit
Basis	Spanning forest	Basis	CSF
Set meeting all bases	Disconnecting set	Codependent set	Cut
Min set meeting all bases	Bond	Cocircuit	Bond
Rank	# edges in spanning forest	Rank	# facets in CSF

Here "codependent" means dependent in the dual matroid.

3 Enumerating Cellular Spanning Forests

Our first result generalizes the simplicial and cellular matrix-tree theorems of Duval et al. [2009] and Duval et al. [2011a] (where we required that Σ be $\mathbb{Q}$-acyclic in codimension one). Closely related results have been obtained independently Catanzaro et al. [2012] and Lyons [2009].

Definition 3.1 *Let Σ be a d-dimensional cell complex with rank r. Let $\Gamma \subseteq \Sigma$ be a subcomplex of dimension less than or equal to $d-1$ such that $\Gamma_{(d-2)} = \Sigma_{(d-2)}$. We say that Γ is* relatively acyclic *if the inclusion map $i \colon \Gamma \to \Sigma$ induces isomorphisms $i_* \colon \tilde{H}_k(\Gamma; \mathbb{Q}) \to \tilde{H}_k(\Sigma; \mathbb{Q})$ for all $k < d$.*

By the long exact sequence for relative homology, Γ is relatively acyclic if and only if $\tilde{H}_d(\Sigma; \mathbb{Q}) \to \tilde{H}_d(\Sigma, \Gamma; \mathbb{Q})$ is an isomorphism and $\tilde{H}_k(\Sigma, \Gamma; \mathbb{Q}) = 0$ for all $k < d$. These conditions can occur only if $|\Gamma_{d-1}| = |\Sigma_{d-1}| - r$. This quantity may be zero (in which case the only relatively acyclic subcomplex is $\Sigma_{(d-2)}$). A relatively acyclic subcomplex is precisely the complement of a $(d-1)$-cobase (a basis of the matroid represented over $\mathbb{R}$ by the rows of the boundary matrix ∂) in the terminology of Lyons [2009]. Two special cases are worth noting. First, if $d = 1$, then a relatively acyclic complex consists of one vertex in each connected component. Second, if $\tilde{H}_{d-1}(\Sigma; \mathbb{Q}) = 0$, then Γ is relatively acyclic if and only if it is a cellular spanning forest of $\Sigma_{(d-1)}$.

For a matrix M, denote by $M_{A,B}$ the submatrix with rows A and columns B.

Proposition 3.2 *Let Σ be a d-dimensional cell complex, let $\Upsilon \subseteq \Sigma$ be a cellular spanning forest, and let $\Gamma \subseteq \Sigma$ be a relatively acyclic $(d-1)$-subcomplex. Then*

$$\mathbf{t}_{d-1}(\Upsilon)\, \mathbf{t}_{d-1}(\Sigma, \Gamma) = \mathbf{t}_{d-1}(\Sigma)\, \mathbf{t}_{d-1}(\Upsilon, \Gamma).$$

The proof uses the following observation: the maximal nonsingular square submatrices of ∂ are precisely those whose columns correspond to a cellular spanning tree Υ and whose rows correspond to a relatively acyclic subcomplex Γ, and in this case the determinant of such a matrix is (up to sign) the cardinality of the relative complex (Υ, Γ).

Expanding $\det L_\Gamma$ with the Binet-Cauchy formula and using the fact that $|\det \partial_{\Gamma,\Upsilon}| = \mathbf{t}_{d-1}(\Upsilon, \Gamma)$ (where Υ is a cellular spanning tree and Γ is a subcomplex generated by r ridges) implies the following formula for the complexity of Σ.

Proposition 3.3 *Let Σ be a d-dimensional cell complex and let $\Gamma \subseteq \Sigma$ be a relatively acyclic $(d-1)$-dimensional subcomplex, and let L_Γ be the restriction of $L^{\mathrm{ud}}_{d-1}(\Sigma)$ to the $(d-1)$-cells of Γ. Then*

$$\tau_d(\Sigma) = \frac{\mathbf{t}_{d-1}(\Sigma)^2}{\mathbf{t}_{d-1}(\Sigma,\Gamma)^2} \det L_\Gamma.$$

As an example of the usefulness of homological techniques, suppose that $\tilde{H}_{d-1}(\Sigma;\mathbb{Z})$ is purely torsion (a frequent case). Then the relative homology sequence of the pair (Σ, Γ) gives rise to the exact sequence

$$0 \to \mathbf{T}(\tilde{H}_{d-1}(\Sigma;\mathbb{Z})) \to \mathbf{T}(\tilde{H}_{d-1}(\Sigma,\Gamma;\mathbb{Z})) \to \mathbf{T}(\tilde{H}_{d-2}(\Gamma;\mathbb{Z})) \to \mathbf{T}(\tilde{H}_{d-2}(\Sigma;\mathbb{Z})) \to 0$$

which implies that $\mathbf{t}_{d-1}(\Sigma)/\mathbf{t}_{d-1}(\Sigma,\Gamma) = \mathbf{t}_{d-2}(\Sigma)/\mathbf{t}_{d-2}(\Gamma)$. Thus, Proposition 3.3 becomes the formula $\tau_d(\Sigma) = \frac{\mathbf{t}_{d-2}(\Sigma)^2}{\mathbf{t}_{d-2}(\Gamma)^2} \det L_\Gamma$, which was one of the original versions of the cellular matrix-tree theorem [Duval et al., 2011a, Theorem 2.8(2)] (see also Catanzaro et al. [2012] and Lyons [2009]).

4 Bases of the Cut and Flow Spaces

As before, let Σ be a cell complex of dimension d and rank r; that is, every cellular spanning forest of Σ has r facets), and identify cellular chains and cochains by the standard inner product. We wish to construct combinatorially meaningful bases for the cut space $\operatorname{im} \partial_d^*$ and the flow space $\ker \partial_d$. We first recall the construction in the case of a graph.

There are two natural ways to construct bases of the cut space of a graph. First, if G is a graph on vertex set V and R is a set of ("root") vertices, one in each connected component, then the rows of ∂ corresponding to the vertices $V \setminus R$ form a basis for $\mathrm{Cut}_1(G)$. This observation generalizes easily to cell complexes: a set of r rows of the top-dimensional boundary matrix forms a row basis if and only if the corresponding set of $(d-1)$-cells is the complement of a relatively acyclic $(d-1)$-subcomplex.

Second, for every spanning tree of a graph the signed characteristic vectors of its fundamental bonds form a basis of its cut space (see [Godsil and Royle, 2001, Chapter 14]) In the more general setting of a cell complex Σ, it is relatively straightforward to show that each bond in Σ supports a one-dimensional subspace of $\mathrm{Cut}(\Sigma)$.

Theorem 4.1 *Let Σ be a d-dimensional cell complex with top boundary map ∂, and let $L = L^{\mathrm{du}}_d(\Sigma) = \partial^*\partial$. Let $\Upsilon = \{\sigma_1, \sigma_2, \ldots, \sigma_r\}$ be a cellular spanning tree, and $\sigma = \sigma_i \in \Upsilon$. Then:*

1. *The vector*
$$\bar{\chi}(\Upsilon, \sigma_i) = \sum_{j=1}^{r} (-1)^j (\det L_{\Upsilon\setminus\sigma_i, \Upsilon\setminus\sigma_j}) L\sigma_j \ \in\ C_d(\Sigma)$$
spans the space of all cut-vectors with support contained in the fundamental bond $\mathsf{bo}(\Upsilon, \sigma_i)$.

2. *The set $\{\bar{\chi}(\Upsilon, \sigma_1), \ldots, \bar{\chi}(\Upsilon, \sigma_r)\}$ is a vector space basis for* $\mathrm{Cut}(\Sigma)$.

3. *The coefficients of $\bar{\chi}(\Upsilon, \sigma_i)$ have the following interpretation. Let*

$$\mu = \mu_\Upsilon = \mathbf{t}_{d-1}(\Upsilon) \sum_{\Gamma} \frac{\mathbf{t}_{d-1}(\Sigma, \Gamma)^2}{\mathbf{t}_{d-1}(\Sigma)^2}.$$

Then

$$\bar{\chi}(\Upsilon, \sigma) \;=\; \sum_{\rho \in \mathsf{bo}(\Upsilon,\sigma)} (\det L_{\Upsilon \setminus \sigma \cup \rho, \Upsilon})\rho \;=\; \mu \sum_{\rho \in \mathsf{bo}(\Upsilon,\sigma)} (\pm \mathbf{t}_{d-1}(\Upsilon \setminus \sigma \cup \rho))\rho.$$

The vectors $\chi(\Upsilon, \sigma) = \bar{\chi}(\Upsilon, \sigma)/\mu$ are the cellular analogues of signed characteristic vectors of bonds in graphs. (Note that if indeed $d = 1$, then all the torsion coefficients are 1; μ is just the number of vertices of Σ; and for any edge σ in Υ, the vector $\chi(\Upsilon, \sigma)$ is the usual signed characteristic vector of the fundamental bond $\mathsf{bo}(\Upsilon, \sigma)$.)

Torsion plays a role in the characteristic vectors of bonds, even when Σ is a simplicial complex. For example, let Σ be the the complete 2-dimensional simplicial complex on 6 vertices and let Υ be the triangulation of $\mathbb{RP}^2$ obtained by identifying opposite faces in an icosahedron. Then Υ is a cellular spanning forest of Σ (and in fact Σ has twelve spanning forests of this kind). For any facet $\sigma \in \Upsilon$, we have $\mathsf{bo}(\Upsilon, \sigma) = \Sigma_2 \setminus \Upsilon_2 \cup \{\sigma\}$, and the entries of the calibrated cut-vector include both ± 2 (in position σ) as well as ± 1's (in positions $\Sigma \setminus \Upsilon$).

The analogous theorem for the flow space is as follows.

Theorem 4.2 *Let Σ be a d-dimensional cell complex with top boundary map ∂. For every circuit C in the cellular matroid, the space of flow vectors supported on a subset of C is one-dimensional, spanned by*

$$\varphi(C) = \sum_{\sigma \in C} \pm \mathbf{t}_{d-1}(\Delta \setminus \sigma)\, \sigma.$$

Moreover, for every cellular spanning forest $\Upsilon \subseteq \Sigma$, the set $\{\varphi(\mathsf{ci}(\Upsilon, \sigma)) \colon \sigma \notin \Upsilon\}$ is a vector space basis for $\mathrm{Flow}(\Sigma)$.

The argument is easier for the flow space than for the cut space; for instance, the explicit formula for $\varphi(C)$ is essentially a calculation using Cramer's rule.

In the graph case, these combinatorial bases of the cut and flow spaces are in fact integral bases of the lattices $\mathcal{C}$ and $\mathcal{F}$ respectively. In the cellular case, the possibility of torsion requires additional assumptions. Specifically:

Theorem 4.3 *Let Υ be a cellular spanning forest of Σ.*

1. *If $\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})$ is torsion-free, then $\{\chi(\Upsilon, \sigma) \colon \sigma \in \Upsilon\}$ is an integral basis for the cut lattice $\mathcal{C}(\Sigma)$.*

2. *If $\tilde{H}_{d-1}(\Upsilon; \mathbb{Z}) = \tilde{H}_{d-1}(\Sigma; \mathbb{Z})$, then $\{\hat{\varphi}(\Upsilon, \sigma) \colon \sigma \notin \Upsilon\}$ is an integral basis for the flow lattice $\mathcal{F}(\Sigma)$, where $\hat{\varphi}$ denotes φ divided by the g.c.d. of its coefficients.*

5 Groups and Lattices

In this section, we define the critical, cocritical, and cutflow groups of a cell complex. We identify the relationships between these groups and to the discriminant groups of the cut and flow lattices. The graphical case was studied in detail in Bacher et al. [1997] and Biggs [1999], and is presented concisely in [Godsil and Royle, 2001, Chapter 14]. Throughout the section, all (co)chain and (co)homology groups are assume to have coefficients in $\mathbb{Z}$; in particular, we identify both $C_d(\Sigma;\mathbb{Z})$ and $C^d(\Sigma;\mathbb{Z})$ with $\mathbb{Z}^n$.

Definition 5.1 *The* critical group *of* Σ *is*

$$K(\Sigma) := \mathbf{T}(\ker\partial_{d-1}/\operatorname{im}\partial_d\partial_d^*) = \mathbf{T}(\operatorname{coker}(\operatorname{im}\partial_d\partial_d^*)).$$

Definition 5.2 *The* cutflow group *of* Σ *is* $\mathbb{Z}^n/(\mathcal{C}(\Sigma)\oplus\mathcal{F}(\Sigma))$.

In order to define the cocritical group of a cell complex, we need the notion of an *acyclization*. of Σ, which is a $(d+1)$-dimensional complex Ω such that $\Omega_{(d)} = \Sigma$ and $\tilde{H}_{d+1}(\Omega;\mathbb{Z}) = \tilde{H}_d(\Omega;\mathbb{Z}) = 0$. Algebraically, this construction corresponds to finding an integral basis for $\ker\partial_d(\Sigma)$ and declaring its elements to be the columns of $\partial_{d+1}(\Omega)$ (so in particular $|\Omega_{(d+1)}| = \tilde{\beta}_d(\Sigma)$). The definition of acyclization and the universal coefficient theorem for cohomology together imply that $\tilde{H}^{d+1}(\Omega;\mathbb{Z}) = 0$; that is, $\partial_{d+1}^*(\Omega)$ is surjective.

Definition 5.3 *The* cocritical group $K^*(\Sigma)$ *is*

$$K^*(\Sigma) := C_{d+1}(\Omega;\mathbb{Z})/\operatorname{im}\partial_{d+1}^*\partial_{d+1} = \operatorname{coker} L_{d+1}^{\mathrm{du}}.$$

It is not immediate that the group $K^*(\Sigma)$ is independent of the choice of Ω; we will prove this independence as part of Theorem 5.5. For the moment, it is at least clear that $K^*(\Sigma)$ is finite, since $\operatorname{rank}\partial_{d+1}^* = \operatorname{rank} L_{d+1}^{\mathrm{du}} = \operatorname{rank} C_{d+1}(\Omega;\mathbb{Z})$. In the special case of a graph, the cocritical group coincides with the discriminant group of the lattice generated by the columns of the "intersection matrix" defined by Kotani and Sunada Kotani and Sunada [2000]. (See also [Biggs, 2007, Sections 2, 3].)

Remark 5.4 *As in Duval et al. [2011b], one can define critical and cocritical groups in every dimension by*

$$K_i(\Sigma) = \mathbf{T}(C_i(\Sigma;\mathbb{Z})/\operatorname{im}\partial_{i+1}\partial_{i+1}^*), \qquad K_i^*(\Sigma) = \mathbf{T}(C_i(\Sigma;\mathbb{Z})/\operatorname{im}\partial_i^*\partial_i).$$

If the cellular chain complexes of Σ *and* Ψ *are algebraically dual (for example, if* Σ *and* Ψ *are Poincaré dual cell structures on a compact orientable* d*-manifold), then* $K_i(\Psi) = K_{d-i}^*(\Sigma)$ *for all* i.

Our main theorem states that the critical and cocritical groups are isomorphic to the discriminant groups of the cut and flow lattices respectively, and the cutflow group mediates between the critical and cocritical groups, with an "error term" given by torsion. Specifically:

Theorem 5.5 *Let* Σ *be a cell complex of dimension* d *with* n *facets. Then there are short exact sequences*

$$0 \to \mathbb{Z}^n/(\mathcal{C}\oplus\mathcal{F}) \to \mathcal{C}^\sharp/\mathcal{C} \cong K(\Sigma) \to \mathbf{T}(\tilde{H}^d(\Sigma;\mathbb{Z})) \to 0$$

and

$$0 \to \mathbf{T}(\tilde{H}_{d-1}(\Sigma;\mathbb{Z})) \to \mathbb{Z}^n/(\mathcal{C}\oplus\mathcal{F}) \to \mathcal{F}^\sharp/\mathcal{F} \cong K^*(\Sigma) \to 0.$$

In fact, the error terms $\mathbf{T}(\tilde{H}^d(\Sigma;\mathbb{Z}))$ and $\mathbf{T}(\tilde{H}_{d-1}(\Sigma;\mathbb{Z}))$ are in fact isomorphic, by a special case of the universal coefficient theorem for cohomology [Hatcher, 2002, p. 205, Corollary 3.3].

Corollary 5.6 *If* $\tilde{H}_{d-1}(\Sigma;\mathbb{Z})$ *is torsion-free, then the groups* $K(\Sigma)$*,* $K^*(\Sigma)$*,* $\mathcal{C}^\sharp/\mathcal{C}$*,* $\mathcal{F}^\sharp/\mathcal{F}$*, and* $\mathbb{Z}^n/(\mathcal{C}\oplus\mathcal{F})$ *are all isomorphic to each other.*

Corollary 5.6 includes the case that Σ is a graph, as studied in Bacher et al. [1997] and Biggs [1999]. It also includes the combinatorially important family of Cohen-Macaulay simplicial complexes, as well as cellulations of compact orientable manifolds.

Example 5.7 *Suppose that $\tilde{H}_d(\Sigma;\mathbb{Z}) = \mathbb{Z}$ and that $\tilde{H}_{d-1}(\Sigma;\mathbb{Z})$ is torsion-free. Then the flow lattice is generated by a single element, and it follows from Corollary 5.6 that $K(\Sigma) \cong K^*(\Sigma) \cong \mathcal{F}^\sharp/\mathcal{F}$ is a cyclic group. For instance, if Σ is homeomorphic to a cellular sphere or torus, then the critical group is cyclic of order equal to the number of facets. (The authors had previously proved this fact for simplicial spheres [Duval et al., 2011b, Theorem 3.7], but this approach using the cocritical group makes the statement more general and the proof transparent.)*

Example 5.8 *Let Σ be the standard cellulation $e^0 \cup e^1 \cup e^2$ of the real projective plane, whose cellular chain complex is*

$$\mathbb{Z} \xrightarrow{\partial_2 = 2} \mathbb{Z} \xrightarrow{\partial_1 = 0} \mathbb{Z}.$$

Then $\mathcal{C} = \operatorname{im} \partial_2^* = 2\mathbb{Z}$, $\mathcal{C}^\sharp = \frac{1}{2}\mathbb{Z}$, and $K(\Sigma) = \mathcal{C}^\sharp/\mathcal{C} = \mathbb{Z}_4$. Meanwhile, $\mathcal{F} = \mathcal{F}^\sharp = \mathcal{F}^\sharp/\mathcal{F} = K^*(\Sigma) = 0$. The cutflow group is $\mathbb{Z}_2$. Note that the rows of the Theorem 5.5 are not split in this case.

Example 5.9 *Let $a, b \in \mathbb{Z} \setminus \{0\}$. Let Σ be the cell complex whose cellular chain complex is*

$$\mathbb{Z} \xrightarrow{\partial_2 = [a\ b]} \mathbb{Z} \xrightarrow{\partial_1 = 0} \mathbb{Z}.$$

Topologically, Σ consists of a vertex e^0, a loop e^1, and two facets of dimension 2 attached along e^1 by maps of degrees a and b. Then

$$\mathcal{C}^\sharp/\mathcal{C} = \mathbb{Z}_\tau, \qquad \mathbb{Z}^2/(\mathcal{C} \oplus \mathcal{F}) = \mathbb{Z}_{\tau/g}, \qquad \mathcal{F}^\sharp/\mathcal{F} = \mathbb{Z}_{\tau/g^2},$$

where $\tau = a^2 + b^2$ and $g = \gcd(a, b)$. Note that $\tau = \tau_2(\Sigma)$ is the complexity of Σ (see equation (2)) and that $g = |\tilde{H}_1(\Sigma;\mathbb{Z})|$. The short exact sequence for $K^*(\Sigma)$ of Theorem 5.5 is in general not split (for example, if $a = 6$ and $b = 2$).

For a connected graph, the cardinality of the critical group equals the number of spanning trees. In the cellular case, Examples 5.8 and 5.9 both indicate that $K(\Sigma) \cong \mathcal{C}^\sharp/\mathcal{C}$ should have cardinality equal to the complexity $\tau(\Sigma)$. Indeed, in Theorem 4.2 of Duval et al. [2011b], the authors proved that $|K(\Sigma)| = \tau(\Sigma)$ whenever Σ has a cellular spanning tree Υ such that $\tilde{H}_{d-1}(\Upsilon;\mathbb{Z}) = \tilde{H}_{d-1}(\Sigma;\mathbb{Z}) = 0$ (in particular, Σ must be not merely $\mathbb{Q}$-acyclic, but actually $\mathbb{Z}$-acyclic, in codimension one). We prove that this condition holds true for any cell complex. Together with Theorem 5.5, we obtain:

Theorem 5.10 *Let Σ be a d-dimensional cell complex and let $\mathbf{t} = \mathbf{t}_{d-1}(\Sigma) = |\mathbf{T}(\tilde{H}_{d-1}(\Sigma;\mathbb{Z}))|$. Then*

$$\begin{aligned} |\mathcal{C}^\sharp/\mathcal{C}| = |K(\Sigma)| &= \tau_d(\Sigma), \\ |\mathbb{Z}^n/(\mathcal{C} \oplus \mathcal{F})| &= \tau_d(\Sigma)/\mathbf{t}, \quad \textit{and} \\ |\mathcal{F}^\sharp/\mathcal{F}| = |K^*(\Sigma)| &= \tau_d(\Sigma)/\mathbf{t}^2. \end{aligned}$$

Dually, we can interpret the cardinality of the cocritical group as enumerating cellular spanning forests by relative torsion (co)homology:

Theorem 5.11 *Let Ω be an acyclization of Σ. Then*

$$|K^*(\Sigma)| = \sum_{\Upsilon} |\tilde{H}^{d+1}(\Omega, \Upsilon; \mathbb{Z})|^2 = \sum_{\Upsilon} |\tilde{H}_d(\Omega, \Upsilon; \mathbb{Z})|^2$$

with the sums over all cellular spanning forests $\Upsilon \subseteq \Sigma$.

Note that the groups $\tilde{H}^{d+1}(\Omega, \Upsilon; \mathbb{Z})$ and $\tilde{H}_d(\Omega, \Upsilon; \mathbb{Z})$ are all finite, by definition of acyclization.

Remark 5.12 *Let $\tau^*(\Sigma) = \sum_{\Upsilon} |\tilde{H}_d(\Omega, \Upsilon; \mathbb{Z})|^2$, as in Theorem 5.11. Then combining Theorems 5.10 and Theorems 5.11 gives*

$$\begin{aligned} |\mathcal{C}^\sharp/\mathcal{C}| = |K(\Sigma)| = \tau(\Sigma) \quad &= \tau^*(\Sigma) \cdot \mathbf{t}^2, \\ |\mathcal{F}^\sharp/\mathcal{F}| = |K^*(\Sigma)| = \tau^*(\Sigma) \quad &= \tau(\Sigma)/\mathbf{t}^2, \\ |\mathbb{Z}^n/(\mathcal{C} \oplus \mathcal{F})| = \tau(\Sigma)/\mathbf{t} &= \tau^*(\Sigma) \cdot \mathbf{t}, \end{aligned}$$

highlighting the duality between the cut and flow lattices.

6 Bounds on combinatorial invariants from lattice geometry

Let $n \geq 1$ be an integer. The *Hermite constant* γ_n is defined as the maximum value of

$$\left(\min_{x \in \mathcal{L} \setminus \{0\}} \langle x, x \rangle \right) (|\mathcal{L}^\sharp/\mathcal{L}|)^{-1/n} \tag{3}$$

over all lattices $\mathcal{L} \subseteq \mathbb{R}^n$, where $\langle \cdot, \cdot \rangle$ is the standard inner product. The Hermite constant arises both in the study of quadratic forms and in sphere packing; see [Lagarias, 1995, Section 4]. It is known that γ_n is finite for every n, although the precise values are known only for $1 \leq n \leq 8$ and $n = 24$ Cohn and Kumar [2009].

As observed in Kotani and Sunada [2000], if $\mathcal{L} = \mathcal{F}$ is the flow lattice of a connected graph, then the shortest vector in $\mathcal{F}$ is the characteristic vector of a cycle of minimum length; therefore, the numerator in equation (3) is the girth of G. Meanwhile, $|\mathcal{F}^\sharp/\mathcal{F}|$ is the number of spanning trees. We now generalize this theorem to cell complexes.

Definition 6.1 *Let Σ be a cell complex. The* girth *and the* connectivity *are defined as the cardinalities of, respectively the smallest circuit and the smallest cocircuit of the cellular matroid of Σ.*

Theorem 6.2 *Let Σ be a cell complex of dimension d with girth g and connectivity k, and top boundary map of rank r. Let $b = \operatorname{rank} \tilde{H}_{d-1}(\Sigma; \mathbb{Z})$. Then*

$$k\tau(\Sigma)^{-1/r} \leq \gamma_r \qquad \textit{and} \qquad g\tau^*(\Sigma)^{-1/b} \leq \gamma_b.$$

Proof: Every nonzero vector of the cut lattice (resp., the flow lattice) contains a cocircuit (resp., a circuit) in its support. Therefore,

$$\min_{x \in \mathcal{C} \setminus \{0\}} \langle x, x \rangle \geq k \qquad \text{and} \qquad \min_{x \in \mathcal{F} \setminus \{0\}} \langle x, x \rangle \geq g.$$

Meanwhile, $|\mathcal{C}^\sharp/\mathcal{C}| = \tau$ and $|\mathcal{F}^\sharp/\mathcal{F}| = \tau^*$ by Theorem 5.10. The desired inequalities now follow from applying the definition of Hermite's constant to the cut and flow lattices respectively. □

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Denominator vectorsr and compatibility degrees in cluster algebras of finite type

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Abstract. We present two simple descriptions of the denominator vectors of the cluster variables of a cluster algebra of finite type, with respect to any initial cluster seed: one in terms of the compatibility degrees between almost positive roots defined by S. Fomin and A. Zelevinsky, and the other in terms of the root function of a certain subword complex. These descriptions only rely on linear algebra, and provide simple proofs of the known fact that the d-vector of any non-initial cluster variable with respect to any initial cluster seed has non-negative entries and is different from zero.

Résumé. Nous présentons deux descriptions élémentaires des vecteurs dénominateurs des algèbres amassées de type fini pour tout amas initial: l'une en termes de degrés de compatibilitié entre racines presque positives définis par S. Fomin et A. Zelevinsky, et l'autre en termes de la fonction racine d'un certain complexe de sous-mots. Ces descriptions ne reposent que sur l'algèbre linéaire et fournissent des preuves simples du fait (connu) que le d-vecteur de toute variable d'amas, qui n'est pas dans l'amas initial, a des entrées positives ou nulles et est différent du vecteur nul.

Keywords: Finite type cluster algebras, d-vectors, subword complexes.

1 Introduction

Cluster algebras were introduced by S. Fomin and A. Zelevinsky in [FZ02, FZ03a]. They are commutative rings generated by a (possibly infinite) set of *cluster variables*, which are grouped into overlapping *clusters*. The clusters can be obtained from any *initial cluster seed* $X = \{x_1, \dots, x_n\}$ by a mutation process. Each mutation exchanges a single variable y to a new variable y' satisfying a relation of the form $yy' = M_+ + M_-$, where M_+ and M_- are monomials in the variables involved in the current cluster and distinct from y and y'. The precise content of these monomials M_+ and M_- is controlled by a combinatorial object (a skew-symmetrizable matrix, or equivalently a weighted quiver [Kel12]) which is attached to each cluster and is also transformed during the mutation. We refer to [FZ02] for the precise definition of these joint dynamics. In [FZ02, Thm. 3.1], S. Fomin and A. Zelevinsky proved that given any initial cluster seed $X = \{x_1, \dots, x_n\}$, the cluster variables obtained during this mutation process are

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Laurent polynomials in the variables $x_1, \dots, x_n$. That is to say, every non-initial cluster variable y can be written in the form

$$y = \frac{F(x_1, \dots, x_n)}{x_1^{d_1} \cdots x_n^{d_n}}$$

where $F(x_1, \dots, x_n)$ is a polynomial which is not divisible by any variable x_i for $i \in [n]$. This intriguing property is called *Laurent Phenomenon* in cluster algebras [FZ02]. The *denominator vector* (or *d-vector* for short) of the cluster variable y with respect to the initial cluster seed X is defined as the vector $\mathbf{d}(X, y) := (d_1, \dots, d_n)$. The d-vector of the initial cluster variable x_i is by definition $\mathbf{d}(X, x_i) := -e_i := (0, \dots, -1, \dots, 0)$.

Note that we think of the cluster variables as a set of variables satisfying some algebraic relations. These variables can be expressed in terms of the variables in any initial cluster seed $X = \{x_1, \dots, x_n\}$ of the cluster algebra. Starting from a different cluster seed $X' = \{x'_1, \dots, x'_n\}$ would give rise to an isomorphic cluster algebra, expressed in terms of the variables $x'_1, \dots, x'_n$ of this seed. Therefore, the d-vectors of the cluster variables depend on the choice of the initial cluster seed X in which the Laurent polynomials are expressed. This dependence is explicit in the notation $\mathbf{d}(X, y)$. Note also that since the denominator vectors do not depend on coefficients, we restrict our attention to coefficient-free cluster algebras.

In this paper, we only consider finite type cluster algebras, *i.e.* cluster algebras whose mutation graph is finite. They were classified in [FZ03a, Thm. 1.4] using the Cartan-Killing classification for finite crystallographic root systems. In [FZ03a, Thm. 1.9], S. Fomin and A. Zelevinsky proved that in the cluster algebra of any given finite type, with a bipartite quiver as initial cluster seed,

(i) there is a bijection ϕ from almost positive roots to cluster variables, which sends the negative simple roots to the initial cluster variables;

(ii) the d-vector of the cluster variable $\phi(\beta)$ corresponding to an almost positive root β is given by the vector $(b_1, \dots, b_n)$ of coefficients of the root $\beta = \sum b_i \alpha_i$ on the linear basis Δ formed by the simple roots $\alpha_1, \dots, \alpha_n$; and

(iii) these coefficients coincide with the *compatibility degrees* $(\alpha_i \parallel \beta)$ defined in [FZ03b, Sec. 3.1].

These results were extended to all cluster seeds corresponding to Coxeter elements of the Coxeter group (see *e.g.* [Kel12, Thm. 3.1 & Sec. 3.3]). More precisely, assume that the initial seed is the cluster X_c corresponding to a Coxeter element c (its associated quiver is the Coxeter graph oriented according to c). Then one can define a bijection ϕ_c from almost positive roots to cluster variables such that the d-vector of the cluster variable $\phi_c(\beta)$ corresponding to β, with respect to the initial cluster seed X_c, is still given by the vector $(b_1, \dots, b_n)$ of coordinates of $\beta = \sum b_i \alpha_i$ in the basis Δ of simple roots. Under this bijection, the collections of almost positive roots corresponding to clusters are called *c-clusters* and were studied by N. Reading [Rea07, Sec. 7].

In this paper, we provide similar interpretations for the denominators of the cluster variables of any finite type cluster algebra with respect to *any initial cluster seed* (acyclic or not):

(i) Our first description (Corollary 3.2) uses compatibility degrees: if $\{\beta_1, \dots, \beta_n\}$ is the set of almost positive roots corresponding to the cluster variables in any initial seed $X = \{\phi(\beta_1), \dots, \phi(\beta_n)\}$, then the d-vector of the cluster variable $\phi(\beta)$ corresponding to an almost positive root β, with respect to the initial cluster seed X, is given by the vector of compatibility degrees $((\beta_1 \parallel \beta), \dots, (\beta_n \parallel \beta))$ of [FZ03b, Sec. 3.1]. We also provide a refinement of this result parametrized by a Coxeter element c, using the bijection ϕ_c together with the notion of c-compatibility degrees (Corollary 3.3).

(ii) Our second description (Corollary 3.4) uses the recent connection [CLS13] between the theory of cluster algebras of finite type and the theory of subword complexes, initiated by A. Knutson and E. Miller [KM04]. We describe the entries of the d-vector in terms of certain coefficients given by the root function of a subword complex associated to a certain word.

These results lead to simple alternative proofs of the known fact that, in a cluster algebra of finite type, the d-vector of any non-initial cluster variable with respect to any initial cluster seed is non-negative and not equal to zero (Corollary 3.5).

Finally, we also provide explicit geometric interpretations for all the concepts and results in this paper for the classical types A, B, C and D in Section 4. Our interpretation in type D is new and differs from known interpretations in the literature. It simplifies certain combinatorial and algebraic aspects and makes an additional link between the theory of cluster algebras and pseudotriangulations [RSS08].

The proofs of our results, omitted in this extended abstract, can be found in [CP13].

2 Preliminaries

Let (W, S) be a finite crystallographic Coxeter system of rank n. We consider a root system Φ, with simple roots $\Delta := \{\alpha_1, \dots, \alpha_n\}$, positive roots Φ^+, and almost positive roots $\Phi_{\geq -1} := \Phi^+ \cup -\Delta$. We refer to [Hum90] for a reference on Coxeter groups and root systems.

Let $\mathcal{A}(W)$ denote the cluster algebra associated to type W, as defined in [FZ03a]. Each cluster is formed by n cluster variables, and is endowed with a weighted quiver (an oriented and weighted graph on S) which controls the cluster dynamics. Since we will not make extensive use of it, we believe that it is unnecessary to recall here the precise definition of the quiver and cluster dynamics, and we refer to [FZ02, Kel12] for details. For illustrations, we recall geometric descriptions of these dynamics in types A, B, C, and D in Section 4.

Let c be a Coxeter element of W, and $\mathrm{c} := (c_1, \cdots, c_n)$ be a reduced expression of c. The element c defines a particular weighted quiver $\mathcal{Q}_c$: the Coxeter graph of the Coxeter system (W, S) directed according to the order of appearance of the simple reflections in c. We denote by X_c the cluster seed whose associated quiver is $\mathcal{Q}_c$. Let $\mathrm{w}_\circ(\mathrm{c}) := (w_1, \cdots, w_N)$ denote the c-*sorting word* for $w_\circ$, *i.e.* the lexicographically first subword of the infinite word c^∞ which represents a reduced expression for the longest element $w_\circ \in W$. We consider the word $\mathrm{Q_c} := \mathrm{c}\mathrm{w}_\circ(\mathrm{c})$ and denote by $m := n + N$ the length of this word.

2.1 Cluster variables, almost positive roots, and positions in the word $\mathrm{Q_c}$

We recall here the above-mentioned bijections between cluster variables, almost positive roots and positions in the word $\mathrm{Q_c}$. We will see in the next sections that both the clusters and the d-vectors (expressed on any initial cluster seed X) can also be read off in these different contexts. Figure 1 summarizes these different notions and the corresponding notations. We insist that the choice of the Coxeter element c and the choice of the initial cluster X are not related. The former provides a labeling of the cluster variables by the almost positive roots or by the positions in $\mathrm{Q_c}$, while the latter gives an algebraic basis to express the cluster variables and to assign them d-vectors.

First, there is a natural bijection between cluster variables and almost positive roots, which can be parametrized by the Coxeter element c. Start from the initial cluster seed X_c associated to the weighted quiver $\mathcal{Q}_c$ corresponding to the Coxeter element c. Then the d-vectors of the cluster variables of $\mathcal{A}(W)$ with respect to the initial seed X_c are given by the almost positive roots $\Phi_{\geq -1}$. This defines a bijection ϕ_c from almost positive roots to cluster variables. Notice that this bijection depends on the choice of the

$$\begin{tikzcd}
 & \begin{array}{c}\text{cluster variables } x,y \in \mathcal{A}(W)\\ \text{cluster } X = \{x_1,\dots,x_n\}\\ \text{compatibility degree } d(x,y)\\ \text{denominator vector } \mathbf{d}(X,y)\end{array} \arrow[loop above, "\tau"] & \\
\begin{array}{c}\text{almost positive roots } \alpha,\beta \in \Phi_{\geq -1}\\ c\text{-cluster } B = \{\beta_1,\dots,\beta_n\}\\ c\text{-compatibility degree } (\alpha \parallel_c \beta)\\ \text{compatibility vector } \mathbf{d}_c(B,\beta)\end{array} \arrow[ur, "\phi_c"] \arrow[loop left, "\tau_c"] & & \begin{array}{c}\text{positions } i,j \in [m] \text{ in the word } \mathrm{Q_c}\\ \mathrm{c}\text{-cluster } I = \{i_1,\dots,i_n\}\\ \mathrm{c}\text{-compatibility degree } \{i \parallel_{\mathrm{c}} j\}\\ \text{coefficient vector } \mathbf{d}_{\mathrm{c}}(I,j)\end{array} \arrow[ul, "\psi_{\mathrm{c}}"'] \arrow[ll, "\chi_{\mathrm{c}}"'] \arrow[loop right, "\tau_{\mathrm{c}}"]
\end{tikzcd}$$

Fig. 1: Three different contexts for cluster algebras of finite type, their different notions of compatibility degrees, and the bijections between them. See Sections 2.1, 2.2 and 2.3 for definitions.

Coxeter element c. When c is a bipartite Coxeter element, it is the bijection ϕ of S. Fomin and A. Zelevinsky [FZ03a, Thm. 1.9] mentioned above. Transporting the structure of the cluster algebra $\mathcal{A}(W)$ through the bijection ϕ_c, we say that a subset B of almost positive roots forms a *c-cluster* iff the corresponding subset of cluster variables $\phi_c(B)$ forms a cluster of $\mathcal{A}(W)$. The collection of c-clusters forms a simplicial complex on the set $\Phi_{\geq -1}$ of almost positive roots called the *c-cluster complex*. This complex was described in purely combinatorial terms by N. Reading in [Rea07, Sec. 7]. Given an initial c-cluster seed $B := \{\beta_1,\dots,\beta_n\}$ in $\Phi_{\geq -1}$ and an almost positive root β, we define the d-vector of β with respect to B as $\mathbf{d}_c(B,\beta) := \mathbf{d}\big(\phi_c(B),\phi_c(\beta)\big)$. If c is a bipartite Coxeter element, then we speak about classical clusters and omit c in the previous notation to write $\mathbf{d}(B,\beta)$.

Second, there is a bijection χ_{c} from the positions in the word $\mathrm{Q_c} = \mathrm{c}\mathrm{w}_\circ(\mathrm{c})$ to the almost positive roots as follows. The letter c_i of c is sent to the negative root $-\alpha_{c_i}$, while the letter w_i of $\mathrm{w}_\circ(\mathrm{c})$ is sent to the positive root $w_1\cdots w_{i-1}(\alpha_{w_i})$. To be precise, note that this bijection depends not only on the Coxeter element c, but also on its reduced expression c. This bijection was defined by C. Ceballos, J.-P. Labbé and C. Stump in [CLS13, Thm. 2.2].

Composing the two maps described above provides a bijection ψ_{c} from positions in the word $\mathrm{Q_c}$ to cluster variables (precisely defined by $\psi_{\mathrm{c}} := \phi_c \circ \chi_{\mathrm{c}}$). Transporting the structure of $\mathcal{A}(W)$ through the bijection ψ_{c}, we say that a subset I of positions in $\mathrm{Q_c}$ forms a *c-cluster* iff the corresponding cluster variables $\psi_{\mathrm{c}}(I)$ form a cluster of $\mathcal{A}(W)$. Moreover, given an initial c-cluster seed $I \subseteq [m]$ in $\mathrm{Q_c}$ and a position $j \in [m]$ in $\mathrm{Q_c}$, we define the d-vector of j with respect to I as $\mathbf{d}_{\mathrm{c}}(I,j) := \mathbf{d}\big(\psi_{\mathrm{c}}(I),\psi_{\mathrm{c}}(j)\big)$. It turns out that the c-clusters can be read off directly in the word $\mathrm{Q_c}$ as follows.

Theorem 2.1 ([CLS13, Thm. 2.2 & Coro. 2.3]) *A subset I of positions in $\mathrm{Q_c}$ forms a* c*-cluster in $\mathrm{Q_c}$ if and only if the subword of $\mathrm{Q_c}$ formed by the complement of I is a reduced expression for $w_\circ$.*

Remark 2.2 *The previous theorem relates c-cluster complexes to subword complexes as defined by A. Knutson and E. Miller [KM04]. Given a word Q on the generators S of W and an element $\pi \in W$, the* subword complex *$\mathcal{SC}(\mathrm{Q},\rho)$ is the simplicial complex whose faces are subwords P of Q such that the complement $\mathrm{Q} \smallsetminus \mathrm{P}$ contains a reduced expression of π. See [CLS13] for more details on this connection.*

2.2 The rotation map

In this section we introduce a rotation map τ_{c} on the positions in the word $\mathrm{Q_c}$, and naturally extend it to a map on almost positive roots and cluster variables using the bijections of Section 2.1 (see Figure 1). The rotation map plays the same role for arbitrary finite type as the rotation of the polygons associated to the classical types A, B, C and D, see e.g. [CLS13, Thm. 8.10] and Section 4.

Definition 2.3 (Rotation maps) *The rotation $\tau_{\mathrm{c}} : [m] \longrightarrow [m]$ is the map on the positions in the word $\mathrm{Q_c}$ defined as follows. If $q_i = s$, then $\tau_{\mathrm{c}}(i)$ is defined as the position in $\mathrm{Q_c}$ of the next occurrence of s if possible, and as the first occurrence of $w_\circ s w_\circ$ otherwise.*

Using the bijections χ_{c} (resp. ψ_{c}) from the positions in the word $\mathrm{Q_c}$ to almost positive roots (resp. to cluster variables), this rotation can also be regarded as a map on almost positive roots (resp. on cluster variables). For simplicity, we abuse notation and also write τ_{c} for the composition $\chi_{\mathrm{c}} \circ \tau_{\mathrm{c}} \circ \chi_{\mathrm{c}}^{-1}$ and τ for the composition $\psi_{\mathrm{c}} \circ \tau_{\mathrm{c}} \circ \psi_{\mathrm{c}}^{-1}$. These maps can also be expressed purely in terms of almost positive roots or cluster variables, see [CP13].

Lemma 2.4 *The rotation map preserves clusters:*

(i) a subset $I \subset [m]$ of positions in the word $\mathrm{Q_c}$ is a c-cluster if and only if $\tau_{\mathrm{c}}(I)$ is a c-cluster;

(ii) a subset $B \subset \Phi_{\geq -1}$ of almost positive roots is a c-cluster if and only if $\tau_c(B)$ is a c-cluster; and

(iii) a subset X of cluster variables is a cluster if and only if $\tau(X)$ is a cluster.

Remark 2.5 *Let c be a bipartite Coxeter element, with sources corresponding to the positive vertices $(+)$ and sinks corresponding to the negative vertices $(-)$. Then, the rotation τ_c on the set of almost positive roots is the product of the maps $\tau_+, \tau_- : \Phi_{\geq -1} \to \Phi_{\geq -1}$ defined in [FZ03b, Sec. 2.2]. We refer the interested reader to that paper for the definitions of τ_+ and τ_-.*

2.3 Three descriptions of c-compatibility degrees

In this section we introduce three notions of compatibility degrees on the set of cluster variables, almost positive roots, and positions in the word $\mathrm{Q_c}$. We will see in Section 3 that these three notions coincide under the bijections of Section 2.1, and will use it to describe three different ways to compute d-vectors for cluster algebras of finite type. We refer again to Figure 1 for a summary of our notations in these three situations.

On cluster variables. Let $X = \{x_1, \dots, x_n\}$ be a set of cluster variables of $\mathcal{A}(W)$ forming a cluster, and let

$$y = \frac{F(x_1, \dots, x_n)}{x_1^{d_1} \cdots x_n^{d_n}} \tag{1}$$

be a cluster variable of $\mathcal{A}(W)$ expressed in terms of the variables $\{x_1, \dots, x_n\}$ such that $F(x_1, \dots, x_n)$ is a polynomial which is not divisible by any variable x_j for $j \in [n]$. Recall that the d-vector of y with respect to X is $\mathbf{d}(X, y) = (d_1, \dots, d_n)$.

Lemma 2.6 *For cluster algebras of finite type, the i-th component of the d-vector $\mathbf{d}(X, y)$ is independent of the cluster X containing the cluster variable x_i.*

Definition 2.7 (Compatibility degree on cluster variables) *For any two cluster variables x and y, we denote by $d(x, y)$ the x-component of the d-vector $\mathbf{d}(X, y)$ for any cluster X containing the variable x. We refer to $d(x, y)$ as the* compatibility degree *of y with respect to x.*

On almost positive roots. Extending the definition of [FZ03b, Sec. 3.1], we now define compatibility degrees on almost positive roots.

Definition 2.8 (c-compatibility degree on almost positive roots) *The c-compatibility degree on the set of almost positive roots is the unique function*

$$\begin{array}{ccc} \Phi_{\geq -1} \times \Phi_{\geq -1} & \longrightarrow & \mathbb{Z} \\ (\alpha, \beta) & \longmapsto & (\alpha \,\|_c\, \beta) \end{array}$$

characterized by the following two properties:

$$(-\alpha_i \,\|_c\, \beta) = b_i, \qquad \text{for all } i \in [n] \text{ and } \beta = \sum b_i \alpha_i \in \Phi_{\geq -1}, \tag{2}$$

$$(\alpha \,\|_c\, \beta) = (\tau_c \alpha \,\|_c\, \tau_c \beta), \quad \text{for all } \alpha, \beta \in \Phi_{\geq -1}. \tag{3}$$

Remark 2.9 *This definition is motivated by the classical compatibility degree defined by S. Fomin and A. Zelevinsky in [FZ03b, Sec. 3.1]. Namely, if c is a bipartite Coxeter element, then the c-compatibility degree $(\cdot \,\|_c\, \cdot)$ coincides with the compatibility degree $(\cdot \,\|\, \cdot)$ of [FZ03b, Sec. 3.1] except that $(\alpha \,\|_c\, \alpha) = -1$ while $(\alpha \,\|\, \alpha) = 0$ for any $\alpha \in \Phi_{\geq -1}$. Throughout this paper, we ignore this difference: we still call classical compatibility degree, and denote by $(\cdot \,\|\, \cdot)$, the c-compatibility degree for a bipartite Coxeter element c.*

On positions in the word $\mathrm{Q_c}$. We recall now the notion of root functions associated to c-clusters in $\mathrm{Q_c}$, and use them in order to define a c-compatibility degree on the set of positions in $\mathrm{Q_c}$. This description relies only on linear algebra and is one of the main contributions of this paper. The root function was defined by C. Ceballos, J.-P. Labbé, and C. Stump in [CLS13, Def. 3.2] and was extensively used by V. Pilaud and C. Stump in the construction of Coxeter brick polytopes [PS11].

Definition 2.10 ([CLS13]) *The root function $\mathsf{r}(I, \cdot) : [m] \longrightarrow \Phi$ associated to a* c*-cluster $I \subseteq [m]$ in $\mathrm{Q_c}$ is defined by $\mathsf{r}(I, j) := \sigma^{\mathrm{c}}_{[j-1] \smallsetminus I}(\alpha_{q_j})$, where σ^{c}_X denotes the product of the reflections $q_x \in \mathrm{Q_c}$ for $x \in X$ in this order. The root configuration of I is the multiset $\mathsf{R}(I) := \{\!\{ \mathsf{r}(I, i) \mid i \in I \}\!\}$.*

As proved in [CLS13, Sec. 3.1], the root function $\mathsf{r}(I, \cdot)$ encodes exchanges in the c-cluster I. Namely, any $i \in I$ can be exchanged with the unique $j \notin I$ such that $\mathsf{r}(I, j) = \pm \mathsf{r}(I, j)$ (see [CLS13, Lem. 3.3]), and the root function can be updated during this exchange (see [CLS13, Lem. 3.6]). It was moreover shown in [PS11, Sec. 6] that the root configuration $\mathsf{R}(I)$ forms a basis for $\mathbb{R}^n$ for any given initial c-cluster I in $\mathrm{Q_c}$. It enables us to decompose any other root on this basis to get the following coefficients, which will play a central role in the remainder of the paper.

Definition 2.11 (c-compatibility degree on positions in $\mathrm{Q_c}$) *Fix any initial* c*-cluster $I \subseteq [m]$ of $\mathrm{Q_c}$. For any position $j \in [m]$, we decompose the root $\mathsf{r}(I, j)$ on the basis $\mathsf{R}(I)$ as follows:*

$$\mathsf{r}(I, j) = \sum_{i \in I} \rho_i(j) \mathsf{r}(I, i).$$

For $i \in I$ and $j \in [m]$, we define the c-compatibility degree as the coefficient

$$\{i \,\|_{\mathrm{c}}\, j\} = \begin{cases} \rho_i(j) & \text{if } j > i, \\ -\rho_i(j) & \text{if } j \leq i. \end{cases}$$

Lemma 2.12 *The coefficients $\{i \,\|_{\mathrm{c}}\, j\}$ are independent of the* c*-cluster $I \subseteq [m]$ of $\mathrm{Q_c}$ containing i.*

3 Main results: Three descriptions of d-vectors

The following result states that the three notions of compatibility degrees above are essentially the same. The proof can be found in [CP13].

Theorem 3.1 *The three notions of compatibility degrees on the set of cluster variables, almost positive roots, and positions in the word Q_c coincide under the bijections of Section 2.1. More precisely, for every pair of positions i, j in the word Q_c we have*

$$d(\psi_\mathrm{c}(i), \psi_\mathrm{c}(j)) = (\chi_\mathrm{c}(i) \parallel_c \chi_\mathrm{c}(j)) = \{i \parallel_\mathrm{c} j\}.$$

In particular, if c is a bipartite Coxeter element, then these coefficients coincide with the classical compatibility degrees of S. Fomin and A. Zelevinsky [FZ03b, Sec. 3.1] (except for Remark 2.9).

The following three statements are the main results of this paper and are direct consequences of Theorem 3.1. The first statement describes the denominator vectors in terms of the compatibility degrees of [FZ03b, Sec. 3.1].

Corollary 3.2 *Let $B := \{\beta_1, \dots, \beta_n\} \subseteq \Phi_{\geq -1}$ be a (classical) cluster in the sense of S. Fomin and A. Zelevinsky [FZ03a, Thm. 1.9], and let $\beta \in \Phi_{\geq -1}$ be an almost positive root. Then the d-vector $\mathbf{d}(B, \beta)$ of the cluster variable $\phi(\beta)$ with respect to the initial cluster seed $\phi(B) = \{\phi(\beta_1), \dots, \phi(\beta_n)\}$ is given by*

$$\mathbf{d}(B, \beta) = \big((\beta_1 \parallel \beta), \dots, (\beta_n \parallel \beta)\big),$$

where $(\beta_i \parallel \beta)$ is the compatibility degree of β with respect to β_i as defined by S. Fomin and A. Zelevinsky [FZ03b, Sec. 3.1] (except for Remark 2.9).

The next statement extends this result to any Coxeter element c of W.

Corollary 3.3 *Let $B := \{\beta_1, \dots, \beta_n\} \subseteq \Phi_{\geq -1}$ be a c-cluster in the sense of N. Reading [Rea07, Sec. 7], and let $\beta \in \Phi_{\geq -1}$ be an almost positive root. Then the d-vector $\mathbf{d}_c(B, \beta)$ of the cluster variable $\phi_c(\beta)$ with respect to the initial cluster seed $\phi_c(B) = \{\phi_c(\beta_1), \dots, \phi_c(\beta_n)\}$ is given by*

$$\mathbf{d}_c(B, \beta) = \big((\beta_1 \parallel_c \beta), \dots, (\beta_n \parallel_c \beta)\big),$$

where $(\beta_i \parallel_c \beta)$ is the c-compatibility degree of β with respect to β_i as defined in Definition 2.8.

Finally, the third statement describes the denominator vectors in terms of the coefficients $\{i \parallel_\mathrm{c} j\}$ obtained from the word Q_c.

Corollary 3.4 *Let $I \subseteq [m]$ be a* c*-cluster and $j \in [m]$ be a position in Q_c. Then the d-vector $\mathbf{d}_\mathrm{c}(I, j)$ of the cluster variable $\psi_\mathrm{c}(j)$ with respect to the initial cluster seed $\psi_\mathrm{c}(I) = \{\psi_\mathrm{c}(i) \mid i \in I\}$ is given by*

$$\mathbf{d}_\mathrm{c}(I, j) = \big(\{i \parallel_\mathrm{c} j\}\big)_{i \in I}.$$

These corollaries lead to simple proofs of the following statement.

Corollary 3.5 *For cluster algebras of finite type, the d-vector of a cluster variable that is not in the initial seed is non-negative and not equal to zero.*

This corollary was conjectured by S. Fomin and A. Zelevinsky for arbitrary cluster algebras [FZ07, Conj. 7.4]. In the case of cluster algebras of finite type, this conjecture also follows from [CCS06, Thm. 4.4 & Rem. 4.5] and from [BMR07, Thm. 2.2], where the authors show that the d-vectors can be computed as the dimension vectors of certain indecomposable modules.

4 Geometric interpretations in types A, B, C and D

In this section, we present geometric interpretations for the classical types A, B, C, and D of the objects discussed in this paper: cluster variables, clusters, mutations, compatibility degrees, and d-vectors. These interpretations are classical in types A, B and C when the initial cluster seed corresponds to a bipartite Coxeter element, and can already be found in [FZ03b, Sec. 3.5] and [FZ03a, Sec. 12]. In contrast, our interpretation in type D slightly differs from that of S. Fomin and A. Zelevinsky since we prefer to use pseudotriangulations (we motivate this choice in Remark 4.1). Moreover, these interpretations are extended here to any initial cluster seed, acyclic or not.

We can associate to each classical finite type a geometric configuration, so that there is a correspondence between:

(i) cluster variables and *diagonals* (or centrally symmetric pairs of diagonals) in the geometric picture;

(ii) clusters and *geometric clusters*: triangulations in type A, centrally symmetric triangulations in types B and C, and centrally symmetric pseudotriangulations in type D (*i.e.* maximal crossing-free sets of centrally symmetric pairs of chords in the geometric picture);

(iii) cluster mutations and *geometric flips* (we can also express geometrically the exchange relations on cluster variables);

(iv) compatibility degrees and *crossing numbers* of (centrally symmetric pairs of) diagonals;

(v) d-vectors and *crossing vectors* of (centrally symmetric pairs of) diagonals.

Due to space limitations, we only detail the cases of types A_n and D_n. A similar analysis in types B_n and C_n, as well as explicit examples of computation for all classical finite types can be found in [CP13].

4.1 Type A_n

Consider the Coxeter group $A_n = \mathfrak{S}_{n+1}$, generated by the simple transpositions $\tau_i := (i\ \ i+1)$ for $i \in [n]$. The corresponding geometric picture is a convex regular $(n+3)$-gon. Cluster variables, clusters, exchange relations, compatiblity degrees, and d-vectors in the cluster algebra $\mathcal{A}(A_n)$ can be interpreted geometrically as follows:

(i) Cluster variables correspond to *(internal) diagonals* of the $(n+3)$-gon. We denote by $\chi(\delta)$ the cluster variable corresponding to a diagonal δ.

(ii) Clusters correspond to *triangulations* of the $(n+3)$-gon.

(iii) Cluster mutations correspond to *flips* between triangulations. See Figure 2.

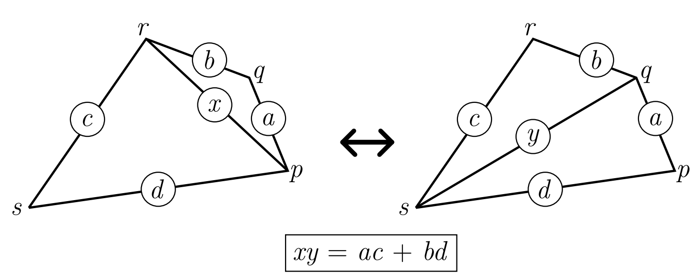

Fig. 2: Flip in type A.

Moreover, flipping diagonal pr to diagonal qs in a quadrilateral $\{p,q,r,s\}$ results in the exchange relation

$$\chi(pr)\cdot\chi(qs) = \chi(pq)\cdot\chi(rs) + \chi(ps)\cdot\chi(qr).$$

In this relation, we set $\chi(\delta) = 1$ when δ is a boundary edge of the $(n+3)$-gon.

(iv) Given any two diagonals θ, δ, the compatibility degree $d(\chi(\theta), \chi(\delta))$ between the corresponding cluster variables $\chi(\theta)$ and $\chi(\delta)$ is given by the *crossing number* $[\,\theta \,\|\, \delta\,]$ of the diagonals θ and δ. By definition, $[\,\theta \,\|\, \delta\,]$ is equal to -1 if $\theta = \delta$, to 1 if the diagonals $\theta \neq \delta$ cross, and to 0 otherwise.

(v) Given any initial seed $T := \{\theta_1, \dots, \theta_n\}$ and any diagonal δ, the d-vector of the cluster variable $\chi(\delta)$ with respect to the initial cluster seed $\chi(T)$ is the *crossing vector* $\mathbf{d}(T,\delta) := ([\,\theta_1 \,\|\, \delta\,], \dots, [\,\theta_n \,\|\, \delta\,])$ of δ with respect to T.

4.2 Type D_n

Consider the Coxeter group D_n of even signed permutations of $[n]$, generated by the simple transpositions $\tau_i := (i \;\; i+1)$ for $i \in [n-1]$ and by the operator τ_0 which exchanges 1 and 2 and invert their signs. Note that τ_0 and τ_1 play symmetric roles in D_n (they both commute with all the other simple generators except with τ_2). This Coxeter group can be folded in type C_{n-1}, which provides a geometric interpretation of the cluster algebra $\mathcal{A}(D_n)$ on a $2n$-gon with bicolored long diagonals [FZ03b, Sec. 3.5][FZ03a, Sec. 12.4]. In this section, we present a new interpretation of the cluster algebra $\mathcal{A}(D_n)$ in terms of pseudotriangulations. The precise connection to the classical interpretation of type D_n cluster algebras [FZ03b, Sec. 3.5][FZ03a, Sec. 12.4] is given in Remark 4.1.

We consider a regular convex $2n$-gon, together with a disk D (placed at the center of the $2n$-gon), whose radius is small enough such that D only intersects the long diagonals of the $2n$-gon. We denote by $\mathcal{D}_n$ the resulting configuration, see Figure 3. The *chords* of $\mathcal{D}_n$ are all the diagonals of the $2n$-gon, except the long ones, plus all the segments tangent to the disk D and with one endpoint among the vertices of the $2n$-gon. Note that each vertex p is adjacent to two of the latter chords; we denote by p^{L} (resp. by p^{R}) the chord emanating from p and tangent on the left (resp. right) to the disk D. Cluster variables, clusters, exchange relations, compatiblity degrees, and d-vectors in the cluster algebras $\mathcal{A}(D_n)$ can be interpreted geometrically as follows:

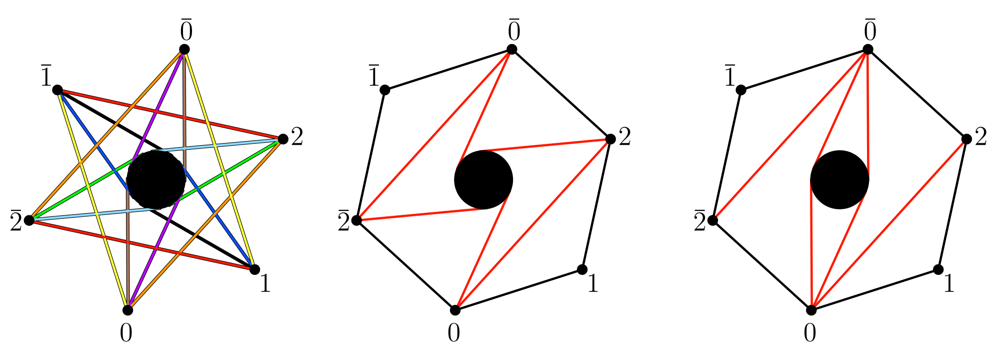

Fig. 3: The configuration $\mathcal{D}_3$ with its 9 centrally symmetric pairs of chords (left). A centrally symmetric pseudotriangulation T of $\mathcal{D}_3$ (middle). The centrally symmetric pseudotriangulation of $\mathcal{D}_3$ obtained from T by flipping the chords 2^{R} and $\bar{2}^{\mathrm{R}}$.

(i) Cluster variables correspond to *centrally symmetric pairs of (internal) chords* of the geometric configuration $\mathcal{D}_n$. See Figure 3 (left). To simplify notations, we identify a chord δ, its centrally symmetric copy $\bar{\delta}$, and the pair $\{\delta, \bar{\delta}\}$. We denote by $\chi(\delta) = \chi(\bar{\delta})$ the cluster variable corresponding to the pair of chords $\{\delta, \bar{\delta}\}$.

(ii) Clusters of $\mathcal{A}(D_n)$ correspond to *centrally symmetric pseudotriangulations* of $\mathcal{D}_n$ (*i.e.* maximal centrally symmetric crossing-free sets of chords of $\mathcal{D}_n$). Each pseudotriangulation of $\mathcal{D}_n$ contains exactly $2n$ chords, and partitions $\operatorname{conv}(\mathcal{D}_n) \smallsetminus D$ into *pseudotriangles* (*i.e.* interiors of simple closed curves with three convex corners related by three concave chains). See Figure 3 (middle) and (right). We refer to [RSS08] for a complete survey on pseudotriangulations.

(iii) Cluster mutations correspond to *flips* of centrally symmetric pairs of chords between centrally symmetric pseudotriangulations of $\mathcal{D}_n$. A flip in a pseudotriangulation T replaces an internal chord e by the unique other internal chord f such that $(T \smallsetminus e) \cup f$ is again a pseudotriangulation of T. To be more precise, deleting e in T merges the two pseudotriangles of T incident to e into a pseudoquadrangle $⌑$ (*i.e.* the interior of a simple closed curve with four convex corners related by four concave chains), and adding f splits the pseudoquadrangle $⌑$ into two new pseudotriangles. The chords e and f are the two unique chords which lie both in the interior of $⌑$ and on a geodesic between two opposite corners of $⌑$. We refer again to [RSS08] for more details.

For example, the two pseudotriangulations of Figure 3 (middle) and (right) are related by a centrally symmetric pair of flips. We have represented different types of flips between centrally symmetric pseudotriangulations of the configuration $\mathcal{D}_n$ in Figure 4.

As in types A, B, and C, the exchange relations between cluster variables during a cluster mutation can be understood in the geometric picture. More precisely, flipping e to f in the pseudoquadrangle $⌑$ with convex corners $\{p, q, r, s\}$ (and simultaneously $\bar{e}$ to $\bar{f}$ in the centrally symmetric pseudoquadrangle $\bar{⌑}$) results in the exchange relation

$$\Pi(⌑, p, r) \cdot \Pi(⌑, q, s) = \Pi(⌑, p, q) \cdot \Pi(⌑, r, s) + \Pi(⌑, p, s) \cdot \Pi(⌑, q, r),$$

where

- $\Pi(⌑, p, r)$ denotes the product of the cluster variables $\chi(\delta)$ corresponding to all chords δ of the geodesic from p to r in $⌑$ — and similarly for $\Pi(⌑, q, s)$,
- $\Pi(⌑, p, q)$ denotes the product of the cluster variables $\chi(\delta)$ corresponding to all chords δ of the concave chain from p to q in $⌑$ — and similarly for $\Pi(⌑, q, r)$, $\Pi(⌑, r, s)$, and $\Pi(⌑, p, s)$.

For example, the four flips in Figure 4 result in the following relations:

$$\begin{aligned}
\chi(pr) \cdot \chi(qs) &= \chi(pq) \cdot \chi(rs) + \chi(ps) \cdot \chi(qr), \\
\chi(pr) \cdot \chi(q^{\mathrm{R}}) &= \chi(pq) \cdot \chi(r^{\mathrm{R}}) + \chi(p^{\mathrm{R}}) \cdot \chi(qr), \\
\chi(pr) \cdot \chi(q\bar{p}) &= \chi(pq) \cdot \chi(r\bar{p}) + \chi(\bar{p}^{\mathrm{L}}) \cdot \chi(p^{\mathrm{R}}) \cdot \chi(qr), \\
\chi(\bar{p}^{\mathrm{L}}) \cdot \chi(p^{\mathrm{R}}) \cdot \chi(q^{\mathrm{R}}) &= \chi(pq) \cdot \chi(\bar{p}^{\mathrm{R}}) + \chi(q\bar{p}) \cdot \chi(p^{\mathrm{R}}).
\end{aligned}$$

Note that the last relation will always simplify by $\chi(p^{\mathrm{R}}) = \chi(\bar{p}^{\mathrm{R}})$.

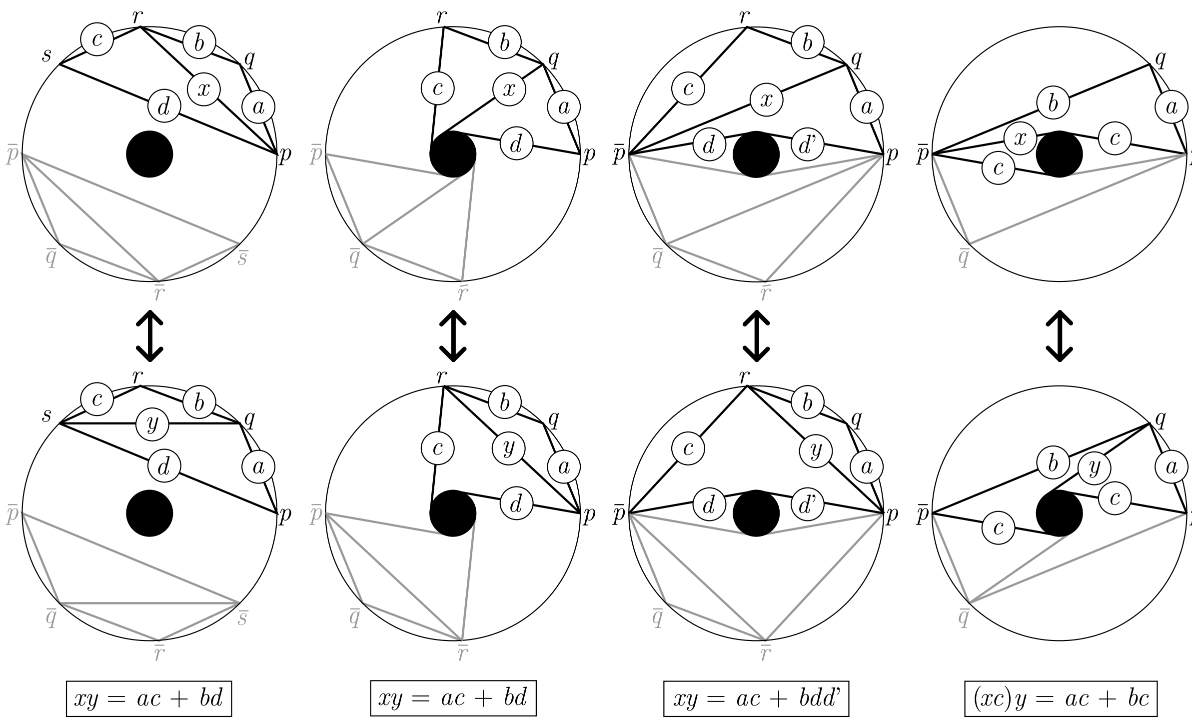

Fig. 4: Different types of flips in type D.

(iv) Given any two centrally symmetric pairs of chords θ, δ, the compatibility degree $d(\chi(\theta), \chi(\delta))$ between the corresponding cluster variables $\chi(\theta)$ and $\chi(\delta)$ is given by the *crossing number* $[\,\theta \,\|\, \delta\,]$ of the pairs of chords θ and δ. By definition, $[\,\theta \,\|\, \delta\,]$ is equal to -1 if $\theta = \delta$, and to the number of times that a representative diagonal of the pair δ crosses the chords of θ if $\theta \neq \delta$.

(v) Given any initial centrally symmetric seed pseudotriangulation $T := \{\theta_1, \dots, \theta_n\}$ and any centrally symmetric pair of chords δ, the d-vector of the cluster variable $\chi(\delta)$ with respect to the initial cluster seed $\chi(T)$ is the *crossing vector* $\mathbf{d}(T, \delta) := ([\,\theta_1 \,\|\, \delta\,], \dots, [\,\theta_n \,\|\, \delta\,])$ of δ with respect to T.

Remark 4.1 *Our geometric interpretation of type D cluster algebras slightly differs from that of S. Fomin and A. Zelevinsky in [FZ03b, Sec. 3.5][FZ03a, Sec. 12.4]. Namely, to obtain their interpretation, we can just remove the disk in the configuration $\mathcal{D}_n$ and replace the centrally symmetric pairs of chords $\{p^{\mathrm{L}}, \bar{p}^{\mathrm{L}}\}$ and $\{p^{\mathrm{R}}, \bar{p}^{\mathrm{R}}\}$ by long diagonals $p\bar{p}$ colored in red and blue respectively. Long diagonals of the same color are then allowed to cross, while long diagonals of different colors cannot. Flips and exchange relations can then be worked out, with special rules for colored long diagonals, see [FZ03b, Sec. 3.5][FZ03a, Sec. 12.4]. Although our presentation is only slightly different from the classical presentation, we believe that it has certain advantages:*

(i) Since there is no color code, it simplifies certain combinatorial and algebraic aspects (e.g. the notion of crossing, the d-vector, the cluster mutations, and the exchange relations are simpler to express).

(ii) It makes an additional link between cluster algebras and pseudotriangulations.

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Descent sets for oscillating tableaux

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Abstract. The descent set of an oscillating (or up-down) tableau is introduced. This descent set plays the same role in the representation theory of the symplectic groups as the descent set of a standard tableau plays in the representation theory of the general linear groups. In particular, we show that the descent set is preserved by Sundaram's correspondence.

This gives a direct combinatorial interpretation of the branching rules for the defining representations of the symplectic groups; equivalently, for the Frobenius character of the action of a symmetric group on an isotypic subspace in a tensor power of the defining representation of a symplectic group.

Résumé. Dans cet article, nous définissons la notion d'ensemble de descentes pour un tableau oscillant. Ces descentes sont analogues aux descentes d'un tableau standard dans la théorie des représentations des groupes généraux linéaires. Nous montrons que la correspondance de Sundaram préserve cet ensemble et nous donnons une interprétation combinatoire directe des règles de branchement pour la représentation des groupes symplectiques. Enfin, nous décrivons combinatoirement les caractères de Frobenius associés à l'action du groupe symétrique sur les composantes isotypiques du produit tensoriel des représentations d'un groupe symplectique.

Keywords: oscillating tableaux, quasisymmetric expansion, Frobenius character

1 Introduction

There is a well developed combinatorial theory associated with the polynomial representations of the algebraic groups $\mathrm{GL}(n)$. For a textbook treatment we refer to Stanley's book [6]. At its core is the Robinson-Schensted correspondence

$$\mathrm{RS} : \{1, \dots, n\}^r \to \bigcup_{\substack{\mu \in P(r) \\ \ell(\mu) \leq n}} SSYT(\mu, n) \times SYT(\mu),$$

where the union is over the set of partitions μ of r into at most n parts, $SSYT(\mu, n)$ is the set of semistandard Young tableaux of shape μ and entries no larger than n and $SYT(\mu)$ is the set of standard Young tableaux of shape μ.

This correspondence can be regarded as a combinatorial counterpart of the direct sum decomposition of the r-th tensor power of the defining representation V of $\mathrm{GL}(n)$ as a $\mathrm{GL}(n) \times \mathfrak{S}_r$ module, where $\mathrm{GL}(n)$

acts diagonally and the symmetric group acts by permuting tensor coordinates:

$$\otimes^r V \cong \bigoplus_{\substack{\mu \in P(r) \\ \ell(\mu) \leq n}} V(\mu) \otimes S(\mu), \tag{1}$$

where $V(\mu)$ and $S(\mu)$ are irreducible representations of $\mathrm{GL}(n)$ and $\mathfrak{S}_r$ and respectively. Namely, the character of $V(\mu)$ is given by the Schur function associated to μ,

$$s_\mu(x_1, \ldots, x_n) = \sum_{T \in SSYT(\mu,n)} \mathbf{x}^T. \tag{2}$$

and there is a basis of $S(\mu)$ indexed by the elements of $SYT(\mu)$.

A remarkable property of the Robinson-Schensted correspondence, due to Schützenberger [5, Remarque 2], is that the descent set of a word equals the descent set of the standard Young tableau to which it is mapped. In this article we prove an analogous result for the symplectic groups $\mathrm{Sp}(2n)$, Theorem 5.1. As a corollary we obtain Theorem 3.8, which is a symplectic version of the expansion of Schur functions in terms of the fundamental quasisymmetric functions L_α:

$$s_\mu = \sum_{T \in SYT(\mu)} L_{\mathrm{co}\,\mathrm{Des}(T)}, \tag{3}$$

where we associate with a subset $D = \{d_1 < d_2 < \cdots < d_{k-1}\}$ of $\{1, \ldots, r-1\}$ the composition $\mathrm{co}\, D = \{d_1, d_2 - d_1, \ldots, r - d_{k-1}\}$ of r, and

$$L_{\mathrm{co}\,D} = \sum_{\substack{i_1 \leq \cdots \leq i_r \\ i_j < i_{j+1} \text{ if } j \in D}} x_{i_1} x_{i_2} \cdots x_{i_r}.$$

2 The general setting

Let V be a finite dimensional rational representation of a complex reductive algebraic group G. Let Λ be the set of isomorphism classes of irreducible rational representations of G and let $V(\mu)$ be the representation corresponding to $\mu \in \Lambda$. For example, when G is the general linear group $\mathrm{GL}(n)$ we can identify Λ with the set of non-decreasing sequences of integers of length n. Also, when G is the symplectic group $\mathrm{Sp}(2n)$ we can identify Λ with the set of partitions with at most n parts. In both cases, the trivial representation corresponds to the empty partition and the defining representation corresponds to the partition 1.

Let $\mathfrak{S}_r$ be the symmetric group permuting r elements. We identify the isomorphism classes of its irreducible representations with $P(r)$ and denote the representation corresponding to $\lambda \in P(r)$ by $S(\lambda)$.

For each $r \geq 0$, the tensor power $\otimes^r V$ is a rational representation of $G \times \mathfrak{S}_r$, where G acts diagonally and $\mathfrak{S}_r$ acts by permuting tensor coordinates. This representation is completely reducible. Decomposing it as a representation of G we obtain the following analogue of Equation (1):

$$\otimes^r V \cong \bigoplus_{\mu \in \Lambda} V(\mu) \otimes U(r, \mu). \tag{4}$$

The isotypic space $U(r,\mu)$ inherits the action of $\mathfrak{S}_r$ and therefore decomposes as

$$U(r,\mu) \cong \bigoplus_{\lambda\in P(r)} A(\lambda,\mu)\otimes S(\lambda). \tag{5}$$

Thus, the Frobenius character of $U(r,\mu)$ is

$$\operatorname{ch} U(r,\mu) = \sum_{\lambda\in P(r)} a(\lambda,\mu) s_\lambda, \tag{6}$$

where $a(\lambda,\mu)=\dim A(\lambda,\mu)$ and s_λ is the Schur function associated to λ.

When V is the defining representation of $\mathrm{GL}(n)$ the coefficient $a(\lambda,\mu)$ equals 1 for $\lambda=\mu$ and vanishes otherwise. Thus, in this case the Frobenius character is simply s_μ. For the defining representation of the symplectic group $\mathrm{Sp}(2n)$ the coefficients $a(\lambda,\mu)$ were determined by Sundaram [7]. In general it is a difficult problem to determine these characters explicitly.

We would like to advertise a new approach to describe the Frobenius character, using descent sets. It appears that the proper setting for a general definition of descent set is the combinatorial theory of crystal graphs. This theory is an off-shoot of the representation theory of Drinfeld-Jimbo quantised enveloping algebras. However, for our purposes a few notions from this theory suffice. For a textbook treatment we refer to the book by Hong and Kang [2].

For each rational representation V of a connected reductive algebraic group there is a crystal. A crystal is a combinatorial framework for the representation. The vector space V is replaced by a set of cardinality $\dim(V)$. The raising and lowering operators, which are certain linear operators on V, are replaced by partial functions on the set. It is common practice to represent these partial functions by directed graphs. Each vertex of the crystal has a weight and the sum of these weights is the character of the representation, see Equations (2) and (9). Isomorphic modules correspond to crystal graphs that are isomorphic as coloured digraphs and the module is irreducible if and only if the graph is connected.

For example, the crystal graph corresponding to the defining representation of $\mathrm{GL}(n)$ is

$$1 \xrightarrow{1} 2 \xrightarrow{2} \cdots \xrightarrow{n-2} n-1 \xrightarrow{n-1} n. \tag{7}$$

while the crystal graph corresponding to the defining representation of $\mathrm{Sp}(2n)$ is

$$1 \xrightarrow{1} 2 \xrightarrow{2} \cdots \xrightarrow{n-2} n-1 \xrightarrow{n-1} n \xrightarrow{n} -n \xrightarrow{n-1} -(n-1) \xrightarrow{n-2} \cdots \xrightarrow{2} -2 \xrightarrow{1} -1. \tag{8}$$

A vertex in a crystal graph with no in-coming arcs is a highest weight vertex. Each connected component contains a unique highest weight vertex. The weight of this vertex is the weight of the representation it corresponds to. Finally, there is a (relatively) simple way to construct the crystal graph of a tensor product of two modules given their individual crystal graphs. Thus, the highest weight vertices of the crystal corresponding to $\otimes^r V$ can be regarded as words of length r with letters being vertices of the crystal corresponding to V.

Many aspects of the classical theory can be generalised at least to the crystals corresponding to the classical groups. In particular, explicit analogues of the Robinson-Schensted correspondence were found for the defining representations of the symplectic groups $\mathrm{Sp}(2n)$ as well as for the odd and even orthogonal groups, see [4].

Analogous to the expansion in Equation (3) of s_μ we can now state the fundamental property we require for a general descent set:

Definition 2.1. A function Des which assigns a subset of $\{1, 2, \ldots, r-1\}$ to each highest weight vertex of the crystal graph corresponding to $\otimes^r V$ is a *descent function* if it satisfies

$$\operatorname{ch} U(r, \mu) = \sum_{w} L_{\operatorname{co}\operatorname{Des}(w)},$$

where the sum is over all highest weight vertices of weight μ.

In particular, when V is the defining representation of the $\mathrm{GL}(n)$, the highest weight vertices of the corresponding crystal graph can be taken to be the reversals of the Yamanouchi words (or lattice permutations) in $\{1, \ldots, n\}^r$. It is straightforward to check that the restriction of the Robinson-Schensted correspondence to these words is a bijection to standard Young tableaux such that the weight of the word is mapped to the shape of the tableau. Therefore,

$$\operatorname{ch} U(r, \mu) = s_\mu = \sum_{T \in SYT(\mu)} L_{\operatorname{co}\operatorname{Des}(T)} = \sum_{w} L_{\operatorname{co}\operatorname{Des}(w)},$$

the last summation being over all highest weight vertices of weight μ. Thus the usual descent set of a word, $\operatorname{Des}(w_1 w_2 \ldots w_r) = \{k \mid w_k > w_{k+1}\}$, is a descent set in the sense of Definition 2.1. We remark that in terms of the crystal graph (7) above, a highest weight vertex $w_1 w_2 \ldots w_r$ has a descent at position k if and only if there is a (nontrivial) directed path from w_{k+1} to w_k in the crystal graph.

In the remaining sections we will show that we can define a descent function for the symplectic group in almost the same way. In the next section we define a notion of descent set for oscillating tableaux which will permit us to give the desired combinatorial interpretation of $\operatorname{ch} U(r, \mu)$ in Theorem 3.8. Section 4 describes the tools necessary to prove this theorem, namely symplectic Littlewood-Richardson tableaux and a correspondence due to Sundaram. In Section 5 we give the proof by analysing this correspondence in detail.

3 Oscillating tableaux and descents

In the case of the defining representation of the symplectic group $\mathrm{Sp}(2n)$ the vertices of the crystal graph corresponding to $\otimes^r V$ are words $w_1 w_2 \ldots w_r$ in $\{\pm 1, \ldots, \pm n\}^r$. The weight of a vertex is the tuple $(\mu_1, \ldots, \mu_n)$, where μ_i is the number of letters i minus the number of letters $-i$ in w. The vertex is a highest weight vertex if for any $k \leq r$, the weight of $w_1, \ldots, w_k$ is a partition, i.e., $\mu_1 \geq \mu_2 \cdots \geq \mu_n$.

Definition 3.1. A highest weight vertex $w_1 w_2 \ldots w_r$ in the crystal graph corresponding to $\otimes^r V$ has a *descent* at position k if there is a (nontrivial) directed path from w_k to w_{k+1} in the crystal graph (8).

The *descent set* of w is

$$\operatorname{Des}(w) = \{k \mid k \text{ is a descent of } w\}.$$

We can now state our main result:

Theorem 3.2. *Let V be the defining representation of the symplectic group $\mathrm{Sp}(2n)$. Then the Frobenius character of $\otimes^r V$ is*

$$\operatorname{ch} U(r, \mu) = \sum_{w} L_{\operatorname{co}\operatorname{Des}(w)},$$

where the sum is over all highest weight vertices of the corresponding crystal graph.

To prove this theorem we will first rephrase it in terms of n-symplectic oscillating tableaux, also known as up-down-tableaux, which are in bijection with the highest weight vertices of $\otimes^r V$:

Definition 3.3. An *oscillating tableau* of length r and (final) *shape* μ is a sequence of partitions

$$(\emptyset = \mu^0, \mu^1, \ldots, \mu^r = \mu)$$

such that the Ferrers diagrams of two consecutive partitions differ by exactly one cell.

The k-th step (going from μ^{k-1} to μ^k) is an *expansion* if a cell is added and a *contraction* if a cell is deleted. We will refer to the cell that is added or deleted in the k-th step as b_k.

The oscillating tableau $O = (\mu^0, \mu^1, \ldots, \mu^r)$ is n-*symplectic* if every partition μ^i has at most n non-zero parts.

The oscillating tableau corresponding to a highest weight vertex $w_1 w_2 \ldots w_r$ is given by the sequence of weights of its initial factors $w_1, w_1 w_2, w_1 w_2 w_3, \ldots, w_1 w_2 \ldots w_r$.

Example 3.4. The 1-symplectic oscillating tableaux of length three are

$$(\emptyset, 1, 2, 3), \quad (\emptyset, 1, 2, 1), \quad \text{and} \quad (\emptyset, 1, \emptyset, 1).$$

The corresponding words are

$$1\,1\,1, \quad 1\,1\,\text{-}1, \quad \text{and} \quad 1\,\text{-}1\,1.$$

As a running example, we will use the oscillating tableau

$$O = (\emptyset, 1, 11, 21, 2, 1, 2, 21, 211, 21)$$

which has length 9 and shape 21. It is 3-symplectic (since no partition has four parts) but it is not 2-symplectic (since there is a partition with three parts). The corresponding word is 1 2 1 -2 -1 1 2 3 -3 with descent set $\{1, 3, 4, 6, 7, 8\}$.

Definition 3.5. An oscillating tableau O has a *descent* at position k in any of the following three cases:

1. Step k is an expansion and step $k+1$ is a contraction,
2. steps k and $k+1$ are both expansions and b_k is in a row strictly above b_{k+1}.
3. steps k and $k+1$ are both contractions and b_k is in a row strictly below b_{k+1},

where we view all Ferrers diagrams in English notation. The *descent set* of O is

$$\mathrm{Des}(O) = \{k \mid k \text{ is a descent of } O\}.$$

Example 3.6. The descent set of the oscillating tableau O from Example 3.4 is

$$\mathrm{Des}(O) = \{1, 3, 4, 6, 7, 8\}.$$

The definition for the descent set of an oscillating tableau is such that the bijection between highest weight vertices and oscillating tableaux has the following property:

Proposition 3.7. *The descent set of an n-symplectic oscillating tableau coincides with the descent set of the corresponding highest weight vertex of the crystal graph of $\otimes^r V$.*

Thus it suffices to prove the following variant of Theorem 3.2:

Theorem 3.8. *Let V be the defining representation of the symplectic group $\mathrm{Sp}(2n)$. Then the Frobenius character of $\otimes^r V$ is*

$$\operatorname{ch} U(r,\mu) = \sum_O L_{\operatorname{co}\operatorname{Des}(O)},$$

where the sum is over all n-symplectic oscillating tableaux of length r and shape μ.

Let us motivate Definition 3.5 in a second way. Note first that a standard Young tableau T can be regarded as an oscillating tableau O where every step is an expansion and the cell containing k in T is added during the k-th step in O. In this case

$$\operatorname{Des}(O) = \operatorname{Des}(T),$$

so the definition of descents for oscillating tableaux is an extension of the one for standard tableaux.

In Sundaram's correspondence an arbitrary oscillating tableau O is first transformed into a fixed-point-free involution ι and a partial Young tableau T, that is, a filling of a Ferrers shape with all entries distinct and increasing in rows and columns. There is a natural extension of descents for these objects:

Definition 3.9. The *descent set* of an involution ι (or in fact any permutation) defined on a set A is

$$\operatorname{Des}(\iota) = \{k : k, k+1 \in A,\ \iota(k) > \iota(k+1)\}.$$

For a partial Young tableau T whose set of entries is A, the *descent set* is

$$\operatorname{Des}(T) = \{k : k, k+1 \in A,\ k+1 \text{ is in a row below } k\}.$$

The definition for descents in oscillating tableaux is constructed so that the descent set of the oscillating tableau contains the union of the descent set of the associated partial tableau and permutation.

4 The correspondences of Berele and Sundaram

One of our main tools for proving Theorem 3.8 will be a bijection Sun due to Sundaram [7, 8]. In combination with a correspondence due to Berele [1], which is a combinatorial counterpart of the isomorphism in Equation (4) in the case where V is the defining representation of $\mathrm{Sp}(2n)$, Sundaram's bijection can be regarded as a combinatorial counterpart of the isomorphism in Equation (5). In this section we define the objects involved, the bijection itself will be described in detail in the next section.

Definition 4.1. Let u be a word with letters in $\mathbb{N}$. Then u is a *Yamanouchi word* (or *lattice permutation*) if in any initial factor $u_1 u_2 \dots u_k$ and for each i, there are at least as many occurrences of i in u as there are occurrences of $i+1$.

The *weight* β of a lattice permutation u is the partition $\beta = (\beta_1 \geq \beta_2 \geq \cdots)$, where β_i is the number of occurrences of the letter i in u.

For a skew semistandard Young tableau S the *reverse reading word* $w^{rev}(S)$ is the reversal of the word obtained by concatenating the rows from bottom to top.

A skew semistandard Young tableau S of shape λ/μ is an *n-symplectic Littlewood-Richardson tableau of weight β* if

- its reverse reading word is a lattice permutation of weight β, where β is a partition with all columns having even height, and
- entries in row $n+i+1$ of λ are strictly larger than $2i+1$ for $0 \le i \le \frac{1}{2}\ell(\beta)$.

The number of n-symplectic Littlewood-Richardson tableaux of shape λ/μ and weight β is denoted by $c^{\lambda}_{\mu,\beta}(n)$.

For $\ell(\lambda) \le n+1$ and β a partition with all columns having even height, the number $c^{\lambda}_{\mu,\beta}(n)$ is the usual Littlewood-Richardson coefficient. This is trivial for $\ell(\lambda) \le n$, and follows from the correspondence Sun described below for $\ell(\lambda) = n+1$.

Note that we are only interested in the case that the length of μ is at most n. In this case the restriction on the size of the entries is equivalent to the condition given by Sundaram that $2i+1$ appears no lower than row $n+i$ for $0 \le i \le \frac{1}{2}\ell(\beta)$.

As an example, when $\mu = (1)$ there is a single 1-symplectic Littlewood-Richardson tableau of weight $\beta = (1,1)$:

$$\begin{array}{|c|c|}\hline & 1 \\ \hline 2 \\ \cline{1-1}\end{array}$$

We alert the reader that there is a typo both in [7, Definition 9.5] and [8, Definition 3.9], where the range of indices is stated as $1 \le i \le \frac{1}{2}\ell(\beta)$. With this definition,

$$\begin{array}{|c|}\hline \\ \hline 1 \\ \hline 2 \\ \hline\end{array}$$

would also be 1-symplectic, since $\beta = (1,1)$ and 3 does not appear at all. However, there are only two 1-symplectic oscillating tableaux of length 3 and shape (1), and there are two standard Young tableaux of shape $(2,1)$. Thus, if the tableau above would also be 1-symplectic, Theorem 4.2 below would fail.

We now have all the definitions in place to explain the domain and range of the correspondence Sun.

Theorem 4.2 ([7, Theorem 9.4])**.** *There is an explicit bijection* Sun *between n-symplectic oscillating tableaux of length r and shape μ and pairs (Q,S), where*

- *Q is a standard tableau of shape λ, with $|\lambda| = r$, and*
- *S is an n-symplectic Littlewood-Richardson tableau of shape λ/μ and weight β, where $|\beta| = r - |\mu|$ and has even columns.*

For completeness, let us point out the relation between Sundaram's bijection and Berele's correspondence. This correspondence involves the following objects due to King [3], indexing the irreducible representations of the symplectic group $\mathrm{Sp}(2n)$:

Definition 4.3. An *n-symplectic semistandard tableau of shape μ* is a filling of μ with letters from $1 < -1 < 2 < -2 < \cdots < n < -n$ such that

- entries in rows are weakly increasing,
- entries in columns are strictly increasing, and

- the entries in row i are greater than or equal to i, in the above ordering.

Denoting the set of n-symplectic oscillating tableaux of length r and final shape μ by $Osc(r,n,\mu)$ and the set of n-symplectic semistandard tableaux of shape μ by $K(\mu,n)$, Berele's correspondence is a bijection

$$\mathrm{Ber} : \{\pm 1, \dots, \pm n\}^r \to \bigcup_{\substack{\mu\in P(r)\\ \ell(\mu)\le n}} K(\mu,n)\times Osc(r,n,\mu).$$

In analogy to Equation (2), the character of the representation $V(\mu)$ of the symplectic group $\mathrm{Sp}(2n)$ is

$$sp_\mu(x_1^{\pm 1}, \dots, x_n^{\pm 1}) = \sum_{T\in K(\mu,n)} \mathbf{x}^T. \tag{9}$$

Now consider an n-symplectic oscillating tableau as a word in the ordered alphabet $1 < -1 < 2 < -2 < \cdots < n < -n$, as described just after Definition 3.3. We can then apply the Robinson-Schensted correspondence to obtain a semistandard Young tableau P_{RS} in this alphabet and a (usual) standard Young tableau Q_{RS}. Alternatively, we can compose Berele's correspondence with Sundaram's bijection to obtain a triple $(P_{\mathrm{Ber}}, Q_{\mathrm{Sun}}, S_{\mathrm{Sun}})$. It then turns out that $Q_{\mathrm{RS}} = Q_{\mathrm{Sun}}$. This implies that for each standard Young tableau Q_{Sun} we have a correspondence $P_{\mathrm{RS}} \mapsto (P_{\mathrm{Ber}}, S_{\mathrm{Sun}})$. Moreover, this correspondence is independent of the choice of Q_{Sun}. One can then show the following theorem:

Theorem 4.4 ([7, Theorem 12.1])**.** *For all λ, μ the coefficient $a(\lambda,\mu)$ is given by*

$$a(\lambda,\mu) = \sum_\beta c^\lambda_{\mu,\beta}(n)$$

where the sum is over the partitions β of $|\lambda| - |\mu|$ having only columns of even length.

5 Correspondences and the proof of the main result

In light of Sundaram's results, we claim that to prove Theorem 3.8 it suffices to demonstrate the following:

Theorem 5.1. *Let $\mathrm{Sun}(O) = (Q,S)$. Then*

$$\mathrm{Des}(O) = \mathrm{Des}(Q).$$

Proof of Theorem 3.8: We have

$$\begin{aligned}
\sum_O L_{\mathrm{co\,Des}(O)} &= \sum_{(Q,S)} L_{\mathrm{co\,Des}(Q)} && \text{(by Theorems 4.2 and 5.1)}\\
&= \sum_{|Q|=r}\sum_\beta c^{\mathrm{sh}\,Q}_{\mu,\beta}(n) L_{\mathrm{co\,Des}(Q)} && \text{(by Definiton 4.1)}\\
&= \sum_{|Q|=r} a(\mathrm{sh}(Q),\mu) L_{\mathrm{co\,Des}(Q)} && \text{(by Theorem 4.4)}\\
&= \sum_{\lambda\in P(r)} a(\lambda,\mu) \sum_{\mathrm{sh}(Q)=\lambda} L_{\mathrm{co\,Des}(Q)}\\
&= \mathrm{ch}\, U(r,\mu) && \text{(by Equations (3) and (6))}
\end{aligned}$$

which is the desired conclusion. □

In order to prove Theorem 5.1, we will need to analyse the bijection Sun in detail. Sundaram described Sun as the composition of several bijections. Specifically, Sun is the composition

$$O \overset{\mathrm{Sun}_1}{\mapsto} (\iota, T) \mapsto (\mathrm{RS}(\iota), T) \overset{\mathrm{Sun}_2}{\mapsto} (Q, S)$$

where Sun_1 and Sun_2 are described below and RS denotes the Robinson-Schensted correspondence. We will prove Theorem 5.1 by tracking their effect on the descent set.

5.1 Sundaram's first bijection

We now describe Sundaram's first bijection which we will denote Sun_1. It maps an oscillating tableau O to a pair (ι, T) where ι is a fixed-point-free involution, T is a partial Young tableau (that is, a filling of a Ferrers shape with all entries distinct and increasing in rows and columns), and the entries of ι and T are complementary sets.

Let $O = (\emptyset = \mu^0, \mu^1, \dots, \mu^r)$ be an oscillating tableau. We then construct a sequence of pairs (ι_k, T_k) for $0 \le k \le r$, such that $\mathrm{sh}(T_k) = \mu^k$ and the set of entries of the pair (ι_k, T_k) is $\{1, \dots, k\}$, all entries being distinct.

Both ι_0 and T_0 are empty. For $k > 0$ the pair (ι_k, T_k) is constructed from the pair (ι_{k-1}, T_{k-1}) and the k-th step in the oscillating tableau:

- If the k-th step is an expansion then $\iota_k = \iota_{k-1}$ and T_k is obtained from T_{k-1} by putting k in cell b_k.
- Otherwise, if the k-th step is a contraction then take T_{k-1} and bump out (using Robinson-Schensted column deletion) the entry in cell b_k to get a letter x and the standard tableau T_k. In other words, let x and T_k be such that column inserting x into T_k yields T_{k-1}. The involution ι_k is then given by adjoining the transposition (k, x) to ι_{k-1}.

The result of the bijection is the final pair (ι_r, T_r).

Lemma 5.2 (Sundaram [7, Lemma 8.7])**.** *The map* Sun_1 *is a bijection between oscillating tableaux of length r and shape μ and pairs (ι, T) where*

- *ι is a fixed-point-free involution of a set $A \subseteq \{1, \dots, r\}$, and*
- *T is a partial tableau of shape μ such that its set of entries is $\{1, \dots, r\} \setminus A$.*

Example 5.3. Starting with the oscillating tableau O from Example 3.4 we get the following sequence of pairs (T_k, i_k), where in the diagram below we only list each pair of the involution once when it is produced by the algorithm.

k:	0	1	2	3	4	5	6	7	8	9
T_k:	$\emptyset$	$\begin{array}{\|c\|}\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 1\\\hline 2\\\hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 1&3\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 3\\\hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 3&6\\\hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 3&6\\\hline 7\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 3&6\\\hline 7\\\cline{1-1} 8\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 3&6\\\hline 7\\\cline{1-1}\end{array}$
ι_k:	$\emptyset$				$(2,4)$	$(1,5)$				$(8,9)$

So the final output is the involution

$$\iota = (1,5)(2,4)(8,9) = \begin{matrix} 1 & 2 & 4 & 5 & 8 & 9 \\ 5 & 4 & 2 & 1 & 9 & 8 \end{matrix}$$

and the partial tableau

$$T = \begin{array}{|c|c|}\hline 3 & 6 \\ \hline 7 & \\ \cline{1-1}\end{array}$$

with $\iota \uplus T = \{1, \ldots, 9\}$.

We wish to define the descent set of a pair (ι, T) in such a way that this map preserves descents. Let us define for any pair A and B

$$\mathrm{Des}(A/B) = \{k : k \in A \text{ and } k+1 \in B\}$$

and define

$$\mathrm{Des}(\iota, T) = \mathrm{Des}(\iota) \uplus \mathrm{Des}(T) \uplus \mathrm{Des}(T/\iota),$$

where we use Definition 3.9 for the descent sets of ι and T. Thus, in our running example, $\mathrm{Des}(T/\iota) = \{3, 7\}$ and $\mathrm{Des}(\iota, T) = \{1, 3, 4, 6, 7, 8\}$, which coincides with $\mathrm{Des}(O)$.

We can now take our first step in proving Theorem 5.1.

Proposition 5.4. *Let O be an oscillating tableau and suppose that* $\mathrm{Sun}_1(O) = (\iota, T)$. *Then*

$$\mathrm{Des}(O) = \mathrm{Des}(\iota, T).$$

Proof: We proceed by analysing the effect of two successive steps in the oscillating tableau.

If step k is an expansion and step $k+1$ is a contraction then $k+1 \in \iota$ and $\iota(k+1) < k+1$. Now k either ends up in T or in ι. In the former case, $k \in \mathrm{Des}(T/\iota)$. In the latter case $\iota(k) > k \geq \iota(k+1)$ and $k \in \mathrm{Des}(\iota)$. In both cases this gives a descent of (ι, T).

If step k is a contraction and step $k+1$ is an expansion then $k \in \iota$ and $\iota(k) < k$. Now either $k+1$ ends up in T or in ι. In the former case $k \in \mathrm{Des}(\iota/T)$ rather than $\mathrm{Des}(T/\iota)$. In the latter case $\iota(k) < k < k+1 < \iota(k+1)$. Neither of these cases gives a descent of (ι, T).

If steps k and $k+1$ are both contractions then $k, k+1 \in \iota$. If b_k is strictly below b_{k+1} then, by well-known properties of RS, the element removed when bumping out b_k will be in a lower row than the one obtained when bumping out b_{k+1}. Thus $\iota(k) > \iota(k+1)$ and $k \in \mathrm{Des}(\iota, T)$ as desired. By a similar argument, if b_k is weakly above b_{k+1} then $\iota(k) < \iota(k+1)$ and $k \notin \mathrm{Des}(\iota, T)$.

Now suppose steps k and $k+1$ are both expansions. If b_k is in a row strictly above b_{k+1}, then any column deletion will keep k in a row strictly above $k+1$. It follows that at the end we have one of three possibilities. The first is that $k, k+1 \in T$ and, as was just observed, we must have $k \in \mathrm{Des}(T)$. If either element is removed, then $k+1$ must be removed first because the row condition forces $k+1$ to always be in a column weakly left of k. So at the end we either have $k \in T$ and $k+1 \in \iota$, or we have $\iota(k) > \iota(k+1)$ and in both cases $k \in \mathrm{Des}(\iota, T)$.

If steps k and $k+1$ are both expansions and b_k is in a row weakly below b_{k+1}, then any column deletion will keep k in a row weakly below $k+1$. Again there are three possibilities. The first is that $k, k+1 \in T$ and, as was just observed, we must have $k \notin \mathrm{Des}(T)$. If either element is removed, then k must be removed first because the row condition forces k to always be in a column strictly left of $k+1$. So at the end either we have $k+1 \in T$ and $k \in \iota$ or we have $\iota(k+1) > \iota(k)$, and neither are descents of (ι, T).

This completes the check of all the cases and the proof. $\square$

5.2 Robinson-Schensted

Next, we apply the Robinson-Schensted correspondence to the fixed-point-free involution ι to obtain a partial Young tableau I with the same set of entries. Let us first recall the following fact:

Lemma 5.5 ([6, Exercise 7.28a])**.** *Let $\pi \in \mathfrak{S}_r$ and let $\mathrm{RS}(\pi) = (P, Q)$. Then π is a fixed-point-free involution if and only if $P = Q$ and all columns of Q have even length.*

Combining Proposition 5.4 with this lemma we obtain:

Proposition 5.6. *Let O be an oscillating tableau, let $\mathrm{Sun}_1(O) = (\iota, T)$ and let $\mathrm{RS}(\iota) = (I, I)$. Then*

$$\mathrm{Des}(O) = \mathrm{Des}(I) \uplus \mathrm{Des}(T) \uplus \mathrm{Des}(T/I).$$

Example 5.7. The fixed point free involution from Example 5.3 is mapped to the tableau

$$I = \begin{array}{|c|c|}\hline 1 & 8 \\ \hline 2 & 9 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}$$

5.3 Sundaram's second bijection

Finally, we need a bijection Sun_2 that transforms the pair of partial Young tableaux (I, T) to pairs (Q, S) as in Theorem 4.2. Let Q be the standard Young tableau of shape λ obtained by *column-inserting* the reverse reading word of the tableau I into the tableau T. Also construct a skew semistandard Young tableau S as follows: whenever a letter of $w^{rev}(I)$ is inserted, record its row index in I in the cell which is added. Then $\mathrm{Sun}_2(I, T) = (Q, S)$.

Sundaram actually defines this map for pairs (I, T) of semistandard Young tableaux. But we will not need this level of generality.

Lemma 5.8 (Sundaram [7, Theorem 8.11, Theorem 9.4])**.** *The map Sun_2 is a bijection from pairs of partial Young tableaux (I, T) of shapes β and μ, respectively, such that $I \uplus T = \{1, \dots, r\}$ to pairs (Q, S) such that*

- *Q is a standard Young tableau of shape $\lambda \in P(r)$ and*
- *S is an n-symplectic Littlewood-Richardson tableau of shape λ/μ and weight β, for some n.*

Example 5.9. The tableau I from Example 5.7 has $w^{\mathrm{rev}}(I) = 819245$. Inserting this word into the tableau T from Example 5.3 and recording the row indices yields the pair of tableaux

$$(Q, S) = \left(\begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 7 \\ \cline{1-2} 4 & 8 \\ \cline{1-2} 5 & 9 \\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|}\hline & & 1 \\ \hline & 2 \\ \cline{1-2} 1 & 3 \\ \cline{1-2} 2 & 4 \\ \cline{1-2}\end{array} \right).$$

Note that the $\mathrm{Des}(Q) = \{1, 3, 4, 6, 7, 8\} = \mathrm{Des}(O)$ as desired.

To prove Theorem 5.1 we need two properties of column insertion:

Lemma 5.10. *Let π be a permutation and suppose that $\mathrm{RS}^{col}(\pi) = T$. Then*

- $\mathrm{RS}^{col}\left(w^{rev}(T)\right) = T$
- $\mathrm{Des}(T) = \{k : k$ *is left of* $k+1$ *in the* 1*-line notation of* $\pi\}$.

Proof of Theorem 5.1: By the first assertion of Lemma 5.10 we have that Q can also be obtained by column-inserting the concatenation of $w^{\mathrm{rev}}(T)$ and $w^{\mathrm{rev}}(I)$. By the second assertion of Lemma 5.10, the descent set of Q equals $\mathrm{Des}(I) \uplus \mathrm{Des}(T) \uplus \mathrm{Des}(T/I)$, which in turn equals the descent set of O by Proposition 5.6. $\square$

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Double-dimers, the Ising model and the hexahedron recurrence

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Abstract. We define and study a recurrence relation in $\mathbb{Z}^3$, called the hexahedron recurrence, which is similar to the octahedron recurrence (Hirota bilinear difference equation) and cube recurrence (Miwa equation). Like these examples, solutions to the hexahedron recurrence are partition functions for configurations on a certain graph, and have a natural interpretation in terms of cluster algebras. We give an explicit correspondence between monomials in the Laurent expansions arising in the recurrence with certain double-dimer configurations of a graph. We compute limit shapes for the corresponding double-dimer configurations. The Kashaev difference equation arising in the Ising model star-triangle relation is a special case of the hexahedron recurrence. In particular this reveals the cluster nature underlying the Ising model. The above relation allows us to prove a Laurent phenomenon for the Kashaev difference equation.

Resumé. Nous définissons une relation sur $\mathbb{Z}^3$ appellée “hexahedron recurrence”, qui est un cousin des relations bilinéaires “octaédrale” et “cubique”. Comme ces exemples, ses solutions peuvent être décrits comme fonctions de partition pour certaines configurations d'arêtes sur un graphe planaire, et ont une interprétation naturelle en termes de clusters. Nous trouvons une correspondance explicite entre le termes dans les développements de Laurent dans ce récurrences et certains double-recouvrements par dimères du graphe sous-jacent. On calcule les formes limites.

L'équation de Kashaev paraissant dans l'opération triangle-étoile du modèle d'Ising est un cas spéciale de notre récurrence. Ce fait révèle la nature “cluster” du modèle d'Ising, et nous permette de montrer la propriété de Laurent pour l'équation de Kashaev.

Keywords: cluster algebra, urban renewal, Laurent property, Y-Delta

1 Introduction

We study the hexahedron recurrence and its specialization to the Kashaev recurrence. In this introductory section we review known facts about the related cube and octahedron recurrences and state the main definitions and results for the hexahedron recurrence.

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A function $f : \mathbb{Z}^3 \to \mathbb{C}$ is said to satisfy the **octahedron recurrence** (or **Hirota bilinear difference equation**) if

$$f_{(1)} f_{(23)} = f_{(2)} f_{(13)} + f_{(3)} f_{(12)} \,. \tag{1}$$

Here, $f_{(S)}$ represents f evaluated at the translate of v by the basis vectors in S, e.g., $f_{(12)}(v) = f(v + e_1 + e_2)$. This is a specialization to $\mathbb{Z}^3$ of a transformation on that replaces a vertex v of a locally finite graph by a vertex w with slightly different local connections and replaces the value $f(v)$ by $f(w) := [f(v_1)f(v_2) + f(v_3)f(v_4)]/f(v)$. Such a move is an example of a cluster algebra mutation (Fomin and Zelevinsky 2002b). The Laurent phenomenon for cluster algebras implies that the new values of f produced by iterating such moves, which are *a fortiori* rational functions of the original values, are in fact Laurent polynomials in those values (Fomin and Zelevinsky 2002a). The octahedron recurrence goes back to Dodgson (1866) who observed that it was satisfied by subdeterminants and used it as a means of recursively computing determinants.

The function $g : \mathbb{Z}^3 \to \mathbb{C}$ is said to satisfy the **cube recurrence** (or **Miwa equation**) if

$$g_{(123)} g = g_{(1)} g_{(23)} + g_{(2)} g_{(13)} + g_{(3)} g_{(12)} . \tag{2}$$

This recurrence also has its roots in the 19^{th} century. Consider a resistor network containing somewhere in it a node, v, of degree 3. It was observed by Kennelly (1899) that replacing the three resistors incident to v by three resistors making pairwise connections between the neighbors of v leaves the effective resistance of the network unchanged, provided that the new resistances are related to the old resistance via (using a different parametrization) the cube recurrence.

The main object of study in this paper is the **hexahedron recurrence**. The quantities related by the hexahedron recurrence are variables indexed by the vertices and faces of the cubic lattice $\mathbb{Z}^3$. Let $Z^3_{1/2}$ denote the even vertices of $\frac{1}{2}\mathbb{Z}^3$, that is, those whose coordinates sum to an integer. Each non-integer point of $Z^3_{1/2}$ is the center of a square face of the cubic lattice, perpendicular to one of the three coordinate axes.

Let

$$h, h^{(x)}, h^{(y)}, h^{(z)} : \mathbb{Z}^3 \to \mathbb{C}$$

be four functions on the three-dimensional lattice. We think of $h(v)$ as sitting on the vertex v. Similarly, $h^{(x)}(v)$ sits on the face center perpendicular to the x-axis having vertices $v, v+e_2, v+e_3$ and $v+e_2+e_3$; the values of $h^{(y)}(v)$ and $h^{(z)}(v)$ are similarly situated at face centers of the other two types. We may identify these four functions with a single function f on $Z^3_{1/2}$, where for integers v,

$$f(v) = h(v), f(v+(0,1/2,1/2)) = h^{(x)}(v), f(v+(1/2,0,1/2)) = h^{(y)}(v) \text{ and } f(v+(1/2,1/2,0)) = h^{(z)}(v) \,.$$

Definition 1 (hexahedron recurrence) *We say that four functions $h, h^{(x)}, h^{(y)}$ and $h^{(z)}$ satisfy the hexahedron recurrence if the following equations are satisfied for all $v \in \mathbb{Z}^3$.*

$$h^{(x)}_{(1)} h^{(x)} h = h^{(x)} h^{(y)} h^{(z)} + h_{(1)} h_{(2)} h_{(3)} + h h_{(1)} h_{(23)} \tag{3}$$

$$h^{(y)}_{(2)} h^{(y)} h = h^{(x)} h^{(y)} h^{(z)} + h_{(1)} h_{(2)} h_{(3)} + h h_{(2)} h_{(13)} \tag{4}$$

$$h^{(z)}_{(3)} h^{(z)} h = h^{(x)} h^{(y)} h^{(z)} + h_{(1)} h_{(2)} h_{(3)} + h h_{(3)} h_{(12)} \tag{5}$$

$$
\begin{aligned}
h_{(123)}h^2h^{(x)}h^{(y)}h^{(z)} &= \left(h^{(x)}h^{(y)}h^{(z)}\right)^2 \qquad (6)\\
&\quad +h^{(x)}h^{(y)}h^{(z)}\left[2h_{(1)}h_{(2)}h_{(3)}+hh_{(1)}h_{(23)}+hh_{(2)}h_{(13)}+hh_{(3)}h_{(12)}\right]\\
&\quad +\left(h_{(1)}h_{(2)}+hh_{(12)}\right)\left(h_{(1)}h_{(3)}+hh_{(13)}\right)\left(h_{(2)}h_{(3)}+hh_{(23)}\right). \qquad (7)
\end{aligned}
$$

Here again $h_{(1)} = h_{v+e_1}$ and so on.

Statistical mechanical interpretations

The octahedron recurrence on $\mathbb{Z}^3$ expresses f_v as a Laurent polynomial in the values f_w as w ranges over variables in a region lying underneath v in an initial plane. This Laurent polynomial is a generating function for a statistical mechanical ensemble: its monomials are in bijection with perfect matchings of the **Aztec diamond** graph, associated with the region in the initial graph lying underneath v (see, e.g., Speyer 2007). Setting the initial indeterminates all equal to one allows us to count perfect matchings; in general the indeterminates represent multiplicative weights, which we may change in certain natural ways to study further properties of the ensemble of perfect matchings.

The cube recurrence (2) also has a combinatorial interpretation. Its monomials are in bijection with **cube groves**. These were first defined and studied by Carroll and Speyer (2004). In a cube grove, each edge of a large triangular box in the planar triangular lattice is either present or absent. The allowed configurations are those in which there are no cycles and no islands (thus they are **essential spanning forests**), and the connectivity of boundary points has a prescribed form. Both Aztec diamond matchings and cube groves have limiting shapes. Specifically, as the size of the box goes to infinity, there is a boundary, which is an algebraic curve, outside of which there is no entropy and inside of which there is positive entropy per site. The hexahedron recurrence has a statistical mechanical interpretation as well. In Section 2 we define the **double-dimer** model on a finite bipartite graph; in Section 4 we prove the following theorem.

Theorem *The monomials of the Laurent polynomial h_{nnn} are in bijection with taut double-dimer coverings of the graph $\Gamma(U_{-3n})$.*

The remainder of the paper is spent investigating the properties of the double-dimer ensemble. In Section 5 we analyze the limiting shape under several natural, periodic specifications of the initial varibles. In Section 6 we find a specialization of the initial variables under which urban renewal becomes the Ising Y-Δ transformation, which is a transformation of the Ising model changing the interaction strengths in a different way from how they change under the resistor network Y-Δ transformation, but changing the graph in exactly the same way.

2 Dimer model

Definitions

Let Γ be a finite bipartite graph with positive edge weights $\nu : E \to \mathbb{R}_+$. A **dimer cover** or **perfect matching** is a collection of edges with the property that every vertex is an endpoint of exactly one edge. The “dimers” of a dimer cover are the chosen edges (terminology suggesting a collection of bi-atomic molecules packed into the graph). We let $\Omega_d(\Gamma)$ be the set of dimer covers and we define the probability measure μ_d on Ω_d giving a dimer cover $m \in \Omega_d$ a probability proportional to the product of its edge weights. A **double-dimer configuration** is a union of two dimer covers: it is a covering of the graph with loops and doubled edges. The **double dimer measure** μ_{dd} is the probability measure defined by taking the union of two μ_d-independent dimer covers.

It is convenient to parametrize the measure μ_d by parameters other than the edge weight function ν. Given a function A on the faces of a planar bipartite graph, define

$$\nu_A(e) = \frac{1}{A(f)A(g)}\,, \tag{8}$$

where f and g are the two faces containing e.

Urban renewal

Certain local rearrangements of Γ preserve the dimer measure μ_d, see Ciucu (1998). Some are obvious. For example, given a vertex v of degree 2 (with equal edge weights) one can **contract** its two edges, erasing v and merging its two neighbors into one vertex. Another local rearrangement is called **urban renewal**. It involves taking a quadrilateral face, call it 0, and adding "legs". This is shown in Figure 1, ignoring for the moment the specific values $a_0, \ldots, a_4$ shown for the pre-weights $A(0), \ldots, A(4)$. Let us

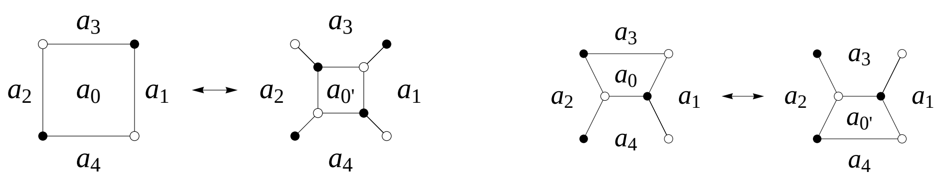


Fig. 1: Two versions of urban renewal; the central variable a_0 changes from a_0 to $a_5 = \dfrac{a_1a_2 + a_3a_4}{a_0}$.

designate the faces around face 0 by the numbers 1, 3, 2 and 4. Each of these faces gains two new edges. In the new graph Γ', there are faces $1', 2', 3'$ and $4'$ each with two more edges than the corresponding face $1, 2, 3, 4$. There is a face $0'$ which is also square. Each other face f of Γ corresponds to a face f' of Γ' with the same number of edges as f. There are four new neighboring relations among faces: $1', 2', 3'$ and $4'$ are neighbors in cyclic order, in addition to any neighboring relations that may have held before. The point of urban renewal is to give a corresponding adjustment of the weights that preserves μ_d. This is most easily done in terms of the A variables.

Proposition 2 (urban renewal) *Suppose 0 is a quadrilateral face of* Γ. *Let* Γ' *be constructed from* Γ *as above. Define the new pre-weight function* $A : F' \to \mathbb{C}$ *by* $A(f') = A(f)$ *if* $f \neq 0$ *and*

$$A(0') := \frac{A(1)A(2) + A(3)A(4)}{A(0)}\,.$$

Let μ' *denote the dimer measure on* Γ' *with face weights* $X^{A'}$ *and* μ *the dimer measure on* Γ *with face weights* X^A. *Then* μ *and* μ' *may be coupled so that the sample pair* (m, m') *agrees on every edge other than the four edges bounding face 0 in* Γ *and the eight edges touching face* $0'$ *in* Γ'. $\square$

The transformation of (Γ, A) to (Γ', A') under urban renewal is, in the language of cluster algebras, a mutation operation. It follows (Fomin and Zelevinsky 2002a) that that the final variables after any number of urban renewals are Laurent polynomials in the original variables $\{A(f) : f \in F\}$.

Superurban renewal transformation for the dimer model

Figure 2 defines **superurban renewal** as a sequence of six urban renewals on a planar face-weighted bipartite graph at a hexagonal face for which with at least one alternating set of neighbors is quadrilateral.

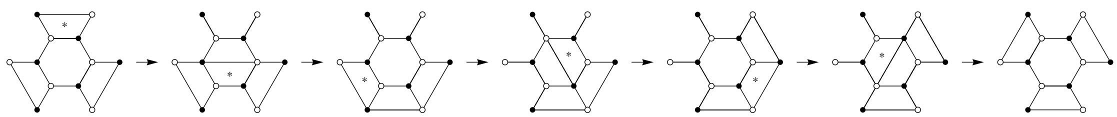


Fig. 2: Superurban renewal: the stars indicate which face undergoes urban renewal

The end result of superurban renewal is the transformation of face-weighted graphs shown in Figure 3.

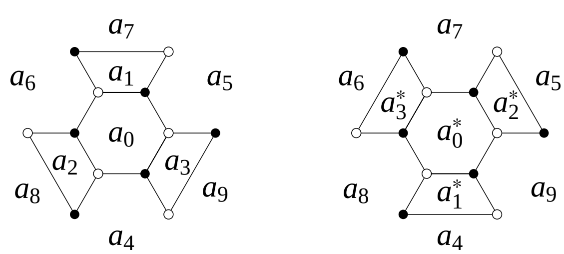


Fig. 3: Result of superurban renewal

Computing the result of the six operations yields the following equations for the four new quantities a_0^*, a_1^*, a_1^* and a_3^* in Figure 3 in terms of the old quantities $a_0 - a_9$.

$$a_1^* = \frac{a_1a_2a_3 + a_4a_5a_6 + a_0a_4a_7}{a_0a_1} \tag{9}$$

$$a_2^* = \frac{a_1a_2a_3 + a_4a_5a_6 + a_0a_5a_8}{a_0a_2} \tag{10}$$

$$a_3^* = \frac{a_1a_2a_3 + a_4a_5a_6 + a_0a_6a_9}{a_0a_3} \tag{11}$$

$$a_0^* = \frac{a_1^2a_2^2a_3^2 + a_1a_2a_3(2a_4a_5a_6 + a_0a_4a_7 + a_0a_5a_8 + a_0a_6a_9) + (a_5a_6 + a_0a_7)(a_4a_5 + a_0a_9)(a_4a_6 + a_0a_8)}{a_0^2a_1a_2a_3} \tag{12}$$

Because superurban renewal is built from urban renewal, the Laurent property for urban renewal stemming from its cluster algebra representation immediately yields

Proposition 3 (Laurent property for superurban renewal) *Under iterated superurban renewal, all new variables are Laurent polynomials in the original variables.* □

Just as urban renewal is the basis for the octahedron recurrence, we will see that superurban renewal is the basis for the hexahedron recurrence (see Section 3). In section 6 we show how superurban renewal specializes to the Y-Delta transformation for the Ising model.

3 Stepped surfaces and the operation of adding a cube

A **stepped solid** in $\mathbb{R}^3$ is a union U of lattice cubes that is downwardly closed (closed under moves in the $-x, -y$ and $-z$ directions). A **stepped surface** is the topological boundary of a stepped solid. Every stepped surface is the union of lattice squares and every lattice square has vertex set of the form $\{v, v+e_i, v+e_j, v+e_i+e_j\}$ for some $v \in \mathbb{Z}^3$ and some integers $1 \leq i < j \leq 3$. For each stepped surface ∂U, the **associated graph** $\Gamma(U)$ is obtained by starting with the planar dual graph and replacing each vertex by a small quadrilateral. Figure 4 shows the **4-6-12** graph, which is the graph $\Gamma(U)$ when $U = Z_2$ is the union of all cubes lying entirely within the region $\{(x,y,z) : x+y+z \leq 2\}$.

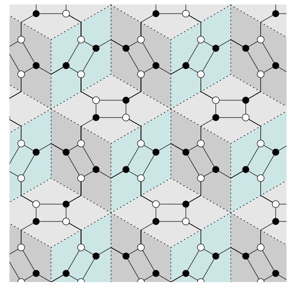


Fig. 4: The 4-6-12 graph is drawn on the stepped surface U_0 bounding the union of cubes up to level 2

Proposition 4 (superurban renewal is adding a cube) *Let U be a downwardly closed stepped solid with stepped surface ∂U and associated graph $\Gamma(U)$. Suppose that (i,j,k) is a point of ∂U which is a local minimum with respect to the height function $i+j+k$. Let U^{+ijk} be the union of U with the cube $[i,i+1] \times [j,j+1] \times [k,k+1]$.*

1. *The face in $\Gamma(U)$ corresponding to (i,j,k) is a hexagon with alternating neighbors quadrilateral.*
2. *The graph $\Gamma(U^{+ijk})$ is obtained from the graph $\Gamma(U)$ by superurban renewal at this hexagon.*
3. *The variables associated with each face of $\Gamma(U)$ transform under superurban renewal (9)–(12) according to the hexahedron recurrence (3)–(7), provided we interpret $h(i,j,k) = A(i,j,k)$, $h^{(x)}(i,j,k) = A(i,j+1/2,k+1/2)$ and so forth.*

□

We now know that adding a cube to a downwardly closed stepped solid corresponds to superurban renewal on the associated graph, which corresponds to the use of the hexahedron recurrence to write the top variable in terms of lower variables.

Let U_0 be the union of cubes in the negative orthant. The associated graph $\Gamma(U_0)$ is called the **cubic corner graph** and is shown on the left of Figure 5. Let $\mathcal{L}$ be the lattice of all downwardly closed subsets of U_0 containing all but finitely many cubes of U_0. For each $U \in \mathcal{L}$, one may add a finite sequence of cubes resulting in U_0. Therefore, a finite sequence of superurban renewals represents $A(0,0,0)$ in terms of the variables labeling faces and vertices of the stepped surface ∂U that are in the union of the removed lattice cubes. Denote this set of variables by $\mathcal{I} = \mathcal{I}(U)$.

Proposition 5 (i) *The rational function F representing $A(0,0,0)$ in terms of the variables in $\mathcal{I}$ is a Laurent polynomial.* (ii) *If $U' \subseteq U$ in $\mathcal{L}$ and the representation of each variable $w \in \mathcal{I}(U)$ in terms of variables in $\mathcal{I}(U')$ is substituted into F, the resulting Laurent polynomial is the representation of $A(0,0,0)$ in terms of variables in $\mathcal{I}(U')$.*

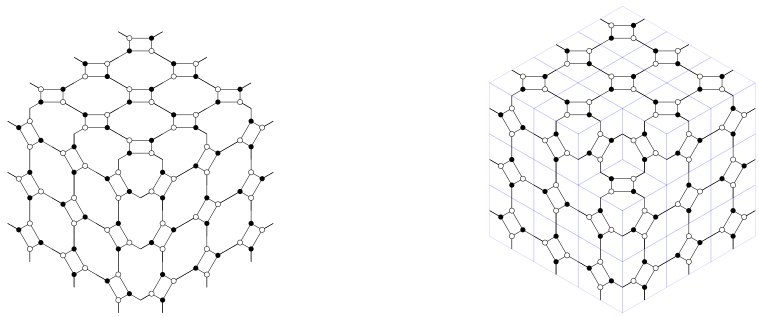


Fig. 5: The cubic corner graph, before and after removing the top cube

Proof: By Proposition 4, the expression F is obtained by a sequence of superurban renewals. By definition, each of these is a sequence of six urban renewals, hence part (i) follows from the Laurent property for urban renewal. Part (ii) is a consequence of the lack of relations among the variables in any stepped surface. □

Two classes of examples of stepped surfaces play a role in our combinatorial interpretations of these formulae. The first are the parallel surfaces Z_{-n} defined to be the set of all lattice cubes lying in the halfspace $x + y + z \leq -n$. The associated graph $\Gamma(\partial Z_{-n})$ is isomorphic to the 4-6-12 graph of Figure 4. Its labels are precisely $\mathcal{I}(Z_{-n})$ of $\mathbb{Z}^3$ at levels $-n-2, -n-1$ and $-n$ together with the half integer points at level $-n-1$. The second are the surfaces U_{-n} defined to be those cubes of U_0 lying entirely within the halfspace $\{(x, y, z) : x + y + z \leq -n\}$. This solid and its associated graph are illustrated for $n = -1$ (only the top cube removed) on the right of Figure 5.

The graphs U_{-n} differ from U_0 by finitely many cubes so they are better for recurrences, while the graphs Z_{-n} are translation invariant so they are better for exhibiting translation invariant formulae. The hexahedron recurrence imposes no relations on $\mathcal{I}(-n)$, hence from the point of view of determining $A(0, 0, 0)$ as a function of the variables in $\mathcal{I}(-n)$, we may use U_{-n} instead, thus guaranteeing a finite recursion.

We define a double-dimer configuration m_0 on the cubic corner graph $\Gamma(U_0)$ as in Figure 6. This configuration m_0 plays the role of our initial configuration. This configuration has the following property. If we erase a finite piece of m_0, there is a unique way to complete it to a double-dimer configuration which has the same boundary connections, that is, connections between far-away points. For $U \in \mathcal{L}$, we say that a double-dimer configuration on $\Gamma(U)$ is **taut** if it has the same boundary connections as m_0, that is, it is identical to m_0 far from the origin and there is a bijection between its bi-infinite paths and those of m_0 which is the identity near ∞. There are a finite number of taut configurations. See Figure 6 for one such on $\Gamma(U_{-1})$.

4 Main formula

Given a taut dimer configuration m, let $c(m)$ denote the number of loops in m and define $c(m; i, j, k) := L(i, j, k) - 2 - d(m; i, j, k)$ where $L(i, j, k)$ is the number of edges in the face (i, j, k) and $d(m; i, j, k)$ is the number of dimers lying along the face (i, j, k) in the matching m. In the configuration m_0, all quadrilateral faces have 2 dimers and all octagonal faces have 6 dimers, so the only face (i, j, k) with

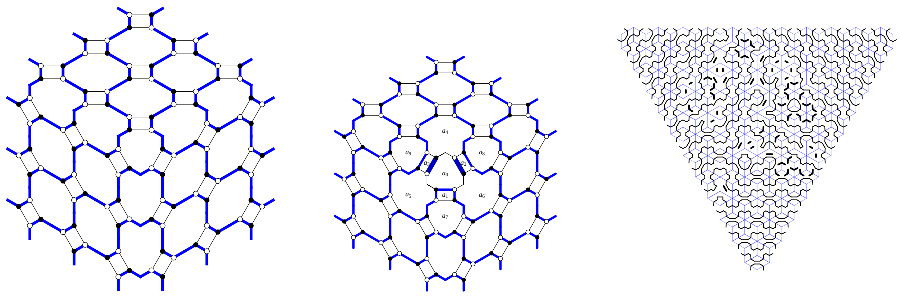


Fig. 6: Left: the initial double-dimer configuration m_0; Center: a taut configuration on $\Gamma(U_{-1})$; Right: a taut configuration on $\Gamma(U_{-12})$.

$c(m_0; i, j, k) \neq 0$ is the hexagonal face which has 3 dimers and $c(m_0; 0, 0, 0) = 6 - 2 - 3 = 1$. Any taut configuration differs from m_0 in finitely many places, hence has finitely many variables appearing in it.

Theorem 6 *Fix any $U \in \mathcal{L}$ and let $\mathcal{I}(U)$ be the labels of $\Gamma(U)$. Use the notation $m \preceq U$ to signify that m is a taut double-dimer configuration on $\Gamma(U)$. Then the representation of $A(0,0,0)$ as a Laurent polynomial in the variables in $\mathcal{I}(U)$ is given by*

$$A(0,0,0) = \sum_{m \preceq U} 2^{c(m)} \prod_{(i,j,k) \in \mathcal{I}(U)} A(i,j,k)^{c(m;i,j,k)} . \tag{13}$$

Specializing to U_{-n} and $A(i,j,k) = 1$ for all i,j,k with $-n-2 \leq i+j+k \leq -n$ gives the formula

$$A(0,0,0) = \sum_{m \preceq U_{-n}} 2^{c(m)} .$$

For example, the middle configuration of Figure 6 has monomial $\dfrac{a_4^2 a_5 a_6 a_7}{a_0 a_1 a_2 a_3}$.

Proof: We induct on U. It is true for $U = U_2$: there is one configuration, m_0, with $c(m_0; i, j, k) = 1$ if $i = j = k = 0$ and zero otherwise. For the induction, we need to see that the conclusion remains true if we remove a maximal cube, that is, when we execute a superurban renewal. Checking this for each type of boundary connection is straightforward. In one instance, the ratio of monomials on the right side and left side of the equation is

$$\frac{a_5^2 a_1^{2-4} a_2^{2-4} a_3^{2-4} a_4^{2-4}}{2a_0^{-2} a_1^{-1} a_2^{-1} a_3^{-1} a_4^{-1} + a_0^{-2} a_1^{-2} a_2^{-2} a_3^0 a_4^0 + a_0^{-2} a_1^0 a_2^0 a_3^{-2} a_4^{-2}} = \frac{a_5^2 a_0^2}{a_1 a_2 a_3 a_4 (2 + \frac{a_3 a_4}{a_1 a_2} + \frac{a_1 a_2}{a_3 a_4})} = 1 .$$

The other cases are similar. □

5 Limit shapes

Isotropic solutions

In this section we specialize values of the initial variables in several natural ways and study the behavior of the resulting ensembles. Let $\mathcal{I} = \mathcal{I}(2)$ denote the integer vertices (i,j,k) with $i + j + k = 0, 1, 2$

together with the half integer vertices with $i+j+k=1$. Say that the function f is **isotropic** if it depends only on $i+j+k$ and whether (i,j,k) is integral. If f is isotropic on $\mathcal{I}$ then the hexahedron recurrence extends f to an isotropic function on all of $Z^3_{1/2}$. Letting A_n denote the common value on integral points with $i+j+k=n$ and B_n be the value at nonintegral points with $i+j+k=n+1$, it is easy to find isotropic solutions to the recurrence. The simplest is

$$A_n = 3^{n^2/2}, B_n = 2 \cdot 3^{(n+1)^2/2}\,. \tag{14}$$

Another interesting solution is obtained by setting the initial variables A_0, A_1, A_2, B_0 all equal to 1. This yields $A_3 = 14, B_1 = 3$ and $B_2 = 14$ and leads to the following proposition.

Proposition 7 *The number of taut double-dimer configurations of* $\Gamma(U_{-n})$*, weighted by* $2^{c(m)}$*, is equal to*

$$14^{\frac{n}{2}\left(\frac{n}{2}+1\right)+\frac{1}{4}\delta_{\mathrm{odd}}(n)}\,.$$

□

Recurrence for the derivative

Isotropic initial conditions allow simplification of the formal derivative of the four hexahdron recurrence equations with respect to a parameter t. Let $g_{(v)}$ denote the formal derivative of $\log f_{(v)}$ with respect to a formal parameter t. Taking the logarithmic derivative of the four recurrence relations and plugging in the initial conditions (14) gives the linear system

$$g_{(123)} = -g + \frac{1}{3}(g_{(1)} + g_{(2)} + g_{(3)} + g_{(23)} + g_{(13)} + g_{(12)})$$

and similar equations for $g^{(x)}_{(1)}, g^{(y)}_{(2)}$ and $g^{(z)}_{(3)}$. The first of these equations gives a self-contained recurrence for the logarithmic derivatives at the integer points. In other words, letting $F(x,y,z) = \sum g_{i,j,k} x^i y^j z^k$, we see that the solution to the recurrence above with boundary conditions $g(0,0,0) = 1$, $g(i,j,k) = 0$ for other points (i,j,k) with $i+j+k \le 0$ and satisfying the recurrence everywhere except at $(-1-1-1)$, we see that

$$F(x,y,z) = \frac{G(x,y,z)}{H(x,y,z)} = \frac{1}{1 + xyz - \frac{1}{3}(x+y+z+xy+xz+yz)}\,.$$

This is the same as the recurrence as is satisfied by the cube grove placement probabilities (Petersen and Speyer 2005). The boundary of the dual cone is known as the “arctic circle”, which is the inscribed circle in the triangular region $\{x+y+z = n, x, y, z \ge 0\}$. Outside of this, the placement probabilities decay exponentially while inside the arctic circle they do not. Inside, the limit function is homogeneous of degree -1 and is asymptotically equal to the inverse of the distance to the arctic circle in the plane normal to the diagonal direction (Baryshnikov and Pemantle 2011). We can conclude from this that with high probability, a random configuration from Γ_n is equal to m_0 outside a neighborhood of size $o(n)$ of the arctic circle and that there is positive local entropy everywhere inside the arctic circle.

Different periodic initial conditions lead to different limiting shapes. As a somewhat generic example, let $A_0 = 1, B_0 = 1, A_1 = 2, A_2 = 3$. The resulting linear system is more complicated but may be solved and yields a recursion with characteristic polynomial

$$H = 63x^2y^2z^2 - 62(x^2yz + xy^2z + xyz^2) - (x^2y^2 + x^2z^2 + y^2z^2) + 62(xy + xz + yz) + (x^2 + y^2 + z^2) - 63\,.$$

The generating function $F(x,y,z) = \sum_{i,j,k} g(i,j,k)x^i y^j z^k$ is a rational function with denominator H. Asymptotics for $g(i,j,k)$ may be computed from the generating functions by the methods of Baryshnikov and Pemantle (2011). We briefly sketch the computation.

Step one is to compute the leading homogeneous part $\overline{H}$ of H at $(1,1,1)$, namely the terms of lowest degree in the Taylor expansion there. In this case $\overline{H}$ is the symmetric function

$$\overline{H} = 62(x^2y + xy^2 + x^2z + y^2z + xz^2 + yz^2) + 132xyz\,.$$

The arctic boundary is the algebraic dual of this cubic curve, which may be computed by setting $z = ax + by$ and setting $\partial\overline{H}/\partial x = \partial\overline{H}/\partial y = 0$, yielding

$$\begin{aligned} P^*(a,b) \quad = \quad & 923521 + 5125974\,ba - 3044572\,ab^2 - 2085370\,ab^5 - 3044572\,b^3a - 3044572\,a^2b + 45167\,a^2b^4 \\ & +5125974\,b^4a + 6191514\,a^2b^2 + 2233364\,b^3a^3 + 45167\,a^4b^2 - 3044572\,a^2b^3 - 2085370\,a^5b \\ & -3044572\,a^3b + 5125974\,a^4b - 3044572\,b^2a^3 - 2085370\,a - 2085370\,b + 45167\,a^2 + 45167\,b^2 \\ & +45167\,b^4 + 2233364\,b^3 + 2233364\,a^3 - 2085370\,b^5 + 45167\,a^4 - 2085370\,a^5 + 923521\,b^6 + 923521\,a^6\,. \end{aligned}$$

The arctic boundary is shown in Figure 7 after the change to barycentric coordinates $\alpha = a/(1-a-b), \beta = b/(1-a-b)$.
The meaning of this curve is that it represents the regions of sub-exponential decay of coefficients of the generating function: the triangle represents the set of directions in the positive orthant in $\mathbb{Z}^3$; a direction (α,β,γ) outside or on the arctic curve means that the coefficient $G_{\lfloor n\alpha\rfloor,\lfloor n\beta\rfloor,\lfloor n\gamma\rfloor}$ decays exponentially fast with n. The coefficients within the "temperate region" decay polynomially. The coefficients in the facet region near the center decay exponentially towards a constant nonzero value.

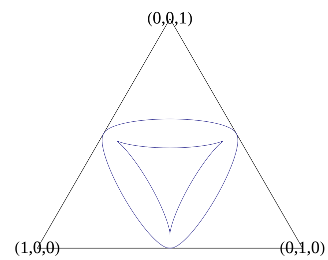


Fig. 7: The arctic boundary is a degree-6 curve.

The case for general initial conditions A_0, A_1, A_2, B_0 is not much different. Dividing out by a constant, the leading homogeneous term of the characteristic polynomial is in general given by

$$\overline{H} = x^2y + xy^2 + x^2z + y^2z + xz^2 + yz^2 + \lambda xyz$$

with $\lambda \in (2,3]$. This is irreducible for λ in the open interval $(2,3)$. The picture varies continuously with λ. When $\lambda = 3$, which corresponds to the initial conditions (14), the outer curve is the inscribed circle and the facet has shrunk to a point; in fact $\overline{H}$ factors in this case and is the same as the characteristic polynomial for cube groves. As λ approaches 2, the outer curve approaches an inscribed triangle and the facet expands to fill up the entire temperate region. The limiting characteristic polynomial at $\lambda = 2$ is the product $(x+y)(x+z)(y+z)$ of linear factors but is not attained by any initial conditions.

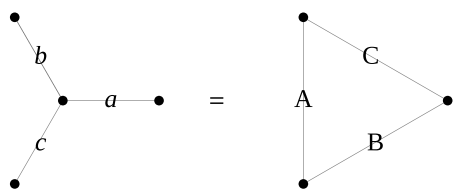
6 Ising model, the Ising-Y-Delta move, and Kashev's equation

In this section we will show how the Ising-Y-Delta move for the Ising model is a special case of the hexahedron recurrence. We begin by recalling the definition of the Ising model. Let $G = (V,E)$ be a finite graph with $c : E \to \mathbb{R}_+$ a positive weight function on edges. The Ising model is a probability measure μ on the configuration space $\Omega = \{\pm 1\}^V$. A configuration of spins $\sigma \in \Omega$ has probability

$$\mu(\sigma) = \frac{1}{Z} \prod_{e=\{v,v'\}\in E} c(e)^{(1+\sigma(v)\sigma(v'))/2}, \tag{15}$$

where the product is over all edges in E and the partition function Z is the sum of the unweighted probabilities $\prod c(e)^{(1+\sigma(v)\sigma(v'))/2}$ over all configurations σ. In other words, the probability of a configuration is proportional to the product of the edge weights of those edges where the spins are equal. The Ising model originated as a thermodynamical ensemble with energy function $H(\sigma) = -\sum_e \sigma(v)\sigma(v')J(e)$: take $J(e) = (T/2)\log c(e)$ where T is Boltzmann's constant times the temperature.

The Ising-Y-Delta move on the weighted graph $G = (V, E, c)$ transforms the graph the same way as does the Y-Delta move for electrical networks but transforms the edge weights differently. The transformation is depicted in Figure 8. Its key property is preservation of the associated measure.



$$A = \sqrt{\frac{(abc+1)(a+bc)}{(b+ac)(c+ab)}} \tag{16}$$

$$B = \sqrt{\frac{(abc+1)(b+ac)}{(a+bc)(c+ab)}} \tag{17}$$

$$C = \sqrt{\frac{(abc+1)(c+ab)}{(a+bc)(b+ac)}}. \tag{18}$$

Fig. 8: The Y-Delta move.

When placed on a lattice, these relations have an interpretation as a recurrence for stepped surfaces. Previously we associated a graph $\Gamma(U)$ with each stepped surface ∂U; now we associate a planar graph $\Upsilon(U)$. The vertices of $\Upsilon(U)$ are taken to be the even vertices of ∂U and the edges of $\Upsilon(U)$ are the digaonals of the faces of ∂U whose endpoints are even. If $f : \mathbb{Z}^d \to \mathbb{R}^+$ is a positive function, define edge weight $w(e)$ on an edge e of $\Upsilon(U)$ to be the positive solution to $(w - 1/w)^2/4 = b$ where $b = f(v)f(v')/(f(u)f(u'))$, where $e = \{v, v'\}$ and where u and u' are the other two vertices of the face of ∂U on which e lies. The following lemma is known as **Kashaev's difference equation**.

Lemma 8 (Kashaev (1996)) *Let $U \subseteq U'$ be stepped solids differing by a single cube.*

1. *The graph $\Upsilon(U')$ differs from $\Upsilon(U)$ by a Y-Delta move: Y to Delta if the bottom vertex of the added cube was even and Delta to Y otherwise.*
2. *If e is a weight function on the edges of $\Upsilon(U)$, extended by the Ising-Y-Delta relations to the edges of $\Upsilon(U')$, and if f is a function on the vertices of $\mathbb{Z}^d$ inducing e on the edges of $\Upsilon(U)$ and $\Upsilon(U')$ then at the eight vertices of the added cube, f satisfies the relations*

$$\begin{aligned} f^2 f_{(123)}^2 + f_{(1)}^2 f_{(23)}^2 + f_{(2)}^2 f_{(13)}^2 + f_{(3)}^2 f_{(12)}^2 - 2f_{(1)}f_{(2)}f_{(23)}f_{(13)} \\ -2f_{(1)}f_{(3)}f_{(23)}f_{(12)} - 2f_{(3)}f_{(2)}f_{(12)}f_{(13)} - 2ff_{(123)}(f_{(1)}f_{(23)} \\ +f_{(2)}f_{(13)} + f_{(3)}f_{(12)}) - 4ff_{(23)}f_{(13)}f_{(12)} - 4f_{(123)}f_{(1)}f_{(2)}f_{(3)} &= 0\,. \end{aligned} \tag{19}$$

□

Kashaev's equation for $f : \mathbb{Z}^3 \to \mathbb{C}$ may be embedded in the hexahedron recurrence by extending f to $Z^3_{1/2}$ as follows.

Proposition 9 *Suppose $f : Z^3_{1/2} \to \mathbb{C}$ satisfies the following relation for integer (i, j, k):*

$$\begin{aligned} f(i+1/2, j+1/2, k)^2 &= f(i,j,k)f(i+1,j+1,k) + f(i,j+1,k)f(i+1,j,k) \\ f(i+1/2, j, k+1/2)^2 &= f(i,j,k)f(i+1,j,k+1) + f(i,j,k+1)f(i+1,j,k) \\ f(i, j+1/2, k+1/2)^2 &= f(i,j,k)f(i,j+1,k+1) + f(i,j,k+1)f(i,j+1,k)\,. \end{aligned}$$

Then f satisfies Kashaev's relation (19) at integer points if and only if f satisfies the hexahedron relations (3)–(7), where as usual we interpret $h = f$, $h^{(x)} = f_{(0,1/2,1/2)}$, and so forth. □

Specializing the hexahedron recurrence creates redundancy, which may be exploited to produce simpler forms of the hexahedron/Kashaev recurrence. The following result may be proved.

Theorem 10 *Let $X(v) := f^{(x)}_{(v)}$, $Y(v) := f^{(y)}_{(v)}$, and $Z(v) := f^{(z)}_{(v)}$. Then $f_{i,j,k}$ may be written as a Laurent polynomial in the initial variables $\{f_{i,j,k}\}_{0 \le i+j+k \le 2}$ and $\{X_{i,j,k}, Y_{i,j,k}, Z_{i,j,k}\}_{i+j+k=0}$ with the X, Y, Z variables appearing only with power 0 or 1.* □

Open question What are the natural combinatorial structures counted by $f_{i,j,k}$?

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Ehrhart h^*-vectors of hypersimplices

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Abstract. We consider the Ehrhart h^*-vector for the hypersimplex. It is well-known that the sum of the h_i^* is the normalized volume which equals an Eulerian number. The main result is a proof of a conjecture by R. Stanley which gives an interpretation of the h_i^* coefficients in terms of descents and excedances. Our proof is geometric using a careful book-keeping of a shelling of a unimodular triangulation. We generalize this result to other closely related polytopes.

Résumé. Nous considérons que la Ehrhart h^*-vecteur pour la hypersimplex. il est bien connu que la somme de la h_i^* est le volume normalisé qui est égal à un nombre eulérien. Le résultat principal est une preuve de la conjecture par R. Stanley qui donne une interprétation des coefficients h_i^* en termes de descentes et excedances. Notre preuve est géom etrique àl'aide d'un attention la comptabilité d'un bombardement d'une triangulation unimodulaire. Nous généralisons ce résultat à d'autres polytopes étroitement liés.

Keywords: Hypersimplex, Ehrhart h^*-vector, Shellable triangulation, Eulerian statistics

1 Introduction

Hypersimplices appear naturally in algebraic and geometric contexts. For example, they can be considered as moment polytopes for torus actions on Grassmannians or weight polytopes of the fundamental representations of the general linear groups GL_n. Fix two integers $0 < k \leq n$. The (k,n)-th *hypersimplex* is defined as follows

$$\overline{\Delta}_{k,n} = \{(x_1, \dots, x_n) \mid 0 \leq x_1, \dots, x_n \leq 1;\, x_1 + \cdots + x_n = k\},$$

or equivalently,

$$\Delta_{k,n} = \{(x_1, \dots, x_{n-1}) \mid 0 \leq x_1, \dots, x_{n-1} \leq 1;\, k-1 \leq x_1 + \cdots + x_{n-1} \leq k\}.$$

They can be considered as the slice of the hypercube $[0,1]^{n-1}$ located between the two hyperplanes $\sum_{i=1}^{n-1} x_i = k-1$ and $\sum_{i=1}^{n-1} x_i = k$.

For a permutation $w \in \mathfrak{S}_n$, we call $i \in [n-1]$ a *descent* of w, if $w(i) > w(i+1)$. We define $\operatorname{des}(w)$ to be the number of descents of w. We call $A_{k,n-1}$ the Eulerian number, which equals the number of permutations in $\mathfrak{S}_{n-1}$ with $\operatorname{des}(w) = k-1$. The following result is well-known (see for example, [9, Exercise 4.59 (b)]).

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Theorem 1.1 (Laplace) *The normalized volume of $\Delta_{k,n}$ is the Eulerian number $A_{k,n-1}$.*

Let $S_{k,n}$ be the set of all points $(x_1,\dots,x_{n-1}) \in [0,1]^{n-1}$ for which $x_i < x_{i+1}$ for exactly $k-1$ values of i (including by convention $i=0$). Foata asked whether there is some explicit measure-preserving map that sends $S_{k,n}$ to $\Delta_{k,n}$. Stanley [6] gave such a map, which gave a triangulation of the hypersimplex into $A_{k,n-1}$ unit simplices and provided a geometric proof of Theorem 1.1. Sturmfels [10] gave another triangulation of $\Delta_{k,n}$, which naturally appears in the context of Gröbner bases. Lam and Postnikov [5] compared these two triangulations together with the alcove triangulation and the circuit triangulation. They showed that these four triangulations are identical. We call a triangulation of a convex polytope *unimodular* if every simplex in the triangulation has normalized volume one. It is clear that the above triangulations of the hypersimplex are unimodular.

Let $\mathcal{P} \in \mathbf{Z}^N$ be any n-dimensional integral polytope (its vertices are given by integers). Then Ehrhart's theorem tells us that the function

$$i(\mathcal{P},r) := \#(r\mathcal{P} \cap \mathbf{Z}^N)$$

is a polynomial in r, and

$$\sum_{r\geq 0} i(\mathcal{P},r)t^r = \frac{h^*(t)}{(1-t)^{n+1}},$$

where $h^*(t)$ is a polynomial in t with degree $\leq n$. We call $h^*(t)$ the h^*-*polynomial* of $\mathcal{P}$, and the vector $(h_0^*,\dots,h_n^*)$, where h_i^* is the coefficient of t^i in $h^*(t)$, is called the h^*-*vector* of $\mathcal{P}$. We denote its term by $h_i^*(\mathcal{P})$. It is known that the sum $\sum_{i=0}^n h_i^*(\mathcal{P})$ equals the normalized volume of $\mathcal{P}$.

Katzman [3] proved the following formula for the h^*-vector of the hypersimplex $\Delta_{k,n}$. In particular, we see that $\sum_{i=0}^n h_i^*(\Delta_{k,n}) = A_{k,n-1}$. Write $\binom{n}{r}_\ell$ to denote the coefficient of t^r in $(1+t+t^2+\cdots+t^{\ell-1})^n$. Then the h^*-vector of $\Delta_{k,n}$ is $(h_0^*(\Delta_{k,n}),\dots,h_{n-1}^*(\Delta_{k,n}))$, where for $d=0,\dots,n-1$

$$h_d^*(\Delta_{k,n}) = \sum_{i=0}^{k-1}(-1)^i\binom{n}{i}\binom{n}{(k-i)d-i}_{k-i}. \tag{1}$$

Moreover, since all the $h_i^*(\Delta_{k,n})$ are nonnegative integers ([7]) (this is not clear from (1)), it will be interesting to give a combinatorial interpretation of the $h_i^*(\Delta_{k,n})$.

The *half-open hypersimplex* $\Delta'_{k,n}$ is defined as follows. If $k>1$,

$$\Delta'_{k,n} = \{(x_1,\dots,x_{n-1}) \mid 0 \leq x_1,\dots,x_{n-1} \leq 1;\ k-1 < x_1+\cdots+x_{n-1} \leq k\},$$

and

$$\Delta'_{1,n} = \Delta_{1,n}.$$

We call $\Delta'_{k,n}$ "half-open" because it is basically the normal hypersimplex with the "lower" facet removed. From the definitions, it is clear that the volume formula and triangulations of the usual hypersimplex $\Delta_{k,n}$ also work for the half-open hypersimplex $\Delta'_{k,n}$, and it is nice that for fixed n, the half-open hypersimplices $\Delta'_{k,n}$, for $k=1,\dots,n-1$, form a disjoint union of the hypercube $[0,1]^{n-1}$. From the following formula for the h^*-polynomial of the half-open hypersimplices, we can compute the h^*-polynomial of the usual hypersimplices inductively. Also, we can compute its Ehrhart polynomial.

For a permutation w, we call i an *excedance* of w if $w(i) > i$ (a *reversed excedance* if $w(i) < i$). We denote by $\mathrm{exc}(w)$ the number of excedances of w. The main theorems of the paper are the following.

Theorem 1.2 *The h^*-polynomial of the half-open hypersimplex $\Delta'_{k,n}$ is given by*

$$\sum_{\substack{w\in \mathfrak{S}_{n-1}\\ \mathrm{exc}(w)=k-1}} t^{\mathrm{des}(w)}.$$

We prove this theorem first by a generating function method (in Section 2) and second by a geometric method, i.e., giving a shellable triangulation of the hypersimplex (in Sections 4). In Section 3, we will provide some background.

We can define a different shelling order on the triangulation of $\Delta'_{k,n}$, and get another expression of its h^*-polynomial using descents and a new permutation statistic called *cover* (see its definition in Lemma 5.4).

Theorem 1.3 *The h^*-polynomial of $\Delta'_{k,n}$ is*

$$\sum_{\substack{w\in\mathfrak{S}_{n-1}\\ \mathrm{des}(w)=k-1}} t^{\mathrm{cover}(w)}.$$

Combining Theorem 1.3 with Theorem 1.2, we have the equal distribution of $(\mathrm{exc}, \mathrm{des})$ and $(\mathrm{des}, \mathrm{cover})$:

Corollary 1.4

$$\sum_{w\in\mathfrak{S}_n} t^{\mathrm{des}(w)} x^{\mathrm{cover}(w)} = \sum_{w\in\mathfrak{S}_n} t^{\mathrm{exc}(w)} x^{\mathrm{des}(w)}.$$

Finally, we study the generalized hypersimplex $\Delta_{k,\alpha}$ (see definition in Section 6). This polytope is related to algebras of Veronese type. For example, it is known [1] that every algebra of Veronese type coincides with the Ehrhart ring of a polytope $\Delta_{k,\alpha}$. We can extend the second shelling to the generalized hypersimplex $\Delta'_{k,\alpha}$ (defined in (6)), and express its h^*-polynomial in terms of a colored version of descents and covers (see Theorem 6.2). This extended abstract is based on [4], where you can find more details.

2 Proof of Theorem 1.2 by generating functions

Here is a proof of this theorem using generating functions.

Proof: Suppose we can show that

$$\sum_{r\geq 0}\sum_{k\geq 0}\sum_{n\geq 0} i(\Delta'_{k+1,n+1}, r) u^n s^k t^r = \sum_{n\geq 0}\sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{des}(\sigma)} s^{\mathrm{exc}(\sigma)} \frac{u^n}{(1-t)^{n+1}}. \tag{2}$$

By considering the coefficient of $u^n s^k$ in (2), we have

$$\sum_{r\geq 0} i(\Delta'_{k+1,n+1}, r) t^r = (1-t)^{-(n+1)} \left(\sum_{\substack{w\in \mathfrak{S}_n\\ \mathrm{exc}(w)=k}} t^{\mathrm{des}(w)} \right),$$

which implies Theorem 1.2. By the following equation due to Foata and Han [2, Equation (1.15)],

$$\sum_{n\geq 0}\sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{des}(\sigma)}s^{\mathrm{exc}(\sigma)}\frac{u^n}{(1-t)^{n+1}} = \sum_{r\geq 0} t^r \frac{1-s}{(1-u)^{r+1}(1-us)^{-r}-s(1-u)},$$

we only need to show that

$$\sum_{k\geq 0}\sum_{n\geq 0} i(\Delta'_{k+1,n+1}, r)u^n s^k = \frac{1-s}{(1-u)^{r+1}(1-us)^{-r}-s(1-u)}.$$

By the definition of the half-open hypersimplex, we have, for any $r \in \mathbf{Z}_{\geq 0}$,

$$r\Delta'_{k+1,n+1} = \{(x_1,\ldots,x_n) \mid 0 \leq x_1,\ldots,x_n \leq r, rk+1 \leq x_1+\cdots+x_n \leq (k+1)r\},$$

if $k > 0$, and for $k = 0$,

$$r\Delta'_{1,n+1} = \{(x_1,\ldots,x_n) \mid 0 \leq x_1,\ldots,x_n \leq r, 0 \leq x_1+\cdots+x_n \leq r\}.$$

So

$$i(\Delta'_{k+1,n+1}, r) = ([x^{kr+1}]+\cdots+[x^{(k+1)r}])\left(\frac{1-x^{r+1}}{1-x}\right)^n, \tag{3}$$

if $k > 0$, and when $k = 0$, we have

$$i(\Delta'_{1,n+1}, r) = ([x^0]+[x]+\cdots+[x^r])\left(\frac{1-x^{r+1}}{1-x}\right)^n, \tag{4}$$

where the notation $[x^i]f(x)$ for some power series $f(x)$ denotes the coefficient of x^i in $f(x)$. Notice that the case of $k = 0$ is different from $k > 0$ and $i(\Delta'_{1,n+1}, r)$ is obtained by evaluating $k = 0$ in (3) plus an extra term $[x^0]\left(\frac{1-x^{r+1}}{1-x}\right)^n$. Since the coefficient of x^k of a function $f(x)$ equals the constant term of $\frac{f(x)}{x^k}$, we have

$$\begin{aligned}([x^{kr+1}]+\cdots+[x^{(k+1)r}])\left(\frac{1-x^{r+1}}{1-x}\right)^n &= [x^0]\left(\frac{1-x^{r+1}}{1-x}\right)^n (x^{-kr-1}+\cdots+x^{-(k+1)r})\\ &= [x^{kr}]\left(\frac{1-x^{r+1}}{1-x}\right)^n (x^{-kr-1}+\cdots+x^{-(k+1)r})x^{kr}\\ &= [x^{kr}]\frac{(1-x^r)(1-x^{r+1})^n}{(1-x)^{n+1}x^r}.\end{aligned}$$

So we have, for $k > 0$,

$$\begin{aligned}\sum_{n\geq 0} i(\Delta'_{k+1,n+1}, r)u^n &= \sum_{n\geq 0}[x^{kr}]\frac{(1-x^r)(1-x^{r+1})^n}{(1-x)^{n+1}x^r}u^n\\ &= [x^{kr}]\frac{(1-x^r)}{(1-x)x^r}\sum_{n\geq 0}\left(\frac{(1-x^{r+1})u}{1-x}\right)^n\\ &= [x^{kr}]\frac{x^r-1}{x^r(u-ux^{r+1}-1+x)}.\end{aligned}$$

For $k=0$, based on the difference between (3) and (4) observed above, we have:

$$\begin{aligned}\sum_{n\geq 0} i(\Delta'_{1,n+1},r)u^n &= \sum_{n\geq 0}[x^0]\frac{(1-x^r)(1-x^{r+1})^n}{(1-x)^{n+1}x^r}u^n + \sum_{n\geq 0}[x^0]\left(\frac{1-x^{r+1}}{1-x}\right)^n u^n \\ &= \left([x^0]\frac{x^r-1}{x^r(u-ux^{r+1}-1+x)}\right)+\frac{1}{1-u}.\end{aligned}$$

So

$$\sum_{k\geq 0}\sum_{n\geq 0} i(\Delta'_{k+1,n+1},r)u^n s^k = \left(\sum_{k\geq 0}[x^{kr}]\frac{x^r-1}{x^r(u-ux^{r+1}-1+x)}s^k\right)+\frac{1}{1-u}.$$

Let $y=x^r$. We have

$$\sum_{k\geq 0}\sum_{n\geq 0} i(\Delta'_{k+1,n+1},r)u^n s^k = \sum_{k\geq 0}[x^{kr}]\frac{y-1}{y(u-uxy-1+x)}s^k+\frac{1}{1-u}.$$

Expand $\frac{y-1}{y(u-uxy-1+x)}$ in powers of x, we have

$$\begin{aligned}\frac{y-1}{y(u-uxy-1+x)} &= \frac{y-1}{y}\cdot\frac{1}{u-1-(uxy-x)} \\ &= \frac{y-1}{y(u-1)}\cdot\frac{1}{1-\frac{x(uy-1)}{u-1}} \\ &= \frac{1-y}{y(1-u)}\sum_{i\geq 0}\left(\frac{(1-uy)x}{1-u}\right)^i.\end{aligned}$$

Since we only want the coefficient of x^i such that r divides i, we get

$$\begin{aligned}\frac{1-y}{y(1-u)}\sum_{j\geq 0}\left(\frac{(1-uy)x}{1-u}\right)^{rj} &= \frac{1-y}{y(1-u)}\cdot\frac{1}{1-\frac{(1-uy)^r x^r}{(1-u)^r}} \\ &= \frac{1-y}{y(1-u)}\cdot\frac{(1-u)^r}{(1-u)^r-(1-uy)^r x^r} \\ &= \frac{(1-u)^{r-1}(1-y)}{y(1-u)^r-y^2(1-yu)^r}.\end{aligned}$$

So

$$\sum_{k\geq 0}\sum_{n\geq 0} i(\Delta'_{k+1,n+1},r)u^n s^k = \left(\sum_{k\geq 0}s^k[y^k]\frac{(1-u)^{r-1}(1-y)}{y(1-u)^r-y^2(1-yu)^r}\right)+\frac{1}{1-u}.$$

To remove all negative powers of y, we do the following expansion

$$\begin{aligned}\frac{(1-u)^{r-1}(1-y)}{y(1-u)^r - y^2(1-yu)^r} =& \frac{1-y}{(1-u)y} \cdot \frac{1}{1-\frac{y(1-yu)^r}{(1-u)^r}} \\ =& \sum_{i\geq 0}\left(\frac{y^{i-1}(1-uy)^{ri}}{(1-u)^{ri+1}} - \frac{y^{i}(1-uy)^{ri}}{(1-u)^{ri+1}}\right) \\ =& \frac{1}{1-u}y^{-1} + \text{ nonnegative powers of } y.\end{aligned}$$

Notice that $\sum_{k\geq 0} s^k[y^k]\frac{(1-u)^{r-1}(1-y)}{y(1-u)^r-y^2(1-yu)^r}$ is obtained by taking the sum of nonnegative powers of y in $\frac{(1-u)^{r-1}(1-y)}{y(1-u)^r-y^2(1-yu)^r}$ and replacing y by s. So

$$\sum_{k\geq 0} s^k[y^k]\frac{(1-u)^{r-1}(1-y)}{y(1-u)^r-y^2(1-yu)^r} = \frac{(1-u)^{r-1}(1-s)}{s(1-u)^r - s^2(1-su)^r} - \frac{1}{s(1-u)}.$$

Therefore,

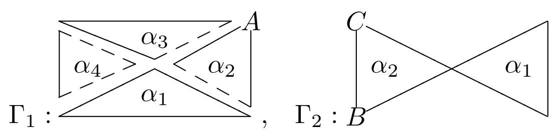
$$\begin{aligned}\sum_{k\geq 0}\sum_{n\geq 0} i(\Delta'_{k+1,n+1}, r)u^n s^k =& \frac{(1-u)^{r-1}(1-s)}{s(1-u)^r - s^2(1-su)^r} - \frac{1}{s(1-u)} + \frac{1}{1-u} \\ =& \frac{1-s}{(1-u)^{r+1}(1-us)^{-r} - s(1-u)}.\end{aligned}$$

□

3 Background

3.1 Shellable triangulation and the h^*-polynomial

Let Γ be a triangulation of an n-dimensional polytope $\mathcal{P}$, and let $\alpha_1, \dots, \alpha_s$ be an ordering of the simplices (maximal faces) of Γ. We call $(\alpha_1, \dots, \alpha_s)$ a *shelling* of Γ [7], if for each $2 \leq i \leq s$, $\alpha_i \cap (\alpha_1 \cup \cdots \cup \alpha_{i-1})$ is a union of facets ($(n-1)$-dimensional faces) of α_i. For example, (ignore the letters A, B, and C for now) Γ_1 is a shelling, while any order starting with Γ_2 cannot be a shelling.



An equivalent condition (see e.g., [8]) for a shelling is that every simplex has a *unique minimal non-face*, where by a "non-face", we mean a face that has not appeared in previous simplices. For example, for $\alpha_2 \in \Gamma_1$, the vertex A is its unique minimal non-face, while for $\alpha_2 \in \Gamma_2$, both B and C are minimal and have not appeared before α_2. We call a triangulation with a shelling a *shellable triangulation*. Given a shellable triangulation Γ and a simplex $\alpha \in \Gamma$, define the *shelling number* of α (denoted by $\#(\alpha)$) to be the number of facets shared by α and some simplex preceding α in the shelling order. For example, in Γ_1, we have

$$\#(\alpha_1) = 0,\ \#(\alpha_2) = 1,\ \#(\alpha_3) = 1,\ \#(\alpha_4) = 2.$$

The benefit of having a shelling order for Theorem 1.2 comes from the following result.

Theorem 3.1 ([7] Shelling and Ehrhart polynomial) *Let Γ be a unimodular shellable triangulation of an n-dimensional polytope $\mathcal{P}$. Then*

$$\sum_{r\geq 0} i(\mathcal{P}, r)t^r = \left(\sum_{\alpha\in\Gamma} t^{\#(\alpha)}\right)(1-t)^{-(n+1)}.$$

3.2 Excedances and descents

Let $w \in \mathfrak{S}_n$. Define its *standard representation of cycle notation* to be a cycle notation of w such that the first element in each cycle is its largest element and the cycles are ordered with their largest elements increasing. We define the *cycle type* of w to be the composition of n: $\mathrm{C}(w) = (c_1, \ldots, c_k)$ where c_i is the length of the ith cycle in its standard representation. The *Foata map* $F\colon w \to \hat{w}$ maps w to $\hat{w}$ obtained from w by removing parentheses from the standard representation of w. For example, consider a permutation $w\colon [5] \to [5]$ given by $w(1) = 5$, $w(2) = 1$, $w(3) = 4$, $w(4) = 3$ and $w(5) = 2$ or in one line notation $w = 51432$. Its standard representation of cycle notation is $(43)(521)$, so $\hat{w} = 43521$. The inverse Foata map $F^{-1}\colon \hat{w} \to w$ allows us to go back from $\hat{w}$ to w as follows: first insert a left parenthesis before every left-to-right maximum and then close each cycle by inserting a right parenthesis accordingly. In the example, the left-to-right maximums of $\hat{w} = 43521$ are 4 and 5, so we get back $(43)(521)$. Based on the Foata map, we have the following result for the equal distribution of excedances and descents.

Theorem 3.2 (Excedances and descents) *The number of permutations in $\mathfrak{S}_n$ with k excedances equals the number of permutations in $\mathfrak{S}_n$ with k descents.*

3.3 Triangulation of the hypersimplex

We start from a unimodular triangulation $\{t_w \mid w \in \mathfrak{S}_n\}$ of the hypercube, where

$$t_w = \{(y_1, \ldots, y_n) \in [0,1]^n \mid y_{w_1} \leq y_{w_2} \leq \cdots \leq y_{w_n}\}.$$

It is easy to see that t_w has the following $n+1$ vertices: $v_0 = (0, \ldots, 0)$, and $v_i = (y_1, \ldots, y_n)$ given by $y_{w_1} = \cdots = y_{w_{n-i}} = 0$ and $y_{w_{n-i+1}} = \cdots = y_{w_n} = 1$. It is clear that $v_{i+1} = v_i + e_{w_{n-i}}$. Now define the following map ϕ ([6],[5]) that maps t_w to s_w, a simplex in $\Delta_{k+1,n+1}$, sending $(y_1, \ldots, y_n)$ to $(x_1, \ldots, x_n)$, where

$$x_i = \begin{cases} y_i - y_{i-1}, & \text{if } (w^{-1})_i > (w^{-1})_{i-1}, \\ 1 + y_i - y_{i-1}, & \text{if } (w^{-1})_i < (w^{-1})_{i-1}, \end{cases} \tag{5}$$

where we set $y_0 = 0$. For each point $(x_1, \ldots, x_n) \in s_w$, set $x_{n+1} = k + 1 - (x_1 + \cdots + x_n)$. Since v_{i+1} and v_i only differ in $y_{w_{n-i}}$, by (5), $\phi(v_i)$ and $\phi(v_{i+1})$ only differ in $x_{w_{n-i}}$ and $x_{w_{n-i}+1}$. More explicitly, we have

Lemma 3.3 *Denote w_{n-i} by r. For $\phi(v_i)$, we have $x_r x_{r+1} = 01$ and for $\phi(v_{i+1})$, we have $x_r x_{r+1} = 10$. In other words, from $\phi(v_i)$ to $\phi(v_{i+1})$, we move a 1 from the $(r+1)$th coordinate forward by one coordinate.*

Proof: First, we want to show that for $\phi(v_i)$, we have $x_r = 0$ and $x_{r+1} = 1$. We need to look at the segment $y_{r-1}y_ry_{r+1}$, of v_i. We know that $y_r = 0$, so there are four cases for $y_{r-1}y_ry_{r+1}$: 000, 001, 100, 101. If $y_{r-1}y_ry_{r+1} = 000$ for v_i, then $y_{r-1}y_ry_{r+1} = 010$ for v_{i+1}. Therefore, $w^{-1}_{r-1} < w^{-1}_r > w^{-1}_{r+1}$. Then by (5), we have $x_rx_{r+1} = 01$. Similarly, we can check in the other three cases that $x_rx_{r+1} = 01$ for $\phi(v_i)$.

Similarly, we can check the four cases for $y_{r-1}y_ry_{r+1}$: 010, 011, 110, 111 in $\phi(v_{i+1})$ and get $x_rx_{r+1} = 10$ in all cases. $\square$

Let $\mathrm{des}(w^{-1}) = k$. It follows from Lemma 3.3 that the sum of the coordinates $\sum_{i=1}^n x_i$ for each vertex $\phi(v_i)$ of s_w is either k or $k+1$. So we have the triangulation [6] of the hypersimplex $\Delta_{k+1,n+1}$: $\Gamma_{k+1,n+1} = \{s_w \mid w \in \mathfrak{S}_n,\ \mathrm{des}(w^{-1}) = k\}$.

Now we consider a graph $G_{k+1,n+1}$ on the set of simplices in the triangulation of $\Delta_{k+1,n+1}$. There is an edge between two simplices s and t if and only if they are adjacent (they share a common facet). We can represent each vertex of $G_{k+1,n+1}$ by a permutation and describe each edge of $G_{k+1,n+1}$ in terms of permutations [5]. We call this new graph $\Gamma_{k+1,n+1}$. It is clear that $\Gamma_{k+1,n+1}$ is isomorphic to $G_{k+1,n+1}$.

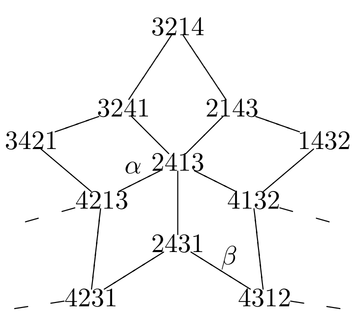
Proposition 3.4 ([5, Lemma 6.1 and Theorem 7.1]) *The graph $\Gamma_{k+1,n+1}$ can be described as follows: its vertices are permutations $u = u_1 \dots u_n \in \mathfrak{S}_n$ with $\mathrm{des}(u^{-1}) = k$. There is an edge between u and v, if and only if one of the following two holds:*

1. *(type one edge) $u_i - u_{i+1} \neq \pm 1$ for some $i \in \{1, \dots, n-1\}$, and v is obtained from u by exchanging u_i, u_{i+1}.*

2. *(type two edge) $u_n \neq 1, n$, and v is obtained from u by moving u_n to the front of u_1, i.e., $v = u_nu_1 \dots u_{n-1}$; or this holds with u and v switched.*

Example 3.5 *Here is the graph $\Gamma_{3,5}$ for $\Delta'_{3,5}$.*

$\Gamma_{3,5}$:



In the above graph, the edge α between $u = 2413$ and $v = 4213$ is a type one edge with $i = 1$, since $4 - 2 \neq \pm 1$ and one is obtained from the other by switching 2 and 4; the edge β between $u = 4312$ and $v = 2431$ is a type two edge, since $u_4 = 2 \neq 1, 4$ and $v = u_4u_1u_2u_3$. The dotted line attached to a simplex s indicates that s is adjacent to some simplex t in $\Delta_{2,5}$. Since we are considering the half-open hypersimplices, the common facet $s \cap t$ is removed from s.

4 Proof (outline) of Theorem 1.2 by a shellable triangulation

We want to show that the h^*-polynomial of $\Delta'_{k+1,n+1}$ is

$$\sum_{\substack{w \in \mathfrak{S}_n \\ \mathrm{exc}(w)=k}} t^{\mathrm{des}(w)}.$$

Compare this to Theorem 3.1: if $\Delta'_{k+1,n+1}$ has a shellable unimodular triangulation $\Gamma_{k+1,n+1}$, then its h^*-polynomial is

$$\sum_{\alpha \in \Gamma_{k+1,n+1}} t^{\#(\alpha)}.$$

We will define a shellable unimodular triangulation $\Gamma_{k+1,n+1}$ for $\Delta'_{k+1,n+1}$, label each simplex $\alpha \in \Gamma_{k+1,n+1}$ by a permutation $w_\alpha \in \mathfrak{S}_n$ with $\mathrm{exc}(w_\alpha) = k$. Then show that $\#(\alpha) = \mathrm{des}(w_\alpha)$.

We start from the triangulation $\Gamma_{k+1,n+1}$ studied in Section 3.3. By Proposition 3.4, each simplex is labeled by a permutation $u \in \mathfrak{S}_n$ with $\mathrm{des}(u^{-1}) = k$. Based on the Foata map defined in Section 3.2, we can use a sequence of maps and get a graph $S_{k+1,n+1}$ with vertices being permutations in $\mathfrak{S}_n$ with k excedances. Applying the above maps to vertices of $\Gamma_{k+1,n+1}$, we call the new graph $S_{k+1,n+1}$. We will define the shelling order on the simplices in the triangulation by orienting each edge in the graph $S_{k+1,n+1}$. If we orient an edge (u, v) such that the arrow points to u, then in the shelling, let the simplex labeled by u be after the simplex labeled by v. We can orient each edge of $S_{k+1,n+1}$ such that the directed graph is acyclic. This digraph therefore defines a partial order on the simplices of the triangulation. We can prove that any linear extension of this partial order gives a shelling order. Given any linear extension obtained from the digraph, the shelling number of each simplex is the number of incoming edges. Let w_α be the permutation in $S_{k+1,n+1}$ corresponding to the simplex α. Then we can show that for each simplex, its number of incoming edges equals $\mathrm{des}(w_\alpha)$. We will leave out the details here.

5 Proof of Theorem 1.3: second shelling

We want to show that the h^*-polynomial of $\Delta'_{k+1,n+1}$ is also given by

$$\sum_{\substack{w \in \mathfrak{S}_n \\ \mathrm{des}(w)=k}} t^{\mathrm{cover}(w)},$$

we will define cover in a minute. Compare this to Theorem 3.1: if $\Delta'_{k+1,n+1}$ has a shellable unimodular triangulation $\Gamma_{k+1,n+1}$, then its h^*-polynomial is

$$\sum_{\alpha \in \Gamma_{k+1,n+1}} t^{\#(\alpha)}.$$

Similar to the proof of Theorem 1.2, we will define shellable unimodular triangulation for $\Delta'_{k+1,n+1}$, but this shelling is different from the one we use for Theorem 1.2. Label each simplex $\alpha \in \Gamma_{k+1,n+1}$ by a permutation $w_\alpha \in \mathfrak{S}_n$ with $\mathrm{des}(w_\alpha) = k$. Then show that $\#(\alpha) = \mathrm{cover}(w_\alpha)$.

We start from the graph $\Gamma_{k+1,n+1}$ studied in Section 3.3. Define a graph $M_{k+1,n+1}$ such that $w \in V(M_{k+1,n+1})$ if and only if $w^{-1} \in V(\Gamma_{k+1,n+1})$ and $(w,u) \in E(M_{k+1,n+1})$ if and only if $(w^{-1},u^{-1}) \in E(\Gamma_{k+1,n+1})$. By Proposition 3.4, we have

$$V(M_{k+1,n+1}) = \{w \in \mathfrak{S}_n \mid \mathrm{des}(w) = k\},$$

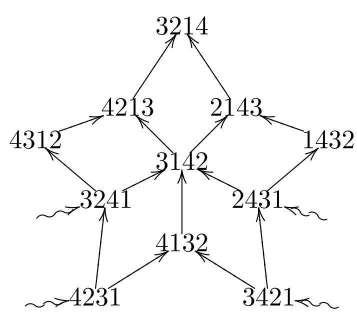
and $(w,u) \in E(M_{k+1,n+1})$ if and only if w and u are related in one of the following ways:

1. type one: exchanging the letters i and $i+1$ if these two letters are not adjacent in w and u
2. type two: one is obtained by subtracting 1 from each letter of the other (1 becomes $n-1$).

Now we want to orient the edges of $M_{k+1,n+1}$ to make it a digraph. Consider $e = (w,u) \in E(M_{k+1,n+1})$.

1. if e is of type one, and i is before $i+1$ in w, i.e., $\mathrm{inv}(w) = \mathrm{inv}(u) - 1$, then orient the edge as $w \leftarrow u$.
2. if edge (w,u) is of type two, and v is obtained by subtracting 1 from each letter of u (1 becomes $n-1$), then orient the edge as $w \leftarrow u$.

Example 5.1 *Here is the directed graph $M_{3,5}$ for $\Delta'_{3,5}$:*



Lemma 5.2 *There is no cycle in the directed graph $M_{k+1,n+1}$.*

Therefore, $M_{k+1,n+1}$ defines a poset on $V(M_{k+1,n+1})$ and $M_{k+1,n+1}$ is the Hasse diagraph of the poset, which we still denote as $M_{k+1,n+1}$.

For an element in the poset $M_{k+1,n+1}$, the larger its rank is, the further its corresponding simplex is from the origin. More precisely, notice that each $v = (x_1,\ldots,x_n) \in V_{k+1,n+1} = \Delta_{k+1,n+1} \cap \mathbf{Z}^n$ has $|v| = \sum_{i=1}^n x_i = k$ or $k+1$. For $u \in M_{k+1,n+1}$, by which we mean $u \in V(M_{k+1,n+1})$, define $A_u = \#\{v \text{ is a vertex of the simplex } s_{u^{-1}} \mid |v| = k+1\}$.

Proposition 5.3 *Let $w > u$ in the above poset $M_{k+1,n+1}$. Then $A_w \geq A_u$.*

This proposition follows from a lemma proving that $A_u = u_n$, and the definition of the two types of directed edges.

We define the *cover* of a permutation $w \in M_{k+1,n+1}$ to be the number of permutations $v \in M_{k+1,n+1}$ it covers, i.e., the number of incoming edges of w in the graph $M_{k+1,n+1}$. From the above definition, we have the following, (in the half-open setting):

Lemma 5.4 *1. If $w_1 = 1$, then* $\operatorname{cover}(w) = \#\{i \in [n-1] \mid (w^{-1})_i + 1 < (w^{-1})_{i+1}\}$*;*

2. if $w_1 \neq 1$, then $\operatorname{cover}(w) = \#\{i \in [n-1] \mid (w^{-1})_i + 1 < (w^{-1})_{i+1}\} + 1$.

Proposition 5.5 *Any linear extension of the above ordering gives a shelling order on the triangulation of* $\Delta'_{k+1,n+1}$.

It is clear that the shelling number of the simplex corresponding to w is $\operatorname{cover}(w)$. Then by Theorem 3.1 and Proposition 5.5, we have a proof of Theorem 1.3.

6 The h^*-polynomial for generalized half-open hypersimplex

We want to extend Theorem 1.3 to the hyperbox $B = [0, a_1] \times \cdots \times [0, a_n]$. Write $\alpha = (a_1, \ldots, a_n)$, $a_i \in \mathbb{Z}_{>0}$ and define the generalized half-open hypersimplex as

$$\Delta'_{k,\alpha} = \{(x_1, \ldots, x_n) \mid 0 \leq x_i \leq a_i; k - 1 < x_1 + \cdots + x_n \leq k\}. \tag{6}$$

Note that the above polytope is a multi-hypersimplex studied in [5]. For a nonnegative integral vector $\beta = (b_1, \ldots, b_n)$, let $C_\beta = \beta + [0,1]^n$ be the cube translated from the unit cube by the vector β. We call β the *color* of C_β.

We extend the triangulation of the unit cube to B by translation and assign to each simplex in B a *colored permutation*

$$w_\beta \in \mathfrak{S}_\alpha = \{w \in \mathfrak{S}_n \mid b_i < a_i, i = 1, \ldots, n\}.$$

Let $F_i = \{x_i = 0\} \cap [0,1]^n$ for $i = 1, \ldots, n$. Define the *exposed facets* for the simplex $s_{u^{-1}}$ in $[0,1]^n$ to be $\operatorname{Expose}(u) = \{i \mid s_{u^{-1}} \cap F_i \text{ is a facet of } s_{u^{-1}}\}$.

We can compute $\operatorname{Expose}(u)$ explicitly as follows.

Lemma 6.1 *Set $u_0 = 0$. Then* $\operatorname{Expose}(u) = \{i \in [n] \mid u_{i-1} + 1 = u_i\}$.

Now we want to extend the second shelling on the unit cube to the larger rectangle. In this extension, F_i will be removed from C_β if $b_i \neq 0$. Therefore, for the simplex s_{w_β}, we will remove the facet $F_i \cap s_{w_\beta}$ for each $i \in \operatorname{Expose}(w) \cap \{i \mid b_i \neq 0\}$ as well as the $\operatorname{cover}(w_\beta)$ facets for neighbors within C_β. We call this set $\operatorname{Expose}(w) \cap \{i \mid b_i \neq 0\}$ the *colored exposed facet (cef)*, denoted by $\operatorname{cef}(w_\beta)$, for each colored permutation $w_\beta = (w, \beta)$.

Based on the above extended shelling, with some modifications of Proposition 5.5, we can show that the above order is a shelling order. Then, by Theorem 3.1 and the fact that the shelling number for w_β is $\operatorname{cover}(w_\beta) + \operatorname{cef}(w_\beta)$, we have the following theorem.

Theorem 6.2 *The h^*-polynomial for $\Delta'_{k,\alpha}$ is*

$$\sum_{\substack{w_\beta \in \mathfrak{S}_\alpha \\ \operatorname{des}(w) + |\beta| = k-1}} t^{\operatorname{cover}(w_\beta) + \operatorname{cef}(w_\beta)}.$$

We have some interesting identities about exc, des, cover and Expose.

Proposition 6.3 *For any $k \in [n-1]$, we have*

1. $\#\{w \in \mathfrak{S}_n \mid \operatorname{exc}(w) = k, \operatorname{des}(w) = 1\} = \binom{n}{k+1}$.

2. $\{w \in \mathfrak{S}_n \mid \mathrm{des}(w) = k, \mathrm{cover}(w) = 1\} = \{w \in \mathfrak{S}_n \mid \#\,\mathrm{Expose}(w) = n - (k+1)\}$.
3. $\#\{w \in \mathfrak{S}_n \mid \mathrm{des}(w) = k, \mathrm{cover}(w) = 1, \mathrm{Expose}(w) = S\} = 1$, *for any* $S \subset [n]$ *with* $|S| = n - (k+1)$.
4. $\#\{w \in \mathfrak{S}_n \mid \mathrm{des}(w) = k, \mathrm{cover}(w) = 1\} = \binom{n}{k+1}$.

Proposition 6.4 *For any* $1 < k < n$, *we have*

1. $\#\{w \in \mathfrak{S}_n \mid \mathrm{exc}(w) = 1, \mathrm{des}(w) = k\} = \binom{n+1}{2k}$.
2. $\#\{w \in \mathfrak{S}_n \mid \mathrm{des}(w) = 1, \#\,\mathrm{Expose}(w) = n - 2k$ *or* $n + 1 - 2k\} = 1$
3. $\{w \in \mathfrak{S}_n \mid \mathrm{des}(w) = 1, \#\,\mathrm{Expose}(w) = n - 2k$ *or* $n + 1 - 2k\} \subset \{w \in \mathfrak{S}_n \mid \mathrm{cover}(w) = k\}$.
4. $\#\{w \in \mathfrak{S}_n \mid \mathrm{des}(w) = 1, \mathrm{cover}(w) = k\} = \binom{n}{2k} + \binom{n}{2k-1} = \binom{n+1}{2k}$.

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Euler flag enumeration of Whitney stratified spaces †

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Abstract. We show the $\mathbf{cd}$-index exists for Whitney stratified manifolds by extending the notion of a graded poset to that of a quasi-graded poset. This is a poset endowed with an order-preserving rank function and a weighted zeta function. This allows us to generalize the classical notion of Eulerianness, and obtain a $\mathbf{cd}$-index in the quasi-graded poset arena. We also extend the semi-suspension operation to that of embedding a complex in the boundary of a higher dimensional ball and study the shelling components of the simplex.

Résumé. Nous montrons le $\mathbf{cd}$-index existe pour les manifolds de Whitney stratifiées en élargissant la notion d'un poset gradué à celle que un poset quasi-gradué. C'est un poset doté avec une fonction de rang que préservant l'ordre du poset et une fonction de zêta qu'est pondérée. Ceci nous permet de généraliser la notion classique de "Eulerianness", et obtenir un $\mathbf{cd}$-index dans l'arène des posets quasi-gradués. Nous tenons également à l'opération de semi-suspension pour que d'intégrer une complexe dans la frontière d'une balle de dimension supérieur et étudions les composants des shelling d'un simplex.

Keywords: Eulerian condition, quasi-graded poset, semisuspension, weighted zeta function, Whitney's conditions A and B.

1 Introduction

In this paper we extend the theory of face incidence enumeration of polytopes, and more generally, chain enumeration in graded Eulerian posets, to that of Whitney stratified spaces and quasi-graded posets.

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The idea of enumeration using the Euler characteristic was suggested throughout Rota's work and influenced by Schanuel's categorical viewpoint [21, 23, 24, 25]. In order to carry out such a program that is topologically meaningful and which captures the broadest possible classes of examples, two key insights are required. First, the notion of grading in the face lattice of a polytope must be relaxed. Secondly, the usual zeta function in the incidence algebra must be extended to include the Euler characteristic as an important instance.

The *flag f-vector* of a graded poset counts the number of chains passing through a prescribed set of ranks. In the case of a polytope, it records all of the face incidence data, including that of the f-vector. Bayer and Billera proved that the flag f-vector of any Eulerian poset satisfies a collection of linear equalities now known as the *generalized Dehn–Sommerville relations* [2]. These linear equations may be interpreted as natural redundancies among the components of the flag f-vector. Bayer and Klapper removed these redundancies by showing that the space of flag f-vectors of Eulerian posets has a natural basis with Fibonacci many elements consisting of certain non-commutative polynomials in the two variables $\mathbf{c}$ and $\mathbf{d}$ [3]. The coefficients of this $\mathbf{cd}$-*index* were later shown by Stanley to be non-negative in the case of spherically-shellable posets [27]. Other milestones for the $\mathbf{cd}$-index include its inherent coalgebraic structure [11], its appearance in the proofs of inequalities for flag vectors [5, 9, 10, 20], its use in understanding the combinatorics of arrangements of subspaces and sub-tori [6, 14], and most recently, its connection to the Bruhat graph and Kazhdan–Lusztig theory [4, 13].

In this article we extend the $\mathbf{cd}$-index and its properties to a more general situation, that of quasi-graded posets and Whitney stratified spaces. A quasi-grading on a poset P consists of a strictly order-preserving "rank" function $\rho : P \rightarrow \mathbb{N}$ and a weighted zeta function $\bar{\zeta}$ in the incidence algebra $I(P)$ such that $\bar{\zeta}(x, x) = 1$ for all $x \in P$. See Section 2. A quasi-graded poset $(P, \rho, \bar{\zeta})$ will be said to be *Eulerian* if the function $(-1)^{\rho(y)-\rho(x)} \cdot \bar{\zeta}(x, y)$ is the inverse of $\bar{\zeta}(x, y)$ in the incidence algebra of P. This reduces to the classical definition of Eulerian if $(P, \rho, \bar{\zeta})$ is a ranked poset with the standard zeta function ζ.

Theorem 3.1 states that the $\mathbf{cd}$-index is defined for Eulerian quasi-graded posets. The existence of the $\mathbf{cd}$-index for Eulerian quasi-graded posets is equivalent to the statement that the flag $\bar{f}$-vector of an Eulerian quasi-graded poset satisfies the generalized Dehn–Sommerville relations (Theorem 3.2).

Eulerian ranked posets arise geometrically as the face posets of regular cell decompositions of a sphere, whereas Eulerian quasi-graded posets arise geometrically from the more general case of Whitney stratifications. A *Whitney stratification* X of a compact topological space W is a decomposition of W into finitely many smooth manifolds which satisfy Whitney's "no-wiggle" conditions on how the strata fit together. See Section 4. These conditions guarantee (a) that X does not exhibit Cantor set-like behavior and (b) that the closure of each stratum is a union of strata. The faces of a convex polytope and the cells of a regular cell complex are examples of Whitney stratifications, but in general, a stratum in a stratified space need not be contractible. Moreover, the closure of a stratum of dimension d does not necessarily contain strata of dimension $d - 1$, or for that matter, of any other dimension. Natural Whitney stratifications exist for real or complex algebraic sets, analytic sets, semi-analytic sets and for quotients of smooth manifolds by compact group actions.

The strata of a Whitney stratification (of a topological space W) form a poset, where the order relation

$A < B$ is given by $A \subset \overline{B}$. Moreover, this set admits a natural quasi-grading which is defined by $\rho(A) = \dim(A) + 1$ and $\bar{\zeta}(A, B) = \chi(\mathrm{link}(A) \cap B)$ whenever $A < B$ are strata and χ is the Euler characteristic. See Definition 4.4. This is the setting for our Euler-characteristic enumeration.

Theorem 4.5 states that *the quasi-graded poset of strata of a Whitney stratified set is Eulerian* and therefore its $\mathbf{cd}$-index is defined and its flag $\bar{f}$-vector satisfies the generalized Dehn–Sommerville relations. Due to space constraints, the background and results needed for this proof are omitted. Please refer to the full-length article for details.

It is important to point out that, unlike the case of polytopes, the coefficients of the $\mathbf{cd}$-index of Whitney stratified manifolds can be negative. See Examples 4.1 and 4.8. It is our hope that by applying topological techniques to stratified manifolds, a combinatorial interpretation for the coefficients of the $\mathbf{cd}$-index will be discovered. This may ultimately explain Stanley's non-negativity results for spherically shellable posets [27] and Karu's results for Gorenstein* posets [20], and settle the conjecture that non-negativity holds for regular cell complexes.

In his proof that the $\mathbf{cd}$-index of a polytope is non-negative, Stanley introduced the notion of semisuspension. Given a polytopal complex that is homeomorphic to a ball, the semisuspension adds another facet whose boundary is the boundary of the ball. The resulting spherical CW-complex has the same dimension, and the intervals in its face poset are Eulerian [27].

It is precisely the setting of Whitney stratified manifolds, and the larger class of Whitney stratified spaces, which is critical in order to study face enumeration of the semisuspension in higher dimensional spheres and more general topologically interesting examples. In Section 5 the nth semisuspension and its $\mathbf{cd}$-index is studied. In Theorem 5.2, by using the method of quasi-graded posets, we are able to give a short proof (that completely avoids the use of shellings) of a key result of Billera and Ehrenborg [5] that was needed for their proof that the n-dimensional simplex minimizes the $\mathbf{cd}$-index among all n-dimensional polytopes. Furthermore, we establish the Eulerian relation for the nth semisuspension (Theorem 5.3).

In Section 6 the $\mathbf{cd}$-index of the nth semisuspension of a non-pure shellable simplicial complex is determined. The $\mathbf{cd}$-index of the shelling components are shown to satisfy a recursion involving a derivation which first appeared in [11]. By relaxing the notion of shelling, we furthermore show that the shelling components satisfy a Pascal type recursion. This yields new expressions for the shelling components and illustrates the power of leaving the realm of regular cell complexes for that of Whitney stratified spaces.

2 Quasi-graded posets and their $\mathbf{ab}$-index

Recall the *incidence algebra* of a poset is the set of all functions $f : I(P) \to \mathbb{C}$ where $I(P)$ denotes the set of intervals in the poset. The multiplication is given by $(f \cdot g)(x, y) = \sum_{x \leq z \leq y} f(x, z) \cdot g(z, y)$ and the identity is given by the delta function $\delta(x, y) = \delta_{x,y}$, where the second delta is the usual Kronecker delta function $\delta_{x,y} = 1$ if $x = y$ and zero otherwise. A poset is said to be *ranked* if every maximal chain in the poset has the same length. This common length is called the *rank* of the poset. A poset is said to be *graded* if it is ranked and has a minimal element $\widehat{0}$ and a maximal element $\widehat{1}$. For other poset terminology, we refer the reader to Stanley's treatise [26].

We introduce the notion of a quasi-graded poset. This extends the notion of a ranked poset.

Definition 2.1 *A quasi-graded poset $(P, \rho, \bar{\zeta})$ consists of (i) a finite poset P (not necessarily ranked), (ii) a strictly order-preserving function ρ from P to $\mathbb{N}$, that is, $x < y$ implies $\rho(x) < \rho(y)$, and (ii) a function $\bar{\zeta}$ in the incidence algebra $I(P)$ of the poset P, called the* weighted zeta function*, such that $\bar{\zeta}(x, x) = 1$ for all elements x in the poset P.*

Observe that we do not require the poset to have a minimal element or a maximal element. Since $\bar{\zeta}(x, x) \neq 0$ for all $x \in P$, the function $\bar{\zeta}$ is invertible in the incidence algebra $I(P)$ and we denote its inverse by $\bar{\mu}$.

For $x \leq y$ in a quasi-graded poset $P = (P, \rho, \bar{\zeta})$, the *rank difference function* is given by $\rho(x, y) = \rho(y) - \rho(x)$. We say that a quasi-graded poset $(P, \rho, \bar{\zeta})$ with minimal element $\widehat{0}$ and maximal element $\widehat{1}$ has rank n if $\rho(\widehat{0}, \widehat{1}) = n$. The interval $[x, y]$ is itself a quasi-graded poset together with the rank function $\rho_{[x,y]}(w) = \rho(w) - \rho(x)$ and the weighted zeta function $\bar{\zeta}$.

Let $(P, \rho, \bar{\zeta})$ be a quasi-graded poset with unique minimal element $\widehat{0}$ and unique maximal element $\widehat{1}$. The assumption of a quasi-graded poset having a $\widehat{0}$ and $\widehat{1}$ will be essential in order to define its $\mathbf{ab}$-index and $\mathbf{cd}$-index. For a chain $c = \{x_0 < x_1 < \cdots < x_k\}$ in the quasi-graded poset P, define $\bar{\zeta}(c)$ to be the product

$$\bar{\zeta}(c) = \bar{\zeta}(x_0, x_1) \cdot \bar{\zeta}(x_1, x_2) \cdots \bar{\zeta}(x_{k-1}, x_k). \tag{2.1}$$

Similarly, for the chain c define its *weight* to be

$$\mathrm{wt}(c) = (\mathbf{a} - \mathbf{b})^{\rho(x_0,x_1)-1} \cdot \mathbf{b} \cdot (\mathbf{a} - \mathbf{b})^{\rho(x_1,x_2)-1} \cdot \mathbf{b} \cdots \mathbf{b} \cdot (\mathbf{a} - \mathbf{b})^{\rho(x_{k-1},x_k)-1},$$

where $\mathbf{a}$ and $\mathbf{b}$ are non-commutative variables each of degree 1. The $\mathbf{ab}$*-index of a quasi-graded poset* $(P, \rho, \bar{\zeta})$ is

$$\Psi(P, \rho, \bar{\zeta}) = \sum_{c} \bar{\zeta}(c) \cdot \mathrm{wt}(c), \tag{2.2}$$

where the sum is over all chains starting at the minimal element $\widehat{0}$ and ending at the maximal element $\widehat{1}$, that is, $c = \{\widehat{0} = x_0 < x_1 < \cdots < x_k = \widehat{1}\}$. When the rank function ρ and the weighted zeta function are clear from the context, we will write the shorter $\Psi(P)$. Observe that if a quasi-graded poset $(P, \rho, \bar{\zeta})$ has rank $n + 1$ then its $\mathbf{ab}$-index is homogeneous of degree n.

The $\mathbf{ab}$-index depends on the rank difference function $\rho(x, y)$ but not on the rank function itself. Hence we may uniformly shift the rank function without changing the $\mathbf{ab}$-index. Later we will use the convention that $\rho(\widehat{0}) = 0$.

The $\mathbf{ab}$-index of a quasi-graded poset is a coalgebra homomorphism. Define a coproduct $\Delta : \mathbb{Z}\langle\mathbf{a}, \mathbf{b}\rangle \longrightarrow \mathbb{Z}\langle\mathbf{a}, \mathbf{b}\rangle \otimes \mathbb{Z}\langle\mathbf{a}, \mathbf{b}\rangle$ by $\Delta(1) = 0$ and for an $\mathbf{ab}$-monomial $u = u_1 u_2 \cdots u_k$ by $\Delta(u) = \sum_{i=1}^{k} u_1 \cdots u_{i-1} \otimes$

$u_{i+1} \cdots u_k$ and extend Δ extend to $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ by linearity. It is straightforward to see that this coproduct is coassociative. The coproduct Δ first appeared in [11].

Theorem 2.2 *Let $(P, \rho, \bar{\zeta})$ be a quasi-graded poset. Then the following identity holds:*

$$\Delta(\Psi(P, \rho, \bar{\zeta})) = \sum_{\widehat{0} < x < \widehat{1}} \Psi([\widehat{0}, x], \rho, \bar{\zeta}) \otimes \Psi([x, \widehat{1}], \rho, \bar{\zeta}).$$

A quasi-graded poset is said to be *Eulerian* if for all pairs of elements $x \leq z$ we have that

$$\sum_{x \leq y \leq z} (-1)^{\rho(x,y)} \cdot \bar{\zeta}(x, y) \cdot \bar{\zeta}(y, z) = \delta_{x,z}. \tag{2.3}$$

In other words, the function $\bar{\mu}(x, y) = (-1)^{\rho(x,y)} \cdot \bar{\zeta}(x, y)$ is the inverse of $\bar{\zeta}(x, y)$ in the incidence algebra. In the case $\bar{\zeta}(x, y) = \zeta(x, y)$, we refer to relation (2.3) as the *classical Eulerian relation*.

Theorem 2.3 (Alexander duality for quasi-graded posets) *Let $(P, \rho, \bar{\zeta})$ be an Eulerian quasi-graded poset with $\widehat{0}$ and $\widehat{1}$ of rank $n + 1$. Let Q and R be two subposets of P such that $Q \cup R = P$ and $Q \cap R = \{\widehat{0}, \widehat{1}\}$. Then*

$$(\bar{\zeta}|_Q)^{-1}(\widehat{0}, \widehat{1}) = (-1)^n \cdot (\bar{\zeta}|_R)^{-1}(\widehat{0}, \widehat{1}).$$

3 The $\mathbf{cd}$-index and quasi-graded posets

Bayer and Billera determined all the linear relations which hold among the flag f-vector of (classical) Eulerian posets, known as the generalized Dehn–Sommerville relations [2]. Bayer and Klapper showed that the space of flag f-vectors of Eulerian posets have a natural basis of Fibonacci dimension as expressed by the $\mathbf{cd}$-index [3]. Stanley later gave a more elementary proof of the existence of the $\mathbf{cd}$-index for Eulerian posets and showed the coefficients are non-negative for spherically-shellable posets [27]. Generalizing the classical result of Bayer and Klapper for graded Eulerian posets, we have the analogue for quasi-graded posets.

Theorem 3.1 *For an Eulerian quasi-graded poset $(P, \rho, \bar{\zeta})$ its $\mathbf{ab}$-index $\Psi(P, \rho, \bar{\zeta})$ can be written uniquely as a polynomial in the non-commutative variables $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$. Furthermore, if the function $\bar{\zeta}$ is integer-valued then the $\mathbf{cd}$-index only has integer coefficients.*

A different way to express the existence of the $\mathbf{cd}$-index is as follows.

Theorem 3.2 *The flag $\bar{f}$-vector of an Eulerian quasi-graded poset of rank $n+1$ satisfies the generalized Dehn–Sommerville relations. More precisely, for a subset $S \subseteq \{1, \ldots, n\}$ and $i, k \in S \cup \{0, n+1\}$ with $i < k$ and $S \cap \{i+1, \ldots, k-1\} = \emptyset$, the following relation holds:*

$$\sum_{j=i}^{k} (-1)^j \cdot \bar{f}_{S \cup \{j\}} = 0. \tag{3.1}$$

4 Whitney stratified sets

Example 4.1 Consider the non-regular CW-complex Ω consisting of one vertex v, one edge e and one 2-dimensional cell c such that the boundary of c is the union $v \cup e$, that is, boundary of the complex Ω is a one-gon. Its face poset is the four element chain $\mathcal{F}(\Omega) = \{\widehat{0} < v < e < c\}$. This is not an Eulerian poset. The classical definition of the $\mathbf{ab}$-index, in other words, using $\bar{\zeta}(x, y) = 1$ for all $x \leq y$, yields that the $\mathbf{ab}$-index of Ω is $\mathbf{a}^2$. Note that $\mathbf{a}^2$ cannot be written in terms of $\mathbf{c}$ and $\mathbf{d}$.

Observe that the edge e is attached to the vertex v twice. Hence it is natural to change the value of $\bar{\zeta}(v, e)$ to be 2. The face poset $\mathcal{F}(\Omega)$ is now Eulerian, its $\mathbf{ab}$-index is given by $\bar{\zeta}(\Omega) = \mathbf{a}^2 + \mathbf{b}^2$ and hence its $\mathbf{cd}$-index is $\bar{\zeta}(\Omega) = \mathbf{c}^2 - \mathbf{d}$.

The motivation for the value 2 in Example 4.1 is best expressed in terms of the Euler characteristic of the link. The link of the vertex v in the edge e is two points whose Euler characteristic is 2. In order to view this example in the right topological setting, we review the notion of a Whitney stratification. For more details, see [8], [17], [18, Part I §1.2], and [22].

A subset S of a topological space M is *locally closed* if S is a relatively open subset of its closure $\overline{S}$. Equivalently, for any point $x \in S$ there exists a neighborhood $U_x \subseteq S$ such that the closure $\overline{U_x} \subseteq S$ is closed in M. Another way to phrase this is a subset $S \subset M$ is locally closed if and only if it is the intersection of an open subset and a closed subset of M.

Definition 4.2 *Let W be a closed subset of a smooth manifold M which has been decomposed into a finite union of locally closed subsets*

$$W = \bigcup_{X \in \mathcal{P}} X.$$

Furthermore suppose this decomposition satisfies the condition of the frontier*:*

$$X \cap \overline{Y} \neq \emptyset \Longleftrightarrow X \subseteq \overline{Y}.$$

This implies the closure of each stratum is a union of strata, and it provides the index set $\mathcal{P}$ with the partial ordering: $X \subseteq \overline{Y} \Longleftrightarrow X \leq_{\mathcal{P}} Y$. This decomposition of W is a Whitney stratification *if*

1. *Each $X \in \mathcal{P}$ is a (locally closed, not necessarily connected) smooth submanifold of M.*

2. *If $X <_{\mathcal{P}} Y$ then Whitney's conditions (A) and (B) hold: Suppose $y_i \in Y$ is a sequence of points converging to some $x \in X$ and that $x_i \in X$ converges to x. Also assume that (with respect to some local coordinate system on the manifold M) the secant lines $\ell_i = \overline{x_i y_i}$ converge to some limiting line ℓ and the tangent planes $T_{y_i} Y$ converge to some limiting plane τ. Then the following inclusions hold:*

$$\text{(A) } T_x X \subseteq \tau \qquad \text{and} \qquad \text{(B) } \ell \subseteq \tau.$$

Remark 4.3 An example of an algebraic set W with a decomposition into smooth manifolds that is *not* locally trivial is provided by *Whitney's cusp*. See [22, Example 2.6] and [30].

We next state the key definition for developing face incidence enumeration for Whitney stratified spaces.

Definition 4.4 *Let W be a Whitney stratified closed subset of a smooth manifold M. Define the face poset $\mathcal{F} = \mathcal{F}(W)$ of W to be the quasi-graded poset consisting of the poset of strata $\mathcal{P}$ adjoined with a minimal element $\widehat{0}$. The rank function is given by $\rho(X) = \dim(X) + 1$ if $X > \widehat{0}$ and $\rho(\widehat{0}) = 0$. The weighted zeta function is $\bar{\zeta}(X, Y) = \chi(\text{link}_Y(X))$ if $X > \widehat{0}$ and $\bar{\zeta}(\widehat{0}, Y) = \chi(Y)$.*

Theorem 4.5 *Let W be Whitney stratified closed subset of a smooth manifold M. Then the face poset of W is an Eulerian quasi-graded poset.*

We now give a few examples of Whitney stratifications beginning with the classical polygon.

Example 4.6 Consider a two dimensional cell c with its boundary subdivided into n vertices $v_1, \ldots, v_n$ and n edges $e_1, \ldots, e_n$. There are three ways to view this as a Whitney stratification.

(1) Declare each of the $2n + 1$ cells to be individual strata. This is the classical view of an n-gon. Here the weighted zeta function is the classical zeta function, that is, always equal to 1 (assuming $n \geq 2$).

(2) Declare the union of the n edges to be one stratum $e = \cup_{i=1}^{n} e_i$, that is, we have the $n + 2$ strata $v_1, \ldots, v_n, e, c$. Here the non-one values of the weighted zeta function are given by $\bar{\zeta}(\widehat{0}, e) = n$ and $\bar{\zeta}(v_i, e) = 2$.

(3) Lastly, we can have the three strata $v = \cup_{i=1}^{n} v_i$, $e = \cup_{i=1}^{n} e_i$ and c. Now non-one values of the weighted zeta function are given by $\bar{\zeta}(\widehat{0}, v) = \bar{\zeta}(\widehat{0}, e) = n$ and $\bar{\zeta}(v, e) = 2$.

In contrast, we cannot have $v, e_1, \ldots, e_n, c$ as a stratification, since the link of a point p in e_i depends on the point p in v chosen.

The $\mathbf{cd}$-index of each of the three Whitney stratifications in Example 4.6 are the same, that is, the $\mathbf{cd}$-index of an n-gon is given by $\mathbf{c}^2 + (n-2) \cdot \mathbf{d}$ for $n \geq 1$.

The last stratification in the previous example can be extended to any simple polytope.

Example 4.7 Let P be an n-dimensional simple polytope. Recall that the simple condition implies that every interval $[x,y]$, where $\widehat{0} < x \leq y$, is a Boolean algebra. We obtain a different stratification of the ball by joining all the facets together to one strata. Note that the $\mathbf{cd}$-index does not change, since the information is carried in the weighted zeta function. We continue by joining all the subfacets together to one strata. Again the $\mathbf{cd}$-index remains unchanged. In the end we obtain a stratification where the union of all the i-dimensional faces forms the ith strata x_{i+1}. The face poset of this stratification is the $(n+2)$-element chain $C = \{\widehat{0} = x_0 < x_1 < \cdots < x_{n+1} = \widehat{0}\}$, with the rank function $\rho(x_i) = i$ and weighted zeta function $\bar{\zeta}(\widehat{0}, x_i) = f_{i-1}(P)$ and $\bar{\zeta}(x_i, x_j) = \binom{n+1-i}{n+1-j}$. We have $\Psi(C, \rho, \bar{\zeta}) = \Psi(P)$.

A similar stratification can be obtained for any regular polytope, since the isomorphism type of any upper interval $[x, \widehat{1}]$ only depends on the rank $\rho(x)$.

The next example is a higher dimensional analogue of the one-gon in Example 4.1.

Example 4.8 Consider the subdivision Ω_n of the n-dimensional ball $\mathbb{B}^n$ consisting of a point p, an $(n-1)$-dimensional cell c and the interior b of the ball. If $n \geq 2$, the face poset is $\{\widehat{0} < p < c < b\}$ with the elements having ranks 0, 1, n and $n+1$, respectively. In the case $n = 1$, the two elements p and c are incomparable. The weighted zeta function is given by $\bar{\zeta}(\widehat{0}, p) = \bar{\zeta}(\widehat{0}, c) = \bar{\zeta}(\widehat{0}, b) = 1$, $\bar{\zeta}(p, c) = 1 + (-1)^n$, and $\bar{\zeta}(p, b) = \bar{\zeta}(c, b) = 1$. When n is even, the $\mathbf{cd}$-index evaluates to

$$\Psi(\Omega_n) = \frac{1}{2} \cdot \left[(\mathbf{c}^2 - 2\mathbf{d})^{n/2} + \mathbf{c} \cdot (\mathbf{c}^2 - 2\mathbf{d})^{(n-2)/2} \cdot \mathbf{c} \right], \tag{4.1}$$

and when n is odd

$$\Psi(\Omega_n) = \frac{1}{2} \cdot \left[\mathbf{c} \cdot (\mathbf{c}^2 - 2\mathbf{d})^{(n-1)/2} + (\mathbf{c}^2 - 2\mathbf{d})^{(n-1)/2} \cdot \mathbf{c} \right]. \tag{4.2}$$

As a remark, these $\mathbf{cd}$-polynomials played an important role in proving that the $\mathbf{cd}$-index of a polytope is coefficient-wise minimized on the simplex, namely, $\Psi(\Omega_n) = (-1)^{n-1} \cdot \alpha_n$, where α_n are defined in [5]. See Theorem 5.2 for a generalization of one of the main identities in [5].

5 The semisuspension

Let Γ be a polytopal complex, that is, a regular cell complex whose cells are polytopes. Assume the dimension of Γ is k. Let $n > k$ be an integer. We define the *nth semisuspension* of Γ, denoted $\mathrm{Semi}(\Gamma, n)$, to be the family of CW-complexes obtained by embedding Γ in the boundary of an n-dimensional ball $\mathbb{B}^n$, if they exist. Note that one really has a family of embeddings. For example, one can embed a circle into the boundary of a 4-dimensional ball so that the result is any given knot. Nevertheless, we will show the face poset of $\mathrm{Semi}(\Gamma, n)$ is well-defined. Furthermore, in the case Γ is homeomorphic to a k-dimensional ball, the semisuspension $\mathrm{Semi}(\Gamma, n)$ is unique.

Theorem 5.1 *Let Γ and Δ be two polytopal complexes such that their union $\Gamma \cup \Delta$ is a polytopal complex* *of dimension less than n. Then the following inclusion-exclusion relation holds:*

$$\Psi(\mathrm{Semi}(\Gamma, n)) + \Psi(\mathrm{Semi}(\Delta, n)) = \Psi(\mathrm{Semi}(\Gamma \cap \Delta, n)) + \Psi(\mathrm{Semi}(\Gamma \cup \Delta, n)).$$

The next theorem generalizes Proposition 4.3 in [5] which considered the case when $F_1, \ldots, F_r$ is the initial line shelling segment of an n-dimensional polytope. Their proof is based on shelling, whereas our proof of Theorem 5.2 is an application of inclusion-exclusion.

Theorem 5.2 *Let Γ be a polytopal complex of dimension less than n. Assume that Γ has facets $F_1, \ldots, F_r$. Then the* $\mathbf{cd}$*-index of the semisuspension* $\mathrm{Semi}(\Gamma, n)$ *is given by*

$$\Psi(\mathrm{Semi}(\Gamma, n)) = -\sum_{F} \widetilde{\chi}(\mathrm{link}_\Gamma(F)) \cdot \Psi(F) \cdot \Psi(\Omega_{n-\dim(F)}),$$

where the sum is over all possible intersections F of the facets $F_1, \ldots, F_r$.

Let Γ be a regular subdivision of an n-dimensional ball $\mathbb{B}^n$ such that the interior of the ball is one the faces. Let Λ be a regular subdivision of Γ such that the interior of the ball is yet again a face of Λ. For a face F of Γ we define $\Lambda|_F$ to be the subdivision of F induced by Λ. There are two extremal cases. When F is the empty set, let $\Lambda|_F$ be the empty subdivision of the empty face. In this case the semisuspension $\mathrm{Semi}(\Lambda|_F, n)$ is the $(n-1)$-dimensional sphere and the interior of the n-dimensional ball. The second extremal case is when $F = \widehat{1}$, and we let $\Lambda|_F$ and $\mathrm{Semi}(\Lambda|_F, n)$ denote the subdivision Λ of the n-dimensional sphere.

Theorem 5.3 *Let Γ be a regular subdivision of the n-ball $\mathbb{B}^n$ and let Λ be a regular subdivision of Γ such that both subdivisions have the interior of the ball as a face. Then the alternating sum of* $\mathbf{cd}$*-indices of semisuspensions is equal to zero, that is,*

$$\sum_{F \in \Gamma} (-1)^{\rho(F, \widehat{1})} \cdot \Psi(\mathrm{Semi}(\Lambda|_F, n)) = 0.$$

6 Shelling components for non-pure simplicial complexes

We now turn our attention to computing the $\mathbf{cd}$-index of the nth semisuspension of a (non-pure) shellable simplicial complex. The first step is to define the shelling components. For $i \leq k$ let $\Delta_{k,i}$ be the simplicial complex consisting of $i+1$ facets of the k-dimensional simplex. Define the quasi-graded poset $P_{n,k,i}$ for $0 \leq i \leq k \leq n$ to be the face poset of the semisuspension $P_{n,k,i} = \mathcal{F}(\mathrm{Semi}(\Delta_{k,i}, n))$. Define the shelling component $\check{\Phi}(n, k, i)$ to be the difference $\check{\Phi}(n, k, i) = \Psi(P_{n,k,i}) - \Psi(P_{n,k,i-1})$ for $1 \leq i \leq k \leq n$ and $\check{\Phi}(n, k, 0) = \Psi(P_{n,k,0})$ for $0 \leq k \leq n$. The polynomials $\check{\Phi}(n, n, i)$ (the case $k = n$) were introduced by Stanley [27]. Let G be the derivation on $\mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$ defined by $G(\mathbf{c}) = \mathbf{d}$ and $G(\mathbf{d}) = \mathbf{cd}$.

Theorem 6.1 *The shelling components of the simplex satisfy the recursion*

$$G(\check{\Phi}(n,k,i)) = \check{\Phi}(n+1,k+1,i+1)$$

with the boundary conditions $\check{\Phi}(n,k,0) = \Psi(B_k)\cdot\Psi(\Omega_{n-k+1})$*, for* $k \geq 1$ *and* $\check{\Phi}(n,0,0)$ *is* $(\mathbf{c}^2-2\mathbf{d})^{n/2}$ *or* $(\mathbf{c}^2-2\mathbf{d})^{(n-1)/2}\cdot\mathbf{c}$ *depending on parity of* n*.*

Recall that Björner and Wachs [7] extended the notion of shellability to non-pure complexes. They generalized the h-vector to the h-triangle.

Theorem 6.2 *Let* Δ *be a non-pure shellable simplicial complex of dimension at most* n*. Then the* $\mathbf{cd}$*-index of the semisuspension of* Δ *is given by*

$$\Psi(\mathrm{Semi}(\Delta,n)) = \sum_{k=0}^{n}\sum_{i=0}^{k} h_{k,i}\cdot\check{\Phi}(n,k,i),$$

where the h*-triangle entry* $h_{k,i}$ *is the number of facets of shelling type* (k,i)*.*

7 Concluding remarks

As was mentioned in the introduction, finding the linear inequalities that hold among the entries of the $\mathbf{cd}$-index of a Whitney stratified manifold expands the program of determining linear inequalities for flag vectors of polytopes. Since the coefficients may be negative, one must ask what should the new minimization inequalities be. Observe that Kalai's convolution [19] still holds. More precisely, let M and N be two linear functionals defined on the $\mathbf{cd}$-coefficients of any m-dimensional, respectively, n-dimensional manifold. If both M and N are non-negative then their convolution is non-negative on any $(m+n+1)$-dimensional manifold.

Other inequality questions are: Can Ehrenborg's lifting technique [9] be extended to stratified manifolds? Is there an associated Stanley–Reisner ring for the barycentric subdivision of a stratified space, and if so, what is the right version of the Cohen–Macaulay property [28]? Finally, what non-linear inequalities hold among the $\mathbf{cd}$-coefficients?

One interpretation of the coefficients of the $\mathbf{cd}$-index is due to Karu [20] who, for each $\mathbf{cd}$-monomial, gave a sequence of operators on sheaves of vector spaces to show the non-negativity of the coefficients of the $\mathbf{cd}$-index for Gorenstein* posets [20]. Is there a signed analogue of Karu's construction to explain the negative coefficients occurring in the $\mathbf{cd}$-index of quasi-graded posets?

Observe that the shelling components $\check{\Phi}(n,k,i)$ can have negative coefficients. For which values of n, k and i do we know that they are non-negative? Is there a combinatorial interpretation of the $\mathbf{cd}$-polynomials $\check{\Phi}(n,k,i)$? In the case $n=k$ such interpretations are known in terms of André permutations and Simsun permutations [11, 15, 16].

Given a Whitney stratified space, its face poset with rank function given by dimension and weighted zeta function involving the Euler characteristic (see Definition 4.4 and Theorem 4.5) yields an Eulerian quasi-graded poset. Conversely, given an Eulerian quasi-graded poset $(P, \rho, \bar{\zeta})$ can one construct an associated Whitney stratified space? It is clear that for $x \prec y$ with $\rho(x) + 1 = \rho(y)$ one must require $\bar{\zeta}(x, y)$ to be a positive integer since $\mathrm{link}_y\, x$ is a 0-dimensional space consisting of a collection of one or more points. What other conditions on an Eulerian quasi-graded poset are necessary so that it is the face poset of a Whitney stratified space?

As always when the $\mathbf{ab}$-index is defined one also has the companion quasisymmetric function. This quasisymmetric function can be defined by the (almost) isomorphism γ in [12, Section 3]. More directly, for a chain $c = \{\widehat{0} = x_0 < x_1 < \cdots < x_k = \widehat{1}\}$, define the composition $\rho(c) = (\rho(x_0, x_1), \rho(x_1, x_2), \ldots, \rho(x_{k-1}, x_k))$. Then the quasisymmetric function of a quasi-graded poset $(P, \rho, \bar{\zeta})$ is given by $F(P, \rho, \bar{\zeta}) = \sum_c \bar{\zeta}(c) \cdot M_{\rho(c)}$, where M is the monomial quasisymmetric function. It is straightforward to observe that F can be viewed as a Hopf algebra morphism as follows.

$$\begin{aligned}
\Delta(F(P, \rho, \bar{\zeta})) &= \sum_{\widehat{0} \leq x \leq \widehat{1}} F([\widehat{0}, x], \rho, \bar{\zeta}) \otimes F([x, \widehat{1}], \rho, \bar{\zeta}), \\
F(P \times Q, \rho_{P \times Q}, \bar{\zeta}_{P \times Q}) &= F(P, \rho_P, \bar{\zeta}_P) \times F(Q, \rho_Q, \bar{\zeta}_Q).
\end{aligned}$$

See [1] for results on generalized Dehn–Sommerville relations in the setting of combinatorial Hopf algebras.

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Fully commutative elements and lattice walks

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Abstract. An element of a Coxeter group W is fully commutative if any two of its reduced decompositions are related by a series of transpositions of adjacent commuting generators. These elements were extensively studied by Stembridge in the finite case. In this work we deal with any finite or affine Coxeter group W, and we enumerate fully commutative elements according to their Coxeter length. Our approach consists in encoding these elements by various classes of lattice walks, and we then use recursive decompositions of these walks in order to obtain the desired generating functions. In type A, this reproves a theorem of Barcucci et al.; in type $\widetilde{A}$, it simplifies and refines results of Hanusa and Jones. For all other finite and affine groups, our results are new.

Résumé. Un élément d'un groupe de Coxeter W est dit totalement commutatif si deux de ses décompositions réduites peuvent toujours être reliées par une suite de transpositions de générateurs adjacents qui commutent. Ces éléments ont été étudiés en détail par Stembridge dans le cas où W est fini. Dans ce travail, nous considérons W fini ou affine, et énumérons les éléments totalement commutatifs selon leur longueur de Coxeter. Notre approche consiste à encoder ces éléments par diverses classes de chemins du plan que nous décomposons récursivement pour obtenir les fonctions génératrices voulues. Pour le type A cela redonne un théorème de Barcucci et al.; pour $\widetilde{A}$, cela simplifie et précise des résultats de Hanusa et Jones. Pour tous les autres groupes finis et affines, nos résultats sont nouveaux.

Keywords: Fully commutative elements, Coxeter groups, generating functions, lattice walks, heaps.

Introduction

Let W be a Coxeter group. An element $w \in W$ is said to be *fully commutative* if any reduced expression for w can be obtained from any other one by transposing adjacent pairs of commuting generators. Fully commutative elements were extensively studied by Stembridge in a series of papers [9, 10, 11] where, among others, he classified the Coxeter groups having a finite number of fully commutative elements and enumerated them in each case. In the symmetric group, the fully commutative elements are the 321-avoiding permutations, counted by the Catalan numbers. A nice q-analogue of the Catalan numbers arises when these permutations are enumerated according to their inversion number. This has been done by Barcucci et al. [1], where an elegant expression for the corresponding generating function as a ratio of q-Bessel functions is provided. In the case of the affine symmetric group, a similar q-analogue has been recently found by Hanusa and Jones [7].

The main goal of the present paper is the computation of the generating function $\sum_{w\in W^{FC}} q^{\ell(w)}$, when W is any finite or affine irreducible Coxeter group. Here W^{FC} denotes the subset of fully commutative

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elements of W, and ℓ denotes the Coxeter length. It is known that fully commutative elements in Coxeter groups index a basis for a quotient of the associated Iwahori-Hecke algebra ([4, 5]), so that our formulas give the graded dimensions of these algebras for finite and affine W.

Our investigation in the finite case (Section 2) is based on the *spinal analysis* of Stembridge [9, §2.2], which he uses to find and solve recurrences for the number of elements in W^{FC} when this set is finite. Here we reformulate his results in terms of certain lattice walks (cf. Definition 6), which allows us to take the Coxeter length into account. We then use recursive decompositions of these various families of walks to compute the length generating functions of W^{FC} for all finite W. In type A this gives a simple proof of a result of Barcucci et al. [1]; to our knowledge, the results are new in all other finite types.

In the affine case (Section 3), not tackled by Stembridge, we can also associate lattice walks to fully commutative elements. As in the finite case, decompositions of these walks lead to the computation of the length generating function of W^{FC} for any affine Coxeter group W. The main result of this section is that, in each case, the sequence of coefficients of this generating function is ultimately periodic (Theorem 14). This was already shown in type $\widetilde{A}$ by Hanusa and Jones [7, §5]; however our method gives a much simpler expression for the generating function, and settles positively a question in [7] regarding the beginning of the periodicity. For all other affine types our results are new. In a last section we mention various possible extensions of this work, for instance to involutions.

Finally we point out that the GAP package GBNP was extremely useful at many stages of this work.

1 Fully commutative elements, heaps and walks

1.1 Fully commutative elements in Coxeter groups

Let W be a Coxeter group with finite generating set S, and Coxeter matrix $M = (m_{st})_{s,t\in S}$. That is, M is a symmetric matrix with $m_{ss} = 1$ and, for $s \neq t$, $m_{st} = m_{ts} \in \{2, 3, \ldots\} \cup \{\infty\}$. The relations among the generators are of the form $(st)^{m_{st}} = 1$ if $m_{st} < \infty$. The pair (W, S) is called a Coxeter system. We can write the relations as $sts\cdots = tst\cdots$, each side having length m_{st}; these are usually called *braid relations*; when $m_{st} = 2$, this is a commutation relation. The Dynkin diagram Γ associated to (W, S) is the graph with vertex set S and, for each pair s, t with $m_{st} \geq 3$, an edge between s and t labeled by m_{st} (when $m_{st} = 3$ the edge is left unlabeled).

For $w \in W$ we denote by $\ell(w)$ the minimum length of any expression $w = s_1 \cdots s_l$ with $s_i \in S$. Such expressions with minimum length are called *reduced*, and we denote by $\mathcal{R}(w)$ the set of all reduced expressions of w. A fundamental result in Coxeter group theory is that any expression in $\mathcal{R}(w)$ can be obtained from any other one using only braid relations (see [8]). If an element w satisfies the stronger condition that any reduced expression for w can be obtained from any other one by using only commutation relations, then it is said to be *fully commutative* (which we shall sometimes abbreviate in FC). The following characterization of FC elements is particularly useful.

Proposition 1 ([9], Prop. 2.1) *An element $w \in W$ is fully commutative if and only if for all s, t such that $3 \leq m_{st} < \infty$, there is no expression in $\mathcal{R}(w)$ that contains the factor $\underbrace{sts\cdots}_{m_{st}}$.*

1.2 Heaps

We follow Stembridge [9] in this section. Fix a word $\mathbf{s} = (s_{a_1}, \ldots, s_{a_l})$ in S^*, the free monoid generated by S. We define a partial ordering of the indices $\{1, \ldots, l\}$ by $i \prec j$ if $i < j$ and $m(s_i, s_j) \geq 3$ and extend by transitivity. We denote by $H_{\mathbf{s}}$ this poset together with the "labeling" $i \mapsto s_{a_i}$: this is the *heap* of $\mathbf{s}$. In the Hasse diagram of $H_{\mathbf{s}}$, elements with the same labels will be drawn in the same column. The size $|H|$ of a heap H is the length l of the corresponding word, and for each $s_a \in S$ we let $|H|_a$ be the number of elements in H with label s_a. In Figure 1, we fix a Dynkin diagram on the left, and we give two examples of words with the corresponding heaps.

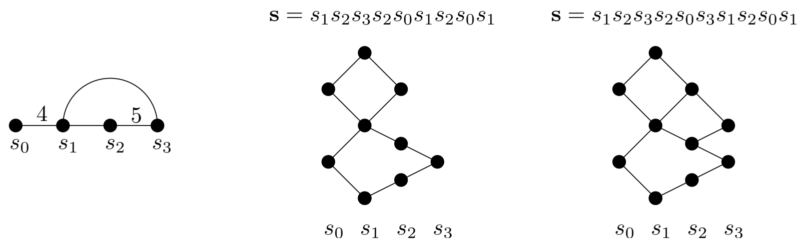


Fig. 1: Two words and their respective heaps.

If we consider heaps up to poset isomorphism which preserve the labeling, then heaps encode precisely *commutativity classes*, that is, if the word $\mathbf{s}'$ is obtainable from $\mathbf{s}$ by transposing commutating generators then there exists a poset isomorphism between $H_{\mathbf{s}}$ and $H_{\mathbf{s}'}$. In particular, if w is fully commutative, the heaps of the reduced words are all isomorphic, and thus we can define the heap of w, denoted by H_w.

A *linear extension* of a poset H is a linear ordering π of H such that $\pi(i) < \pi(j)$ implies $i \prec j$. Now let $\mathcal{L}(H_{\mathbf{s}})$ be the set of words $(s_{\pi(1)}, \ldots, s_{\pi(l)})$ where π goes through all linear extensions of $\mathbf{s}$.

Proposition 2 ([9], Theorem 3.2) *Let w be a fully commutative element. Then $\mathcal{L}(H_{\mathbf{s}})$ is equal to $\mathcal{R}(w)$ for some (equivalently every)* $\mathbf{s} \in \mathcal{R}(w)$.

We say that a chain $i_1 \prec \cdots \prec i_m$ in a poset H is convex if the only elements u satisfying $i_1 \preceq u \preceq i_m$ are the elements i_j of the chain. The next result characterizes *FC heaps*, namely the heaps representing the commutativity classes of FC elements.

Proposition 3 ([9], Proposition 3.3) *A heap H is the heap of some FC element if and only if (a) there is no convex chain $i_1 \prec \cdots \prec i_{m_{st}}$ in H such that $s_{i_1} = s_{i_3} = \cdots = s$ and $s_{i_2} = s_{i_4} = \cdots = t$ where $3 \leq m_{st} < \infty$, and (b) there is no covering relation $i \prec j$ in H such that $s_i = s_j$.*

Therefore it is equivalent to characterize FC elements in the Coxeter system (W, S) and heaps verifying the conditions of Proposition 3. The heap on the right of Figure 1 is a FC heap, whereas the one on the left is not since it contains the convex chain with labels (s_2, s_1, s_2) while $m_{s_1 s_2} = 3$. In the next section we will exhibit a class of heaps which play an important role for the Coxeter systems we will be interested in.

1.3 Alternating heaps and walks

In all this section, we fix $m_{01}, m_{12}, \ldots, m_{n-1n} \in \{3, 4, \ldots\} \cup \{\infty\}$ and we consider the Coxeter system (W, S) corresponding to the linear Dynkin diagram $\Gamma_n = \Gamma_n((m_{i\,i+1})_i)$ of Figure 2.

Definition 4 (Alternating words and heaps) *A reduced word* $\mathbf{s} \in S^*$ *is alternating if, for $i = 0, \ldots, n-1$, the occurrences of s_i alternate with those of s_{i+1}. A heap is called alternating if it is of the form $H_{\mathbf{s}}$ for an alternating word* $\mathbf{s}$*; a FC element is alternating if its heap is.*

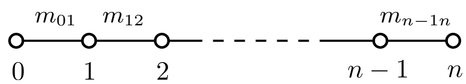


Fig. 2: The linear Dynkin diagram Γ_n.

$\mathbf{s} = 851078627108$

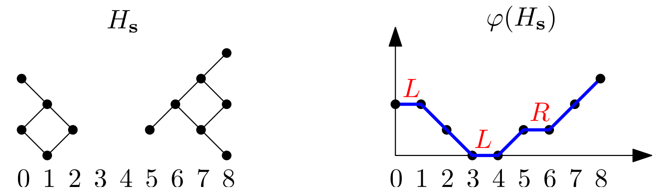


Fig. 3: An alternating word $\mathbf{s}$ with corresponding heap $H_{\mathbf{s}}$, and its encoding by a walk.

An alternating heap is presented in Figure 3. Not all alternating heaps correspond to FC elements, but a characterization is easy by Proposition 3.

Proposition 5 *An alternating heap H is FC if and only if* $\begin{cases} m_{01} > 3 & \text{or} \quad |H|_0 \leq 1 \\ m_{n-1n} > 3 & \text{or} \quad |H|_n \leq 1. \end{cases}$

The alternating word/heap in Figure 3 is FC if and only if $m_{01}, m_{n-1n} > 3$.

Definition 6 (Walks) *We call* walk *P of length n a sequence of points $(P_0, P_1, \ldots, P_n)$ in $\mathbb{N}^2$ with n steps in the set $\{(1,1), (1,-1), (1,0)\}$, where the horizontal steps $(1,0)$ are labeled either by L or R. Walks are furthermore required to start at a point P_0 with abscissa 0. A walk is said to satisfy condition $(*)$ if all horizontal steps of the form $(i,0) \to (i+1,0)$ have label L.*

The set of all walks of length n will be denoted by $\mathcal{G}_n$. The subset of walks ending at $P_n = (n,0)$ will be denoted by $\mathcal{Q}_n$, and the subset of $\mathcal{Q}_n$ with $P_0 = (0,i)$ will be denoted by $\mathcal{M}_n^{(i)}$. For short we will write $\mathcal{M}_n = \mathcal{M}_n^{(0)}$ for walks starting and ending on the x-axis. To each family $\mathcal{F}_n \subseteq \mathcal{G}_n$ corresponds subfamilies $\mathcal{F}_n^* \subseteq \mathcal{F}_n$ consisting of those walks in $\mathcal{F}_n$ which satisfy the condition $(*)$, and $\check{\mathcal{F}}_n \subseteq \mathcal{F}_n$ consisting of those walks hit the x-axis at some point.

Remark: Write U, D to represent steps $(1,1), (1,-1)$, and L, R to represent steps $(1,0)$ with these labels; then we can encode our walks by the data of P_0 and a word in $\{U, D, L, R\}$. Consider the injective transformation on such words $U \mapsto UU, D \mapsto DD, L \mapsto UD, R \mapsto DU$. This restricts to a bijection from $\mathcal{M}_n^*$ to Dyck walks, i.e. walks from $(0,0)$ to $(2n,0)$ with steps U, D staying above the x-axis.

The *total height* ht of a walk is the sum of the heights of its points: if $P_i = (i, h_i)$ then $\mathrm{ht}(P) = \sum_{i=0}^{n} h_i$. To each family $\mathcal{F}_n \subseteq \mathcal{G}_n$ we associate the series $F_n(q) = \sum_{P \in \mathcal{F}_n} q^{\mathrm{ht}(P)}$, and we define the generating functions in the variable x by

$$F(x) = \sum_{n \geq 0} F_n(q) x^n \quad \text{and} \quad F^*(x) = \sum_{n \geq 0} F_n^*(q) x^n.$$

We now define a bijective encoding of alternating heaps by walks. In the next section we will exploit this bijection in order to compute some generating functions.

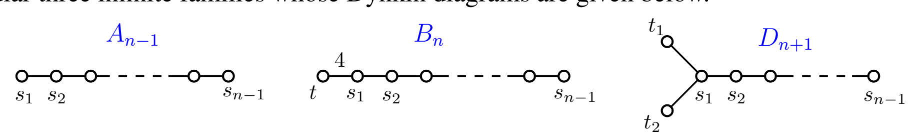
Definition 7 (Map φ) *Let H be an alternating heap of type Γ_n. To each $s_i \in S$ we associate a point $P_i = (i, |H|_i)$. If $|H|_{i+1} = |H|_i > 0$ we label the ith step by L (*resp. *R) if among all elements of H indexed by i and $i+1$, the lowest element has index $i+1$ (*resp. *i). If $|H|_{i+1} = |H|_i = 0$, we label the ith step by L. We define $\varphi(H)$ as the walk $(P_0, P_1, \dots, P_n)$ with the labels on its steps.*

Theorem 8 *The map $H \mapsto \varphi(H)$ is a bijection between alternating heaps of type Γ_n and $\mathcal{G}_n^*$. The size $|H|$ of the heap is the total height of $\varphi(H)$.*

Proof: The map is clearly well defined since for any alternating heap H, we have $-1 \leq |H|_i - |H|_{i+1} \leq 1$. Fix $(P_0, P_1, \dots, P_n) \in \mathcal{G}_n^*$. If the step $P_i = (i, h_i) \to P_{i+1} = (i+1, h_{i+1})$ is equal to $(1,1)$ (resp. $(1,-1)$), then we define a convex chain $\mathcal{C}_i$ of length h_{i+1} (resp. h_i) as $(s_{i+1}, s_i, \dots, s_{i+1})$ (resp. $(s_i, s_{i+1}, \dots, s_i)$). If the step $P_i \to P_{i+1}$ is labeled by L (resp. R), then we define a chain $\mathcal{C}_i$ of length $h_i = h_{i+1}$ as $(s_i, s_{i+1}, \dots, s_i, s_{i+1})$ (resp. $(s_{i+1}, s_i, \dots, s_{i+1}, s_i)$). We define H as the transitive closure of the chains $\mathcal{C}_0, \dots, \mathcal{C}_{n-1}$. The heap H is uniquely defined, alternating, and satisfies $\varphi(H) = (P_0, P_1, \dots, P_n)$. Since $h_i = |H|_i$ the result follows. $\square$

2 Finite types

The irreducible Coxeter systems corresponding to finite groups are completely classified ([8]). There are in particular three infinite families whose Dynkin diagrams are given below.



The elements of W^{FC} were enumerated by Stembridge using recurrence relations. Here we reinterpret in many cases the decompositions leading to these recurrences as describing certain lattice walks. The advantage of this approach is that we can take into account the Coxeter length via Theorem 8 and then find recursive decomposions of the walks leading to equations for the generating functions.

For each group W we define $W^{FC}(q) = \sum_{w \in W^{FC}} q^{\ell(w)}$.

2.1 Type A

In this case, we show that it is possible to derive from Theorem 8 the generating function $A^{FC}(x) = \sum_{n \geq 1} A_{n-1}^{FC}(q) x^n$, which was computed in a different way in [1].

Proposition 9 *We have $A_{n-1}^{FC}(q) = M_n^*(q)$, and equivalently $A^{FC}(x) = M^*(x) - 1$.*

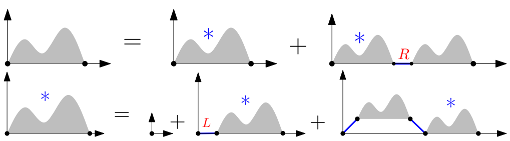
Proof: We are in the setting of Section 1.3 with a diagram Γ_{n-2} and $m_{i,i+1} = 3$ for $i = 0, \dots, n-3$. The key property in this type is that *all FC elements are alternating* (see Definition 4): this is easy to show (cf. [2, Theorem 2.1]), and is also a particular case of Proposition 15 from Section 3.1. By Proposition 5 and Theorem 8, A_{n-1}^{FC} is then in bijection with walks in $\mathcal{G}_{n-2}^*$ with starting and ending point at height 0 or 1. This set is clearly in bijection with $\mathcal{M}_n^*$: it suffices to add two extra points to $\varphi(H_w)$ on the x-axis (one at the beginning and one at the end). $\square$

Specializing at $q = 1$, this shows that A_{n-1}^{FC} has cardinality the nth Catalan number $Cat_n = \frac{1}{n+1}\binom{2n}{n}$, since $\mathcal{M}_n^*$ is in bijection with Dyck walks by the remark in Section 1.3.

Corollary 10 *The generating function A^{FC} satisfies the following functional equation*

$$A^{FC}(x) = x + xA^{FC}(x) + qxA^{FC}(x)(A^{FC}(qx) + 1). \tag{1}$$

Proof: We have the following walk decompositions with corresponding equations:



$$M(x) = M^*(x) + xM^*(x)M(x), \tag{2}$$

$$M^*(x) = 1 + xM^*(x) + qx^2M^*(x)M(qx). \tag{3}$$

The identity (2) gives $M(x) = M^*(x)/(1 - xM^*(x))$, which can be replaced in (3) to yield after some simplifications:

$$M^*(x) = 1 + xM^*(x) + qx(M^*(x) - 1)M^*(qx). \tag{4}$$

Finally, we see through Proposition 9 that (4) is exactly Equation (1). □

Corollary 10 gives another proof of [1, Eq. (3.0.2)], where $A^{FC}(x)$ is denoted by $C(x, q)$. In this work, Barcucci et al. also proved an expression for $C(x, q)$ as a quotient of q-Bessel type functions, using a recursive rewriting rule for 321-avoiding permutations and a result of Bousquet-Mélou [3]. It is possible to derive their ratio of q-Bessel functions by writing

$$A^{FC}(x) + 1 = M^*(x) = \frac{\sum_{n\geq 0} \alpha_n(q)x^n/(x;q)_{n+1}}{\sum_{n\geq 0} \alpha_n(q)x^n/(x;q)_n}, \tag{5}$$

where $(x; q)_n = (1 - x)\cdots(1 - xq^{n-1})$ stands for the classical q-rising factorial, and $\alpha_n(q)$ is a q-hypergeometric coefficient . Then plugging (5) into (4) yields after a few simplifications and identification $\alpha_n(q) = (-1)^n q^{n(n+1)/2}\alpha_0(q)/(q; q)_n$, as in [1].

2.2 Type B

The goal of this subsection is to extend the previous results to the type B. To this aim, we set $B_0^{FC}(q) = 1$, $B_1^{FC}(q) = 1 + q$, and $B^{FC}(x) = \sum_{n\geq 0} B_n^{FC}(q)x^n$. Then we prove the following result.

Proposition 11 *We have* $\quad B^{FC}(x) = Q^*(x) + \dfrac{x^2q^3}{1 - xq^2}M^*(x)M(qx).$

Moreover $Q^*(x)$ can be explicitly computed by using the two following equations

$$Q(x) = M(x)(1 + xqQ(qx)) \quad \textit{and} \quad Q^*(x) = M^*(x)(1 + xqQ(qx)). \tag{6}$$

Proof: We first compute the generating function for FC elements in B_n corresponding to alternating heaps of type Γ_{n-1}. By using Proposition 5, we see that these FC elements correspond to alternating heaps H of type Γ_{n-1} with $m_{0,1} = 4$ and $m_{i,i+1} = 3$ for $i = 1, \ldots, n - 2$, such that $|H|_{n-1} \leq 1$. Moreover, Theorem 8 implies that $\varphi(H)$ is a path in $\mathcal{G}_{n-1}^*$ with ending point at height 0 or 1. This set is clearly in bijection with $\mathcal{Q}_n^*$: it suffices to add an extra point to $\varphi(H)$ on the x-axis at the end. Adding the trivial FC element corresponding to $n = 0$, one gets the generating function $Q^*(x)$.

It remains to compute the generating function for FC elements in B_n which do not correspond to alternating heaps. Following Stembridge [11, Lemma 2.3], we know that these elements correspond to heaps $H_{\mathbf{s}}$ containing exactly one t, two s_1's, …, two s_m's (for some $m \in \{1, \ldots, n-1\}$) and such that (a) the restriction of $\mathbf{s}$ to $t, s_1, \ldots, s_m$ is $(s_m, \ldots, s_1, t, s_1, \ldots, s_m)$, and (b) the restriction of $\mathbf{s}$ to $s_m, \ldots, s_{n-1}$, where we also delete one of the two s_m's, is an alternating word. Thanks to Theorem 8 the subwords in (b) are in bijection with $\mathcal{M}^{(1)*}_{n-m}$. This shows that the generating function of the non alternating FC elements in B_n is $M^{(1)*}(x) \times xq^2/(1-xq^2)$. We derive $B^{FC}(x)$ by using the relation $M^{(1)*}(x) = xqM^*(x)M(qx)$, which can easily be obtained by splitting the path at the first intersection with the x-axis. Similar decompositions finally yield the equations (6). $\square$

Note that the elements in B_n^{FC} corresponding to alternating heaps are called *fully commutative top elements of* B_n in [10], and *commutative elements of the Weyl group* $\mathcal{C}_n$ in [4]. Therefore $Q^*(x)$ gives a generating function for these particular elements.

2.3 Type D

In this subsection, we will see how it is possible to derive from the previous results in type B analogous expressions in type D. We set $D_1^{FC}(q) = 1$, $D_2^{FC}(q) = (1+q)^2$, and $D^{FC}(x) = \sum_{n\geq 0} D_{n+1}^{FC}(q)x^n$, and we prove the following result.

Proposition 12 *We have* $$D^{FC}(x) = 2Q^*(x) - M^*(x) + \frac{xq^2}{1-xq^2}M^*(x)M(qx).$$

Proof: Reformulating Stembridge [11, §3.3], each element of D_{n+1}^{FC} can be obtained from one of B_n^{FC} by exactly one of the following rules. Consider an alternating element w in B_n^{FC} and $\mathbf{s} \in \mathcal{R}(w)$. If $\mathbf{s}$ contains no occurrence of t it yields an element of D_{n+1}^{FC}. If $\mathbf{s}$ contains at least one occurrence of t, then we can replace its subword $(t, \ldots, t)$ by $(t_1, t_2, \ldots)$ or $(t_2, t_1, \ldots)$, giving rise to two elements in D_{n+1}^{FC}. If $\mathbf{s}$ contains exactly one occurrence of t, we obtain a FC element of type D by replacing t by t_1t_2 $(= t_2t_1)$. By the proof of Proposition 11, the generating function corresponding to these three families is given by

$$M^*(x) + 2(Q^*(x) - M^*(x)) + qM^{(1)*}(x).$$

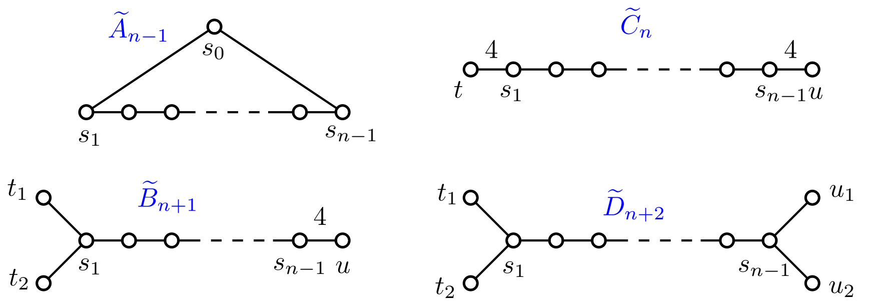
The remaining elements of D_{n+1}^{FC} are simply obtained from non alternating elements of B_n^{FC} by again replacing t by t_1t_2, therefore yielding $M^{(1)*}(x) \times xq^3/(1-xq^2)$, and the results follows. $\square$

2.4 The exceptional cases

The exceptional types are $I_2(m), H_3, H_4, F_4, E_6, E_7$, and E_8. For the dihedral group $W = I_2(m)$ one easily has $I_2(m)^{FC}(q) = 1 + 2q + 2q^2 + \cdots + 2q^{m-1} = 1 + 2q(1-q^{m-1})/(1-q)$. For the remaining cases, we used the computer to find the generating polynomials $W^{FC}(q)$; for instance, $E_8(q)$ has degree 29 in q. Note that the number of FC elements in a Coxeter group may be finite even though the group itself is infinite: Stembridge [9] discovered that there are three families $E_n(n > 8), F_n(n > 4), H_n(n > 4)$ of infinite groups with a finite number of FC elements. Extending Stembridge [11] with similar walk techniques, it is possible to enumerate such elements according to their length.

3 Affine types

The Dynkin diagrams of the infinite families of affine groups are represented below.



Lemma 13 *Suppose $F(q) = \sum_{i\geq 0} a_i q^i = P(q)/(1-q^N)$ where $P(q)$ is a polynomial of degree d. Then one has $a_{i+N} = a_i$ for all $i \geq d$. Furthermore the average value over a period $(a_i + a_{i+1} + \cdots + a_{i+N-1})/N$ is equal to $P(1)/N$ for $i \geq d$.*

We will show that this lemma applies to all generating functions $W^{FC}(q)$ when W is affine.

Theorem 14 *For each irreducible affine group W, the sequence of coefficients of $W^{FC}(q)$ is ultimately periodic, with period recorded in the following table ($\widetilde{F}_4^{FC}, \widetilde{E}_8^{FC}$ are finite sets):*

AFFINE TYPE	$\widetilde{A}_{n-1}$	$\widetilde{C}_n$	$\widetilde{B}_{n+1}$	$\widetilde{D}_{n+2}$	$\widetilde{E}_6$	$\widetilde{E}_7$	$\widetilde{G}_2$	$\widetilde{F}_4, \widetilde{E}_8$
PERIODICITY	n	$n+1$	$(n+1)(2n+1)$	$n+1$	4	9	5	1

The periods for $\widetilde{B}_{n+1}$ look surprising at first sight, but they are experimentally close to the actual minimal periods: for $n+1 = 3, 4, 5, 6$, the formula gives $15, 28, 45, 66$ while the minimal periods are $15, 7, 45, 33$.

The proof of Theorem 14 will be detailed in type $\widetilde{A}_n$, and simply outlined for other types.

3.1 Type $\widetilde{A}_n$

The generating function $\widetilde{A}_{n-1}^{FC}(q)$ was computed by Hanusa and Jones [7]; we will compare our results to theirs at the end of this section.

Proposition 15 *An element $w \in \widetilde{A}_{n-1}$ is fully commutative if and only if, in any reduced decomposition of w, the occurrences of s_i and s_{i+1} alternate for all $i \in \{0, \ldots, n-1\}$, where we set $s_n = s_0$.*

Proof: The condition is clearly sufficient by using Proposition 1. To show that it is necessary, assume $w \in \widetilde{A}_{n-1}^{FC}$ and $w_1 \cdots w_l$ is a reduced word for w ($w_k \in S$). We will show that between two consecutive s_i there is necessarily a s_{i+1}, which suffices to prove the proposition. Assume then for the sake of contradiction that there exist $i \in \{0, \ldots, n-1\}$ and $j_1 < j_2$ such that $w_{j_1} = w_{j_2} = s_i$ and that for all j satisfying $j_1 < j < j_2$ one has $w_j \neq s_i, s_{i+1}$. Among all possible i, j_1, j_2, pick one with $j_2 - j_1$ minimal. Now consider the number m of indices j with $j_1 < j < j_2$ and $w_j = s_{i-1}$. If $m = 0$, then by successive commutations we see that the word is not reduced, which is excluded. If $m = 1$, then by successive commutations one obtains a factor $s_i s_{i-1} s_i$ which is excluded by Proposition 1. If $m \geq 2$, then two consecutive occurrences of s_{i-1} contradict the minimality of $j_2 - j_1$. This finishes the proof. □

We are not exactly in the case of Section 1.3 since the Dynkin diagram is not linear, but one can nonetheless define walks from the "alternating" words described in Proposition 15: given $w \in \widetilde{A}_{n-1}^{FC}$ and $i = 0, \dots, n-1$, draw a step from $P_i = (i, |w|_{s_i})$ to $P_{i+1} = (i+1, |w|_{s_{i+1}})$ as in the definition of φ; here $|w|_s$ is the number of occurrences of s in any reduced decomposition of w. This forms a path noted $\varphi'(w)$ of length n, with both P_0 and P_n at height $|w|_{s_0}$.

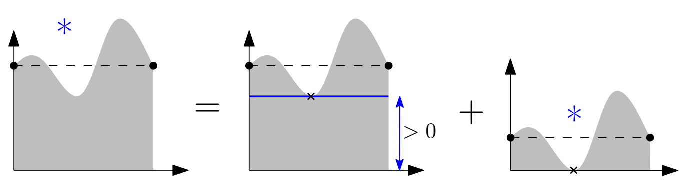
So we have $\varphi'(w) \in \mathcal{O}_n$, defined as the set of paths P in $\mathcal{G}_n$ whose starting and ending point are at the same height. Define also $\mathrm{ht}'(P) = \sum_{i<n} h_i$ where $P_i = (i, h_i)$, so that we count just once the height of the final and initial point (one should think of these walks as being on a cylinder, with these two points coinciding). We also define $O_n(q)$ to be the generating function of $\mathcal{O}_n$ with respect to the modified total height ht'. Finally, denote by $\mathcal{E}_n$ the set of horizontal walks in $\mathcal{O}_n^*$ with all vertices at height $h > 0$ and all steps with the same label L or R.

Theorem 16 *The map* $\varphi' : \widetilde{A}_{n-1}^{FC} \to \mathcal{O}_n^* \setminus \mathcal{E}_n$ *is a bijection such that* $\ell(w) = \mathrm{ht}'(\varphi'(w))$.

Proof (Sketch): The walks in $\mathcal{E}_n$ are not of the form $\varphi'(w)$: indeed, given an element $w \in \widetilde{A}_{n-1}^{FC}$ and any reduced word for it, consider the positions $j_0, j_1, \dots, j_{n-1}$ of the leftmost $s_0, s_1, \dots, s_{n-1}$ respectively. Suppose $\varphi'(w)$ consists of horizontal steps labeled R: then this would imply $j_0 < j_1 < \cdots < j_{n-1} < j_0$ which is a contradiction. The case of labels L is similar, so the image of φ' is included in $\mathcal{O}_n^* \setminus \mathcal{E}_n$. The function φ' is injective, since, as for Theorem 8, the walk $\varphi'(w)$ allows us to reconstruct the heap H_w.

For the surjectivity we omit the proof in this abstract: one has to check that when trying to reconstruct the possible heap of w from a walk $P \in \mathcal{O}_n^* \setminus \mathcal{E}_n$ as in the proof of Theorem 8, one always gets the Hasse diagram of a poset in the end. □

As an immediate corollary we have that $\widetilde{A}_{n-1}^{FC}(q) = O_n^*(q) - 2q^n/(1-q^n)$. Now we have to count walks in $\mathcal{O}_n^*$, and to this end we decompose them according to their lowest point: this is pictured below and gives the equation $O_n^*(q) = \check{O}_n^*(q) + q^n \check{O}_n(q)/(1-q^n)$.

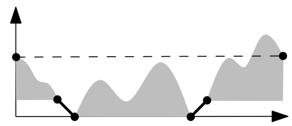


By Theorem 16, we thus have the generating function

$$\widetilde{A}_{n-1}^{FC}(q) = O_n^*(q) - 2\frac{q^n}{1-q^n} = \frac{q^n(\check{O}_n(q) - 2)}{1-q^n} + \check{O}_n^*(q). \tag{7}$$

Note that $\check{O}_n^*(q)$ and $\check{O}_n(q)$ are *polynomials*, both of degree $\lceil n/2 \rceil \lfloor n/2 \rfloor$. By Lemma 13, the coefficients of $\widetilde{A}_{n-1}^{FC}(q)$ are periodic of period n, and the average value over a period is $\frac{1}{n}\left(\binom{2n}{n} - 2\right)$. Indeed we have $\check{O}_n(1) = \binom{2n}{n}$: to see this, shift any path from $\check{\mathcal{O}}_n$ so that it starts at the origin, and use the transformations $U \mapsto UU, D \mapsto DD, L \mapsto UD, R \mapsto DU$ defined after Definition 6. This is a bijection from $\check{\mathcal{O}}_n$ to paths from the origin to $(2n, 0)$ using steps U or D, and there are obviously $\binom{2n}{n}$ such paths.

We still have to explain how to compute $\check{O}_n(q)$ and $\check{O}^*_n(q)$. Walks in $\check{\mathcal{O}}_n$ (*resp.* in $\check{\mathcal{O}}^*_n$) either belong to $\mathcal{M}_n$ (*resp.* $\mathcal{M}^*_n$) or can be decomposed as in the picture on the right. Using standard techniques, this translates into the equations:



$$\check{O}(x) = M(x)\left(1 + qx^2\frac{\partial(xM)}{\partial x}(xq)\right) \quad \text{and} \quad \check{O}^*(x) = M^*(x)\left(1 + qx^2\frac{\partial(xM)}{\partial x}(xq)\right), \tag{8}$$

which allows us to compute easily the generating functions $\widetilde{A}^{FC}_{n-1}(q)$.

Link with [7]: Hanusa and Jones [7] compute the same generating function $\widetilde{A}^{FC}_{n-1}(q)$ by using the realization of $\widetilde{A}_{n-1}$ as the affine symmetric group $\widetilde{S}_n$. It is known [6] that FC elements in this representation correspond to 321-avoiding permutations, extending the finite case. The expressions in [7] are much more complicated than ours, and more difficult to derive. The authors show that periodicity of the coefficients start at the index $2\lfloor n/2\rfloor\lceil n/2\rceil$ but conjecture that $1 + \lfloor(n-1)/2)\rfloor\lceil(n-1)/2\rceil$ is the right beginning. Lemma 13 applied to (7) gives the slightly worse result of $n + \lfloor n/2\rfloor\lceil n/2\rceil$, but we can actually prove their conjecture. Consider the operation Up on $\mathcal{O}^*_n$ which simply adds 1 to the height of each point: this sends a walk P with $\mathrm{ht}'(P) = k$ to a walk Q with $\mathrm{ht}'(Q) = n + k$. Now it is easy to see that if $k > \lfloor(n-1)/2)\rfloor\lceil(n-1)/2\rceil$, then no path P with $\mathrm{ht}'(P) = k$ has an horizontal step at height zero. From this one deduces that if $k > \lfloor(n-1)/2)\rfloor\lceil(n-1)/2\rceil$, then Up is a bijection between paths P with $\mathrm{ht}'(P) = k$ and paths P with $\mathrm{ht}'(P) = n + k$, which proves the conjecture of [7].

3.2 *Other affine types*

Type $\widetilde{C}_n$: Starting from a certain length (depending on n), we can show that FC elements can be classified in two families: (a) alternating elements, which correspond to walks in $\mathcal{G}^*_n$ by Proposition 3 and the bijection φ from Theorem 8, and (b) exceptional elements, whose reduced words appear as factors of the infinite periodic word $(ts_1s_2\cdots s_{n-1}us_{n-1}\cdots s_2s_1)^\infty$. The heaps of the words in (b) are totally ordered, and for any length $> n+1$ these words are not alternating and there $2n$ of them. Using a decomposition of $\mathcal{G}^*_n$ extending the one for $\mathcal{O}^*_n$, we can then write

$$\widetilde{C}^{FC}_n(q) = \frac{q^{n+1}\check{G}_n(q)}{1-q^{n+1}} + \frac{2nq^{n+2}}{1-q} + R_n(q), \tag{9}$$

where $R_n(q)$ is a certain polynomial that we are able to compute, and this also holds for $\check{G}_n(q)$ since $\check{G}(x) = M(x)\left(1 + qxQ(qx)\right)^2$ by extending the decomposition we used for $\check{O}(x)$.

Applying Lemma 13 to (9), we have that the coefficients of $\widetilde{C}^{FC}_n(q)$ are ultimately periodic of period $n+1$, and that the average value over a period is $2n + 4^n/(n+1)$. Indeed, $\check{G}_n(1) = 4^n$, which can be immediately seen by shifting down walks of $\check{\mathcal{G}}_n$ so that they start at the origin.

Types $\widetilde{B}_{n+1}$ and $\widetilde{D}_{n+2}$: For a large enough length (depending on n), there are for both types only two families of FC elements (as in type $\widetilde{C}_n$):

(a) Those coming from alternating heaps of type $\widetilde{C}_n$, as in Section 2.3 relating B_n and D_{n+1}. In type $\widetilde{B}_{n+1}$, apply the replacements $(t,t,t,\ldots) \mapsto (t_1,t_2,t_1,\ldots)$ or $(t_2,t_1,t_2,\ldots)$. In type $\widetilde{D}_{n+2}$, apply this map together with the same one with u, u_1, u_2 instead of t, t_1, t_2. The length generating function of these FC elements is clearly ultimately periodic with period $n+1$ in both types $\widetilde{B}_{n+1}$ and $\widetilde{D}_{n+2}$.

(b) Exceptional elements similar to those of type $\widetilde{C}_n$. In type $\widetilde{B}_{n+1}$, these have for reduced words the factors of $(t_1t_2s_1s_2\dots s_{n-1}us_{n-1}\dots s_2s_1)^\infty$ where t_1, t_2 are allowed to commute. For a large enough length i, there are $2n+3$ such elements unless i is divisible by $2n+1$, in which case there are $2n+4$ such factors. In type $\widetilde{D}_{n+2}$, these exceptional elements have for reduced words the factors of $(t_1t_2s_1s_2\dots s_{n-1}u_1u_2s_{n-1}\dots s_2s_1)^\infty$ where u_1, u_2 are also allowed to commute. For a large enough length i, there are $2n+6$ such factors unless i is divisible by $n+1$, in which case there are $2n+8$ factors.

From this, it follows that the coefficients of $\widetilde{B}_{n+1}^{FC}(q)$ (*resp.* $\widetilde{D}_n^{FC}(q)$) are ultimately periodic with period $(n+1)(2n+1)$ (*resp.* $n+1$).

Exceptional types: The remaining irreducible affine types are $\widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8, \widetilde{F}_4$, and $\widetilde{G}_2$. The number of FC elements in $\widetilde{F}_4$ or $\widetilde{E}_8$ is actually finite, since they correspond in Stembridge's classification [9, Theorem 5.1] to types F_5 and E_9: the corresponding polynomials $\widetilde{F}_4(q)$ and $\widetilde{E}_8(q)$ have degrees respectively 18 and 44. The number of FC elements according to the length is ultimately periodic in the remaining types $\widetilde{E}_6, \widetilde{E}_7, \widetilde{G}_2$, the periods being $4, 9$ and 5 respectively: this is easily seen in type $\widetilde{G}_2$ and requires a finer analysis in the two other types. In all three cases we can actually compute the whole series $W^{FC}(q)$.

4 Further questions

Involutions: As explained by Stembridge in [11], a FC element w in a Coxeter group is an involution if and only if $\mathcal{R}(w)$ is *palindromic*, meaning that it includes the mirror image of some (equivalently, all) of its members. Thanks to Theorem 8, the alternating FC involutions correspond to Motzkin type walks having no horizontal steps at height greater than zero. Therefore, denoting by $\overline{A}^{FC}(x) = \sum_{n\geq 1} \overline{A}_{n-1}^{FC}(q)x^n$, $\overline{B}^{FC}(x) = \sum_{n\geq 0} \overline{B}_n^{FC}(q)x^n$ and $\overline{D}^{FC}(x) = \sum_{n\geq 0} \overline{D}_{n+1}^{FC}(q)x^n$ the generating functions for FC involutions in types A, B and D, we can prove that:

$$
\begin{aligned}
\overline{A}^{FC}(x) &= \frac{\mathrm{Cat}(x)}{1-x\mathrm{Cat}(x)} - 1,\\
\overline{B}^{FC}(x) &= \frac{\mathrm{Cat}^+(x)}{1-x\mathrm{Cat}(x)} + \frac{x^2q^3}{1-xq^2}\frac{\mathrm{Cat}(x)\mathrm{Cat}(qx)}{1-x\mathrm{Cat}(x)},\\
\overline{D}^{FC}(x) &= 2\cdot\frac{\mathrm{Cat}^o(x)}{1-x\mathrm{Cat}(x)} + \frac{\mathrm{Cat}}{1-x\mathrm{Cat}(x)} + \frac{xq^2}{1-xq^2}\frac{\mathrm{Cat}(x)\mathrm{Cat}(qx)}{1-x\mathrm{Cat}(x)},
\end{aligned}
$$

where $\mathrm{Cat}(x)$ (*resp.* $\mathrm{Cat}^+(x)$, *resp.* $\mathrm{Cat}^o(x)$) denotes the generating function for q weighted Dyck walks. (*resp.* suffixes of Dyck walks, *resp.* suffixes of Dyck walks starting at an odd height). Note that all these generating functions can be computed by the functional equations $\mathrm{Cat}(x) = 1+qx^2\mathrm{Cat}(x)\mathrm{Cat}(qx)$, $\mathrm{Cat}^+(x) = \mathrm{Cat}(x)(1+xq\mathrm{Cat}^+(qx))$, and $\mathrm{Cat}^o(x) = xq\mathrm{Cat}(x)\mathrm{Cat}(qx)(1+xq^2\mathrm{Cat}^o(q^2x))$.

Other problems: One may try to find a formula for $Q^*(x)$, maybe in terms of certain q-Bessel functions as for $M^*(x)$, since this would give formulas for $B^{FC}(x)$ and $D^{FC}(x)$.

It would be interesting to study other statistics on the sets W^{FC}; for instance, the number of descents, which has the advantage of being defined for any Coxeter group.

In the hyperplane arrangement associated to an affine group W , elements correspond to the regions (named *alcoves*). One may wonder where the alcoves corresponding to FC elements are located, and how the periodicity of Theorem 14 is involved.

Finally, a natural problem is to determine which Coxeter groups W are such that $W^{FC}(q)$ has ultimately periodic coefficients. This is work in progress: there are apparently exactly two exceptional, non-affine groups with such a periodicity.

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Matroids over a ring

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Abstract. We introduce the notion of a matroid M over a commutative ring R, assigning to every subset of the ground set an R-module according to some axioms. When R is a field, we recover matroids. When $R = \mathbb{Z}$, and when R is a DVR, we get (structures which contain all the data of) quasi-arithmetic matroids, and valuated matroids, respectively. More generally, whenever R is a Dedekind domain, we extend the usual properties and operations holding for matroids (e.g., duality), and we compute the Tutte-Grothendieck group of matroids over R.

Résumé. Nous introduisons la notion de matroïde M sur un anneau commutatif R, qui assigne à chaque partie d'un ensemble E un R-module selon certains axiomes. Quand R est un corps, on retrouve les matroïdes. Lorsque $R = \mathbb{Z}$, et lorsque R est un anneau de valuation discrète, nous obtenons (structures qui contiennent toutes les données) respectivement des matroïdes quasi-arithmétiques et des matroïdes valués. En plus de généralité, quand R est un anneau de Dedekind, nous étendons les propriétés et operations habituelles pour les matroïdes (par exemple, la dualité), et nous calculons le groupe de Tutte-Grothendieck des matroïdes sur R.

Keywords: arithmetic matroids, valuated matroids, tropical flag variety, Dedekind domains, Tutte polynomial, Tutte-Grothendieck group

1 Introduction

The notion of a *matroid* axiomatizes the linear algebra of a list of vectors. Matroid theory has proved to be a versatile language to deal with many problems on the interfaces of combinatorics and algebra. In the years since 1935, when Whitney first introduced matroids, a number of enriched variants thereof have arisen, among them oriented matroids [2], valuated matroids [8], complex matroids [1], and (quasi-) arithmetic matroids [13, 5]. Each of these structures retains some information about a vector configuration, or an equivalent object, which is richer than the purely linear algebraic information that matroids retain.

As a running motivating example, let us focus on quasi-arithmetic matroids. A quasi-arithmetic matroid endows a matroid with a multiplicity function, whose values (in the representable case) are the cardinalities of certain finite abelian groups, namely, the torsion parts of the quotients of an ambient lattice $\mathbb{Z}^n$ by the sublattices spanned by subsets of vectors. From a list of vectors with integer coordinates one may produce objects like a toric arrangement, a partition function, and a zonotope. In order to have a combinatorial structure from which these objects may be read off, one needs to keep track of arithmetic properties of the vectors, and this is what quasi-arithmetic matroids provide.

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It is natural to ask how well these generalizations of matroids can be unified under one framework. Such a unification was sought by Dress in his program of matroids with coefficients, represented for example in his work with Wenzel [8] wherein valuated matroids are matroids with coefficients in a "fuzzy ring".

In the present paper we suggest a different approach to such unification, by defining the notion of a *matroid M over a commutative ring R*. Such an M assigns, to every subset A of a ground set, a finitely generated R-module $M(A)$ according to some axioms (Definition 2.1). We find this definition to have multiple agreeable features. For one, by building on the well-studied setting of modules over commutative rings, we get a theory where the considerable power and development of commutative algebra can be easily brought to bear. For another, unlike arithmetic and valuated matroids, a matroid over R is not defined as a matroid decorated with extra data; there are only two axioms, and we suggest that they are comparably simple to the matroid axioms themselves. In particular, a *representable* matroid over R is precisely a vector configuration in a finitely generated R-module.

When R is a field, a matroid M over R is nothing but a matroid: the data $M(A)$ is a vector space, which contains only the information of its dimension, and this directly encodes the rank function of M. When $R = \mathbb{Z}$, every module $M(A)$ is an abelian group, and by extracting its torsion subgroup we get a quasi-arithmetic matroid. When R is a discrete valuation ring (DVR), we may similarly extract a valuated matroid. More generally, whenever R is a Dedekind domain, we can extend the usual properties and operations holding for matroids, such as duality.

The idea of matroids over rings was suggested by features of the theory of quasi-arithmetic matroids. Some significant information about an integer vector configuration is not retained in the multiplicity function, as many finite abelian groups can have the same cardinality. Recording the whole structure of these groups is desirable in several situations, for example, in developing a combinatorial intersection theory for the arrangements of subtori arising as characteristic varieties. The properties of the multiplicity function of a quasi-arithmetic matroid turn out to be just shadows of group-theoretic properties.

One of the most-loved invariants of matroids is their Tutte polynomial $\mathbf{T}_M(x, y)$. It thus comes as no surprise that the Tutte polynomial has been considered for generalizations of matroids as well. A quasi-arithmetic matroid $\hat{M}$ has an associated arithmetic Tutte polynomial $\mathbf{M}_{\hat{M}}(x, y)$, which has proved to be a useful tool in studying toric arrangements, partition functions, zonotopes, and graphs ([13, 7, 3]). More strongly, the authors of [3] define a *Tutte quasi-polynomial* of an integer vector configuration, interpolating between $\mathbf{T}_M(x, y)$ and $\mathbf{M}_{\hat{M}}(x, y)$, which is no longer an invariant of the quasi-arithmetic matroid (as it depends on the groups, not just their cardinalities).

Among its properties, the Tutte polynomial of a classical matroid is the universal deletion-contraction invariant. In more algebraic language, following [4], the class of a matroid in the Tutte-Grothendieck group for deletion-contraction relations is exactly its Tutte polynomial. While the arithmetic Tutte polynomial and Tutte quasi-polynomial are deletion-contraction invariants, neither is universal for this property. Our generalization of the Tutte polynomial for matroids over a Dedekind ring R is also the class in the Tutte-Grothendieck group, so it retains the universality of the usual Tutte polynomial, and we obtain the two generalizations of Tutte just mentioned as evaluations of it.

This paper is organized as follows. In Section 2 we give the basic definitions for matroids over a commutative ring, including representability, and we explain how they generalize the classical ones.

We introduce the assumption that R is a Dedekind domain, and do some groundwork, in Section 3. This assumption on R remains for the most part in force from this section onward. Its first application comes in Section 4, where we establish the existence and the properties of the dual of a matroid over a Dedekind domain R.

In Section 5 we develop the local theory, with a structure theorem for matroids over a DVR. We show connections with the Hall algebra and with the tropical Plücker relations for the flag variety. Finally, we describe how to recover valuated matroids.

The global theory is developed in Section 6. We describe the structure of a matroid over a Dedekind ring R in terms of the structure of all its localizations (completely described in the previous section) plus some global information coming from the Picard group of R . This also explains the connection between matroids over $\mathbb{Z}$ and quasi-arithmetic matroids.

Finally, in Section 7 we compute the Tutte-Grothendieck group. In particular, given a matroid over $\mathbb{Z}$, we present its Tutte quasi-polynomial as an evaluation of its class in $K(\mathbb{Z}\text{-Mat})$.

This paper is an extended abstract of the article [10], to which the interested reader is suggested to refer for many details and for all the proofs, which are omitted here.

2 Matroids over a ring

By R-Mod we mean the category of finitely generated R-modules over a commutative ring R. We write "f.g." for "finitely generated" throughout.

Definition 2.1 *Let R be a commutative ring. A* matroid over R *on the ground set E is a function M assigning to each subset $A \subseteq E$ a finitely-generated R-module $M(A)$ satisfying the following axioms:*

(M1) For any $A \subseteq E$ and $b \in E \setminus A$, there exists a surjection $M(A) \twoheadrightarrow M(A \cup \{b\})$ whose kernel is a cyclic submodule of $M(A)$.

(M2) For any $A \subseteq E$ and $b \neq c \in E \setminus A$, there exists a pushout

$$\begin{tikzcd} M(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & M(A \cup \{b\}) \arrow[d] \\ M(A \cup \{c\}) \arrow[r] & M(A \cup \{b, c\}) \end{tikzcd}$$

where all four morphisms are surjections with cyclic kernel.

Polymatroids can be defined similarly (see [10, Definition 2.2]). Clearly, Axiom (M1) is redundant if $|E| \geq 2$, and the choice of the modules $M(A)$ is only relevant up to isomorphism. For concision, we will hereafter let $M(Ab)$ abbreviate $M(A \cup \{b\})$, $M(Abc)$ stand for $M(A \cup \{b, c\})$, and so forth.

The fundamental way of producing matroids over R is from vector configurations in an R-module. Given a f.g. R-module N and a list $X = x_1, \ldots, x_n$ of elements of N, the matroid M_X of X associates to the sublist A of X the quotient

$$M_X(A) = N \Big/ \left(\sum_{x \in A} Rx \right). \tag{2.1}$$

For each $x \in X$ there is a quotient map from $M_X(A)$ to $M_X(A \cup \{x\})$, which quotients out by the image of Rx in $M_X(A)$; this system of maps satisfies axioms (M1) and (M2).

The following definition captures this concisely. Let $\mathcal{B}(E)$ be the category of the Boolean poset of subsets of E, where inclusions of sets are the morphisms.

Definition 2.2 *A matroid M over R is* representable *(or* realizable*) if it is the map on objects of some functor $F : \mathcal{B}(E) \to R$-*Mod*, and axioms (M1) and (M2) are satisfied by choosing the morphisms $F(A \to Ab)$. A* representation *(or* realization*) of M is a choice of such an F.*

So M_X is a representable matroid, and X gives a representation thereof. We have chosen to cast Definition 2.2 as we did, as opposed to in a more down-to-earth way involving M_X, to emphasize the way in which a representable matroid is a matroid. A representation of a matroid over R is a functor from $\mathcal{B}(E)$, with both objects and morphisms having images. A general matroid over R is what is gotten by retaining only the objects as data, discarding the morphisms and merely requiring that they can be resupplied to look like a represented matroid over R in any square of covering relations in $\mathcal{B}(E)$.

Fact 2.3 *If a matroid M over R is representable, corresponding to the functor F, then it is the matroid M_X of a vector configuration $(N, X = \{x_a\})$, where N is $F(\emptyset)$, and x_a is a generator of* $\ker F(\emptyset \to \{a\})$ *for each $a \in E$. Indeed, in this above setting, the pushout axiom (M2) applied to F guarantees that equation* (2.1) *holds for all $A \subseteq E$.*

Our having chosen to call these objects "matroids over R" is appropriate, as they are a generalization of matroids in the classical sense, as we show in Proposition 2.5. There is one hitch in the equivalence, corresponding to the ability to choose a vector configuration that does not span its ambient space. Accordingly, let us say that a matroid M over R is *full-rank* if no nontrivial projective module is a direct summand of $M(E)$. Lemma 2.4 shows that very little is lost in restricting to full-rank matroids.

Before getting there we must generalize some standard operations on matroids. Let M and M' be matroids over R on respective ground sets E and E'. We define their *direct sum* $M \oplus M'$ on the ground set $E \amalg E'$ by

$$(M \oplus M')(A \amalg A') = M(A) \oplus M'(A').$$

If i is an element of E, we define two matroids over R on the ground set $E \setminus \{i\}$: the *deletion* of i in M, denoted $M \setminus i$, by

$$(M \setminus i)(A) = M(A)$$

and the *contraction* of i in M, denoted $M \setminus i$, by

$$(M/i)(A) = M(A \cup \{i\}).$$

It is easily seen that the class of representable matroids is closed under minors and direct sums.

If N is an R-module, let the *empty matroid* for N be the matroid over R on the ground set $\emptyset$ which maps $\emptyset$ to N. By a *projective empty matroid* we mean an empty matroid for a projective module.

Lemma 2.4 *Every matroid M over R is the direct sum of a full-rank matroid over R and a projective empty matroid.*

Recall that the *corank* $\operatorname{cork}(A)$ of a set A in a classical matroid is equal to $\operatorname{rk}(E) - \operatorname{rk}(A)$, where $\operatorname{rk}(E)$ is the rank of the matroid.

Proposition 2.5 *Let $\mathbb{K}$ be a field. Full-rank matroids M over $\mathbb{K}$ are equivalent to (classical) matroids. If M is a full-rank matroid over $\mathbb{K}$, then* $\dim M(A)$ *is the corank of A in the corresponding classical matroid. Furthermore, a matroid over $\mathbb{K}$ is representable if and only if, as a classical matroid, it is representable over $\mathbb{K}$.*

The proof of this fact is simple, and relies on the fact that finitely generated modules over $\mathbb{K}$ are the finite-dimensional $\mathbb{K}$-vector spaces, which are completely classified up to isomorphism by dimension. So we may replace $M(A)$ by its $\mathbb{K}$-dimension without losing information.

Let $R \to S$ be a map of rings. Then every matroid over S is naturally also a matroid over R. Furthermore, given such a map $R \to S$, the tensor product $-\otimes_R S$ is a functor R-Mod $\to$ S-Mod. One can use this to perform base change of matroids over R. If M is a matroid over R, define $M \otimes_R S$ be the composition of M with $-\otimes_R S$, so that for every A

$$(M \otimes_R S)(A) = M(A) \otimes_R S.$$

Proposition 2.6 *If M is a matroid over R, then $M \otimes_R S$ is a matroid over S.*

Two special cases of this construction will be of fundamental importance for our theory.

1. For every prime ideal $\mathfrak{m}$ of R, let $R_\mathfrak{m}$ be the localization of R at $\mathfrak{m}$. We call $M_\mathfrak{m} \doteq M \otimes_R R_\mathfrak{m}$ the *localization* of M at $\mathfrak{m}$.
2. If R is an integral domain, let $\mathrm{Frac}(R)$ be the fraction field of R. Then we call $M_{\text{gen}} \doteq M \otimes_R \mathrm{Frac}(R)$ the *generic matroid* of M.

Our approach will be much based on studying the matroid M via these localizations.

Notice that every matroid over $R_\mathfrak{m}$ induces a matroid over the residue field $R_\mathfrak{m}/(\mathfrak{m})$; the latter, as well as M_{gen}, is by Proposition 2.5 equivalent to a classical matroid (except that it may be not full-rank).

3 Dedekind domains

In several ways, Definition 2.1 yields a theory best parallelling the theory of classical matroids just when R is a Dedekind domain. The reason for that is explained in [10, Lemma 3.1 and Example 3.2].

We next recall some structural results about modules over a Dedekind domain R. Given an R-module N, let $N_{\mathrm{tors}} \subseteq N$ denote the submodule of its torsion elements, and N_{proj} denote the projective module N/N_{tors}. Then N is the direct sum of N_{tors} and of a projective module isomorphic to N_{proj}. We recall the following fact.

Proposition 3.1 *[9, exercises 19.4–6] Every torsion R-module may be written uniquely up to isomorphism as a sum of submodules $R/\mathfrak{m}^k$ for $\mathfrak{m}$ a maximal prime of R and $k \in \mathbb{Z}_{>0}$.*

Every nonzero projective R-module is uniquely isomorphic to $R^h \oplus I$ for some $h \geq 0$ and nonzero ideal I, up to differing isomorphic choices of I. For ideals I and J, we have $I \oplus J \cong R \oplus (I \otimes J)$.

We recall the following definitions. The *Picard group* of R, $\mathrm{Pic}(R)$, is the set of the isomorphism classes of f.g. projective modules of rank 1, with product induced by the tensor product. If P is a projective module of rank n, the exterior algebra $\Lambda^n P$ is a f.g. projective module of rank $\binom{n}{n} = 1$. We call *determinant*, and denote by $\det(P)$, its class in $\mathrm{Pic}(R)$.

We will also find useful a description of the *algebraic K-theory group* $K_0(R)$ of f.g. R-modules: that is, the abelian group generated by isomorphism classes $[N]$ of f.g. R-modules, modulo the relations $[N] = [N'] + [N'']$ for any exact sequence

$$0 \to N' \to N \to N'' \to 0.$$

Proposition 3.2 *There is an isomorphism of groups*

$$\Phi : K_0(R) \longrightarrow \mathbb{Z} \oplus \mathrm{Pic}(R).$$

In fact, when P is a projective module, the map above is simply given by $\Phi([P]) = (\mathrm{rk}(P), \det(P))$.

In virtue of the isomorphism above, from now on we will denote by $\det(N)$ the class of any f.g. R-module N in the Picard group, i.e. the second summand of $\Phi([N])$. In the same way, by $\mathrm{rk}(N)$ we denote the first summand of $\Phi([N])$: this coincides with the rank of N_{proj}, i.e. with the dimension of $N \otimes \mathrm{Frac}(R)$.

Note in particular that Φ extends the usual map from invertible ideals to $\mathrm{Pic}(R)$.

The potential nontriviality of this summand $\mathrm{Pic}(R) \subseteq K_0(R)$ has global consequences for matroids over R: see Proposition 4.3 below.

4 Duality for matroids over Dedekind domains

In this section R will be a Dedekind domain. Let M be a matroid over R, on ground set E. Fix a free module F that surjects on $M(\emptyset)$. For any $A \subseteq E$ and maximal flag of subsets $\emptyset = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_{|A|} = A$, we obtain a composite surjection

$$F \to M(\emptyset) \to M(A_1) \to \cdots \to M(A).$$

Using the horseshoe lemma, we may assemble minimal projective resolutions of each step of this composition into a projective resolution of $F/M(A)$, yielding a projective resolution

$$P(A)_\bullet : \quad 0 \to P_2(A) \to P_1(A) \xrightarrow{d_1} F \to M(A) \to 0$$

of $M(A)$. As usual, we write $^\vee$ for the contravariant functor $\mathrm{Hom}(—, R)$.

Definition 4.1 *Define the module $M^*(E \setminus A)$ as the cokernel of the map dual to d_1 in $P(A)_\bullet$, that is*

$$M^*(E \setminus A) \doteq \mathrm{coker}\left(F^\vee \xrightarrow{d_1^\vee} P_1(A)^\vee\right).$$

This is well-defined ([10, Lemma 4.2]). We define M^*, the dual *matroid over R to M, to be the collection of these modules $M^*(E \setminus A)$.*

Theorem 4.2 *If R is a Dedekind domain, and M is a matroid over R, then its dual M^* is a full-rank matroid over R. Furthermore, M is the direct sum of M^{**} and the projective empty matroid for $M(E)_{\mathrm{proj}}$; in particular, if M is full-rank, $M^{**} = M$.*

If M is representable, also M^* is.

The last statement above gives a generalization of the classical Gale duality of vector configurations.

Furthermore, duality of matroids over rings is well-behaved with respect to deletion, contraction, direct sums, and tensor products, as shown in [10, Proposition 4.9].

Proposition 4.3 *Let M be a matroid over R. The element*

$$\det(M) \doteq \det(M(A)_{\mathrm{proj}}) + \det(M^*(E \setminus A)_{\mathrm{proj}}) + \det(M(A)_{\mathrm{tors}})$$

of $\mathrm{Pic}(R)$ is independent of the choice of $A \subseteq E$.

5 Structure of matroids over a DVR

In this section and the next we record some structure theorems for matroids over R in terms of structure theorems for the modules over R themselves. Our analysis of general Dedekind domains in the next section will make much use of base changing to localizations of R, so we begin here with the local case.

For the whole of this section, R will be a DVR with maximal ideal $\mathfrak{m}$. We first recall the structure theory of f.g. R-modules: any indecomposible f.g. R-module is isomorphic to either R or $R/\mathfrak{m}^n$ for some integer $n \geq 1$. We will sometimes formally subsume R into the latter family by writing it as $R/\mathfrak{m}^\infty$. So, if N is a f.g. R-module and $i \geq 1$ is an integer, define

$$d_i(N) \doteq \dim_{R/\mathfrak{m}}(\mathfrak{m}^{i-1}N/\mathfrak{m}^i N),$$

and $d_{\leq i}(N) \doteq \sum_{j=1}^{i} d_j(N)$, and for convenience $d_i(N) = d_{\leq i}(N) = 0$ if $i \leq 0$. Let $d_\bullet(N)$ denote the infinite sequence of these. We have

$$d_i(R/\mathfrak{m}^n) = \begin{cases} 1 & 0 < i \leq n \\ 0 & i > n \end{cases},$$

where n may be ∞. The following is a quick consequence.

Proposition 5.1 *Isomorphism types of f.g. R-modules are in bijection with nonincreasing infinite sequences $d_\bullet$ of nonnegative integers indexed by positive integers, the bijection being given by*

$$N \longleftrightarrow d_\bullet(N).$$

This bijection permits a straightforward identification of those isomorphism classes of modules which permit maps satisfying axioms (M1) and (M2).

Theorem 5.2 *Let N and N' be f.g. R-modules. There exists a surjection $\phi : N \to N'$ with cyclic kernel if and only if*

(L1) for each $n \geq 1$,

$$d_n(N) - d_n(N') \in \{0, 1\}.$$

Let $M(\emptyset)$, $M(1)$, $M(2)$, and $M(12)$ be f.g. R-modules. There exist four surjections with cyclic kernels forming a pushout square

$$\begin{tikzcd} M(\emptyset) \arrow[r, "\phi"] \arrow[d, "\psi"'] \arrow[dr, phantom, "\lrcorner"] & M(1) \arrow[d, "\psi'"] \\ M(2) \arrow[r, "\phi'"'] & M(12) \end{tikzcd}$$

if and only if (L1) is satisfied for each pair $M(A), M(Ab)$, and moreover

(L2a) for each $n \geq 1$,

$$d_{\leq n}(M(\emptyset)) - d_{\leq n}(M(1)) - d_{\leq n}(M(2)) + d_{\leq n}(M(12)) \geq 0;$$

(L2b) for any $n \geq 1$ such that $d_n(M(1)) \neq d_n(M(2))$, equality holds above:

$$d_{\leq n}(M(\emptyset)) - d_{\leq n}(M(1)) - d_{\leq n}(M(2)) + d_{\leq n}(M(12)) = 0.$$

Condition (L2a) asserts that $A \mapsto -d_{\leq n}(M(A))$ is a *submodular* function.

In the case that N and N' have finite length, condition (L1) follows from facts about the Hall algebra [11]. Indeed, it is equivalent that N have finite length and that $d_i(N)$ stabilize to 0 for $i \gg 0$. In this case d_i is a partition, and its conjugate partition is the one usually used to label N. For a cyclic module, this conjugate partition has a single row. Then, under the specialization taking the Hall polynomials to the Littlewood-Richardson coefficients, condition (L1) is a consequence of the Pieri rule.

The structure of matroids over R in fact has interesting tropical-geometric import (for background on tropical geometry, see [12]). The first inkling of this is in three-element matroids:

Proposition 5.3 *Let M be a matroid over R on the ground set $[3]$, and let n be a natural or ∞. Then, among the three quantities*

$$d_{\leq n}(M(1)) + d_{\leq n}(M(23)), d_{\leq n}(M(2)) + d_{\leq n}(M(13)), d_{\leq n}(M(3)) + d_{\leq n}(M(12)),$$

the minimum is achieved at least twice.

Let M be a matroid over R with ground set E. For $A \subseteq E$, define p_A to be $d_{\leq n}(M(A))$. Proposition 5.3 applied to the 3-element minors of M can be taken to say that the *tropicalizations* of the relations

$$p_{Ab}p_{Acd} - p_{Ac}p_{Abd} + p_{Ad}p_{Abc} = 0 \tag{5.1}$$

hold of the numbers $p_\bullet$, where we continue abbreviating $A \cup \{b, c\}$ as Abc and similarly.

The relations (5.1) are among the Plücker relations for the full flag variety (of type A). We can show [10, Proposition 5.6] that the $p_\bullet$ satisfy some of the other Plücker relations, which imply that for every r the point $(p_A : |A| = r)$ lies on the *Dressian* $\mathrm{Dr}(r, n)$. The Dressian is one Grassmannian-like space in tropical geometry: it is the parameter space for tropical linear spaces. That is, there is a tropical linear space determined by $(p_A : |A| = r)$. Corollary 5.4 follows.

Corollary 5.4 *Let M be a matroid over a DVR $(R, \mathfrak{m})$. Then the function $A \mapsto \dim_{R/\mathfrak{m}} M(A)$ makes the generic matroid of M into a valuated matroid, in the sense of Dress and Wenzel [8].*

To be precise, our sign convention is the opposite of the one adopted in [8]; for perfect agreement we would have to negate this function. But our sign convention is frequently adopted in tropical geometry.

Conjecture 5.5 *The collection of the p_A satisfies every tropical Plücker relation, i.e. gives a point on the Dressian analogue of the tropical full flag variety.*

We expect that Conjecture 5.5 follows directly from Proposition 5.3, and needs no further matroidal arguments. The main obstruction to proving 5.5 seems to be only that the tropical full flag variety has been little studied.

6 Global structure of matroids over a Dedekind domain

Throughout this section R will be a Dedekind domain. Let us recall that given a R-module N, by $\det(N)$ we will denote its class in the Picard group $\mathrm{Pic}(R)$, as defined in Section 3. The next Theorem gives a complete characterization of the structure of matroids over R, in terms of their localizations for which we have Theorem 5.2.

Theorem 6.1 *Let N and N' be f.g. R-modules. There exists a surjection $N \to N'$ with cyclic kernel if and only if there exists such a surjection $N_\mathfrak{m} \to N'_\mathfrak{m}$ after localizing at each maximal prime $\mathfrak{m}$ of R, and*

- *if* $\mathrm{rk}(N) - \mathrm{rk}(N') = 0$ *then* $\det(N_{\mathrm{proj}}) = \det(N'_{\mathrm{proj}})$, *whereas*
- *if* $\mathrm{rk}(N) - \mathrm{rk}(N') = 1$ *then* $\det(N) = \det(N')$.

Let $M(\emptyset)$, $M(1)$, $M(2)$, and $M(12)$ be f.g. R-modules. There exist four surjections with cyclic kernels forming a pushout square

$$\begin{tikzcd} M(\emptyset) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & M(1) \arrow[d] \\ M(2) \arrow[r] & M(12) \end{tikzcd}$$

if and only if the same is true after localizing at each maximal prime $\mathfrak{m}$, and the above conditions on classes are true of each $(N, N') = (M(A), M(Ab))$.

6.1 Quasi-arithmetic matroids

If M is a matroid over $\mathbb{Z}$, then we can define a corank function of M as the corank function of the generic matroid $M \otimes_\mathbb{Z} \mathbb{Q}$ described above, that is $\mathrm{cork}(A) = \mathrm{rk}_\mathbb{Z}(M(A)_{\mathrm{proj}})$.

As before, we let $M(A)_{\mathrm{tors}}$ denote the torsion submodule (subgroup, in this case) of $M(A)$. Then we define

$$m(A) \doteq |M(A)_{\mathrm{tors}}|.$$

Corollary 6.2 *The triple (E, cork, m) is a* quasi-arithmetic matroid*, i.e (E, cork) defines a matroid, and m satisfies the following properties:*

(A1) Let be $A \subseteq E$ and $b \in E$; if b is dependent on A, then $m(A \cup \{b\})$ divides $m(A)$; otherwise $m(A)$ divides $m(A \cup \{b\})$;

(A2b) if $A \subseteq B \subseteq E$ and B is a disjoint union $B = A \cup F \cup T$ such that for all $A \subseteq C \subseteq B$ we have $\mathrm{rk}(C) = \mathrm{rk}(A) + |C \cap F|$, then

$$m(A) \cdot m(B) = m(A \cup F) \cdot m(A \cup T).$$

Furthermore it satisfies the following property:

(A2a) if $A, B \subseteq E$ and $\mathrm{rk}(A \cup B) + \mathrm{rk}(A \cap B) = \mathrm{rk}(A) + \mathrm{rk}(B)$, then $m(A) \cdot m(B)$ divides $m(A \cup B) \cdot m(A \cap B)$

In fact properties (A1), (A2a), (A2b) follow from (L1), (L2a), (L2b) respectively. This corollary establishes that matroids over $\mathbb{Z}$ recover many of the essential features of the second author's theory of *arithmetic matroids* from [5]. To be precise, the objects we have recaptured are *quasi-arithmetic matroids*: see [10, Remark 6.4]. In fact the two objects are not truly equivalent, in that the information contained in matroids over $\mathbb{Z}$ is richer, because there are many finite abelian groups with the same cardinality.

7 The Tutte-Grothendieck group

In this section we continue to let R be a Dedekind domain. All matroids over R in this section are full-rank. As we defined the operations of deletion and contraction in Section 2, any element may be deleted or contracted. However, suppose $a \in E$ is a (*generic*) *coloop* of a matroid M over R, that is a coloop of the generic matroid, equivalently an element such that $M(E \setminus \{a\})$ has a nontrivial projective summand. In this case, $M \setminus a$ is not full-rank. The dual of this situation is the case where a is a (*generic*) *loop*, i.e. a loop of the generic matroid, and one contracts a.

Essentially following Brylawski [4], define the *Tutte-Grothendieck group* of matroids over R, which we here denote $K(R\text{-Mat})$, to be the abelian group generated by a symbol $\mathbf{T}_M$ for each unlabelled full-rank matroid M over R with nonempty ground set, modulo the relations

$$\mathbf{T}_M = \mathbf{T}_{M\setminus a} + \mathbf{T}_{M/a}$$

whenever a is not a generic loop or coloop (so that we avoid the above situations). We have omitted empty matroids for technical reasons, though they cause no essential problem; the interested reader can refer to [10, Remark 7.2]. By "unlabelled", we mean that we consider two matroids M and M' over R to be identical if there is a bijection $\sigma : E \xrightarrow{\sim} E'$ of their ground sets such that $M(A) \cong M'(\sigma(A))$ for each subset A of E.

The ring $K(R\text{-Mat})$ turns out to be best understood in terms of the monoid ring of the monoid of R-modules under direct sum, as in Theorem 7.1 below. Define $\mathbb{Z}[R\text{-Mod}]$ to be the ring with a $\mathbb{Z}$-linear basis $\{u^N\}$ with an element u^N for each f.g. R-module N up to isomorphism, and product given by $u^N u^{N'} = u^{N\oplus N'}$.

Theorem 7.1 *The Tutte-Grothendieck group $K(R\text{-Mat})$ is a ring without unity, with product given by $\mathbf{T}_M \cdot \mathbf{T}_{M'} = \mathbf{T}_{M\oplus M'}$. As a ring it injects into $\mathbb{Z}[R\text{-Mod}] \otimes \mathbb{Z}[R\text{-Mod}]$, and under this injection, the class of M maps to*

$$\mathbf{T}_M = \sum_{A\subseteq E} X^{M(A)} Y^{M^*(E\setminus A)}, \tag{7.1}$$

where $\{X^N\}$ and $\{Y^N\}$ are the respective bases of the two tensor factors $\mathbb{Z}[R\text{-Mod}]$.

If we include empty matroids, the ring $\mathbb{K}(R\text{-Mat})$ is the subring of $\mathbb{Z}[R\text{-Mod}] \otimes \mathbb{Z}[R\text{-Mod}]$ generated by X^P and Y^P as P ranges over rank 1 projective modules, and $(XY)^N$ as N ranges over torsion modules.

Here $(XY)^N$ abbreviates $X^N Y^N$. We immediately compare Theorem 7.1 with the case of matroids over a field, where the Tutte-Grothendieck invariant is the familiar Tutte polynomial $\mathbf{T}_M$. If R is a field, then $\mathbb{Z}[R\text{-Mod}]$ is the univariate polynomial ring $\mathbb{Z}[u]$, and then $\mathbb{Z}[R\text{-Mod}] \otimes \mathbb{Z}[R\text{-Mod}]$ is, appropriately, a bivariate polynomial ring. If we call the generators of the two tensor factors $x - 1$ and $y - 1$ rather than X and Y, then equation (7.1) in fact gives the classical Tutte polynomial, since $\dim M(A)$ is the corank of A and $\dim M^*(E \setminus A)$ is its nullity.

Since decomposing a matroid M over a ring into $M \setminus i$ and M/i is not a unique decomposition in the sense of [4], and the irreducibles for direct sum are not all single-element matroids, Theorem 7.1 does not follow directly from the bidecomposition methods of [4].

7.1 Arithmetic Tutte polynomial and quasi-polynomial

In this subsection, M is a matroid over $\mathbb{Z}$. We show that the arithmetic Tutte polynomial of its associated quasi-arithmetic matroid $\hat{M}$, and its Tutte quasi-polynomial, are each images of $\mathbf{T}_M$ under ring homomorphisms. When $R = \mathbb{Z}$, the Picard group is trivial, and

$$\mathbf{T}_M = \sum_{A \subseteq E} (X^R)^{\mathrm{cork}_M(A)} (Y^R)^{\mathrm{nullity}_M(A)} (XY)^{M(A)_{\mathrm{tors}}}.$$

where we use the notation $\mathrm{nullity}_M(A) = \mathrm{cork}_{M^*}(E \setminus A) = \dim M^*(E \setminus A)$.

We may define a specialization of $\mathbf{T}_M$ by specializing X^R to $(x-1)$, Y^R to $(y-1)$, and $(XY)^N$ to the cardinality of N for each torsion module N. This specialization is the arithmetic Tutte polynomial $\mathbf{M}_{\hat{M}}(x,y)$ of the quasi-arithmetic matroid $\hat{M}$ defined by M:

$$\mathbf{M}_{\hat{M}}(x,y) = \sum_{A \subseteq E} m(A)(x-1)^{\mathrm{rk}(E)-\mathrm{rk}(A)}(y-1)^{|A|-\mathrm{rk}(A)}.$$

This polynomial proved to have several applications to toric arrangements, partition functions, zonotopes, and graphs with labeled edges (see [13, 6, 5]). Notice that an ordinary matroid $\tilde{M}$ can be trivially made into an arithmetic matroid $\hat{M}$ by setting all the multiplicities to be equal to 1, and then $\mathbf{M}_{\hat{M}}(x,y)$ is nothing but the classical Tutte polynomial $\mathbf{T}_{\tilde{M}}(x,y)$.

The polynomial $\mathbf{M}_{\hat{M}}(x,y)$ is not the universal deletion-contraction invariant of $\hat{M}$: for instance, the ordinary Tutte polynomial $\mathbf{T}_{\tilde{M}}(x,y)$ of the matroid $\tilde{M}$ obtained from $\hat{M}$ by forgetting of its arithmetic data is also a deletion-contraction invariant of $\hat{M}$, which is not determined by $\mathbf{M}_{\hat{M}}(x,y)$. This led the authors of [3] to define a *Tutte quasi-polynomial* $\mathbf{Q}_M(x,y)$, interpolating between $\mathbf{T}_{\tilde{M}}(x,y)$ and $\mathbf{M}_{\hat{M}}(x,y)$. This invariant is stronger, but still not universal, and more importantly, it is not an invariant of the arithmetic matroid, as it depends on the groups $M(A)_{\mathrm{tors}}$ and not just on their cardinalities. In fact $\mathbf{Q}_M(x,y)$ is an invariant of the matroid over $\mathbb{Z}$, and we show explicitly how to compute it from the universal invariant.

For every positive integer q, let us define a function V_q as $V_q((XY)^{\mathbb{Z}/p^k}) = 1$ if p^k divides q, while $V_q((XY)^{\mathbb{Z}/p^k}) = p^{k-j}$ if p^k does not divide q and $j \geq 0$ is the largest integer such that p^j divides q. We will extend this to define $V_q((XY)^N)$ multiplicatively for any torsion abelian group N. Then we define a specialization of $\mathbf{T}_M$ to the ring of quasipolynomials by specializing X^R to $(x-1)$, Y^R to $(y-1)$, and $(XY)^N$ to $V_{(x-1)(y-1)}((XY)^N)$. This gives

$$\mathbf{Q}_M(x,y) = \sum_{A \subseteq E} \frac{|M(A)_{\mathrm{tors}}|}{|(x-1)(y-1)M(A)_{\mathrm{tors}}|}(x-1)^{\mathrm{rk}(E)-\mathrm{rk}(A)}(y-1)^{|A|-\mathrm{rk}(A)}.$$

Since $(q+|G|)G = qG$ holds for any finite group G, it follows that $\mathbf{Q}_M(x,y)$ is a quasi-polynomial in $q = (x-1)(y-1)$. In particular, when $|M(A)_{\mathrm{tors}}|$ divides $(x-1)(y-1)$, then the group $(x-1)(y-1)M(A)_{\mathrm{tors}}$ is trivial and $\mathbf{Q}_M(x,y)$ coincides with $\mathbf{M}_{\hat{M}}(x,y)$; while when $|M(A)_{\mathrm{tors}}|$ is coprime with

$(x-1)(y-1)$, then $\mathbf{Q}_M(x,y)$ coincides with $\mathbf{T}_{\tilde{M}}(x,y)$. Then in some sense $\mathbf{Q}_M(x,y)$ interpolates between the two polynomials.

Notice that while $\mathbf{M}_{\hat{M}}$ and $\mathbf{T}_{\tilde{M}}(x,y)$ only depend on the induced quasi-arithmetic matroid $\hat{M}$, $\mathbf{T}_M$ and $\mathbf{Q}_M(x,y)$ are indeed invariants of the matroid over $\mathbb{Z}$, M. Also the *chromatic quasi-polynomial* and the *flow quasi-polynomial* defined in [3] are actually invariants of the matroid over $\mathbb{Z}$: by [3, Theorem 9.1] they are specializations of $\mathbf{Q}_M(x,y)$, and hence of the universal invariant $\mathbf{T}_M$.

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Moments of Askey-Wilson polynomials

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Abstract. New formulas for the n^{th} moment $\mu_n(a,b,c,d;q)$ of the Askey-Wilson polynomials are given. These are derived using analytic techniques, and by considering three combinatorial models for the moments: Motzkin paths, matchings, and staircase tableaux. A related positivity theorem is given and another one is conjectured.

Résumé. Nous présentons de nouvelles formules pour les n-moments $\mu_n(a,b,c,d;q)$ des polynômes Askey-Wilson. Ils sont calculés avec des techniques analytiques, et en considérant trois modèles combinatoires pour les moments: des chemins de Motzkin, des couplages, et des tableaux escalier. Un théorème de positivité liée est donné et un autre est conjecturé.

Keywords: Askey-Wilson polynomials, moments of orthogonal polynomials, Motzkin paths, hypergeometric series

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1 Introduction

The monic Askey-Wilson polynomials $P_n = P_n(x;a,b,c,d;q)$ are polynomials in x of degree n which depend upon five parameters a, b, c, d, and q. They may be defined by the three-term recurrence $P_{n+1} = (x-b_n)P_n - \lambda_n P_{n-1}$ with $P_{-1}=0$ and $P_0 = 1$ for $b_n = \frac{1}{2}(a+a^{-1}-(A_n+C_n))$ and $\lambda_n = \frac{1}{4}A_{n-1}C_n$, where

$$A_n = \frac{(1-abq^n)(1-acq^n)(1-adq^n)(1-abcdq^{n-1})}{a(1-abcdq^{2n-1})(1-abcdq^{2n})},$$
$$C_n = \frac{a(1-q^n)(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{(1-abcdq^{2n-2})(1-abcdq^{2n-1})}.$$

We refer to [6] for the standard basic hypergeometric notation and for information about the Askey-Wilson polynomials.

In view of the above three-term recurrence relation, the Askey-Wilson polynomials are orthogonal polynomials. An explicit absolutely continuous measure [1, Theorem 2.2], may be given for these polynomials. (We assume here that $\max\{|a|,|b|,|c|,|d|\} < 1$.) It is supported on $x \in [-1,1]$, and with $x=\cos\theta$, the measure is

$$w(\cos\theta,a,b,c,d;q) = \frac{(e^{2i\theta})_\infty(e^{-2i\theta})_\infty}{(ae^{i\theta})_\infty(ae^{-i\theta})_\infty(be^{i\theta})_\infty(be^{-i\theta})_\infty(ce^{i\theta})_\infty(ce^{-i\theta})_\infty(de^{i\theta})_\infty(de^{-i\theta})_\infty}.$$

The measure has total mass given by the *Askey-Wilson integral*,

$$I_0 = \frac{(q)_\infty}{2\pi}\int_0^\pi w(\cos\theta, a, b, c, d; q)d\theta = \frac{(abcd)_\infty}{(ab)_\infty(ac)_\infty(ad)_\infty(bc)_\infty(bd)_\infty(cd)_\infty}. \tag{1}$$

The purpose of this paper is to study the n^{th} moment $\mu_n(a,b,c,d;q)$ of the measure $w(x;a,b,c,d;q)$ for the Askey-Wilson polynomials

$$\mu_n(a,b,c,d;q) = C\int_{-1}^1 x^n w(x;a,b,c,d;q)\frac{dx}{\sqrt{1-x^2}}.$$

for some normalization constant C. With the normalization of $\mu_0(a,b,c,d;q)=1$, (the explicit C may be found from (1)), the n^{th} moment is known to be a rational function of a, b, c, d and q. We shall give new explicit expressions for $\mu_n(a,b,c,d;q)$ and study three combinatorial models for $\mu_n(a,b,c,d;q)$. One unusual feature of these results is the mixture of binomial and q-binomial terms in the explicit formulas. We shall see why this occurs, both analytically and combinatorially.

The simplest expression for $\mu_n(a,b,c,d;q)$ is a double sum (see [4, Theorem 1.12])

$$\mu_n(a,b,c,d;q) = \frac{1}{2^n}\sum_{m=0}^n \frac{(ab,ac,ad;q)_m}{(abcd;q)_m} q^m \sum_{j=0}^m \frac{q^{-j^2}a^{-2j}(aq^j+q^{-j}/a)^n}{(q,q^{1-2j}/a^2;q)_j(q,q^{2j+1}a^2;q)_{m-j}}. \tag{2}$$

However this expression is not obviously symmetric in a, b, c, and d, even though the Askey-Wilson polynomials $P_n(x;a,b,c,d;q)$ and the moments $\mu_n(a,b,c,d;q)$ are symmetric. Nor does it exhibit the correct poles of $\mu_n(a,b,c,d;q)$ as a rational function. For $d=0$ the moments are polynomials in a, b, c, and q. We give new expressions for the moments $\mu_n(a,b,c,d;q)$, which are symmetric and polynomial when $d=0$, see Theorem 2.3. We also give a symmetric version for all a,b,c,d in Theorem 2.7, although the polynomial dependence in q is not clear. We give new expressions for the moments $\mu_n(a,b,c,d;q)$ in the special case $b=-a$, $d=-c$, see Theorem 2.4 and Theorem 2.5. We prove a new positivity theorem in Corollary 4.4, and conjecture another one in Conjecture 1.

Our second goal is to combinatorially study the moments $\mu_n(a,b,c,d;q)$ as functions of a, b, c, and d. We use three combinatorial models for this purpose. The moments for any set of orthogonal polynomials may be given as weighted Motzkin paths [13]. In this case $\mu_n(a,b,c,d;q)$ is the generating function for Motzkin paths with weights which are rational functions of a,b,c,d and q. We use this setup and a generalization of an idea of D. Kim [11] to combinatorially prove Theorem 3.1 and Corollary 3.2 in Section 3. A special combinatorial model for the Askey-Wilson integral, which evaluated the normalization constant C, was given in [7]. We modify this model appropriately to give a combinatorial model for some non-normalized moments in Theorem 4.1. We also give new explicit rational expressions for $\mu_n(a,b,c,d;q)$ so that $(abcd)_n\mu_n(a,b,c,d;q)$ are clearly polynomials in a, b, c, d and q, see Theorem 4.2 and Theorem 4.3.

A third combinatorial model was given by Corteel and Williams [5]. They give a combinatorial interpretation for the polynomial $2^n(abcd)_n\mu_n(a,b,c,d;q)$ using a rational transformation over the complex numbers of the parameters a, b, c, and d to parameters α, β, γ, and δ. In these new parameters $2^n(abcd)_n\mu_n(a,b,c,d;q)$ is a polynomial with positive integer coefficients with a combinatorial meaning. (See Section 5). Using their ideas, we explicitly find coefficients of certain terms in $2^n(abcd)_n\mu_n(a,b,c,d;q)$ as Catalan numbers in Theorem 5.2.

2 Askey-Wilson moments

In this section we consider the moments $\mu_n(a,b,c,d;q)$ as functions of the parameters a, b, c, d and q. Our goal is to give new explicit formulas for these moments, using simple series and integral evaluations. First we note an elementary fact.

Proposition 2.1. *$2^n(abcd;q)_n\mu_n(a,b,c,d;q)$ is a polynomial in a,b,c,d,q with integer coefficients.*

Note that the expression (2) has some removable singularities as functions of a and q. The method of proof of (2), which appears in [4], is to use an appropriate q-Taylor expansion. Theorem 2.2 and Theorem 2.3 below follow from (2) by series manipulations which are not given here.

Using (2) we have the following theorem, which exhibits $2^n(abcd;q)_n\mu_n(a,b,c,d;q)$ as a symmetric polynomial in b, c, and d.

Theorem 2.2. *The Askey-Wilson moments are*

$$2^n\mu_n(a,b,c,d;q) = \sum_{m=0}^{n} \frac{(ab,ac,ad;q)_m}{(abcd;q)_m}(-q)^m \sum_{s=0}^{n+1}\left(\binom{n}{s}-\binom{n}{s-1}\right) \times \sum_{p=0}^{n-2s-m} a^{-n+2s+2p}\begin{bmatrix} m+p \\ m\end{bmatrix}_q \begin{bmatrix} n-2s-p \\ m\end{bmatrix}_q q^{(-n+2s+p)m+\binom{m}{2}}.$$

If k is not a nonnegative integer, we define $\binom{n}{k}=0$.

Theorem 2.3. *The Askey-Wilson moments for $d=0$ are*

$$2^n\mu_n(a,b,c,0;q) = \sum_{k=0}^{n}\left(\binom{n}{\frac{n-k}{2}}-\binom{n}{\frac{n-k}{2}-1}\right) \times \sum_{u+v+w+2t=k} a^u b^v c^w (-1)^t q^{\binom{t+1}{2}} \begin{bmatrix} u+v+t \\ v\end{bmatrix}_q \begin{bmatrix} v+w+t \\ w\end{bmatrix}_q \begin{bmatrix} w+u+t \\ u\end{bmatrix}_q,$$

where the second sum is over all integers $0\le u,v,w\le k$ and $-k\le t\le k/2$ satisfying $u+v+w+2t=k$.

A special case of the Askey-Wilson polynomials has a different expression for the moments. Consider $b=-a$ and $d=-c$, so that the Askey-Wilson measure is an even function. In this case $b_n=0$, so the odd moments are zero, and the $2n^{\text{th}}$ moment has a shorter alternative expression, again proven by q-Taylor series and integration.

Theorem 2.4. *The non-zero Askey-Wilson moments for $b=-a$ and $d=-c$ are*

$$4^n\mu_{2n}(a,-a,c,-c;q) = \sum_{m=0}^{n}\frac{(-a^2;q)_{2m}(a^2c^2;q^2)_m}{(qa^2c^2;q^2)_m}q^{2m}\sum_{j=0}^{m}\frac{a^{-4j}q^{-2j^2}(aq^j+a^{-1}q^{-j})^{2n}}{(q^2,a^4q^{2+4j};q^2)_{m-j}(q^2,a^{-4}q^{2-4j};q^2)_j}.$$

Zeng [14] found a formula equivalent to Theorem 2.4 when $c=0$. Theorem 2.4 has a version with differences of binomial coefficients, similar to Theorem 2.2.

Theorem 2.5. *The non-zero Askey-Wilson moments for $b=-a$ and $d=-c$ are*

$$4^n\mu_{2n}(a,-a,c,-c;q)=\sum_{m=0}^{n}\frac{(-a^2;q)_{2m}(a^2c^2;q^2)_m}{(qa^2c^2;q^2)_m}(-q^2)^m$$
$$\times\sum_{s=0}^{2n+2}\left(\binom{2n+1}{s}-\binom{2n+1}{s-1}\right)\sum_{p=0}^{n-m-s}a^{-2n+4p+2s}\begin{bmatrix}m+p\\ m\end{bmatrix}_{q^2}\begin{bmatrix}n-p-s\\ m\end{bmatrix}_{q^2}q^{-2m(n-p-s)+m(m-1)}.$$

There is a positivity conjecture for the moments $\mu_{2n}(a,-a,c,-c;q)$. These moments are not polynomials, but Proposition 2.1 implies that $4^n(a^2c^2;q)_{2n}\mu_{2n}(a,-a,c,-c;q)$ is a polynomial. Half of the apparent poles of $\mu_{2n}(a,-a,c,-c;q)$ do not occur.

Proposition 2.6. *The Askey-Wilson moments*

$$\tau_{2n}(a^2,c^2)=4^n(qa^2c^2;q^2)_n\mu_{2n}(a,-a,c,-c;q)/(1-q)^n$$

are polynomials in a^2, c^2 and q with integer coefficients. Moreover the sum of the coefficients in $\tau_{2n}(a^2,c^2)$ is $2^{2n}(2n-1)(2n-3)\cdots 1$.

Conjecture 1. The coefficients of $\tau_{2n}(a^2,c^2)$ are non-negative integers.

If $q=0$, one may show that $\tau_{2n}(a^2,c^2)$ is a non-negative polynomial by a combinatorial method. The sum of the coefficients is 2^{2n}. It is a generating function for certain non-crossing complete matchings.

Although simple, (2) does not clearly demonstrate the symmetry or polynomiality of $\mu_n(a,b,c,d;q)$ in all four parameters a, b, c and d. We next give such a formula, which generalizes Theorem 2.2.

Let A be an arbitrary parameter. Let

$${}_8W_7(m)={}_8W_7(A^2/q;A/a,A/b,A/c,A/d,q^{-m};q;abcdq^m).$$

From the definition of the ${}_8W_7$ one may show that $(aA,bA,cA,dA;q)_m\,{}_8W_7(m)$ is a symmetric polynomial in a,b,c and d.

Using Watson's transformation [6, (III.17)] of an ${}_8W_7$ to a ${}_4\phi_3$, the following apparent rational function of A and q is in fact a polynomial in each of the parameters: a, b, c, d, A, and q.

$$\frac{(aA,bA,cA,dA;q)_m}{(A^2;q)_m}{}_8W_7(m)=\sum_{j=0}^{m}\begin{bmatrix}m\\ j\end{bmatrix}_q(cd)^j(A/c,A/d;q)_j(ab;q)_j(Aaq^j,Abq^j,cd;q)_{m-j}.$$

The next result, proven using q-Taylor series and integration, gives a symmetric polynomial version for $2^n(abcd;q)_n\mu_n(a,b,c,d;q)$. Theorem 2.7 is independent of A.

Theorem 2.7.

$$2^n\mu_n(a,b,c,d;q)=\sum_{m=0}^{n}\frac{(aA,bA,cA,dA;q)_m}{(A^2,abcd;q)_m}(-q)^m{}_8W_7(m)\sum_{s=0}^{n+1}\left(\binom{n}{s}-\binom{n}{s-1}\right)$$
$$\times\sum_{p=0}^{n-2s-m}A^{-n+2s+2p}\begin{bmatrix}m+p\\ m\end{bmatrix}_q\begin{bmatrix}n-2s-p\\ m\end{bmatrix}_q q^{m(-n+2s+p)+\binom{m}{2}}.$$

If $A=a$, then Theorem 2.7 becomes Theorem 2.2. Theorem 2.7 has one defect: not all of the powers of q are positive due to the $q^{m(-n+2s+p)}$ term. The individual terms are Laurent polynomials in q.

If $A^2=q$, the p-sum in Theorem 2.7 is evaluable by the q-Vandermonde sum [6, II.6].

3 Weighted Motzkin paths

The first combinatorial model uses weighted Motzkin paths to find the moments. The weights are given by the complicated rational functions in the three-term recurrence relation. However if $c = d = 0$, these rational weights become simple polynomial weights, and the combinatorial model provided by Motzkin paths simplifies, as does the formula for the moments.

Theorem 3.1. *The Askey-Wilson moments for $c = d = 0$ are*

$$2^n \mu_n(a,b,0,0;q) = \sum_{k=0}^{n} \left(\binom{n}{\frac{n-k}{2}} - \binom{n}{\frac{n-k}{2}-1} \right) \sum_{u+v+2t=k} a^u b^v (-1)^t q^{\binom{t+1}{2}} \begin{bmatrix} u+v+t \\ u,v,t \end{bmatrix}_q, \tag{3}$$

where the second sum is over all nonnegative integers u, v, t satisfying $u + v + 2t = k$.

Theorem 3.1 is equivalent to a result of Josuat-Vergès [9, Theorem 6.1.1], and has an attractive special case.

Corollary 3.2. *We have*

$$2^n \mu_n(a,q/a,0,0;q) = \sum_{k=0}^{n} \left(\binom{n}{\frac{n-k}{2}} - \binom{n}{\frac{n-k}{2}-1} \right) (q/a)^k \sum_{i=0}^{k} a^{2i} q^{i(k-i-1)}.$$

In this section we sketch combinatorial proofs of Theorem 3.1 and Corollary 3.2.

The main idea is as follows. We first interpret the moment $2^n \mu_n(a,b,0,0;q)$ as a weighted sum of Motzkin paths. Then using Penaud's decomposition [12] we can decompose a weighted Motzkin path into a pair of paths: a Dyck prefix and another weighted Moztkin path. We map the new weighted Moztkin path to a new object: doubly striped skew shapes. These objects are a generalization of striped skew shapes introduced by D. Kim [11] in order to prove the moment formula for Al-Salam-Carlitz polynomials. We then find a sign-reversing involution on doubly striped skew shapes and show that the fixed points have a weighted sum equal to a q-trinomial coefficient. This completes the sketch of the proof of Theorem 3.1. For Corollary 3.2, we find a further cancellation on the doubly striped skew shapes which leaves only one fixed point for given size.

A *Motzkin path* is a lattice path in $\mathbb{N} \times \mathbb{N}$ from $(0,0)$ to $(n,0)$ consisting of up steps $(1,1)$, down steps $(1,-1)$, and horizontal steps $(1,0)$. We say that the *level* of a step is i if it is an up step or a down step between the lines $y = i-1$ and $y = i$, or it is a horizontal step on the line $y = i$. A *weighted Motzkin path* is a Motzkin path in which each step has a certain weight. The *weight* $\mathrm{wt}(p)$ of a weighted Motzkin path p is the product of the weights of all steps. Note that the level of an up step or a down step is at least 1 and the level of a horizontal step may be 0.

Let $\mathrm{Mot}_n(a,b)$ denote the set of weighted Motzkin paths of length n such that the weight of an up step of level i is either q^i or -1, the weight of a down step of level i is either abq^{i-1} or -1, and the weight of a horizontal step of level i is either aq^i or bq^i. Then by Viennot's theory [13] we have

$$2^n \mu_n(a,b,0,0;q) = \sum_{P \in \mathrm{Mot}_n(a,b)} \mathrm{wt}(P). \tag{4}$$

We define $\mathrm{Mot}^*_n(a,b)$ to be the set of weighted Motzkin paths in $\mathrm{Mot}_n(a,b)$ such that there is no peak of weight 1, that is, an up step of weight -1 immediately followed by a down step of weight -1.

By the same idea that is a variation of Penaud's decomposition as in [8, Proposition 5.1], we have

$$\sum_{P\in \mathrm{Mot}_n(a,b)} \mathrm{wt}(P) = \sum_{k=0}^{n} \left(\binom{n}{\frac{n-k}{2}} - \binom{n}{\frac{n-k}{2}-1} \right) \sum_{P\in \mathrm{Mot}_k^*(a,b)} \mathrm{wt}(P). \tag{5}$$

By (4) and (5), Theorem 3.1 and Corollary 3.2 can be restated as follows.

Theorem 3.3. *We have*

$$\sum_{P\in \mathrm{Mot}_k^*(a,b)} \mathrm{wt}(P) = \sum_{u+v+2t=k} a^u b^v (-1)^t q^{\binom{t+1}{2}} \begin{bmatrix} u+v+t \\ u,v,t \end{bmatrix}_q,$$

$$\sum_{P\in \mathrm{Mot}_k^*(a,q/a)} \mathrm{wt}(P) = (q/a)^k \sum_{i=0}^{k} a^{2i} q^{i(k-i-1)}.$$

We note that the second identity in Theorem 3.3 is equivalent to a result of Corteel et al. [3, Proposition 5], which is used to prove a formula [3, Theorem 1] for the moments of q-Laguerre polynomials. Their proof of [3, Theorem 1] is combinatorial except for the proof of [3, Proposition 5]. In this section we prove the above theorem combinatorially, thus providing the first combinatorial proof of their result.

In order to give combinatorial proofs of the two formulas in Theorem 3.3 we introduce doubly striped skew shapes. These objects are a generalization of striped skew shapes introduced by D. Kim [11].

A *doubly striped skew shape* of size $m \times n$ is a quadruple (λ, μ, W, B) of partitions $\mu \subset \lambda \subset (n^m)$ and a set W of white stripes and a set B of black stripes with $W \cap B = \emptyset$. Here, a *white stripe* is a diagonal set S of λ/μ such that λ/μ contains neither the cell to the left of the leftmost cells of S nor the cell below the rightmost cell of S, where a diagonal set means a set of cells in row $r+i$ and column $s+i$ for $i = 1, 2, \dots, p$ for some integers r, s, p. Similarly, a *black stripe* is a diagonal set S of λ/μ such that λ/μ contains neither the cell above the leftmost cell of S and the cell to the right of the rightmost cell S. We will call a cell in a white stripe (resp. black stripe) a *white dot* (resp. *black dot*).

Let $\mathrm{DSS}(m,n)$ denote the set of doubly striped skew shapes of size $m \times n$. We define the *weight* of $(\lambda, \mu, W, B) \in \mathrm{DSS}(m,n)$ to be

$$\mathrm{wt}_{a,b}(\lambda, \mu, W, B) = a^m b^n (-1)^{|W|+|B|} q^{|\lambda/\mu| - \|W\| - \|B\|} (q/ab)^{|W|}, \tag{6}$$

where $\|W\|$ and $\|B\|$ are the total numbers of white dots and black dots respectively.

We define a map $\rho : \mathrm{Mot}_k^*(a,b) \to \bigcup_{i=0}^{k} \mathrm{DSS}(i, k-i)$ as follows. Let $P \in \mathrm{Mot}_k^*(a,b)$. We will construct an upper path and a lower path which determine two partitions λ and μ respectively. The upper path and the lower path start at the origin. For each step of P, we add one step to the two lattice paths as follows. If the step is an up step of weight abq^{i-1}, we add a north step of weight a to the upper path and an east step of weight b to the lower path. If the step is an up step of weight -1, we add a north step of weight a to the upper path and an east step of weight b to the lower path, and we make a white stripe between these two steps, see Figure 1. Similarly we add one step to the upper and lower paths for the other types of steps in P as shown in Figure 1. Then we define $\rho(P)$ to be the resulting diagram. See Figure 2 for an example of ρ. It is easy to see that ρ is a weight-preserving bijection, which implies

$$\sum_{P\in \mathrm{Mot}_k^*(a,b)} \mathrm{wt}(P) = \sum_{i=0}^{k} \sum_{S\in \mathrm{DSS}(i,k-i)} \mathrm{wt}_{a,b}(S).$$

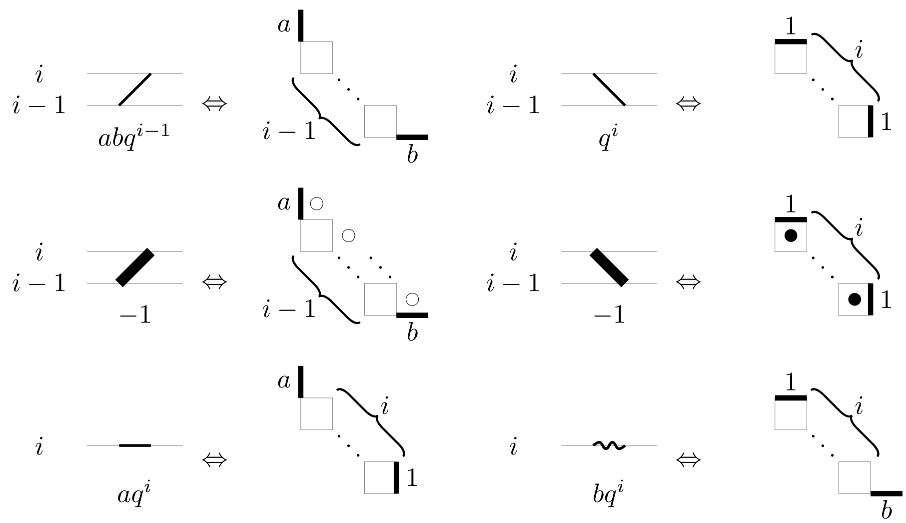

Fig. 1: Converting a weighted Motzkin path to a doubly striped skew shape.

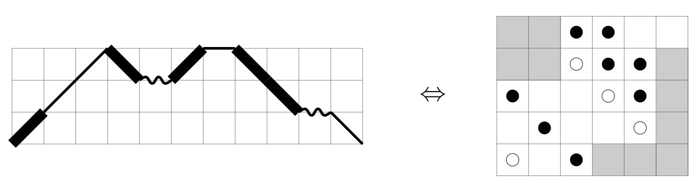

Fig. 2: An example of the bijection ρ. The steps of weight -1 are the thick steps.

Thus Theorem 3.3 is equivalent to the following proposition.

Proposition 3.4. *For $k \geq 0$, we have*

$$\sum_{i=0}^{k} \sum_{S \in \mathrm{DSS}(i,k-i)} \mathrm{wt}_{a,b}(S) = \sum_{u+v+2t=k} a^u b^v (-1)^t q^{\binom{t+1}{2}} \begin{bmatrix} u+v+t \\ u,v,t \end{bmatrix}_q, \tag{7}$$

$$\sum_{i=0}^{k} \sum_{S \in \mathrm{DSS}(i,k-i)} \mathrm{wt}_{a,q/a}(S) = (q/a)^k \sum_{i=0}^{k} a^{2i} q^{i(k-i-1)}. \tag{8}$$

Our proof of Proposition 3.4 is based on D. Kim's sign-reversing involution on striped skew shapes.

4 Matchings

In this section we provide the second combinatorial approach to compute a non-normalized n^{th} moment, using an idea in [7]. The orthogonality relation for Askey-Wilson polynomials is

$$\int_0^\pi P_n(\cos\theta, a, b, c, d; q) P_m(\cos\theta, a, b, c, d; q) w(\cos\theta, a, b, c, d; q) d\theta = 0, \qquad n \neq m.$$

Let

$$I_n := \frac{(q)_\infty}{2\pi} \int_0^\pi (\cos\theta)^n w(\cos\theta, a, b, c, d; q) d\theta.$$

Then $I_n = \mu_n(a, b, c, d; q) I_0$.

The integral I_n, which is a multiple of the n^{th} moment, is the q-exponential generating function for a set of complete matchings. Ismail, Stanton, and Viennot [7] evaluated the Askey-Wilson integral I_0 by interpreting the weight function $w(\cos\theta, a, b, c, d; q)$ as a generating function of four q-Hermite polynomials. By generalizing the method in [7], we obtain the following theorem.

Theorem 4.1. *We have*

$$I_n = \left(\frac{\sqrt{1-q}}{2}\right)^n \sum_{n_1,n_2,n_3,n_4 \geq 0} \frac{\widetilde{a}^{n_1} \widetilde{b}^{n_2} \widetilde{c}^{n_3} \widetilde{d}^{n_4}}{[n_1]_q! [n_2]_q! [n_3]_q! [n_4]_q!} \widetilde{f}_n(n_1, n_2, n_3, n_4; q), \tag{9}$$

where $\widetilde{a} = a/\sqrt{1-q}$, $\widetilde{b} = b/\sqrt{1-q}$, $\widetilde{c} = c/\sqrt{1-q}$, $\widetilde{d} = d/\sqrt{1-q}$, and

$$\widetilde{f}_n(n_1, \ldots, n_k; q) = \sum_{\sigma \in \mathcal{CM}(n; n_1, n_2, \ldots, n_k)} q^{\mathrm{cr}(\sigma)},$$

where $\mathcal{CM}(n; n_1, n_2, \ldots, n_k)$ is the set of complete matchings on $[n] \uplus [n_1] \uplus \cdots \uplus [n_k]$ such that homogeneous edges are contained in $[n]$, see below for the precise definition.

A *matching* is a set partition of $\{1, 2, \ldots, n\}$ in which every block has size 1 or 2. A block of size 2 is called an *edge* and a block of size 1 is called a *fixed point*. A *complete matching* is a matching without fixed points. For a matching π we define the *crossing number* $\mathrm{cr}(\pi)$ to be the number of pairs (e_1, e_2) of blocks $e_1, e_2 \in \pi$ such that $e_1 = \{i_1, j_1\}, e_2 = \{i_2, j_2\}$ with $i_1 < i_2 < j_1 < j_2$ or $e_1 = \{i_1, j_1\}, e_2 = \{j_2\}$ with $i_1 < i_2 < j_1$.

For fixed integers $n_0, n_1, n_2, \dots, n_k$, let $S_i = \{m_{i-1}+1, m_{i-1}+2, \dots, m_i\}$ for $i = 0, 1, 2, \dots, k$, where $m_{-1} = 0$ and $m_i = n_0 + n_1 + \cdots + n_i$ for $i = 0, 1, \dots, k$. We define $\mathcal{CM}(n_0; n_1, n_2, \dots, n_k)$ to be the set of complete matchings π on $\bigcup_{i=0}^k S_i$ such that if an edge of π is contained in S_i, then $i = 0$.

We can use Theorem 4.1 to find new explicit formulas for the moments, and also explain the mixed binomial and q-binomial coefficients. The reason is that the generating function for a crossing number of matchings (not necessarily complete) always has such a formula.

Let $\mathcal{M}(n,m)$ denote the set of matchings on $\{1, 2, \dots, n\}$ with m fixed points. Note that $\mathcal{M}(n,m) = \emptyset$ unless $n \equiv m \mod 2$. Josuat-Vergès [10, Proposition 5.1] showed the following (see also [2, Proposition 15]): if $n \equiv m \mod 2$, we have

$$\sum_{\pi \in \mathcal{M}(n,m)} q^{\operatorname{cr}(\pi)} = \frac{1}{(1-q)^{(n-m)/2}} \sum_{k \geq 0} \left(\binom{n}{\frac{n-k}{2}} - \binom{n}{\frac{n-k}{2}-1} \right) (-1)^{(k-m)/2} q^{\binom{(k-m)/2+1}{2}} \begin{bmatrix} \frac{k+m}{2} \\ \frac{k-m}{2} \end{bmatrix}_q . \tag{10}$$

Let

$$P(n,m) = \sum_{\pi \in \mathcal{M}(n,m)} q^{\operatorname{cr}(\pi)}, \qquad \overline{P}(n,m) = (1-q)^{(n-m)/2} P(n,m).$$

An explicit formula for $\widetilde{f}_0(n_1, \dots, n_k; q)$ in [7, Theorem 3.2] allows one give new formulas for the moments, which are explicit rational functions with the correct singularities.

Theorem 4.2. *We have*

$$2^n \mu_n(a,b,c,d;q) = \sum_{\alpha,\beta,\gamma,\delta \geq 0} a^\alpha b^\beta c^\gamma d^\delta \overline{P}(n, \alpha+\beta+\gamma+\delta) \begin{bmatrix} \alpha+\beta+\gamma+\delta \\ \alpha,\beta,\gamma,\delta \end{bmatrix}_q \frac{(ad)_{\beta+\gamma}(ac)_\beta(bd)_\gamma}{(abcd)_{\beta+\gamma}},$$

$$2^n \mu_n(a,b,c,0;q) = \sum_{\alpha,\beta,\gamma \geq 0} a^\alpha b^\beta c^\gamma \overline{P}(n, \alpha+\beta+\gamma) \begin{bmatrix} \alpha+\beta+\gamma \\ \alpha,\beta,\gamma \end{bmatrix}_q (ac)_\beta,$$

$$2^n \mu_n(a,b,0,0;q) = \sum_{\alpha,\beta \geq 0} a^\alpha b^\beta \overline{P}(n, \alpha+\beta) \begin{bmatrix} \alpha+\beta \\ \alpha \end{bmatrix}_q .$$

Theorem 4.3. *We have*

$$\begin{aligned} 2^n \mu_n(a,b,c,d;q) = \sum_{k=0}^{n} \left(\binom{n}{\frac{n-k}{2}} - \binom{n}{\frac{n-k}{2}-1} \right) &\sum_{\alpha+\beta+\gamma+\delta+2t=k} a^\alpha b^\beta c^\gamma d^\delta \frac{(ac)_\beta (bd)_\gamma}{(abcd)_{\beta+\gamma}} \\ &\times (-1)^t q^{\binom{t+1}{2}} \begin{bmatrix} \alpha+\beta+\gamma+t \\ \alpha \end{bmatrix}_q \begin{bmatrix} \beta+\gamma+\delta+t \\ \beta,\gamma,\delta+t \end{bmatrix}_q \begin{bmatrix} \delta+\alpha+t \\ \delta \end{bmatrix}_q , \end{aligned}$$

in the second sum, $\alpha, \beta, \gamma, \delta \geq 0$ and $-k \leq t \leq k/2$.

In Theorem 4.3, if $c = 0$, then $\gamma = 0$ in the sum. Thus we get Theorem 2.3.

When $ac = bd = q$ in Theorem 4.3, the formula can be simplified and we obtain a positivity theorem of the moments.

Corollary 4.4. *We have*

$$2^n \mu_n(a,b,q/a,q/b;q) = \sum_{k=0}^{n} \left(\binom{n}{\frac{n-k}{2}} - \binom{n}{\frac{n-k}{2}-1} \right) \frac{1}{[k+1]_q} \sum_{\substack{|A|+|B|\le k \\ A+B\equiv k \mod 2}} a^A b^B q^{\frac{k-A-B}{2}},$$

where the second sum is over all integers A and B such that $|A|+|B| \le k$ and $A+B \equiv k \mod 2$. Thus $[n+1]_q! 2^n \mu_n(a,b,q/a,q/b;q)$ is a Laurent polynomial in a and b whose coefficients are positive polynomials in q.

Our proof of Corollary 4.4 involves hypergeometric series summations. Since the formula is simple it will be very interesting to find a combinatorial proof of it.

Problem 1. Find a combinatorial proof of Corollary 4.4.

The Laurent polynomiality in Corollary 4.4 seems to be generalized further.

Conjecture 2. $2^n[n+i+j-1]_q!\mu_n(a,b,q^i/a,q^j/b;q)$ is a Laurent polynomial in a,b and polynomial in q with nonnegative coefficients.

5 Staircase tableaux

In this section we review the third combinatorial model, called staircase tableaux, for the moments of Askey-Wilson polynomials. The staircase tableaux were first introduced in [5] and further studied in [4]. Using the staircase tableaux we shall find the coefficient of the first few highest terms in $2^n(abcd;q)_n\mu_n(a,b,c,d;q)$.

A *staircase tableau of size* n is a filling of the Young diagram of the staircase partition $(n, n-1, \dots, 1)$ with $\alpha, \beta, \gamma, \delta$ such that every diagonal cell is nonempty, all cells above an α or γ in the same column are empty, and all cells to the left of a β or δ in the same row are empty. Here a *diagonal cell* is a cell in the ith row and $(n+1-i)$th column for some $i \in \{1,2,\dots,n\}$. We denote by $\mathcal{T}(n)$ the set of staircase tableaux of size n.

Each empty cell s of $T \in \mathcal{T}(n)$ is labeled uniquely as follows. Here, for brevity let $\mathrm{right}(s)$ be the first nonempty cell to the right of s in the same row, and $\mathrm{below}(s)$ the first nonempty cell below s in the same column. If $\mathrm{right}(s)$ has a β, then s is labeled with u. If $\mathrm{right}(s)$ has a δ, then s is labeled with q. If $\mathrm{right}(s)$ has an α or γ, and $\mathrm{below}(s)$ has an α or δ, then s is labeled with u. If $\mathrm{right}(s)$ has an α or γ, and $\mathrm{below}(s)$ has an β or γ, then s is labeled with q. See Figure 3 for an example of a staircase tableau and the labeling of its empty cells.

For $T \in \mathcal{T}(n)$, we define $\mathrm{b}(T)$ to be the number of α's and δ's on the diagonal cells, and $A(T)$, $B(T)$, $C(T)$, $D(T)$, $E(T)$ to be the number of α's, β's, γ's, δ's, empty cells labeled with q in T respectively. For example, if T is the staircase tableau in Figure 3, we have $\mathrm{b}(T)=3$, $A(T)=2$, $B(T)=3$, $C(T)=3$, $D(T)=3$, and $E(T)=11$.

Corteel et al. [4] showed that

$$\mu_n(a,b,c,d;q) = \frac{(1-q)^n}{2^n i^n \prod_{j=0}^{n-1}(\alpha\beta - \gamma\delta q^j)} Z_n(-1;\alpha,\beta,\gamma,\delta;q), \tag{11}$$

		β				γ
	γ			α	α	
				δ		
	δ		γ			
		β				
	δ					
β						

u	u	β	q	u	u	γ
q	γ	q	q	α	α	
q	q	q	q	δ		
q	δ	q	γ			
u	u	β				
q	δ					
β						

Fig. 3: A staircase tableau and the labeling of its empty cells.

where $\alpha = \frac{1-q}{(1+ai)(1+ci)}$, $\beta = \frac{1-q}{(1-bi)(1-di)}$, $\gamma = \frac{ac(1-q)}{(1+ai)(1+ci)}$, $\delta = \frac{bd(1-q)}{(1-bi)(1-di)}$, and

$$Z_n(y;\alpha,\beta,\gamma,\delta;q) = \sum_{T\in\mathcal{T}(n)} y^{\mathrm{b}(T)}\alpha^{A(T)}\beta^{B(T)}\gamma^{C(T)}\delta^{D(T)}q^{E(T)}.$$

Since $\alpha\beta - \gamma\delta q^j = \frac{(1-q)^2(1-abcdq^j)}{(1+ai)(1+ci)(1-bi)(1-di)}$, we can rewrite (11) as follows.

Proposition 5.1. *The Askey-Wilson moments satisfy*

$$\begin{aligned} 2^n(abcd;q)_n\mu_n(a,b,c,d;q) = i^{-n} &\sum_{T\in\mathcal{T}(n)} (-1)^{\mathrm{b}(T)}(1-q)^{A(T)+B(T)+C(T)+D(T)-n}q^{E(T)} \\ &\times (ac)^{C(T)}(bd)^{D(T)}\big((1+ai)(1+ci)\big)^{n-A(T)-C(T)}\big((1-bi)(1-di)\big)^{n-B(T)-D(T)}. \end{aligned}$$

The highest degree term appearing in Proposition 5.1 is $a^nb^nc^nd^nq^{\binom{n}{2}}$. By analyzing staircase tableaux we obtain the coefficients of the first few highest degree terms.

Theorem 5.2. *We have*

$$\begin{aligned} \left[a^nb^nc^nd^nq^{\binom{n}{2}}\right] 2^n(abcd;q)_n\mu_n(a,b,c,d;q) &= \mathrm{Cat}\left(\frac{n}{2}\right), \\ \left[a^{n-1}b^nc^nd^nq^{\binom{n}{2}}\right] 2^n(abcd;q)_n\mu_n(a,b,c,d;q) &= -\,\mathrm{Cat}\left(\frac{n+1}{2}\right), \\ \left[a^{n-1}b^{n-1}c^nd^nq^{\binom{n}{2}}\right] 2^n(abcd;q)_n\mu_n(a,b,c,d;q) &= \mathrm{Cat}\left(\frac{n+2}{2}\right) - \mathrm{Cat}\left(\frac{n}{2}\right), \end{aligned}$$

where $\mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}$ *if n is a nonnegative integer, and* $\mathrm{Cat}(n) = 0$ *otherwise.*

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On r-stacked triangulated manifolds

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Abstract. The notion of r-stackedness for simplicial polytopes was introduced by McMullen and Walkup in 1971 as a generalization of stacked polytopes. In this paper, we define the r-stackedness for triangulated homology manifolds and study their basic properties. In addition, we find a new necessary condition for face vectors of triangulated manifolds when all the vertex links are polytopal.

Résumé. Généralisant les polytopes simpliciaux empilés, McMullen et Walkup ont introduit en 1971 la notion de r-empilement pour les polytopes simpliciaux. Dans cet article, nous définissons la notion de r-empilement pour les variétés homologiques simpliciales et étudions ses propriétés élémentaires. En outre, nous donnons une nouvelle condition pour les f-vecteurs des variétés simpliciales lorsque tous les sommets ont un lien polytopal.

Keywords: stackedness, triangulation, manifold, f-vector, face ring

1 Introduction

A triangulated d-ball is said to be *r-stacked* if it has no interior faces of dimension $\leq d-r-1$, and the boundary of an r-stacked d-ball is called an r-stacked $(d-1)$-sphere. It is known that r-stacked d-balls and $(d-1)$-spheres with $r < \frac{d}{2}$ have many nice combinatorial properties, and they have been used to obtain several important results on polytopes and triangulated spheres. For example, they appeared in Barnette's lower bound theorem [Ba1, Ba2] and in the generalized lower bound conjecture given by McMullen and Walkup [MW]. They also appeared in the proof of the sufficiency of the famous g-theorem by Billera and Lee [BL] (see [KlL]) as well as in the construction of many non-polytopal triangulated spheres given by Kalai [Ka1]. The purpose of this paper is to extend this notion to triangulated manifolds, and establish their fundamental properties.

Throughout the paper, we fix a field $\mathbf{k}$. For a simplicial complex Δ and its face $F \in \Delta$, the *link of F in Δ* is the simplicial complex

$$\mathrm{lk}_\Delta(F) = \{G \in \Delta : F \cup G \in \Delta \text{ and } F \cap G = \emptyset\}.$$

A simplicial complex Δ of dimension d is said to be a $(\mathbf{k}$-$)$*homology d-sphere* if, for all faces $F \in \Delta$ (including the empty face $\emptyset$), one has $\beta_i(\mathrm{lk}_\Delta(F)) = 0$ for $i \neq d - \#F$ and $\beta_{d-\#F}(\mathrm{lk}_\Delta(F)) = 1$, where

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$\beta_i(\Delta) = \dim_{\mathbf{k}} \widetilde{H}_i(\Delta; \mathbf{k})$ is the ith *Betti number* of Δ over $\mathbf{k}$. A simplicial complex is said to be *pure* if all its facets have the same dimension. A *($\mathbf{k}$-)homology d-manifold without boundary* is a d-dimensional pure simplicial complex all whose vertex links are $\mathbf{k}$-homology spheres. A pure d-dimensional simplicial complex Δ is said to be a *($\mathbf{k}$-)homology d-manifold with boundary* if it satisfies

(i) for all $\emptyset \neq F \in \Delta$, $\beta_i(\mathrm{lk}_\Delta(F))$ vanish for $i \neq d - \#F$ and is equal to 0 or 1 for $i = d - \#F$.

(ii) the *boundary* $\partial\Delta = \{F \in \Delta : \beta_i(\mathrm{lk}_\Delta(F)) = 0\} \cup \{\emptyset\}$ of Δ is a $\mathbf{k}$-homology $(d-1)$-manifold without boundary.

Triangulations of topological manifolds are examples of homology manifolds. Also, condition (ii) can be omitted if we replace $\mathbf{k}$ by $\mathbb{Z}$ (see [Mi]).

We say that a homology d-manifold Δ with boundary is *r-stacked* if it has no interior faces (namely, faces which are not in $\partial\Delta$) of dimension $\leq d - r - 1$. Also, a homology manifold without boundary is said to be *r-stacked* if it is the boundary of an r-stacked homology manifold with boundary. We prove the following properties for r-stacked homology manifolds.

(a) *Enumerative criterion:* We give a simple criterion for the r-stackedness in terms of h-vectors and Betti numbers for homology manifolds with boundary (Theorem 3.1). Also, we give a similar result for $(r-1)$-stacked homology $(d-1)$-manifolds without boundary with $r \leq \frac{d}{2}$ when all the vertex links are polytopal (Corollary 5.5). In particular, these results prove that r-stackedness depends only on face numbers and Betti numbers for these manifolds.

(b) *Vanishing of Betti numbers and missing faces:* We show that if a homology manifold (with or without boundary) is r-stacked, then it has zero Betti numbers and no missing faces in certain dimensions (Corollary 3.2 and Theorem 4.4).

(c) *Uniqueness of stacked manifolds:* For an $(r-1)$-stacked $(d-1)$-manifold Δ without boundary, it is shown that if $r \leq \frac{d}{2}$ then there is a unique $(r-1)$-stacked homology manifold Σ such that $\partial\Sigma = \Delta$ (Theorem 4.2).

(d) *Local criterion:* For $r < \frac{d}{2}$, we show that a homology $(d-1)$-manifold without boundary is $(r-1)$-stacked if and only if all its vertex links are $(r-1)$-stacked (Theorem 4.6).

(e) *The $\tilde{g}$-vector — a new necessary condition for face vectors:* Motivated by a recent conjecture given by Bagchi and Datta, we define the $\tilde{g}$-vector of a simplicial complex Δ, and show that it is an M-vector if Δ is an $(r-1)$-stacked homology $(d-1)$-manifolds without boundary when $r \leq \frac{d}{2}$. Moreover, regardless of stackedness, we show that the same result holds for connected orientable rational homology manifolds all whose vertex links are polytopal (Theorem 5.4).

Most of the results listed above are natural extensions of known results for triangulated balls and spheres. However their proofs are not straightforward and we believe that these properties are useful in the study of face numbers of triangulated manifolds. Indeed, the results about the $\tilde{g}$-vector prove a refinment of [BD2, Conjecture 1.6] for orientable homology manifolds all whose vertex links are polytopal.

About (c) and (d), the same results were proved independently by Bagchi and Datta [BD3, Theorem 2.19] with essentially the same proof. Their results also prove vanishing of missing faces in (b).

This paper is organized as follows. In Section 2, we recall basic properties of h'- and h''-vectors which play an important role in the study of face numbers of homology manifolds. In Section 3, we study r-stacked homology manifolds with boundary. In Sections 4 and 5, we study r-stacked homology manifolds without boundary and consider the $\tilde{g}$-vector. Some of the proofs are omitted from this extended abstract, for space limit, and can be found in the full version of this paper, at math arXiv:1209.0868.

2 h'- and h''-vectors

In this section, we recall h'- and h''-vectors and their algebraic meanings. We first recall some basics on simplicial complexes. A simplicial complex Δ on the vertex set V is a collection of subsets of V satisfying that $F \in \Delta$ and $G \subset F$ imply $G \in \Delta$. Elements of Δ are called *faces* of Δ and subsets of V which are not faces of Δ are called *non-faces* of Δ. The maximal faces of Δ (with respect to inclusion) are called the *facets* of Δ and the minimal non-faces of Δ are called the *missing faces* of Δ. The dimension of a face (or a missing face) F is $\#F - 1$, where $\#X$ denotes the cardinality of a finite set X, and a face (or a missing face) of dimension k is called a *k-face* (or a *missing k-face*). Also, the dimension of a simplicial complex is the maximum dimension of its faces. For a simplicial complex Δ of dimension $d-1$, let $f_k = f_k(\Delta)$ be the number of k-faces of Δ for $k = -1, 0, \dots, d-1$, where $f_{-1} = 1$. The vector $f(\Delta) = (f_{-1}, f_0, \dots, f_{d-1})$ is called the f-vector of Δ. Also, the h-vector $h(\Delta) = (h_0(\Delta), h_1(\Delta), \dots, h_d(\Delta))$ of Δ is defined by the relation

$$\sum_{i=0}^{d} h_i(\Delta) t^i = \sum_{i=0}^{d} f_{i-1}(\Delta) t^i (1-t)^{d-i}.$$

Now we define h'- and h''-vectors. For a simplicial complex Δ of dimension $d-1$, its *h'-vector* $h'(\Delta) = (h'_0(\Delta), \dots, h'_d(\Delta))$ and its *h''-vector* $h''(\Delta) = (h''_0(\Delta), \dots, h''_d(\Delta))$ are defined by

$$h'_i(\Delta) = h_i(\Delta) - \binom{d}{i} \sum_{k=1}^{i-1} (-1)^{i-k} \beta_{k-1}(\Delta)$$

for $i = 0, 1, \dots, d$, and by

$$h''_i(\Delta) = h_i(\Delta) - \binom{d}{i} \sum_{k=1}^{i} (-1)^{i-k} \beta_{k-1}(\Delta) = h'_i(\Delta) - \binom{d}{i} \beta_{i-1}(\Delta)$$

for $i = 0, 1, \dots, d-1$ and $h''_d(\Delta) = h'_d(\Delta)$. Note that

$$h''_d(\Delta) = h'_d(\Delta) = \textstyle\sum_{\ell=0}^{d} (-1)^{\ell-d} f_{\ell-1} - \sum_{k=0}^{d-1} (-1)^{d-k} \beta_{k-1}(\Delta) = \beta_{d-1}(\Delta).$$

If one knows the Betti numbers of Δ, then knowing $h(\Delta)$ is equivalent to knowing $h'(\Delta)$ (or $h''(\Delta)$).

h'- and h''-vectors have nice algebraic meanings in terms of Stanley–Reisner rings. Let $S = \mathbf{k}[x_1, \dots, x_n]$ be a polynomial ring over a field $\mathbf{k}$ with $\deg x_i = 1$ for all i. For a simplicial complex Δ on $[n] = \{1, 2, \dots, n\}$, the *Stanley–Reisner ring* of Δ is the quotient ring

$$\mathbf{k}[\Delta] = S/I_\Delta$$

where $I_\Delta = (x_{i_1} \cdots x_{i_k} : \{i_1, \ldots, i_k\} \notin \Delta)$. If Δ has dimension $d-1$ and $\mathbf{k}$ is infinite, there is a sequence $\Theta = \theta_1, \ldots, \theta_d \in S_1$ of linear forms such that $\dim_{\mathbf{k}}(S/(I_\Delta + (\Theta))) < \infty$. This sequence Θ is called a *linear system of parameters* (l.s.o.p. for short) of $\mathbf{k}[\Delta]$. In the rest of this paper, we always assume that $\mathbf{k}$ is infinite.

A simplicial complex Δ of dimension $d-1$ is said to be *Cohen–Macaulay* (over $\mathbf{k}$) if, for all $F \in \Delta$, $\widetilde{H}_i(\mathrm{lk}_\Delta(F); \mathbf{k})$ vanishes for $i \neq d-1-\#F$. Note that any Cohen–Macaulay simplicial complex is pure. A pure simplicial complex is said to be *Buchsbaum* (over $\mathbf{k}$) if all its vertex links are Cohen–Macaulay. Homology manifolds are examples of Buchsbaum simplicial complexes.

Let $\mathfrak{m} = (x_1, \ldots, x_n)$ be the graded maximal ideal of S. For a graded S-module N, let $F_N(t) = \sum_{i \in \mathbb{Z}} (\dim_{\mathbf{k}} N_i) t^i$ be the *Hilbert Series* of N, where N_i is the graded component of N of degree i, and let $\mathrm{Soc}\,(N) = \{f \in N : \mathfrak{m} f = 0\}$ be the *socle* of N. The following results shown in [Sc, p. 137] and [NS1, Theorem 3.5] give algebraic meanings of h'- and h''-vectors.

Lemma 2.1 *Let Δ be a Buchsbaum simplicial complex of dimension $d-1$, $\Theta = \theta_1, \ldots, \theta_d$ an l.s.o.p. of $\mathbf{k}[\Delta]$ and $R = S/(I_\Delta + (\Theta))$. Then*

(i) (Schenzel) $F_R(t) = h'_0(\Delta) + h'_1(\Delta)t + \cdots + h'_d(\Delta)t^d$.

(ii) (Novik–Swartz) $\dim_{\mathbf{k}}(\mathrm{Soc}\,(R))_i \geq \binom{d}{i}\beta_{i-1}(\Delta)$ for all i. In particular, there is an ideal $N \subset \mathrm{Soc}(R)$ such that $F_{R/N}(t) = h''_0(\Delta) + h''_1(\Delta)t + \cdots + h''_d(\Delta)t^d$.

In the rest of this section, we study the relation between the vanishing of h''-numbers and missing faces. For a homogeneous ideal $I \subset S$, let $\mu_k(I)$ be the number of elements of degree k in a minimal generating set of I, namely, $\mu_k(I) = \dim_{\mathbf{k}}(I/\mathfrak{m}I)_k$. Since missing faces of Δ correspond to the minimal generators of I_Δ, $\mu_k(I_\Delta)$ is equal to the number of missing $(k-1)$-faces of Δ.

Lemma 2.2 *Let $I \subset S$ be a homogeneous ideal, $w \in S_1$ a linear form and $k \geq 2$ an integer. If the multiplication $\times w : (S/I)_{k-1} \to (S/I)_k$ is injective then $\mu_k(I) = \mu_k(I + (w))$.*

Proof: It is clear that $\mu_k(I) \geq \mu_k(I + (w))$ for $k \geq 1$ even without injectivity assumption. We show $\mu_k(I) \leq \mu_k(I + (w))$. Let $\sigma_1, \ldots, \sigma_t \in I$ be elements of degree k which are linearly independent in $I/\mathfrak{m}I$. What we must prove is that they are also linearly independent in $(I + (w))/\mathfrak{m}(I + (w))$.

Let $\tau = \lambda_1\sigma_1 + \cdots + \lambda_t\sigma_t \in \mathfrak{m}(I + (w))$, where $\lambda_1, \ldots, \lambda_t \in \mathbf{k}$. We claim $\tau \in \mathfrak{m}I$. Indeed, if $\tau \notin \mathfrak{m}I$ then there are $\rho' \in \mathfrak{m}I$ and $\rho'' \notin I$ such that $\tau = \rho' + w\rho''$, which implies ρ'' is in the kernel of the multiplication map $\times w : (S/I)_{k-1} \to (S/I)_k$, contradicting the assumption. □

Lemma 2.3 *For a homogeneous ideal $I \subset S$, if $(S/I)_j = 0$ for some $j \geq 0$ then $\mu_k(I) = 0$ for $k \geq j+1$.*

Proof: Since $(S/I)_j = 0$, we have $I_k = \mathfrak{m}_k$ for $k \geq j$. Thus $\mu_k(I) = \mu_k(\mathfrak{m}) = 0$ for $j \geq k+1$. □

The following statement appears in [Sc, Corollary 2.5 and Theorem 4.3].

Lemma 2.4 (Schenzel) *Let Δ be a Buchsbaum simplicial complex of dimension $d-1$, $R = \mathbf{k}[\Delta]$, $\Theta = \theta_1, \ldots, \theta_d$ an l.s.o.p. of $\mathbf{k}[\Delta]$, and let $\mathcal{K}(i)$ be the kernel of*

$$\times\theta_i : R/(\theta_1, \ldots, \theta_{i-1})R \to R/(\theta_1, \ldots, \theta_{i-1})R.$$

Then $\dim_{\mathbf{k}} \mathcal{K}(i)_j = \binom{i-1}{j}\beta_{j-1}(\Delta)$ for all i and j.

Proposition 2.5 *Let Δ be a Buchsbaum simplicial complex of dimension $d-1$. If $h''_r(\Delta)=0$ then*

(i) $\beta_k(\Delta)=0$ *for* $k\geq r$.

(ii) Δ *has no missing faces of dimension* $\geq r+1$.

Proof: Let Θ be an l.s.o.p. of $\mathbf{k}[\Delta]$. Since $h''_r(\Delta)=0$, by Lemma 2.1(ii) all elements in $S/(I_\Delta+(\Theta))$ of degree r are contained in the socle of $S/(I_\Delta+(\Theta))$. This fact implies

$$S/(I_\Delta+(\Theta))_k=0 \quad \text{for all } k\geq r+1. \tag{1}$$

Then since $\dim_{\mathbf{k}}(S/(I_\Delta+(\Theta)))_k \geq \binom{d}{k}\beta_{k-1}(\Delta)$ by Lemma 2.1(ii), we have $\beta_k(\Delta)=0$ for $k\geq r$, proving (i). Moreover, this fact and Lemmas 2.2 and 2.4 show $\mu_k(I_\Delta)=\mu_k(I_\Delta+(\Theta))$ for $k\geq r+1$. Since $S/(I_\Delta+(\Theta))_{r+1}=0$ by (1), the statement (ii) follows from Lemma 2.3. □

3 Stacked manifolds with boundary

In this section, we study r-stacked manifolds with boundary. Recall that a homology d-manifold with boundary is said to be *r-stacked* if it has no interior faces of dimension $\leq d-r-1$ and that a homology manifold without boundary is said to be *r-stacked* if it is the boundary of an r-stacked homology manifold with boundary. For a simplicial complex Δ of dimension $d-1$, let

$$g_i(\Delta)=h_i(\Delta)-h_{i-1}(\Delta)$$

for $i=0,1,\dots,d$.

Enumerative criterion

It is known that a homology ball Δ is $(r-1)$-stacked if and only if $h_r(\Delta)=0$. See [Mc, Proposition 2.4]. We first extend this property for stacked manifolds.

Let Δ be a homology $(d-1)$-manifold with boundary. Then the Dehn–Sommerville relations for homology manifolds with boundary [Gr, Corollary 2.2] say

$$g_i(\partial\Delta) \quad = h_i(\Delta)-h_{d-i}(\Delta)+\tbinom{d}{i}(-1)^{d-1-i}\widetilde{\chi}(\Delta), \tag{2}$$

where $\widetilde{\chi}(\Delta)=\sum_{k=-1}^{d-1}(-1)^k f_k(\Delta)$ is the reduced Euler characteristic. By substituting $h_{d-i}(\Delta)=h''_{d-i}(\Delta)+\binom{d}{i}\sum_{k=1}^{d-i}(-1)^{d-i-k}\beta_{k-1}(\Delta)$ and $\widetilde{\chi}(\Delta)=\sum_{k=0}^{d-1}(-1)^k\beta_k(\Delta)$ in (2), we obtain

$$g_i(\partial\Delta) \quad = h_i(\Delta)-h''_{d-i}(\Delta)+\tbinom{d}{i}\textstyle\sum_{k=d-i}^{d-1}(-1)^{d-1-i-k}\beta_k(\Delta). \tag{3}$$

Theorem 3.1 *Let $1\leq r\leq d$ and let Δ be a homology $(d-1)$-manifold with boundary. Then Δ is $(r-1)$-stacked if and only if $h''_r(\Delta)=0$.*

Proof: We first prove that Δ is $(r-1)$-stacked if and only if $g_i(\partial\Delta)=h_i(\Delta)$ for all $i\leq d-r$. Indeed, it is clear that Δ is $(r-1)$-stacked if and only if $f_i(\partial\Delta)=f_i(\Delta)$ for all $i\leq d-r-1$. Consider the equations

$$\sum_{i=0}^{d} f_{i-1}(\Delta)t^i=\sum_{i=0}^{d} h_i(\Delta)t^i(t+1)^{d-i} \tag{4}$$

and

$$\sum_{i=0}^{d-1} f_{i-1}(\partial\Delta)t^i = \sum_{i=0}^{d-1} h_i(\partial\Delta)t^i(t+1)^{d-i-1} \tag{5}$$

$$= \sum_{i=0}^{d-1} h_i(\partial\Delta)\{t^i(t+1)^{d-i} - t^{i+1}(t+1)^{d-(i+1)}\}.$$

By comparing the coefficients of the polynomials in (4) and (5), we conclude that $f_i(\partial\Delta) = f_i(\Delta)$ for $i \le d-r-1$ if and only if $h_i(\partial\Delta) - h_{i-1}(\partial\Delta) = h_i(\Delta)$ for all $i \le d-r$.

We first prove the 'if' part. Suppose $h''_r(\Delta) = 0$. Then we have $h''_k(\Delta) = 0$ for all $k \ge r$, as $h''(\Delta)$ is an M-sequence by Lemma 2.1(ii). Also, $\beta_r(\Delta) = \cdots = \beta_{d-1}(\Delta) = 0$ by Proposition 2.5. Then the Dehn–Sommerville relation (3) shows

$$g_i(\partial\Delta) = h_i(\Delta)$$

for all $i \le d-r$, as desired.

Next, we prove the 'only if' part. Suppose $g_i(\partial\Delta) = h_i(\Delta)$ for all $i \le d-r$. The Dehn–Sommerville relations (3) imply

$$h''_{d-i}(\Delta) = -\binom{d}{i}\beta_{d-i}(\Delta) + \binom{d}{i}\sum_{k=d-i+1}^{d-1}(-1)^{d-1-i-k}\beta_k(\Delta) \tag{6}$$

for all $i \le d-r$. We show by induction on i that $\beta_{d-i}(\Delta) = 0$ and $h''_{d-i}(\Delta) = 0$ for $i \le d-r$: The claim is clear for $i = 0$ by (6). For $i > 0$, by induction the second summand on the right-hand side of (6) vanish. Thus $h''_{d-i}(\Delta) = -\binom{d}{i}\beta_{d-i}(\Delta)$. Since h''-vectors and Betti numbers are non-negative we have $h''_{d-i}(\Delta) = \beta_{d-i}(\Delta) = 0$. □

Vanishing of missing faces

If Δ is an $(r-1)$-stacked triangulated ball then Δ is Cohen–Macaulay and $h_r(\Delta) = 0$. These facts and Lemmas 2.2 and 2.3 say that Δ has no missing faces of dimension $\ge r$ (another proof of this fact was given in [BD3, Lemma 2.10]). Proposition 2.5 and Theorem 3.1 prove an analogue of this fact for manifolds.

Corollary 3.2 *Let Δ be an $(r-1)$-stacked homology manifold with boundary. Then*

(i) $\beta_k(\Delta) = 0$ *for* $k \ge r$.

(ii) Δ *has no missing k-faces of dimension* $\ge r+1$.

Finally, we give a few known examples of stacked manifolds.

Example 3.3 (Kühnel–Lassmann construction [Kü, KüL]) *Let $K_{d,n}$ be the simplicial complex on $[n]$ generated by the facets*

$$\{\{i, i+1, \ldots, i+d-1\} : i = 1, 2, \ldots, n\},$$

where $i+k$ means $i+k-n$ if $i+k > n$. If $n \ge 2d-1$ then $K_{d,n}$ is a homology manifold whose boundary triangulates either $S^1 \times S^{d-3}$ or a non-orientable S^{d-3}-bundle over S^1 depending on the parity of d [KüL]. Since the interior faces of $K_{d,n}$ are those containing one of $\{i, i+1, \ldots, i+d-2\}$ for $i = 1, 2, \ldots, n$, the simplicial complex $K_{d,n}$ is 1-stacked and has the h''-vector $(1, n-d, 0, \ldots, 0)$.

Example 3.4 (Klee–Novik construction [KN]) *Let $X = \{x_1, \dots, x_d\}$ and $Y = \{y_1, \dots, y_d\}$ be disjoint sets. For integers $0 \le i \le d-2$, let $B_{d,i}$ be the simplicial complex on the vertex set $X \cup Y$ generated by the facets*

$$\{\{z_1, \dots, z_d\} : z_i \in \{x_i, y_i\} \text{ and } \#\{k : \{z_k, z_{k+1}\} \not\subset X \text{ and } \{z_k, z_{k+1}\} \not\subset Y\} \le i\}.$$

The simplicial complex $B_{d,i}$ is a combinatorial manifold whose boundary triangulates $S^i \times S^{d-i-2}$ and its h''-vector is given by $h''_k(B_{d,i}) = \binom{d}{k}$ for $k \le i$ and $h''_{i+1}(B_{d,i}) = 0$ [KN, Proposition 5.1]. In particular, these triangulated manifolds are i-stacked by Theorem 3.1.

Remark 3.5 *If Δ is an $(r-1)$-stacked triangulated ball then Δ has no missing r-faces. However, an $(r-1)$-stacked homology manifold with boundary could have missing r-faces. Indeed, the simplicial complex $K_{4,7}$ in Example 3.3 is 1-stacked but has a missing face $\{1, 4, 7\}$.*

4 Stacked manifolds without boundary

In Sections 4 and 5, we study $(r-1)$-stacked $(d-1)$-manifolds without boundary with $r \le \frac{d}{2}$. In this section, we study these manifolds from combinatorial viewpoints.

Uniqueness of stacked manifolds

A homology d-manifold Δ with boundary is said to be a *($\mathbf{k}$-)homology d-ball* if $\widetilde{H}_k(\Delta; \mathbf{k}) = 0$ for all k and $\partial\Delta$ is a ($\mathbf{k}$-)homology $(d-1)$-sphere. For a simplicial complex Δ on $[n]$, let

$$\Delta(r) = \{F \subset [n] : \mathrm{skel}_r(F) \subset \Delta\},$$

where $\mathrm{skel}_r(F) = \{G \subset F : \#G \le r+1\}$ is the *r-skeleton* of F. This simplicial complex can be defined algebraically. For a homogeneous ideal $I \subset S$, let $I_{\le k}$ be the ideal generated by all elements in I of degree $\le k$. Then it is easy to see that $(I_\Delta)_{\le r+1} = I_{\Delta(r)}$.

For an $(r-1)$-stacked homology $(d-1)$-sphere Δ, it was shown by McMullen [Mc, Theorem 3.3] (for polytopes) and by Bagchi and Datta [BD1, Proposition 2.10] (for triangulated spheres) that an $(r-1)$-stacked homology d-ball Σ satisfying $\partial\Sigma = \Delta$ is unique. Moreover, the following result was shown in [BD1, Corollary 3.6] (for polytopes) and in [MN, Lemma 2.1 and Theorem 2.3] (for homology spheres) by a different approach.

Lemma 4.1 *Let $1 \le r \le \frac{d+1}{2}$ and Δ an $(r-1)$-stacked homology $(d-1)$-sphere. If Σ is an $(r-1)$-stacked homology d-ball with $\partial\Sigma = \Delta$ then $\Sigma = \Delta(r-1)$.*

Proof: Observe that Σ has no missing faces of dimension $\ge r$ (see the discussion before Corollary 3.2). Then we have $I_\Sigma = (I_\Sigma)_{\le r}$. Since Σ and Δ have the same $(d-r)$-skeleton and $r-1 \le d-r$, we have $(I_\Sigma)_{\le r} = (I_\Delta)_{\le r}$. Hence

$$I_\Sigma = (I_\Sigma)_{\le r} = (I_\Delta)_{\le r} = I_{\Delta(r-1)},$$

which implies $\Sigma = \Delta(r-1)$. □

The following is an extension of Lemma 4.1.

Theorem 4.2 *Let $1 \le r \le \frac{d}{2}$ and Δ an $(r-1)$-stacked homology $(d-1)$-manifold without boundary. If Σ is an $(r-1)$-stacked homology d-manifold with $\partial\Sigma = \Delta$ then $\Sigma = \Delta(r)$.*

Proof: Since Σ is $(r-1)$-stacked, by Corollary 3.2(ii), Σ has no missing faces of dimension $\geq r+1$, namely, $I_\Sigma = (I_\Sigma)_{\leq r+1}$. Then the statement follows in the same way as in the proof of Lemma 4.1. □

Remark 4.3 *We cannot replace $\Delta(r)$ by $\Delta(r-1)$ in Theorem 4.2 by the same reason as in Remark 3.5. Similarly, the statement fails when $r = \frac{d+1}{2}$.*

Vanishing of missing faces

It was shown by Kalai [Ka2, Proposition 3.6] and Nagel [Na, Corollary 4.6] that if Δ is an $(r-1)$-stacked homology $(d-1)$-sphere and $r \leq \frac{d}{2}$ then Δ has no missing k-faces for $r \leq k \leq d-r$ (they write statements only for polytopes but this fact for homology spheres follows from Nagel's proof since an $(r-1)$-stacked homology $(d-1)$-sphere with $r \leq \frac{d}{2}$ has the weak Lefschetz property and satisfies $h_{r-1} = h_r$). This fact can be generalized as follows.

Theorem 4.4 *Let $1 \leq r < \frac{d}{2}$ and let Δ be an $(r-1)$-stacked homology $(d-1)$-manifold without boundary. Then*

(i) $\beta_k(\Delta) = 0$ *for* $r \leq k \leq d-1-r$.

(ii) Δ *has no missing k-faces with* $r+1 \leq k \leq d-r$.

Proof: Let Σ be an $(r-1)$-stacked homology d-manifold with $\partial\Sigma = \Delta$. Since Σ and Δ have the same $(d-r)$-skeleton, we have $\beta_i(\Delta) = \beta_i(\Sigma)$ for $i < d-r$ and $\mu_j(I_\Delta) = \mu_j(I_\Sigma)$ for $j \leq d-r+1$. Then the statement follows from Corollary 3.2. □

Remark 4.5 *Lemma 4.1 and the above proof give another proof for the fact that if $r \leq \frac{d}{2}$ and if Δ is an $(r-1)$-stacked homology $(d-1)$-sphere then Δ has no missing k-faces for $r \leq k \leq d-r$.*

Local criterion

Next, we discuss a local criterion of stackedness. We say that a homology d-manifold without boundary is *locally r-stacked* if all its vertex links are r-stacked. It is clear from the definition that if a homology manifold Δ is r-stacked then it is locally r-stacked. It was shown by Kalai [Ka2, Proposition 3.5] that if $r < \frac{d}{2}$ then the converse holds for the boundary of a simplicial d-polytope. This property can be extended as follows:

Theorem 4.6 *Let $1 \leq r < \frac{d}{2}$. Then a homology $(d-1)$-manifold without boundary is $(r-1)$-stacked if and only if it is locally $(r-1)$-stacked.*

Proof: The 'only if' part is obvious. The proof of the 'if' part is similar to that of [Mc, Theorem 5.3], however, for space limit, it is omitted. □

Remark 4.7 *Theorem 4.6 fails for $r = \frac{d}{2}$. Indeed, the join Δ of boundaries of two r-simplices is a $(2r-1)$-sphere which is not $(r-1)$-stacked but it is locally $(r-1)$-stacked. Indeed, Δ is not $(r-1)$-stacked since $\Delta(r-1)$ is the power set of $[n]$. Also, Δ is locally $(r-1)$-stacked since, for every vertex v of Δ, $\mathrm{lk}_\Delta(v)$ is the boundary of the join of an $(r-1)$-simplex and the boundary of an r-simplex.*

Remark 4.8 *Theorems 4.2 (for $r < \frac{d}{2}$), 4.4(ii) and 4.6 were also proved independently by Bagchi and Datta [BD3, Theorem 2.19] with essentially the same method.*

5 New necessary condition for face numbers of manifolds

McMullen and Walkup [MW] conjectured that, for the boundary complex Δ of a simplicial d-polytope, one has $h_{r-1}(\Delta) \leq h_r(\Delta)$ for $r \leq \frac{d}{2}$ and if equality holds for some r then Δ is $(r-1)$-stacked. This conjecture is called the generalized lower bound conjecture (GLBC for short). The first part of the GLBC was solved by Stanley [St1] in his proof of the necessity of the g-theorem and the second part of the GLBC was recently proved in [MN]. Recall that a connected homology d-manifold Δ without boundary is said to be *orientable* if $\beta_d(\Delta) = 1$. Motivated by the GLBC, Bagchi and Datta [BD2, Conjecture 1.6] suggested the following conjecture.

Conjecture 5.1 (GLBC for triangulated manifolds) *Let Δ be a connected triangulated $(d-1)$-manifold without boundary. Then*

(i) $h_r(\Delta) \geq h_{r-1}(\Delta) + \binom{d+1}{r} \sum_{j=1}^{r} (-1)^{r-j} \beta_{j-1}(\Delta)$ *for* $r = 1, 2, \ldots, \lfloor \frac{d}{2} \rfloor$.

(ii) if an equality holds for some $r < \frac{d}{2}$ in (i) then Δ is locally $(r-1)$-stacked.

Concerning part (i) of the conjecture, a similar conjecture was given by Swartz [Sw]. Moreover, it was proved by Novik and Swartz that (i) holds for all homology manifolds all whose vertex links satisfy certain algebraic property called the weak Lefschetz property. See [NS3, p. 270, Inequality (9)]. Also, the conjecture is known to be true for orientable manifolds when $r = 2$ [NS1, Theorem 5.2].

Conjecture 5.1 suggests us to study the following invariant of simplicial complexes, which we call the *$\tilde{g}$-vector*. For a simplicial complex Δ of dimension $d-1$, let

$$\tilde{g}_r(\Delta) = h_r(\Delta) - h_{r-1}(\Delta) - \binom{d+1}{r} \sum_{j=1}^{r} (-1)^{r-j} \beta_{j-1}(\Delta)$$

for $r = 0, 1, 2, \ldots, \lfloor \frac{d}{2} \rfloor$, where $\tilde{g}_0(\Delta) = 1$, and let $\tilde{g}(\Delta) = (\tilde{g}_0(\Delta), \tilde{g}_1(\Delta), \ldots, \tilde{g}_{\lfloor \frac{d}{2} \rfloor}(\Delta))$. Then Conjecture 5.1(i) asks if $\tilde{g}_k(\Delta) \geq 0$ for all k when Δ is a connected triangulated manifold without boundary. For an $(r-1)$-stacked homology $(d-1)$-manifold with $r \leq \frac{d}{2}$, its $\tilde{g}$-vector has the following simple but interesting form.

Proposition 5.2 *Let $1 \leq r \leq \frac{d}{2}$ and let Δ be an $(r-1)$-stacked homology d-manifold with boundary. Then $\tilde{g}_i(\partial\Delta) = h''_i(\Delta)$ for $i \leq \frac{d}{2}$.*

Proof: Since Δ and $\partial\Delta$ have the same $\lfloor \frac{d}{2} \rfloor$-skeleton, $\beta_{k-1}(\Delta) = \beta_{k-1}(\partial\Delta)$ for $k \leq \frac{d}{2}$, and, as shown in the proof of Theorem 3.1,

$$g_i(\partial\Delta) = h_i(\Delta)$$

for $i \leq \frac{d}{2}$. Subtracting $\binom{d+1}{i} \sum_{j=1}^{i} (-1)^{i-j} \beta_{j-1}(\Delta)$ from the above equation, we obtain the desired equation. □

Recall that a vector $h = (h_0, h_1, \ldots, h_t) \in \mathbb{Z}^{t+1}$ is said to be an *M-vector* if there is a standard graded $\mathbf{k}$-algebra A such that $h_k = \dim_{\mathbf{k}} A_k$ for $k = 0, 1, \ldots, t$. Lemma 2.1(ii) shows that, in Proposition 5.2, $\tilde{g}(\partial\Delta)$ is not only a non-negative vector but also an M-vector. It is natural to ask if $\tilde{g}(\partial\Delta)$ is an M-vector for *any* homology manifold without boundary. In this section, we prove that this property as well as Conjecture 5.1 hold for orientable homology manifolds all whose links satisfy a certain algebraic condition described below.

We say that a homology $(d-1)$-sphere Δ on $[n]$ has the *weak Lefschetz property* (WLP for short) if there is an l.s.o.p. Θ of $\mathbf{k}[\Delta] = S/I_\Delta$ and a linear form $w \in S_1$ such that the multiplication

$$\times\, w : (S/(I_\Delta + (\Theta)))_{i-1} \to (S/(I_\Delta + (\Theta)))_i \tag{7}$$

is injective for $i \leq \frac{d+1}{2}$ and is surjective for $i \geq \frac{d+1}{2}$. Note that it is known that the boundary complex of a simplicial polytope has the WLP over the rationals.

The following result is due to Swartz [Sw, Theorem 4.26]

Lemma 5.3 (Swartz) *Let Δ be a connected orientable homology $(d-1)$-manifold without boundary on the vertex set $[n]$. Suppose that all the vertex links of Δ have the WLP. Then there is an l.s.o.p. Θ of $\mathbf{k}[\Delta]$ and a linear form w such that the multiplication map*

$$\times w : (S/(I_\Delta + (\Theta)))_{i-1} \to (S/(I_\Delta + (\Theta)))_i$$

is surjective for all $i \geq \frac{d}{2} + 1$.

The main result of this section is the following.

Theorem 5.4 *With the same assumptions and notation as in Lemma 5.3, let $R = S/(I_\Delta + (\Theta))$ and $R' = R/wR$. Then*

(i) there is an ideal $J \subset R'$ such that $\dim_{\mathbf{k}}(R'/J)_i = \tilde{g}_i(\Delta)$ for $i \leq \frac{d}{2}$. In particular, $\tilde{g}(\Delta)$ is an M-vector.

(ii) if $\tilde{g}_r(\Delta) = 0$ for some $r < \frac{d}{2}$ then Δ is locally $(r-1)$-stacked.

Theorem 5.4(i) extends the result of Novik and Swartz [NS3] who proved the non-negativity of $\tilde{g}$-vectors for homology manifolds all whose vertex links have the WLP, and Theorem 5.4(ii) proves that Conjecture 5.1(ii) holds for these manifolds. In particular, Conjecture 5.1 holds for any rational orientable homology manifold all whose vertex links are polytopal, namely, are the boundary complexes of simplicial polytopes. It was conjectured that any homology sphere has the WLP. Thus, if this conjecture is true then Conjecture 5.1 holds for all orientable homology manifolds.

The proof of Theorem 5.4, for space limit, is omitted.

The local criterion for stackedness and Theorem 5.4 imply the following criterion for stackedness.

Corollary 5.5 *Let $r < \frac{d}{2}$ and let Δ be a connected orientable homology $(d-1)$-manifold without boundary. If all the vertex links of Δ have the WLP then Δ is $(r-1)$-stacked if and only if $\tilde{g}_r(\Delta) = 0$.*

Proof: The 'if' part follows from Theorems 4.6 and 5.4. The 'only if' part follows from Theorem 3.1 and Proposition 5.2. □

We end this paper by a few questions.

Conjecture 5.6 *With the same assumptions and notation as in Theorem 5.4,* $\dim_{\mathbf{k}}(\mathrm{Soc} R')_r \geq \binom{d+1}{r}\beta_{r-1}(\Delta)$ *for $r \leq \frac{d}{2}$.*

If the conjecture is true, it will give a necessary condition for h-vectors of triangulated manifolds stronger than Theorem 5.4(i). Indeed, Conjecture 5.6 implies Theorem 5.4(i) since [NS2, Theorem 3.2] implies

$$\dim_{\mathbf{k}} R'_r = \dim_{\mathbf{k}} R_r - \dim_{\mathbf{k}} (R/\mathrm{Soc}(R))_{r-1} = h'_r - h''_{r-1} = \tilde{g}_r + \binom{d+1}{r}\beta_{r-1}$$

for $i \leq \frac{d}{2}$. For $(r-1)$-stacked homology $(d-1)$-manifolds without boundary with $r \leq \frac{d}{2}$, the conjecture follows from Lemma 2.1(ii) by taking (Θ, w) for Δ to be a general l.s.o.p. of $\mathbf{k}[\Sigma]$, where Σ is the $(r-1)$-stacked homology manifold with $\partial\Sigma = \Delta$. The conjecture also holds for triangulations of the product of spheres (under the WLP assumption) since the ideal J in Theorem 5.4 is concentrated in a single degree in this case.

Question 5.7 *Is it true that if Δ is a homology $(2k-1)$-manifold without boundary such that $\tilde{g}_k(\Delta) = 0$ then Δ is $(k-1)$-stacked?*

A similar question was raised by Novik–Swartz [NS1, Problem 5.3] when $k = 2$. However, we do not have an answer even for this case.

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On Orbits of Order Ideals of Minuscule Posets

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Abstract. An action on order ideals of posets considered by Fon-Der-Flaass is analyzed in the case of posets arising from minuscule representations of complex simple Lie algebras. For these minuscule posets, it is shown that the Fon-Der-Flaass action exhibits the cyclic sieving phenomenon, as defined by Reiner, Stanton, and White. A uniform proof is given by investigation of a bijection due to Stembridge between order ideals of minuscule posets and fully commutative Weyl group elements. This bijection is proven to be equivariant with respect to a conjugate of the Fon-Der-Flaass action and an arbitrary Coxeter element.

If P is a minuscule poset, it is shown that the Fon-Der-Flaass action on order ideals of the Cartesian product $P \times [2]$ also exhibits the cyclic sieving phenomenon, only the proof is by appeal to the classification of minuscule posets and is not uniform.

Résumé. Une action sur des idéaux d'ordre d'ensembles partiellement ordonnés, qui ont été considérés par Fon-Der-Flaass, est analysée dans le cas des ensembles ordonnés qui proviennent des représentations minuscules d'algèbres de Lie simples complexes. À propos de ces ensembles ordonnés minuscules, il est démontré que l'action Fon-Der-Flaass offre le phénomène du crible cyclique, tel que défini par Reiner, Stanton et White. Une preuve uniforme est donnée par une étude d'une bijection due à Stembridge entre les idéaux d'ordre d'ensembles ordonnés minuscules et les éléments complètement commutatifs du groupe de Weyl. Il est démontré que cette bijection est équivariante en ce qui concerne un conjugué de l'action Fon-Der-Flaass et un élément de Coxeter arbitraire.

Si P est un ensemble ordonné minuscule, il est démontré que l'action Fon-Der-Flaass sur des idéaux d'ordre du produit cartésien $P \times [2]$ manifeste le phénomène du crible cyclique aussi, mais la preuve de se fait est par appel à la classification des ensembles ordonnés minuscules et n'est pas uniforme.

Keywords: order ideals, antichains, minuscule posets, minuscule representations, fully commutative elements, cyclic sieving phenomenon

1 Introduction

The Fon-Der-Flaass action on order ideals of a poset has been the subject of extensive study since it was introduced in its original form on hypergraphs in Duchet (1974). In this extended abstract of Rush and Shi (2012), we identify a disparate collection of posets characterized by properties from representation theory – the minuscule posets – that exhibits consistent behavior under the Fon-Der-Flaass action. We illustrate the commonality via the cyclic sieving phenomenon of Reiner et al. (2004), which provides a unifying framework for organizing combinatorial data on orbits arising from cyclic actions.

If P is a poset, and $J(P)$ is the set of order ideals of P, partially ordered by inclusion, the Fon-Der-Flaass action Ψ maps an order ideal $I \in J(P)$ to the order ideal $\Psi(I)$ whose maximal elements are the minimal elements of $P \setminus I$. Since Ψ is invertible, it generates a cyclic group $\langle\Psi\rangle$ acting on $J(P)$, but the orbit structure is not immediately apparent.

In Reiner et al. (2004), Reiner, Stanton, and White observed many situations in which the orbit structure of the action of a cyclic group $\langle c\rangle$ on a finite set X may be predicted by a polynomial $X(q) \in \mathbb{Z}[q]$.

Definition. The triple $(X, X(q), \langle c\rangle)$ exhibits the *cyclic sieving phenomenon* if, for any integer d, the number of elements x in X fixed by c^d is obtained by evaluating $X(q)$ at $q = \zeta^d$, where n is the order of c on X and ζ is any primitive n^{th} root of unity.

In the case when $X = J(P)$ and c is the Fon-Der-Flaass action, the natural generating function to consider is the rank-generating function for $J(P)$, which we denote by $J(P; q)$. Here the rank of an order ideal $I \in J(P)$ is given by the cardinality $|I|$ (so that $J(P;q) := \sum_{I\in J(P)} q^{|I|}$).

The minuscule posets are a class of posets arising in the representation theory of Lie algebras that enjoy some astonishing combinatorial properties. We give some background.

Let $\mathfrak{g}$ be a complex simple Lie algebra with Weyl group W and weight lattice Λ. There is a natural partial order on Λ called the *root order* in which one weight μ is considered to be smaller than another weight ω if the difference $\omega - \mu$ may be expressed as a positive linear combination of simple roots. If $\lambda \in \Lambda$ is dominant and the only weights occuring in the irreducible highest weight representation V^λ are the weights in the W-orbit $W\lambda$, then λ is called *minuscule*, and the restriction of the root order to the set of weights $W\lambda$ (which is called the *weight poset*) has two alternate descriptions:

- Let W_J be the maximal parabolic subgroup of W stabilizing λ, and let W^J be the set of minimum-length coset representatives for the parabolic quotient W/W_J. Then there is a natural bijection

$$\begin{array}{rcl} W^J & \longrightarrow & W\lambda \\ w & \longmapsto & w_0 w\lambda \end{array}$$

  (where w_0 denotes the longest element of W), and this map is an isomorphism of posets between the *strong Bruhat order* on W restricted to W^J and the root order on $W\lambda$.

- Let P be the poset of join-irreducible elements of the root order on $W\lambda$. Then P is called the *minuscule poset for* λ, P is ranked, and there is an isomorphism of posets between the weight poset and $J(P)$.

If P is minuscule, Proctor showed (Proctor (1984), Theorem 6) that P enjoys what Stanley calls the *Gaussian* property (cf. Stanley (2012), Exercise 25): There exists a function $f : P \to \mathbb{Z}$ such that, for all positive integers m,

$$J(P \times [m]; q) = \prod_{p\in P} \frac{1 - q^{m+f(p)+1}}{1 - q^{f(p)+1}}.$$

This may be verified case-by-case, but it follows uniformly from the *standard monomial theory* of Lakshmibai, Musili, and Seshadri, as is shown in Proctor (1984). Furthermore, all Gaussian posets are ranked, and if P is Gaussian, we may take f to be the rank function of P.

Thus, for all positive integers m, we are led to consider the triple $(X, X(q), \langle\Psi\rangle)$, where $X = J(P \times [m])$, $X(q) = J(P \times [m]; q)$, and P is any minuscule poset. We are at last ready to state the first two of our main results, answering a question of Reiner.

Theorem 1.1. *Let P be a minuscule poset. If $m = 1$, $(X, X(q), \langle\Psi\rangle)$ exhibits the cyclic sieving phenomenon.*

Theorem 1.2. *Let P be a minuscule poset. If $m = 2$, $(X, X(q), \langle\Psi\rangle)$ exhibits the cyclic sieving phenomenon.*

It turns out that the claim analogous to Theorems 1.1 and 1.2 is false for $m = 3$; computations performed by Kevin Dilks[(i)] reveal that when $m = 3$ and P is the minuscule poset $[3] \times [3]$, the triple $(X, X(q), \langle\Psi\rangle)$ does not exhibit the cyclic sieving phenomenon. However, if P belongs to the third infinite family of minuscule posets (see the classification at the end of the introduction), the same triple exhibits the cyclic sieving phenomenon for all positive integers m. This was proved in Rush and Shi (2011) but is omitted here. The rest of this introduction is devoted to a discussion of Theorems 1.1 and 1.2 and a brief summary of our approach to their proofs.

It should be noted that several special cases of Theorem 1.1 already exist in the literature. When P arises from a Lie algebra with root system of type A, for instance, Theorem 1.1 reduces to a result of Stanley (2009) coupled with Theorem 1.1(b) in Reiner et al. (2004), and it is recorded as Theorem 6.1 in Striker and Williams (2012). The case when the root system is of type B turns out to be handled almost identically, and it is recorded as Corollary 6.3 in Striker and Williams (2012). That being said, our theorem is a generalization of these results, and, in relating Theorem 1.1 to a known cyclic sieving phenomenon for finite Coxeter groups (Theorem 1.6 in Reiner et al. (2004)), we expose the Fon-Der-Flaass action to new algebraic lines of attack.

If P is a finite poset, it is shown in Cameron and Fon-Der-Flaass (1995) that the Fon-Der-Flaass action Ψ may be expressed as a product of the involutive generators $\{t_p\}_{p \in P}$ for a larger group acting on the poset of order ideals $J(P)$. For all $p \in P$ and $I \in J(P)$, $t_p(I)$ is obtained by *toggling I at p*, so that $t_p(I)$ is either the symmetric difference $I \Delta \{p\}$, if this forms an order ideal, or just I, otherwise. In Striker and Williams (2012), Striker and Williams named this group the *toggle group*.

On the other hand, there is a natural labeling of the elements of a minuscule poset P by the Coxeter generators $S = \{s_1, s_2, \ldots, s_n\}$ for the Weyl group W, which is given in Stembridge (1996). In particular, if P is a minuscule poset, there exists a labeling of P such that the linear extensions of the labeled poset (which is called a *minuscule heap*) index the reduced words for the fully commutative element of W representing the topmost coset $w_0 W_J$. This labeling is illustrated in Figure 2 and explained more thoroughly in section 5. It has the following important properties.

First, it realizes the poset isomorphism $J(P) \cong W^J$ explicitly. Given an order ideal $I \in J(P)$ and a linear extension $(x_1, x_2, \ldots, x_t)$ of the partial order restricted to the elements of I, if the corresponding sequence of labels is $(i_1, i_2, \ldots, i_t)$, define $\phi(I)$ to be $s_{i_t} \cdots s_{i_2} s_{i_1}$. Then the map $\phi : J(P) \to W^J$ is an order-preserving bijection.

Second, it indicates a correspondence between Coxeter elements in W and sequences of toggles in $G(P)$: The choice of a linear ordering on the Coxeter generators $S = (s_{i_1}, \ldots, s_{i_n})$ yields a choice of the following.

- An element $t_{(i_1, \ldots, i_n)}$ in the toggle group that executes the following sequence of toggles: first toggle at all elements of P labeled by s_{i_n}, in any order; then toggle at all elements of P labeled by $s_{i_{n-1}}$, in any order;...; then toggle at all elements of P labeled by s_{i_2}, and, finally, toggle at all elements of P labeled by s_{i_1}, and

(i) Computed using code in the computer algebra package `Maple`. The authors also thank Dilks for allowing them the use of his code for subsequent computations.

- A *Coxeter element* $c = s_{i_1}s_{i_2}\cdots s_{i_{n-1}}s_{i_n}$ in the Weyl group, which acts on cosets W/W_J by left translation (i.e., $c(wW_J) = cwW_J$), and thus also acts on W^J.

The theorems that reduce Theorem 1.1 to the cyclic sieving result of Reiner et al. (2004) are as follows.

Theorem 1.3. *For any minuscule poset P and any ordering of $S = (s_{i_1}, \ldots, s_{i_n})$, the actions Ψ and $t_{(i_1,\ldots,i_n)}$ are conjugate in $G(P)$.*

Theorem 1.4. *For any minuscule poset P and any ordering of $S = (s_{i_1}, \ldots, s_{i_n})$, if $\phi : J(P) \to W^J$ is the isomorphism described above, then the following diagram is commutative:*

$$\begin{array}{rcl} J(P) & \xrightarrow{\phi} & W^J \\ t_{(i_1,\ldots,i_n)} \downarrow & & c \downarrow \\ J(P) & \xrightarrow{\phi} & W^J. \end{array}$$

To see that these theorems suffice to demonstrate Theorem 1.1, we quote Theorem 1.6 from Reiner et al. (2004).

Theorem 1.5 (Reiner et al. (2004), Theorem 1.6)**.** *Let W be a finite Coxeter group; let S be the set of Coxeter generators, and let J be a subset of S. Let W^J be the set of minimum-length coset representatives, and let $W^J(q) = \sum_{w\in W^J} q^{l(w)}$, where $l(w)$ denotes the length of w. If $c \in W$ is a regular element in the sense of Springer (1974), then $(W^J, W^J(q), \langle c\rangle)$ exhibits the cyclic sieving phenomenon.*

In Theorem 1.5, if W^J is a distributive lattice, then the length function l also serves as a rank function, so $W^J(q)$ is the rank-generating function. Furthermore, if c is a Coxeter element of W, then $c \in W$ is regular (cf. Springer (1974)).

Overviews of the proofs of Theorem 1.3 and Theorem 1.4 are given in section 6; sections 2, 3, 4, and 5 provide the requisite background. We did not manage to adapt the techniques developed in these sections for the proof of Theorem 1.2, so we appealed to the classification of minuscule posets, from which Theorem 1.2 follows after generating function calculations. Proofs (with details suppressed) may be found in Rush and Shi (2012). Full proofs are available in Rush and Shi (2011).

We close the introduction with a description of the three infinite families and two exceptional cases of minuscule posets and the root systems associated to the Lie algebras from which they arise. The following facts are well-known (cf. for instance, Stembridge (1994)).

- For the root systems of the form A_n, there are n possible minuscule weights, which lead to n associated minuscule posets, namely, all those posets of the form $[j] \times [n+1-j]$ such that $1 \leq j \leq n$. Posets of this form are considered to comprise the *first infinite family*.

- For the root systems of the form B_n, there is 1 possible minuscule weight, which leads to 1 associated minuscule poset, namely, $[n] \times [n]/S_2$. Posets of this form are considered to comprise the *second infinite family*.

- For the root systems of the form C_n, there is 1 possible minuscule weight, which leads to 1 associated minuscule poset, namely, $[2n-1]$. Posets of this form already belong to the first infinite family.

- For the root systems of the form D_n, there are 3 possible minuscule weights, which, because two of the minuscule weights lead to the same minuscule poset, only lead to 2 associated minuscule posets, namely, $[n-1] \times [n-1]/S_2$ and $J^{n-3}([2] \times [2])$. Posets of the latter form are considered to comprise the *third infinite family* (it should be clear that posets of the former form already belong to the second infinite family).

- For the root system E_6, there are 2 possible minuscule weights, which, because both minuscule weights lead to the same minuscule poset, only lead to 1 associated minuscule poset, namely, $J^2([2] \times [3])$. This poset is called the *first exceptional case*.

- For the root system E_7, there is 1 possible minuscule weight, which leads to 1 associated minuscule poset, namely, $J^3([2] \times [3])$. This poset is called the *second exceptional case*.

No other root systems admit minuscule weights.

2 The Fon-Der-Flaass Action

In this section, we introduce and analyze the Fon-Der-Flaass action.

Let $P = (X, <)$ be a partially ordered set, and let $J(P)$ be the set of order ideals of P, partially ordered by inclusion. Following the notation of Cameron and Fon-Der-Flaass (1995), for all order ideals $I \in J(P)$, let

$$Z(I) = \{x \in I : y > x \Longrightarrow y \notin I\},$$

and let

$$U(I) = \{x \notin I : y < x \Longrightarrow y \in I\}.$$

Then the *Fon-Der-Flaass action*, which we denote by Ψ, is defined as follows.

Definition 2.1. For all $I \in J(P)$, $\Psi(I)$ is the unique order ideal satisfying $Z(\Psi(I)) = U(I)$.

Remark 2.2. It is clear from Definition 2.1 that Ψ permutes the order ideals of P.

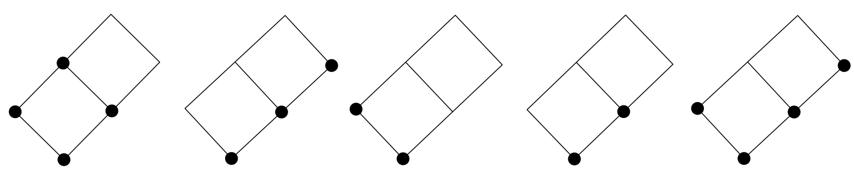

Fig. 1: An orbit of order ideals under the Fon-Der-Flaass action

This definition of the Fon-Der-Flaass action is global. We now give an equivalent definition that decomposes it into a product of local actions, which are more easily understood. Recall from the introduction that for all $p \in P$ and $I \in J(P)$, we let $t_p : J(P) \to J(P)$ be the map defined by $t_p(I) = I \setminus \{p\}$ if $p \in Z(I)$, $t_p(I) = I \cup \{p\}$ if $p \in U(I)$, and $t_p(I) = I$ otherwise. The following theorem is equivalent to Lemma 1 in Cameron and Fon-Der-Flaass (1995).

Theorem 2.3. *Let P be a poset. For all linear extensions $(p_1, p_2, \ldots, p_n)$ of P and order ideals $I \in J(P)$, $\Psi(I) = t_{p_1} t_{p_2} \cdots t_{p_n}(I)$.*

The group $G(P) := \langle t_p \rangle_{p \in P}$ is named the *toggle group* in Striker and Williams (2012). Note that for all x and y, the generators t_x and t_y commute unless x and y share a covering relation.

In the case that the poset P is ranked, it is natural to consider the linear extensions label the elements of P by order of increasing rank. For the purposes of this paper, we shall say that P is ranked if there exists an integer-valued function r on X (called the *rank function*) such that $r(p) = 0$ for all minimal elements $p \in X$ and, for all $x, y \in X$, if x covers y, then $r(x) - r(y) = 1$.

If P is a ranked poset, let the maximum value of r be R. For all $0 \leq i \leq R$, let $P_i = \{p \in P : r(p) = i\}$, and let $t_i = \prod_{p \in P_i} t_p$. We see that t_i is always well-defined because, for all i, t_x and t_y commute for all $x, y \in P_i$. By Theorem 2.3, $\Psi = t_0 t_1 \cdots t_R$. Note that t_i and t_j commute for all $|i - j| > 1$. The following theorem is also a result of Cameron and Fon-Der-Flaass (1995).

Theorem 2.4. *For all permutations σ of $\{0, 1, \ldots, R\}$, $\Psi_\sigma := t_{\sigma(0)} t_{\sigma(1)} \cdots t_{\sigma(R)}$ is conjugate to Ψ in $G(P)$.*

Corollary 2.5. *The action Ψ_σ has the same orbit structure as Ψ for all σ.*

Let $t_{\text{even}} = \prod_{i \text{ even}} t_i$, and let $t_{\text{odd}} = \prod_{i \text{ odd}} t_i$. It should be clear that t_{even} and t_{odd} are well-defined, and it follows from Theorem 2.4 that $t_{\text{even}} t_{\text{odd}}$ is conjugate to Ψ in $G(P)$, as noted in the second paragraph of section 4 in Cameron and Fon-Der-Flaass (1995). This means that the action of toggling at all the elements of odd rank, followed by toggling at all the elements of even rank, is conjugate to the Fon-Der-Flaass action in the toggle group. As we shall see, this holds the key to demonstrating that the induced action of every Coxeter element of W on $J(P)$ under ϕ is conjugate to the Fon-Der-Flaass action as well. Striker and Williams made use of the same argument to obtain the conjugacy of promotion and rowmotion (their name for the Fon-Der-Flaass action) in section 6 of Striker and Williams (2012), so it should be no surprise that our induced actions reduce to promotion in types A and B. In this sense, our proof of Theorem 1.1 may be considered to be a continuation of their work.

3 Minuscule Posets

In this section, we introduce the primary objects of study for this paper – the minuscule posets. We begin with some notation, following Stembridge (1994). Let $\mathfrak{g}$ be a complex simple Lie algebra; let $\mathfrak{h}$ be a Cartan subalgebra; choose a set Φ^+ of positive roots α in $\mathfrak{h}^*$, and let $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be the set of simple roots. Let $(\cdot, \cdot)$ be the inner product on $\mathfrak{h}^*$, and, for each root α, let $\alpha^\vee = 2\alpha/(\alpha, \alpha)$ be the corresponding coroot. Finally, let $\Lambda = \{\lambda \in \mathfrak{h}^* : \alpha \in \Phi \to (\lambda, \alpha^\vee) \in \mathbb{Z}\}$ be the weight lattice.

For all $1 \leq i \leq n$, let s_i be the simple reflection corresponding to the simple root α_i, and let $W = \langle s_i \rangle_{1 \leq i \leq n}$ be the Weyl group of $\mathfrak{g}$. If s is conjugate to a simple reflection s_i in W, we refer to s as an (abstract) reflection.

Let V be a finite-dimensional representation of $\mathfrak{g}$. For each $\lambda \in \Lambda$, let

$$V_\lambda = \{v \in V : h \in \mathfrak{h} \Longrightarrow hv = \lambda(h)v\}$$

be the weight space corresponding to λ, and let Λ_V be the (finite) set of weights λ such that V_λ is nonzero. Recall that there is a standard partial order on Λ called the *root order* defined to be the transitive closure of the relations $\mu < \omega$ for all weights μ and ω such that $\omega - \mu$ is a simple root.

Definition 3.1. The *weight poset* Q_V of the representation V is the restriction of the root order on Λ to Λ_V.

If V is irreducible, Q_V has a unique maximal element, which is called the highest weight of V. This leads to the following definition.

Definition 3.2. Let V be a nontrivial, irreducible, finite-dimensional representation of $\mathfrak{g}$. V is a *minuscule representation* if the action of W on Λ_V is transitive. In this case, the highest weight of V is called the *minuscule weight*.

Theorem 3.3. *If V is minuscule, the weight poset Q_V is a distributive lattice.*

Definition 3.4. If V is minuscule, let P_V be the poset of join-irreducible elements of the weight poset Q_V, so that P_V is the unique poset satisfying $J(P_V) \cong Q_V$. Then P_V is the *minuscule poset* of V, and posets of this form comprise the *minuscule posets*.

Remark 3.5. If V is a minuscule representation and λ is the highest weight of V, we refer to P_V as the *minuscule poset for λ*.

4 Bruhat Posets

In this section, we develop the framework for the proofs of Theorems 1.3 and 1.4. We begin by discussing the Bruhat posets. Then we establish the connection between these objects and the weight posets of minuscule representations.

We continue with the notation of the previous section. Given a Weyl group W, we define a length function l on the elements of W as follows. For all $w \in W$, we let $l(w)$ be the minimum length of a word of the form $s_{i_1} s_{i_2} \dots s_{i_\ell}$ such that $w = s_{i_1} s_{i_2} \dots s_{i_\ell}$ and s_{i_j} is a simple reflection for all $1 \leq j \leq \ell$. This allows us to introduce a well-known partial order on W, known as the (strong) Bruhat order, for which l also serves as a rank function. The Bruhat order is defined to be the transitive closure of the relations $w <_B sw$ for all Weyl group elements w and (abstract) reflections s satisfying $l(w) < l(sw)$.

We are herein concerned not with the Bruhat order on W, but with the restrictions of the Bruhat order to parabolic quotients of W, for these are the orders that give rise to the Bruhat posets.

Definition 4.1. If J is a subset of $\{1, 2, \dots, n\}$, then $W_J := \langle s_i \rangle_{i \in J}$ is the *parabolic subgroup* of W generated by the corresponding simple reflections, and $W^J := W/W_J$ is the *parabolic quotient*.

It is well-known that each coset in W^J has a unique representative of minimum length, so the quotient W^J may be regarded as the subset of W containing only the minimum-length coset representatives. This fact facilitates the definition of an analogous partial order on W^J.

Definition 4.2. The *Bruhat order* $<_B$ on the parabolic quotient W^J is the restriction of the Bruhat order on W to W^J. Posets of the form $(W^J, <_B)$ comprise the *Bruhat posets*.

We may also define the left (weak) Bruhat order on W to be the transitive closure of the relations $w <_L sw$ for all Weyl group elements w and *simple* reflections s satisfying $l(w) < l(sw)$. The analogous partial order on W^J is defined in precisely the same way: $(W^J, <_L)$ is the restriction of $(W, <_L)$ to the minimum-length coset representatives W^J. While the left Bruhat order is not necessary to establish the connection between the minuscule posets and the Bruhat posets, we introduce it here so that our work in this section may be compatible with the theory of fully commutative elements developed in section 5 and exploited in section 6.

We are now ready to state the following theorem, which appears as Proposition 4.1 in Proctor (1984).

Theorem 4.3. *Let V be a minuscule representation with minuscule weight λ, and let $J = \{i : s_i\lambda = \lambda\}$. Then W_J is the stabilizer of λ in the Weyl group W, and the weight poset Q_V is isomorphic to the Bruhat poset $(W^J, <_B)$.*

Definition 4.4. The parabolic quotient W^J is *minuscule* if W_J is the stabilizer of a minuscule weight λ.

The assumption that $\mathfrak{g}$ be simple implies that λ is a fundamental weight. Hence if $\lambda = \omega_j$, then $s_i\lambda = \lambda$ for all $i \neq j$. It follows that if W^J is minuscule, $J = \{1, 2, \ldots, n\} \setminus \{j\}$, so W_J is a maximal parabolic subgroup of W. In general, a minuscule Bruhat poset is obtained precisely when the "missing" element of J is the index of a fundamental weight for which there exists a representation of $\mathfrak{g}$ in which that fundamental weight is minuscule.

We note that Bruhat posets W^J provide a natural setting for identifying instances of the cyclic sieving phenomenon because they come equipped with a group action, namely that of W, and a rank-generating function $W^J(q) := \sum_{w \in W^J} q^{l(w)}$, which is what motivated us to consider them in the first place. We now turn our attention to the labeling of the minuscule poset P_V and the construction of the isomorphism $\phi : J(P_V) \to W^J$, which lie behind the proofs of Theorems 1.3 and 1.4.

5 Fully Commutative Elements

In this section we borrow from Stembridge's theory of fully commutative elements of Weyl groups. In the next section, we shall see how the theory enables us to characterize the relationship between the action of the Weyl group on the elements of these lattices and the action of the toggle group on the order ideals of the corresponding minuscule posets.

Definition 5.1. Let W be a Weyl group, and let $S = \{s_1, s_2, \ldots, s_n\}$ be the set of Coxeter generators. An element $w \in W$ is *fully commutative* if every reduced word for w can be obtained from every other by means of commuting braid relations only (i.e., via relations of the form $s_j s_{j'} = s_{j'} s_j$ for commuting Coxeter generators s_j and $s_{j'}$).

Given a fully commutative element w, we can define a labeled poset P_w that generates all the reduced words of w in the sense that putting labels in the place of poset elements gives a bijection between the linear extensions of P_w and the reduced words of w.

Definition 5.2. Let $s_{i_1} s_{i_2} \cdots s_{i_\ell}$ be a reduced word for w. Let $P_w = (\{1, 2, \ldots, \ell\}, <)$ be a partially ordered set, where the partial order on $\{1, 2, \ldots, \ell\}$ is defined to be the transitive closure of the relations $j > j'$ for all $j < j'$ in integers such that s_{i_j} and $s_{i_{j'}}$ do not commute. Then P_w is the *heap* of w, and, for all $1 \leq j \leq \ell$, s_{i_j} is the *label* of the heap element $j \in P_w$. An example is given in Figure 2.

Let $\mathcal{L}(P_w) := \{\pi : \pi(1) \geq \pi(2) \geq \ldots \geq \pi(\ell)\}$ be the set of *reverse* linear extensions of P_w, and let $\mathcal{L}(P_w, w)$ be the set of labeled reverse linear extensions of P_w, i.e.,

$$\mathcal{L}(P_w, w) := \{s_{i_{\pi(1)}} s_{i_{\pi(2)}} \cdots s_{i_{\pi(\ell)}} : \pi \in \mathcal{L}(P_w)\}.$$

As alluded to above, the set $\mathcal{L}(P_w, w)$ is significant for the following reason.

Proposition 5.3 (Stembridge (1996), Proposition 2.2)**.** *$\mathcal{L}(P_w, w)$ is the set of reduced words for w in W.*

It follows from Proposition 5.3 that, if w is fully commutative, the heaps of the reduced words for w are all equivalent, so we may refer to the heap of w unambiguously. This is also noted in Stembridge (1996). The crucial claim is the next theorem.

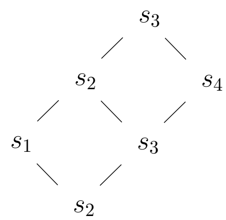

Fig. 2: If W is the Weyl group arising from the root system A_4, then the element $w := s_3s_2s_4s_1s_3s_2$ is fully commutative, and the heap P_w is as displayed above.

Theorem 5.4. *Let $w \in W$ be fully commutative. Then $J(P_w) \cong \{x \in W : x \leq_L w\}$ is an isomorphism of posets.*

We are now ready to define the bijection between $J(P_w)$ and $\{x \in W : x \leq_L w\}$. Given an order ideal $I \in J(P_w)$, let ρ be a linear extension of P_w such that $\rho(j) \in I$ for all $1 \leq j \leq |I|$ and $\rho(j) \notin I$ otherwise.

Theorem 5.5. *The association*

$$I \quad \longmapsto \quad s_{i_{\rho(|I|)}} \cdots s_{i_{\rho(2)}} s_{i_{\rho(1)}}.$$

defines a bijection

$$\phi : J(P_w) \quad \longrightarrow \quad \{x \in W : x \leq_L w\}$$

Remark 5.6. The choice of the symbol ϕ to denote this map is deliberate, for when the heap P_w is minuscule (see Definition 6.1), ϕ is the map described in the introduction.

The following theorem demonstrates the relevance of the theory of fully commutative elements to our main results.

Theorem 5.7 (Stembridge (1996), Theorems 6.1 and 7.1)**.** *If W^J is minuscule, then the following three claims hold:*

(i) If $w \in W^J$, w is fully commutative;

(ii) $(W^J, <_L)$ is a distributive lattice;

(iii) $(W^J, <_B) = (W^J, <_L)$.

6 The Main Results

In this section, we present an overview of our proofs of Theorems 1.3 and 1.4. We start with the following definition and subsequent theorem.

Definition 6.1. If W^J is minuscule, and w_0^J is the longest element of W^J, then the heap $P_{w_0^J}$ is *minuscule*, and heaps of this form comprise the *minuscule heaps*.

Theorem 6.2. *Let V be a minuscule representation of a complex simple Lie algebra $\mathfrak{g}$ with minuscule weight λ and Weyl group W. If $S = \{s_1, s_2, \ldots, s_n\}$ is the set of Coxeter generators and W_J is the maximal parabolic subgroup stabilizing λ, then the following claims hold:*

(i) If w_0^J is the longest element of W^J, then the poset $\{x \in W : x \leq_L w_0^J\}$ and the lattice $(W^J, <_L)$ are identical, and, furthermore, the minuscule heap $P_{w_0^J}$ and the minuscule poset P_V are isomorphic as posets.

(ii) The isomorphism $\phi : J(P_{w_0^J}) \to \{x \in W : x \leq_L w_0^J\} \cong (W^J, <_L) \cong (W^J, <_B)$ defined in Definition 5.5 satisfies the following property: For all $1 \leq k \leq n$, the induced action of the Coxeter generator s_k on $J(P_{w_0^J})$ in the toggle group $G(P_{w_0^J})$ may be expressed in the form $\displaystyle\prod_{\substack{p \in P_{w_0^J} \\ p \text{ is labeled by } s_k}} t_p.$

Example 6.3. In the case when the root system is A_4 and the minuscule weight is ω_2, Figure 3 shows the minuscule heap $P_{s_3s_2s_4s_1s_3s_2}$ (on the left) and the corresponding Bruhat poset $(W^J, <_B)$ (on the right). If I is the order ideal encircled by the solid line, then $\phi(I)$ is the coset representative encircled by the solid line, and $\prod_{p \in P_{s_3s_2s_4s_1s_3s_2} \text{ is labeled by } s_2} t_p(I)$ is the order ideal encircled by the dotted line. Furthermore, $\phi(\prod_{p \in P_{s_3s_2s_4s_1s_3s_2} \text{ is labeled by } s_2} t_p(I)) = s_2\phi(I)$ is the coset representative encircled by the dotted line, thus illustrating the statement *(ii)* in Theorem 6.2.

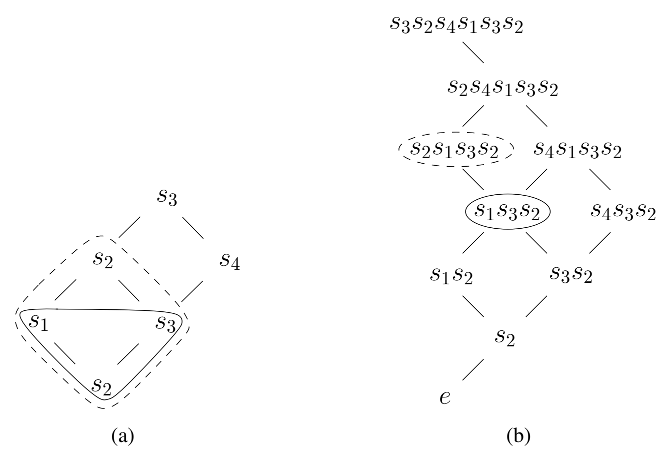

Fig. 3: The map ϕ sends the indicated order ideals to the indicated coset representatives.

We omit the proof of Theorem 6.2, but we will list the three crucial lemmas. For all $1 \leq k \leq n$, let C_k be the set of all heap elements labelled by s_k, and let t'_k be the toggle group element defined by $t'_k = \prod_{p \in C_k} t_p$. From section 5, we know that C_k is totally ordered, and, by definition of $P_{w_0^J}$, no two elements of C_k share a covering relation, so it follows that t'_k is well-defined for all k. The three lemmas are as follows:

Lemma 6.4. *The order ideal $t'_k(I)$ disagrees with I on at most one vertex of $P_{w_0^J}$.*

Lemma 6.5. *There exists an element $p_0 \in P$ such that $p_0 \in Z(I)$ if and only if $s_k w$ is not reduced. In this case, if $s_{i_{l(s_k w)}} \cdots s_{i_2} s_{i_1}$ is a reduced word for $s_k w$, then $s_k s_{i_{l(s_k w)}} \cdots s_{i_2} s_{i_1}$ is a reduced word for w, and $\phi(I \setminus \{p_0\}) = s_{i_{l(s_k w)}} \cdots s_{i_2} s_{i_1}$.*

Lemma 6.6. *There exists an element $p_0 \in P$ such that $p_0 \in U(I)$ if and only if $s_k w$ is reduced and $s_k w \in W^J$. In this case, if $s_{i_{l(w)}} \cdots s_{i_2} s_{i_1}$ is a reduced word for w, then $s_k s_{i_{l(w)}} \cdots s_{i_2} s_{i_1}$ is a reduced word for $s_k w$, and $\phi(I \cup \{p_0\}) = s_k s_{i_{l(w)}} \cdots s_{i_2} s_{i_1}$.*

Theorem 1.4 is a direct consequence of Theorem 6.2. We proceed to our overview of the proof of Theorem 1.3.

6.1 Proof Overview for Theorem 1.3

Let P_V be a minuscule poset, and again label each element of P_V by the label of the corresponding element of $P_{w_0^J}$. Theorem 6.2 embeds the Weyl group W as a subgroup of the toggle group $G(P_V)$. In light of Theorem 1.4, since the Coxeter elements are known to be pairwise conjugate in W, it suffices to exhibit a particular ordering $S = (s_{i_1}, s_{i_2}, \ldots, s_{i_n})$ such that $t_{(i_1,i_2,\ldots,i_n)} = t'_{i_1} t'_{i_2} \cdots t'_{i_n}$ is conjugate to Ψ in $G(P_V)$. However, in section 2, we saw that $t_{\text{even}} t_{\text{odd}}$ is conjugate to Ψ in $G(P_V)$. It suffices to show that there exists an ordering $(s_{i_1}, s_{i_2}, \ldots, s_{i_n})$ such that the toggle group elements $t'_{i_1} t'_{i_2} \cdots t'_{i_n}$ and $t_{\text{even}} t_{\text{odd}}$ are equal.

The crucial lemmas are the following:

Lemma 6.7 (Stanley (2012), Exercise 25(b))**.** *If P is a minuscule poset, then P is a ranked poset.*

Lemma 6.8 (Björner and Brenti (2005), Chapter 1, Exercise 4)**.** *If W is the Weyl group of a complex simple Lie algebra $\mathfrak{g}$, then the Dynkin diagram of the associated root system is acyclic and therefore bipartite.*

Let r be the rank function for P_V. From these lemmas, we note that if $p, p' \in P_V$ and $r(p) \equiv r(p') \pmod 2$, then t_p and $t_{p'}$ commute in $G(P_V)$. Let S_{odd} be the set of all k such that s_k is a simple reflection and the rank of p is odd for all vertices $p \in P_{w_0^J}$ labelled by s_k. Similarly, let S_{even} be the set of all k' such that $s_{k'}$ is a simple reflection and the rank of p is even for all vertices $p \in P_{w_0^J}$ labelled by $s_{k'}$. It follows that $t_{\text{even}} t_{\text{odd}} = \prod_{k' \in S_{\text{even}}} t'_{k'} \prod_{k \in S_{\text{odd}}} t'_k$.

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Permutation patterns, Stanley symmetric functions, and the Edelman-Greene correspondence

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Abstract. Generalizing the notion of a vexillary permutation, we introduce a filtration of S_∞ by the number of Edelman-Greene tableaux of a permutation, and show that each filtration level is characterized by avoiding a finite set of patterns. In doing so, we show that if w is a permutation containing v as a pattern, then there is an injection from the set of Edelman-Greene tableaux of v to the set of Edelman-Greene tableaux of w which respects inclusion of shapes. We also consider the set of permutations whose Edelman-Greene tableaux have distinct shapes, and show that it is closed under taking patterns.

Résumé. Généralisant la notion d'une permutation vexillaire, nous introduisons une filtration de S_∞ par le nombre de tableaux d'Edelman-Greene d'une permutation, et nous montrons que chaque niveau de la filtration se caractérise par un ensemble fini des motifs exclus. Ce faisant, nous montrons que si w est une permutation qui inclut le motif v, il existe une injection de l'ensemble des tableaux d'Edelman-Greene de v dans l'ensemble des tableaux d'Edelman-Greene de w qui respecte l'inclusion de formes. Nous considérons aussi l'ensemble des permutations dont les tableaux d'Edelman-Greene ont des formes distinctes, et nous montrons que c'est clos pour l'inclusion de motifs.

Keywords: Edelman-Greene correspondence, Stanley symmetric functions, Specht modules, pattern avoidance

1 Introduction

Stanley (1984) defined a symmetric function F_w depending on a permutation w, with the property that the coefficient of $x_1 \cdots x_\ell$ in F_w is the number of reduced words of w. Therefore if $F_w = \sum_\lambda a_{w\lambda} s_\lambda$ is written in terms of Schur functions, then

$$|\operatorname{Red}(w)| = \sum_\lambda a_{w\lambda} f^\lambda, \tag{1}$$

where f^λ is the number of standard Young tableaux of shape λ and $\operatorname{Red}(w)$ the set of reduced words of w.

Edelman and Greene (1987) gave an algorithm which realizes (1) bijectively and shows that the $a_{w\lambda}$ are nonnegative.

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Theorem 1.1 *Given a permutation w, there is a set $EG(w)$ of semistandard Young tableaux and a bijection*

$$\mathrm{Red}(w) \leftrightarrow \{(P,Q) : P \in EG(w), Q \text{ a standard tableau of shape } \mathrm{shape}(P)\}. \tag{2}$$

The tableaux $EG(w)$ are those semistandard tableaux whose column word—gotten by reading up columns starting with the leftmost—is a reduced word for w. The shapes of these tableaux precisely give the Schur function expansion of F_w:

$$F_w = \sum_{P \in EG(w)} s_{\mathrm{shape}(P)}. \tag{3}$$

Stanley also characterized those w for which F_w is a single Schur function, or equivalently for which $|EG(w)| = 1$. These are the *vexillary* permutations, those avoiding the pattern 2143. Our main results can be viewed as generalizations of this characterization. The first main theorem shows that $EG(w)$ is well-behaved with respect to pattern containment.

Theorem 1.2 *Let v, w be permutations with w containing v as a pattern. There is an injection $\iota : EG(v) \hookrightarrow EG(w)$ such that if $P \in EG(v)$, then $\mathrm{shape}(P) \subseteq \mathrm{shape}(\iota(P))$. Moreover, if P, P' have the same shape, so do $\iota(P), \iota(P')$.*

An immediate corollary is that the sets $\{w \in \bigcup_{n\geq 0} S_n : |EG(w)| \leq k\}$ respect pattern containment, in the sense that if $|EG(w)| \leq k$ and w contains v, then $|EG(v)| \leq k$. Our second main result is a sort of converse.

Definition 1.3 *Given a positive integer k, a permutation $w \in S_n$ is* k-vexillary *if $|EG(w)| \leq k$.*

Theorem 1.4 *For each integer $k \geq 1$, there is a finite set V_k of permutations such that w is k-vexillary if and only if w avoids all patterns in V_k.*

Stembridge (2001) gives a criterion for the product of Schur functions $s_\lambda s_\mu$ to be *multiplicity-free*, i.e. a sum of distinct Schur functions, which Thomas and Yong (2010) generalize by answering the analogous question for Schubert classes on Grassmannians. Call a permutation w multiplicity-free if F_w is multiplicity-free. Theorem 1.2 shows that the set of multiplicity-free w is closed under patterns.

Theorem 1.5 *If a permutation w contains the pattern v, and w is multiplicity-free, then so is v.*

We have not been able to prove that the property of being multiplicity-free is equivalent to avoiding a finite set of patterns. However, an explicit computation shows that if w in S_{12} is not multiplicity-free, then w properly contains a pattern v which is not multiplicity-free.

Conjecture 1 *A permutation w is multiplicity-free if and only if w avoids every non-multiplicity-free pattern in S_m for $m \leq 11$.*

In Section 2, we recall the connection between Stanley symmetric functions and the representation theory of the symmetric group, along with the Lascoux-Schützenberger recurrence for computing Stanley symmetric functions. We also recall the definitions of pattern avoidance and containment. Section 3 introduces the notion of a James-Peel tree for a general diagram, following James and Peel (1979). In Section 4, we specialize these ideas to permutation diagrams, with the Lascoux-Schützenberger tree as a key tool, and prove Theorem 1.2. In Section 5 we analyze in more detail the relationship between $|EG(w)|$ and $|EG(v)|$ for v a pattern in w, and prove Theorem 1.4. Section 6 is devoted to open problems.

2 Background

2.1 Permutation patterns

We first recall the definitions of pattern avoidance and containment for permutations.

Definition 2.1 *Let v, w be two permutations. We say w* contains *$v \in S_m$ if there are $i_1 < \cdots < i_m$ such that $v(j) < v(k)$ if and only if $w(i_j) < w(i_k)$. If w does not contain v, it* avoids *v. Often we say that w contains or avoids the* pattern *v.*

Example 2.2 *The permutation 2513764 contains the patterns 2143 (e.g. as the subsequence 2174) and 23154. It avoids 1234.*

2.2 Specht modules

Our proof of Theorem 1.2 goes via the representation theory of S_n, specifically the interpretation of F_w as the Frobenius character of a certain *Specht module*, which we discuss next.

Definition 2.3 *A* diagram *is a finite subset of $\mathbb{N} \times \mathbb{N}$.*

We refer to the elements of a diagram as *cells*. The diagrams of greatest interest for us will be *permutation diagrams* (sometimes called Rothe diagrams, from Rothe (1800)). Given $w \in S_n$, define

$$D(w) = \{(i, w(j)) : 1 \leq i < j \leq n, w(i) > w(j)\}. \tag{4}$$

We'll draw $D(w)$ using matrix coordinates:

$$D(42153) = \begin{array}{ccccc} \circ & \circ & \circ & \times & \cdot \\ \circ & \times & \cdot & \cdot & \cdot \\ \times & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \circ & \cdot & \times \\ \cdot & \cdot & \times & \cdot & \cdot \end{array}$$

Members of a diagram will be represented by $\circ$. We'll often augment $D(w)$ by adding $\times$ at the points $(i, w(i))$.

A *filling* of a diagram D is a bijection $T : D \to \{1, \ldots, n\}$, where $n = |D|$. There is a natural left action of S_n on fillings of D by permuting entries. The *row group* $R(T)$ of a filling T is the subgroup of S_n consisting of permutations σ which act on T by permuting entries within their row; the *column group* $C(T)$ is defined analogously. The *Young symmetrizer* of a filling T is an element of $\mathbb{C}[S_n]$, defined by

$$c_T = \sum_{p \in R(T)} \sum_{q \in C(T)} \operatorname{sgn}(q) q p. \tag{5}$$

Definition 2.4 *Given a diagram D and a choice of filling T, the* Specht module *S^D is the S_n-module $\mathbb{C}[S_n]c_T$, where $n = |D|$. The* Schur function *s_D of D is the Frobenius characteristic of S^D.*

The isomorphism type of S^D is independent of the choice of T, and is also unaffected by permuting rows or columns of D.

Definition 2.5 *If a diagram D is gotten from a diagram D' by permuting rows and columns, say D and D' are* equivalent*, and write $D \simeq D'$. This includes inserting or deleting empty rows and columns.*

Over $\mathbb{C}$, the Specht modules of partition diagrams form complete sets of irreducible S_n-representations: see Sagan (2001) or Fulton (1997). In general, it is an open problem to find a reasonable combinatorial algorithm for decomposing S^D into irreducibles. The Littlewood-Richardson rule handles the case where D is a skew shape, and Reiner and Shimozono (1998) and Liu (2009) treat other classes of diagrams.

Definition 2.6 *Given two diagrams D_1, D_2, where $D_1 \subseteq [r] \times [c]$, define their* product $D_1 \cdot D_2$ *to be the diagram*

$$D_1 \cup \{(i+r, j+c) : (i,j) \in D_2\}.$$

One can check that $s_{D_1 \cdot D_2} = s_{D_1} s_{D_2}$. We will use this operation in Section 3.

2.3 Stanley symmetric functions

The *Stanley symmetric function* of a permutation w of length ℓ is

$$F_w = \sum_{a \in \mathrm{Red}(w)} \sum_i x_{i_1} x_{i_2} \cdots x_{i_\ell}, \tag{6}$$

where for each $a \in \mathrm{Red}(w)$, i runs over all integer sequences $1 \leq i_1 \leq \cdots \leq i_\ell$ such that $i_j < i_{j+1}$ if $a_j > a_{j+1}$.

Write D^t for the transpose of D. For a permutation w, let $1^m \times w = 12\cdots m(w(1)+m)(w(2)+m)\cdots$. The results of Billey et al. (1993) show that $F_w = \lim_{m\to\infty} \mathfrak{S}_{1^m \times w^{-1}}$, where $\mathfrak{S}_v$ is a Schubert polynomial. Theorem 31 in Reiner and Shimozono (1995b) and Theorem 20 in Reiner and Shimozono (1998) then imply the following result, which is also implicit in Kraśkiewicz (1995).

Theorem 2.7 *For any permutation w, $F_w = s_{D(w)^t} = s_{D(w^{-1})}$.*

Stanley symmetric functions can be decomposed into Schur functions using a recursion introduced in Lascoux and Schützenberger (1985). Given a permutation w, let r be maximal with $w(r) > w(r+1)$. Then let $s > r$ be maximal with $w(s) < w(r)$. Let t_{ij} denote the transposition $(i\,j)$, and define

$$T(w) = \{wt_{rs}t_{rj} : \ell(wt_{rs}t_{rj}) = \ell(w)\}; \tag{7}$$

or, if the set on the right-hand side is empty, set $T(w) = T(1 \times w)$ where $1 \times w = 1(w(1)+1)(w(2)+1)\cdots$ in one-line notation. The members of $T(w)$ are called *transitions* of w. The *Lascoux-Schützenberger tree* (L-S tree for short) is the finite rooted tree of permutations with root w where the children of a vertex v are:

- None, if v is vexillary (avoids 2143).
- $T(v)$ otherwise.

Monk's rule for Schubert polynomials and the identity $F_w = \lim_{m\to\infty} \mathfrak{S}_{1^m \times w^{-1}}$ lead to the recurrence $F_w = \sum_{v \in T(w)} F_v$. This, together with the finiteness of the Lascoux-Schützenberger tree and Stanley's result that F_v is a Schur function exactly when v is vexillary, imply that

$$F_w = \sum_v s_{\mathrm{shape}(v)^t}, \tag{8}$$

or equivalently

$$s_{D(w)} = \sum_{v} s_{\text{shape}(v)}, \tag{9}$$

where v runs over the leaves of the Lascoux-Schützenberger tree, and $\text{shape}(v)$ denotes the partition whose shape is equivalent to $D(v)$. Here we use the fact that $D(v)$ is equivalent to a partition diagram if and only if v is vexillary.

Remark 2.8 The reduced words of $1 \times w$ are exactly those of w with all letters shifted up by 1, so the same is true of the tableaux in $EG(1 \times w)$ compared to the tableaux in $EG(w)$. In particular, the shapes are the same and $F_w = F_{1\times w}$. Since the Lascoux-Schützenberger tree is finite, there is some m such that in constructing the tree for $1^m \times w$, we never need to make the replacement of v by $1 \times v$. Thus we will ignore this possible step in what follows.

3 James-Peel moves

Let D be a diagram. Given two positive integers a, b, let $R_{a\to b}D$ be the diagram which contains a cell (i, j) if and only if one of the following cases holds:

- $i \neq a, b$ and $(i, j) \in D$.
- $i = b$, and either $(a, j) \in D$ or $(b, j) \in D$.
- $i = a$, and $(a, j), (b, j) \in D$.

That is, $R_{a\to b}D$ is gotten by moving cells in row a to row b if the appropriate position is empty. Similarly, we define $C_{c\to d}D$ by moving cells of D in column c to column d if possible.

The operators $R_{a\to b}$ and $C_{c\to d}$ were introduced in James and Peel (1979), so we will call them *James-Peel moves*. The next theorem uses James-Peel moves to find irreducible factors inside S^D, and can be viewed as a generalization of Pieri's rule.

Definition 3.1 *A subset D' of a diagram D is a* subdiagram *if it is the intersection of some rows and columns with D. That is, there are sets $X, Y \subseteq \mathbb{N}$ such that $D' = (X \times Y) \cap D$.*

Let δ_p denote the partition $(p-1, p-2, \dots, 1)$.

Theorem 3.2 *Suppose D contains $\delta_p \cdot (1)$ as a subdiagram in rows $\alpha(1) < \cdots < \alpha(p)$ and columns $\beta(1) < \cdots < \beta(p)$. There is a filtration*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_p = S^D$$

of S^D by $S_{|D|}$-submodules such that for each $1 \leq j \leq p$, there is a surjection

$$M_j/M_{j-1} \twoheadrightarrow S^{R_{\alpha(p)\to\alpha(p-j+1)}C_{\beta(p)\to\beta(j)}D}.$$

The case $p = 1$ of Theorem 3.2 is Theorem 2.4 of James and Peel (1979). Theorem 3.2 is valid over any field, but for convenience, we will work over $\mathbb{C}$. In this case, complete irreducibility of S_n-modules gives an inclusion

$$\bigoplus_{j=1}^{p} S^{R_{\alpha(p)\to\alpha(p-j+1)}C_{\beta(p)\to\beta(j)}D} \hookrightarrow S^D.$$

Example 3.3 *Take*

$$D = \begin{array}{ccccc} \bullet & \circ & \bullet & \cdot & \cdot \\ \bullet & \circ & \cdot & \cdot & \cdot \\ \cdot & \circ & \cdot & \circ & \bullet \end{array} \simeq D(4317256).$$

The subdiagram $\delta_2 \cdot (1)$ *appears in rows* $1, 2, 3$ *and columns* $1, 3, 5$*, and is shaded in the picture above. Then*

$$R_{3\to 3}C_{5\to 1}D = \begin{array}{cccc} \bullet & \circ & \bullet & \cdot \\ \bullet & \circ & \cdot & \cdot \\ \bullet & \circ & \cdot & \circ \end{array} \quad R_{3\to 2}C_{5\to 3}D = \begin{array}{cccc} \bullet & \circ & \bullet & \cdot \\ \bullet & \circ & \bullet & \circ \\ \cdot & \circ & \cdot & \cdot \end{array} \quad R_{3\to 1}C_{5\to 5}D = \begin{array}{ccccc} \bullet & \circ & \bullet & \circ & \bullet \\ \bullet & \circ & \cdot & \cdot & \cdot \\ \cdot & \circ & \cdot & \cdot & \cdot \end{array}$$

The second and third diagrams here are equivalent to the diagrams of the partitions $(4, 3, 1)$ *and* $(5, 2, 1)$ *respectively. Hence if* D_1 *is the first diagram, Theorem 3.2 gives* $S^{D_1} \oplus S^{(4,3,1)} \oplus S^{(5,2,1)} \hookrightarrow S^D$*. Another application of Theorem 3.2 to the subdiagram of* D_1 *in rows* $1, 3$ *and columns* $3, 4$ *gives* $S^{(4,2,2)} \oplus S^{(3,3,2)} \hookrightarrow S^{D_1}$*.*

In fact, both these inclusions are isomorphisms, as one can check with the Littlewood-Richardson rule since D *is equivalent to the skew shape* $(5, 3, 3)/(2, 1)$*. Alternatively, one can compute the Edelman-Greene tableaux of* 4317256 *and look at their (transposed) shapes:*

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 2 & 3 & 5 \\ \hline 3 & 6 & \\ \cline{1-2} \end{array} \qquad \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 2 & 3 & 5 \\ \hline 3 & \multicolumn{2}{c}{} \\ \cline{1-1} 6 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 2 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2} 6 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 2 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & \multicolumn{2}{c}{} \\ \cline{1-1} 5 & \multicolumn{2}{c}{} \\ \cline{1-1} 6 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \;.$$

Even without the conditions of Theorem 3.2, applying a James-Peel move always gives an inclusion of Specht modules (over $\mathbb{C}$).

Lemma 3.4 *For any positive integers* a, b*,* $S^{R_{a\to b}D} \hookrightarrow S^D$ *and* $S^{C_{a\to b}D} \hookrightarrow S^D$*.*

James-Peel moves and Theorem 3.2 present one possible way to decompose a Specht module into irreducibles. In general it is not known if an arbitrary Specht module can be decomposed by finding some appropriate tree of James-Peel moves, as the surjections in Theorem 3.2 may be not be isomorphisms. The way we prove Theorem 1.2 is to find such a tree for the case of $D(w)$. The usefulness of James-Peel moves for us comes from the fact that they are well-behaved with respect to subdiagram inclusion, and pattern inclusion for permutations corresponds to subdiagram inclusion on the level of permutation diagrams.

To be more precise about this, we make the following definition.

Definition 3.5 *A* James-Peel tree *for a diagram* D *is a rooted tree* $\mathcal{T}$ *with vertices labeled by diagrams and edges labeled by sequences of James-Peel moves, satisfying the following conditions:*

- *The root of* $\mathcal{T}$ *is* D*.*
- *If* B *is a child of* A *with a sequence* $\mathbf{JP}$ *of James-Peel moves labeling the edge* A*—*B*, then* $B = \mathbf{JP}(A)$*.*

- *If A has more than one child, these children arise as a result of applying Theorem 3.2 to A. That is, A contains $\delta_p \cdot (1)$ as a subdiagram in rows $\alpha(1) < \cdots < \alpha(p)$ and columns $\beta(1) < \cdots < \beta(p)$, and each edge leading down from A is labeled $R_{\alpha(p)\to\alpha(p-j+1)}C_{\beta(p)\to\beta(j)}$ for some distinct values $1 \le j \le p$ (perhaps not all such j appear).*

Theorem 3.2 and Lemma 3.4 immediately imply the following statement.

Lemma 3.6 *If D has a James-Peel tree $\mathcal{T}$ with leaves $A_1, \ldots, A_n$, then $\bigoplus_i S^{A_i} \hookrightarrow S^D$.*

Definition 3.7 *A James-Peel tree $\mathcal{T}$ for D is* complete *if its leaves $A_1, \ldots, A_n$ are equivalent to Ferrers diagrams of partitions and $S^D \simeq \bigoplus_i S^{A_i}$.*

In James and Peel (1979), an algorithm is given which constructs a complete James-Peel tree when D is a skew shape. More generally, Reiner and Shimozono (1995a) construct a complete James-Peel tree for any *column-convex* diagram: a diagram D for which $(a, x), (b, x) \in D$ with $a < b$ implies $(i, x) \in D$ for all $a < i < b$. In the next section we construct a complete James-Peel tree for the diagram of a permutation, so it's worth noting that neither of these classes of diagrams contains the other. For example, $D(37154826)$ is not equivalent to any column-convex or row-convex diagram, while the column-convex diagram

$$\begin{array}{cccc} \circ & \circ & \cdot & \cdot \\ \circ & \cdot & \circ & \cdot \\ \circ & \cdot & \cdot & \circ \end{array}$$

is not equivalent to the diagram of any permutation.

Notice that if w contains a pattern v, then $D(v)$ is (up to reindexing) a subdiagram of $D(w)$. Specifically, if v appears in positions $i_1, \ldots, i_k$ of w, then $D(v)$ is the subdiagram of $D(w)$ induced by the rows $i_1, \ldots, i_k$ and columns $w(i_1), \ldots, w(i_k)$.

Let $I(D)$ denote the multiset of partitions of $n = |D|$ such that $S^D \simeq \bigoplus_{\lambda \in I(D)} S^\lambda$.

Lemma 3.8 *Suppose D' is a subdiagram of D and that D' has a complete James-Peel tree. Then there is an injection $\iota : I(D') \hookrightarrow I(D)$ such that if $\lambda \in I(D')$, then $\lambda \subseteq \iota(\lambda)$. Moreover, ι depends only on shape, in the sense that if λ appears m times in $I(D')$, then $\iota(\lambda)$ appears at least m times in $I(D')$.*

In particular, taking $D = D(w)^t$ and $D' = D(v)^t$ for v a pattern in w, Lemma 3.8 together with the equalities

$$s_{D(w^{-1})} = F_w = \sum_{P \in EG(w)} s_{\mathrm{shape}(P)} \tag{10}$$

will immediately imply Theorem 1.2 once we show that $D(w)$ always has a complete James-Peel tree.

The main idea of the proof of Lemma 3.8 is to view the James-Peel tree $\mathcal{T}'$ for D' as a James-Peel tree $\mathcal{T}$ for D, via the inclusion $D' \subseteq D$. One can write down an explicit formula for ι, although in general there will be many possible choices.

Example 3.9 *Take $D = D(w)^t$ where $w = 4317256$ as in Example 3.3, so $F_w = s_{D(w)^t} = s_{332} + s_{3311} + s_{3221} + s_{32111}$. Write $w^i = fl(w(1) \cdots \widehat{w(i)} \cdots w(n))$. We give $F_{w^i} = s_{D(w^i)^t}$ for several i, and line up each term s_λ in F_{w^i} with the corresponding term $s_{\iota(\lambda)}$ in F_w, for a particular choice of ι, so $D' = D(w^i)^t$.*

$$\begin{array}{lllllllll} F_w & = & s_{332} & + & s_{3311} & + & s_{3221} & + & s_{32111} \\ F_{w^1} & = & s_{221} & + & & & s_{2111} & & \\ F_{w^4} & = & s_{221} & & & & & & \\ F_{w^5} & = & s_{32} & + & s_{311} & + & s_{221} & + & s_{2111} \end{array}$$

Note that if F_w is multiplicity-free, as in this example, an injection $\iota : I(D(v)^t) \hookrightarrow I(D(w)^t)$ uniquely defines an injection $EG(v) \hookrightarrow EG(w)$.

4 Transitions as James-Peel moves

Recall the following notation from Section 2.3. Given a permutation w, take r maximal with $w(r) > w(r+1)$, then $s > r$ maximal with $w(s) < w(r)$. The set of transitions of w is $T(w) = \{wt_{rs}t_{rj} : \ell(wt_{rs}t_{rj}) = \ell(w)\}$, or else $T(1 \times w)$ if the set on the right is empty.

Upon taking diagrams of permutations, one finds that transitions correspond to certain sequences of James-Peel moves.

Lemma 4.1 *Given a permutation w, let r, s be as above and take $v = wt_{rs}t_{rj} \in T(w)$. Then*

$$D(v) = R_{r\to j}C_{w(s)\to w(j)}D(w) = C_{w(s)\to w(j)}R_{r\to j}D(w). \tag{11}$$

Theorem 4.2 *For a permutation w, the diagram $D(w)$ has a complete James-Peel tree.*

Proof sketch: If w has p transitions, then $D(w)$ contains $\delta_p \cdot (1)$ as a certain subdiagram. Lemma 4.1 shows that the diagrams arising from applying Theorem 3.2 to this subdiagram are, up to equivalence, exactly the diagrams $D(v)$ as v runs over $T(w)$. This implies that replacing permutations by diagrams in the Lascoux-Schützenberger tree and labeling edges with James-Peel moves according to Lemma 4.1 yields a James-Peel tree for $D(w)$. This tree is complete because $s_{D(w)} = \sum_v s_{D(v)}$ for v running over the leaves of the L-S tree. □

Lemma 3.8 and the discussion following it now imply Theorem 1.2.

Corollary 4.3 *If a permutation w is k-vexillary and v is a pattern in w, then v is k-vexillary.*

Remark 4.4 Theorem 1.2 shows the existence of an injection $EG(v) \hookrightarrow EG(w)$ which respects inclusion of shapes for v a pattern contained in w, but an explicit map on tableaux is lacking. The Edelman-Greene correspondence shows that this is equivalent to an injection $\mathrm{Red}(v) \hookrightarrow \mathrm{Red}(w)$ which is an inclusion on the shapes of Edelman-Greene insertion tableaux. The characterization in Tenner (2006) of vexillary permutations yields an explicit injection in the case where v is vexillary.

5 k-vexillary permutations

In this section we show that the property of k-vexillarity is characterized by avoiding a finite set of patterns for any k. The key step is to remove some inessential moves from the James-Peel tree for $D(w)$, namely those which only permute rows or columns.

If D is an arbitrary diagram, and σ, τ are permutations, let $(\sigma, \tau)D$ be the diagram $\{(\sigma(i), \tau(j)) : (i, j) \in D\}$. Given a James-Peel tree $\mathcal{T}$ for D, let $(\sigma, \tau)\mathcal{T}$ denote the James-Peel tree for $(\sigma, \tau)D$ gotten by replacing every James-Peel move $R_{x \to y}$ labeling an edge of $\mathcal{T}$ by $R_{\sigma(x) \to \sigma(y)}$, and every move $C_{x \to y}$ by $C_{\tau(x) \to \tau(y)}$, and relabeling vertices accordingly. Whenever a move labeling an edge e of a James-Peel tree just permutes rows or columns, we can eliminate that move from the tree at the cost of relabeling rows and columns of James-Peel moves below e.

Definition 5.1 *Given a James-Peel tree $\mathcal{T}$ of a diagram D, the* reduced James-Peel tree $\operatorname{red}(\mathcal{T})$ *of D is defined inductively as follows.*

- *If D has no children in $\mathcal{T}$, then* $\operatorname{red}(\mathcal{T}) = \mathcal{T}$.
- *If D has just one child F, and $D = (\sigma, \tau)F$ for some $\sigma, \tau \in S_\infty$, let $\mathcal{T}_1$ be the subtree of $\mathcal{T}$ below F with root F. Then* $\operatorname{red}(\mathcal{T}) = (\sigma, \tau) \operatorname{red}(\mathcal{T}_1)$.
- *If D has at least two children $F_1, F_2, \ldots, F_p$ or D has one child F_1 not equivalent to D, let $\mathcal{T}_i$ be the subtree of $\mathcal{T}$ below F_i with root F_i. Then* $\operatorname{red}(\mathcal{T})$ *is $\mathcal{T}$ with each $\mathcal{T}_i$ replaced by* $\operatorname{red}(\mathcal{T}_i)$.

Note that $\operatorname{red}(\mathcal{T})$ is still a James-Peel tree for D. As equivalent diagrams have isomorphic Specht modules, if $\mathcal{T}$ is complete then so is $\operatorname{red}(\mathcal{T})$.

Definition 5.2 *A rooted tree is* bushy *if every non-leaf vertex has at least two children.*

If a James-Peel tree has a vertex A with just one child B, but A and B are not equivalent, the tree cannot be complete. This implies:

Lemma 5.3 *If $\mathcal{T}$ is a complete James-Peel tree, then* $\operatorname{red}(\mathcal{T})$ *is bushy.*

A bushy tree cannot have too many more edges than leaves.

Lemma 5.4 *The largest number of edges possible in a bushy tree with k leaves is $2k - 2$.*

Let $JP(w)$ be the James-Peel tree for $D(w)$ constructed in Theorem 4.2, and $RJP(w) = \operatorname{red}(JP(w))$. Suppose $\mathcal{T}$ is a subtree of $RJP(w)$ with root $D(w)$. Let $R(\mathcal{T})$ be the union of $\{a, b\}$ over all moves $R_{a \to b}$ appearing in $\mathcal{T}$, and $C(\mathcal{T})$ the union of $\{c, d\}$ over all $C_{c \to d}$ appearing in $\mathcal{T}$. Write $R(\mathcal{T}) \cup w^{-1}C(\mathcal{T}) = \{i_1 < \cdots < i_r\}$, and define a permutation

$$w_{\mathcal{T}} = fl(w(i_1) \cdots w(i_r)).$$

Remark 5.5 *In Section 2 we noted that, for convenience, w could be replaced by $1^m \times w$ to remove the necessity of sometimes replacing v by $1 \times v$ in the Lascoux-Schützenberger tree. The definition of $w_{\mathcal{T}}$ above is then an abuse of notation, since we are really taking a subsequence of $1^m \times w$. However, rows and columns $1, \ldots, m$ of $D(w)$ are empty, so are not affected at all by the James-Peel moves in $RJP(w)$ or $\mathcal{T}$. This means that the subsequence defining $w_{\mathcal{T}}$ occurs entirely after the mth position of $1^m \times w$, so we are free to shift it down by m and consider it as a subsequence of w. This applies also to Theorems 5.8 and 5.9 below.*

Each edge of $JP(w)$ has a label RC for some row and column moves R, C. Some of these moves end up being equivalences, and so are lost in $RJP(w)$. Thus, $RJP(w)$ has edges with labels of the form RC, R, or C—in fact, each internal vertex always has one R-edge, one C-edge, and the remaining edges are RC-edges. We are interested in controlling the number of letters in $w_{\mathcal{T}}$, which is at most $|R(\mathcal{T})| + |C(\mathcal{T})|$. This motivates the next definition.

Definition 5.6 *A subtree $\mathcal{T}$ of $RJP(w)$ with root $D(w)$ is* colorful *if each non-leaf vertex of $\mathcal{T}$ has at least the two children corresponding to its R-edge and its C-edge. Thus, colorful implies bushy.*

Lemma 5.7 *Say $\mathcal{T}$ is a subtree of $RJP(w)$ rooted at $D(w)$ with k leaves. Then $k \leq |EG(w_{\mathcal{T}})| \leq |EG(w)|$. If $\mathcal{T}$ is colorful, then $w_{\mathcal{T}} \in S_m$ for some $m \leq 4k-4$.*

Proof sketch: Up to relabeling rows and columns to account for flattening, the tree $\mathcal{T}$ is a James-Peel tree for $D(w_{\mathcal{T}})$ (not necessarily complete), so $k \leq EG(w_{\mathcal{T}})$. Theorem 1.2 implies $|EG(w_{\mathcal{T}})| \leq |EG(w)|$.

If F is a vertex of $\mathcal{T}$ with p children, a careful count shows that the edge labels leading down from F contribute at most p elements to each of $|R(\mathcal{T})|$ and $|C(\mathcal{T})|$. Summing over all vertices leads to the result. □

In particular, taking $\mathcal{T} = RJP(w)$ in Lemma 5.7 gives the following result.

Theorem 5.8 *Any permutation w contains a pattern $v \in S_m$ such that $|EG(w)| = |EG(v)|$, for some $m \leq 4|EG(w)|-4$.*

More generally, Lemma 5.7 lets us show that k-vexillarity is characterized by avoiding a *finite* set of patterns.

Theorem 5.9 *Say w is a permutation with $|EG(w)| > k$. Then w contains a pattern $v \in S_m$ such that $|EG(v)| > k$, for some $m \leq 4k$.*

Proof: By Lemma 5.7, it is enough to exhibit a colorful subtree of $RJP(w)$ with $k+1$ leaves. Such a tree is not difficult to construct by choosing one edge at a time. □

Corollary 5.10 *A permutation w is k-vexillary if and only if it avoids all non-k-vexillary patterns in S_m for $1 \leq m \leq 4k$.*

For $k = 2$, we can explicitly find all non-2-vexillary patterns in S_m for $1 \leq m \leq 8$ and eliminate those containing a smaller non-2-vexillary pattern to find a minimal list.

Theorem 5.11 *A permutation w is 2-vexillary if and only if it avoids all of the following 35 patterns.*

21543	231564	315264	5271436	26487153	54726183	64821537
32154	241365	426153	5276143	26581437	54762183	64872153
214365	241635	2547163	5472163	26587143	61832547	65821437
214635	312645	4265173	25476183	51736284	61837254	65827143
215364	314265	5173264	26481537	51763284	61873254	65872143

This process is also feasible for $k = 3$, in which case we need to look at non-3-vexillary patterns up through S_{12}. Here we find that the bound in Corollary 5.10 is not sharp.

Theorem 5.12 *A permutation w is 3-vexillary if and only if it avoids a list of 91 patterns in $S_6 \cup S_7 \cup S_8$.*

Searching through all non-4-vexillary permutation in S_{16} is currently beyond our computational capabilities. However, one does find that every non-4-vexillary permutation in S_{13} contains a proper non-4-vexillary pattern.

Conjecture 2 *A permutation w is 4-vexillary if and only if it avoids a list of 2346 patterns in $S_6 \cup S_7 \cup \cdots \cup S_{12}$.*

Unless n is large compared to k, our pattern characterizations are less efficient for checking that $w \in S_n$ is k-vexillary than the Lascoux-Schützenberger tree. On the other hand, pattern characterizations give an easy way to compare theorems. As an example, the *essential set* $Ess(w)$ of a permutation w is the set of southeast corners of connected components of $D(w)$. Fulton (1992) introduced the essential set and showed that the rank conditions for the Schubert variety indexed by w need only be checked at cells in the essential set. Making $Ess(w)$ into a poset by $(i_1, j_1) \leq (i_2, j_2)$ if $i_1 \geq i_2$ and $j_1 \leq j_2$, Fulton also showed that w is vexillary if and only if $Ess(w)$ is a chain.

Similarly, one can check that the property that $Ess(w)$ is a union of two chains is characterized by w avoiding a specific set of patterns. All of these patterns turn out to be non-3-vexillary, and then Theorem 1.2 shows that if w is 3-vexillary, $Ess(w)$ is a union of two chains.

6 Future work

We were led to Theorem 1.2 by trying to prove Lemma 3.8 for arbitrary diagrams and subdiagrams. Lemma 3.8 holds when the subdiagram is (isomorphic to) a permutation diagram, a skew shape, or a column-convex diagram, since these diagrams all admit complete James-Peel trees. The algorithm given by Reiner and Shimozono in Reiner and Shimozono (1998) for decomposing Specht modules shows that the conclusion of Lemma 3.8 also holds when D is percent-avoiding and $D' = D \cap \{i : a \leq i \leq b\} \times \{j : c \leq j \leq d\}$ for some a, b, c, d.

We have no simple characterizations of the lists of patterns arising from Corollary 5.10 and Theorems 5.11 and 5.12. One necessary condition for w to be non-k-vexillary but contain only k-vexillary patterns is that every $w(i)$ participates in some 2143 pattern. Otherwise, the ith row and $w(i)$th column of $D(w)$ are contained in or contain every other row and column, and so they do not participate in the James-Peel moves of $RJP(w)$. This is far from sufficient, however.

In Billey and Lam (1998), vexillary signed permutations of types B, C, D in the hyperoctahedral group are defined as those whose Stanley symmetric function is equal to a single Schur P- or Q-function (P in types B, D, and Q in type C), and it is shown that the vexillary signed permutations are again characterized by avoiding a finite set of patterns. Computer calculations show that Corollary 4.3 with $k = 2$ holds in B_9 for types B, C and in D_8; moreover, the 2-vexillary patterns in B_9 of types B, C are characterized by avoiding sets of patterns in $B_3 \cup \cdots \cup B_8$. The main obstacle to extending our proofs to these other root systems is the apparent lack of an analogue of the Specht module of a diagram. In a recent preprint, Anderson and Fulton (2012) give a different definition of vexillary permutations in types B, C, D, and one might ask if there is a reasonable notion of k-vexillary in their setting.

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Rainbow supercharacters and a poset analogue to q-binomial coefficients

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Abstract. This paper introduces a variation on the binomial coefficient that depends on a poset and interpolates between q-binomials and 1-binomials: a total order gives the usual q-binomial, and a poset with no relations gives the usual binomial coefficient. These coefficients arise naturally in the study of supercharacters of the finite groups of unipotent upper-triangular matrices, whose representation theory is dictated by the combinatorics of set partitions. In particular, we find a natural set of modules for these groups, whose characters have degrees given by q-binomials, and whose decomposition in terms of supercharacters are given by poset binomial coefficients. This results in a non-trivial family of formulas relating poset binomials to the usual q-binomials.

Résumé. Cet article présente une variation des coefficients binomiaux qui dépend d'un ordre partiel et qui interpole entre les q-binômes et les 1-binômes: un ordre total donne les q-binômes habituelles, et un ordre partiel sans relations donne les coefficients binomiaux habituelles. Ces coefficients apparaissent naturellement dans l'étude des supercaractères de groupes finis de matrices unipotentes, triangulaires supérieures, dont la représentation est dictée par la théorie de la combinatoire des partitions d'ensembles. En particulier, nous trouvons un ensemble naturel de modules pour ces groupes, dont les caractères ont des degrés donnés par les q-binômes, et dont les décompositions en termes de supercaractères sont donnés par les coefficients binômiaux d'ordres partiels. Cela donne une famille non-trivial de formules qui relient les binômes d'ordres partiels aux q-binômes habituelles.

Keywords: q-binomials, posets, supercharacters, set partitions

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1 Introduction

The supercharacters of the finite groups of unipotent uppertriangular matrices U_n have seen an increased amount of attention in recent years. While the representation theory of these groups is well-known to be wild, André [2] gave a way to coarsen our notion of an irreducible module in such a way the resulting theory is much more tractable (this notion is generalized to arbitrary finite groups in [5]). More specifically, if $\mathrm{cf}(U_n)$ is the space of class functions of U_n, then André defines a subspace $\mathrm{scf}(U_n)$ which retains many of the nice properties enjoyed by $\mathrm{cf}(U_n)$. Of computational importance is that while the dimension of $\mathrm{cf}(U_n)$ is unknown, the dimension of $\mathrm{scf}(U_n)$ is the nth Bell number.

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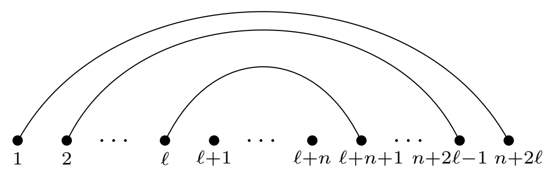
At an American Institute of Mathematics workshop, the participants [1] showed that gluing all the supercharacter theories together

$$\mathrm{scf} = \bigoplus_{n\geq 0} \mathrm{scf}(U_n)$$

gives a Hopf algebra isomorphic to the Hopf algebra of symmetric functions in noncommuting variables. The space scf has two natural bases: one comes from the supercharacters themselves, and certain unions of conjugacy classes (called superclasses) imposed by the supercharacters. The isomorphism in [1] relates the superclass basis to the usual basis of monomial symmetric functions (given, for example, in [6]). While other bases of scf have been studied (such as [3]), the supercharacter basis remains somewhat mysterious in its relation to symmetric functions. One obstruction is that the coproduct – coming from the restriction functor on representations – is not well-understood.

This paper studies the restriction functor on a particular family of supercharacters, which we call "rainbow" supercharacters. They correspond to set partitions of $\{1, 2, \ldots, n\}$ of the form $\{\{1, n+2\ell\}, \{2, n+2\ell-1\}, \ldots, \{\ell, \ell+n+1\}, \{\ell+1\}, \ldots, \{\ell+n\}\}$ for some $n, \ell \in \mathbb{Z}_{\geq 0}$, which we view as



We are interested in computing the restriction of the corresponding supercharacter to the subgroup

$$U_n \cong U_{[l+1,l+n]} = \left(\begin{array}{c|c|c} \mathrm{Id}_\ell & 0 & 0 \\ \hline 0 & U_n & 0 \\ \hline 0 & 0 & \mathrm{Id}_\ell \end{array}\right) \subseteq U_{2\ell+n}. \tag{1}$$

The restriction problem for these characters is already quite complicated. While there is a recursive algorithm for computing the restriction [7], this algorithm does not give an obvious interpretation for the coefficients.

The main result of this paper observes that these supercharacters in fact restrict to a space

$$\mathbb{Z}_{\geq 0}\text{-span}\{\psi_0^{(n)}, \psi_1^{(n)}, \ldots, \psi_n^{(n)}\} \subseteq \mathrm{scf}(U_n),$$

spanned by a natural family of characters whose modules are easy to find. Furthermore, they have a combinatorial decomposition in terms of supercharacters with coefficients given by a poset based generalization of q-binomial coefficients. These two facts then give an easy way to compute the coefficients of the restrictions of characters.

The methods from this paper seem to lend themselves to more general statements, and we hope these results will extend to the general restriction problem.

2 Preliminaries

This section reviews our version of set partitions and the particular supercharacter theory that we will use for our main results.

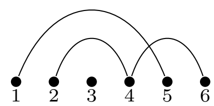
2.1 Set partition combinatorics

A *set partition* λ *of* $\{1, 2, \ldots, n\}$ is a set of pairs $i \frown j$ with $1 \leq i < j \leq n$ such that if $i \frown j, k \frown l \in \lambda$, then $i = k$ if and only if $j = l$. We typically refer to $i \frown j$ as an *arc* of λ, i as the *left endpoint* of $i \frown j$ and j as the *right endpoint* of $i \frown j$. Let

$$\mathcal{S}_n = \{\text{set partitions of } \{1, 2, \ldots, n\}\}.$$

We typically view these set partitions diagrammatically as a family of arcs on a row of n nodes so that if $i \frown j \in \lambda$, then there is an arc connecting the ith node to the jth node. For example,

$$\lambda = \{1 \frown 5, 2 \frown 4, 4 \frown 6\} \longleftrightarrow$$



We can obtain the more traditional version of set partitions as follows. For $\lambda \in \mathcal{S}_n$, define the *blocks* $\mathrm{bl}(\lambda)$ of λ to be the set of equivalence classes on $\{1, 2, \ldots, n\}$ given by the transitive closure of $i \sim j$ if $i \frown j \in \lambda$. For example,

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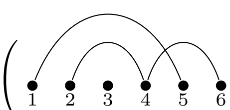


$$) = \{\{1, 5\}, \{2, 4, 6\}, \{3\}\}.$$

A *crossing* in a set-partition λ is a pair of arcs $i \frown k, j \frown l \in \lambda$ such that $i < j < k < l$. A *nesting* in a set partition λ is a pair of arcs $i \frown l, j \frown k \in \lambda$ such that $i < j < k < l$. For $\lambda, \mu \in \mathcal{S}_n$, let

$$\begin{aligned} \mathrm{nst}_\mu^\lambda &= \{(i \frown l, j \frown k) \in \lambda \times \mu \mid i < j < k < l\} \\ \mathrm{crs}(\lambda) &= \{(i \frown k, j \frown l) \in \lambda \times \lambda \mid i < j < k < l\}. \end{aligned}$$

Define the set of noncrossing set-partitions to be

$$\mathcal{NCS}_n = \{\lambda \in \mathcal{S}_n \mid \mathrm{crs}(\lambda) = \emptyset\}.$$

Define a function

$$\mathrm{uncr} : \mathcal{S}_n \longrightarrow \mathcal{NCS}_n$$

where $\mathrm{uncr}(\lambda)$ is the unique set partition in $\mathcal{NCS}_n$ that has the same left endpoints and the same right endpoints as λ (in $\mathrm{uncr}(\lambda)$, the first right end-point from the left must be connected to the closest left endpoint, etc.).

Remark 2.1 One can also obtain $\mathrm{uncr}(\lambda)$ from λ by iteratively uncrossing each crossing $\{i \frown k, j \frown l\}$ into a nesting $\{i \frown l, j \frown k\}$. Since this map changes neither the set of left endpoints nor the set of right endpoints, the order in which we "resolve" the crossings does not matter.

2.2 Supercharacters of U_n

A *supercharacter theory* $\mathrm{scf}(G)$ of a finite group G is a subspace of the space of class functions $\mathrm{cf}(G)$ of G such that $\mathrm{scf}(G)$ is a $\mathbb{C}$-subalgebra under the two products

$$\begin{array}{rccc} \circ: & \mathrm{cf}(G)\otimes\mathrm{cf}(G) & \longrightarrow & \mathrm{cf}(G) \\ & \chi\otimes\psi & \mapsto & \begin{array}{rl} \chi\circ\psi: G \to & \mathbb{C} \\ g \mapsto & \frac{1}{|G|}\sum_{h\in G}\chi(h)\overline{\psi(h^{-1}g)} \end{array} \end{array}$$

and

$$\begin{array}{rccc} \odot: & \mathrm{cf}(G)\otimes\mathrm{cf}(G) & \longrightarrow & \mathrm{cf}(G) \\ & \chi\otimes\psi & \mapsto & \begin{array}{rl} \chi\odot\psi: G \to & \mathbb{C} \\ g \mapsto & \chi(g)\psi(g). \end{array} \end{array}$$

Remark 2.2 This definition gives a supercharacter theory as a special kind of S-ring. There is an alternate definition by Diaconis–Isaacs [5] which stresses the partitions of G and $\mathrm{Irr}(G)$ that arise out of the two distinguished bases, described below.

Every such a supercharacter theory has two distinguished bases (up to scalar multiples) given by

$$\begin{aligned} \mathrm{scf}(G) &= \mathbb{C}\text{-span}\{\kappa_A \mid A\in\mathcal{K}\} \\ &= \mathbb{C}\text{-span}\{\sum_{\psi\in B}\psi(1)\psi \mid B\in\mathcal{X}\}, \end{aligned}$$

where $\mathcal{K}$ is a partition of G, $\mathcal{X}$ is a partition of $\mathrm{Irr}(G)$, and for $g\in G$,

$$\kappa_A(g) = \begin{cases} 1 & \text{if } g\in A, \\ 0, & \text{if } g\notin A \end{cases}$$

We typically call the blocks of $\mathcal{K}$ *superclasses*. For each $B\in\mathcal{X}$ we choose constants $c_B\in\mathcal{Q}_{>0}$ such that

$$\chi^B = c_B\sum_{\psi\in B}\psi(1)\psi$$

is a character. The resulting χ^B are called *supercharacters*.

Let U_n be the subgroup of unipotent upper-triangular matrices of the general linear group $\mathrm{GL}_n(\mathbb{F}_q)$ over the finite field $\mathbb{F}_q$ with q elements, and let $T_n\subseteq\mathrm{GL}_n(\mathbb{F}_q)$ be the subgroup of diagonal matrices. While there are many supercharacter theories of U_n, we will focus on the supercharacter theory $\mathrm{scf}(U_n)$ obtained by the following superclasses. Let $u,v\in U_n$ be in the same superclass if there exist $a,b\in U_n$ and $t\in T_n$ such that

$$u = ta(v-\mathrm{Id}_n)bvt^{-1}+\mathrm{Id}_n.$$

Remark 2.3 This supercharacter theory is slightly coarser than the canonical supercharacter theory for algebra groups given by [5], since we let T_n act by outer automorphisms on the usual algebra group supercharacter theory.

The dimension of $\mathrm{scf}(U_n)$ turns out to be Bell number $|\mathcal{S}_n|$. In fact, for every superclass of U_n there exists $\mu \in \mathcal{S}_n$ and a distinguished element u_μ in the superclass such that

$$(u_\mu)_{ij} = \begin{cases} 1, & \text{if } i \frown j \in \mu \text{ or } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

For each $\lambda, \mu \in \mathcal{S}_n$, the supercharacter corresponding to λ is given by

$$\chi^\lambda(\mu) = \chi^\lambda(u_\mu) = \begin{cases} \dfrac{(q-1)^{|\lambda-\mu|} q^{\dim(\lambda)-|\lambda|}(-1)^{|\lambda\cap\mu|}}{q^{\mathrm{nst}_\mu^\lambda}} & \text{if } i \frown l \in \lambda \text{ and } i < j < l \\ & \text{implies } i \frown j, j \frown l \notin \mu, \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

where

$$\dim(\lambda) = \sum_{i \frown j \in \lambda} j - i.$$

In particular, note that the trivial character $\mathbb{1}$ is the supercharacter associated with the empty partition $\emptyset_n \in \mathcal{S}_n$, and

$$\chi^\lambda(1) = (q-1)^{|\lambda|} q^{\dim(\lambda)-|\lambda|}.$$

Our primary interest is in rainbow characters. That is, for $\ell, n \in \mathbb{Z}_{\geq 0}$, consider

$$\chi^{\{1 \frown (2\ell+n), 2 \frown (2\ell+n-1), \ldots, \ell \frown \ell+n+1\}},$$

and it follows from the character formula (2), that if U_n is as in (1) and $\mu \in \mathcal{S}_n$, then

$$\begin{aligned} \mathrm{Res}^{U_{2\ell+n}}_{U_n}(\chi^{\{1 \frown (2\ell+n), 2 \frown (2\ell+n-1), \ldots, \ell \frown (\ell+n+1)\}})(\mu) &= q^{2\binom{\ell}{2}+\ell(n-|\mu|)}(q-1)^\ell \\ &= q^{2\binom{\ell}{2}} \mathrm{Res}^{U_{2\ell+n}}_{U_n}(\underbrace{\chi^{\ell \frown (\ell+n+1)} \odot \cdots \odot \chi^{\ell \frown (\ell+n+1)}}_{\ell \text{ terms}}), \end{aligned}$$

where $\odot$ is the point-wise product on functions. We will therefore focus on the somewhat notationally simpler restriction

$$\mathrm{Res}^{U_{n+2}}_{U_n}(\underbrace{\chi^{1 \frown (n+2)} \odot \cdots \odot \chi^{1 \frown (n+2)}}_{\ell \text{ terms}}).$$

3 Poset q-Binomials

This section defines the analogue of q-binomials central to this paper. We then restrict our attention to the particular family of posets arising from set partitions that is our focus in the next section.

3.1 A poset variation on q-binomial coefficients

Let $\mathcal{P}$ be a poset on n objects. Given a subset $A \subseteq \mathcal{P}$, let

$$\mathrm{wt}(A) = \sum_{a \in A} \mathrm{wt}(a), \qquad \text{where} \qquad \mathrm{wt}(a) = \#\{b \in \mathcal{P} \mid b \succ_{\mathcal{P}} a\}.$$

For $n, k \in \mathbb{Z}_{\geq 0}$, the $\mathcal{P}$-binomial coefficient is

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathcal{P}} = \sum_{\substack{S \subseteq \mathcal{P} \\ |S|=k}} q^{\mathrm{wt}(S)}. \tag{3}$$

Note that $\mathrm{wt}(a)$ can also be expressed in terms of the size of the upper ideal containing a.

These coefficients satisfy many recursive relations.

Proposition 3.1 *Let $a \in \mathcal{P}$, $\mathcal{P}' = \mathcal{P} - \{a\}$ and $A = \{a' \mid a' \prec a\}$. Then*

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathcal{P}} = \sum_{j=0}^{k} \binom{|A|}{j} q^j \left(q^{\mathrm{wt}(b)} \begin{bmatrix} n - |A| - 1 \\ k - j - 1 \end{bmatrix}_{\mathcal{P}'} + \begin{bmatrix} n - |A| - 1 \\ k - j \end{bmatrix}_{\mathcal{P}'} \right).$$

However, perhaps the most useful one corresponds to removing some minimal element in the poset.

Corollary 3.2 *Let b be a minimal element of a poset $\mathcal{P}$. Let $\mathcal{P}' = \mathcal{P} - \{b\}$. Then*

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathcal{P}} = q^{\mathrm{wt}(b)} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{\mathcal{P}'} + \begin{bmatrix} n-1 \\ k \end{bmatrix}_{\mathcal{P}'}.$$

Examples.

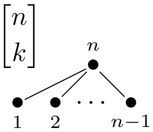
- If $\mathcal{P}$ is the poset with no relations on n elements, then

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathcal{P}} = \sum_{\substack{S \subseteq \mathcal{P} \\ |S|=k}} q^0 = \binom{n}{k}.$$

- If $\mathcal{P}$ is a total order (say $1 < 2 < \cdots < n$), then

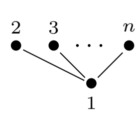
$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathcal{P}} = \sum_{\substack{S \subseteq \mathcal{P} \\ |S|=k}} q^{\mathrm{wt}(S)} = q^{1+2+\cdots+(k-1)} \sum_{1 \leq a_1 < a_2 < \cdots < a_k \leq n} q^{n-a_1-(k-1)+n-a_2-(k-2)+\cdots+n-a_k-0} = q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q.$$

- For the posets with nearly no relations we have

$$\begin{bmatrix} n \\ k \end{bmatrix}$$


$$= \sum_{\substack{S \subseteq \mathcal{P} \\ |S|=k, n \in S}} q^{k-1} + \sum_{\substack{S \subseteq \mathcal{P} \\ |S|=k, n \notin S}} q^k = q^{k-1} \left(\binom{n-1}{k-1} + q \binom{n-1}{k} \right).$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}$$


$$= \sum_{\substack{S \subseteq \mathcal{P} \\ |S|=k, 1 \in S}} q^{n-1} + \sum_{\substack{S \subseteq \mathcal{P} \\ |S|=k, 1 \notin S}} q^0 = q^{n-1} \binom{n-1}{k-1} + \binom{n-1}{k}.$$

- However, note that in general $\begin{bmatrix} n \\ k \end{bmatrix}_{\mathcal{P}} \neq \begin{bmatrix} n \\ n-k \end{bmatrix}_{\mathcal{P}}$, though the coefficient sequences of the polynomial of one is the reverse coefficient sequence of the other. That is, if

$$b_{\mathcal{P}}(n,k,r,s) = \sum_{\substack{A \sqcup B = \{1,2,\ldots,n\} \\ |A|=k, |B|=n-k}} r^{\mathrm{wt}(A)} s^{\mathrm{wt}(B)},$$

then

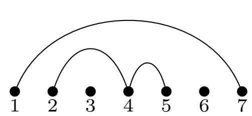
$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathcal{P}} = b_{\mathcal{P}}(n,k,q,1) \qquad \text{and} \qquad \begin{bmatrix} n \\ n-k \end{bmatrix}_{\mathcal{P}} = b_{\mathcal{P}}(n,k,1,q).$$

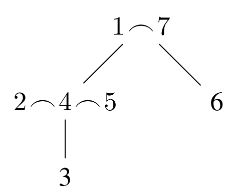
3.1.1 Main example

Let $\lambda \in \mathcal{NCS}_n$. Since there are no crossings in λ, we can define a poset depending on λ based on whether blocks are nested or not. Let $\mathcal{P}(\lambda)$ be the poset on $\mathrm{bl}(\lambda)$ given by $a \prec b$ if either

- $|a| > 1$ and there exist $j, k \in a$ and $i, l \in b$ such that $i < j < k < l$, or
- $a = \{j\}$ and there exist $i, k \in b$ such that $i < j < k$.

For example,

$\lambda =$ 

gives $\mathcal{P}(\lambda) =$ 

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Note that for each $\lambda \in \mathcal{NCS}_n$ the poset $\mathcal{P}(\lambda)$ is a forest where each connected component has a unique maximal element. In fact, all such forests arise in this way (as n and λ vary).

We obtain a function

$$\mathcal{S}_n \xrightarrow{\mathrm{uncr}} \mathcal{NCS}_n \xrightarrow{\mathcal{P}} \left\{ \begin{matrix} \text{forests where each} \\ \text{connected component has} \\ \text{a unique maximal element} \end{matrix} \right\}.$$

The following lemma describes a recursion on these binomial coefficients that correspond to small changes in the arcs of the corresponding set partitions (in particular, it allows us to make arcs smaller).

Lemma 3.3 *Fix* $1 \leq i < l-1 < n$. *Let* $\lambda' \in \mathcal{S}_n$ *be such that*

$$\lambda = \lambda' \cup \{i \frown l\} \in \mathcal{S}_n, \quad \mu = \lambda' \cup \{i \frown (l-1)\} \in \mathcal{S}_n, \quad \textit{and} \quad \nu = \lambda' \cup \{i \frown (l-1) \frown l\} \in \mathcal{S}_n.$$

Then for $0 \leq k \leq n - |\lambda|$,

$$q^{\mathrm{nst}_\lambda^\lambda} \begin{bmatrix} n-|\lambda| \\ k \end{bmatrix}_{\mathcal{P}(\mathrm{uncr}(\lambda))} = (q-1) q^{\mathrm{nst}_\nu^\nu} \begin{bmatrix} n-|\nu| \\ k-1 \end{bmatrix}_{\mathcal{P}(\mathrm{uncr}(\nu))} + q^{\mathrm{nst}_\mu^\mu} \begin{bmatrix} n-|\mu| \\ k \end{bmatrix}_{\mathcal{P}(\mathrm{uncr}(\mu))}.$$

3.2 A related module for U_n

Let

$$U_{k\times n} = \bigsqcup_{\substack{A\subseteq\{1,2,\ldots,n\}\\|A|=k}} \{u_A \mid u \in U_n\},$$

where u_A is the submatrix of u obtained by taking only the rows indexed by A (to obtain a $k \times n$ matrix).

Note that

$$|U_{k\times n}| = \begin{bmatrix} n \\ k \end{bmatrix}_{\mathcal{T}} = q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q,$$

where $\mathcal{T}$ is a total order on n elements. Define

$$V^{k\times n} = \mathbb{C}\text{-span}\{v \mid v \in U_{k\times n}\},$$

which is a right module for U_n under right multiplication.

Remark 3.1 These modules interpolate between the trivial module $V^{0\times n}$ and the regular module $V^{n\times n}$.

Proposition 3.4 *For $\mu \in \mathcal{S}_n$, $0 \le k \le n$, and $u \in U_n$ of superclass type μ*

$$\mathrm{tr}(V^{k\times n}, u) = \begin{bmatrix} n - |\mu| \\ k \end{bmatrix}_{\mathcal{T}}.$$

For each $0 \le k \le n$, there exists a injection

$$\iota_k : U_{k\times n} \longrightarrow U_n$$

given by $\iota_k(u)$ is the unique matrix obtained by augmenting u with rows consisting of all zeroes except for one 1. For example,

$$\iota_2 \begin{pmatrix} 1 & a & b & c & d \\ 0 & 0 & 1 & e & f \end{pmatrix} = \begin{pmatrix} 1 & a & b & c & d \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & e & f \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let $\theta : \mathfrak{u}_n^* \longrightarrow U_n$ be the bijection given by

$$\theta(\gamma)_{ij} = \begin{cases} \gamma(e_{ij}) & \text{if } 1 \le i < j < n, \\ 1 & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

where e_{ij} is the $n \times n$ matrix with a 1 in the (i, j) entry and zeroes elsewhere. Then the $U_n \times U_n$-orbits in $\mathfrak{u}_n^*$ partition U_n via θ.

Let $\mathfrak{u}_n^*$ be the dual space to the $\mathbb{F}_q$-vector space

$$\mathfrak{u}_n = U_n - \mathrm{Id}_n.$$

The superclasses of U_n are parametrized by $U_n \times U_n$-orbits in $\mathfrak{u}_n$. The supercharacters are parametrized by $(U_n \times U_n) \rtimes T_n$-orbits in $\mathfrak{u}_n^*$, where the action is given by

$$(ta\gamma bt^{-1})(u - \mathrm{Id}_n) = \gamma(a^{-1}t^{-1}(u - \mathrm{Id}_n)tb^{-1}), \qquad \text{for } a, b, u \in U_n, t \in T_n \text{ and } \gamma \in \mathfrak{u}_n^*.$$

For $\lambda \in \mathcal{S}_n$, let $\gamma_\lambda \in \mathfrak{u}_n^*$ be given by

$$\gamma_\lambda(u - \mathrm{Id}_n) = \sum_{i \frown j \in \lambda} u_{ij}.$$

Then each γ_λ is in a different orbit of $\mathfrak{u}_n^*$ (and they therefore are orbit representatives).

For $\lambda \in \mathcal{S}_n$ with $|\lambda| \leq k$, let

$$U_{k\times n}^{\lambda} = \{u \in U_{k\times n} \mid \theta^{-1}(\iota_k(u)) \in U_n\gamma_\lambda U_n\}.$$

From our partition of U_n given by

$$U_n = \bigsqcup_{\lambda \in \mathcal{S}_n} \mathrm{Id}_n + \theta(U_n\gamma_\lambda U_n),$$

we obtain a partition

$$U_{k\times n} = \bigsqcup_{\substack{\lambda \in \mathcal{S}_n \\ |\lambda| \leq k}} U_{k\times n}^{\lambda}. \tag{4}$$

The following lemma says how these cardinalities behave as we uncross partitions.

Lemma 3.5 *Let $\{i \frown l, j \frown m\} \in \lambda$ be a crossing with $i < j < l < m$ and $l - j$ maximal. Let $\mu = (\lambda - \{i \frown l, j \frown m\}) \cup \{i \frown m, j \frown l\}$. Then*

$$|U_{k\times n}^{\lambda}| = \frac{1}{q}|U_{k\times n}^{\mu}| \qquad \textit{and} \qquad q^{\mathrm{nst}_\lambda^\lambda}\begin{bmatrix} n - |\lambda| \\ k - |\lambda| \end{bmatrix}_{\mathcal{P}(\mathrm{uncr}(\lambda))} \chi^\lambda(1) = \frac{1}{q} q^{\mathrm{nst}_\mu^\mu}\begin{bmatrix} n - |\mu| \\ k - |\mu| \end{bmatrix}_{\mathcal{P}(\mathrm{uncr}(\mu))} \chi^\mu(1).$$

We can now give a representation theoretic interpretation for the $\mathcal{P}$-binomials when $\mathcal{P}$ is a forest.

Proposition 3.6 *For $0 \leq k \leq n$ and $\lambda \in \mathcal{S}_n$ with $|\lambda| \leq k$,*

$$|U_{k\times n}^{\lambda}| = q^{\mathrm{nst}_\lambda^\lambda}\begin{bmatrix} n - |\lambda| \\ k - |\lambda| \end{bmatrix}_{\mathcal{P}(\mathrm{uncr}(\lambda))} \chi^\lambda(1).$$

Observing that the $U_{k\times n}^{\lambda}$ decompose $U_{k\times n}$, we obtain the following corollary that will be a key step in connecting the poset binomial coefficients to supercharacters.

Corollary 3.7 *For $0 \leq k \leq n$,*

$$\sum_{\lambda \in \mathcal{S}_n} q^{\mathrm{nst}_\lambda^\lambda}\begin{bmatrix} n - |\lambda| \\ k - |\lambda| \end{bmatrix}_{\mathcal{P}(\mathrm{uncr}(\lambda))} \chi^\lambda(1) = \begin{bmatrix} n \\ k \end{bmatrix}_{\mathcal{T}},$$

where $\mathcal{T}$ is a total order on n elements.

4 Decomposition in terms supercharacters of U_n

Corollary 3.7 suggests a possible decomposition for the character of the module $V^{k\times n}$ into supercharacters. Let

$$\psi_k^{(n)} = \sum_{\lambda\in\mathcal{S}_n} q^{\mathrm{nst}_\lambda^\lambda} \begin{bmatrix} n-|\lambda| \\ k-|\lambda| \end{bmatrix}_{\mathcal{P}(\mathrm{uncr}(\lambda))} \chi^\lambda.$$

This section seeks to prove that

$$\psi_k^{(n)}(u) = \mathrm{tr}(V^{k\times n}, u) \qquad \text{for all } u \in U_n.$$

4.1 Main Theorem

The main result is to prove the following simple character formula for the $\psi_k^{(n)}$.

Theorem 4.1 *For $0 \le k \le n$ and $\mu \in \mathcal{S}_n$,*

$$\psi_k^{(n)}(\mu) = \begin{bmatrix} n-|\mu| \\ k \end{bmatrix}_{\mathcal{T}},$$

where $\mathcal{T}$ is a total order on $n-|\mu|$ elements.

Remark 4.1 It would be desirable to have some simple counting argument to prove Theorem 4.1. However, there seems to be some unpredictable cancellation that happens that makes this challenging. For example, for $n=3$ and $k=2$, this theorem implies

$$\binom{3}{2} + \binom{2}{1}(q-1) + \binom{2}{1}(q-1) - (q+1)q + (q-1)^2 = q.$$

We outline the proof by stating three key lemmas, where the proof of the first lemma is the most involved. The basic idea is to show that the $\psi_k^{(n)}(\mu)$ are invariant under adjustments of the arcs in μ. We are then able to reduce the problem to the case when $\mu = \emptyset$, and then use Corollary 3.7.

Let

$$\mathcal{J}_l = \{\lambda \in \mathcal{S}_n \mid l \frown (l+1) \in \lambda\}.$$

Define bijections

$$\varphi_l^R : \mathcal{S}_n - \mathcal{J}_l \longrightarrow \mathcal{S}_n - \mathcal{J}_l \qquad \text{and} \qquad \varphi_l^L : \mathcal{S}_n - \mathcal{J}_l \longrightarrow \mathcal{S}_n - \mathcal{J}_l \tag{5}$$

by letting $\varphi_l^R(\lambda)$ (respectively $\varphi_l^L(\lambda)$) be the set partition obtained from λ by applying the transposition $(l, l+1)$ to all the right (respectively left) endpoints of the arcs in λ. Lemma 4.2 begins by showing that the $\psi_k^{(n)}$ is invariant under small shifts on the arcs.

Lemma 4.2 *Suppose $1 \le l \le n-1$ and $0 \le k \le n$. For $\nu \in \mathcal{S}_n - \mathcal{J}_l$,*

$$\psi_k^{(n)}(\varphi_l^R(\nu)) = \psi_k^{(n)}(\nu) = \psi_k^{(n)}(\varphi_l^L(\nu)).$$

Lemma 4.3 uses Lemma 4.2 to show that $\psi_k^{(n)}$ depends only on the number of arcs associated to a superclass.

Lemma 4.3 *Suppose* $0 \leq k \leq n$ *and* $\mu, \nu \in \mathcal{S}_n$ *with* $|\mu| = |\nu|$. *Then*

$$\psi_k^{(n)}(\mu) = \psi_k^{(n)}(\nu).$$

Lemma 4.4 gives a recurrence on $\psi_k^{(n)}$ in terms of $\psi_k^{(n-1)}$. This allows us to reduce the theorem to evaluating at the identity. Thus, Theorem 4.1 is proved by Corollary 3.7.

Lemma 4.4 *Suppose* $\nu \in \mathcal{S}_n$, $\mu \in \mathcal{S}_{n-1}$ *such that* $|\mu| = |\nu| - 1$. *Then*

$$\psi_k^{(n)}(\nu) = \psi_k^{(n-1)}(\mu).$$

4.2 Some consequences

We first obtain a family of combinatorial identities by applying (2) to Theorem 4.1. For $\lambda \in \mathcal{S}_n$, let

$$\mathrm{cflt}(\lambda) = \{j \frown k \mid 1 \leq j < k < l \text{ with } j \frown l \in \lambda \text{ or } i < j \leq k \leq n \text{ with } i \frown k \in \lambda\}.$$

Note that while the statement of the corollary seems to have negative powers of q, in fact, each summand is a polynomial in q.

Corollary 4.5 *For* $n, k, \in \mathbb{Z}_{\geq 0}$, $\mu \in \mathcal{S}_n$, *and a total order* $\mathcal{T}$ *on* $\{1, 2, \ldots, n - |\mu|\}$,

$$\sum_{\substack{\lambda \in \mathcal{S}_n \\ \mu \cap \mathrm{cflt}(\lambda) = \emptyset}} (-1)^{|\lambda \cap \mu|} \frac{q^{\mathrm{nst}_\lambda^\lambda + \dim(\lambda)}}{q^{\mathrm{nst}_\mu^\lambda + |\lambda|}} (q-1)^{|\lambda - \mu|} \begin{bmatrix} n - |\lambda| \\ k - |\lambda| \end{bmatrix}_{\mathcal{P}(\mathrm{uncr}(\lambda))} = \begin{bmatrix} n - |\mu| \\ k \end{bmatrix}_{\mathcal{T}}.$$

At $q = 1$, this corollary is uninteresting but at $q = 0$ we obtain a sum over compositions of n.

Corollary 4.6 *For* $n, k, \in \mathbb{Z}_{\geq 0}$,

$$\sum_{\lambda \vDash n} (-1)^{n - \ell(\lambda)} \binom{\ell(\lambda)}{\ell(\lambda) - n + k} = \begin{cases} 0 & \text{if } k > 1, \\ 1 & \text{if } k \in \{0, 1\}. \end{cases}$$

However, our main motivation is representation theoretic. By using (2) and a result on q-derivatives of generating functions of q-binomials [4], we obtain the following theorem.

Theorem 4.7

$$\mathrm{Res}_{U_{[2,n+1]}}^{U_{n+2}} (\underbrace{\chi^{1 \frown (n+2)} \odot \cdots \odot \chi^{1 \frown (n+2)}}_{k \text{ terms}}) = (q-1)^k \sum_{j=0}^{k} \left(\prod_{i=0}^{j-1} (q^{k-i} - 1) \right) \psi_j^{(n)},$$

where $U_{[2,n+1]}$ *is as in (1).*

Since we have a decomposition of the $\psi_k^{(n)}$ in terms of supercharacters, we also obtain a formula for the decomposition of the restriction in terms of supercharacters.

Corollary 4.8 *For* $\lambda \in \mathcal{S}_n$,

$$\mathrm{Res}_{U_{[2,n+1]}}^{U_{n+2}} ((\chi^{1 \frown (n+2)})^{\odot k}) = (q-1)^k \sum_{\lambda \in \mathcal{S}_n} \left(\sum_{j=|\lambda|}^{\min(k,n)} q^{\mathrm{nst}_\lambda^\lambda} \begin{bmatrix} n - |\lambda| \\ j - |\lambda| \end{bmatrix}_{\mathcal{P}(\mathrm{uncr}(\lambda))} \prod_{i=0}^{j-1} (q^{k-i} - 1) \right) \chi^\lambda.$$

Acknowledgements

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Relating Edelman-Greene insertion to the Little map

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Abstract. The Little map and the Edelman-Greene insertion algorithm, a generalization of the Robinson-Schensted correspondence, are both used for enumerating the reduced decompositions of an element of the symmetric group. We show the Little map factors through Edelman-Greene insertion and establish new results about each map as a consequence. In particular, we resolve some conjectures of Lam and Little.

Résumé. La correspondance de Little et l'algorithme d'Edelman-Greene généralisant la correspondance de Robinson-Schensted sont utilisés pour l'énumération des décompositions réduites associées aux éléments du groupe symétrique. Nous démontrons que la correspondance de Little peut être réduite à celle d'Edelman-Greene. En particulier, nous obtenons de nouvelle réponses à quelques conjectures de Lam et Little.

Keywords: Young tableaux; reduced decompositions in the symmetric group; Edelman-Greene insertion; Lascoux-Schützenberger tree; Knuth moves; Stanley symmetric functions

1 Introduction

1.1 Preliminaries

In this paper, we clarify the relationship between two algorithmic bijections, due respectively to Edelman and Greene (1987) and to Little (2003), both of which deal with reduced decompositions in the symmetric group, S_n. It is well known that S_n can be viewed as a Coxeter group with the presentation

$$S_n = \langle s_1, s_2, \ldots, s_{n-1} \mid s_i^2 = 1,\ s_i s_j = s_j s_i \text{ for } |i-j| \geq 2,\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \rangle$$

where w_i can be viewed as the transposition $(i\ i{+}1)$. Let $\sigma = \sigma_1 \sigma_2 \ldots \sigma_n \in S_n$. A *reduced decomposition* or *reduced expression* of σ is a minimal-length sequence $s_{w_1}, s_{w_2}, \ldots, s_{w_m}$ such that $\sigma = s_{w_1} s_{w_2} \ldots s_{w_m}$. The word $w = w_1 w_2 \ldots w_m$ is called a *reduced word* of σ. It is convenient to refer to a reduced decomposition by its corresponding reduced word and we will conflate the two often. The set of all reduced decompositions of σ is denoted $\mathrm{Red}(\sigma)$. An *inversion* in σ is a pair (i, j) with $i < j$ and $\sigma_i > \sigma_j$. Let $l(\sigma)$ be the number of inversions in σ. Since each transposition s_i either introduces or removes an inversion, for $w = w_1 \ldots w_m$ a reduced word of σ, we see $m = l(\sigma)$.

The enumerative theory of reduced decompositions was first studied in Stanley (1984), where using algebraic techniques it is shown for the reverse permutation $\sigma = n \dots 21$ that

$$|\mathrm{Red}(\sigma)| = \frac{\binom{n}{2}!}{(2n-3)(2n-5)^2 \dots 5^{n-2}3^{n-2}}. \tag{1}$$

This is the same as the number of standard Young tableaux with the staircase shape $\lambda = (n-1, n-2, \dots, 1)$. In addition, Stanley conjectured for arbitrary $\sigma \in S_n$ that $|\mathrm{Red}(\sigma)|$ can be expressed as the number of standard Young tableaux of various shapes (possibly with multiplicity). This conjecture was resolved in Edelman and Greene (1987) using a generalization of the Robinson-Schensted insertion algorithm, usually called *Edelman-Greene insertion*. Edelman-Greene insertion maps a reduced word w to the pair of Young tableaux $(P(w), Q(w))$ where the entries of $P(w)$ are row-and-column strict and $Q(w)$ is a standard Young tableau. The same map also provides a bijective proof of (1), as there is only one possibility for $P(w)$.

Algebraic techniques developed in Lascoux and Schützenberger (1985) can be used to compute the exact multiplicity of each shape for given σ. A bijective realization of Lascoux and Schützenberger's techniques in this setting is demonstrated in Little (2003). Permutations with precisely one descent are referred to as *Grassmannian*. There is a simple bijection between reduced words of a Grassmannian permutation σ and standard Young tableaux of a shape determined by σ. The Little map works by applying a sequence of modifications referred to as *Little bumps* to the reduced word w until the modified word's corresponding permutation is Grassmannian so that it can be mapped to a standard Young tableau denoted LS(w).

1.2 Results

Since the Little map's introduction, there has been speculation on its relationship to Edelman-Greene insertion. In the appendix of Garsia (2002), written by Little, Conjecture 4.3.2 asserts that $\mathrm{LS(w)} = \mathrm{Q(w)}$ when the maps are restricted to reduced words which realize the reverse permutation. Similar comments are made in Little (2003). We show the connection is much stronger than previously suspected: this equality is true for every permutation.

Theorem 1.1 *Let w be a reduced word. Then*

$$Q(w) = \mathrm{LS(w)}.$$

The proof is based on an argument from canonical form. We define the *column word*, a reading word of $P(w)$ that plays nice with both Edelman-Greene insertion and Little bumps. We then show the statement's truth is invariant under Coxeter-Knuth moves, transformations that span the space of reduced words with identical $P(w)$.

Given Theorem 1.1, one might suspect the structure of the two maps is intimately related. Specifically, Conjecture 2.5 of Lam (2010) proposes that Little bumps relate to Edelman-Greene insertion in a way that is analogous to the role dual Knuth transformations play for the Robinson-Schensted-Knuth algorithm.

Let v and w be reduced words. We say v and w *communicate* if there exists a sequence of Little bumps changing v to w. This is an equivalence relation as Little bumps are invertible.

Theorem 1.2 (Lam's Conjecture) *Let v and w be two reduced words. Then v and w communicate if and only if $Q(v) = Q(w)$.*

1.3 Structure of the paper

In the second section, we review those parts of Edelman and Greene (1987); Little (2003) which we need: we define Edelman-Greene insertion and the Little map, as well as generalized Little bumps. Additionally, we state some properties of these maps that are important to our work. The third section defines Coxeter-Knuth transformations and studies their interaction with Little bumps and action on $Q(w)$. We conclude in the fourth section by proving our main results and resolving several conjectures of Little. Due to space considerations, several proofs have been omitted. The curious reader may find these details in Hamaker and Young (2012).

2 Two Maps

2.1 Edelman-Greene insertion

In order to define Edelman-Greene insertion, we must first define a rule for inserting a number into a tableau. Let $n \in \mathbb{N}$ and T be a tableau with rows $R_1, R_2, \dots, R_k$ where $R_i = r_1^i \leq r_2^i \leq \cdots \leq r_{l_i}^i$. We define the insertion rule for Edelman-Greene insertion, following Edelman and Greene (1987).

1. If $n \geq r_{l_1}^1$ or if R_i is empty, adjoin k to the end of R_i.
2. If $n < r_{l_1}^1$, let j be the smallest number such that $n < r_j^1$.
   (a) If $r_j^1 = n+1$ and $r_{j-1}^1 = n$, insert $n+1$ into $T' = R_2, \dots, R_k$ and leave R_1 unchanged.
   (b) Otherwise, replace r_j^1 with n and insert it into $T' = R_2, \dots, R_k$.

Aside from 2(a), this is the RSK insertion rule. For $w = w_1 \dots w_m$ a word (not necessarily reduced), we define $\mathrm{EG}(w) = (P(w), Q(w))$ via the following sequence of tableaux (see Figure 1 for an example). We obtain $P_1(w)$ by inserting w_m into the empty tableau. Then $P_j(w)$ is obtained by inserting w_{m-j+1} into $P_{j-1}(w)$. Note we are inserting the entries of w from right to left. At each step, one additional box is added. In $Q(w)$, the entry of each box records the time of the step in which it was added. From this, we can conclude that $Q(w)$ is a standard Young tableau. Note the fourth insertion in Figure 1 follows 2(a). For w is a reduced word of some σ, it is shown that the entries of $P(w)$ are strictly increasing across rows and down columns in Edelman and Greene (1987). Additionally, we can recover σ from $P(w)$ with no additional information.

2.2 Grassmannian permutations

Recall a permutation σ is Grassmannian if it has exactly one descent. We can then write

$$\sigma = a_1 a_2 \dots a_k b_1 b_2 \dots b_{n-k}$$

where $\{a_i\}_{i=1}^k$ and $\{b_j\}_{j=1}^{n-k}$ are increasing sequences with $a_k > b_1$. A word w is *Grassmannian* if it is the reduced word of a Grassmannian permutation. From the Grassmannian word $w = w_1 \dots w_m$ we construct a tableau $\mathrm{Tab}(w)$ as follows. Index the columns of $\mathrm{Tab}(w)$ by $b_1, \dots, b_{n-k}$ and the rows by $a_k, a_{k-1}, \dots, a_1$. Since all inversions in σ feature an a_i and a b_j, each w_l in w represents the swap between an a_i and a b_j. For w_l, we enter $m+1-l$ in the column indexed by a_i and b_j. If a_i swaps with b_j, we see it must later swap with each smaller b. This shows entries are increasing across rows.

Fig. 1: Edelman-Greene insertion for $w = 4, 2, 1, 2, 3, 2, 4$

$$P_1 = \begin{array}{|c|}\hline 4 \\ \hline\end{array} \quad Q_1 = \begin{array}{|c|}\hline 1 \\ \hline\end{array} \qquad P_2 = \begin{array}{|c|}\hline 2 \\ \hline 4 \\ \hline\end{array} \quad Q_2 = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \qquad P_3 = \begin{array}{cc} 2 & 3 \\ 4 & \end{array} \quad Q_3 = \begin{array}{cc} 1 & 3 \\ 2 & \end{array} \qquad P_4 = \begin{array}{cc} 2 & 3 \\ 3 & \\ 4 & \end{array} \quad Q_4 = \begin{array}{cc} 1 & 3 \\ 2 & \\ 4 & \end{array}$$

$$P_5 = \begin{array}{cc} 1 & 3 \\ 2 & \\ 3 & \\ 4 & \end{array} \quad Q_5 = \begin{array}{cc} 1 & 3 \\ 2 & \\ 4 & \\ 5 & \end{array} \qquad P_6 = \begin{array}{cc} 1 & 2 \\ 2 & 3 \\ 3 & \\ 4 & \end{array} \quad Q_6 = \begin{array}{cc} 1 & 3 \\ 2 & 6 \\ 4 & \\ 5 & \end{array} \qquad P_7 = P(w) = \begin{array}{ccc} 1 & 2 & 4 \\ 2 & 3 & \\ 3 & & \\ 4 & & \end{array} \quad Q_7 = Q(w) = \begin{array}{ccc} 1 & 3 & 7 \\ 2 & 6 & \\ 4 & & \\ 5 & & \end{array}$$

Likewise, if b_j swaps with a_i, it must later swap with each larger a so entries increase down columns. From this, we can conclude that Tab(w) is a standard Young tableau whose shape is determined by σ. For a given Grassmannian permutation σ, this map is a bijection as the process is easily reversed. Multiple Grassmannian permutations may correspond to the same shape. However, they will only differ by some fixed points at the beginning and end of the permutation.

2.3 Little bumps and the Little map

We now describe the method in Little (2003) for transforming an arbitrary reduced word into the reduced word of a Grassmannian permutation. Let $w = w_1 \dots w_m$ be a reduced word and $w^{(i)} = w_1 \dots w_{i-1} w_{i+1} \dots w_m$. We construct

$$w^{(i-)} = \begin{cases} w_1 \dots w_{i-1}(w_i - 1)w_{i+1} \dots w_m & \text{if } w_i > 1 \\ (w_1 + 1) \dots (w_{i-1} + 1) w_i (w_{i+1} + 1) \dots (w_m + 1) & \text{if } w_i = 1 \end{cases}$$

by decrementing w_i by one or incrementing each other entry if $w_i = 1$.

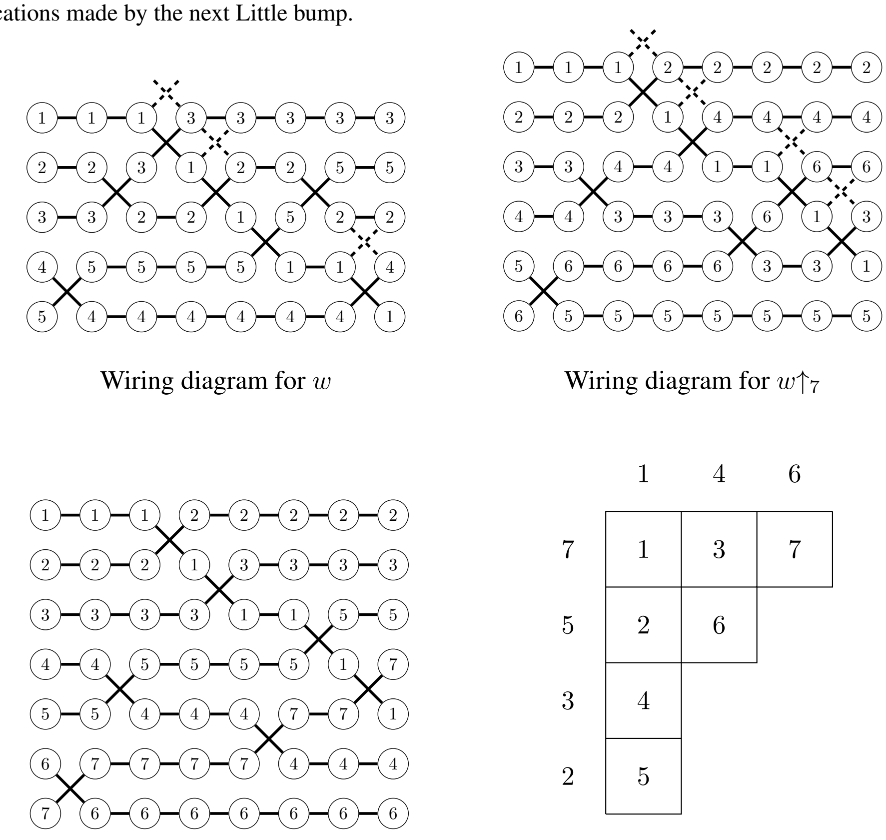
Let w be a reduced word so that $w^{(i)}$ is also reduced. Note $w^{(i-)}$ may not be reduced, as $w_i - 1$ may swap the same values as some w_j with $j \neq i$. However, this is the only way $w^{(i)-}$ can fail to be reduced as $w^{(i)}$ is reduced and we have added one additional swap. Removing w_j from $w^{(i-)}$, we obtain a new reduced word $w^{(i-)(j)}$. Repeating this process of decrementation, we can construct $w^{(i-)(j-)}$ and so on until we are left with a reduced word $v = v_1 \dots v_m$. We refer to this process as a *Little bump* beginning at position i and say $v = w{\uparrow}_i$, where i is the initial index the bump was started at. To see that this process terminates, we refer to the following lemma.

Lemma 2.1 (Lemma 5, Little (2003)) *Let w be a reduced word such that $w^{(i)}$ is reduced. Let $i_1, i_2, \dots$ be the sequence of indices decremented in $w{\uparrow}_i$. Then the entries of $i_1, i_2, \dots$ are unique.*

Since w is finite, we see the process terminates so that $w{\uparrow}_i$ is well-defined. We highlight a property of Little bumps observed in Little (2003), that they preserve the descent structure of w.

Corollary 2.2 *Let $w = w_1 \dots w_m$ and $v = v_1 \dots v_m$ be a reduced words and $\uparrow$ be a Little bump such that $v = w\uparrow$. Then $v_i > v_{i+1}$ if and only if $w_i > w_{i+1}$ for all i.*

Fig. 2: The Little map for the reduced decomposition $w_4w_2w_1w_2w_3w_2w_4$ of $\sigma = 35241$. The dashed crosses show the modifications made by the next Little bump.



Wiring diagram for w — Wiring diagram for $w{\uparrow_7}$ — Wiring diagram for $w{\uparrow_7}{\uparrow_7}$ — $\mathrm{Tab}(w{\uparrow_7}{\uparrow_7}) = \mathrm{LS(w)}$

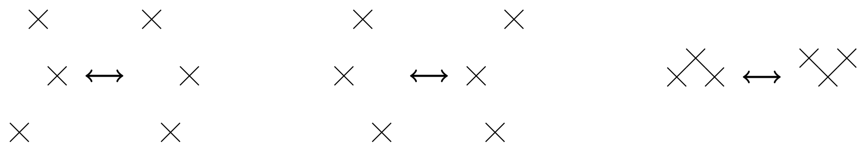
Proof: Let $w_i > w_{i+1}$. As each w_i is decremented at most once, we see $v_i \geq v_{i+1}$, but $v_i \neq v_{i+1}$. Thus $v_i > v_{i+1}$. By the same reasoning, if $w_i < w_{i+1}$, we see $v_i < v_{i+1}$. □

Let w be a reduced word of $\sigma \in S_n$. We define the Little map $\mathrm{LS(w)}$.

1. If w is a Grassmannian word, then $\mathrm{LS(w)} = \mathrm{Tab(w)}$
2. If w is not a Grassmannian word, identify the swap location i of the last inversion (lexicographically) in σ and output $\mathrm{LS(w{\uparrow_i})}$.

It is a corollary of work in Lascoux and Schützenberger (1985) and Little (2003) that LS terminates. We then see that $w \mapsto \mathrm{LS(w)}$ where $\mathrm{LS(w)}$ is a standard Young tableau. An example can be seen in Figure 2, where the word w is represented by its *wiring diagram*: an arrangement of horizontal, parallel wires spaced one unit apart, labelled 1 through n on the left-hand side, in which the letter in the word w are represented by crossings of wires.

Fig. 3: The three types of Coxeter-Knuth moves acting on wiring diagrams.



(a) Type 1 (b) Type 2 (c) Type 3

3 The action of Coxeter-Knuth moves

3.1 Basics of Coxeter-Knuth moves

First introduced in Edelman and Greene (1987), Coxeter-Knuth moves are perhaps the most important tool for studying Edelman-Greene insertion. They are modifications of the second and third Coxeter relations. Let $a < b < c$ and x be integers. The three *Coxeter-Knuth moves* are the modifications

1. $acb \leftrightarrow cab$

2. $bac \leftrightarrow bca$

3. $x(x+1)x \leftrightarrow (x+1)x(x+1)$

applied to three consecutive entries of a reduced word. Let $w = w_1 w_2 \dots w_m$ be a reduced word of σ and α_i denote a Coxeter-Knuth move on the entries $w_{i-1}w_i w_{i+1}$. Since $a < b < c$, if α_i is of type one or two we have $w\alpha_i$ a reduced word of σ as well by the second Coxeter relation. If α_i is of type three then $w\alpha_i$ is a reduced word of σ by the third Coxeter relation. We say two reduced words v and w are *Coxeter-Knuth equivalent* if there exists a sequence $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_k}$ of Coxeter-Knuth moves such that

$$v = w\alpha_{i_1} \dots \alpha_{i_k}.$$

Note that two Coxeter-Knuth equivalent reduced words must correspond to reduced decompositions of the same permutation. We can see their action on wiring diagrams in Figure 3.

Coxeter-Knuth moves play a role in the study of Edelman-Greene insertion analogous to that of Knuth moves in the study of RSK insertion.

Theorem 3.1 (Theorem 6.24 in Edelman and Greene (1987)) *Let v and w be a reduced words. Then $P(v) = P(w)$ if and only if v and w are Coxeter-Knuth equivalent.*

3.2 The action of Coxeter-Knuth moves on $Q(w)$

In order to understand the relationships of Coxeter-Knuth moves and Little bumps, we must first understand in greater detail how Coxeter-Knuth moves relate to Edelman-Greene insertion. From Theorem 3.1, we understand how Coxeter-Knuth moves relate to $P(w)$. We must also understand their action on $Q(w)$. For T a standard Young tableau with n entries, let $Tt_{i,j}$ be the Young tableau obtained by swapping the entries labeled $n-i$ and $n-j$.

Lemma 3.2 *Let $w = w_1 \dots w_m$ be a reduced word and α be a Coxeter-Knuth move on $w_{i-1}w_iw_{i+1}$. If α is a Coxeter-Knuth move of type one or three, then*

$$Q(w\alpha) = Q(w)t_{i-1,i}.$$

If α is a Coxeter-Knuth move of type two, then α_i acts on $Q(w)$ as above or

$$Q(w\alpha) = Q(w)t_{i,i+1}.$$

The proof of Lemma 3.2 is based on and can be recovered with little additional effort from the argument presented for Theorem 6.24 in Edelman and Greene (1987). We omit the proof for space considerations.

3.3 Coxeter-Knuth moves and Little bumps

We now set out to show that Coxeter-Knuth moves commute with Little bumps. This requires two results. The first is that the order we perform a Coxeter-Knuth move α and a Little bump $\uparrow$ does not affect the resulting reduced word.

Lemma 3.3 *Let $w = w_1 \dots w_m$ be a reduced word, α a Coxeter-Knuth move on $w_{i-1}w_iw_{i+1}$, and $\uparrow_{j,k}$ be a Little bump begun at the swap between the j and kth trajectories. Then*

$$(w\alpha)\uparrow_{j,k} = (w\uparrow_{j,k})\alpha.$$

Proof: Let $v = w\uparrow_{j,k}$ and $v' = (w\alpha)\uparrow_{j,k}$. Recall from Lemma 2.1 and Corollary 2.2 that $w_j - v_j \in \{0, 1\}$ and v has the same descent structure of w.

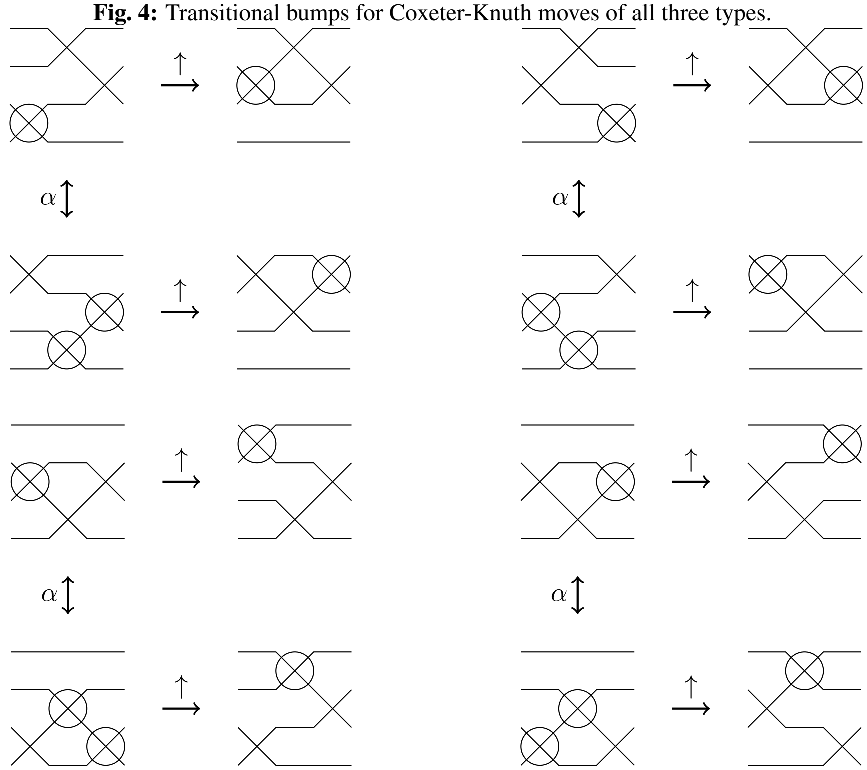
1. Let α be a Coxeter-Knuth move of the first type, i.e. $w_{i-1}w_iw_{i+1} \mapsto w_iw_{i-1}w_{i+1}$ with w_{i+1} strictly between w_{i-1} and w_i. Since a Little bump decrements an entry of w by at most one, one can check that if w_{i+1} differs from w_i or w_{i-1} by more than one, there is a Coxeter-Knuth move of type one on $v_{i-1}v_iv_{i+1}$. In the event that they differ by exactly one and the smallest entry is decremented, we see in Figure 4 that after the bump they differ by a Coxeter-Knuth move of the third type.

2. Let α be a Coxeter-Knuth move of the second type, i.e. $w_{i-1}w_iw_{i+1} \mapsto w_{i-1}w_{i+1}w_i$ with w_{i-1} strictly between w_{i+1} and w_i. Since a Little bump decrements an entry of w by at most one, one can check that if w_{i-1} differs from w_i or w_{i+1} by more than one, there is a Coxeter-Knuth move of type two on $v_{i-1}v_iv_{i+1}$. In the event that they differ by exactly one and the smallest entry is bumped, we see in Figure 4 that after the bump they differ by a Coxeter-Knuth move of the third type.

3. Let α be a Coxeter-Knuth move of the third type. Note the middle entry cannot be bumped unless all three entries are bumped. In the event fewer entries (but not zero) are bumped, we see in Figure 4 that there will be a Coxeter-Knuth move of the first or second type remaining.

We next show that the rest of the Little bump proceeds in the same manner once the crossings involved in the Coxeter-Knuth move have been bumped. To see this, we need only observe that the last bumped swap is between the same two trajectories. This can be verified readily by examining Figures 4.

Fig. 4: Transitional bumps for Coxeter-Knuth moves of all three types.



The preceding argument assumes that the bumping path does not return to the crossings involved in the Coxeter-Knuth move. It is possible that the bumping path passes through the crossings involved in the Coxeter-Knuth path twice (but no more than that, by Lemma 2.1). However, the same argument applies, showing that all three crossings are bumped regardless of whether the Coxeter-Knuth move is performed before or after the bump.

□

We now show that the action of a Coxeter-Knuth move on $Q(w)$ remains the same after applying a Little bump. Combined with Lemma 3.3, this shows that the order in which Coxeter-Knuth moves and Little bumps are performed on a reduced word w does not effect either the resulting reduced word or the resulting recording tableau.

Lemma 3.4 *Let w be a reduced word, α be a Coxeter-Knuth move and $\uparrow$ a Little bump. Then $Q(w\alpha) = Q(w)t_{i,i+1}$ if and only if $Q(w{\uparrow}\alpha) = Q(w{\uparrow})t_{i,i+1}$.*

The proof of Lemma 3.4 reduces to a simple observation. The only problematic case is when α is a Coxeter-Knuth move on $w_{i-1}w_iw_{i+1}$ of type two that acts on $Q(w)$ as $t_{i,i+1}$. Here, the truncated word

$w|_i = w_i w_{i+1} w_{i+2} \dots w_n$ and $w\alpha|_i = w_{i+1} w_i w_{i+2} \dots w_n$ have the same insertion tableau. Therefore, they are related by Coxeter-Knuth moves, and the action of this sequence of moves can be shown to be preserved by Little bumps. We omit the details of this argument.

4 Proof of Results

4.1 The Grassmannian case

Before proving Theorem 1.1, we need to establish the base case where w is a Grassmannian word. In order to do so, we must understand which entries are exchanging places with each swap. For $w = w_1 \dots w_m$ a reduced word, we define $\sigma_i = s_{w_1} s_{w_2} \dots s_{w_i}$ where σ_0 is the identity permutation. The kth *trajectory* of w is the sequence $\{\sigma_i(k)\}_{i=0}^{m}$. For w a Grassmannian word of $\sigma = a_1 a_2 \dots a_k b_1 b_2 \dots b_{n-k}$, observe that the jth column of Tab(w) lists the times for all swaps featuring b_j. Since all such swaps increase the value of b_j, we can reconstruct its trajectory from the number and location of these swaps. Similarly, we can reconstruct the trajectory of each a_i from the $k+1-i$th row of Tab(w). We will find it convenient to identify the kth trajectory of a Grassmannian word with the indices $\{i_1, i_2, \dots, i_{t_k}\} \subset [n]$ of the swaps featuring k. Since insertion takes place from right to left, we label the entries such that $i_1 > i_2 > \dots > i_{t_k}$.

Lemma 4.1 *Let $w = w_1 \dots w_m$ be a reduced decomposition of a Grassmannian permutation σ. Then Tab$(w) = Q(w)$.*

The proof of Lemma 4.1 follows by showing that for $\sigma = a_1 a_2 \dots a_{n-k} b_1 b_2 \dots b_k$ a Grassmannian permutation with sole descent $a_{n-k} b_1$, the trajectory of each b_j will insert into the jth column. This is shown inductively, as the trajectory of each b_i will block off the trajectory of b_{i+1}. The entries of b_{i+1} must then be inserted further to the right of entries in b_i. A trajectory unobstructed will insert into a single column, so we can conclude each trajectory will insert one at a time into its own column. We omit the details of this argument.

4.2 The column reading word

The only ingredient missing from our argument is a canonical form that is invariant under Little bumps.

Definition 4.2 *For T a Young tableau with columns $C^1, C^2 \dots, C^m$ where $C^i = c^i_1, c^i_2, \dots, c^i_k$ with c^i_j being the (j, i)th entry of T. We define the* column reading word *of T to be the word*

$$\tau(T) = C^m C^{m-1} \dots C^1.$$

If T is row and column strict then $P(\tau(T)) = T$ and each column of $Q(\tau(T))$ has consecutive entries. For w a reduced word, we define $\tau(w)$ to be $\tau(P(w))$. By the previous observation, w and $\tau(w)$ are Coxeter-Knuth equivalent.

For example, the tableau in Figure 2 has columns 1245, 36 and 7, so its column word is 7361245. One can think of the column reading word as closely related to the bottom-up reading word. Since insertion takes place from right to left, the column reading word is in some sense its transpose.

Lemma 4.3 *Let w be a reduced word and $\uparrow$ a Little bump on w. Then*

$$Q(\tau(w)) = Q(\tau(w)\uparrow).$$

Proof: Let w be a reduced word, $\tau(w) = C^m C^{m-1} \dots C^1$ and $\tau(w)\uparrow = D^m D^{m-1} \dots D^1$ (note D^k is not *a priori* a column of $P(\tau(w)\uparrow)$). Since $\tau(w)$ and $\tau(w)\uparrow$ have the same descent structure, we see C^1 and D^1 insert identically. As each entry of $\tau(w)\uparrow$ is decremented at most once and $P(\tau(w))$ is row and column strict, we see

$$d_i^k \leq c_i^k \leq d_i^k + 1 \leq d_i^{k+1},$$

so d_i^{k+1} will not bump any d_j^k with $j \leq i$. Therefore, any entry of D^k will stay in the kth column of $P(\tau(w)\uparrow)$ for all k, that is the entries of the kth column of $P(\tau(w)\uparrow)$ are D^k. Thus $\tau(w)\uparrow$ is a column reading word with identical column sizes, so $Q(\tau(w)) = Q(\tau(w)\uparrow)$.

□

4.3 Proof of Theorem 1.1 and its corollaries

Combining Lemma 4.3 with Lemmas 3.3 and 3.4, we can conclude the following:

Theorem 4.4 *Let w be a reduced word and $\uparrow$ be a Little bump on w. Then*

$$Q(w) = Q(w\uparrow).$$

Proof: Let w be a reduced word. There exists a sequence $\alpha_1, \alpha_2, \dots, \alpha_k$ of Coxeter-Knuth moves such that $w = \tau(w)\alpha_1 \dots \alpha_k$. As $Q(\tau(w)) = Q(\tau(w)\uparrow)$ by Lemma 4.3, we compute

$$\begin{aligned} Q(w) = Q(\tau(w)\alpha_1 \dots \alpha_k) &= Q((\tau(w)\uparrow)\alpha_1 \dots \alpha_k) && (2)\\ &= Q((\tau(w)\alpha_1 \dots \alpha_k)\uparrow) = Q(w\uparrow) && (3)\end{aligned}$$

where the third equality follows by Lemmas 3.3 and 3.4. □

Proof of Theorem 1.1: Let w be a reduced word and $\uparrow_1, \dots, \uparrow_k$ be the sequence of canonical Little bumps. By Theorem 4.4 and Lemma 4.1, we see

$$Q(w) = Q(w\uparrow_1 \dots \uparrow_k) = \text{Tab}(w\uparrow_1 \dots \uparrow_k) = \text{LS(w)}.$$

□

We now demonstrate several consequences, including Lam's Conjecture. The first is Conjecture 11 from Little (2005), which first appeared as Conjecture 4.3.3 in the appendix of Garsia (2002).

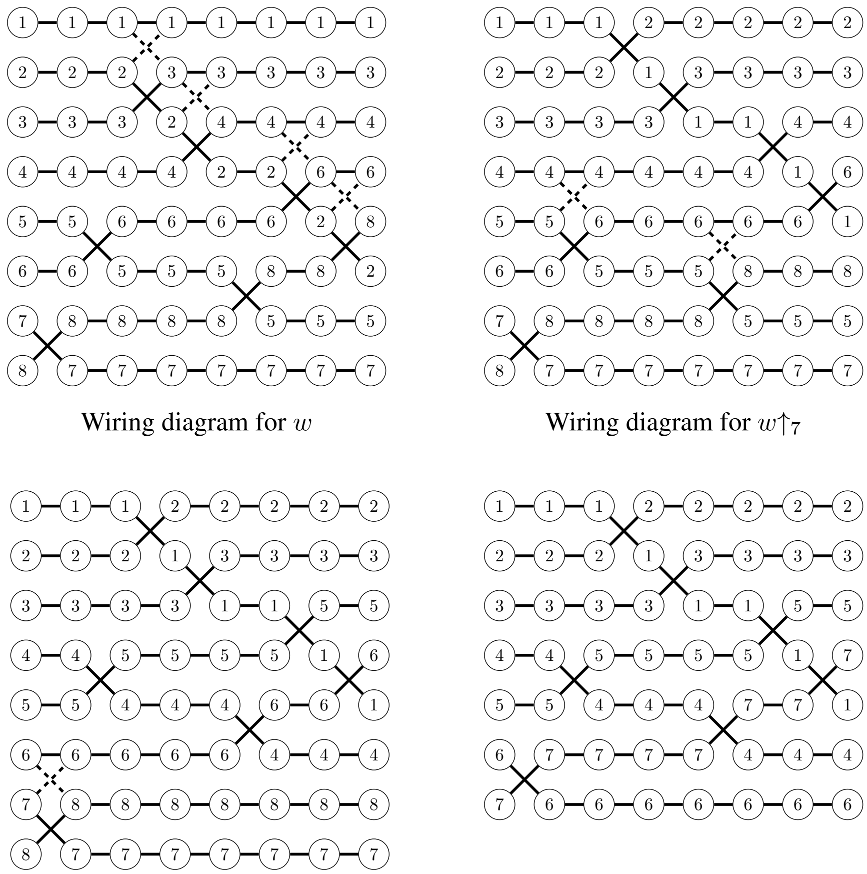
Corollary 4.5 *Let w be a reduced word and let $\uparrow_1, \uparrow_2, \dots, \uparrow_m$ be any sequence of Little bumps such that*

$$v = w\uparrow_1 \dots \uparrow_m$$

is a Grassmannian word. Then Tab(v) = LS(w).

This follows from Theorem 4.4. We can extend this result further. Let λ be a partition with w a Grassmannian word of shape λ. The permutation σ associated to w can be characterized by the number of initial fixed points. A Grassmannian permutation is *minimal* if it has no initial fixed points. Note the minimal Grassmannian permutation of a given shape is unique in S_∞. Recall two reduced words *communicate* if there exists a sequence of Little bumps and inverse Little bumps changing one to the other.

Fig. 5: Removing a fixed point from the Grassmannian word $w = 7523645$ via the canonical sequence of bumps.



Wiring diagram for w

Wiring diagram for $w{\uparrow_7}$

Wiring diagram for $w{\uparrow_7}{\uparrow_5}$

Wiring diagram for $w{\uparrow_7}{\uparrow_5}{\uparrow_1}$

Proof of Theorem 1.2: Let v and w be reduced words. Suppose first that v and w communicate. Then by Theorem 4.4, we have that $Q(v) = Q(w)$.

Conversely, suppose that $Q(v) = Q(w)$. By applying the canonical sequence of Little bumps, w can be changed to the Grassmannian word w' and v to the Grassmannian word v'. Since Little bumps are invertible, $Q(w) = Q(w')$ and $Q(v) = Q(v')$, we can conclude that v and w communicate if Grassmannian permutations of the same shape communicate. To show this, we demonstrate a sequence of Little bumps that will remove a fixed point at the beginning of an arbitrary Grassmannian permutation. Let $\sigma = a_1 \dots a_k b_1 \dots b_{n-k}$ be a Grassmannian permutation with $a_k b_1$ its sole descent. Our sequence is constructed by initiating a little bump at the last swap featuring each b_j, beginning with b_1. See Figure 5 for an example. Therefore, any Grassmannian permutation communicates with the minimal permutation of that shape. From this, we can conclude any two Grassmannian permutations with the same shape communicate.

□

Additionally, we show how to embed Robinson-Schensted insertion and RSK in the Little map. In doing so, we recover the main results of Little (2005) in a much simplified form. This embedding was first predicted as Conjecture 4.3.1 in the appendix of Garsia (2002). For w a word, let $\vec{w}$ be the reverse of w.

Theorem 4.6 *Let $\sigma = \sigma_1 \dots \sigma_n \in S_n$, so that $w(\sigma) = (2\sigma_n - 1) \dots (2\sigma_1 - 1)$ is a reduced word, and let $RS(\sigma) = (P'(\sigma), Q'(\sigma))$ be the output of Robinson-Schensted insertion applied to σ. Upon applying the transformation $k \mapsto k - 1/2$ to the entries of* $\mathrm{LS}(\mathrm{w})$*, we obtain $Q'(\sigma)$. We can obtain $P'(\sigma)$ by applying the same transformation to* $\mathrm{LS}(\mathrm{w}(\sigma^{-1})$.

Proof: Since $\mathrm{LS}(\mathrm{w}) = \mathrm{Q}(\mathrm{w})$ and there are no special bumps, Edelman-Greene insertion will perform the same insertion process on w as Robinson-Schensted insertion performs on σ. Therefore, upon applying the transformation $k \mapsto k - 1/2$, we see $\mathrm{LS}(\mathrm{w}(\sigma)) = \mathrm{Q}(\mathrm{w}(\sigma)) = \mathrm{Q}'(\sigma)$. Since $\mathrm{RS}(\sigma^{-1}) = (Q'(\sigma), P'(\sigma))$ (see *e.g.* Stanley (2001)), we can obtain $P'(\sigma)$ by applying the same transformation to $\mathrm{LS}(\mathrm{w}(\sigma^{-1}))$.

□

We can embed RSK in Robinson-Schensted insertion (see Section 7 of Little (2005) for a description of this process), so Theorem 4.6 recovers an embedding of RSK into the Little map as well.

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Spanning forests in regular planar maps

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Abstract. We address the enumeration of p-valent planar maps equipped with a spanning forest, with a weight z per face and a weight u per component of the forest. Equivalently, we count regular maps equipped with a spanning *tree*, with a weight z per face and a weight $\mu := u + 1$ per *internally active* edge, in the sense of Tutte. This enumeration problem corresponds to the limit $q \to 0$ of the q-state Potts model on the (dual) p-angulations.

Our approach is purely combinatorial. The generating function, denoted by $F(z, u)$, is expressed in terms of a pair of series defined by an implicit system involving doubly hypergeometric functions. We derive from this system that $F(z, u)$ is *differentially algebraic*, that is, satisfies a differential equation (in z) with polynomial coefficients in z and u. This has recently been proved for the more general Potts model on 3-valent maps, but via a much more involved and less combinatorial proof.

For $u \geq -1$, we study the singularities of $F(z, u)$ and the corresponding asymptotic behaviour of its n^{th} coefficient. For $u > 0$, we find the standard asymptotic behaviour of planar maps, with a subexponential factor $n^{-5/2}$. At $u = 0$ we witness a phase transition with a factor n^{-3}. When $u \in [-1, 0)$, we obtain an extremely unusual behaviour in $n^{-3}/(\log n)^2$. To our knowledge, this is a new "universality class" of planar maps.

Keywords: Planar maps — Spanning forests — Exact and asymptotic enumeration

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1 Introduction

A *planar map* is a proper embedding of a connected graph in the sphere. The enumeration of planar maps has received a continuous attention since the early 1960s, first in combinatorics with the pioneering work of Tutte, then in theoretical physics, where maps are considered as random surfaces modelling the effect of *quantum gravity*, and more recently in probability theory. General planar maps have been studied, as well as sub-families obtained by imposing constraints of higher connectivity or prescribing the degrees of vertices or faces (*e.g.*, triangulations). Precise definitions are given below.

Several robust enumeration methods have been designed, from Tutte's recursive approach (*e.g.* [25]), which leads to functional equations for generating functions of maps, to the beautiful bijections initiated by Schaeffer [22] and further developed by physicists and combinatorics alike [5, 9], via a powerful approach based on matrix integrals [15]. See for instance [8] for a more complete (though non-exhaustive) bibliography.

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Beyond planar maps, which are now well understood, the attention has also focussed on two more general objects: maps on higher genus surfaces, and maps equipped with an additional structure. The latter question is particularly relevant in physics, where a surface on which nothing happens ("pure gravity") is of little interest. For instance, one has studied maps equipped with a polymer, with an Ising model, with a proper coloring, with a loop model, with a spanning tree, percolation on planar maps... Due to the lack of space, we cannot give the relevant bibliography here.

In particular, several papers have been devoted in the past 20 years to the study of the *Potts model* on families of planar maps [1, 7, 14, 16, 18, 26]. In combinatorial terms, this means counting maps equipped with a colouring in q colours, according to the size (the number of edges) and the number of *monochromatic edges* (edges whose endpoints have the same colour). Up to a change of variables, this also means counting maps weighted by their *Tutte polynomial* (a bivariate combinatorial invariant which has numerous interesting specializations). It has recently been proved that the associated generating function is *differentially algebraic*, that is, satisfies a (non-linear) differential equation (with respect to the size variable) with polynomial coefficients [3, 4, 8]. This holds for general planar maps and for triangulations (or dualy, for cubic maps).

The method that yields these differential equations is extremely involved, and does not shed much light on the structure of q-coloured maps. Moreover, one has not been able, so far, to derive from these equations the asymptotic behaviour of the number of coloured maps, nor the location of phase transitions (see however [7] for recent progress in this direction).

The aim of this paper is to remedy these problems — so far for a one-variable specialization of the Tutte polynomial, obtained by setting to 1 one of its variables, or by taking (in an adequate way) the limit $q \to 0$ in the Potts model. Combinatorially, we are simply counting maps (in this paper, p-valent maps) equipped with a spanning forest. We call them *forested maps* (Figure 1). This problem has already been studied in [12] via a random matrix approach (but with no explicit solution) and, in a special case, in [9], which was in fact the starting point of the present paper. Our enumeration keeps track of the number of faces (variable z) and the number of trees in the forest (minus one; variable u). The case $u = 0$ thus corresponds to maps equipped with a spanning *tree*, solved a long time ago by Mullin [21].

We first obtain in Section 3, in a purely combinatorial manner, a system of functional equations defining the associated generating function $F(z, u)$. We then derive from this system that $F(z, u)$ is differentially algebraic in z, and give explicit differential equations for 3- and 4-valent maps (Section 4). Section 5 is a combinatorial interlude explaining why the series occurring in our system of equations still have non-negative coefficients when $u \in [-1, 0]$. These results are needed in Section 6, which is devoted to asymptotic results: when $u > 0$, forested maps follow the standard asymptotic behaviour of planar maps ($\mu^n n^{-5/2}$) but then a phase transition occurs at $u = 0$, and a very unusual asymptotic behaviour in $\mu^n n^{-3}(\log n)^{-2}$ holds when $u \in [-1, 0)$. To our knowledge, this is the first time a class of planar maps is shown to exhibit this behaviour. This proves in particular that $F(z, u)$ is not *D-finite*, that is, does not

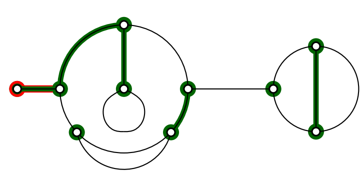

Fig. 1: A (quasi-cubic) forested map with 6 faces and 5 trees.

satisfy any *linear* differential equation in z for $u \in [-1,0)$ (nor for a generic value of u). This contrasts with the case $u = 0$, for which the generating function of maps equipped with a spanning forest is known to be D-finite.

2 Preliminaries

2.1 Planar maps and trees

A *planar map* is a proper embedding of a connected graph (possibly with loops and multiple edges) in the oriented sphere, considered up to continuous deformation. A *face* is a connected component of the complement of the map. Each edge consists of two *half-edges*, each incident to an endpoint of the edge. A *corner* is an ordered pair (e_1, e_2) of half-edges incident to the same vertex, such that e_2 immediately follows e_1 in counterclockwise order. The *degree* of a vertex is the number of corners incident to it. A vertex of degree p is called *p-valent*. One-valent vertices are also called *leaves*. A map is *p-valent* if all its vertices are p-valent. A *rooted map* is a map with a marked corner (e_1, e_2), indicated by an arrow in our figures. The *root vertex* is the vertex incident to the root. The *root edge* is the edge supporting e_2.

A *(plane) tree* is a planar map with a unique face. A tree is p-valent if all non-leaf vertices have degree p. A *leaf-rooted* (resp. *corner-rooted*) tree is a tree with a marked leaf (resp. corner). A corner-rooted and two leaf-rooted trees appear in Figure 2(b). The number of p-valent leaf-rooted (resp. corner-rooted) trees with k leaves is denoted by t_k (resp. t^c_k). These numbers are well-known [23, Thm. 5.3.10]: they are 0 unless $k = (p-2)\ell + 2$ with $\ell \geq 1$, and in this case,

$$t_k = \frac{((p-1)\ell)!}{\ell!((p-2)\ell+1)!} \qquad \text{and} \qquad t^c_k = p\frac{((p-1)\ell)!}{(\ell-1)!((p-2)\ell+2)!}. \tag{1}$$

These numbers should in principle be denoted $t_{k,p}$ and $t^c_{k,p}$, but we consider p as a fixed integer ($p \geq 3$).

Let M be a rooted planar map with vertex set V. A *spanning forest* of M is a graph $F = (V, S)$ where S is a subset of edges of M forming no cycle. Each tree of F is called a *component*, and the *root component* is the tree containing the root vertex. We say that the pair (M, F) is a *forested map*. Let us denote by $F(z, u)$ the generating function of rooted p-valent forested maps counted by faces (variable z) and non-root components (variable u). For instance, when $p = 3$, the first terms of $F(z, u)$ are

$$F(z,u) = (6 + 4\,u)\,z^3 + \left(140 + 234\,u + 144\,u^2 + 32\,u^3\right) z^4 + \cdots \tag{2}$$

The term $6z^3$ means that there are 6 rooted cubic maps with 3 faces and a distinguished spanning tree.

2.2 Forest counting, the Tutte polynomial and related models

Let $G = (V, E)$ be a graph with vertex set V and edge set E. The *Tutte polynomial* of G is the following polynomial in two indeterminates (see e.g. [6]):

$$\mathrm{T}_G(\mu, \nu) := \sum_{S \subseteq E} (\mu - 1)^{\mathrm{c}(S) - \mathrm{c}(G)} (\nu - 1)^{\mathrm{e}(S) + \mathrm{c}(S) - \mathrm{v}(G)}, \tag{3}$$

where the sum is over all spanning subgraphs of G (equivalently, over all subsets S of edges) and $\mathrm{v}(.)$, $\mathrm{e}(.)$ and $\mathrm{c}(.)$ denote respectively the number of vertices, edges and connected components.

When $\nu = 1$, the only subgraphs that contribute to (3) are the forests. Hence the above defined generating function of forested maps can be written as

$$F(z,u) = \sum_{M\ p-\text{valent}} z^{\mathrm{f}(M)}\, \mathrm{T}_M(u+1,1). \tag{4}$$

Even though this is not clear from (3), the Tutte polynomial $\mathrm{T}_G(\mu,\nu)$ has non-negative coefficients in μ and ν. This was proved combinatorially by Tutte [24], who showed that $\mathrm{T}_G(\mu,\nu)$ counts spanning *trees* of G according to two parameters, called *internal* and *external* activities (see [2] for an alternative description). It follows that $F(z,\mu-1)$ is also the generating function of p-valent planar maps M equipped with a spanning tree T, counted by the face number of M (by z) and the internal activity of T (by μ).

Using duality properties of the Tutte polynomial, and various combinatorial interpretations of $\mathrm{T}_G(1,\nu)$, we can also describe $F(z,u)$ in terms of the dual p-*angulations* equipped: either with a connected (spanning) subgraph; or with a *recurrent configuration of the sandpile model* [13, 20]; or with a q-state Potts model, taken in the limit $q \to 0$. Precise statements and details are given in the complete version of this paper.

2.3 Formal power series

Let $A = A(z) \in \mathbb{K}[[z]]$ be a power series in one variable with coefficients in a field $\mathbb{K}$. We say that A is ***D-finite*** if it satisfies a (non-trivial) linear differential equation with coefficients in $\mathbb{K}[z]$ (the ring of polynomials in z). More generally, it is ***D-algebraic*** if there exist a positive integer k and a non-trivial polynomial $P \in \mathbb{K}[x, x_0, \dots, x_k]$ such that $P\left(z, A, \frac{\partial A}{\partial z}, \dots, \frac{\partial^k A}{\partial z^k}\right) = 0$.

A k-variate power series $A = A(z_1, \dots, z_k)$ with coefficients in $\mathbb{K}$ is ***D-finite*** if its partial derivatives (of all orders) span a finite dimensional vector space over $\mathbb{K}(z_1, \dots, z_k)$.

3 Generating functions for forested maps

In this section, we give a system of equations that defines the generating function $F(z,u)$ of p-valent forested maps. In fact, it gives an expression of the series $zF'_z(z,u)$ that counts forested maps with a marked face. We also give two simpler systems for two variants of $F(z,u)$, not involving a derivative.

3.1 p-Valent maps

Theorem 3.1 *Fix $p \geq 3$. Let θ, ϕ_1 and ϕ_2 be the following doubly hypergeometric series:*

$$\theta(x,y) := \sum_{i\geq 0}\sum_{j\geq 0} t^{c}_{2i+j} \binom{2i+j}{i,i,j} x^i y^j,$$

$$\phi_1(x,y) := \sum_{i\geq 1}\sum_{j\geq 0} t_{2i+j} \binom{2i+j-1}{i-1,i,j} x^i y^j, \quad \phi_2(x,y) := \sum_{i\geq 0}\sum_{j\geq 0} t_{2i+j+1} \binom{2i+j}{i,i,j} x^i y^j, \tag{5}$$

where t_k and t^c_k are given by (1) *and $\binom{a+b+c}{a,b,c}$ denotes the trinomial coefficient $(a+b+c)!/(a!b!c!)$.*

There exists a unique pair (R,S) of series in z with constant term 0 and coefficients in $\mathbb{Q}[u]$ that satisfy

$$R = z + u\,\phi_1(R,S), \tag{6}$$
$$S = u\,\phi_2(R,S). \tag{7}$$

The generating function $F(z,u)$ of p-valent forested maps is characterized by $F(0,u) = 0$ and

$$F'_z(z,u) = \theta(R,S). \tag{8}$$

Remarks
1. These equations allow us to compute the first terms of the expansion of $F(z,u)$ in z, for any $p \geq 3$.
2. When p is even, then $t_{2i+1} = 0$ for all i and the series S vanishes, which greatly simplifies the system.
3. When $u = 0$, an even more drastic simplification occurs: not only $S = 0$, but also $R = z$, so that (8) becomes an explicit expression of F', which we can readily integrate:

$$F(z,0) = \sum_{i\geq 0} t^c_{2i} \binom{2i}{i} \frac{z^{i+1}}{i+1} = \sum_{\ell \geq 1} \frac{p((p-1)\ell)!}{(\ell-1)!(1+(p-2)\ell/2)!(2+(p-2)\ell/2)!} z^{2+(p-2)\ell/2}, \tag{9}$$

where we require ℓ to be even if p is odd. This series counts p-valent maps equipped with a spanning tree, and this expression was already proved by Mullin [21, Eq. (5.8)].

In order to prove Theorem 3.1, we first relate $F(z,u)$ to the generating function of planar maps counted by the distribution of their vertex degrees. More precisely, let $M^\diamond \equiv M^\diamond(z,u;g_1,g_2,\ldots;h_1,h_2,\ldots)$ be the generating function of rooted planar maps with a marked face, where u counts non-root vertices, z counts faces, g_k counts non-root vertices of degree k and h_k root vertices (!) of degree k.

Lemma 3.2 *The series $F(z,u)$ is related to $M^\diamond$ through:*

$$zF'_z(z,u) = M^\diamond(z,u;t_1,t_2,\ldots;t^c_1,t^c_2,\ldots). \tag{10}$$

Proof: Start from a p-valent forested map (M,F) and contract each tree of F that is incident to k half-edges (not in F) into a k-valent vertex (Figure 2). This operation can be seen as an extension of Mullin's construction for maps equipped with a spanning tree [21]. It also appears in [12] and in [9, Appendix A], where the authors study 4-valent forested maps such that the root edge is not in the forest.

To recover the forested map (M,F) from the contracted map M', one has to remember, for the root vertex of M', from which corner-rooted tree it came, and for each non-root vertex, from which *leaf-rooted* tree it came (the reason why a leaf-rooted, rather than corner-rooted tree suffices is related to the fact that the rooting of M induces a total order on its half-edges). Each vertex of M' gives a connected component of F. $\square$

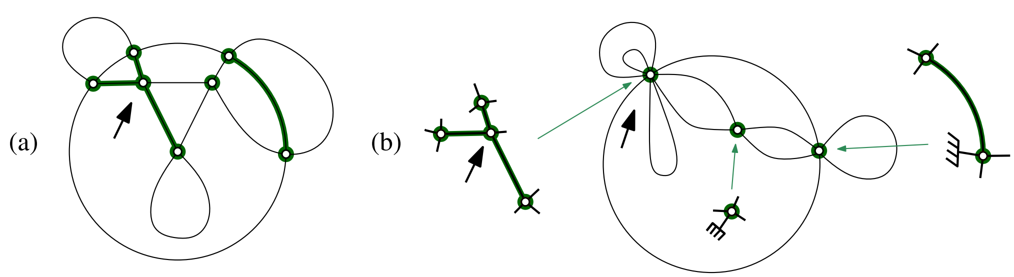

Fig. 2: (a) A 4-valent forested map with 9 faces and 2 non-root components. The arrow indicates the root. (b) The same map, after contraction of the forest, with the collection of rooted trees that stems from (a).

In a recent paper [11, Eq. (2.6)], Bouttier and Guitter have characterized the series $M^\diamond$ by a system of equations, established bijectively. Their system, specialized as in Lemma 3.2, gives Theorem 3.1. $\square$

Remark. In [11, Eq. (1.4)], the authors also give a complicated expression for the generating function $M \equiv M(g_1, g_2, \ldots; h_1, h_2, \ldots)$ that counts rooted planar maps (no marked face) by the distribution of degrees of non-root vertices and the degree of the root vertex. By the above argument, this yields a closed form expression of the series $F(z,u)$ itself. However, we have not been able to use this expression (for instance to construct a differential equation for F) without differentiating it first.

3.2 Two variants

A map is said to be *quasi-p-valent* if all its vertices have degree p, except one vertex that is a leaf. Such maps exist only when p is odd (Figure 1). These maps are naturally rooted at their leaf. Let $G(z,u)$ denote the generating function of quasi-p-valent forested maps counted by faces (z) and non-root components (u).

By relating $G(z,u)$ to the generating function of one-leg maps determined in [10], we obtain:

Proposition 3.3 *The generating function of quasi-p-valent forested maps is*

$$G(z,u) = (1+\bar u)\left(zS - u \sum_{i\geq 2}\sum_{j\geq 0} t_{2i+j-1}\binom{2i+j-2}{i-2,i,j} R^i S^j \right),$$

where $\bar u = 1/u$, the series R and S are defined by (6–7)*, and the numbers t_k by* (1)*. Also,*

$$G'_z(z,u) = (1+\bar u)S.$$

We finally consider p-valent forested maps such that the root edge is outside the forest. Let $H(z,u)$ denote the associated generating function. We can relate $H(z,u)$ to the generating function of general planar maps that has been determined in [10]. This yields the following proposition.

Proposition 3.4 *The generating function $H(z,u)$ of p-valent forested maps such that the root edge is outside the forest is*

$$H(z,u) = \bar u z R + \bar u z S^2 - \bar u z^2 \\ - 2S \sum_{i\geq 2}\sum_{j\geq 0} t_{2i+j-1}\binom{2i+j-2}{i-2,i,j} R^i S^j - \sum_{i\geq 3}\sum_{j\geq 0} t_{2i+j-2}\binom{2i+j-3}{i-3,i,j} R^i S^j$$

where $\bar u = 1/u$, the series R and S are defined by (6–7)*, and the numbers t_k by* (1)*.*

When p is even, then $S = 0$ and the first double sum disappears. In this case, we also have a very simple expression of $H'_z(z,u)$:

$$H'_z(z,u) = 2\bar u(R - z).$$

4 Differential equations

The equations established in the previous section imply that series counting regular forested maps are D-algebraic. We prove this and compute explicitly a few differential equations.

Theorem 4.1 *The generating function $F(z,u)$ of p-valent forested maps is D-algebraic (with respect to z). The same holds for the series $G(z,u)$ and $H(z,u)$ of Propositions 3.3 and 3.4.*

Proof: We start from the expression of $F' \equiv F'_z$ given in Theorem 3.1. We first observe that the doubly hypergeometric series θ, ϕ_1, ϕ_2 are D-finite. This follows from the closure properties of D-finite multivariate series [19]. Then, by differentiating (6) and (7) with respect to z, we obtain rational expressions of R' and S' in terms of u and the partial derivatives $\partial\phi_\ell/\partial x$ and $\partial\phi_\ell/\partial y$, evaluated at (R,S) (for $\ell = 1, 2$).

Let $\mathbb{K}$ be the field $\mathbb{Q}(u)$. Using (8) and the previous point, it is now easy to prove by induction that for all $k \geq 1$, there exists a rational expression of $F^{(k)}(z,u)$ in terms of

$$\left\{ \frac{\partial^{i+j}\phi_\ell}{\partial x^i \partial y^j}(R,S), \frac{\partial^{i+j}\theta}{\partial x^i \partial y^j}(R,S) \right\}_{i\geq 0, j\geq 0, \ell\in\{1,2\}}$$

with coefficients in $\mathbb{K}$. But since θ, ϕ_1 and ϕ_2 are D-finite, the above set of series spans a vector space of finite dimension, say d, over $\mathbb{Q}(R,S)$. Therefore there exist d elements $\varphi_1, \ldots, \varphi_d$ in this space, and rational functions $A_k \in \mathbb{K}(x, y, x_1, \ldots, x_d)$, such that $F^{(k)}(z,u) = A_k(R, S, \varphi_1, \ldots, \varphi_d)$ for all $k \geq 1$.

Since the transcendence degree of $\mathbb{K}(R, S, \varphi_1, \ldots, \varphi_d)$ over $\mathbb{K}$ is (at most) $d+2$, the $d+3$ series $F', F'', \ldots, F^{(d+3)}$ are algebraically dependant over $\mathbb{K}$. This implies that F' (and thus F) is D-algebraic.

The proof is similar for the series $G(z,u)$ and $H(z,u)$. □

4.1 The 4-valent case

We specialize the above argument to the case $p = 4$. As mentioned below Theorem 3.1, the series S vanishes. The series $F(z,u)$ is characterized by

$$F'_z = \theta(R), \qquad R = z + u\,\phi(R), \tag{11}$$

with

$$\theta(x) = 4\sum_{i\geq 2} \frac{(3(i-1))!}{i!^2(i-2)!} x^i \quad \text{and} \quad \phi(x) = \sum_{i\geq 2} \frac{(3(i-1))!}{i!(i-1)!^2} x^i.$$

The series $\theta(x)$, $\phi(x)$ and their derivatives live in a 3-dimensional vector space over $\mathbb{Q}(x)$ spanned (for instance) by 1, $\theta(x)$ and $\phi(x)$. This follows from the following equations, which are easily checked:

$$x(27x-1)\phi''(x) + 6\phi(x) + 6x = 0, \qquad 3\theta(x) = 2(27x-1)\phi'(x) - 42\phi(x) + 12x.$$

By the argument described above, we can then express F' and all its derivatives as rational functions of u, R, $\phi(R)$ and $\theta(R)$. But since $R = z + u\phi(R)$, this means a rational function of u, z, R and $\theta(R)$. We compute these expressions for F', F'' and F''', eliminate R and $\theta(R)$ from these three equations, and this gives a differential equation of order 2 and degree 7 satisfied by F':

$$\begin{gathered}
9\,F'^2F''^5u^6+36\,F'^2F''^3F'''\,u^5z+144\,F'^2F''^4u^5-12\,(21\,z-1)F'\,F''^5u^5+432\,F'^2F''^2F'''\,u^4z-48\,(24\,z-1)F'\,F''^3F'''\,u^4z\\
+864\,F'^2F''^3u^4-96\,(27\,z-2)F'\,F''^4u^4+4\,(27\,z-1)(15\,z-1)F''^5u^4+1728\,F'^2F''\,F'''\,u^3z-288\,(21\,z-2)F'\,F''^2F'''\,u^3z\\
+10368\,F'\,F'''^2u^2z^3+16\,(27\,z-1)(21\,z-1)F''^3F'''\,u^3z+2304\,F'^2F''^2u^3-288\,(31\,z-4)F'\,F''^3u^3\\
-64\left(6\,uz-162\,z^2+33\,z-1\right)F''^4u^3+2304\,F'^2F'''\,u^2z-2304\,(6\,z-1)F'\,F''\,F'''\,u^2z\\
-192\left(8\,uz-54\,z^2+29\,z-1\right)F''^2F'''\,u^2z-768\,(2\,u+189\,z-7)F'''^2uz^3+2304\,F'^2F''\,u^2-3072\,(3\,z-1)F'\,F''^2u^2\\
-192\left(24\,uz-27\,z^2+55\,z-2\right)F''^3u^2-1536\,(21\,z-2)F'\,F'''\,uz-768\left(12\,uz+81\,z^2+24\,z-1\right)F''\,F'''\,uz+1536\,(9\,z+2)F'\,F''\,u\\
-512\left(39\,uz+81\,z^2+51\,z-2\right)F''^2u+36864\,F'\,z-1024\left(12\,uz-162\,z^2+33\,z-1\right)F'''\,z-1024\,(36\,uz+27\,z-1)F''-24576\,z=0.
\end{gathered}$$

We do not know if F itself satisfies a differential equation of order 2. For the series H of Proposition 3.4 however, a similar approach gives an equation of order 2 and degree 3:

$$3\,(u+1)u^2{H'}^2H'' + 12\,u^2zH'H'' + 6\,(u-8)u{H'}^2 + 240\,H \\ + 4\,(6\,uz - 54\,z + 1)H' + 4\,(3\,uz^2 + 30\,uH + 27\,z^2 - z)H'' + 24\,z^2 = 0.$$

4.2 The cubic case

The cubic case is heavier, since we now have to deal with series ϕ_1 and ϕ_2 in two variables. We obtain for $F'(z,u)$ a differential equation of order 2 and degree 17. For the generating function $G(z,u)$ of quasi-cubic forested maps, the degree is only 5:

$$0 = (3\,u^4z{W'}^4 - u^3(5\,Wu - uz + z){W'}^3 + 4\,(u+1)(5\,Wu - uz + z)^2)W'' \\ - 48\,u^2z(u+1){W'}^3 + 8\,u(u+1)(5\,Wu - uz + z){W'}^2 + 4\,(u-1)(u+1)(5\,Wu - uz + z)W',$$

where $G = (W + z\bar{u})/2$. Introducing the series W is natural in the solution of the Potts model presented in [4], where the above equation was first obtained. It makes the equation more compact.

5 Combinatorics of forested trees

As shown by (4), the series $F(z,u)$ that counts p-valent forested maps has non-negative coefficients in $(1+u)$. We say that it is $(u+1)$-*positive*. More precisely, $F(z,\mu-1)$ counts p-valent maps equipped with a spanning tree weighted by its *internal activity* (by μ). This will lead us to study the asymptotic behaviour of the coefficient of z^n in $F(z,u)$ not only for $u \geq 0$, but for $u \geq -1$. However, our main tool, namely the *singularity analysis* of [17], is much easier when applied to series with non-negative coefficients, and we will need to know that a few other series, related to F, are also $(u+1)$-positive. We prove this thanks to a combinatorial argument that applies to several classes of *forested trees*.

5.1 Positivity in $(1+u)$

Let T be a tree having at least one edge, and $\mathcal{F}$ a set of spanning forests of T. We define a property of $\mathcal{F}$ that guarantees that the generating function $A_{\mathcal{F}}(u)$ that counts forests of $\mathcal{F}$ by the number of components is $(u+1)$-positive (after division by u).

Let $F \in \mathcal{F}$, and let e be an edge of T. By *flipping* e in the forest F, we mean adding e to F if it is not in F, and removing it from F otherwise. This gives a new forest F' on T. We say that e is *flippable* for F if F' still belongs to $\mathcal{F}$. We say that $\mathcal{F}$ is *stable* if for each $F \in \mathcal{F}$ (i) every edge of T not belonging to F is flippable, and (ii) flipping a flippable edge gives a new forest with the same set of flippable edges.

Lemma 5.1 *Assume $\mathcal{F}$ is stable. Then all elements of $\mathcal{F}$ have the same number, say f, of flippable edges, and the generating function of forests of $\mathcal{F}$, counted by components, is $A_{\mathcal{F}}(u) = u(1+u)^f$.*

Proof: The stability of $\mathcal{F}$ implies that the forest $F_{\max}$ consisting of all edges of T belongs to $\mathcal{F}$. Moreover, we can obtain $F_{\max}$ from any forest F of $\mathcal{F}$ by adding iteratively flippable edges. By Condition (ii), this implies that any forest F of $\mathcal{F}$ has the same set of flippable edges as $F_{\max}$. It also means that, to construct a forest F of $\mathcal{F}$, it suffices to choose, for each flippable edge of $F_{\max}$, whether it belongs to F or not. Since $F_{\max}$ has a unique component, and since deleting an edge from a forest increases by 1 the number of components, the expression of $A_{\mathcal{F}}(u)$ follows. □

5.2 Enriched blossoming trees

Define R and S by (6–7), and $\tilde{S}$ by $\tilde{S} = u\,\phi_2(z, \tilde{S})$, where ϕ_2 is given by (5). We now give combinatorial interpretations of these three series in terms of forested trees.

We consider leaf-rooted plane trees, which we draw hanging from their root as in Figure 3. A vertex of degree d is seen as the parent of $d-1$ children. A *subtree* consists of a vertex and all its descendants. A *blossoming tree* is a leaf-rooted plane tree with two kinds of childless vertices: *leaves*, represented by white arrows, and *buds*, represented by black arrows. The edges that carry leaves and buds, as well as the root edge, are considered as half-edges. Each leaf is assigned a *charge* $+1$ while each bud is assigned a charge -1. The *charge* of a subtree is the difference between the number of leaves and buds that it contains.

Definition 5.2 *Let $p \geq 3$. A p-valent blossoming tree equipped with a spanning forest F is an* enriched R- (resp. S-) tree *if (i) its total charge is 1 (resp. 0) and (ii) any subtree rooted at an edge* not in *F has charge 0 or 1. It is an* enriched $\tilde{\text{S}}$-tree *if each component of F is incident to as many leaves as buds (in this case it is also an enriched S-tree).*

Proposition 5.3 *The series R, S and $\tilde{S}$ count enriched R-, S- and $\tilde{\text{S}}$-trees by the number of leaves (z) and the number of components in the forest (u).*

Proof: The readers who are familiar with the R- and S-trees of [10] will recognize that our enriched R- and S-trees are obtained from them by inflating each vertex of degree k into a (leaf rooted) p-valent tree with k leaves. Thus (6–7) follows from [10] by specializing the indeterminate g_k to t_k.

For the other readers (and for the series $\tilde{S}$), the equations follow from a recursive decomposition of enriched trees. For instance, an enriched R-tree is either reduced to a single leaf, or consists of a root component (say, with k incident edges) in which each non-root incident edge is replaced either by a bud, or an enriched R-tree, or an enriched S-tree. If there are i attached enriched R-trees, we must have $i-1$ buds for the total charge to be 1, and j S-trees with $k-1 = 2i-1+j$. This gives (6). □

Proposition 5.4 *Let T be a p-valent blossoming tree with charge 1 (resp. 0), having at least one edge, and $\mathcal{F}$ the set of spanning forests of T that make it an enriched R- (resp. S-) tree. Then $\mathcal{F}$ is stable, in the sense of Section 5.1. The same holds if T is a p-valent blossoming tree with charge 0, and $\mathcal{F}$ the set of spanning forests of T that make it an enriched $\tilde{\text{S}}$-tree.*

Proof: An edge is flippable if and only if the attached subtree has charge 0 or 1. □

By combining this proposition with Lemma 5.1, and Proposition 5.3, we obtain:

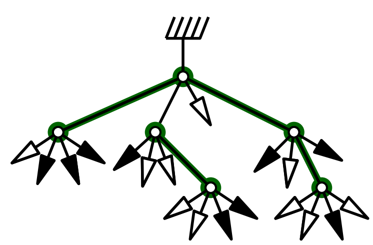

Fig. 3: An enriched 5-valent R-tree. It has 10 leaves (white; charge +1) and 9 buds (black; charge -1).

Corollary 5.5 *The series $\bar{u}(R-z)$, $\bar{u}S$ and $\bar{u}\tilde{S}$ are $(u+1)$-positive.*

6 Asymptotic results

Let $p \geq 3$, and let $F(z,u) = \sum_n f_n(u) z^n$ be the generating function of p-valent forested maps, given by Theorem 3.1. That is, $f_n(u)$ counts p-valent forested maps with n faces by the number of non-root components. As recalled in Section 2.2, $f_n(\mu - 1)$ also counts p-valent maps with n faces equipped with a spanning tree, with a weight μ on each *internally active* edge, or p-angulations equipped with a *recurrent sandpile configuration* weighted (by μ) by its *level* [13, 20]. This explains why we will study the asymptotic behaviour of $f_n(u)$ for any $u \geq -1$.

Here, we first state our results for 4-valent maps and discuss the proof and its difficulties. The fact that p is even simplifies the system of Theorem 3.1 and makes 4-valent maps the most tractable case. We then briefly describe the (analogous) results obtained for cubic maps, with the new difficulties raised by the system of two equations defining the series R and S.

Theorem 6.1 *Let $p = 4$, and take $u \in [-1, +\infty)$. Let $f_n(u)$ be the coefficient of z^n in $F(z,u)$. There exists a positive constant κ_u, depending on u only, such that*

$$f_n(u) \sim \begin{cases} \kappa_u \rho_u^{-n} n^{-3} (\log n)^{-2} & \text{if } u \in [-1, 0), \\ \kappa_u \rho_u^{-n} n^{-3} & \text{if } u = 0, \\ \kappa_u \rho_u^{-n} n^{-5/2} & \text{if } u > 0. \end{cases}$$

Moreover, the radius ρ_u of $F(z,u)$ is an affine function of u when $u \in [-1, 0]$:

$$\rho_u = \frac{1+u}{27} - u\frac{\sqrt{3}}{12\pi}. \tag{12}$$

For $u > 0$, the subexponential term $n^{-5/2}$ is typical of rooted maps. The behaviour for negative values of u is much more surprising. In fact, we prove that in this case, the singular behaviour of $F'_z(z,u)$ at its unique dominant singularity $\rho \equiv \rho_u$ involves a term $(1 - z/\rho)/\log(1 - z/\rho)$. Since this cannot be the singular behaviour of a D-finite series [17, p. 520 and 582], we have the following corollary.

Corollary 6.2 *For $u \in [-1, 0)$, the generating function $F(z,u)$ of 4-valent forested maps is D-algebraic, but not D-finite. The same holds when u is an indeterminate.*

Note also the simplicity of the radius for $u \leq 0$. In particular, $F(z, -1)$ counts 4-valent maps equipped with a spanning tree having no *internal activity*, and this series has a transcendental radius $\sqrt{3}/(12\pi)$.

Recall that $F(z, 0)$ is explicit (see (9)). The estimate of $f_n(0)$ follows from Stirling's formula.

Proof of Theorem 6.1 (sketched): Recall the equations (11) that define the series $F(z,u)$. The key point is to study the singular behaviour of the series R defined implicitly by (11). When $u > 0$, we are in the *smooth implicit function schema* discussed for instance in [17, p. 467]: the series $R(z,u)$ becomes singular when $1 = u\phi'(R)$, and this occurs before R reaches the radius $1/27$ of ϕ and θ. It follows that R, and also F'_z, have a square root singularity at ρ. One then proves that F'_z is analytic in a Δ-domain and concludes, using singularity analysis, that $n f_n(u) \sim \kappa_u \rho_u^{-n} n^{-3/2}$. In brief, the singularities of ϕ and θ are not felt when $u > 0$.

When $u < 0$ however, the conditions of the implicit function theorem do not fail, but R reaches at its radius the singularity of ϕ, namely $1/27$. Since $\phi(1/27) = \sqrt{3}/(12\pi) - 1/27$, Eq. (11) gives the value (12) of ρ_u. The singular behaviour of R now depends on the singular behaviour of ϕ, and is found to be in $(1-z/\rho)/\log(1-z/\rho)$. Similarly, we need to know the singular behaviour of θ to derive the behaviour of F': it is found to behave like R, and we conclude via singularity analysis that $nf_n \sim \kappa_u \rho_u^{-n} n^{-3} (\log n)^{-2}$.

Several ingredients make the case $u < 0$ significantly harder than the case $u > 0$: one of them is that the series R has no longer non-negative coefficients (this is however partially alleviated by Corollary 5.5); it is also harder to prove that R has a unique dominant singularity; finally, we obviously need to know the singular behaviours of ϕ and θ (but these can be found in the literature). □

We have also worked out the asymptotic behaviour of $f_n(u)$ in the cubic case ($p = 3$). This is harder than the 4-valent case, since we now have to deal with a *system* of equations defining R and S (however, $\phi_1(x, y)$ and $\phi_2(x, y)$ can be expressed in terms of hypergeometric functions of $x/(1 - 4y)^2$, which simplifies things a bit). Of course the case $u < 0$ is again harder than the case $u > 0$. Lemma 5.1 and Corollary 5.5 are crucial in the study of this case. The results are the same as in Theorem 6.1, except that the radius for negative u is now a quadratic (rather than affine) function of u, with $\rho_{-1} = \pi^2/384$.

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Structure and enumeration of $(3 + 1)$*-free posets (extended abstract)*

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Abstract. A poset is $(3 + 1)$-free if it does not contain the disjoint union of chains of length 3 and 1 as an induced subposet. These posets are the subject of the $(3 + 1)$-free conjecture of Stanley and Stembridge. Recently, Lewis and Zhang have enumerated *graded* $(3+1)$-free posets, but until now the general enumeration problem has remained open. We enumerate all $(3 + 1)$-free posets by giving a decomposition into bipartite graphs, and obtain generating functions for $(3 + 1)$-free posets with labelled or unlabelled vertices.

Résumé. Un poset sans $(3 + 1)$ est un poset qui n'a pas de sous-poset induit formé de deux chaînes disjointes de longeur 3 et 1. Ces posets sont l'objet de la conjecture $(3+1)$ de Stanley et Stembridge. Récemment, Lewis et Zhang on énuméré les posets *étagés* sans $(3 + 1)$, mais en général la question d'énumération est restée ouverte jusqu'à maintenant. Nous énumérons tous les posets sans $(3 + 1)$ en donnant une décomposition de ces posets en graphes bipartis, et obtenons des fonctions génératrices qui les énumèrent, qu'ils soient étiquetés ou non.

Keywords: (3+1)-free posets, trace monoid, generating functions, chromatic symmetric function

1 Introduction

A poset P is $(i + j)$*-free* if it contains no induced subposet that is isomorphic to the poset consisting of two disjoint chains of lengths i and j. In particular, P is $(3 + 1)$-free if there are no vertices $a, b, c, d \in P$ such that $a < b < c$ and d is incomparable to a, b, and c.

Posets that are $(3+1)$-free play a role in the study of Stanley's chromatic symmetric function [12, 13], a symmetric function associated with a poset that generalizes the chromatic polynomial of a graph. Namely, a well-known conjecture of Stanley and Stembridge [16] is that the chromatic symmetric function of a $(3 + 1)$-free poset has positive coefficients in the basis of elementary symmetric functions. As evidence toward this conjecture, Stanley [12] verified the conjecture for the class of 3-free posets, and Gasharov [5] has shown the weaker result that the chromatic symmetric function of a $(3+1)$-free poset is Schur-positive.

To make more progress toward the Stanley–Stembridge conjecture, a better understanding of $(3 + 1)$-free posets is needed. Reed and Skandera [9, 10] have given structural results and a characterization of $(3 + 1)$-free posets in terms of their antiadjacency matrix. In addition, certain families of $(3 + 1)$-free posets have been enumerated. For example, the number of $(3 + 1)$-and-$(2 + 2)$-free posets with n

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vertices is the nth Catalan number [15, Ex. 6.19(ddd)]; Atkinson, Sagan and Vatter [1] have enumerated the permutations that avoid the patterns 2341 and 4123, which give rise to the $(3+1)$-free posets of dimension two; and Lewis and Zhang [7] have made significant progress by enumerating *graded* $(3+1)$-free posets in terms of bicoloured graphs[(i)] using a new structural decomposition. However, until now the general enumeration problem for $(3+1)$-free posets remained open [14, Ex. 3.16(b)].

In this paper, we give generating functions for $(3+1)$-free posets with unlabelled and labelled vertices in terms of the generating functions for bicoloured graphs with unlabelled and labelled vertices, respectively. As in the graded case, the two problems are equally hard, although the enumeration problem for bicoloured graphs has received more attention.

In the unlabelled case, let $p_{\text{unl}}(n)$ be the number of $(3+1)$-free posets with n unlabelled vertices, and let $S(c,t)$ be the unique formal power series solution (in c and t) of the cubic equation

$$S(c,t) = 1 + \frac{c}{1+c} S(c,t)^2 + tS(c,t)^3. \tag{1}$$

We show that the ordinary generating function for unlabelled $(3+1)$-free posets is

$$\sum_{n\geq 0} p_{\text{unl}}(n) x^n = S\big(x/(1-x), 1 - 2x - B_{\text{unl}}(x)^{-1}\big), \tag{2}$$

where $B_{\text{unl}}(x) = 1 + 2x + 4x^2 + 8x^3 + 17x^4 + \cdots$ is the ordinary generating function for unlabelled bicoloured graphs. Before our investigation, the On-Line Encyclopedia of Integer Sequences [11] had 22 terms in the entry [11, A049312] for the coefficients of $B_{\text{unl}}(x)$, but only 7 terms in the entry [11, A079146] for the numbers $p_{\text{unl}}(n)$. Using (2), we have closed this gap; the numbers $p_{\text{unl}}(n)$ for $n = 0, 1, 2, \ldots, 22$ are

1, 1, 2, 5, 15, 49, 173, 639, 2469, 9997, 43109, 205092, 1153646, 8523086, 91156133, 1446766659, 32998508358, 1047766596136, 45632564217917, 2711308588849394, 219364550983697100, 24151476334929009951, 3618445112608409433287.

Similarly, in the labelled case, let $p_{\text{lbl}}(n)$ be the number of $(3+1)$-free posets with n labelled vertices. We show that the exponential generating function for labelled $(3+1)$-free posets is

$$\sum_{n\geq 0} p_{\text{lbl}}(n) \frac{x^n}{n!} = S\big(e^x - 1, 2e^{-x} - 1 - B_{\text{lbl}}(x)^{-1}\big), \tag{3}$$

where $B_{\text{lbl}}(x) = \sum_{n\geq 0} \sum_{i=0}^{n} \binom{n}{i} 2^{i(n-i)} \frac{x^n}{n!}$ is the exponential generating function for labelled bicoloured graphs. Such bicoloured graphs are easy to count, but before our investigation the OEIS had only 9 terms in the entry [11, A079145] for $p_{\text{lbl}}(n)$. Using (3), arbitrarily many terms $p_{\text{lbl}}(n)$ can be computed.

Our main tool is a new decomposition of $(3+1)$-free posets into *parts* (called *clone sets* and *tangles*). This *tangle decomposition* is compatible with the automorphism group, in the sense that for a $(3+1)$-free poset P, $\mathrm{Aut}(P)$ breaks up as the direct product of the automorphism groups of its parts. The tangle decomposition also generalizes a decomposition of Reed and Skandera [10] for $(3+1)$-and-$(2+2)$-free

[(i)] Throughout this paper, a *bicoloured* graph is a bipartite graphs with a specified ordered bipartition. For example, there are 2 bicoloured graphs with 1 vertex, 6 bicoloured graphs with 2 labelled vertices, and 4 bicoloured graphs with 2 unlabelled vertices.

posets given by *altitudes* of vertices. In terms of generating functions, the restriction of our results to $(3+1)$-and-$(2+2)$-free posets corresponds to the specialization $t = 0$ in (1). Indeed, one can see that $S(x/(1-x), 0)$ satisfies the functional equation for the Catalan generating function, which is consistent with the enumeration result stated earlier for $(3+1)$-and-$(2+2)$-free posets [15, Ex. 6.19(ddd)].

Remark 1.1. Using the tangle decomposition it is possible to quickly generate all $(3+1)$-free posets of a given size up to isomorphism in a straightforward way (see Theorem 3.10). With this approach, we were able to list all $(3+1)$-free posets on up to 11 vertices in a few minutes on modest hardware. Note that this technique can accommodate the generation of interesting subclasses of $(3+1)$-free posets (*e.g.*, $(2+2)$-free, weakly graded, strongly graded, co-connected, fixed height) or constructing these posets from the bottom up, level by level (which can help compute invariants like the chromatic symmetric function).

Remark 1.2. Comparing the list of numbers above with data provided by Joel Brewster Lewis for the number of graded $(3+1)$-free posets [11, A222863, A222865], it appears that, asymptotically, almost all $(3+1)$-free posets are graded. We prove this in the full version of this paper [6], building on the asymptotic analysis of Lewis and Zhang for the graded $(3+1)$-free posets. In fact, almost all $(3+1)$-free posets are 3-free, so their Hasse diagrams are bicoloured graphs.

OUTLINE. In Section 2, we describe the tangle decomposition of a $(3+1)$-free poset into clone sets and tangles and use it to compute the poset's automorphism group. In Section 3, we describe the relationships between the different clone sets and tangles of a $(3+1)$-free poset as parts of a structure called the *skeleton* and enumerate the possible skeleta. In Section 4, we enumerate tangles in terms of bicoloured graphs, and as a result we obtain generating functions for $(3+1)$-free posets.

2 The tangle decomposition

Throughout the paper, we assume that P is a $(3+1)$-free poset. We write $a \parallel b$ if vertices a and b in a poset are incomparable. In this section, we describe the tangle decomposition of a $(3+1)$-free poset.

Given a vertex $a \in P$, we write $D_a = \{x \in P : x < a\}$ and $U_a = \{x \in P : x > a\}$ for the (strict) downset and upset of a. The set $\mathcal{J}(P)$ of all downsets of P (that is, all downward closed subsets of P, not just those of the form D_a for some $a \in P$) forms a distributive lattice, and in particular a poset, under set inclusion. Similarly, the set of upsets of P forms a poset under set inclusion, but it will be convenient for us to consider instead the complements $P \setminus U_a \in \mathcal{J}(P)$ of the upsets of vertices $a \in P$.

Definition 2.1. The *view* $v(a)$ from a vertex $a \in P$ is the pair $(D_a, P \setminus U_a) \in \mathcal{J}(P) \times \mathcal{J}(P)$. If $v(a) = v(b)$, then we say a and b are *clones* and write $a \approx b$.

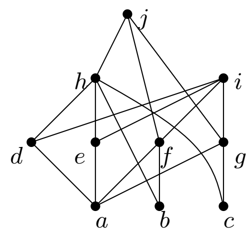
Note that the set $v(P)$ of views of all vertices of P inherits a poset structure from the set $\mathcal{J}(P) \times \mathcal{J}(P)$, where $v(a) \leq v(b)$ if and only if $D_a \subseteq D_b$ and $U_a \supseteq U_b$.

Also note that two vertices $a, b \in P$ are clones precisely when they are *interchangeable*, in the sense that the permutation of the vertices of P which only exchanges a and b is an automorphism of P.

Example 2.2. Figure 1 shows a $(3+1)$-free poset and its view poset. Since $v(d) = v(e)$, we have $d \approx e$.

Remark 2.3. The notion of clones is related to the notion of *trimming* of Lewis and Zhang [7]. Also, Zhang [18] has used techniques involving clones and $(2+2)$-avoidance to prove enumeration results about families of graded posets.

Definition 2.4. Let $a, b \in P$. We write $a \mathrel{\overset{\bullet\,\bullet}{\bowtie}} b$ if $D_a \parallel D_b$, and we write $a \mathrel{\underset{\bullet\,\bullet}{\bowtie}} b$ if $U_a \parallel U_b$.



$$\begin{aligned}
v(j) &= (\{abcdefgh\}, \{abcdefghij\}) \\
v(i) &= (\{abcdefg\}, \{abcdefghi\}) \\
v(h) &= (\{abcde\}, \{abcdefghi\}) \\
v(g) &= (\{ac\}, \{abcdefgh\}) \\
v(f) &= (\{ab\}, \{abcdefgh\}) \\
v(e) &= (\{a\}, \{abcdefg\}) \\
v(d) &= (\{a\}, \{abcdefg\}) \\
v(c) &= (\{\}, \{abcdef\}) \\
v(b) &= (\{\}, \{abcde\}) \\
v(a) &= (\{\}, \{abc\})
\end{aligned}$$

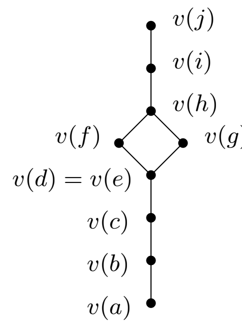
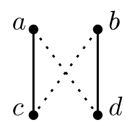


Fig. 1: Left: the Hasse diagram of a $(3+1)$-free poset P with 10 vertices. Centre: the list of views of the vertices of P. Right: the view poset $v(P)$.

The idea behind the notation is the following. If $a \mathrel{\overset{\uparrow\uparrow}{\bowtie}} b$, then there is some vertex $c \in D_a \setminus D_b$, so that $c < a$ and $c \not< b$, and there is some $d \in D_b \setminus D_a$, so that $d \not< a$ and $d < b$. Then, it can be checked that a, b, c, d are distinct vertices, and that they are incomparable except for the two relations $c < a$ and $d < b$. Hence we have the following induced $(2+2)$ subposet with a and b on the top:



Dually, if $a \mathrel{\underset{\downarrow\downarrow}{\bowtie}} b$ then there is an induced $(2+2)$ subposet with a and b on the bottom.

Example 2.5. In Figure 1, we have $f \mathrel{\overset{\uparrow\uparrow}{\bowtie}} g$ and $b \mathrel{\underset{\downarrow\downarrow}{\bowtie}} c$, but *not* $a \mathrel{\underset{\downarrow\downarrow}{\bowtie}} b$, since $U_b \subseteq U_a$.

The following lemma records basic properties of the relations $\approx$, $\mathrel{\overset{\uparrow\uparrow}{\bowtie}}$, and $\mathrel{\underset{\downarrow\downarrow}{\bowtie}}$ and their interactions.

Lemma 2.6. *Let P be a $(3+1)$-free-poset, and let a, b, c be any vertices of P.*

(i) If $a \approx b$ and $b \approx c$, then $a \approx c$.

(ii) If $a \mathrel{\overset{\uparrow\uparrow}{\bowtie}} b$ and $b \approx c$, then $a \mathrel{\overset{\uparrow\uparrow}{\bowtie}} c$.

(iii) If $a \mathrel{\underset{\downarrow\downarrow}{\bowtie}} b$ and $b \approx c$, then $a \mathrel{\underset{\downarrow\downarrow}{\bowtie}} c$.

(iv) If $a \mathrel{\overset{\uparrow\uparrow}{\bowtie}} b$, then $U_a = U_b$.

(v) If $a \mathrel{\underset{\downarrow\downarrow}{\bowtie}} b$, then $D_a = D_b$.

(vi) We have $v(a) \parallel v(b)$ if and only if $a \mathrel{\overset{\uparrow\uparrow}{\bowtie}} b$ or $a \mathrel{\underset{\downarrow\downarrow}{\bowtie}} b$.

(vii) It is not the case that both $a \mathrel{\overset{\uparrow\uparrow}{\bowtie}} b$ and $b \mathrel{\underset{\downarrow\downarrow}{\bowtie}} c$.

Now, consider a graph Γ on the vertices of P with edge set $\{(a, b) : a \mathrel{\overset{\uparrow\uparrow}{\bowtie}} b\}$. We say that a subset $A \subseteq P$ is the *top of a tangle* if $|A| \geq 2$ and A, when viewed as a subset of $V(\Gamma)$, is a connected component of Γ. Analogously, a subset $B \subseteq P$ is the *bottom of a tangle* if $|B| \geq 2$ and B is a connected component under the relation $\mathrel{\underset{\downarrow\downarrow}{\bowtie}}$.

By conclusion (vii) of Theorem 2.6, if A is the top of a tangle and B is the bottom of a tangle, then $A \cap B = \emptyset$. Let us say that a top of a tangle A and a bottom of a tangle B are *matched* if there is an induced $(2+2)$ subposet whose top two vertices are in A, and whose bottom two vertices are in B.

Proposition 2.7. *In a* $(3+1)$*-free poset* P*, every top of a tangle is matched to a unique bottom of a tangle, and every bottom of a tangle is matched to a unique top of a tangle. That is, there is a perfect matching between tops of tangles and bottoms of tangles of* P*.*

Theorem 2.7 justifies the terms 'top of a tangle' and 'bottom of a tangle' and the following definition.

Definition 2.8. A *tangle* is a matched pair $T = (A, B)$ of a top of a tangle A and a bottom of a tangle B.

In other words, a tangle is a subposet of P that is connected by induced $(2+2)$ subposets. In particular, P is $(2+2)$-free exactly when it has no tangles.

Example 2.9. Very often, a two-level poset which is not connected consists of a single tangle. For example, let P be the poset with vertices $\{a_1, a_2, a_3, c_1, c_2\} \cup \{b, d\}$ and relations $a_i > c_j$, $b > d$. Then, the connected components of P are $\{a_1, a_2, a_3, c_1, c_2\}$ and $\{b, d\}$. Every subset of the form $\{a_i, b, c_j, d\}$ forms an induced $(2+2)$ subposet, so $\{a_1, a_2, a_3, b\}$ is the top of a tangle, $\{c_1, c_2, d\}$ is the bottom of a tangle, and the whole poset P is a single tangle.

Example 2.10. In the poset P of Figure 1, the connected component of f under $\bowtie$ is $\{f, g\}$, and the connected component of b under $\bowtie$ is $\{b, c\}$. Therefore P contains the tangle $T = (\{f, g\}, \{b, c\})$. One can check that in fact this is the only tangle of P.

Definition 2.11. Let $T_1 = (A_1, B_1), \ldots, T_s = (A_s, B_s)$ be the tangles of P. A *clone set* is an equivalence class, under $\approx$, of vertices in $P \setminus \bigcup_{j=1}^{s} (A_j \cup B_j)$. We refer to tangles and clone sets as *parts* of P. The set of parts is the *tangle decomposition* of P.

Example 2.12. The tangle decomposition of the poset in Figure 1 appears in Figure 2. It consists of six parts—five clone sets and one tangle.

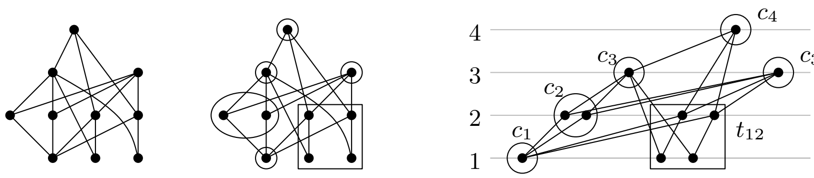


Fig. 2: Left: the Hasse diagram of the poset P from Figure 1. Centre: the tangle decomposition of P into its parts. Right: a compatible listing of the parts. Clone sets are enclosed in circles, and tangles are enclosed in boxes.

The tangle decomposition provides a decomposition of a $(3+1)$-free poset from which the automorphism group, among other properties, can be computed. To show this, it will be useful to have a different characterization of the tops of tangles, bottoms of tangles, and clone sets of P which gives a natural ordering of these subsets of P, as follows. A *co-connected component* of a poset Q is a connected component of the incomparability graph of Q.

Proposition 2.13. *Let* $v(P) \subseteq \mathcal{J}(P) \times \mathcal{J}(P)$ *be the poset of views of all vertices of the* $(3+1)$*-free poset* P*. Then, there is a listing* $(S_1, S_2, \ldots, S_k)$ *of the co-connected components of* $v(P)$ *such that for every* $x \in S_i$ *and every* $y \in S_{i+1}$*, we have* $x < y$*. Moreover, the preimages* $v^{-1}(S_i)$ *for* $i = 1, 2, \ldots, k$ *are exactly the tops of tangles, bottoms of tangles, and clone sets of* P*.*

Let $\mathrm{Aut}(P)$ be the automorphism group of the poset P. Any part X_i of P gives an induced subposet of P, and we write $\mathrm{Aut}(X_i)$ for its automorphism group as a poset. In particular, if X_i is a clone set with k vertices, then $\mathrm{Aut}(X_i)$ is the symmetric group on these k vertices; if X_i is a tangle, then it can be seen as a bicoloured graph (with colour classes 'top' and 'bottom'), and $\mathrm{Aut}(X_i)$ is the group of colour-preserving automorphisms of this graph.

Theorem 2.14. *Let P be a $(3+1)$-free poset, decomposed into its clone sets $C_1, C_2, \ldots, C_r$ and its tangles $T_1, T_2, \ldots, T_s$. Then, the automorphism group of P is*

$$\mathrm{Aut}(P) = \prod_{i=1}^{r} \mathrm{Aut}(C_i) \times \prod_{j=1}^{s} \mathrm{Aut}(T_j).$$

Note that the tangle decomposition of a $(3+1)$-free poset P into its parts generalizes the decomposition considered by Reed and Skandera [10] of a $(3+1)$-and-$(2+2)$-free poset given by the *altitude* $\alpha(a) = |D_a| - |U_a|$ of the vertices $a \in P$, since the altitude $\alpha(a)$ is a function of the view $v(a)$. Of course, even in a $(3+1)$-free poset P with an induced $(2+2)$ subposet, the altitude is well-defined, and it gives a finer decomposition of P than the tangle decomposition. However, the altitude decomposition is too fine, as the example in Figure 3 shows. Namely, there is an automorphism τ which swaps the two vertices with altitude -1, the two vertices with altitude -2, and two of the three vertices with altitude 2, as illustrated. But there is no automorphism which acts nontrivially on a single block of the altitude decomposition.

In contrast, for the tangle decomposition, every automorphism of the poset can be factored as a product of automorphisms which only act nontrivially on a single part.

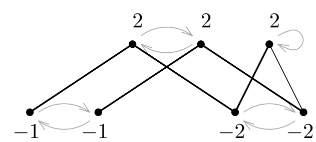


Fig. 3: A poset P consisting of a single tangle. The vertices are labelled by their altitude α, and the arrows describe an automorphism τ of P.

3 Skeleta

Any finite poset P can be decomposed into *levels* as follows: take L_1 to be the set of minimal vertices of P, L_2 to be the set of subminimal vertices (that is, the set of minimal vertices of $P \setminus L_1$), and so on up to the set L_h of sub$^{(h-1)}$minimal vertices of P, where h is the *height* of P. We say that the *level* of a vertex $a \in P$ is $\ell(a)$, where $a \in L_{\ell(a)}$.

If P is $(3+1)$-free, then the only interesting part of the poset structure occurs between adjacent levels, as the following proposition shows.

Proposition 3.1 (Lewis and Zhang [7]). *Let P be a $(3+1)$-free poset and $a, b \in P$ be two vertices with $\ell(a) \leq \ell(b) - 2$. Then, we have $a < b$.*

Note that the covering relations of P may include relations $a < b$ for which $\ell(a) = \ell(b) - 2$. This occurs in Figure 1, for example, where $b < h$, $c < h$, $f < j$, and $g < j$ are covering relations.

The following proposition gives a partial converse of Theorem 3.1.

Proposition 3.2 (Reed and Skandera [10]). *Let P be a poset such that for any two vertices $a, b \in P$ with $\ell(a) \leq \ell(b) - 2$, we have $a < b$. Then, P is $(3+1)$-free if and only if for any two vertices $c, d \in P$ with $\ell(c) = \ell(d)$, we have $U_c \subseteq U_d$ or $D_c \subseteq D_d$ (and symmetrically, $U_c \supseteq U_d$ or $D_c \supseteq D_d$).*

Note that the vertices of a clone set C_i all have the same downset, so they are on the same level. Also, any copy of the $(2+2)$ poset must be contained in two adjacent levels, so any tangle T_j must be contained in two adjacent levels. Thus, we can speak of the level of a clone set or the (adjacent) levels of a tangle.

By construction, the poset structure between two parts of P is fairly restricted. If C_i and C_j are distinct clone sets, then C_i is either completely above, completely below, or completely incomparable with C_j (meaning that every vertex of C_i has the same relationship with every vertex of C_j). If C_i is a clone set and T_j is a tangle, then C_i can be

- completely above T_j;
- completely above the bottom of T_j and incomparable with the top;
- completely below the top of T_j and incomparable with the bottom;
- completely below T_j; or
- completely incomparable with T_j.

Similarly, there are only six possible ways for two tangles T_i and T_j to relate to each other. The following theorem shows how all of these relationships between different parts of P can be put together.

Theorem 3.3. *Let P be a $(3+1)$-free poset, decomposed into clone sets $C_1, \ldots, C_r$ and tangles $T_1, \ldots, T_s$. Then, there exists a listing $(X_1, \ldots, X_{r+s})$ of the clone sets and the tangles of P such that, for any two vertices $a \in X_i$ and $b \in X_j$ with $i \neq j$, we have $a < b$ exactly when*

(i) $\ell(a) \leq \ell(b) - 2$*; or*

(ii) $\ell(a) = \ell(b) - 1$ *and* $i < j$.

Definition 3.4. A listing which satisfies the conditions of Theorem 3.3 is called a *compatible* listing.

Example 3.5. A compatible listing for the poset in Figure 1 is $\big(\{a\}, \{d, e\}, \{h\}, (\{f, g\}, \{b, c\}), \{j\}, \{i\}\big)$, as shown in Figure 2.

idea for Theorem 3.3. For each level, we can get a partial listing of the parts which intersect L_i according to their positions on the view poset $v(P)$. Then, the listing for L_i and L_{i+1} can be interleaved in a unique way to respect condition (ii), so it follows that all of them can be reconciled into a single compatible listing. $\square$

Note that the listing $(X_1, X_2, \ldots, X_{r+s})$ from Theorem 3.3 is not unique in general. In particular, if $(\ldots, X_i, X_{i+1}, \ldots)$ is a compatible listing, then the listing $(\ldots, X_{i+1}, X_i, \ldots)$ obtained by swapping the parts X_i and X_{i+1} is compatible exactly when X_i and X_{i+1} contain no vertices on the same or on adjacent levels of P. We call such a swap *valid*.

Example 3.6. In Figure 2 we can swap the clone set $\{j\}$ on level 4 with the tangle $(\{f, g\}, \{b, c\})$ on levels 1 and 2 to obtain another compatible listing for the poset.

Therefore the natural setting for compatible listings is that of free partially commuting monoids [3], also known as trace monoids [4].

Definition 3.7. Let Σ be the countable alphabet

$$\Sigma = \{c_1, c_2, \ldots, c_i, \ldots\} \cup \{t_{12}, t_{23}, \ldots, t_{i\,i+1}, \ldots\},$$

let Σ^* be the free monoid generated by Σ, and let M be the free partially commuting monoid with commutation relations

$$\begin{aligned} c_i c_j &= c_j c_i, && \text{if } |i-j| \geq 2, \\ c_i t_{j\,j+1} &= t_{j\,j+1} c_i, && \text{if } i \leq j-2 \text{ or } i \geq j+3, \\ t_{i\,i+1} t_{j\,j+1} &= t_{j\,j+1} t_{i\,i+1}, && \text{if } |i-j| \geq 3. \end{aligned}$$

Definition 3.8. If P is a $(3+1)$-free poset, then for each compatible listing $(X_1, X_2, \ldots, X_{r+s})$ of its clone sets and tangles, we can obtain a word in Σ^* by replacing each clone set at level i by the letter c_i and each tangle straddling levels $\{i, i+1\}$ by the letter $t_{i\,i+1}$. It can be seen that any two compatible listings for P are related by a sequence of valid swaps, so the set of these words is an equivalence class under the commutation relations for M (see, *e.g.*, [4, Chapter 1]), and the corresponding element of M is called the *skeleton* of P.

Example 3.9. The two representatives in Σ^* for the skeleton of the poset in Figure 2 are $c_1c_2c_3t_{12}c_4c_3$ and $c_1c_2c_3c_4t_{12}c_3$.

The point of a skeleton is that it exactly captures the relationships between different parts of P. More precisely, two posets with the same skeleton and isomorphic parts are themselves isomorphic; conversely, given a skeleton, any set of parts (with the right number of clone sets and tangles) can be plugged into the skeleton. Together, Theorem 3.10, Theorem 3.11, and Theorem 3.12 below show this and give a characterization of the elements of M which are skeleta.

Corollary 3.10. *Let P be a $(3+1)$-free poset. Then, P is uniquely determined (up to isomorphism) by its skeleton together with, for each letter c_i or $t_{i\,i+1}$ of the skeleton, the cardinality of the corresponding clone set or the isomorphism class of the corresponding tangle.*

Theorem 3.11. *Let m be an element of the monoid M. Then, m is the skeleton of some $(3+1)$-free poset if and only if*

(i) every representative $w \in \Sigma^*$ for m starts with the letter c_1 or t_{12}; and

(ii) no representative $w \in \Sigma^*$ for m contains a factor of the form c_ic_i, $i \geq 1$.

Note that condition (i) of Theorem 3.11 corresponds to the requirement that every vertex of P on level L_{i+1} be greater than some vertex on the previous level L_i, while condition (ii) forbids pairs of clone sets that could be merged into a single clone set.

Theorem 3.12. *Let m be an element of the monoid M. Then, there exists a representative $w_0 \in \Sigma^*$ for m for which every pair of consecutive letters is either*

$$\begin{aligned} &c_i c_j && \textit{for } i \geq j-1\textit{; or} \\ &c_i t_{j\,j+1} && \textit{for } i \geq j-1\textit{; or} \\ &t_{i\,i+1} c_j && \textit{for } i \geq j-2\textit{; or} \\ &t_{i\,i+1} t_{j\,j+1} && \textit{for } i \geq j-2. \end{aligned}$$

Furthermore,

(i) this representative w_0 *is unique and is the lexicographically maximal representative for* m *with respect to the total order* $\{c_1 < t_{12} < c_2 < t_{23} < \cdots\}$ *on* Σ*;*

(ii) if w_0 *starts with* c_1 *or* t_{12}*, then every representative* $w \in \Sigma^*$ *for* m *starts with* c_1 *or* t_{12}*; and*

(iii) if w_0 *does not contain a factor of the form* $c_i c_i$*,* $i \geq 1$*, then no representative* $w \in \Sigma^*$ *for* m *contains a factor of this form.*

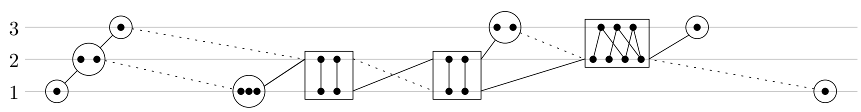
Example 3.13. Of the two representatives given in Theorem 3.9, $c_1c_2c_3c_4t_{12}c_3$ is lexicographically maximal.

Using this characterization of skeleta, we can enumerate them, and this will allow us to obtain generating functions for $(3+1)$-free posets.

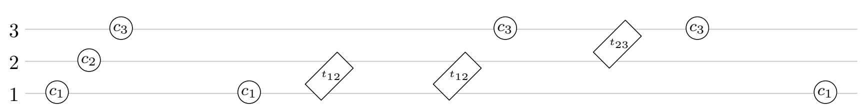
Theorem 3.14. *There is a bijection between skeleta of* $(3+1)$*-free posets and certain decorated Dyck paths. (See Figure 4 for an example.)*

Proof. Given the lexicographically maximal representative w_0 for a skeleton, we can obtain a decorated Dyck path that starts at $(0,0)$, ends at $(2n,0)$ for some $n \geq 0$, and never goes below the x-axis as follows: replace each letter c_i by a $(1,1)$ step ending at height i, each letter $t_{i\,i+1}$ by a $(2,2)$ step ending at height $i+1$, and add $(1,-1)$ down steps as necessary. We call the result *decorated* since a $(2,2)$ step can be seen as a pair of consecutive decorated $(1,1)$ steps. Since w_0 not contain c_ic_i as a factor, the decorated Dyck path obtained from w_0 contains no sequence $(1,1),(1,-1),(1,1)$ of consecutive undecorated steps

Consider the 26-vertex $(3+1)$-free poset P with 10 parts shown in the compatible listing below. Only some of the comparability and incomparability relations between parts are drawn, but the others can be determined from Theorem 3.3.



The word $w_0 = c_1c_2c_3c_1t_{12}t_{12}c_3t_{23}c_3c_1$, shown below in a suggestive manner, is the lexicographically maximal representative for the skeleton of P.



The decorated Dyck path associated with w_0 is the following.

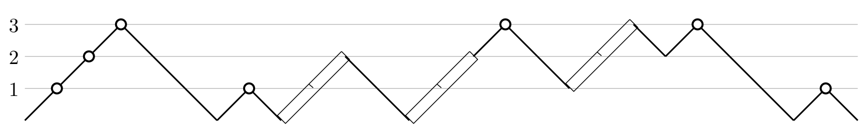


Fig. 4: An example of the bijection given in Theorem 3.14.

(up-down-up). Conversely, every decorated Dyck path avoiding this sequence can be obtained from a skeleton. □

Theorem 3.15. *Let $S(c,t) \in \mathbb{Q}[[c,t]]$ be the ordinary generating function for skeleta with respect to the number of clone sets and the number of tangles, that is, the formal power series*

$$S(c,t) = \sum_{r,s\geq 0} (\text{\# of distinct skeleta with } r \text{ clone sets and } s \text{ tangles})\, c^r t^s.$$

Then, $S(c,t)$ is uniquely determined by the equation

$$S(c,t) = 1 + \frac{c}{1+c} S(c,t)^2 + tS(c,t)^3. \tag{4}$$

idea. See Figure 5. □

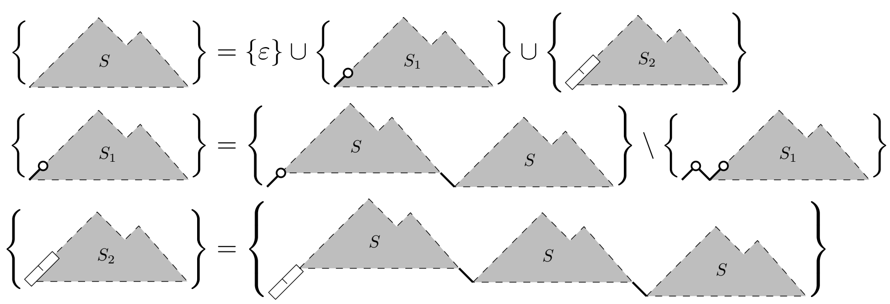


Fig. 5: Equations relating the sets counted by $S(c,t)$, $S_1(c,t)$, and $S_2(c,t)$, where $S_1(c,t)$ and $S_2(c,t)$ are the generating functions for decorated Dyck paths beginning with $(1,1)$ and $(2,2)$, respectively.

4 Enumeration

In this section, we carry out the enumeration of unlabelled and labelled $(3+1)$-free posets by reducing it to the enumeration of unlabelled and labelled bicoloured graphs. Our approach is to consider such a bicoloured graph as a $(3+1)$-free poset in the natural way (with colour classes 'top' and 'bottom') and to apply the machinery of Section 3, as shown in the following lemma.

Lemma 4.1. *The ordinary generating function for skeleta of bicoloured graphs is given by*

$$\sum_{r_1,r_2,s\geq 0} \begin{pmatrix} \text{\# of skeleta of bicoloured graphs with } r_1 \\ \text{clone sets on level } 1,\ r_2 \text{ clone sets on level} \\ 2, \text{ and } s \text{ tangles} \end{pmatrix} c_1^{r_1} c_2^{r_2} t_{12}^s$$
$$= \left(1 - \frac{c_1}{1+c_1} - \frac{c_2}{1+c_2} - t_{12}\right)^{-1}. \tag{5}$$

Now that we have an explicit expression for the generating function of skeleta of bicoloured graphs, we can perform appropriate substitutions to get equations relating the generating functions for tangles and for bicoloured graphs.

Theorem 4.2. *Let* $B_{\text{unl}}(x,y) \in \mathbb{Q}[[x,y]]$ *be the ordinary generating function for unlabelled bicoloured graphs, up to isomorphism. Then, the ordinary generating function for unlabelled tangles is*

$$T_{\text{unl}}(x,y) = 1 - x - y - B_{\text{unl}}(x,y)^{-1}.$$

Proof. This follows from Theorem 4.1 by plugging in the values $c_1 = x/(1-x)$ and $c_2 = y/(1-y)$ for the clone sets of unlabelled vertices and $t = T_{\text{unl}}(x,y)$ for the tangles in (5). □

Theorem 4.3. *Let* $B_{\text{lbl}}(x,y) \in \mathbb{Q}[[x,y]]$ *be the exponential generating function for labelled bicoloured graphs, that is, the formal power series*

$$B_{\text{lbl}}(x,y) = \sum_{i,j\geq 0} 2^{ij}\frac{x^i y^j}{i!j!}.$$

Then, the exponential generating function for labelled tangles is

$$T_{\text{lbl}}(x,y) = e^{-x} + e^{-y} - 1 - B_{\text{lbl}}(x,y)^{-1}.$$

Proof. This follows from Theorem 4.1 by plugging in the values $c_1 = e^x - 1$ and $c_2 = e^y - 1$ for the clone sets of labelled vertices and $t = T_{\text{lbl}}(x,y)$ for the tangles in (5). □

With these expressions for the generating functions $T_{\text{unl}}(x,y)$ and $T_{\text{lbl}}(x,y)$ in hand, the following corollaries of Theorem 3.15 yield the equations (2) and (3) from the introduction.

Corollary 4.4. *Let* $S(c,t)$ *be the generating function of Theorem 3.15 for skeleta. Then, the ordinary generating function for unlabelled* $(3+1)$*-free posets is*

$$\sum_{n\geq 0} p_{\text{unl}}(n)x^n = S\big(x/(1-x), T_{\text{unl}}(x,x)\big).$$

Corollary 4.5. *Let* $S(c,t)$ *be the generating function of Theorem 3.15 for skeleta. Then, the exponential generating function for labelled* $(3+1)$*-free posets is*

$$\sum_{n\geq 0} p_{\text{lbl}}(n)\frac{x^n}{n!} = S\big(e^x - 1, T_{\text{lbl}}(x,x)\big).$$

Remark 4.6. François Bergeron has pointed out to us that the results of this section can be generalized to obtain the cycle index series (see [2]) for the species of $(3+1)$-free posets.

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The immaculate basis of the non-commutative symmetric functions (Extended Abstract)

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Abstract We introduce a new basis of the non-commutative symmetric functions whose elements have Schur functions as their commutative images. Dually, we build a basis of the quasi-symmetric functions which expand positively in the fundamental quasi-symmetric functions and decompose Schur functions according to a signed combinatorial formula.

Résumé. Nous introduisons une nouvelle base des fonctions symétriques non commutatives dont les images commutatives sont des fonctions de Schur. Nous construisons la base duale des fonctions quasi-symétriques qui s'expriment de façon positive en fonction de la base fondamental et décomposer les fonctions de Schur.

Keywords: non-commutative symmetric functions, quasi-symmetric functions, tableaux, Schur functions

1 Introduction

The Schur functions s_λ are indexed by integer partitions and form an additive basis for the algebra of symmetric functions Sym. Schur functions play an important role throughout mathematics, in particular in algebraic geometry (as representatives of Schubert classes for the Grassmannian) and representation theory (they are the characters of the irreducible representations of the general linear group). Another important basis for Sym is the (complete) homogeneous symmetric functions h_λ.

The algebras of non-commutative symmetric functions NSym and quasi-symmetric functions QSym are dual Hopf algebras. These algebras have been of great importance to algebraic combinatorics. As seen in [ABS], they are universal in the category of combinatorial Hopf algebras. They also represent the Grothendieck rings for the projective and finite dimensional representation theory of the 0-Hecke algebra [KT]. An important basis for NSym is formed by the (complete) homogeneous non-commutative symmetric functions H_α, indexed by compositions. The *forgetful map* $\chi : \mathsf{NSym} \longrightarrow \mathsf{Sym}$ maps the homogeneous non-commutative symmetric functions to their symmetric counterparts (see (1)).

The main goal of this abstract is to define and outline the properties of a new basis, the *immaculate basis* $\mathfrak{S}_\alpha$ of NSym, which emulates the role of the Schur functions. This new basis projects onto the Schur basis under the forgetful map and it shares many of the same properties and constructions of the classical basis of Schur functions of Sym. More specifically:

Bernstein operators. One way to construct Schur functions is by iterating the Bernstein row adding operator, which acts on Schur functions by adding a row to the corresponding Ferrers shape. These operators can be described in an algebraic way, which we deform in order to obtain a non-commutative Bernstein operator. This deformed operator now acts on immaculate functions by adding a row to the corresponding composition. Thus, a repeated iteration of these operators will build the immaculate functions, as in Definition 3.3.

Pieri rule. The product of a Schur function and a homogeneous symmetric function corresponding to a partition with only one part can be expressed, via the classical Pieri rule, as a multiplicity-free sum over a specific set of Schur functions. More specifically, this sum is over all ways to add a *horizontal strip* to the original shape. In Theorem 3.5 we show that in a similar way, the product of an immaculate function and a homogeneous non-commutative symmetric function corresponding to a composition with only one part can be expressed as a multiplicity-free sum of immaculate functions. This sum is over all ways to add an analog of a horizontal strip for composition shapes.

Immaculate tableaux and the immaculate Kostka matrix. By iterating the Pieri rule, one can obtain an expansion of the homogeneous symmetric functions in terms of Schur functions, where each coefficient is a *Kostka number*, or number of semistandard Young tableaux of a specified shape and content. In a similar fashion, we introduce *immaculate tableaux*, and by iterating the immaculate Pieri rule, one obtains an expansion of the homogeneous non-commutative functions in terms of the immaculate functions, where each coefficient is the number of immaculate semistandard tableaux of a specified shape and content (Theorem 3.10).

Positive expansion for ribbons. Another important basis of NSym is formed by *ribbon noncommutative functions* R_α. In Theorem 3.15 we expand the Ribbon functions positively in terms of immaculate functions, indexed by certain descent sets on standard immaculate tableaux.

Moreover, the immaculate basis gives rise to a dual basis in the quasi-symmetric function algebra. The *dual immaculate basis* also shares interesting properties with the Schur basis. In particular, by duality arguments, one is able to express the dual immaculate basis in terms of other known bases of QSym.

Jacobi-Trudi determinant formula. The Schur functions can be expanded in terms of the homogeneous symmetric functions by the use of the Jacobi-Trudi determinant. By considering a non-commutative version of this determinant, we expand the immaculate functions in terms of the homogeneous non-commutative symmetric functions, thus obtaining a lifting of the Jacobi-Trudi formula in NSym, as in Theorem 3.17.

Generating series of immaculate tableaux and monomial expansion. The most well known construction for a Schur function is by its expression as a generating series over the set of semistandard Young tableaux, and thus, as a positive sum of monomial (quasi-)symmetric functions. In Theorem 3.21 we express the dual immaculate functions as a generating series over the set of semistandard immaculate tableaux, and thus, as a positive sum of monomial quasi-symmetric functions.

Positive fundamental expansion. The Schur functions can also be expressed as a positive sum of fundamental quasi-symmetric functions, by considering descents on standard Young tableaux. By a duality argument, in Theorem 3.22, we express the dual immaculate functions as a positive sum of fundamental quasi-symmetric functions, by considering descents on standard immaculate tableaux.

Expansion of Schur functions. In Theorem 3.23, we show that the Schur functions expand in the dual immaculate basis via signed combinatorics developed in [ELW].

Littlewood-Richardson rule. In the classical case, the product of two Schur functions can be expressed as a sum of Schur functions, where each coefficient is a *Littlewood-Richardson number*, namely, the number of *Yamanouchi tableaux* of a certain skew shape. Although the product of any two immaculate functions is not in general immaculate positive, we give a combinatorial formula for the coefficients in the product of any immaculate function with an immaculate function corresponding to a partition as the positive sum of immaculate functions, where each coefficient counts the number of *immaculate Yamanouchi tableaux* of a certain skew shape, thus obtaining an analogue of the Littlewood-Richardson rule (Theorem 3.25).

Murnaghan-Nakayama rule. The product of a Schur function and a power sum can be expressed as a sum over Schur functions, over the set of shapes that are obtained by adding a ribbon to the original Ferrers shape. In NSym, an analogue of the power sums basis Ψ_α, was defined in [GKLLRT]. In Theorem 3.27 we express the product of an immaculate function and a noncommutative power sum Ψ_n.

Indecomposable modules. There exists a collection of indecomposable modules for the 0-Hecke algebra with the property that the module indexed by the composition α has the dual immaculate function indexed by α as its characteristic. In the interest of space, we will not pursue this below, but refer the reader to [BBSSZ2].

This text is an extended abstract of the preprints [BBSSZ1], [BBSSZ2] and [BBSSZ3], where complete proofs can be found.

Remark 1.1 *Although our basis of* NSym *is similar to the dual basis of quasi-symmetric Schur functions of [HLMvW] (whose properties were developed in [BLvW]), they are in fact different bases.*

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This research was facilitated by computer exploration using the open-source mathematical software `Sage` [sage] and its algebraic combinatorics features developed by the `Sage-Combinat` community [sage-combinat].

2 Background

2.1 Compositions and combinatorics

A *partition* of a non-negative integer n is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ of non-negative integers satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$, and is denoted $\lambda \vdash n$. Partitions are of particular importance to algebraic combinatorics; among other things, partitions of n index a basis for the symmetric functions of degree n, Sym_n, and the character ring for the representations of the symmetric group. These concepts are intimately connected; we assume the reader is well versed in this area (see for instance [Sagan] for background details).

A *composition* of a non-negative integer n is a list $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_m]$ of positive integers which sum to n, written $\alpha \models n$. The entries α_i of the composition are referred to as the parts of the composition. The size of the composition is the sum of the parts and will be denoted $|\alpha| := n$. The length of the composition is the number of parts and will be denoted $\ell(\alpha) := m$. In this paper we study dual graded Hopf algebras whose bases at level n are indexed by compositions of n.

Compositions of n correspond to subsets of $\{1, 2, \ldots, n-1\}$. We will follow the convention of identifying $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_m]$ with the subset $D(\alpha) = \{\alpha_1, \alpha_1+\alpha_2, \alpha_1+\alpha_2+\alpha_3, \ldots, \alpha_1+\alpha_2+\cdots+\alpha_{m-1}\}$.

If α and β are both compositions of n, say that $\alpha \leq \beta$ in refinement order if $D(\beta) \subseteq D(\alpha)$. For instance, $[1, 1, 2, 1, 3, 2, 1, 4, 2] \leq [4, 4, 2, 7]$, since $D([1, 1, 2, 1, 3, 2, 1, 4, 2]) = \{1, 2, 4, 5, 8, 10, 11, 15\}$ and $D([4, 4, 2, 7]) = \{4, 8, 10\}$.

We introduce a new notion which will arise in our Pieri rule (Theorem 3.5); we say that $\alpha \subset_i \beta$ if:

1. $|\beta| = |\alpha| + i$,
2. $\alpha_j \leq \beta_j$ for all $1 \leq j \leq \ell(\alpha)$,
3. $\ell(\beta) \leq \ell(\alpha) + 1$.

For a composition $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_\ell]$ and a positive integer m, we let $[m, \alpha]$ denote the composition $[m, \alpha_1, \alpha_2, \ldots, \alpha_\ell]$.

2.2 Schur functions and creation operators

We let h_i and e_i denote the complete homogeneous and elementary symmetric functions of degree i respectively. We next define a Schur function indexed by an arbitrary sequence of integers. The resulting family of symmetric functions indexed by partitions λ are the usual Schur basis of the symmetric functions.

Definition 2.1 *For an arbitrary integer tuple $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell) \in \mathbb{Z}^\ell$, we define*

$$s_\alpha := \det \begin{bmatrix} h_{\alpha_1} & h_{\alpha_1+1} & \cdots & h_{\alpha_1+\ell-1} \\ h_{\alpha_2-1} & h_{\alpha_2} & \cdots & h_{\alpha_2+\ell-2} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\alpha_\ell-\ell+1} & h_{\alpha_\ell-\ell+2} & \cdots & h_{\alpha_\ell} \end{bmatrix} = \det |h_{\alpha_i+j-i}|_{1\leq i,j\leq \ell}$$

where we use the convention that $h_0 = 1$ and $h_{-m} = 0$ for $m > 0$.

With this definition, switching two adjacent rows of the defining matrix has the effect of changing the sign of the determinant. It is also equal to the Schur function indexed by a different integer tuple:

$$s_{\alpha_1,\alpha_2,\ldots,\alpha_r,\alpha_{r+1},\ldots,\alpha_\ell} = -s_{\alpha_1,\alpha_2,\ldots,\alpha_{r+1}-1,\alpha_r+1,\ldots,\alpha_\ell}\,.$$

Proposition 2.2 *If α is a composition of n with length equal to k, then $s_\alpha = 0$ if and only if there exists $i, j \in \{1, 2, \ldots, k\}$ with $i \neq j$ such that $\alpha_i - i = \alpha_j - j$. If $s_\alpha \neq 0$, then there is a unique permutation σ such that $(\alpha_{\sigma_1} + 1 - \sigma_1, \alpha_{\sigma_2} + 2 - \sigma_2, \ldots, \alpha_{\sigma_k} + k - \sigma_k)$ is a partition. In this case,*

$$s_\alpha = (-1)^\sigma s_{\alpha_{\sigma_1}+1-\sigma_1,\alpha_{\sigma_2}+2-\sigma_2,\ldots,\alpha_{\sigma_k}+k-\sigma_k}\,.$$

Sym is a self dual Hopf algebra. It has a pairing (the Hall scalar product) defined by

$$\langle h_\lambda, m_\mu \rangle = \langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu} .$$

An element $f \in \mathsf{Sym}$ gives rise to an operator $f^\perp : \mathsf{Sym} \to \mathsf{Sym}$ according to the relation:

$$\langle fg, h \rangle = \langle g, f^\perp h \rangle \quad \text{for all } g, h \in \mathsf{Sym} .$$

We define a "creation" operator $\mathbf{B}_m : \mathsf{Sym}_n \to \mathsf{Sym}_{m+n}$ by:

$$\mathbf{B}_m := \sum_{i \geq 0} (-1)^i h_{m+i} e_i^\perp .$$

The following theorem, which states that creation operators construct Schur functions, will become one of the motivations for our new basis of NSym (see Definition 3.3).

Theorem 2.3 *(Bernstein [Ze, pg 69-70]) For all sequences of $\alpha \in \mathbb{Z}^m$,*

$$s_\alpha = \mathbf{B}_{\alpha_1} \mathbf{B}_{\alpha_2} \cdots \mathbf{B}_{\alpha_m}(1).$$

2.3 Non-commutative symmetric functions

The algebra NSym is a non-commutative analogue of Sym that arises by considering an algebra with one non-commutative generator at each positive degree. In addition to the relationship with the symmetric functions, this algebra has links to Solomon's descent algebra in type A [MR], the algebra of quasi-symmetric functions [MR], and representation theory of the type A Hecke algebra at $q = 0$ [KT], and connections to the theory of combinatorial Hopf algebras [ABS]. While we will follow the foundational results and definitions of references such as [GKLLRT, MR], we have chosen to use notation here which is suggestive of analogous results in Sym.

We define NSym as the algebra with generators $\{H_1, H_2, \dots\}$ and no relations. Each generator H_i is defined to be of degree i, giving NSym the structure of a graded algebra. We let NSym_n denote the graded component of NSym of degree n. A basis for NSym_n are the *complete homogeneous functions* $\{H_\alpha := H_{\alpha_1} H_{\alpha_2} \cdots H_{\alpha_m}\}_{\alpha \models n}$ indexed by compositions of n. To make this convention consistent, some formulas will use expressions that have H indexed by tuples of integers and we use the convention that $H_0 = 1$ and $H_{-r} = 0$ for $r > 0$.

There exists a map (sometimes referred to as the forgetful map) which we shall also denote $\chi : \mathsf{NSym} \to \mathsf{Sym}$ defined by sending the basis element H_α to the complete homogeneous symmetric function

$$\chi(H_\alpha) := h_{\alpha_1} h_{\alpha_2} \cdots h_{\alpha_{\ell(\alpha)}} \in \mathsf{Sym} \tag{1}$$

and extend this map to all of NSym linearly.

Similar to the study of Sym and the ring of characters for the symmetric groups, the ring of non-commutative symmetric functions of degree n is isomorphic to the Grothendieck ring of projective representations of the 0-Hecke algebra. We refer the reader to [KT] for details. The element of NSym which corresponds to the projective representation indexed by α is here denoted R_α. The collection of R_α are a basis of NSym, usually called the *ribbon basis* of NSym. They are defined through their expansion in the complete homogeneous basis:

$$R_\alpha = \sum_{\beta \geq \alpha} (-1)^{\ell(\alpha)-\ell(\beta)} H_\beta, \quad \text{or equivalently,} \quad H_\alpha = \sum_{\beta \geq \alpha} R_\beta.$$

NSym has a coproduct structure, which we will not explain in the interest of space.

2.4 Quasi-symmetric functions

The algebra of quasi-symmetric functions, QSym, was introduced in [Ges] (see also subsequent references such as [GR, Sta84]) and this algebra has become a useful tool for algebraic combinatorics since it is dual to NSym as a Hopf algebra and contains Sym as a subalgebra.

As with the algebra NSym, the graded component QSym_n is indexed by compositions of n. The algebra is most readily realized within the ring of power series of bounded degree $\mathbb{Q}[\![x_1, x_2, \dots]\!]$, and the monomial quasi-symmetric function indexed by a composition α is defined as

$$M_\alpha = \sum_{i_1 < i_2 < \cdots < i_m} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_m}^{\alpha_m}. \tag{2}$$

QSym is defined as the algebra with the monomial quasi-symmetric functions as a basis.

We view Sym as a subalgebra of QSym. In fact, the quasi-symmetric monomial functions decompose the usual monomial symmetric functions $m_\lambda \in \mathsf{Sym}$:

$$m_\lambda = \sum_{sort(\alpha)=\lambda} M_\alpha.$$

Similar to NSym, the algebra QSym is isomorphic to the Grothendieck ring of finite-dimensional representations of the 0-Hecke algebra. The irreducible representations of the 0-Hecke algebra form a basis for this ring, and under this isomorphism the irreducible representation indexed by α is identified with an element of QSym, the *fundamental quasi-symmetric function*, denoted F_α. The F_α, for $\alpha \models n$, form a basis of QSym_n, and are defined by their expansion in the monomial quasi-symmetric basis:

$$F_\alpha = \sum_{\beta \leq \alpha} M_\beta.$$

2.5 Identities relating non-commutative / quasi-symmetric functions

The algebras QSym and NSym form graded dual Hopf algebras. The monomial basis of QSym is dual in this context to the complete homogeneous basis of NSym, and the fundamental basis of QSym is dual to the ribbon basis of NSym. NSym and QSym have a pairing $\langle \cdot, \cdot \rangle : \mathsf{NSym} \times \mathsf{QSym} \to \mathbb{Q}$, defined under this duality as either $\langle H_\alpha, M_\beta \rangle = \delta_{\alpha,\beta}$, or $\langle R_\alpha, F_\beta \rangle = \delta_{\alpha,\beta}$.

We will generalize the operation which is dual to multiplication by a quasi-symmetric function using this pairing. For $F, G \in \mathsf{QSym}$, let $F^\perp$ be the operator which acts on elements $H \in \mathsf{NSym}$ according to the relation $\langle H, FG \rangle = \langle F^\perp H, G \rangle$.

3 A new basis for NSym

We are now ready to introduce our new basis of NSym. These functions were discovered while playing with a non-commutative analogue of the Jacobi-Trudi identity (see Theorem 3.17). They may also be defined as the unique functions in NSym which satisfy a right-Pieri rule (see Theorem 3.5).

3.1 Non-commutative immaculate functions

Definition 3.1 *We define the non-commutative Bernstein operators $\mathbb{B}_m$ as:*

$$\mathbb{B}_m = \sum_{i \geq 0} (-1)^i H_{m+i} F_{1^i}^\perp .$$

Using the non-commutative Bernstein operators, we can inductively build functions using creation operators similar to Bernstein's formula (Theorem 2.3) for the Schur functions.

Remark 3.2 *Under the identification of* Sym *inside* QSym*, the generator* e_i *of* Sym *is precisely the function* F_{1^i} *appearing above.*

Definition 3.3 *For any* $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_m) \in \mathbb{Z}^m$*, the* immaculate function $\mathfrak{S}_\alpha \in \mathsf{NSym}$ *is defined as the composition of the operators*

$$\mathfrak{S}_\alpha = \mathbb{B}_{\alpha_1}\mathbb{B}_{\alpha_2}\cdots\mathbb{B}_{\alpha_m}(1)\ .$$

Calculations in the next subsection will show that the elements $\{\mathfrak{S}_\alpha\}_{\alpha\models n}$ form a basis for NSym_n.

Example 3.4 *For* $a, b > 0$*,* $\alpha = (a)$ *has only one part, and* $\mathfrak{S}_a$ *is just the complete homogeneous generator* H_a*. If* $\alpha = (a, b)$ *consists of two parts, then* $\mathfrak{S}_{ab} = \mathbb{B}_a(H_b) = H_aH_b - H_{a+1}H_{b-1}$.

3.2 The right-Pieri rule for the immaculate basis

Theorem 3.5 *For a composition* α*, the* $\mathfrak{S}_\alpha$ *satisfy a multiplicity free right-Pieri rule for multiplication by* H_s*:*

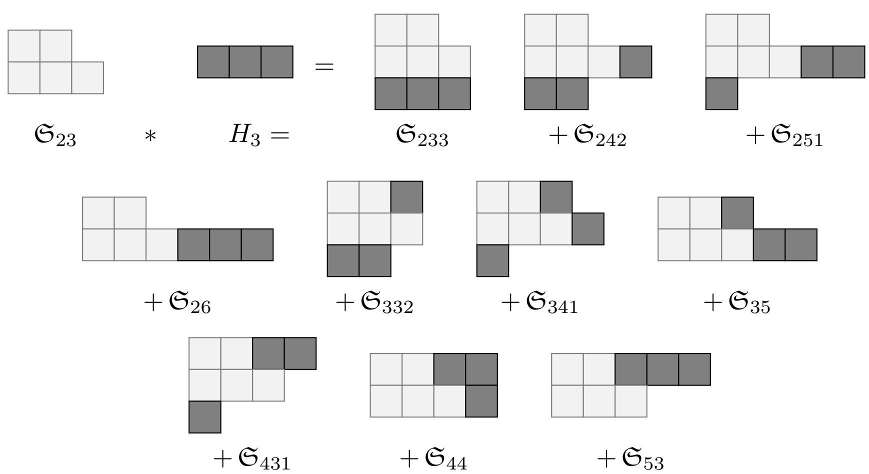
$$\mathfrak{S}_\alpha H_s = \sum_{\alpha \subset_s \beta} \mathfrak{S}_\beta.$$

where the notation $\subset_s$ *is introduced in Section 2.1.*

Remark 3.6 *Products of the form* $H_m\mathfrak{S}_\alpha$ *do not have as nice an expression as* $\mathfrak{S}_\alpha H_m$ *because they generally have negative signs in their expansion and there is no obvious containment of resulting compositions. For example,*

$$H_1\mathfrak{S}_{13} = \mathfrak{S}_{113} - \mathfrak{S}_{221} - \mathfrak{S}_{32}\ .$$

Example 3.7 *The expansion of* $\mathfrak{S}_{23}$ *multiplied on the right by* H_3 *is done below.*



3.3 Relationship to the classical bases of NSym

We will now develop some relations between the classical bases of NSym and the immaculate basis.

3.3.1 Immaculate tableaux

Definition 3.8 *Let α and β be compositions. An* immaculate tableau *of shape α and content β is a labelling of the boxes of the diagram of α by positive integers in such a way that:*

1. *the number of boxes labelled by i is β_i;*
2. *the sequence of entries in each row, from left to right, is weakly increasing;*
3. *the sequence of entries in the* first *column, from top to bottom, is strictly increasing.*

An immaculate tableau is said to be standard *if it has content $1^{|\alpha|}$.*
Let $K_{\alpha,\beta}$ denote the number of immaculate tableaux of shape α and content β.

We re-iterate that besides the first column, there is no relation on other columns of an immaculate tableau. Standard immaculate tableaux of size n are in bijection with set partitions of $\{1, 2, \ldots, n\}$ by ordering the parts in the partition by minimal elements, as was pointed out to us in a discussion with M. Yip.

Example 3.9 *There are five immaculate tableau of shape $[4, 2, 3]$ and content $[3, 1, 2, 3]$:*

$$\begin{array}{|c|c|c|c|}\hline 1&1&1&3\\\hline 2&3\\\cline{1-2} 4&4&4\\\cline{1-3}\end{array}\quad \begin{array}{|c|c|c|c|}\hline 1&1&1&3\\\hline 2&4\\\cline{1-2} 3&4&4\\\cline{1-3}\end{array}\quad \begin{array}{|c|c|c|c|}\hline 1&1&1&4\\\hline 2&3\\\cline{1-2} 3&4&4\\\cline{1-3}\end{array}\quad \begin{array}{|c|c|c|c|}\hline 1&1&1&4\\\hline 2&4\\\cline{1-2} 3&3&4\\\cline{1-3}\end{array}\quad \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\\hline 3&3\\\cline{1-2} 4&4&4\\\cline{1-3}\end{array}$$

3.3.2 Expansion of the homogeneous basis

Theorem 3.10 *The complete homogeneous basis H_α has a positive, uni-triangular expansion in the immaculate basis. Specifically,*

$$H_\beta = \sum_{\alpha \geq_{\mathrm{lex}} \beta} K_{\alpha,\beta} \mathfrak{S}_\alpha,$$

where $K_{\alpha,\beta}$ is the number of immaculate tableaux of shape α and content β.

Example 3.11 *Continuing from Example 3.9, we see that $H_{3123} = \cdots + 5\mathfrak{S}_{423} + \cdots$.*

Corollary 3.12 *The $\{\mathfrak{S}_\alpha : \alpha \models n\}$ form a basis of NSym_n.*

3.3.3 Expansion of the ribbon basis

We will expand the ribbon functions in the immaculate basis. We first need the notion of a descent.

Definition 3.13 *We say that a standard immaculate tableau T has a descent in position i if $(i+1)$ is in a row strictly lower than i. The descent composition of T, $D(T)$, is the composition of the size of T that corresponds to the subset containing all descent positions.*

Example 3.14 *The standard immaculate tableau below has descents in positions $\{2, 5, 11\}$. The descent composition of S is then $[2, 3, 6, 7]$.*

$$S = \begin{array}{|c|c|c|c|c|c|c|}\cline{1-6} 1&2&4&5&10&11\\\hline 3&6&7&8&9\\\hline 12&13&14&15&16&17&18\\\hline\end{array}$$

Let $L_{\alpha,\beta}$ denote the number of standard immaculate tableaux of shape α and descent composition β.

Theorem 3.15 *The ribbon function R_β has a positive expansion in the immaculate basis. Specifically*

$$R_\beta = \sum_{\alpha \geq_\ell \beta} L_{\alpha,\beta} \mathfrak{S}_\alpha.$$

Example 3.16 *There are eight standard immaculate tableaux with descent composition $[2,2,2]$, giving the expansion of R_{222} into the immaculate basis.*

$$\begin{array}{ll} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{array} \qquad \begin{array}{lll} 1 & 2 & \\ 3 & 4 & 6 \\ 5 & & \end{array} \qquad \begin{array}{lll} 1 & 2 & 4 \\ 3 & & \\ 5 & 6 & \end{array} \qquad \begin{array}{lll} 1 & 2 & 4 \\ 3 & 6 & \\ 5 & & \end{array} \quad \begin{array}{lll} 1 & 2 & 6 \\ 3 & 4 & \\ 5 & & \end{array}$$

$$R_{222} = \mathfrak{S}_{222} + \mathfrak{S}_{231} + \mathfrak{S}_{312} + 2\mathfrak{S}_{321}$$

$$\begin{array}{lll} 1 & 2 & 4 \\ 3 & 5 & 6 \end{array} \qquad \begin{array}{llll} 1 & 2 & 4 & 6 \\ 3 & & & \\ 5 & & & \end{array} \qquad \begin{array}{llll} 1 & 2 & 4 & 6 \\ 3 & 5 & & \end{array}$$

$$+ \mathfrak{S}_{33} + \mathfrak{S}_{411} + \mathfrak{S}_{42}$$

3.4 *Jacobi-Trudi rule for* NSym

Another compelling reason to study the immaculate functions is that they also have an expansion in the H_α basis that makes them a clear analogue of the Jacobi-Trudi rule of Definition 2.1.

Theorem 3.17 *For a composition $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_m]$:*

$$\mathfrak{S}_\alpha = \sum_{\sigma \in S_m} (-1)^\sigma H_{\alpha_1+\sigma_1-1, \alpha_2+\sigma_2-2, \ldots, \alpha_m+\sigma_m-m}. \tag{3}$$

Remark 3.18 *This sum is a non-commutative analogue of the determinant of the following matrix:*

$$\begin{bmatrix} H_{\alpha_1} & H_{\alpha_1+1} & \cdots & H_{\alpha_1+\ell-1} \\ H_{\alpha_2-1} & H_{\alpha_2} & \cdots & H_{\alpha_2+\ell-2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{\alpha_\ell-\ell+1} & H_{\alpha_\ell-\ell+2} & \cdots & H_{\alpha_\ell} \end{bmatrix}$$

where we have used the convention that $H_0 = 1$ and $H_{-m} = 0$ for $m > 0$. The non-commutative analogue of the determinant corresponds to expanding this matrix about the first row and multiplying those elements on the left.

Remark 3.19 *One might ask why one would naturally expand about the first row rather than, say, the first column or the last row. What we considered to be the natural analogue of expanding about the first column however is not a basis; the matrix corresponding to $\alpha = (1,2)$ would be 0 under this analogue.*

Of course, the original reason for considering this definition is the property that they are a lift of the symmetric function corresponding to the Jacobi-Trudi matrix.

Corollary 3.20 $\chi(\mathfrak{S}_\alpha) = s_\alpha$.

3.5 The dual immaculate basis

Every basis X_α of NSym_n gives rise to a basis Y_β of QSym_n defined by duality; Y_β is the unique basis satisfying $\langle X_\alpha, Y_\beta\rangle = \delta_{\alpha,\beta}$. The dual basis to the immaculate basis of NSym, denoted $\mathfrak{S}^*_\alpha$, have positive expansions in the monomial and fundamental bases of QSym. Furthermore, they decompose the usual Schur functions of Sym (see Theorem 3.23).

Theorem 3.21 *The dual immaculate functions $\mathfrak{S}^*_\alpha$ are monomial positive. Specifically they expand as*

$$\mathfrak{S}^*_\alpha = \sum_{\beta \leq_\ell \alpha} K_{\alpha,\beta} M_\beta.$$

Theorem 3.22 *The dual immaculate functions $\mathfrak{S}^*_\alpha$ are fundamental positive. Specifically they expand as*

$$\mathfrak{S}^*_\alpha = \sum_{\beta \leq_\ell \alpha} L_{\alpha,\beta} F_\beta.$$

Duality will also yield an explicit expansion of Schur functions into the dual immaculate basis.

Theorem 3.23 *The Schur function s_λ, with $\ell(\lambda) = k$ expands into the dual immaculate basis as follows:*

$$s_\lambda = \sum_{\sigma \in S_k} (-1)^\sigma \mathfrak{S}^*_{\lambda_{\sigma_1}+1-\sigma_1, \lambda_{\sigma_2}+2-\sigma_2, \cdots, \lambda_{\sigma_k}+k-\sigma_k}$$

where the sum is over permutations σ such that $\lambda_{\sigma_i} + i - \sigma_i > 0$ for all $i \in \{1, 2, \ldots, k\}$.

Example 3.24 *Let $\lambda = (2, 2, 2, 1)$. Then s_λ decomposes as:*

$$s_{2221} = \mathfrak{S}^*_{2221} - \mathfrak{S}^*_{1321} - \mathfrak{S}^*_{2131} + \mathfrak{S}^*_{1141},$$

since only the permutations $\sigma \in \{1234, 2134, 1324, 2314\}$ contribute to the sum in the expansion of s_{2221}. There are potentially 24 terms in this sum, but for the partition $(2, 2, 2, 1)$ it is easy to reason that $\sigma_4 = 4$ and $\sigma_1 < 3$.

These combinatorics arise in the paper of Egge, Loehr and Warrington [ELW] when they describe how to obtain a Schur expansion given a quasi-symmetric fundamental expansion. In their language, the terms in this sum correspond to "special rim hook tableau".

3.6 The Littlewood-Richardson rule for immaculate functions

We prove here that the product $\mathfrak{S}_\alpha \mathfrak{S}_\lambda$ expands positively in the immaculate basis, expanding the notion of a Yamanouchi tableau. Recall that a *Yamanouchi word* is a word w such that every left prefix of w contains at least as many occurrences of i as $i + 1$, for all $i \geq 1$. The content of w is the composition whose i-th part is the number of occurrences of i.

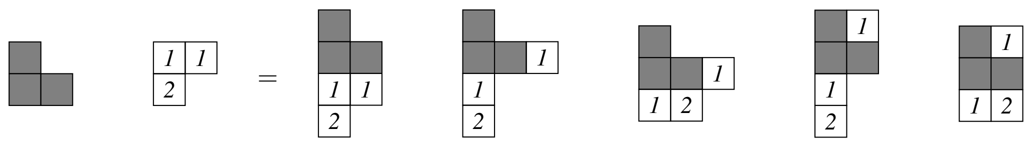
For partitions α and β with $\alpha_i \geq \beta_i$ for all i, denote a *skew composition shape* $\alpha//\beta$ by the shape one obtains by superimposing the bottom left boxes of α and β, and removing the boxes in β. We denote an *immaculate skew tableau* of shape $\alpha//\beta$ as a filling of this shape, satisfying the rules in Definition 3.8. We denote the *reading word* of a skew immaculate tableau T as the word obtained by reading its entries from right to left in each row, starting from the top row and moving down.

Theorem 3.25 *For a composition α and a partition λ, the coefficients $c^{\beta}_{\alpha,\lambda}$ appearing in*

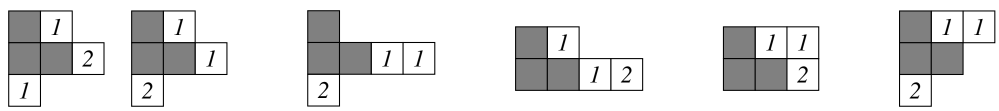
$$\mathfrak{S}_\alpha \mathfrak{S}_\lambda = \sum_\beta c^{\beta}_{\alpha,\lambda} \mathfrak{S}_\beta,$$

are non-negative integers. In particular, $c^{\beta}_{\alpha,\lambda}$ is the number of skew immaculate tableaux of shape $\alpha//\beta$, such that the reading word is a Yamanouchi word of content λ.

Example 3.26 *We give an example with $\alpha = [1, 2]$ and $\lambda = [2, 1]$.*



$$\mathfrak{S}_{12} \quad * \quad \mathfrak{S}_{21} \quad = \quad \mathfrak{S}_{1221} \quad + \quad \mathfrak{S}_{1311} \quad + \quad \mathfrak{S}_{132} \quad + \quad \mathfrak{S}_{2211} \quad + \quad \mathfrak{S}_{222}$$



$$+ \quad 2\mathfrak{S}_{231} \quad + \quad \mathfrak{S}_{141} \quad + \quad \mathfrak{S}_{24} \quad + \quad \mathfrak{S}_{33} \quad + \quad \mathfrak{S}_{321}$$

3.7 The Murnaghan-Nakayama rule for immaculate functions

A non-commutative lifting Ψ_α of the power sum basis elements was given in [GKLLRT]. We now state our version of the Murnaghan-Nakayama rule for immaculate functions.

Theorem 3.27 *For a composition α and a positive integer k,*

$$\mathfrak{S}_\alpha \Psi_k = \sum_{j=1}^{\ell(\alpha)} \mathfrak{S}_{[\alpha_1,\alpha_2,\ldots,\alpha_j+k,\ldots,\alpha_{\ell(\alpha)}]} + \sum_{j=0}^{k-1} \mathfrak{S}_{[\alpha,\underbrace{0\cdots 0}_{j},k]}$$

In other words, the sum is over all ways to add k to one of the parts of the composition obtained by padding α with k zeroes at the end.

Example 3.28 *One may check that*

$$\mathfrak{S}_{132}\Psi_3 = \mathfrak{S}_{432} + \mathfrak{S}_{162} + \mathfrak{S}_{135} + \mathfrak{S}_{1323} + \mathfrak{S}_{13203} + \mathfrak{S}_{132003}.$$

Remark 3.29 *The nicest form of the Murnaghan-Nakayama rule involves weak compositions (possibly allowing zero as an entry) rather than compositions. There is a signed version of the rule which uses only compositions. In the interest of space, we must omit this rule. It will appear in [BBSSZ3].*

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The module of affine descent classes of a Weyl group

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Abstract. The goal of this paper is to introduce an algebraic structure on the space spanned by affine descent classes of a Weyl group, by analogy and in relation to the structure carried by ordinary descent classes. The latter classes span a subalgebra of the group algebra, Solomon's descent algebra. We show that the former span a left module over this algebra. The structure is obtained from geometric considerations involving hyperplane arrangements. We provide a combinatorial model for the case of the symmetric group.

Résumé. Le but de cet article est d'introduire une structure algébrique sur l'espace engendré par les classes de descente affines d'un groupe de Weyl, par rapport à l' structure possédé par les classes de descente finies. Ces dernières engendrent une sous-algèbre de l'algèbre de groupe, l'algèbre de Solomon. Nous montrons que les premières engendrent un module à gauche sur cette algèbre. La structure est obtenue par moyens géométriques impliquant des arrangements d'hyperplans. Un modèle combinatoire est fourni pour le cas du groupe symétrique.

Keywords: Weyl group, Coxeter complex, hyperplane arrangement, Tits product, Solomon's descent algebra, Steinberg torus

1 Introduction

Let W be a finite Coxeter group with simple reflections $S = \{s_1, \ldots, s_n\}$ and corresponding simple roots $\Delta = \{\alpha_1, \ldots, \alpha_n\}$. For $w \in W$, let $D(w)$ denote the set of *right descents* of w, i.e.,

$$D(w) = \{1 \leq i \leq n : \ell(w) > \ell(ws_i)\} = \{1 \leq i \leq n : w\alpha_i < 0\}.$$

For any $J \subseteq [n] := \{1, 2, \ldots, n\}$, let

$$x_J := \sum_{D(w) \subseteq J} w$$

denote the sum, in the group ring $\mathbb{Z}W$, of all elements of W whose descent set is contained in J. As J runs over the subsets of $[n]$, the elements x_J span a subring of $\mathbb{Z}W$, denoted $\mathrm{Sol}(W)$, and called *Solomon's descent algebra* (or ring).

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The descent algebra was introduced by Solomon [22] and has been the object of many subsequent works including [2, 3, 4, 5, 7, 9, 12, 13, 15, 17, 20].

The purpose of this paper is to describe a certain left module over Solomon's descent ring. This module is defined in terms of *affine descent sets*, a notion introduced by Cellini [9], and further studied in [10, 11, 18]. This is reviewed next.

We assume that the finite Coxeter group W is *irreducible* and *crystallographic* [14]. In this case, there is a unique *highest* root $\widetilde{\alpha}$ and a corresponding affine Coxeter group $\widetilde{W}$ generated by W and the affine reflection through this highest root (see Section 2.2). Let $\alpha_0 = -\widetilde{\alpha}$. By analogy with ordinary descent sets, the affine descent set, $\widetilde{D}(w)$, of an element $w \in W$, is defined as follows:

$$\widetilde{D}(w) = \{0 \le i \le n : w\alpha_i < 0\}.$$

Thus, $D(w) \subseteq \widetilde{D}(w)$, and the only difference occurs when w does not take α_0 to a positive root. Notice that every element has at least one affine descent, and no element can have more than n affine descents.

We emphasize that although the construction of the affine descent module (in Section 3) relies heavily on features of the affine group $\widetilde{W}$, the set $\widetilde{D}(w)$ is defined only for elements w of the finite Coxeter group W, and not for general elements of $\widetilde{W}$.

For any proper nonempty subset J of $\overline{[n]} := \{0, 1, \dots, n\}$, let

$$\overline{x}_J := \sum_{\widetilde{D}(w) \subseteq J} w.$$

While the elements $\overline{x}_J$ do not span a subring of $\mathbb{Z}W$, we show that they span a left module over $\mathrm{Sol}(W)$. We remark that Cellini showed that the elements $\sum_{|J|=k} \overline{x}_J$, as k runs from 1 to n, do span a subring (in fact, a commutative nonunital subring) of $\mathbb{Z}W$.

We follow the geometric approach of Tits (in his appendix to Solomon's paper [22]), as developed by Bidigare [6] and Brown [8, Section 4.8]. These works relate the algebraic structure of $\mathrm{Sol}(W)$ to the geometric structure of the Coxeter complex. Specifically, the elements x_J correspond to W-orbits of faces in the Coxeter complex. Work of Dilks, Petersen, and Stembridge [10] shows that the $\overline{x}_J$ correspond to W-orbits in the *Steinberg torus*, an object obtained by taking the quotient of the affine Coxeter complex by the co-root lattice.

Here we show that the faces of the Coxeter complex act on the faces of the affine Coxeter complex, and that this action passes through the quotient to an action on the Steinberg torus.

The action on affine faces admits a simple geometric description. An affine hyperplane arrangement splits the ambient space into a set of faces. The hyperplane *at infinity* is similarly decomposed into a set of faces. The latter set is a monoid under the Tits product and the former a right module over it. In the case of the affine arrangement of W, the faces at infinity constitute the Coxeter complex, affine faces are acted upon by co-root translations, and the quotient by this action is the set the faces of the Steinberg torus. It follows that the set of faces of the Steinberg torus is a right module over the Coxeter complex. The structure is equivariant with respect to the Weyl group, and we may consider the induced structure on orbits. This results in the left module structure of affine descent classes over Solomon's descent ring.

Section 2 describes these geometric aspects, providing background on both finite and affine Coxeter complexes and how they can both be viewed inside the closure of the *Tits cone*. We discuss the Steinberg torus as well, and give combinatorial models for faces of all these complexes in Type A_{n-1}. (For the affine Coxeter complex and the Steinberg torus, this model appears to be new.) Section 3 relates the geometric actions to the module structures.

We thank the referees for pointing out the work of Moszkowski concerning modules over Solomon's descent algebra [16]. We plan to explore possible connections to this work and also to that of Saliola [19, 21] in the future. This extended abstract is a condensed version of a longer article; many details and most of the proofs have been omitted.

2 Products of faces in the Tits cone

Let W be a finite Coxeter group with a crystallographic root system Φ embedded in a real Euclidean space V with inner product $\langle \cdot\,, \cdot \rangle$. (So W is a Weyl group.) For any root $\beta \in \Phi$, let $H_\beta := \{\lambda \in V : \langle \lambda, \beta \rangle = 0\}$ be the hyperplane orthogonal to β and let s_β denote the orthogonal reflection through H_β. If we fix a set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_n\} \subset \Phi$, then $S = \{s_1, \ldots, s_n\}$ denotes the corresponding set of simple reflections.

Having fixed a choice of simple roots, every root β either belongs to the nonnegative span of the simple roots and is designated *positive*, or else belongs to the nonpositive span of the simple roots and is designated *negative*. We write $\beta > 0$ or $\beta < 0$ accordingly. Let $\Pi = \{\beta \in \Phi : \beta > 0\}$ denote the set of positive roots.

2.1 The finite Coxeter complex

The set of hyperplanes $\mathcal{H}(\Phi) := \{H_\beta : \beta \in \Pi\}$ is the *Coxeter arrangement* associated to Φ (we take $\beta > 0$ for convenience since $H_\beta = H_{-\beta}$). A *face* F of the arrangement is any subset of V obtained by choosing, for each $\beta \in \Pi$, either the hyperplane H_β or one of the open half-spaces it bounds, and intersecting all these sets. A face F is determined by its *sign vector*:

$$\sigma(F) = (\sigma_\beta(F))_{\beta \in \Pi},$$

where if λ is any point in F, $\sigma_\beta(F) = +, -$, or 0, according to whether $\langle \lambda, \beta \rangle$ is positive, negative, or zero. Let Σ be the set of faces of $\mathcal{H}(\Phi)$.

We partially order Σ by inclusion of the face closures, i.e., $F \le G \Leftrightarrow \overline{F} \subseteq \overline{G}$. This partial order gives Σ a structure isomorphic to the *Coxeter complex* of W, defined abstractly as the set of cosets of parabolic subgroups of W, ordered by reverse inclusion.

There is a monoid structure on the faces of the Coxeter arrangement, given geometrically as follows. For two faces F and G in Σ, their product FG is the first face of Σ entered upon traveling a small positive distance on a straight line from a point of F to a point in G. See Figure 1. This product is associative and admits the following characterization in terms of sign vectors [1, Proposition 2.82]:

$$\sigma_\beta(FG) = \begin{cases} \sigma_\beta(F) & \text{if } \sigma_\beta(F) \neq 0, \\ \sigma_\beta(G) & \text{if } \sigma_\beta(F) = 0. \end{cases} \tag{1}$$

An alternative characterization of the faces of the Coxeter complex is given in terms of the action of the group W on Σ. The choice of simple roots Δ is equivalent to designating a dominant chamber, namely:

$$C_\emptyset := \{\lambda \in V : \langle \lambda, \alpha \rangle > 0 \text{ for all } \alpha \in \Delta\}.$$

This is the unique face with sign vector $(+, +, \ldots)$. The closure of the dominant chamber is a fundamental domain for the action of W on V, and thus every face has the form wC_J, where $w \in W$, $J \subseteq [n]$, and

$$C_J := \{\lambda \in V : \langle \lambda, \alpha_j \rangle = 0 \text{ for } j \in J, \langle \lambda, \alpha_j \rangle > 0 \text{ for } j \in [n] - J\}.$$

The set J is uniquely determined by the face, but in general the element w is not.

The rays (1-dimensional faces) have the form wC_J where $J = [n] - \{j\}$ for some j. If we assign color j to all such rays, we obtain a *balanced* coloring of Σ; i.e., every maximal face (chamber) has exactly one vertex (extreme ray) of each color. We see that in general a face wC_J has color set $J^c = [n] - J$.

For a positive root β and $\lambda \in C_J$, we have $\langle w\lambda, \beta\rangle = \langle \lambda, w^{-1}\beta\rangle$, so we can characterize sign vectors as follows:

$$\sigma_\beta(wC_J) = \begin{cases} 0 & \text{if } w^{-1}\beta \in \mathrm{Span}\{\alpha_j : j \in J\} \\ + & \text{if } w^{-1}\beta \in \Pi - \mathrm{Span}\{\alpha_j : j \in J\} \\ - & \text{if } -w^{-1}\beta \in \Pi - \mathrm{Span}\{\alpha_j : j \in J\}. \end{cases}$$

In particular, notice that if $w\alpha_i = -\beta < 0$, i.e., if i is a descent of w, then $w^{-1}\beta = -\alpha_i$ and we have $\langle \lambda, -\alpha_i\rangle \leq 0$. Thus the descents of w are encoded among the zeroes and minus signs of $\sigma(wC_J)$. Conversely, if $\sigma_{w\alpha_i}(wC_J) = +$, then i *cannot* be a descent of w.

As seen from (1), the product of a face F with a chamber C always results in another chamber (none of the entries are zero), the *Tits projection* of F onto C.

In particular, we can characterize projections onto the fundamental chamber as follows. Since the fundamental chamber has sign vector $\sigma(C_\emptyset) = (+, +, \ldots)$, Equation (1) gives:

$$\sigma_\beta(FC_\emptyset) = \begin{cases} \sigma_\beta(F) & \text{if } \sigma_\beta(F) \neq 0, \\ + & \text{if } \sigma_\beta(F) = 0. \end{cases}$$

Let w_F denote the unique element of W such that $w_F C_\emptyset = FC_\emptyset$. Since $+$ signs cannot correspond to descents, we have the following.

Proposition 2.1 *For any face F of Σ, $D(w_F) \subseteq \mathrm{col}(F)$. Moreover, for any $w \in W$ and any J with $D(w) \subseteq J \subseteq [n]$, there is a J-colored face F such that $w = w_F$.*

There is a well-known combinatorial model for the Coxeter complex of Type A_{n-1}; see Figure 1. The faces are encoded with *set compositions* of $[n]$, i.e., set partitions with a linear order on the set of blocks. The partial order on faces is given by refinement. The product of two faces is given by refining the first set composition according to the second. For example, $3567|4|12 \cdot 26|35|17|4 = 6|35|7|4|2|1$, where the blocks are separated by bars and the set of blocks is ordered from left to right.

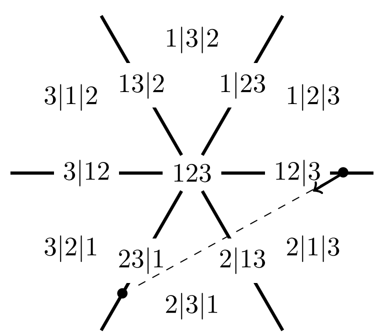

Fig. 1: The product of faces in the Coxeter arrangement of type A_2: $12|3 \cdot 23|1 = 2|1|3$.

The color of a face corresponds to the positions of the vertical bars, and the element w_F is the permutation obtained by writing the elements of the blocks in increasing order and removing the bars. For example, if $F = 3|4|156|2$, $\mathrm{col}(F) = \{1, 2, 5\}$, and $w_F = 341562$. Since the elements of the blocks are written in increasing order, descents can only occur between blocks, i.e., in the locations of the bars.

2.2 The affine Coxeter complex

The affine Weyl group $\widetilde{W}$ is generated by reflections $s_{\beta,k}$ through the affine hyperplanes

$$H_{\beta,k} := \{\lambda \in V : \langle \lambda, \beta \rangle = k\} \qquad (\beta \in \Phi,\ k \in \mathbb{Z}).$$

Alternatively, one may construct $\widetilde{W}$ as the semidirect product $W \ltimes \mathbb{Z}\Phi^\vee$, where $\mathbb{Z}\Phi^\vee$ denotes the lattice generated by all co-roots $\beta^\vee = 2\beta/\langle \beta, \beta \rangle$ ($\beta \in \Phi$). The action of $\widetilde{W}$ on V extends the action of W by linear reflections and the action of $\mathbb{Z}\Phi^\vee$ by translations.

Suppose from now on that Φ is irreducible. Then it has a unique highest root $\widetilde{\alpha}$, and it is well-known that $\widetilde{W}$ is generated by $\widetilde{S} := S \cup \{s_{\widetilde{\alpha},1}\}$ and that $(\widetilde{W}, \widetilde{S})$ is an irreducible Coxeter system.

The *affine Coxeter arrangement* is

$$\widetilde{\mathcal{H}}(\Phi) := \{H_{\beta,k} : \beta \in \Pi, k \in \mathbb{Z}\}.$$

The set of faces of $\widetilde{\mathcal{H}}(\Phi)$ is isomorphic to the affine Coxeter complex of $\widetilde{W}$. We denote it by $\widetilde{\Sigma}$.

A face can again be encoded by a sign vector that records whether the face is "above", "below", or "on" a particular hyperplane. (That the sign vector is now infinite is not a problem. See [1, Section 2.7].) We have $\sigma(F)$, for F a nonempty face in $\widetilde{\Sigma}$, given by

$$\sigma(F) = (\sigma_{\beta,k}(F))_{\beta \in \Pi, k \in \mathbb{Z}}, \tag{2}$$

where $\sigma_{\beta,k}(F)$ is $+, -$, or 0, according to whether $\langle \lambda, \beta \rangle - k$ is positive, negative, or zero. Notice, however, that for a given positive root β, we have a unique j such that

$$(\ldots, \sigma_{\beta,-1}(F), \sigma_{\beta,0}(F), \sigma_{\beta,1}(F), \ldots) = (\ldots, +, +, \sigma_{\beta,j}(F), -, -, \ldots),$$

where $\sigma_{\beta,j}(F)$ is either $+$ or 0. Therefore, for each β we need no more than the pair $(j, \sigma_{\beta,j})$. Thus, let us write instead

$$\sigma(F) = (\,(k_\beta(F), \sigma_\beta(F))\,)_{\beta \in \Pi}, \tag{3}$$

where for any point λ of F, $k_\beta(F) = j$ means $j \le \langle \lambda, \beta \rangle < j + 1$, and $\sigma_\beta(F) = 0$ or $+$ according to whether $\langle \lambda, \beta \rangle$ is equal to or greater than j. We refer to (2) as the *expanded sign vector* of F and (3) as the *compact sign vector* of F.

The product of two faces of $\widetilde{\Sigma}$ is defined exactly as in the finite case. However, we can do more via the Tits cone.

2.3 The Tits cone

The *Tits cone* is a collection of polyhedral cones. We can explicitly realize this cone by embedding V in a vector space of one dimension higher and taking the cone over $\widetilde{\Sigma}$ by a point not in V. The finite Coxeter complex Σ is the boundary of the cone. That is, once linearized by taking the cone point to be the origin, all parallel hyperplanes in $\widetilde{\mathcal{H}}$ converge to a common hyperplane in the space parallel to V and containing the cone point. See Figures 2 and 3.

The faces of Σ can thus be endowed with an expanded sign vector as follows. If $F \in \Sigma$, $\sigma_{\beta,k}(F) = \sigma_\beta(F)$ for all k. The Tits cone shows us how to extend our geometric product to a product of a face $F \in \widetilde{\Sigma}$ with a face $G \in \Sigma$. In terms of expanded sign vectors we have the following.

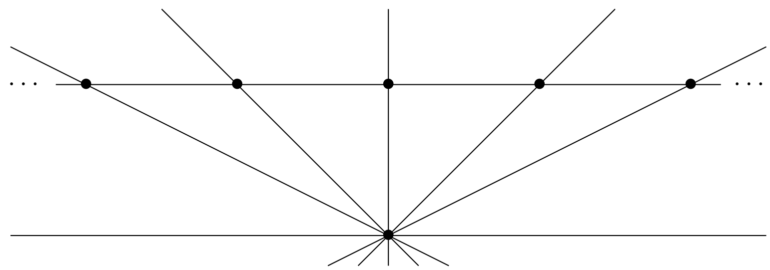

Fig. 2: The Tits cone, with $\widetilde{\Sigma}$ in the interior, Σ on the boundary.

Proposition 2.2 *Let $F \in \widetilde{\Sigma}$ and $G \in \widetilde{\Sigma} \cup \Sigma$.*

$$\sigma_{\beta,k}(FG) = \begin{cases} \sigma_{\beta,k}(F) & \text{if } \sigma_{\beta,k}(F) \neq 0, \\ \sigma_{\beta,k}(G) & \text{if } \sigma_{\beta,k}(F) = 0. \end{cases} \tag{4}$$

This product is associative, i.e., $F(G_1G_2) = (FG_1)G_2$ for all $F \in \widetilde{\Sigma}$, $G_2 \in \Sigma$, and G_1 in either $\widetilde{\Sigma}$ or Σ. The fact that the hyperplane arrangement is infinite is not a problem, since the geometric definition requires only that each face has only a finite number of faces in a small enough neighborhood. Note however that the reverse product, from $G \in \Sigma$ to $F \in \widetilde{\Sigma}$ is ill-defined. Every (full-dimensional) neighborhood of G contains infinitely many hyperplanes, so there is no "first" face to enter in walking toward F.

It is clear from this characterization that both the set $\widetilde{\mathcal{C}}$ of alcoves (maximal simplices in $\widetilde{\Sigma}$) and the set $\mathcal{C}$ of chambers take faces of $\widetilde{\Sigma}$ to alcoves, since both types of products do not leave any 0 entries in the sign vector.

We now interpret the faces of $\widetilde{\Sigma}$ in terms of $\widetilde{W}$ acting on V. The action of $\widetilde{W}$ on alcoves is simply transitive, and the fundamental alcove

$$A_\emptyset := C_\emptyset \cap \{\lambda \in V : \langle \lambda, \widetilde{\alpha} \rangle < 1\}$$

is tied to the choice of $\widetilde{S}$ in the sense that the $\widetilde{W}$-stabilizer of every point in the closure of $A_\emptyset$ (a fundamental domain) is generated by a proper subset of $\widetilde{S}$. The compact sign vector of $A_\emptyset$ is $((0,+),(0,+),\ldots))$.

We index the faces of $A_\emptyset$ by subsets of $\overline{[n]}$ so that the J-th face is

$$A_J := \begin{cases} C_J \cap \{\lambda \in V : \langle \lambda, \widetilde{\alpha} \rangle < 1\} & \text{if } 0 \notin J, \\ C_{J\setminus\{0\}} \cap \{\lambda \in V : \langle \lambda, \widetilde{\alpha} \rangle = 1\} & \text{if } 0 \in J. \end{cases}$$

Note that A_J is the empty face (or the cone point in the Tits cone) when $J = \overline{[n]}$.

Since the closure of $A_\emptyset$ is a fundamental domain for the action of $\widetilde{W}$, each face in this complex has the form $\mu + wA_J$ ($\mu \in \mathbb{Z}\Phi^\vee$, $w \in W$, $J \subseteq \overline{[n]}$). Note that the vertices of $\widetilde{\Sigma}$ are of the form $\mu + wA_{\{j\}^c}$, where $J^c := \overline{[n]} - J$. If we assign color j to each of the vertices $\mu + wA_{\{j\}^c}$, then the vertices of the cell $\mu + wA_J$ are assigned color-set J^c (without repetitions), so this coloring is balanced.

2.4 Translational invariance and the Steinberg torus

Translations are identified with 0-colored vertices, and in terms of compact sign vectors, we find

$$k_\beta(\mu + wA_J) = k_\beta(\mu) + k_\beta(wA_J) \text{ and } \sigma_\beta(\mu + wA_J) = \sigma_\beta(wA_J). \tag{5}$$

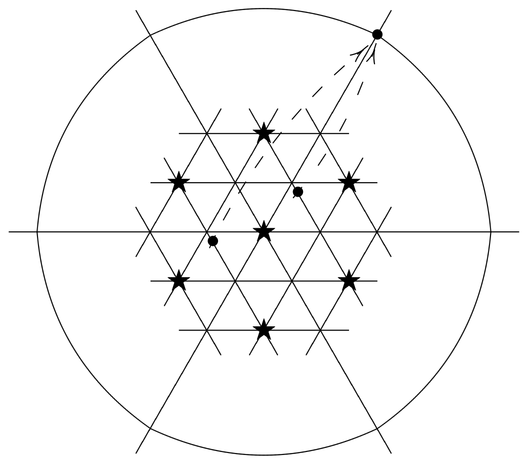

Fig. 3: The product of an affine face with a face at infinity is translation invariant. Elements of the co-root lattice $\mathbb{Z}\Phi^\vee$ are indicated with stars.

Thus for a face F we see that σ_β is completely determined by wA_J, while k_β is almost entirely controlled by μ, since $k_\beta(wA_J)$ can only be $-1, 0$, or 1.

In particular, since products with faces at infinity only change signs from 0 to +, we see that products with faces of Σ are translation invariant. See Figure 3.

Proposition 2.3 *Let $F \in \widetilde{\Sigma}$, $G \in \Sigma$, and $\mu \in \mathbb{Z}\Phi^\vee$. Then $(\mu + F)G = \mu + FG$.*

As mentioned, the product of a face $F \in \widetilde{\Sigma}$ with a chamber $C \in \Sigma$ is an alcove, $\mu + wA_\emptyset$, which we may again refer to as the *Tits projection* of F onto C. By Proposition 2.3, it suffices to characterize projections for faces wA_J, i.e., with $\mu = 0$.

Just as with faces wC_J in Σ, we find that for $i > 0$, if $\alpha_i = w^{-1}\beta$ and $(k_\beta(wA_J), \sigma_\beta(wA_J)) = (0, +)$, we know i is *not* an ordinary descent of w.

If $0 \in J$, we also have

$$(k_\beta(wA_J), \sigma_\beta(wA_J)) = \begin{cases} (1, 0) & \text{if } w^{-1}\beta = \widetilde{\alpha} \\ (-1, 0) & \text{if } w^{-1}\beta = -\widetilde{\alpha}. \end{cases}$$

Thus, if $w\alpha_0 = -\beta < 0$, i.e., if 0 is an affine descent of w, then $w^{-1}\beta = -\alpha_0 = \widetilde{\alpha}$, and we get

$$0 < \langle \lambda, w^{-1}\beta \rangle = \langle \lambda, \widetilde{\alpha} \rangle \leq 1.$$

Therefore if $k_\beta(wA_J) = -1$ we know 0 is *not* an affine descent of w.

For $F \in \widetilde{\Sigma}$, let w_F denote the unique element of W such that $\mu_F + w_F A_\emptyset = FC_\emptyset$. While μ_F is uniquely determined by this projection, its exact nature does not concern us as much as w_F. Equation (4) shows that all zeroes in the expanded sign vector become $+$, and following our analysis of affine descents above allows us to generalize Proposition 2.1.

Proposition 2.4 *For any face F of $\widetilde{\Sigma}$, we have $\widetilde{D}(w_F) \subseteq \mathrm{col}(F)$. Moreover, for any $w \in W$, and any J with $\widetilde{D}(w) \subseteq J \subseteq \overline{[n]}$, there is a J-colored face F such that $w = w_F$.*

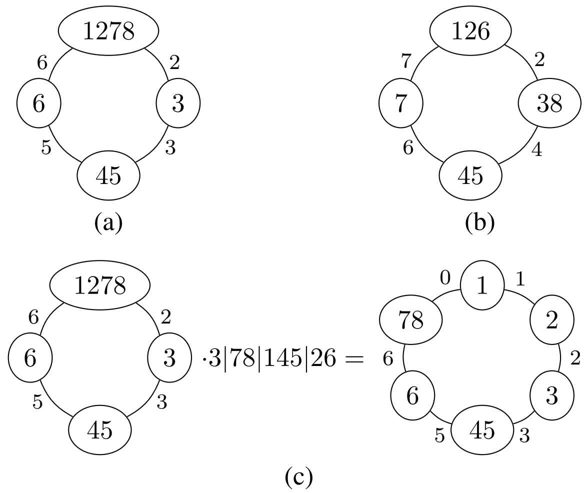

Fig. 4: (a) A spin necklace; (b) another spin necklace; (c) the product of a spin necklace and a set composition.

As the translation subgroup of $\widetilde{W}$, the co-root lattice $\mathbb{Z}\Phi^\vee$ acts as a group of color-preserving automorphisms of $\widetilde{\Sigma}$. Letting T denote the n-torus $V/\mathbb{Z}\Phi^\vee$, it follows that the image of $\widetilde{\Sigma}$ under the projection $\pi : V \to T$ is a balanced Boolean complex,

$$\overline{\Sigma} := \widetilde{\Sigma}/\mathbb{Z}\Phi^\vee.$$

Following [10], we refer to $\overline{\Sigma}$ as the *Steinberg torus*.

An alternative construction of the Steinberg torus is given by identifying maximal opposite faces of the W-invariant convex polytope

$$P_\Phi = \{\lambda \in V : -1 \leq \langle \lambda, \beta \rangle \leq 1 \text{ for all } \beta \in \Phi\}.$$

This polytope is the union of the closures of the alcoves $wA_\emptyset$ $(w \in W)$. Note that there is a bijection with maximal faces: $w \leftrightarrow wA_\emptyset + \mathbb{Z}\Phi^\vee$ for each $w \in W$. Let $\overline{\mathcal{C}}$ denote the set of maximal faces of $\overline{\Sigma}$.

Since products of faces of $\widetilde{\Sigma}$ with faces of Σ are translation invariant (Proposition 2.3), we have a well-defined product FG with $F \in \overline{\Sigma}$ and $G \in \Sigma$. We remark, however, that products of two faces of $\widetilde{\Sigma}$ are *not* translation invariant, and so the projection π does not give a well-defined product of faces of $\overline{\Sigma}$.

We can describe faces of $\widetilde{\Sigma}(A_{n-1})$ and $\overline{\Sigma}(A_{n-1})$ in terms of a combinatorial model similar to set compositions, which we call *labeled spin necklaces*. These objects encode both the color of the face F and the representative w_F a straightforward way. (For $\widetilde{\Sigma}(A_{n-1})$ there is some mild bookkeeping involving the co-root μ_F which we will not describe here.)

First, recall the affine descent set of a permutation $w \in W = S_n$ is the set of *cyclic descents*, i.e., the descent in 0 occurs when $w_n > w_1$. For example, $\widetilde{D}(78345612) = \{2, 6\}$ while $\widetilde{D}(134625) = \{0, 4\}$.

A spin necklace consists of a cyclically ordered set partition $(B_1, \ldots, B_k)$ of $[n]$, together with labeled edges $(e_1, \ldots, e_k)$, with e_i joining B_i to B_{i+1} clockwise (and indices modulo k). The labels are the elements of the color set written in increasing order and the block B_i consists of the elements between positions $e_{i-1} + 1$ and e_i in w_F (read cyclically). Note that the difference of consecutive edge labels is the size of the intermediate block: $e_i - e_{i-1} \equiv |B_i| \mod n$. If F is such that $w_F = 78345612$ and $\mathrm{col}(F) = \{2,3,5,6\}$, its spin necklace is shown in Figure 4 (a).

This restriction on the edge labels, along with the fact that $\widetilde{D}(w_F) \subseteq \mathrm{col}(F)$, allows us to uniquely recover F from a given spin necklace, e.g., the necklace in Figure 4 (b) has $w_F = 26384571$ and $\mathrm{col}(F) = \{2,4,6,7\}$.

The partial order on faces corresponds to refinement of spin necklaces. The product of a face F in $\widetilde{\Sigma}$ or in $\overline{\Sigma}$ with a face G in Σ is similar to the case of two faces of Σ. The only difference between taking $F \in \widetilde{\Sigma}$ versus $F \in \overline{\Sigma}$ is that in the former case μ may change if $0 \notin \mathrm{col}(F)$. We omit the details of this change, and describe only the change in the spin necklaces.

Proposition 2.5 *Let F be a face of $\widetilde{\Sigma}(A_{n-1})$ or $\overline{\Sigma}(A_{n-1})$ with spin necklace $((e_1, B_1), \ldots, (e_k, B_k))$. Let $G = C_1|\cdots|C_l$ be a face of $\Sigma(A_{n-1})$. Then the spin necklace in the product of F and G has its blocks given by all pairwise intersections of the blocks, $B_{i,j} = B_i \cap C_j$, with edge labels $e_{i,j}$ such that $e_{i,1} = e_i$ and $e_{i,j+1} = e_{i,j} + |B_{i,j}|$.*

For example, see Figure 4 (c).

3 Modules over Solomon's descent ring

Let $\mathbb{Z}\Sigma$ denote the monoid ring of Σ and consider the subring $(\mathbb{Z}\Sigma)^W$ of W-invariants. Bidigare [6] showed that the latter is anti-isomorphic to Solomon's descent ring. We follow here the proof of this fact by Brown [8, Section 9.6], and obtain counterparts for $\widetilde{\Sigma}$ and $\overline{\Sigma}$.

The product given in Equation (1) gives the set Σ the structure of a monoid. The product in (4) turns the set $\widetilde{\Sigma}$ into a right Σ-module.

From the translational invariance of Proposition 2.3, it follows that the Steinberg torus $\overline{\Sigma}$ is a quotient right Σ-module of $\widetilde{\Sigma}$. The projection $\pi : \widetilde{\Sigma} \to \overline{\Sigma}$ is thus a morphism of right Σ-modules.

The Weyl group W acts on both $\mathbb{Z}\Phi^\vee$ and $\widetilde{\Sigma}$, and these actions and the action of $\mathbb{Z}\Phi^\vee$ on $\widetilde{\Sigma}$ are related by the *semilinearity* condition

$$w \cdot (\mu + F) = w \cdot \mu + w \cdot F \tag{6}$$

for $w \in W$, $\mu \in \mathbb{Z}\Phi^\vee$, and $F \in \widetilde{\Sigma}$. It follows that W acts on $\overline{\Sigma}$ and that π is a morphism of left W-modules.

The Weyl group W also acts on the monoid Σ and we have

$$w \cdot (FG) = (w \cdot F)(w \cdot G) \tag{7}$$

for $w \in W$, G in Σ and F in either Σ, $\widetilde{\Sigma}$, or $\overline{\Sigma}$.

We linearize the sets $\widetilde{\Sigma}$ and $\overline{\Sigma}$, obtaining abelian groups $\mathbb{Z}\widetilde{\Sigma}$ and $\mathbb{Z}\overline{\Sigma}$. We emphasize that $\mathbb{Z}\widetilde{\Sigma}$ consists of *finite* linear combinations of elements of $\widetilde{W}$. For this reason, 0 is the only element of $\mathbb{Z}\widetilde{\Sigma}$ invariant under the action of $\widetilde{W}$. We consider the action of W on the groups $\mathbb{Z}\widetilde{\Sigma}$ and $\mathbb{Z}\overline{\Sigma}$, and the corresponding subgroups of W-invariant elements. It follows from (7) that $(\mathbb{Z}\widetilde{\Sigma})^W$ is a right module over the ring $(\mathbb{Z}\Sigma)^W$, and also that the map $\pi : \widetilde{\Sigma} \to \overline{\Sigma}$ restricts to a morphism of right $(\mathbb{Z}\Sigma)^W$-modules $\pi : (\mathbb{Z}\widetilde{\Sigma})^W \to (\mathbb{Z}\overline{\Sigma})^W$.

The set of chambers $\mathcal{C}$ is a two-sided ideal of the monoid Σ. The right action of Σ on $\mathcal{C}$ is trivial: $CF = C$ for every $C \in \mathcal{C}$ and $F \in \Sigma$. The product of a face of Σ (or a face of $\widetilde{\Sigma}$, or of $\overline{\Sigma}$) and a chamber of Σ is a chamber of Σ (or an alcove of $\widetilde{\Sigma}$, or a maximal face of $\overline{\Sigma}$). This gives rise to three maps

$$\mathbb{Z}\Sigma \to \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}) \qquad \mathbb{Z}\widetilde{\Sigma} \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\widetilde{\mathcal{C}}) \qquad \mathbb{Z}\overline{\Sigma} \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\overline{\mathcal{C}}) \tag{8}$$

denoted in every case by Φ and given by $\Phi(F)(C) := FC$ (and extended by $\mathbb{Z}$-linearity).

The abelian group $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C})$ is a ring under composition, while both $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\widetilde{\mathcal{C}})$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\overline{\mathcal{C}})$ are right $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C})$-modules in the same manner. Associativity for the product of Σ (or for the right action of Σ on $\widetilde{\Sigma}$, or on $\overline{\Sigma}$) translates into the fact that $\Phi(FG) = \Phi(F) \circ \Phi(G)$ for $G \in \Sigma$ and F in either Σ, $\widetilde{\Sigma}$, or $\overline{\Sigma}$. This says that the first map in (8) is a morphism of rings, while the other two maps are morphisms of right Σ-modules, where $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\widetilde{\mathcal{C}})$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\overline{\mathcal{C}})$ are viewed as right $\mathbb{Z}\Sigma$-modules by restriction via $\Phi : \mathbb{Z}\Sigma \to \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C})$.

The sets $\mathcal{C}$, $\widetilde{\mathcal{C}}$, and $\overline{\mathcal{C}}$ are stable under the action of W, and hence the groups $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C})$, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\widetilde{\mathcal{C}})$, and $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\overline{\mathcal{C}})$ are acted upon by W from the left. The action is $(w \cdot f)(C) = w \cdot f(w^{-1} \cdot C)$ for $w \in W$, $C \in \mathcal{C}$, and f in either $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C})$, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\widetilde{\mathcal{C}})$, or $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\overline{\mathcal{C}})$. Equation (7) implies that $\Phi(w \cdot F) = w \cdot \Phi(F)$ for $w \in W$ and F in either Σ, $\widetilde{\Sigma}$, or $\overline{\Sigma}$. It follows that each map Φ restricts as follows:

$$(\mathbb{Z}\Sigma)^W \to \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C})^W, \qquad (\mathbb{Z}\widetilde{\Sigma})^W \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\widetilde{\mathcal{C}})^W, \qquad (\mathbb{Z}\overline{\Sigma})^W \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\overline{\mathcal{C}})^W.$$

These maps are still denoted by Φ. The first one is a morphism of rings and the other two are morphisms of right $(\mathbb{Z}\Sigma)^W$-modules.

Since the action of W on $\mathcal{C}$ is free and transitive, we have isomorphims

$$\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C})^W = \mathrm{End}_{\mathbb{Z}W}(\mathbb{Z}\mathcal{C}) \cong \mathbb{Z}\mathcal{C}, \quad \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\widetilde{\mathcal{C}})^W = \mathrm{Hom}_{\mathbb{Z}W}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\widetilde{\mathcal{C}}) \cong \mathbb{Z}\widetilde{\mathcal{C}},$$
$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\overline{\mathcal{C}})^W = \mathrm{Hom}_{\mathbb{Z}W}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\overline{\mathcal{C}}) \cong \mathbb{Z}\overline{\mathcal{C}},$$

given in every case by $f \mapsto f(C_\emptyset)$, where $C_\emptyset$ is the fundamental chamber of Σ.

We may further identify $\mathcal{C}$ with W by means of $w \leftrightarrow w \cdot C_\emptyset$, where $C_\emptyset$ is the fundamental chamber of Σ. Consider the composite isomorphism of abelian groups

$$\mathrm{End}_{\mathbb{Z}W}(\mathbb{Z}\mathcal{C}) \cong \mathbb{Z}W. \tag{9}$$

A group element $u \in W$ corresponds to the endomorphism f such that $f(C_\emptyset) = u \cdot C_\emptyset$. If another element $v \in W$ corresponds to the endomorphism g, then $(f \circ g)(C_\emptyset) = f(v \cdot C_\emptyset) = v \cdot f(C_\emptyset) = vu \cdot C_\emptyset$, so $f \circ g$ corresponds to vu. Therefore, the isomorphism of rings (9) reverses products.

Similarly, we have $\widetilde{\mathcal{C}} \cong \widetilde{W}$ and $\overline{\mathcal{C}} \cong W$ via the actions of these groups on the fundamental alcoves of these complexes. This gives rise to isomorphisms of right $\mathrm{End}_{\mathbb{Z}W}(\mathbb{Z}\mathcal{C})$-modules

$$\mathrm{Hom}_{\mathbb{Z}W}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\widetilde{\mathcal{C}}) \cong \mathbb{Z}\widetilde{W} \qquad \text{and} \qquad \mathrm{Hom}_{\mathbb{Z}W}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\overline{\mathcal{C}}) \cong \mathbb{Z}W$$

where now $\mathbb{Z}\widetilde{W}$ and $\mathbb{Z}W$ are first viewed as left $\mathbb{Z}W$-modules by multiplication, and then as right $\mathrm{End}_{\mathbb{Z}W}(\mathbb{Z}\mathcal{C})$-modules via the antimorphism (9).

Composing the maps Φ with the preceding isomorphisms we obtain three maps

$$(\mathbb{Z}\Sigma)^W \to \mathbb{Z}W, \qquad (\mathbb{Z}\widetilde{\Sigma})^W \to \mathbb{Z}\widetilde{W}, \qquad (\mathbb{Z}\overline{\Sigma})^W \to \mathbb{Z}W, \tag{10}$$

denoted in every case by Ψ and given by $\Psi\left(\sum_F a_F\, F\right) = \sum_F a_F\, FC_\emptyset$, where in each case $\sum_F a_F\, F$ stands for a W-invariant element of $\mathbb{Z}\Sigma$, $\mathbb{Z}\widetilde{\Sigma}$, or $\mathbb{Z}\overline{\Sigma}$.

The first map in (10) is an anti-morphism of rings and the other two are morphisms of right $(\mathbb{Z}\Sigma)^W$-modules, where $\mathbb{Z}\widetilde{W}$ and $\mathbb{Z}W$ are first viewed as left $\mathbb{Z}W$-modules by multiplication, and then as right $(\mathbb{Z}\Sigma)^W$-modules via the antimorphism $\Psi : (\mathbb{Z}\Sigma)^W \to \mathbb{Z}W$.

The actions of W on Σ, $\widetilde{\Sigma}$ and $\overline{\Sigma}$ are color-preserving. Therefore $(\mathbb{Z}\Sigma)^W$ is a free abelian group with basis

$$\sigma_J := \sum_{F \in \Sigma_J} F,$$

where J runs over the subsets of $[n]$. Similarly, $(\mathbb{Z}\widetilde{\Sigma})^W$ and $(\mathbb{Z}\overline{\Sigma})^W$ are free abelian groups with bases

$$\widetilde{\sigma}_{J,\mu} := \sum_{F \in \widetilde{\Sigma}_{J,\mu}} F \qquad \text{and} \qquad \overline{\sigma}_J := \sum_{F \in \overline{\Sigma}_J} F,$$

where for $J \subseteq \overline{[n]}$ and $\mu \in \mathbb{Z}\Phi^\vee$, we let $\widetilde{\Sigma}_{J,\mu}$ denote the set of faces in the orbit of $\mu + A_{J^c}$ and $\overline{\Sigma}_J$ denotes the set of J-colored faces of $\overline{\Sigma}$.

For each $J \subseteq [n]$, define elements $x_J \in \mathbb{Z}W$ by

$$x_J := \sum_{w \in W : D(w) \subseteq J} w.$$

Similarly, for $\mu \in \mathbb{Z}\Phi^\vee$ and $J \subseteq \overline{[n]}$, define $\widetilde{x}_{J,\mu} \in \mathbb{Z}\widetilde{W}$ and $\overline{x}_J \in \mathbb{Z}W$ by

$$\widetilde{x}_{J,\mu} := \sum_{w \in W :\, \widetilde{D}(w) \subseteq J} (w, w \cdot \mu), \qquad \text{and} \qquad \overline{x}_J := \sum_{w \in W :\, \widetilde{D}(w) \subseteq J} w.$$

As J varies, the sets $\{w \in W : D(w) = J\}$ and $\{w \in W : \widetilde{D}(w) = J\}$ are disjoint. Therefore, each set $\{x_J\}$, $\{\widetilde{x}_{J,\mu}\}$, and $\{\overline{x}_J\}$ is linearly independent.

Proposition 3.1 *The maps Ψ behave as follows:*

$$\Psi(\sigma_J) = x_J, \qquad \Psi(\widetilde{\sigma}_{J,\mu}) = \widetilde{x}_{J,\mu}, \qquad \Psi(\overline{\sigma}_J) = \overline{x}_J.$$

In particular, Ψ is injective in every case.

Defining $\mathrm{Sol}(W) = \mathrm{Span}\{x_J : J \subseteq [n]\}$, $\widetilde{\mathrm{Sol}}(W) = \mathrm{Span}\{\widetilde{x}_{J,\mu} : J \subseteq \overline{[n]}, \mu \in \mathbb{Z}\Phi^\vee\}$, and $\overline{\mathrm{Sol}}(W) = \{\overline{x}_J : J \subseteq \overline{[n]}\}$, we have our main result.

Theorem 3.2 *The map Ψ gives the followings anti-isomorphisms:*

$$(\mathbb{Z}\Sigma)^W \to \mathrm{Sol}(W), \qquad (\mathbb{Z}\widetilde{\Sigma})^W \to \widetilde{\mathrm{Sol}}(W), \qquad \textit{and} \qquad (\mathbb{Z}\overline{\Sigma})^W \to \overline{\mathrm{Sol}}(W).$$

In particular, $\widetilde{\mathrm{Sol}}(W)$ and $\overline{\mathrm{Sol}}(W)$ are left $\mathrm{Sol}(W)$-modules.

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The partition algebra and the Kronecker product (Extended abstract)

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Abstract. We propose a new approach to study the Kronecker coefficients by using the Schur–Weyl duality between the symmetric group and the partition algebra.

Résumé. Nous proposons une nouvelle approche pour l'étude des coéfficients de Kronecker via la dualité entre le groupe symétrique et l'algèbre des partitions.

Keywords: Kronecker coefficients, tensor product, partition algebra, representations of the symmetric group

1 Introduction

A fundamental problem in the representation theory of the symmetric group is to describe the coefficients in the decomposition of the tensor product of two Specht modules. These coefficients are known in the literature as the *Kronecker coefficients*. Finding a formula or combinatorial interpretation for these coefficients has been described by Richard Stanley as 'one of the main problems in the combinatorial representation theory of the symmetric group'. This question has received the attention of Littlewood [Lit58], James [JK81, Chapter 2.9], Lascoux [Las80], Thibon [Thi91], Garsia and Remmel [GR85], Kleshchev and Bessenrodt [BK99] amongst others and yet a combinatorial solution has remained beyond reach for over a hundred years.

Murnaghan discovered an amazing limiting phenomenon satisfied by the Kronecker coefficients; as we increase the length of the first row of the indexing partitions the sequence of Kronecker coefficients obtained stabilises. The limits of these sequences are known as the *reduced Kronecker coefficients*.

The novel idea of this paper is to study the Kronecker and reduced coefficients through the Schur–Weyl duality between the symmetric group, $\mathfrak{S}_n$, and the partition algebra, $P_r(n)$. The key observation being that the tensor product of Specht modules corresponds to the restriction of simple modules in $P_r(n)$ to a Young subalgebra. The combinatorics underlying the representation theory of both objects is based on

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partitions. The duality results in a Schur functor, $\mathrm{F} : \mathfrak{S}_n\text{-mod} \to P_r(n)\text{-mod}$, which acts by *first row removal* on the partitions labelling the simple modules. We exploit this functor along with the following three key facts concerning the representation theory of the partition algebra: (a) it is semisimple for large n (b) it has a stratification by symmetric groups (c) its non-semisimple representation theory is well developed.

Using our method we explain the limiting phenomenon of tensor products and bounds on stability, we also re-interpret the Kronecker and reduced Kronecker coefficients and the passage between the two in terms of the representation theory of the partition algebra. One should note that our proofs are surprisingly elementary.

The paper is organised as follows. In Sections 2 and 3 we recall the combinatorics underlying the representation theories of the symmetric group and partition algebra. In Section 4 we show how to pass the Kronecker problem through Schur–Weyl duality and phrase it as a question concerning the partition algebra. We then summarise results concerning the Kronecker and reduced Kronecker coefficients that have a natural interpretation (and very elementary proofs) in this setting. Section 5 contains an extended example.

2 Symmetric group combinatorics

The combinatorics underlying the representation theory of the symmetric group, $\mathfrak{S}_n$, is based on partitions. A *partition* λ of n, denoted $\lambda \vdash n$, is defined to be a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of non-negative integers such that the sum $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell$ equals n. The *length* of a partition is the number of nonzero parts, we denote this by $\ell(\lambda)$. We let Λ_n denote the set of all partitions of n.

With a partition, λ, is associated its *Young diagram*, which is the set of nodes

$$[\lambda] = \left\{ (i,j) \in \mathbb{Z}^2_{>0} \mid j \leq \lambda_i \right\}.$$

Given a node specified by $i, j \geq 1$, we say the node has *content* $j - i$. We let $\mathrm{ct}(\lambda_i)$ denote the content of the last node in the ith row of $[\lambda]$, that is $\mathrm{ct}(\lambda_i) = \lambda_i - i$.

Over the complex numbers, the irreducible *Specht* modules, $\mathbf{S}(\lambda)$, of $\mathfrak{S}_n$ are indexed by the partitions, λ, of n. An explicit construction of these modules is given in [JK81].

2.1 The classical Littlewood–Richardson rule

The Littlewood–Richardson rule is a combinatorial description of the coefficients in the restriction of a Specht module to a Young subgroup of the symmetric group. Through Schur–Weyl duality, the rule also computes the coefficients in the decomposition of a tensor product of two simple modules of $\mathrm{GL}_n(\mathbb{C})$.

The following is a simple restatement of this rule as it appears in [JK81, Section 2.8.13].

Theorem 2.1.1 (The Littlewood–Richardson Coefficients) *For $\lambda \vdash r_1$, $\mu \vdash r_2$ and $\nu \vdash r_1 + r_2$,*

$$\mathbf{S}(\nu)\downarrow^{\mathfrak{S}_{r_1+r_2}}_{\mathfrak{S}_{r_1}\times\mathfrak{S}_{r_2}} \cong \bigoplus_{\lambda\vdash r_1, \mu\vdash r_2} c^{\nu}_{\lambda,\mu} \mathbf{S}(\lambda) \boxtimes \mathbf{S}(\mu)$$

where the $c^{\nu}_{\lambda,\mu}$ are the Littlewood–Richardson coefficients.

The Littlewood–Richardson rule calculates the coefficients, $c^{\nu}_{\lambda,\mu}$, by counting tableaux, see [Mac95, Chapter I.9]. By transitivity of induction we have that the Littlewood–Richardson rule determines the structure of the restriction of a Specht module to any Young subgroup. Of particular importance in this paper is the three-part case

$$\mathbf{S}(\nu)\downarrow^{\mathfrak{S}_{r_1+r_2+r_3}}_{\mathfrak{S}_{r_1}\times\mathfrak{S}_{r_2}\times\mathfrak{S}_{r_3}} \cong \bigoplus_{\substack{\lambda\vdash r_1,\\ \mu\vdash r_2,\nu\vdash r_3}} \left(\sum_{\xi\vdash r_1+r_2} c^{\xi}_{\lambda,\mu}c^{\nu}_{\xi,\eta}\right)\mathbf{S}(\lambda)\boxtimes\mathbf{S}(\mu)\boxtimes\mathbf{S}(\eta).$$

We therefore set $c^{\nu}_{\lambda,\mu,\eta} = \sum_{\xi} c^{\xi}_{\lambda,\mu}c^{\nu}_{\xi,\eta}$.

2.2 Tensor products of Specht modules of the symmetric group

In this section we define the Kronecker coefficients and the reduced Kronecker coefficients as well as set some notation. Let λ and μ be two partitions of n, then

$$\mathbf{S}(\lambda)\otimes\mathbf{S}(\mu) = \bigoplus_{\nu\vdash n} g^{\nu}_{\lambda,\mu}\mathbf{S}(\nu),$$

the coefficients $g^{\nu}_{\lambda,\mu}$ are known as the *Kronecker coefficients*. These coefficients satisfy an amazing stability property illustrated in the following example.

Example 2.2.1 *We have the following tensor products of Specht modules:*

$$\begin{aligned}
\mathbf{S}(1^2)\otimes\mathbf{S}(1^2) &= \mathbf{S}(2)\\
\mathbf{S}(2,1)\otimes\mathbf{S}(2,1) &= \mathbf{S}(3)\oplus\mathbf{S}(2,1)\oplus\mathbf{S}(1^3)\\
\mathbf{S}(3,1)\otimes\mathbf{S}(3,1) &= \mathbf{S}(4)\oplus\mathbf{S}(3,1)\oplus\mathbf{S}(2,1^2)\oplus\mathbf{S}(2^2)
\end{aligned}$$

at which point the product stabilises, i.e. for all $n \geq 4$, we have

$$\mathbf{S}(n-1,1)\otimes\mathbf{S}(n-1,1) = \mathbf{S}(n)\oplus\mathbf{S}(n-1,1)\oplus\mathbf{S}(n-2,1^2)\oplus\mathbf{S}(n-2,2).$$

Let $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_\ell)$ be a partition and n be an integer, define $\lambda_{[n]} = (n-|\lambda|,\lambda_1,\lambda_2,\ldots,\lambda_\ell)$. Note that all partitions of n can be written in this form.

For $\lambda_{[n]},\mu_{[n]},\nu_{[n]} \in \Lambda_n$ we let

$$g^{\nu_{[n]}}_{\lambda_{[n]},\mu_{[n]}} = \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathfrak{S}_n}(\mathbf{S}(\lambda_{[n]})\otimes\mathbf{S}(\mu_{[n]}),\mathbf{S}(\nu_{[n]}))),$$

denote the multiplicity of $\mathbf{S}(\nu_{[n]})$ in the tensor product $\mathbf{S}(\lambda_{[n]})\otimes\mathbf{S}(\mu_{[n]})$. Murnaghan showed (see [Mur38, Mur55]) that if we allow the first parts of the partitions to increase in length then we obtain a limiting behaviour as follows. For $\lambda_{[N]},\mu_{[N]},\nu_{[N]} \in \Lambda_N$ and N sufficiently large we have that

$$g^{\nu_{[N+k]}}_{\lambda_{[N+k]},\mu_{[N+k]}} = \overline{g}^{\nu}_{\lambda,\mu}$$

for all $k \geq 1$; the integers $\overline{g}^{\nu}_{\lambda,\mu}$ are called the *reduced Kronecker coefficients*. Bounds for this stability have been given in [Bri93, Val99, Kly04, BOR11].

Remark 2.2.2 *The reduced Kronecker coefficients are also the structural constants for a linear basis for the polynomials in countably many variables known as the* character polynomials*, see [Mac95].*

3 The partition algebra

The partition algebra was originally defined by Martin in [Mar91]. All the results in this section are due to Martin and his collaborators, see [Mar96] and references therein.

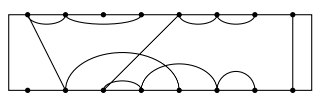
3.1 Definitions

For $r \in \mathbb{Z}_{>0}$, $\delta \in \mathbb{C}$, we let $P_r(\delta)$ denote the complex vector space with basis given by all set-partitions of $\{1, 2, \ldots, r, \bar{1}, \bar{2}, \ldots, \bar{r}\}$. A part of a set-partition is called a *block*. For example,

$$d = \{\{1, 2, 4, \bar{2}, \bar{5}\}, \{3\}, \{5, 6, 7, \bar{3}, \bar{4}, \bar{6}, \bar{7}\}, \{8, \bar{8}\}, \{\bar{1}\}\},$$

is a set-partition (for $r = 8$) with 5 blocks.

A set-partition can be represented *uniquely* by an (r, r)-*partition diagram* consisting of a frame with r distinguished points on the northern and southern boundaries, which we call vertices. We number the northern vertices from left to right by $1, 2, \ldots, r$ and the southern vertices similarly by $\bar{1}, \bar{2}, \ldots, \bar{r}$. Any block in a set-partition is of the form $A \cup B$ where $A = \{i_1 < i_2 < \ldots < i_p\}$ and $B = \{\bar{j}_1 < \bar{j}_2 < \ldots < \bar{j}_q\}$ (and A or B could be empty). We draw this block by putting an arc joining each pair (i_l, i_{l+1}) and $(\bar{j}_l, \bar{j}_{l+1})$ and if A and B are non-empty we draw a strand from i_1 to $\bar{j}_1$, that is we draw a single propagating line on the leftmost vertices of the block. Blocks containing a northern and a southern vertex will be called *propagating blocks*; all other blocks will be called *non-propagating blocks*. For d as in the example above, the partition diagram of d is given by:

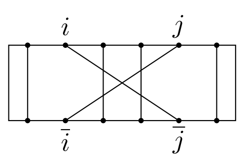
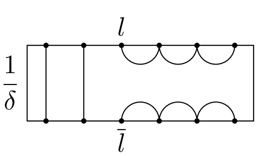


We can generalise this definition to (r, m)-partition diagrams as diagrams representing set-partitions of $\{1, \ldots, r, \bar{1}, \ldots, \bar{m}\}$ in the obvious way.

We define the product $x \cdot y$ of two diagrams x and y using the concatenation of x above y, where we identify the southern vertices of x with the northern vertices of y. If there are t connected components consisting only of middle vertices, then the product is set equal to δ^t times the diagram with the middle components removed. Extending this by linearity defines a multiplication on $P_r(\delta)$.

Assumption: *We assume throughout the paper that* $\delta \neq 0$.

The following elements of the partition algebra will be of importance.

$s_{i,j} =$  $e_l = \frac{1}{\delta}$ 

In particular, note that e_r corresponds to the set-partition $\{1, \bar{1}\}\{2, \bar{2}\} \cdots \{r-1, \overline{r-1}\}\{r\}\{\bar{r}\}$.

3.2 Filtration by propagating blocks and standard modules

Fix $\delta \in \mathbb{C}^{\times}$ and write $P_r = P_r(\delta)$. Note that the multiplication in P_r cannot increase the number of propagating blocks. More precisely, if x, respectively y, is a partition diagram with p_x, respectively

p_y, propagating blocks then $x \cdot y$ is equal to $\delta^t z$ for some $t \geq 0$ and some partition diagram z with $p_z \leq \min\{p_x, p_y\}$. This gives a filtration of the algebra P_r by the number of propagating blocks. This filtration can be realised using the idempotents e_l. We have

$$\mathbb{C} \cong P_r e_1 P_r \subset \ldots \subset P_r e_{r-1} P_r \subset P_r e_r P_r \subset P_r.$$

It is easy to see that

$$e_r P_r e_r \cong P_{r-1} \tag{3.2.1}$$

and that this generalises to $P_{r-l} \cong e_{r-l+1} P_r e_{r-l+1}$ for $1 \leq l \leq r$. We also have

$$P_r/(P_r e_r P_r) \cong \mathbb{C}\mathfrak{S}_r. \tag{3.2.2}$$

Using equation (3.2.2), we get that any $\mathbb{C}\mathfrak{S}_r$-module can be *inflated* to a P_r-module. We also get from equations (3.2.1) and (3.2.2), by induction, that the simple P_r-modules are indexed by the set $\Lambda_{\leq r} = \bigcup_{0 \leq i \leq r} \Lambda_i$.

For any $\nu \in \Lambda_{\leq r}$ with $\nu \vdash r - l$, we define a P_r-module, $\Delta_r(\nu)$, by

$$\Delta_r(\nu) = P_r e_{r-l+1} \otimes_{P_{r-l}} \mathbf{S}(\nu),$$

where the action of P_r is given by left multiplication. (Note that we have identified P_{r-l} with $e_{r-l+1} P_r e_{r-l+1}$ using the isomorphism given in equation (3.2.1).)

For $\delta \notin \{0, 1, \ldots, 2r-2\}$ the algebra $P_r(\delta)$ is semisimple and the set $\{\Delta_r(\nu) : \nu \in \Lambda_{\leq r}\}$ forms a complete set of non-isomorphic simple modules.

In general, the algebra $P_r(\delta)$ is quasi-hereditary with respect to the partial order on $\Lambda_{\leq r}$ given by $\lambda < \mu$ if $|\lambda| > |\mu|$ (see [Mar96]). The modules $\Delta_r(\nu)$ are the standard modules, each of which has a simple head $L_r(\nu)$, and the set $\{L_r(\nu) : \nu \in \Lambda_{\leq r}\}$ forms a complete set of non-isomorphic simple modules.

3.3 Non-semisimple representation theory of the partition algebra

We assume that $\delta = n \in \mathbb{Z}_{>0}$ (as otherwise the algebra is semisimple).

Definition 3.3.1 *Let λ and μ be partitions. We say that (μ, λ) is an n-pair, and write $\mu \hookrightarrow_n \lambda$, if $\mu \subset \lambda$ and the Young diagram of λ differs from the Young diagram of μ by a horizontal row of boxes of which the last (rightmost) one has content $n - |\mu|$.*

Example 3.3.2 *For example, $((2,1),(4,1))$ is a 6-pair. We have that $6 - |\mu| = 3$ and the Young diagrams (with contents) are as follows:*

$$\begin{array}{|c|c|}\hline 0 & 1 \\ \hline -1 & \\ \cline{1-1}\end{array} \subset \begin{array}{|c|c|c|c|}\hline 0 & 1 & 2 & 3 \\ \hline -1 & & & \\ \cline{1-1}\end{array}$$

note that they differ by $\begin{array}{|c|c|}\hline 2 & 3 \\ \hline\end{array}$.

Recall that the set of simple (or standard) modules for $P_r(n)$ are labelled by the set $\Lambda_{\leq r}$. This set splits into $P_r(n)$-blocks. The set of labels in each block forms a maximal chain of n-pairs

$$\lambda^{(0)} \hookrightarrow_n \lambda^{(1)} \hookrightarrow_n \lambda^{(2)} \hookrightarrow_n \ldots \hookrightarrow_n \lambda^{(t)}.$$

Moreover, for $1 \leq i \leq t$ we have that $\lambda^{(i)}/\lambda^{(i-1)}$ consists of a strip of boxes in the ith row. Now we have an exact sequence of $P_r(n)$-modules

$$0 \to \Delta_r(\lambda^{(t)}) \to \ldots \to \Delta_r(\lambda^{(2)}) \to \Delta_r(\lambda^{(1)}) \to \Delta_r(\lambda^{(0)}) \to L_r(\lambda^{(0)}) \to 0$$

with the image of each homomorphism being simple. Each standard module $\Delta_r(\lambda^{(i)})$ (for $0 \leq i \leq t-1$) has Loewy structure

$$\begin{matrix} L_r(\lambda^{(i)}) \\ L_r(\lambda^{(i+1)}) \end{matrix}$$

and so in the Grothendieck group we have

$$[L_r(\lambda^{(i)})] = \sum_{j=i}^{t} (-1)^{j-i} [\Delta_r(\lambda^{(j)})]. \tag{3.3.1}$$

Note that each block is totally ordered by the size of the partitions.

Proposition 3.3.3 *Let $\nu \in \Lambda_{\leq r}$ and assume that $\nu_{[n]}$ is a partition. Then we have that (i) ν is the minimal element in its $P_r(n)$-block, and (ii) ν is the unique element in its block if and only if $n+1-\nu_1 > r$.*

Proof: (i) Observe that for $\nu_{[n]}$ to be a partition we must have $n - |\nu| \geq \nu_1$. This implies that $\mathrm{ct}(\nu_1) = \nu_1 - 1 \leq n - |\nu| - 1$. So we have $\nu \hookrightarrow_n \mu$ for some partition μ with μ/ν being a single strip in the first row. Thus we have $\nu = \nu^{(0)}$ and $\mu = \nu^{(1)}$.
(ii) Now as $\nu^{(1)}/\nu$ is a single strip in the first row with last box having content $n - |\nu|$, we have that $|\nu^{(1)}/\nu| = n - |\nu| + 1 - \nu_1$ and thus $|\nu^{(1)}| = n + 1 - \nu_1$. Thus if $n + 1 - \nu_1 > r$ then $\nu^{(1)} \notin \Lambda_{\leq r}$ and we have that ν is the only partition in its $P_r(n)$-block. □

4 Schur–Weyl duality

Classical Schur–Weyl duality is the relationship between the general linear and symmetric groups over tensor space. To be more specific, let V_n be an n-dimensional complex vector space and let $V_n^{\otimes r}$ denote its rth tensor power.

We have that the symmetric group $\mathfrak{S}_r$ acts on the right by permuting the factors. The general linear group, GL_n, acts on the left by matrix multiplication on each factor. These two actions commute and moreover GL_n and $\mathfrak{S}_r$ are full mutual centralisers in $\mathrm{End}(V_n^{\otimes r})$.

The partition algebra, $P_r(n)$, plays the role of the symmetric group, $\mathfrak{S}_r$, when we restrict the action of GL_n to the subgroup of permutation matrices, $\mathfrak{S}_n$.

4.1 Schur-Weyl duality between $\mathfrak{S}_n$ and $P_r(n)$

Let V_n denote an n-dimensional complex space. Then $\mathfrak{S}_n$ acts on V_n via the permutation matrices.

$$\sigma \cdot v_i = v_{\sigma(i)} \qquad \text{for } \sigma \in \mathfrak{S}_n. \tag{4.1.1}$$

Notice that we are simply restricting the GL_n action in the classical Schur-Weyl duality to the permutation matrices. Thus, $\mathfrak{S}_n$ acts diagonally on the basis of simple tensors of $V_n^{\otimes r}$ as follows

$$\sigma \cdot (v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r}) = v_{\sigma(i_1)} \otimes v_{\sigma(i_2)} \otimes \cdots \otimes v_{\sigma(i_r)}.$$

For each (r,r)-partition diagram d and each integer sequence $i_1 \dots, i_r, i_{\bar{1}}, \dots, i_{\bar{r}}$ with $1 \leq i_j, i_{\bar{j}} \leq n$, define

$$\phi_{r,n}(d)^{i_1,\dots,i_r}_{i_{\bar{1}},\dots,i_{\bar{r}}} = \begin{cases} 1 & \text{if } i_t = i_s \text{ whenever vertices } t \text{ and } s \text{ are connected in } d \\ 0 & \text{otherwise.} \end{cases} \tag{4.1.2}$$

A partition diagram $d \in P_r(n)$ acts on the basis of simple tensors of $V_n^{\otimes r}$ as follows

$$\Phi_{r,n}(d)(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r}) = \sum_{i_{\bar{1}},\dots i_{\bar{r}}} \phi_{r,n}(d)^{i_1,\dots,i_r}_{i_{\bar{1}},\dots,i_{\bar{r}}} v_{i_{\bar{1}}} \otimes v_{i_{\bar{2}}} \otimes \cdots \otimes v_{i_{\bar{r}}}.$$

Theorem 4.1.1 (Jones [Jon94]) *$\mathfrak{S}_n$ and $P_r(n)$ generate the full centralisers of each other in $\mathrm{End}(V_n^{\otimes r})$.*

(a) $P_r(n)$ generates $\mathrm{End}_{\mathfrak{S}_n}(V_n^{\otimes r})$, and when $n \geq 2r$, $P_r(n) \cong \mathrm{End}_{\mathfrak{S}_n}(V_n^{\otimes r})$.

(b) $\mathfrak{S}_n$ generates $\mathrm{End}_{\mathfrak{S}_n}(V_n^{\otimes r})$.

We will denote $E_r(n) = \mathrm{End}_{\mathfrak{S}_n}(V_n^{\otimes r})$.

Theorem 4.1.2 ([Mar96] see also [HR05]) *We have a decomposition of $V_n^{\otimes r}$ as a $(\mathfrak{S}_n, P_r(n))$-bimodule*

$$V_n^{\otimes r} = \bigoplus \mathbf{S}(\lambda_{[n]}) \otimes L_r(\lambda)$$

where the sum is over all partitions $\lambda_{[n]}$ of n such that $|\lambda| \leq r$.

Using [GW98, Theorem 9.2.2] we have, for $\lambda_{[n]}, \mu_{[n]}, \nu_{[n]} \vdash n$ with $\lambda \vdash r$ and $\mu \vdash s$,

$$\mathrm{Hom}_{\mathfrak{S}_n}(\mathbf{S}(\nu_{[n]}), \mathbf{S}(\lambda_{[n]}) \otimes \mathbf{S}(\mu_{[n]})) \tag{4.1.3}$$

$$\cong \begin{cases} \mathrm{Hom}_{E_r(n)\otimes E_s(n)}(L_r(\lambda) \boxtimes L_s(\mu), L_{r+s}(\nu){\downarrow}_{E_r(n)\otimes E_s(n)}) & \text{if } \nu \in \Lambda_{\leq r+s} \\ 0 & \text{otherwise.} \end{cases}$$

4.2 Kronecker product via the partition algebra

Going back to the formula in (4.1.3) we need to consider $L_{r+s}(\nu){\downarrow}_{E_r(n)\otimes E_s(n)}$. Now $L_{r+s}(\nu)$ is a simple $P_{r+s}(n)$-module annihilated by $\ker \Phi_{r+s,n}$ and hence also by $\ker \Phi_{r,n} \otimes \ker \Phi_{s,n}$. Thus $L_{r+s}(\nu){\downarrow}_{P_r(s)\otimes P_s(n)}$ is semisimple and has the same simple factors as $L_{r+s}(\nu){\downarrow}_{E_r(s)\otimes E_s(n)}$.

Now combining (4.1.3) with (3.3.1) we have the following result.

Theorem 4.2.1 *Let $\lambda_{[n]}, \mu_{[n]}, \nu_{[n]} \vdash n$ with $\lambda \vdash r$ and $\mu \vdash s$. Then we have*

$$g^{\nu_{[n]}}_{\lambda_{[n]},\mu_{[n]}} = \begin{cases} \sum_{i=0}^{t} (-1)^i [\Delta_{r+s}(\nu^{(i)}){\downarrow}_{P_r(n)\otimes P_s(n)} \colon L_r(\lambda) \boxtimes L_s(\mu)] & \textit{if } \nu \in \Lambda_{\leq (r+s)} \\ 0 & \textit{otherwise} \end{cases}$$

where $\nu = \nu^{(0)} \hookrightarrow_n \nu^{(1)} \hookrightarrow_n \dots \hookrightarrow \nu^{(t)}$ is the $P_{r+s}(n)$-block of ν.

For sufficiently large values of n the partition algebra is semisimple. Therefore Theorem 4.2.1 reproves the limiting behaviour of tensor products observed by Murnaghan. It also offers the following concrete representation theoretic interpretation of the $\overline{g}^{\nu}_{\lambda,\mu}$.

Corollary 4.2.2 *Let $\lambda \vdash r$ and $\mu \vdash s$ and suppose $|\nu| \leq r+s$. Then we have*

$$\overline{g}^{\nu}_{\lambda,\mu} = [\Delta_{r+s}(\nu){\downarrow}_{P_r(n)\otimes P_s(n)} \colon L_r(\lambda) \boxtimes L_s(\mu)].$$

Using the semisimplicity criterion for the partition algebra given in Section 3.2 and Proposition 3.3.3 we immediately obtain

Corollary 4.2.3 *We have that $\overline{g}^{\nu}_{\lambda,\mu} = g^{\nu_{[n]}}_{\lambda_{[n]},\mu_{[n]}}$ if either $|\lambda|+|\mu| \leq \frac{n+1}{2}$, or $|\lambda|+|\mu| < n+1-\nu_1$.*

The second part of Corollary 4.2.3 is a new proof of Brion's bound [Bri93] for the stability of the Kronecker coefficients using the partition algebra.

By constructing an explicit filtration of the restriction of a standard module to a Young subalgebra of the partition algebra and identifying the corresponding subquotients we obtain

Theorem 4.2.4 *Write $m = r+s$ and let $\nu \vdash m-l$, $\lambda \vdash r-l_r$ and $\mu \vdash s-l_s$ for some non-negative integers l, l_r, l_s. Then $\Delta_m(\nu){\downarrow}_{P_r\otimes P_s}$ has a filtration by standard modules with multiplicities given by*

$$[\Delta_m(\nu){\downarrow}_{P_r\otimes P_s} \colon \Delta_r(\lambda) \boxtimes \Delta_s(\mu)] = \sum_{\substack{l_1,l_2 \\ l_1+2l_2=l-l_r-l_s}} \sum_{\substack{\alpha\vdash r-l_r-l_1-l_2 \\ \beta\vdash s-l_s-l_1-l_2 \\ \pi,\rho,\sigma\vdash l_1 \\ \gamma\vdash l_2}} c^{\nu}_{\alpha,\beta,\pi} c^{\lambda}_{\alpha,\gamma,\rho} c^{\mu}_{\beta,\gamma,\sigma} g^{\pi}_{\rho,\sigma}.$$

By Corollary 4.2.2, we recover the formula for the reduced coefficients given in terms of Littlewood–Richardson coefficients and Kronecker coefficients in [BOR11, Lemma 2.1]

Corollary 4.2.5 *Let λ, μ, ν be any partitions with $|\lambda| = r$, $|\mu| = s$ and $|\nu| = r+s-l$. Then we have*

$$\overline{g}^{\nu}_{\lambda,\mu} = \sum_{\substack{l_1,l_2 \\ l=l_1+2l_2}} \sum_{\substack{\alpha\vdash r-l_1-l_2 \\ \beta\vdash s-l_1-l_2}} \sum_{\substack{\pi,\rho,\sigma\vdash l_1 \\ \gamma\vdash l_2}} c^{\nu}_{\alpha,\beta,\pi} c^{\lambda}_{\alpha,\gamma,\rho} c^{\mu}_{\beta,\gamma,\sigma} g^{\pi}_{\rho,\sigma}$$

Remark 4.2.6 *We also recover the Murnaghan–Littlewood Theorem, namely, for partitions λ, μ, ν with $|\lambda|+|\mu| = |\nu|$ we have that $\overline{g}^{\nu}_{\lambda,\mu} = c^{\nu}_{\lambda,\mu}$.*

4.3 Passing between the Kronecker and reduced Kronecker coefficients

In [BOR11] a formula is given for passing between the Kronecker and reduced Kronecker coefficients. We shall now interpret this formula in the Grothendieck group of the partition algebra by showing that it coincides with the formula in Theorem 4.2.1.

Let $\nu_{[n]}$ be a partition of n. We make the convention that $\nu_0 = n - |\nu|$ is the 0th row of $\nu_{[n]}$. For $i \in \mathbb{Z}_{\geq 0}$ define $\nu^{\dagger i}_{[n]}$ to be the partition obtained from $\nu_{[n]}$ by adding 1 to its first $i-1$ rows and erasing its ith row. In particular we have $\nu^{\dagger 0}_{[n]} = \nu$.

Theorem 4.3.1 (Theorem 1.1 of [BOR11]) *Let $\lambda_{[n]}, \mu_{[n]}, \nu_{[n]} \vdash n$. Then*

$$g^{\nu_{[n]}}_{\lambda_{[n]},\mu_{[n]}} = \sum_{i=0}^{l} (-1)^i \bar{g}^{\nu^{\dagger i}_{[n]}}_{\lambda,\mu}$$

where $l = \ell(\lambda_{[n]})\ell(\mu_{[n]}) - 1$.

Relating this to the partition algebra, we have the following.

Proposition 4.3.2 *Let $\nu_{[n]} \vdash n$ and let $\nu = \nu^{(0)} \hookrightarrow_n \nu^{(1)} \hookrightarrow_n \ldots$ be a chain of n-pairs. Then for all $i \geq 0$ we have*

$$\nu^{\dagger i}_{[n]} = \nu^{(i)}$$

Proof: The $i = 0$ case is clear from the definitions. We proceed by induction. Assume that

$$\nu^{\dagger i}_{[n]} = \nu^{(i)}.$$

Then $(\nu^{(i)})_1 = n - |\nu| + 1$, $(\nu^{(i)})_j = \nu_{j-1} + 1$ for $j \leq i$, and $(\nu^{(i)})_j = \nu_j$ for $j > i$. Therefore

$$|\nu^{(i)}| = n - |\nu| + 1 + \sum_{j \neq i} \nu_j + i - 1 = n - \nu_i + i$$

We have that $\nu^{(i+1)}/\nu^{(i)}$ is a skew partition consisting of a strip in the $(i+1)$th row. By definition of an n-pair the content, $\mathrm{ct}(\nu^{(i+1)}_{i+1})$, of the last node is $n - |\nu^{(i)}|$. Therefore

$$\mathrm{ct}(\nu^{(i+1)}_{i+1}) := \nu^{(i+1)}_{i+1} - (i+1) = n - (n - \nu_i + i) = \nu_i - i$$

and $\nu^{(i+1)}_{i+1} = \nu_i + 1$, therefore $\nu^{\dagger(i+1)}_{[n]} = \nu^{(i+1)}$.

□

In Theorem 4.2.1, t is chosen so that $|\nu^{(t)}| \leq |\lambda| + |\mu|$ and $|\nu^{(t+1)}| > |\lambda| + |\mu|$. So Theorem 4.2.1 and 4.3.1 seem to give a different number of terms in the sum. For example consider

$$g^{(2)}_{(1^2),(1^2)} = 1 \quad g^{(1^2)}_{(1^2),(1^2)} = 0$$

these are given as a sum of one, respectively two terms in Theorem 4.2.1, both cases have four terms in Theorem 4.3.1. Now consider

$$\lambda_{[n]} = \mu_{[n]} = \nu_{[n]} = (10, 10, 10)$$

then $\ell(\lambda_{[n]})\ell(\mu_{[n]}) = 9$. We have $\nu^{\dagger 8}_{[n]} = (11^3, 1^5)$ with $|\nu^{\dagger 8}_{[n]}| = 38$. But $r + s = 40$, so we have two more terms in Theorem 4.2.1, corresponding to $\nu^{(9)} = (11^3, 1^6)$ and $\nu^{(10)} = (11^3, 1^7)$. However, we can show that in fact the two theorems give the same sum.

First assume that $\ell(\lambda_{[n]})\ell(\mu_{[n]}) - 1 > t$, then for all $i > t$ we have

$$\bar{g}^{\nu^{\dagger i}_{[n]}}_{\lambda,\mu} = 0$$

as $|\nu^{\dagger i}_{[n]}| > |\lambda| + |\mu|$. And so the two sums coincide.

Now assume that $\ell(\lambda_{[n]})\ell(\mu_{[n]}) - 1 < t$. Then for all $i > \ell(\lambda_{[n]})\ell(\mu_{[n]}) - 1$, we have

$$i \geq \ell(\lambda_{[n]})\ell(\mu_{[n]}) \geq |\lambda_{[n]} \cap (\mu_{[n]})'|.$$

where $(\mu_{[n]})'$ denotes the conjugate partition of $\mu_{[n]}$. (To see this observe that the Young diagram of $\lambda_{[n]} \cap (\mu_{[n]})'$ fits in a rectangle of size $\ell(\lambda_{[n]}) \times \ell(\mu_{[n]})$). Now we have

$$\ell(\nu^{(i)}) \geq i \geq |\lambda_{[n]} \cap (\mu_{[n]})'|.$$

But this implies that $\bar{g}^{\nu^{(i)}}_{\lambda,\mu} = 0$ by [Dvi93].

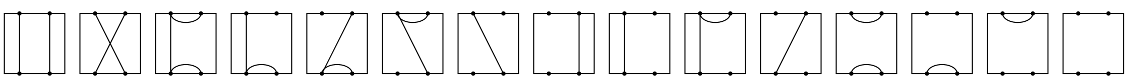
5 Example

In this section, we shall compute the tensor square of the Specht module, $\mathbf{S}(n-1,1)$ for $n \geq 2$, labelled by the first non-trivial hook, via the partition algebra. We have that

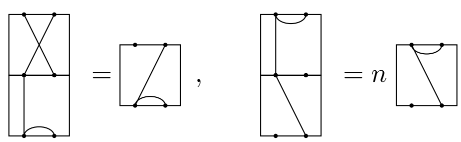
$$\mathrm{Hom}_{\mathfrak{S}_n}(\mathbf{S}(\nu_{[n]}), \mathbf{S}(n-1,1) \otimes \mathbf{S}(n-1,1)) \cong \mathrm{Hom}_{P_1(n)\otimes P_1(n)}(L_1(1) \otimes L_1(1), L_2(\nu){\downarrow}^{P_2(n)}_{P_1(n)\otimes P_1(n)})$$

if $\nu \in \Lambda_{\leq 2}$ and zero otherwise. Therefore, it is enough to consider the restriction of simple modules from $P_2(n)$ to the Young subalgebra $P_1(n) \otimes P_1(n)$.

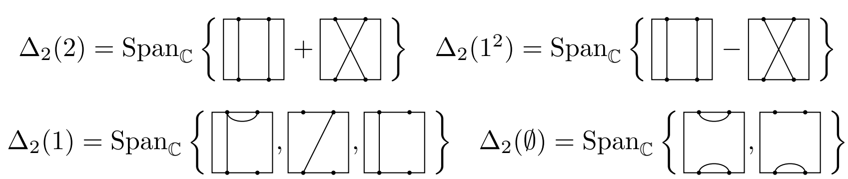
The partition algebra $P_2(n)$ is a 15-dimensional algebra with basis:



and multiplication defined by concatenation. For example:



There are four standard modules corresponding to the partitions of degree less than or equal to 2; these are obtained by inflating the Specht modules from the symmetric groups of degree $0, 1, 2$. These modules have bases:



The action of $P_2(n)$ is given by concatenation. If the resulting diagram has fewer propagating lines than the original, we set the product equal to zero. The algebra $P_1(n) \otimes P_1(n)$ is the 4-dimensional subalgebra spanned by the diagrams with no lines crossing an imagined vertical wall down the centre of the diagram.

The restriction of the standard modules to this subalgebra is as follows:

$$\Delta_2(2)\downarrow_{P_1\otimes P_1}\cong \Delta_1(1)\boxtimes\Delta_1(1), \quad \Delta_2(1^2)\downarrow_{P_1\otimes P_1}\cong \Delta_1(1)\boxtimes\Delta_1(1),$$

$$\Delta_2(1)\downarrow_{P_1\otimes P_1}\cong \Delta_1(1)\boxtimes\Delta_1(1)\oplus\Delta_1(\emptyset)\boxtimes\Delta_1(1)\oplus\Delta_1(1)\boxtimes\Delta_1(\emptyset),$$

$$\Delta_2(\emptyset)\downarrow_{P_1\otimes P_1}\cong \Delta_1(1)\boxtimes\Delta_1(1)\oplus\Delta_1(\emptyset)\boxtimes\Delta_1(\emptyset).$$

In particular, note that $\bar{g}^{\nu}_{(1),(1)} = [\Delta_2(\nu)\downarrow_{P_1\otimes P_1}\colon \Delta_1(1)\boxtimes\Delta_1(1)] = 1$ for $\nu = \emptyset, 1, 1^2, 2$.

The partition algebra $P_2(n)$ is semisimple for $n > 2$. For $\nu = \emptyset, (1), (1^2)$ or (2) we have that $\nu_{[n]} = (n), (n-1,1), (n-2,1^2)$, or $(n-2,2)$ and $\nu_{[n]}$ is a partition for $n \geq 0, 2, 3, 4$ respectively. Therefore the Kronecker coefficients

$$g^{\nu_{[n]}}_{(n-1,1),(n-1,1)}$$

stabilise for $n \geq 4$ and are non-zero for $n \geq 4$ if and only $\nu_{[n]}$ is one of the partitions above.

Now consider the case $n = 2$. Neither $\nu = (1^2)$, nor (2) correspond to partitions of 2, we therefore consider $\nu = \emptyset$ and (1). We have that $(1) \subset (2)$ is the unique 2-pair of partitions of degree less than or equal to 2 (see Section 3.3). Therefore the only standard $P_2(2)$-module which is not simple is $\Delta_2(1)$ and we have an exact sequence

$$0 \to L_2(2) \to \Delta_2(1) \to L_2(1) \to 0.$$

Thus in the Grothendieck group we have that $[L_2(1)] = [\Delta_2(1)] - [\Delta_2(2)]$. Therefore we have that $[L_2(1)\downarrow_{P_1(2)\otimes P_1(2)}\colon L_1(1)\boxtimes L_1(1)] = 0$. We conclude that $g^{(1^2)}_{(1^2),(1^2)} = 0$ and $g^{(2)}_{(1^2),(1^2)} = 1$ as expected.

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Transition matrices for symmetric and quasisymmetric Hall-Littlewood polynomials (Extended Abstract)

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Abstract We introduce explicit combinatorial interpretations for the coefficients in some of the transition matrices relating to skew Hall-Littlewood polynomials $P_{\lambda/\mu}(x;t)$ and Hivert's quasisymmetric Hall-Littlewood polynomials $G_\gamma(x;t)$. More specifically, we provide the following:

1. G_γ-expansions of the P_λ, the monomial quasisymmetric functions, and Gessel's fundamental quasisymmetric functions F_α, and
2. an expansion of the $P_{\lambda/\mu}$ in terms of the F_α.

The F_α expansion of the $P_{\lambda/\mu}$ is facilitated by introducing the set of *starred tableaux*. In the full version of the article we also provide G_γ-expansions of the quasisymmetric Schur functions and the peak quasisymmetric functions of Stembridge.

Résumé. Nous introduisons des interprétations combinatoires explicites pour les coefficients dans l'expansion de quelques matrices de transition en relation avec les polynômes skew de Hall-Littlewood $P_{\lambda/\mu}(x;t)$ et les fonctions quasisimmetriques de Hall-Littlewood $G_\gamma(x;t)$. Plus spécifiquement, nous donnons les suivants:

1. G_γ-expansions pour le P_λ, les fonctions monomiales quasisimmetriques et les fonctions fondamentales quasisymmetriques de Gessel's F_α et
2. une expansion des $P_{\lambda/\mu}$ en termes des F_α.

L'expansion des $P_{\lambda/\mu}$ en termes des F_α est facilitée grâce a l'introduction de l'ensemble de *tableaux étoilés*. Dans la version complete de cette article, nous donnons aussi G_γ-expansions pour les fonctions quasisymmetriques de Schur et les fonctions quasisymmetriques de pic de Stembridge.

Keywords: symmetric functions, quasisymmetric functions, Hall-Littlewood polynomials, standardization, Young tableaux, noncommutative symmetric functions

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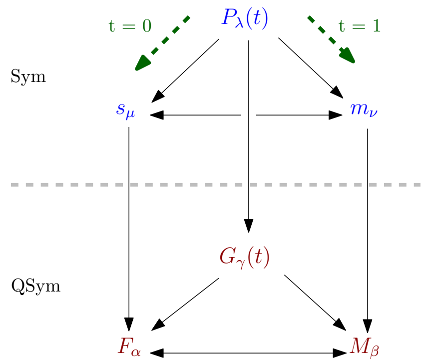

Fig. 1: Prism of bases and transitions.

1 Introduction

The ring of symmetric functions Sym and the ring of quasisymmetric functions QSym both play important roles in algebra and combinatorics. Much of the combinatorial richness arising from these rings stems from their various distinguished bases and the relationships between these bases. The goal of this paper is to present explicit, combinatorial descriptions of several such transition matrices relating to the Hall-Littlewood polynomials. Figure 1 illustrates the bases discussed.

In the top triangle in Figure 1 are included two classical bases for the ring of symmetric functions: the *Schur functions* s_μ and the *monomial symmetric functions* m_ν. The s_μ and m_ν are closely related to a third, one-parameter family of symmetric functions $P_\lambda(x;t)$, known as *Hall-Littlewood polynomials*. More specifically, P_λ equals s_λ at $t = 0$, and it equals m_λ at $t = 1$. The P_λ arose out of a problem studied by P. Hall. Hall had used his eponymous algebra (isomorphic to the algebra of symmetric functions) to encode the structure of finite abelian p-groups. However, at the time there was no known explicit basis of symmetric functions with the same structure constants as that of the natural basis for Hall's algebra. D. E. Littlewood [11] solved this problem in 1961 with his introduction of the $P_\lambda(x;t)$.

The bottom triangle of Figure 1 consists of quasisymmetric analogues of the above bases. In the context of quasisymmetric functions, the *monomial quasisymmetric functions*, M_β, are a very natural analogue of the m_ν. Moreover, there do exist quasisymmetric Schur functions [6]. However, for reasons described in the next paragraph, we anchor the lower-left portion of the bottom triangle in Figure 1 by Gessel's *fundamental quasisymmetric functions*, denoted here by F_α. By defining an action of the Hecke algebra on polynomials which leaves the quasisymmetric functions invariant, Hivert [7] has constructed the *quasisymmetric Hall-Littlewood polynomials* $G_\gamma(x;t)$. (See also work of Lascoux, Novelli, and Thibon [8] for constructions of quasisymmetric and noncommutative symmetric functions with extra parameters.) Similarly to what happens in the top triangle, specialization of the G_γ at $t = 0$ (which corresponds to the southwest-pointing arrow in Figure 1) yields F_γ, while specialization at $t = 1$ yields M_γ.

We now motivate our choice of the F_α as the desired quasisymmetric analogue of the Schur functions. The Schur functions are the prototypical example of a symmetric function with combinatorial expansions in terms of both a collection of *semistandard objects* (i.e., semistandard Young tableaux) and of *standard objects* (i.e., standard Young tableaux). The first case is that of the classical expansion in terms of monomials weighted by the Kostka numbers. The second expansion (due to Gessel [3]) expresses the Schur functions in terms of fundamental quasisymmetric functions F_α. This expansion, which follows from the technique of *standardization*, is indicated by the vertical line connecting s_μ and F_α in Figure 1. Such standardizations have been used recently to give F-expansions of various symmetric functions including plethysms of Schur functions [14], the modified Macdonald polynomials [4, 5], the Lascoux-Leclerc-Thibon (LLT) polynomials [10], and (conjecturally) the image of a Schur function under the Bergeron-Garsia nabla operator [13].

Given Hivert's construction, the following question arises. Is there an expansion of the P_λ in terms of the G_γ which would interpolate between the F-expansion of the s_λ at $t = 0$ and the M-expansion of the m_μ at $t = 1$? The main purpose of this paper is to provide such an expansion, and also to provide other change-of-basis matrices between different bases of the Hall algebra and the algebra of quasisymmetric functions, as explained below. In terms of Figure 1, we provide the middle vertical edge as well as the two downward directed edges in the bottom face (namely, from $G_\gamma(t)$ to both F_α and M_β).

G-expansion of the P Basis. In Theorem 5.6 we give an explicit combinatorial expansion of the Hall-Littlewood polynomials $P_\lambda(x;t)$ in terms of the Hivert quasisymmetric Hall-Littlewood polynomials $G_\gamma(x;t)$. This provides the desired t-interpolation between Gessel's F-expansion of Schur polynomials (i.e., $t = 0$) and the obvious expansion of m_λ's into M_α's (i.e., $t = 1$).

F-expansion of the P Basis. One of the main tools for our calculations is the definition of a new class of tableaux, called *starred tableaux*. With these, we give in Theorem 4.1 a combinatorial expansion of the skew Hall-Littlewood polynomials $P_{\lambda/\mu}(x;t)$ in terms of the fundamental quasisymmetric functions F_α. A minor variation to our method gives a corresponding expansion for the dual Hall-Littlewood polynomials $Q_{\lambda/\mu}$.

G-expansion of the F and M Bases. In Theorems 5.1 and 5.3 we give explicit combinatorial expansions for the F_α and the M_β in terms of the G_γ. These are inverse matrices to those found in [7].

The structure of this extended abstract is as follows. The bases discussed are defined in §2 while the known transition matrices are summarized in §3. The expansions of the Hall-Littlewood polynomials in terms of the F_α and $G_\gamma(x;t)$ are presented in §4 and §5, respectively.

This text is an extended abstract of the preprint [12], where complete proofs can be found. Furthermore, in [12] we give explicit combinatorial expansions for the peak quasisymmetric functions K_α and the quasisymmetric Schur functions $\mathcal{S}_\beta$ in terms of the G_γ.

2 Review of Symmetric and Quasisymmetric Bases

This section reviews the definitions of the symmetric and quasisymmetric functions appearing in Figure 1. Logically, the precise definitions of the various bases are not needed in this paper, as the expansions found in §4 and §5 are derived from the known transition matrices of §3. However, the material of this section is included for completeness.

2.1 Compositions and Partitions

Given $n \in \mathbb{N}$, a *composition of* n is a sequence $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ of positive integers (called *parts*) with $\alpha_1 + \cdots + \alpha_k = n$. Define the *length* $\ell(\alpha)$ to be the number of parts of α, and the *size* $|\alpha|$ to be the sum of its parts. For example, the composition $\alpha = (2, 4, 1)$ has $\ell(\alpha) = 3$ and $|\alpha| = 7$. We may abbreviate the notation, writing α as 241, when no confusion can arise. Let Comp_n be the set of compositions of n, and let Comp be the set of all compositions. A composition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \in \mathrm{Comp}_n$ is called a *partition of* n iff $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$. We write Par_n for the set of partitions of n and Par for the set of all partitions.

For $n \in \mathbb{N}^+$, there are 2^{n-1} compositions of n and 2^{n-1} subsets of $[n-1] = \{1, 2, \ldots, n-1\}$. One can define natural bijections between these sets of objects as follows. Given $\alpha \in \mathrm{Comp}_n$ as above, let

$$\mathrm{sub}(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \ldots, \alpha_1 + \cdots + \alpha_{k-1}\} \subseteq [n-1].$$

The inverse bijection sends any subset $T = \{t_1 < t_2 < \cdots < t_m\} \subseteq [n-1]$ to

$$\mathrm{comp}(T) = (t_1, t_2 - t_1, t_3 - t_2, \ldots, t_m - t_{m-1}, n - t_m) \in \mathrm{Comp}_n .$$

Given $\alpha, \beta \in \mathrm{Comp}_n$, we say β is *finer* than α, denoted $\beta \succeq \alpha$, iff $\mathrm{sub}(\alpha) \subseteq \mathrm{sub}(\beta)$. Informally, β is finer than α if we can chop up some of the parts of α into smaller pieces (without reordering anything) and obtain β. For example, $(1,1,1,1) \succeq (1,2,1) \succeq (3,1) \succeq (4)$.

2.2 Symmetric Polynomials

Let K be a field of characteristic zero, and let $\mathfrak{S}_N$ denote the symmetric group on N letters. A polynomial $f \in K[x_1, \ldots, x_N]$ is called *symmetric* iff

$$f(x_{w(1)}, x_{w(2)}, \ldots, x_{w(N)}) = f(x_1, x_2, \ldots, x_N) \text{ for all } w \in \mathfrak{S}_N.$$

Write Sym_N for the ring of symmetric polynomials in N variables. For each $n \geq 0$, let Sym_N^n be the subspace of Sym_N consisting of zero and the homogeneous polynomials of degree n. For $N \geq n$, bases of the vector space Sym_N^n are naturally indexed by partitions of n.

Given $\lambda \in \mathrm{Par}_n$ of length $k \leq N$, the *monomial symmetric polynomial* $m_\lambda(x_1, \ldots, x_N)$ is the sum of all distinct monomials that can be obtained by permuting subscripts in $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k}$. For $N \geq n$, $\{m_\lambda(x_1, \ldots, x_N) : \lambda \in \mathrm{Par}_n\}$ is readily seen to be a basis of Sym_N^n.

Now suppose $N \geq n$ and $\nu \in \mathrm{Par}_n$ is a partition with *distinct* parts. If necessary, we append parts of size zero to the end of ν to make ν have length N. The *monomial antisymmetric polynomial indexed by* ν *in* N *variables* is

$$a_\nu(x_1, \ldots, x_N) = \sum_{w \in \mathfrak{S}_N} \prod_{i=1}^{N} \mathrm{sgn}(w) x_{w(i)}^{\nu_i} = \det ||x_i^{\nu_j}||_{1 \leq i,j \leq N}.$$

Letting $\delta_N = (N-1, N-2, \ldots, 2, 1, 0)$, $a_{\delta_N}(x_1, \ldots, x_N) = \prod_{1 \leq i < j \leq N}(x_i - x_j)$ is the Vandermonde determinant. Given $\lambda \in \mathrm{Par}_n$, the *Schur symmetric polynomial indexed by* λ *in* N *variables* is

$$s_\lambda(x_1, \ldots, x_N) = \frac{a_{\lambda+\delta_N}(x_1, \ldots, x_N)}{a_{\delta_N}(x_1, \ldots, x_N)}.$$

It can be shown that this rational function is both a polynomial and symmetric. Moreover $\{s_\lambda : \lambda \in \mathrm{Par}_n\}$ is a basis of Sym_N^n [15, §I.3, p. 40].

For the rest of the paper, let t be an indeterminate, and let K be any field containing $\mathbb{Q}(t)$ as a subfield. Following [15, §III.1, pp. 204–7], we define the *Hall-Littlewood symmetric polynomials* as follows. Fix $\lambda \in \mathrm{Par}_n$ and $N \geq n$. Extend λ to have length N by appending parts of size zero if needed. Define

$$R_\lambda(x_1, \ldots, x_N; t) = \frac{\sum_{w \in \mathfrak{S}_N} \mathrm{sgn}(w) x_{w(1)}^{\lambda_1} \cdots x_{w(N)}^{\lambda_N} \prod_{1 \leq i < j \leq N}(x_{w(i)} - t x_{w(j)})}{\prod_{1 \leq i < j \leq N}(x_i - x_j)}.$$

Define $[m]_t = 1+t+t^2+\cdots+t^{m-1}$, $[0]_t = 0$, $[m]!_t = \prod_{i=1}^{m}[i]_t$, and $[0]!_t = 1$. Given that λ has m_0 parts equal to 0, m_1 parts equal to 1, and so on, it can be shown that R_λ is divisible by $[m_0]!_t[m_1]!_t\cdots[m_N]!_t$. We then define the Hall-Littlewood polynomial

$$P_\lambda(x_1,\ldots,x_N;t) = \frac{R_\lambda(x_1,\ldots,x_N;t)}{[m_0]!_t[m_1]!_t\cdots[m_N]!_t}.$$

It can be shown [15, §III.2, p. 209] that for $N \geq n$, the set $\{P_\lambda(x_1,\ldots,x_N;t) : \lambda \in \mathrm{Par}_n\}$ is a basis for Sym_N^n. Moreover, setting $t = 0$ in P_λ gives s_λ, whereas setting $t = 1$ in P_λ gives m_λ. Thus, the Hall-Littlewood basis "interpolates" between the Schur basis and the monomial basis. One can define Schur polynomials and Hall-Littlewood polynomials more concretely by giving combinatorial descriptions of their expansions in terms of monomial symmetric polynomials. See §3.1 below.

2.3 Quasisymmetric Polynomials

A polynomial $f \in K[x_1,\ldots,x_N]$ is called *quasisymmetric* iff for every composition $\alpha = (\alpha_1,\ldots,\alpha_k)$ with at most N parts and every $1 \leq i_1 < i_2 < \cdots < i_k \leq N$, the monomials $x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_k^{\alpha_k}$ and $x_{i_1}^{\alpha_1}x_{i_2}^{\alpha_2}\cdots x_{i_k}^{\alpha_k}$ have the same coefficient in f. Write QSym_N for the ring of quasisymmetric polynomials in N variables. For each $n \geq 0$, let QSym_N^n be the subspace of QSym_N consisting of zero and the homogeneous polynomials of degree n. For $N \geq n$, linear bases of QSym_N^n are naturally indexed by compositions of n. Symmetric polynomials are quasisymmetric, so Sym_N^n is a subspace of QSym_N^n.

For $\alpha \in \mathrm{Comp}_n$ of length $k \leq N$, the *monomial quasisymmetric polynomial* $M_\alpha(x_1,\ldots,x_N)$ is the sum of all monomials $x_{i_1}^{\alpha_1}x_{i_2}^{\alpha_2}\cdots x_{i_k}^{\alpha_k}$ for which $1 \leq i_1 < i_2 < \cdots < i_k \leq N$. For $N \geq n$, $\{M_\alpha(x_1,\ldots,x_N) : \alpha \in \mathrm{Comp}_n\}$ is readily seen to be a basis of QSym_N^n.

For $\alpha \in \mathrm{Comp}_n$ of length at most N, define Gessel's *fundamental quasisymmetric polynomial* [3] by

$$F_\alpha(x_1,\ldots,x_N) = \sum x_{w_1}x_{w_2}\cdots x_{w_n},$$

where we sum over all subscript sequences $w = w_1w_2\cdots w_n$ such that $1 \leq w_1 \leq w_2 \leq \cdots \leq w_n \leq N$ and for all $j \in \mathrm{sub}(\alpha)$, $w_j < w_{j+1}$. In other words, strict increases in the subscripts are required in the "breaks" between parts of the composition α. A routine inclusion-exclusion argument shows that for $N \geq n$, $\{F_\alpha(x_1,\ldots,x_N) : \alpha \in \mathrm{Comp}_n\}$ is a basis of QSym_N^n. Note that some authors index fundamental quasisymmetric polynomials by pairs n, T where $T \subseteq [n-1]$. Additionally, various letters (F, L, Q, etc.) have been used to denote these polynomials.

As in the symmetric case, we would like to have quasisymmetric Hall-Littlewood polynomials (depending on a parameter t) that interpolate between F_α (when $t = 0$) and M_α (when $t = 1$). We sketch the definition of one such family of polynomials, introduced and studied by Hivert [7]. Quasisymmetric functions arise as the invariants of a certain action of $\mathfrak{S}_n$ on polynomials. From this action, one can define divided difference operators in a degenerate Hecke algebra $H_n(0)$ which can then be lifted to $H_n(q)$. Hivert's quasisymmetric Hall-Littlewood polynomials thereby arise from a corresponding t-analogue $\boxdot_\omega$ of the Weyl symmetrizer. For a composition α of length $k \leq N$, define

$$G_\alpha(x_1,\ldots,x_N;t) = \frac{1}{[k]!_t[N-k]!_t}\boxdot_\omega(x_1^{\alpha_1}\cdots x_k^{\alpha_k}).$$

As in the case of symmetric Hall-Littlewood polynomials, there is a more concrete combinatorial definition of G_α giving its expansion into monomials. We discuss this definition in §3.2.

3 Known Transition Matrices

In the theory of symmetric and quasisymmetric polynomials, much combinatorial information is encoded in the transition matrices between various bases. Given two bases $B = \{B_\lambda : \lambda \in \mathrm{Par}_n\}$ and $C = \{C_\lambda : \lambda \in \mathrm{Par}_n\}$ of Sym_N^n, the *transition matrix* $\mathcal{M}(B,C)$ is the unique matrix (with entries in K and rows and columns indexed by partitions of n) such that

$$B_\lambda = \sum_{\mu \in \mathrm{Par}_n} \mathcal{M}(B,C)_{\lambda,\mu} C_\mu.$$

Given a third basis D, it follows readily that $\mathcal{M}(B,D) = \mathcal{M}(B,C)\,\mathcal{M}(C,D)$ and $\mathcal{M}(C,B) = \mathcal{M}(B,C)^{-1}$. We define $\mathcal{M}(B,C)$ similarly if B and C are bases of QSym_N^n, but here the rows and columns of the matrix are indexed by compositions of n. Finally, if B is a basis of Sym_N^n and C is a basis of QSym_N^n, then $\mathcal{M}(B,C)$ is a rectangular matrix expressing each B_λ as a K-linear combination of the C_α's.

This section gives formulas for previously known matrices associated to some of the edges in Figure 1.

3.1 $\mathcal{M}(s,m)$, $\mathcal{M}(s,P)$, and $\mathcal{M}(P,m)$

The expansion of Schur polynomials into monomials uses semistandard tableaux. For later work, we will also need tableaux of skew shape. Suppose $\lambda, \mu \in \mathrm{Par}$ satisfy $\mu \subseteq \lambda$, i.e., $\mu_i \le \lambda_i$ for all i. Define the *skew diagram*

$$\lambda/\mu = \{(i,j) \in \mathbb{N}^+ \times \mathbb{N}^+ : 1 \le i \le \ell(\lambda), \mu_i < j \le \lambda_i\}.$$

We will draw skew diagrams using the English convention where the longest rows are at the top. For $N \in \mathbb{N}^+$, a *semistandard tableau* (SSYT) of shape λ/μ with entries in $[N] = \{1, 2, \ldots, N\}$ is a function $T : \lambda/\mu \to [N]$ that is weakly increasing along rows and strictly increasing down columns. Writing $n = |\lambda/\mu|$, a *standard tableau* (SYT) of shape λ/μ is a bijection $S : \lambda/\mu \to [n]$ that is also a SSYT. Let $\mathrm{SSYT}_N(\lambda/\mu)$ be the set of all SSYT of shape λ/μ with entries in $[N]$, and let $\mathrm{SYT}(\lambda/\mu)$ be the set of all SYT of shape λ/μ. For any $T \in \mathrm{SSYT}_N(\lambda/\mu)$, the *content monomial* x^T is defined to be $\prod_{c \in \lambda/\mu} x_{T(c)}$.

The *skew Schur polynomial* in N variables can now be defined as

$$s_{\lambda/\mu}(x_1, \ldots, x_N) = \sum_{T \in \mathrm{SSYT}_N(\lambda/\mu)} x^T.$$

The ordinary Schur polynomial s_λ is obtained by taking $\mu = (0)$. For $\lambda, \nu \in \mathrm{Par}_n$ and $N \ge n$, it follows that $\mathcal{M}(s,m)_{\lambda,\nu}$ is the Kostka number $K_{\lambda,\nu}$, namely the number of SSYT of shape λ and content ν.

Lascoux and Schützenberger [9] first discovered a combinatorial formula for the *t-Kostka matrix* $\mathcal{M}(s,P)$ involving the famous *charge* statistic. Given a permutation $w = w_1 w_2 \cdots w_n$ of $[n]$, let $\mathrm{IDes}(w)$ be the set of $k < n$ such that $k+1$ appears to the left of k in w, and let $\mathrm{chg}(w) = \sum_{k \in \mathrm{IDes}(w)} (n-k)$.

Next, let v be a word of partition content (i.e., for all $k \ge 1$, the number of $(k+1)$'s in v is no greater than the number of k's). Extract one or more permutations from v as follows. Scan v from left to right marking the first 1, then the first 2 after that, etc., returning to the beginning of v when the right end is reached. Do this until the largest symbol has been marked. Remove the marked symbols from v (in the order they appear) to get the first permutation. Continue to extract permutations in this way until all symbols of v have been used, and let $\mathrm{chg}(v)$ be the sum of the charges of the associated permutations. Finally, given a SSYT T of partition content, let $w(T)$ be the word obtained by reading symbols row by row from top to bottom, reading each row from right to left. Then define $\mathrm{chg}(T) = \mathrm{chg}(w(T))$.

Theorem 3.1 *[9] For all $\lambda, \mu \in \mathrm{Par}_n$, $\mathcal{M}(s,P)_{\lambda,\mu} = \sum t^{\mathrm{chg}(T)}$ summed over all $T \in \mathrm{SSYT}_n(\lambda)$ of content μ.*

Macdonald [15, §III.5, p. 229] gives a formula for the monomial expansion of *skew Hall-Littlewood polynomials* $P_{\lambda/\mu}(x_1,\ldots,x_N;t)$, which yields $\mathcal{M}(P,m)$ and $\mathcal{M}(P,M)$ by taking $\mu = (0)$. We introduce the following combinatorial model for Macdonald's formula.

Let λ/μ be a skew shape with $N \geq \ell(\lambda)$. For $T \in \mathrm{SSYT}_N(\lambda/\mu)$, define the set of *special cells* as

$$\mathrm{Sp}(T) = \{(i,j) \in \lambda/\mu : j > 1 \text{ and for all } u \text{ with } (u,j-1) \in \lambda/\mu, T((u,j-1)) \neq T((i,j))\}.$$

Define the *weight* of a special cell (i,j) to be

$$\begin{aligned} \mathrm{wt}((i,j)) \quad &= \quad |\{(u,j-1) \in \lambda/\mu : u \geq i \text{ and } T((u,j-1)) < T((i,j))\}| \\ &\quad + |\{(u,j-1) \in \mu/(0) : u \geq i\}|. \end{aligned}$$

In other words, a cell c with entry $v = T(c)$ is special for T iff c is not in column 1 and there are no v's in the column of T just left of c's column. In this case, the weight of c is the number of cells weakly below c in the column just left of c that either have entries less than v or are part of the diagram for μ. Now define the set of *starred semistandard tableaux*

$$\mathrm{SSYT}_N^*(\lambda/\mu) = \{(T,E) : T \in \mathrm{SSYT}_N(\lambda/\mu) \text{ and } E \subseteq \mathrm{Sp}(T)\}.$$

A starred tableau $T^* = (T,E)$ has *sign* $\mathrm{sgn}(T^*) = (-1)^{|E|}$, *t-weight* $\mathrm{tstat}(T^*) = \sum_{c\in E} \mathrm{wt}(c)$, *x-weight* $x^{T^*} = x^T$, and *overall weight* $\mathrm{sgn}(T^*)t^{\mathrm{tstat}(T^*)}x^{T^*}$.

For $T \in \mathrm{SSYT}_N(\lambda/\mu)$, Macdonald defines $\psi_T(t) = \prod_{c\in\mathrm{Sp}(T)}(1 - t^{\mathrm{wt}(c)})$. Then Macdonald's monomial expansion of the skew Hall-Littlewood polynomials is

$$P_{\lambda/\mu}(x_1,\ldots,x_N;t) = \sum_{T\in\mathrm{SSYT}_N(\lambda/\mu)} \psi_T(t)x^T.$$

Expanding the product in $\psi_T(t)$ using the distributive law, we get $\sum_{E\subseteq\mathrm{Sp}(T)}\prod_{c\in E}(-t^{\mathrm{wt}(c)})$. Comparing to the overall weight of starred tableaux, we find that

$$P_{\lambda/\mu}(x_1,\ldots,x_N;t) = \sum_{T^*\in\mathrm{SSYT}_N^*(\lambda/\mu)} \mathrm{sgn}(T^*)t^{\mathrm{tstat}(T^*)}x^{T^*}. \tag{1}$$

Example 3.2 *Let $\lambda = (8,6,5,4)$, $\mu = (0)$, $N \geq 8$, and*

$$T = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & \underline{2} & 2 & \underline{4} & \underline{5} & 5 \\ \hline 2 & 2 & 3 & 3 & \underline{6} & \underline{8} \\ \cline{1-6} 3 & 3 & \underline{4} & 4 & \underline{7} \\ \cline{1-5} 5 & 5 & 5 & 5 \\ \cline{1-4} \end{array} \qquad T^* = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2^* & 2 & 4^* & 5 & 5 \\ \hline 2 & 2 & 3 & 3 & 6 & 8^* \\ \cline{1-6} 3 & 3 & 4 & 4 & 7 \\ \cline{1-5} 5 & 5 & 5 & 5 \\ \cline{1-4} \end{array}.$$

In T, the special cells are indicated by the underlined entries. Specifically,

$$\mathrm{Sp}(T) = \{(1,4),(1,6),(1,7),(2,5),(2,6),(3,3),(3,5)\}.$$

These special cells have respective weights $1,1,1,3,2,1,2$. So T contributes the term $(1-t)^4(1-t^2)^2(1-t^3)x^T$ to P_λ. A typical starred tableau is $T^* = (T,\{(1,4),(1,6),(2,6)\})$. The overall weight of the object T^* is $(-1)^3t^{1+1+2}x_1^3x_2^4x_3^4x_4^3x_5^6x_6x_7x_8 = -t^4x^T$.

3.2 $\mathcal{M}(s, F)$, $\mathcal{M}(G, F)$, and $\mathcal{M}(G, M)$

The fundamental quasisymmetric expansion of Schur polynomials is a sum over standard tableaux, rather than semistandard tableaux. Given $\lambda \in \mathrm{Par}_n$ and $S \in \mathrm{SYT}(\lambda)$, define the *descent set* $\mathrm{Des}(S)$ to be the set of $k < n$ such that $k+1$ appears in a lower row of S than k. Define the descent composition $\mathrm{Des}'(S) = \mathrm{comp}(\mathrm{Des}(S))$ to be the composition associated to this subset of $[n-1]$. Gessel first proved [3] that for $N \geq n = |\lambda|$,

$$s_\lambda(x_1, \ldots, x_N) = \sum_{S \in \mathrm{SYT}(\lambda)} F_{\mathrm{Des}'(S)}(x_1, \ldots, x_N). \tag{2}$$

In other words, $\mathcal{M}(s, F)_{\lambda,\alpha}$ is the number of standard tableaux with shape λ and descent set $\mathrm{sub}(\alpha)$.

Let $\alpha, \beta \in \mathrm{Comp}_n$ with β finer than α. Say $\ell(\alpha) = k$ and $\ell(\beta) = m$. By definition, there exist indices $0 = i_0 < i_1 < \cdots < i_k = m$ such that $\alpha_j = \beta_{i_{j-1}+1} + \cdots + \beta_{i_j}$ for $1 \leq j \leq k$. The *refining composition* $\mathrm{Bre}(\beta, \alpha) = (i_1 - i_0, i_2 - i_1, \ldots, i_k - i_{k-1})$ records the number of parts of β derived from each part of α. Define $s(\alpha, \beta) = \sum_{j=1}^{k} j(i_j - i_{j-1} - 1)$. Note that in the notation $\mathrm{Bre}(\beta, \alpha)$ from [7], the finer composition is listed *first*, but in the function s (and g, ξ defined in §5.1), we list the finer composition *second*. This ordering is more convenient when working with transition matrices.

Theorem 3.3 *[7, Theorem 6.6] For all $N \geq n$ and $\alpha \in \mathrm{Comp}_n$,*

$$G_\alpha(x_1, \ldots, x_N; t) = \sum_{\beta \succeq \alpha} (-1)^{\ell(\beta)-\ell(\alpha)} t^{s(\alpha,\beta)} F_\beta(x_1, \ldots, x_N).$$

In other words, $\mathcal{M}(G, F)_{\alpha,\beta} = (-1)^{\ell(\beta)-\ell(\alpha)} t^{s(\alpha,\beta)}$ if $\beta \succeq \alpha$ and 0 otherwise.

Example 3.4 *Take $\beta = (1, 2, 2, 1, 4, 3, 1, 2, 1, 1)$ and $\alpha = (5, 5, 3, 1, 4)$. Then $\mathrm{Bre}(\beta, \alpha) = (3, 2, 1, 1, 3)$ and $s(\alpha, \beta) = 1 \cdot 2 + 2 \cdot 1 + 3 \cdot 0 + 4 \cdot 0 + 5 \cdot 2 = 14$. So $\mathcal{M}(G, F)_{\alpha,\beta} = (-1)^5 t^{14}$.*

Using $\mathcal{M}(G, M) = \mathcal{M}(G, F)\,\mathcal{M}(F, M)$, one can prove the following result giving the monomial expansion of Hivert's quasisymmetric Hall-Littlewood polynomials.

Theorem 3.5 *[7, eq. (105)] For all $\alpha \in \mathrm{Comp}_n$ and $N \geq n$,*

$$G_\alpha(x_1, \ldots, x_N; t) = \sum_{\beta \succeq \alpha} M_\beta(x_1, \ldots, x_N; t) \prod_{i=1}^{\ell(\mathrm{Bre}(\beta,\alpha))} (1 - t^i)^{\mathrm{Bre}(\beta,\alpha)_i - 1}.$$

4 F-expansion of Skew Hall-Littlewood Polynomials

Recall from §3.1 the combinatorial formula (1) for the monomial expansion of the skew Hall-Littlewood polynomials $P_{\lambda/\mu}(x_1, \ldots, x_N; t)$. This section converts this formula to an expansion of these polynomials in terms of the fundamental quasisymmetric basis. In particular, this provides a combinatorial interpretation for the entries of $\mathcal{M}(P, F)$. We remark that one can also obtain $\mathcal{M}(P, F) = \mathcal{M}(P, s)\,\mathcal{M}(s, F)$ by combining Carbonara's combinatorial formula for $\mathcal{M}(P, s)$ in terms of special tournaments [1] with Gessel's formula (2) for $\mathcal{M}(s, F)$. However, this produces a quite complicated interpretation for the coefficients in $\mathcal{M}(P, F)$ as signed combinations of standard tableaux and special tournaments. The new interpretation developed below is much simpler.

To state our result, we need a few more definitions. Given a skew diagram λ/μ with n cells, let $\mathrm{SYT}^*(\lambda/\mu)$ be the set of starred standard tableaux $S^* = (S, E)$ such that S is a standard tableau of shape λ/μ. In this case, observe that $\mathrm{Sp}(S)$ consists of all cells in the diagram not in column 1. So E can be an arbitrary subset of cells of λ/μ not in column 1. Define the *ascent set* of S^*, denoted $\mathrm{Asc}(S^*)$, to be the set of all $k < n$ such that either (a) $k+1$ appears in S in a lower row than k, or (b) there exist u, i, j with $S((u, j-1)) = k$, $S((i, j)) = k+1$, and $(i, j) \in E$. The second alternative says that $k+1$ appears in a cell of E located in the next column after the column containing k. Define $\mathrm{Asc}'(S^*) = \mathrm{comp}(\mathrm{Asc}(S^*))$ to be the associated composition.

Theorem 4.1 *For all skew shapes λ/μ with $n \leq N$ cells,*

$$P_{\lambda/\mu}(x_1, \ldots, x_N; t) = \sum_{S^* \in \mathrm{SYT}^*(\lambda/\mu)} \mathrm{sgn}(S^*) t^{\mathrm{tstat}(S^*)} F_{\mathrm{Asc}'(S^*)}(x_1, \ldots, x_N).$$

We derive a similar formula for the skew Hall-Littlewood polynomials $Q_{\lambda/\mu}$ in [12].

Example 4.2 *Using Theorem 4.1, we can make the following calculation. Each term corresponds to the starred standard tableau shown below it:*

$$\begin{array}{ccccccccc} P_{21}(t) & = & F_{21} & - & tF_{111} & + & F_{12} & - & t^2 F_{111}. \\ & & \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1} \end{array} & & \begin{array}{|c|c|}\hline 1 & 2^* \\ \hline 3 & \\ \cline{1-1} \end{array} & & \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1} \end{array} & & \begin{array}{|c|c|}\hline 1 & 3^* \\ \hline 2 & \\ \cline{1-1} \end{array} \end{array}$$

Remark 4.3 *Carbonara [1] expressed the entries of the inverse t-Kostka matrix $\mathcal{M}(P, s)$ as signed, weighted sums of special tournament matrices. An alternative description can be obtained by following $\mathcal{M}(P, F)$ by the projection from QSym to Sym given in [2]. The entry of $\mathcal{M}(P, s)_{\lambda,\mu}$ is again described as a sum of signed, weighted objects. However, in this description the objects are pairs (S^*, T) where $S^* \in \mathrm{SYT}^*(\lambda)$ and T is a "flat special rim-hook tableau" of shape μ and content $\mathrm{Asc}'(S^*)$.*

In addition to working for skew Hall-Littlewood polynomials, this new description may have computational advantages. For $n = 4$, there are 37 special tournament matrices that contribute to the calculation of $\mathcal{M}(P, s)$. However, only 23 pairs (S^*, T) are now needed. We note that these pairs do not correspond to a subclass of special tournament matrices in any simple way. Carbonara's description computes the value $\mathcal{M}(P, s)_{4,22} = 0$ via the fact that there are no special tournament matrices with parameters $\lambda = (4)$ and $\mu = (2, 2)$. There are two such pairs (S^*, T), albeit of opposite sign and equal weight.

5 New Transition Matrices involving the Hivert G-basis

This section discusses combinatorial formulas for the transition matrices $\mathcal{M}(F, G)$, $\mathcal{M}(M, G)$, and $\mathcal{M}(P, G)$.

5.1 $\mathcal{M}(F, G)$

Let $\alpha, \beta \in \mathrm{Comp}_n$ with β finer than α. Define $\xi_{\alpha,\beta}(j)$ to be j if β_j and β_{j+1} are formed from the same part of α and 0 otherwise. Set $g(\alpha, \beta) = \sum_{j=1}^{\ell(\beta)-1} \xi_{\alpha,\beta}(j)$.

Theorem 5.1 *For all $\alpha, \beta \in \mathrm{Comp}_n$,*

$$\mathcal{M}(F, G)_{\alpha,\beta} = \begin{cases} t^{g(\alpha,\beta)}, & \textit{if } \beta \succeq \alpha; \\ 0, & \textit{otherwise}. \end{cases}$$

The idea of the proof is to show that $\mathcal{M}(G,F)\,\mathcal{M}(F,G) = I$, where $\mathcal{M}(F,G)$ is defined above and $\mathcal{M}(G,F)$ is defined in Theorem 3.3. Equivalently, we must show that for all compositions $\beta \succeq \alpha$,

$$\sum_{\gamma:\,\beta\succeq\gamma\succeq\alpha} (-1)^{\ell(\gamma)-\ell(\alpha)} t^{s(\alpha,\gamma)} t^{g(\gamma,\beta)}$$

is 1 for $\beta = \alpha$ and 0 otherwise. We prove this in [12] using a sign-reversing involution that cancels all negative objects.

Example 5.2 *Using Theorem 5.1, we calculate* $F_3 = G_3 + tG_{21} + tG_{12} + t^3G_{111}$*,* $F_{21} = G_{21} + tG_{111}$*,* $F_{12} = G_{12} + t^2G_{111}$*, and* $F_{111} = G_{111}$*.*

5.2 $\mathcal{M}(M,G)$

Theorem 5.3 *For all* $\alpha, \beta \in \mathrm{Comp}_n$ *with* $\beta \succeq \alpha$*,*

$$\mathcal{M}(M,G)_{\alpha,\beta} = (-1)^{\ell(\beta)-\ell(\alpha)} \prod_{j:\,\xi_{\alpha,\beta}(j)=j} (1-t^j).$$

For other α, β*,* $\mathcal{M}(M,G)_{\alpha,\beta} = 0$*.*

The idea of the proof is to use $\mathcal{M}(M,G) = \mathcal{M}(M,F)\,\mathcal{M}(F,G)$ to see that the α,β-entry of $\mathcal{M}(M,G)$ is

$$(-1)^{\ell(\beta)-\ell(\alpha)} \sum_{\gamma:\beta\succeq\gamma\succeq\alpha} (-1)^{\ell(\gamma)-\ell(\beta)} t^{g(\gamma,\beta)}.$$

We then use a counting argument to rewrite the sum as the product $\prod_{j:\xi_{\alpha,\beta}(j)=j}(1-t^j)$.

Example 5.4 *Consider* $\alpha = 22$ *and* $\beta = 1111$*. Then* $\xi_{\alpha,\beta}(1) = 1$*,* $\xi_{\alpha,\beta}(2) = 0$ *and* $\xi_{\alpha,\beta}(3) = 3$*. So* $\mathcal{M}(M,G)_{\alpha,\beta} = (-1)^2(1-t)(1-t^3)$*.*

Example 5.5 *We calculate* $M_3 = G_3 - (1-t)G_{21} - (1-t)G_{12} + (1-t)(1-t^2)G_{111}$*,* $M_{21} = G_{21} - (1-t)G_{111}$*,* $M_{12} = G_{12} - (1-t^2)G_{111}$*, and* $M_{111} = G_{111}$*.*

5.3 $\mathcal{M}(P,G)$

By multiplying $\mathcal{M}(P,F)$ (as given in §4) and $\mathcal{M}(F,G)$ (as given in §5.1), we obtain the formula

$$\mathcal{M}(P,G)_{\lambda,\beta} = \sum_{\substack{S^*=(S,E)\in\mathrm{SYT}^*(\lambda)\\ \mathrm{Asc}'(S^*)\preceq\beta}} (-1)^{|E|} t^{\mathrm{tstat}(S^*)+g(\mathrm{Asc}'(S^*),\beta)}. \tag{3}$$

However, this can be simplified. In order to do so, we introduce some new notation.

For $S \in \mathrm{SYT}(\lambda)$, define $\mathrm{Sp}(S)$ and $\mathrm{wt}(c)$ as in §3.1. For $E \subseteq \mathrm{Sp}(S)$, define $\mathrm{Asc}(S^*) = \mathrm{Asc}((S,E))$ as in §4. We define the following subset of $\mathrm{Sp}(S)$:

$$\mathrm{Esp}(S) = \{c \in \mathrm{Sp}(S) : \mathrm{Asc}((S,\{c\})) \neq \mathrm{Asc}((S,\emptyset))\}.$$

For each $j \in \mathrm{sub}(\beta)$, let $n_j = n_j(\beta)$ be the number of elements of $\mathrm{sub}(\beta)$ that are at most j. Let $c_j = c_j(S)$ be the unique cell of S in which j appears. Let $n'_j = n_j$ if $j \in \mathrm{sub}(\beta) \setminus \mathrm{Des}(S)$ and 0 otherwise. Recall $\mathrm{Des}(S)$ was defined in §3.2.

Theorem 5.6 *For all $\lambda \in \mathrm{Par}_n$ and $\beta \in \mathrm{Comp}_n$,*

$$\mathcal{M}(P,G)_{\lambda,\beta} = \sum_{\substack{S \in \mathrm{SYT}(\lambda) \\ \mathrm{Des}(S) \subseteq \mathrm{sub}(\beta)}} \prod_{\substack{j \in \mathrm{sub}(\beta): \\ c_{j+1} \in \mathrm{Esp}(S)}} (t^{n_j} - t^{\mathrm{wt}(c_{j+1})}) \prod_{j : c_{j+1} \in \mathrm{Sp}(S) \setminus \mathrm{Esp}(S)} t^{n'_j}(1 - t^{\mathrm{wt}(c_{j+1})}). \quad (4)$$

The idea of the proof is to group together summands in (3) indexed by the starred tableaux $S^* = (S, E)$ with the same underlying standard tableau S. A careful case analysis leads to the sum of products in (4).

Remark 5.7 *If $n_j = \mathrm{wt}(c_{j+1})$ for some $j \in \mathrm{sub}(\beta)$ with $c_{j+1} \in \mathrm{Esp}(S)$, then S can be omitted from the sum in* (4)*.*

Example 5.8 *Let $\lambda = 32$ and $\beta = 1211$. Note that $\mathrm{sub}(\beta) = \{1, 3, 4\}$ and so $n_1 = 1$, $n_3 = 2$, and $n_4 = 3$. In Table 1 we list the five elements of $\mathrm{SYT}(32)$ (referred to from left to right as $S_1, \dots, S_5$) along with pertinent data. The row labeled $\prod_1$ (resp. $\prod_2$) gives the contributions from the first (resp. second) product in* (4)*. Since $\mathrm{Des}(S_2), \mathrm{Des}(S_3) \not\subseteq \mathrm{sub}(\beta)$, $\prod_1$ and $\prod_2$ have been left blank for these two tableaux. (For reference, the corresponding products for these tableaux are $(t-t) \cdot t^2(1-t)(1-t)$ and $(t-t)(t^2-t)(t^3-t^2) \cdot 1$, respectively.) Note that Remark 5.7 applies to S_1 with $j = n_j = 1$. So the only contributions are from the last two columns and we find that*

$$\mathcal{M}(P,G)_{32,1211} = (t^2 - t)(1-t) + (t^3 - t^2)(1-t) = -t^4 + t^3 + t^2 - t.$$

Tab. 1: Computation of $\mathcal{M}(P,G)_{32,1211}$.

S	$\begin{array}{\|c\|c\|c\|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&3&4\\\hline 2&5\\\cline{1-2}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}$
$\mathrm{Des}(S)$	$\{3\}$	$\{2,4\}$	$\{2\}$	$\{1,4\}$	$\{1,3\}$
$\mathrm{Sp}(S)$	$\{c_2,c_3,c_5\}$	$\{c_2,c_4,c_5\}$	$\{c_2,c_4,c_5\}$	$\{c_3,c_4,c_5\}$	$\{c_3,c_4,c_5\}$
$\mathrm{Esp}(S)$	$\{c_2,c_3,c_5\}$	$\{c_2\}$	$\{c_2,c_4,c_5\}$	$\{c_3,c_4\}$	$\{c_3,c_5\}$
$\prod_1$	$(t-t)(t^3-t)$			(t^2-t)	(t^3-t^2)
$\prod_2$	1			$(1-t)$	$(1-t)$

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Type A molecules are Kazhdan-Lusztig

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Abstract. Let (W, S) be a Coxeter system. A W-graph is an encoding of a representation of the corresponding Iwahori-Hecke algebra. Especially important examples include the W-graph corresponding to the action of the Iwahori-Hecke algebra on the Kazhdan-Lusztig basis as well as this graph's strongly connected components (cells). In 2008, Stembridge identified some common features of the Kazhdan-Lusztig graphs ("admissibility") and gave combinatorial rules for detecting admissible W-graphs. He conjectured, and checked up to $n = 9$, that all admissible A_n-cells are Kazhdan-Lusztig cells. The current paper provides a possible first step toward a proof of the conjecture. More concretely, we prove that the connected subgraphs of A_n-cells consisting of simple (i.e. directed both ways) edges do fit into the Kazhdan-Lusztig cells.

Résumé. Soit (W, S) un système de Coxeter. Un W-graphe est un objet qui décrit certaines représentations de l'algèbre de Iwahori-Hecke. Des exemples particulièrement importants sont les W-graphes correspondant à l'action de l'algèbre de Iwahori-Hecke sur la base de Kazhdan-Lusztig ainsi que ses composantes fortement connexes (cellules). En 2008, Stembridge a identifié quelques caractéristiques communes des graphes de Kazhdan-Lusztig et a donné une caractérisation combinatoire de tous ces W-graphes. Il a conjecturé, et a vérifié jusqu'à $n = 9$, que toutes ces A_n-cellules sont des cellules de Kazhdan-Lusztig. Le présent article fournit la premirè étape d'une démonstration possible de cette conjecture. Plus concrètement, nous montrons que les sous-graphes connexes de A_n-cellules composés d'arêtes s'insèrent dans les cellules de Kazhdan-Lusztig.

Keywords: Iwahori-Hecke algebra, W-graphs, W-molecules, dual equivalence graphs, Kazhdan-Lusztig cells

1 Introduction

Let (W, S) be a Coxeter system. A W-graph is a graph with additional structure that encodes a representation of the corresponding Iwahori-Hecke algebra. Kazhdan and Lusztig (1979) introduced such graphs for the regular representation, and showed that the strongly connected components (called "cells") also yield representations. Stembridge identified several common features of the Kazhdan-Lusztig graphs, namely, they are bipartite, (nearly) edge-symmetric, and their edge weights are non-negative integers (collectively he called these properties "admissibility"). He proceeded to describe, via four combinatorial rules, when an admissible graph is a W-graph (Stembridge (2008a)). One hopes that the characterization will allow one to construct the Kazhdan-Lusztig cells without having to compute Kazhdan-Lusztig polynomials (a notoriously difficult task). A piece of evidence suggesting that the definition of a general admissible W-cell approximates a Kazhdan-Lusztig cell is a more recent result of Stembridge that there are only finitely many admissible W-cells for each W (Stembridge (2012)).

There are no known examples of admissible A_n-cells besides the Kazhdan-Lusztig cells (Stembridge experimentally checked it up to $n = 9$). A possible strategy of proof is as follows:

1. An A_n-cell is a strongly connected directed graph. Consider the subgraphs which are connected via two-sided edges (of course these are strongly connected on their own, but a cell may contain several of them). The subgraphs satisfy combinatorial rules slightly weaker than those satisfied by a cell; a graph satisfying these rules is called *a molecule*. The first step is to show that any A_n-molecule *is Kazhdan-Lusztig*, i.e. it appears in the Kazhdan-Lusztig graph.

2. It is known that a Kazhdan-Lusztig A_n-cell is connected via two-sided edges, and these edges are well understood (they are called *dual Knuth moves*). The second step is to prove that no cell may have multiple molecules. The fact that no two Kazhdan-Lusztig A_n molecules may be connected inside a cell has been experimentally checked for $n \leqslant 12$ (Stembridge (2011)).

3. The last part is to prove that there can be only one A_n-graph with a given underlying molecule. For Kazhdan-Lusztig molecules this has been checked for $n \leqslant 13$ (Stembridge (2011)).

In this paper we complete the first part of the above program, namely, we prove that any A_n-molecule is Kazhdan-Lusztig. Together with the above computations, this result implies that all A_n-cells up to $n = 12$ are Kazhdan-Lusztig. The main ingredient of the proof is the axiomatization of graphs on tableaux generated by dual Knuth moves (Assaf (2008)). Five of the axioms follow easily from the molecules axioms, but the last one presents a challenge.

The paper is structured as follows. Section 2 introduces W-molecules. Section 3 discusses Assaf's dual equivalence graphs and relates them to molecules. The last section contains an outline of the proof of the main theorem that the simple part of a type A molecule is a dual equivalence graph.

2 Molecules

This section summarizes the required W-molecules terminology as described by Stembridge (2008a,b).

Let (W, S) be a simply-laced Coxeter system. A significant part of this section extends to multiply-laced types; see the above two papers. The papers are mostly concerned with W-graphs, i.e. graphs that encode certain representations of the corresponding Iwahori-Hecke algebra. It turns out that the simple (i.e. directed both ways) edges of these graphs are much easier to understand than other edges. For example, there is a very explicit description of them for the case of cells arising in the Kazhdan-Luztig W-graph (we will give it in Section 3.1). Thus we consider subgraphs connected by simple edges. These subgraphs are not W-graphs (i.e. they do not encode representations), but they satisfy certain combinatorial rules which are slightly weaker than Stembridge's W-graph rules. We begin this paper by formalizing the definitions and presenting the rules.

2.1 Definitions

An *(admissible) S-labeled graph* is a tuple $G = (V, m, \tau)$, where V is a set (vertices), $m : V \times V \to \mathbb{Z}^{\geqslant 0}$, and $\tau : V \to 2^S$ such that

1. as a directed graph (with edges given by pairs of vertices with non-zero m value), G is bipartite,

2. if $\tau(u) \subseteq \tau(v)$ then $m(u, v) = 0$,

3. if $\tau(u)$ and $\tau(v)$ are incomparable, then $m(u, v) = m(v, u)$.

The function τ is referred to as the τ-invariant.

By a *simple edge* we mean a pair of vertices (v_1, v_2) such that neither $m(v_1, v_2)$ nor $m(v_2, v_1)$ are 0 (in the graphs that we consider both weights will be 1). By an *arc* $v_1 \to v_2$ we mean a pair of vertices (v_1, v_2) such that $m(v_1, v_2) \neq 0$, but $m(v_2, v_1) = 0$. Notice if $u \to v$ is an arc, then $\tau(u) \supset \tau(v)$. If (u, v) is a simple edge then $\tau(u)$ and $\tau(v)$ are incomparable, and $m(u, v) = m(v, u)$.

A simple edge (u, v) *activates* a bond $i - j$ in the Coxeter graph if precisely one of $\tau(u)$ and $\tau(v)$ contains i, and precisely the other one contains j.

For distinct $i, j \in S$, a path $u \to v_1 \to v_2 \to \cdots \to v_{r-1} \to v$ in G is *alternating of type* (i, j) if

- $i, j \in \tau(u)$ and $i, j \notin \tau(v)$,
- $i \in \tau(v_k), j \notin \tau(v_k)$ for odd k,
- $i \notin \tau(v_k), j \in \tau(v_k)$ for even k.

Let $N_{ij}^r(G; u, v)$ denote the weighted count of such paths:

$$N_{ij}^r(G; u, v) := \sum_{v_1, \ldots, v_{r-1}} m(u, v_1) m(v_1, v_2) \ldots m(v_{r-1}, v).$$

Definition 2.1 *An S-labeled graph is called* a molecular graph *if it satisfies*

(SR) If (u, v) is a simple edge then $m(u, v) = m(v, u) = 1$.

(CR) If $u \to v$ is an edge, i.e. $m(u, v) \neq 0$, then every $i \in \tau(u) \setminus \tau(v)$ is bonded to every $j \in \tau(v) \setminus \tau(u)$.

(BR) Suppose $i - j$ is a bond in the Coxeter graph of (W, S). Any vertex u with $i \in \tau(u)$ and $j \notin \tau(u)$ is adjacent to precisely one edge which activates $i - j$.

(LPR2) For any $i, j \in S$ for any $u, v \in V$ with $i, j \in \tau(u)$, $i, j \notin \tau(v)$ and $\tau(v) \setminus \tau(u) \neq \varnothing$, we have

$$N_{ij}^2(G; u, v) = N_{ji}^2(G; u, v).$$

(LPR3) Let $k, i, j, l \in S$ be a copy of A_4 in the Coxeter graph: $k - i - j - l$. For any $u, v \in V$ with $i, j \in \tau(u)$, $i, j \notin \tau(v)$, $k, l \notin \tau(u)$, $k, l \in \tau(v)$, we have

$$N_{ij}^3(G; u, v) = N_{ji}^3(G; u, v).$$

The rules are called, respectively, simplicity rule, compatibility rule, bonding rule, and local polygon rules.

Definition 2.2 *A molecular graph is called* a molecule *if there is a path of simple edges between any pair of vertices.*

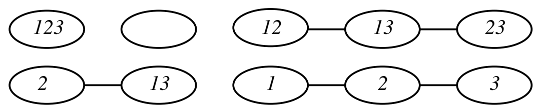
Example 2.3 *It is easy to classify all the S_4 molecules. Because of admissibility, a vertex whose τ-invariant is $\varnothing$ cannot be connected to any other vertex by a simple edge. Similarly for a vertex whose τ-invariant is $\{1, 2, 3\}$.*

Suppose we have a vertex v_1 whose τ-invariant is $\{1\}$. By BR, it is connected by a simple edge to a vertex v_2 whose τ-invariant contains 2, but not 1. By CR, $3 \notin \tau(v_2)$, and hence $\tau(v_2) = \{2\}$. By BR, v_2

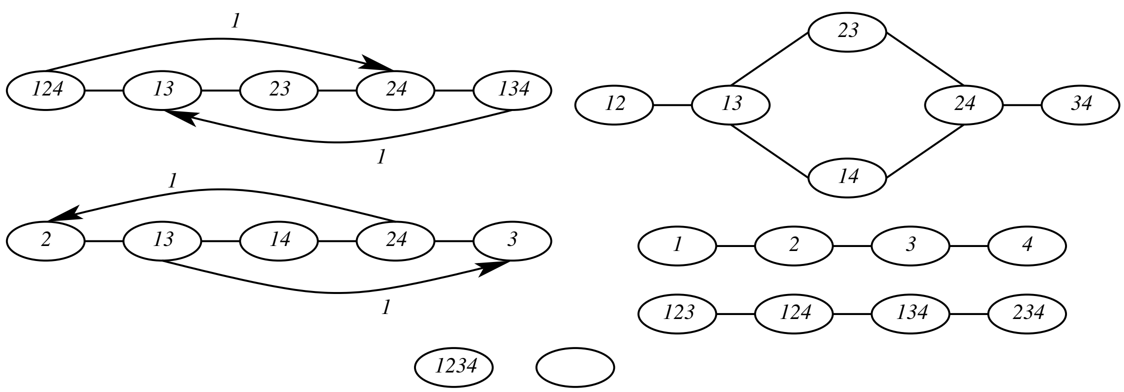
is connected by a simple edge to a vertex v_3 whose τ-invariant contains 3, but not 2. We already know $v_3 \neq v_1$. By BR, $\tau(v_3) = \{3\}$. There are no other simple edges possible, and this is a complete molecule. The same analysis works for v_1 having τ-invariants of $\{3\}, \{1,2\}, \{2,3\}$.

Suppose we have a vertex v_1 whose τ-invariant is $\{2\}$. By BR, it is connected by a simple edge to a vertex v_2 whose τ-invariant contains 1, but not 2. The case of $\tau(v_2) = \{1\}$ was described above, so the only choice is $\tau(v_2) = \{1,3\}$. This yields a complete molecule. The same argument works for v_1 having τ-invariant of $\{1,3\}$.

This completes the classification:



Example 2.4 *It takes some more work to classify the S_5 molecules (see the paper of Stembridge (2008a))*



The *simple part* of a molecule is the graph formed by erasing all the arcs. We usually view it as an undirected graph. A *morphism of molecules* $\varphi : M \to N$ is a map between the vertex sets which

1. is a graph morphism of the simple parts,
2. preserves τ-invariants.

Notice that a morphism does not need to respect arcs, aside from ones whose weights are determined by the local polygon rules from the simple edges.

2.2 Restriction

Let $J \subseteq S$ and let W_J be the corresponding parabolic subgroup.

Let $M = (V, m, \tau)$ be a W-molecular graph. The *W_J-restriction* of M is $N = (V, m', \tau')$, with

1. for all $v \in V$, $\tau'(v) = \tau(v) \cap J$,
2. for all $u, v \in V$,

$$m'(u,v) = \begin{cases} 0, & \text{if } \tau'(u) \subseteq \tau'(v), \\ m(u,v), & \text{otherwise.} \end{cases}$$

The W_J-restriction of M is a W_J-molecular graph. A W_J*-submolecule* of M is a W_J-molecule (i.e. component connected by simple edges) of the W_J-restriction of M. There is a natural inclusion map of a W_J-submolecule into the original molecular graph. Sometimes, abusing notation, we refer to the image of this map as a W_J-submolecule. The sense in which we use the word should be clear from the context.

3 Dual equivalence graphs

This section summarizes the relevant definitions and results of Assaf (2008); they are restated and slightly specialized to make the similarity with W-molecules more apparent.

Fix $n \in \mathbb{Z}^{>0}$. Let (W, S) be a Coxeter system of type A_n. Identify S in a natural way with $\{1, \dots, n\}$. Define a_i to be the edge of the Coxeter graph (throughout the paper we will refer to these edges as *bonds*) which links i and $i+1$. Then $B := \{a_1, \dots, a_{n-1}\}$ is the set of all edges of the Coxeter graph. For examples with small n we will use the notation $a, b, c, \dots$ instead.

Definition 3.1 *A* signed colored graph of type $n+1$ *is a tuple* (V, E, τ, β)*, where* (V, E) *is a finite undirected simple graph,* $\tau : V \to 2^S$*, and* $\beta : E \to 2^B$.

Denote by E_i the set of edges with label i (i.e. such that the corresponding value of β contains i); we call these i*-colored* edges. This is a slight reindexing from Assaf's original definition; in the original E_i was the set of edges whose label contains $i-1$.

3.1 "Standard" dual equivalence graphs

We start by constructing a family, indexed by partitions, of signed colored graphs.

Let λ be a partition of $n+1$. Let $SYT(\lambda)$ be the set of standard Young tableaux of shape λ. The left descent set of a tableau T is

$$\tau(T) := \{1 \leqslant i \leqslant n : i \text{ is located in a higher row than } i+1 \text{ in } T\}.$$

The left descent sets are shown in red in Example 3.2.

The set of vertices of our graph is $V := SYT(\lambda)$.

By a diagonal of a tableau we mean a $NW - SE$ diagonal. A *dual Knuth move* is the exchange of i and $i+1$ in a standard tableau, provided that either $i-1$ or $i+2$ lies (necessarily strictly) between the diagonals containing i and $i+1$. This corresponds to dual Knuth moves on the symmetric group via the "content reading word" (reading each diagonal from top to bottom, and concatenating in order of increasing height of the diagonals). The set of edges of our graph is the set of pairs of tableaux related by a dual Knuth move:

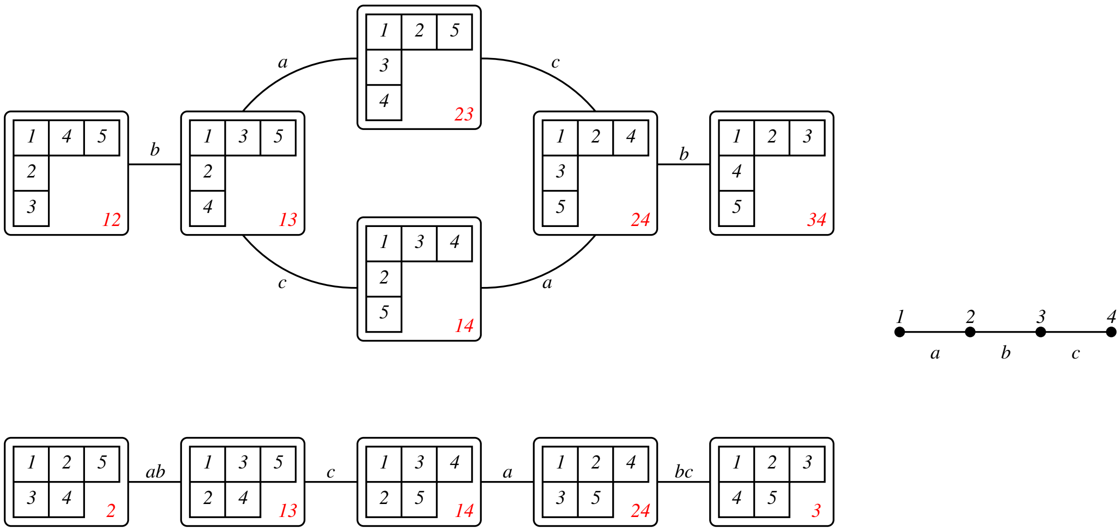
$$E := \{(T, U) : T \text{ and } U \text{ are related by a dual Knuth move}\}.$$

A dual Knuth move between tableaux T and U *activates the bond* a_i if i lies in precisely one of $\tau(T)$ and $\tau(U)$, and $i+1$ lies precisely in the other. Denote this condition by $T \overset{a_i}{-} U$. For $(T, U) \in E$, let

$$\beta(T, U) := \{a_i \in B : T \overset{a_i}{-} U\}.$$

The graph $G_\lambda := (V, E, \tau, \beta)$ is a signed colored graph of type $n+1$.

Example 3.2 *Two standard dual equivalence graphs (corresponding to the shapes* 311 *and* 32*).*



The values of τ are shown in red.

3.2 Axiomatics

A vertex w of a signed colored graph is said to *admit an i-neighbor* if precisely one of i and $i+1$ lies in $\tau(w)$.

Definition 3.3 *A* dual equivalence graph of type $n+1$ *is a signed colored graph* (V, E, τ, β) *such that for any* $1 \leqslant i < n$*:*

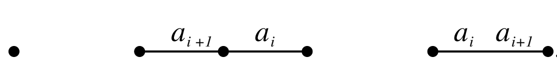
1. *For $w \in V$, w admits an i-neighbor if and only if there exists $x \in V$ which is connected to w by an edge of color i. In this case x must be unique*

2. *Suppose (w, x) is an edge of color i. Then $i \in \tau(w)$ iff $i \notin \tau(x)$, $i+1 \in \tau(w)$ iff $i+1 \notin \tau(x)$, and if $h < i-1$ or $h > i+2$ then $h \in \tau(w)$ iff $h \in \tau(x)$.*

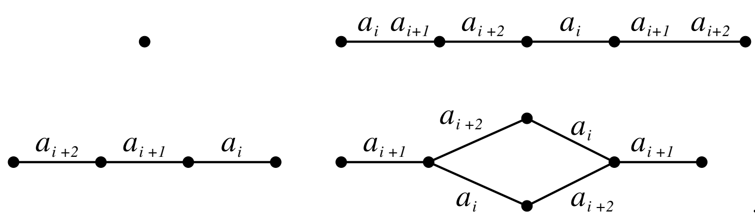
   In other words, going along an i colored edge switches i and $i+1$ in the τ value, and does not affect anything except $i-1, i, i+1$, and $i+2$.

3. *Suppose (w, x) is an edge of color i. If $i-1 \in \tau(w)\Delta\tau(x)$ then ($i-1 \in \tau(w)$ iff $i+1 \in \tau(w)$), where Δ is the symmetric difference. If $i+2 \in \tau(w)\Delta\tau(x)$ then ($i+2 \in \tau(w)$ iff $i \in \tau(w)$).*

4. *If $i < n-2$, consider the subgraph on all the vertices and edges labeled a_i or a_{i+1}. Each of its connected components has the form:*



If $i < n-3$, consider the subgraph on all the vertices and edges labeled a_i, a_{i+1}, or a_{i+2}. Each of its connected components has the form:



5. *Suppose $(w, x) \in E_i, (x, y) \in E_j$, and $|i-j| \geqslant 3$. Then there exists $v \in V$ such that $(w, v) \in E_j, (v, y) \in E_i$.*

6. *Consider a connected component of the subgraph on all the vertices and edges of colors $\leqslant i$. If we erase all the i-colored edges it breaks down into several components. Any two these were connected by an i-colored edge.*

A *weak dual equivalence graph* is a signed colored graph satisfying $1-5$ of the above.

Proposition 3.4 *The graph G_λ (the standard dual equivalence graph) is actually a dual equivalence graph. Moreover, $\{G_\lambda\}_\lambda$ is a complete collection of isomorphism class representatives of dual equivalence graphs.*

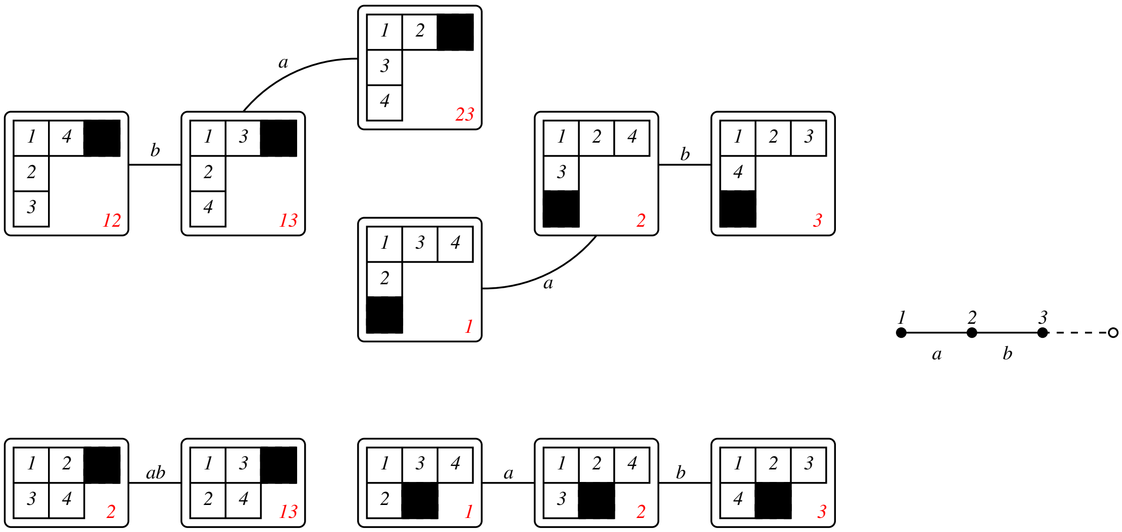
Proof: *The references are to the paper of Assaf (2008). The first statement is Proposition 3.5. The second is a combination of Theorem 3.9 and Proposition 3.11.* $\square$

3.3 Restriction

Suppose G is a signed colored graph of type $n+1$. For $0 \leqslant k < n+1$, a $(k+1)$-*restriction* of G consists of the same vertex set V, the edges colored $\leqslant k-1$, the τ function post-composed with intersection with $\{1, \ldots, k\}$, and the β function post-composed with restriction to $\{a_1, \ldots, a_{k-1}\}$ (see Example 3.5). The $(k+1)$-restriction of G is a signed colored graph of type $k+1$. The property of being a dual equivalence graph (or a weak dual equivalence graph) is preserved by restriction. By a $(k+1)$-*component of G* we mean either the connected component of the restriction, or the induced subgraph of G on vertices corresponding to such connected component. It should be clear from the context which of these we are talking about.

The n-components of G_λ are obtained by fixing the position of $n+1$ in the tableau. Such a component is isomorphic to G_μ, where μ if formed from λ by erasing the outer corner which contained $n+1$. Here is what it looks like on the above examples:

Example 3.5



The condition of being a weak dual equivalence graph is already quite powerful. The following lemma is relevant to us. It essentially says that a weak dual equivalence graph with a nice restriction property is necessarily a cover of a dual equivalence graph.

Lemma 3.6 *Suppose G is a weak dual equivalence graph of type $n+1$. Suppose moreover that each n-component is a dual equivalence graph. Then there is a surjective morphism $\varphi : G \to G_\lambda$ for some partition λ of $n+1$, which restricts to an isomorphism on the n-components.*

Let $C \cong G_\mu$ be an n-component. Then for any partition $\nu \neq \mu$ of n with $\nu \subset \lambda$, there exists a unique n-component $D \cong G_\mu$ with which is connected to C by an $(n-1)$-colored edge. Also, two n-components which are both isomorphic to G_μ are not connected by an $(n-1)$-colored edge.

Proof: *The references are again to the paper of Assaf (2008). The existence of the morphism is shown in Theorem 3.14. Its surjectivity follows by Remark 3.8. The fact that it restricts to an isomorphism on the n-components follows from the proof of Theorem 3.14. The covering properties from the second paragraph are shown in Corollary 3.15, though the last one is not explicitly mentioned.* □

3.4 Molecules and dual equivalence graphs

Proposition 3.7 *The simple part of an A_n molecule, with $\beta(u,v) = \{\text{bonds activated by the edge } (u,v)\}$ is a weak dual equivalence graph.*

Proof: *Axioms (1), (2), (3) follow directly from SR, BR, and CR. The S_4 and S_5 molecules have been computed by Stembridge (2008a) (see Examples 2.3 and 2.4). This shows that (4) is satisfied. The axiom (5) is a weaker version of the local polygon rule.* □

Consider the graph G_λ from section 3.1. It is clear that (viewed as a weighted directed graph with all edges pointing both ways and having weight 1) it is an admissible S-labeled graph for the A_n root system.

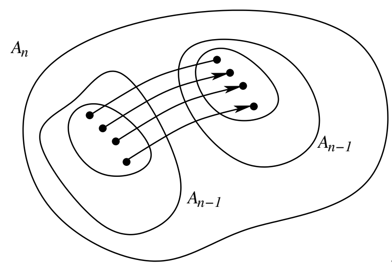
It is well known that it forms the simple part of an A_n-molecule (the left Kazhdan-Lusztig cell) which we call $\overline{G_\lambda}$.

Definition 3.8 *An A_n molecule is a* Kazhdan-Lusztig molecule *if it is isomorphic to $\overline{G_\lambda}$, i.e. if its simple part is a dual equivalence graph.*

4 Main theorem

In this section we show that any A_n molecule is Kazhdan-Lusztig. The proof will proceed by induction on n, so the preliminary results will start with an A_n molecule whose A_{n-1} submolecules are Kazhdan-Lusztig.

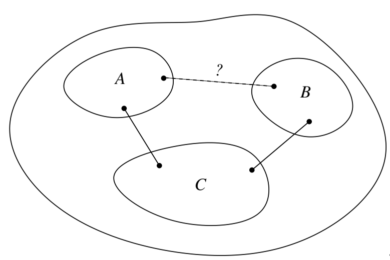
The first of these results states that if two such A_{n-1} molecules are connected by a simple edge, then the connecting A_{n-2} submolecules are isomorphic and there is a "cabling" of edges (possibly arcs) of weight 1 between these A_{n-2} molecules:



.

Lemma 4.1 *Let M be an A_n molecule whose A_{n-1} submolecules are Kazhdan-Lusztig. Suppose A and B are two such submolecules which are joined by a simple edge (in M), namely there exist $x \in A$, $y \in B$ such that the edge $x - y$ is simple. Let A' (resp. B') be the A_{n-2} submolecule of M containing x (resp. y). Then there is an isomorphism ψ between A' and B' such that $\psi(x) = y$. Moreover, if $n \in \tau(x)$ then $m(z \to \psi(z)) = 1$ for all $z \in A'$.*

The second preliminary result shows that if, out of three A_{n-1} submolecules, two pairs (satisfying some conditions) are connected by simple edges, then the third pair is also connected by a simple edge:

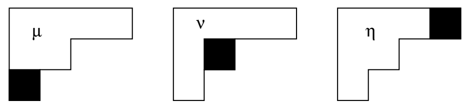


.

The conditions will later be removed to show that any pair of A_{n-1} submolecules of an A_n molecule is connected by a simple edge.

Lemma 4.2 *Let M be an A_n-molecule whose A_{n-1} submolecules are Kazhdan-Lusztig. By Proposition 3.6, there is a surjective morphism $\varphi : M \to \overline{G_\lambda}$ for some partition λ of $n+1$. Let A, B, C be A_{n-1}*

submolecules of M such that A and B are both connected to C by simple edges. Then $A \cong \overline{G_\mu}$, $B \cong \overline{G_\nu}$, $C \cong \overline{G_\eta}$, for some partitions formed by deleting outer corners of λ. The three partitions have to be different by Proposition 3.6. Suppose moreover that the deleted corner for η was the highest of the three, namely:



Then A and B are connected by a simple edge.

We can now finish the proof of the theorem.

Theorem 4.3 *Any A_n molecule is Kazhdan-Lusztig.*

Proof: *We know that the simple part of an A_n molecule is a weak dual equivalence graph. It remains to show that it satisfies the axiom (6), namely that any two A_{n-1} submolecules are connected by a simple edge.*

Proceed by induction on n, the case $n = 1$ being trivial. Let M be an A_n molecule. By inductive assumption, all A_{n-1} molecules are Kazhdan-Lusztig. So, according to 3.6 there is a covering $M \to \overline{G_\lambda}$, for some partition λ of $n+1$.

Choose two of these A_{n-1} submolecules, A and Z. Choose a path of simple edges between them which goes through the fewest number of molecules. If it does not go through other molecules, then we are done. Suppose that is not so. Let A, B, C be the first three molecules on the path (it may happen that $Z = C$). The partitions μ, ν, η corresponding to A, B, and C are formed by removing an outer corner of λ; they are all distinct by Proposition 3.6.

Again using Proposition 3.6, consider the following string of A_{n-1} submolecules connected by simple edges: $A - B - C - A' - B'$, with $A \cong A'$, $B \cong B'$, and some of these possibly equalities. Out of μ, ν, and η choose the partition which is formed by removing the highest box of λ. In the above string, choose a copy of the corresponding A_{n-1} submolecule with submolecules attached on both sides (for example, if $\lambda \setminus \mu$ was highest of the three, then we should choose A'). Then the triple consisting of this submolecule and the two adjacent ones satisfies the condition of the Lemma 4.2 (in the example, it would be the triple $C - A' - B'$). Applying it we get that $A' = A$, and $B' = B$. But then A is connected to C, contradicting our assumption that the path went through a minimal number of molecules.

So any two A_{n-1} molecules are connected by an edge, finishing the proof. $\square$

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Cyclic Sieving of Increasing Tableaux and Small Schröder Paths

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Abstract. An *increasing tableau* is a semistandard tableau with strictly increasing rows and columns. It is well known that the Catalan numbers enumerate both rectangular standard Young tableaux of two rows and also Dyck paths. We generalize this to a bijection between rectangular 2-row increasing tableaux and small Schröder paths. Using the jeu de taquin for increasing tableaux of [Thomas–Yong '09], we then present a new instance of the cyclic sieving phenomenon of [Reiner–Stanton–White '04].

Résumé. Un *tableau croissant* est un tableau semi-standard avec les lignes et les colonnes croissantes au sens strict. Il est bien connu que les nombres de Catalan énumèrent les tableaux de Young standard rectangulaires de deux lignes et aussi les chemins de Dyck. Nous généralisons ceci pour une bijection entre tableaux croissants rectangulaires á 2 lignes et petits chemins de Schröder. Utilisant le jeu de taquin de [Thomas–Yong '09] pour tableaux croissants, nous présentons ensuite une nouvelle instance du phénomène du crible cyclique de [Reiner–Stanton–White '04].

Keywords: increasing tableaux, cyclic sieving phenomenon, K-promotion, Schröder path, Schröder number, non-crossing partition

1 Introduction

An *increasing tableau* is a semistandard tableau such that all rows and columns are strictly increasing and the set of entries is an initial segment of $\mathbb{Z}_{>0}$. For λ a partition of N, we write $|\lambda| = N$. We denote by $\mathrm{Inc}_k(\lambda)$ the set of increasing tableaux of shape λ with maximum value $|\lambda| - k$. Similarly $\mathrm{SYT}(\lambda)$ denotes standard Young tableaux of shape λ. Notice $\mathrm{Inc}_0(\lambda) = \mathrm{SYT}(\lambda)$. We routinely identify a partition λ with its Young diagram; hence for us the notations $\mathrm{SYT}(m \times n)$ and $\mathrm{SYT}(n^m)$ are equivalent.

A *small Schröder path* is a planar path from the origin to $(n, 0)$ that is constructed from three types of line segment: *upsteps* by $(1, 1)$, *downsteps* by $(1, -1)$, and *horizontal steps* by $(2, 0)$, so that the path never falls below the horizontal axis and no horizontal step lies on the axis. The n^{th} *small Schröder number* is defined to be the number of such paths. A *Dyck path* is a small Schröder path without horizontal steps.

Our first result is an extension of the classical fact that Catalan numbers enumerate both Dyck paths and rectangular standard Young tableaux of two rows, $\mathrm{SYT}(2 \times n)$. For $T \in \mathrm{Inc}_k(2 \times n)$, let $\mathrm{maj}(T)$ be the sum of all i in row 1 such that $i + 1$ appears in row 2.

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Theorem 1.1 *There are explicit bijections between* $\mathrm{Inc}_k(2 \times n)$*, small Schröder paths with* k *horizontal steps, and* $\mathrm{SYT}(n-k, n-k, 1^k)$*. This implies the identity*

$$\sum_{T \in \mathrm{Inc}_k(2\times n)} q^{\mathrm{maj}(T)} = q^{n+\frac{1}{2}(k^2+k)} \frac{\begin{bmatrix} n-1 \\ k \end{bmatrix}_q \begin{bmatrix} 2n-k \\ n-k-1 \end{bmatrix}_q}{[n-k]_q}. \tag{1}$$

In particular, the total number of increasing tableaux of shape $2 \times n$ *is the* n^{th} *small Schröder number.*

The "flag-shaped" standard Young tableaux of Theorem 1.1 were previously considered by R. Stanley [Sta96] in relation to polygon dissections.

Suppose X is a finite set, $\mathcal{C}_n = \langle c \rangle$ a cyclic group acting on X, and $f \in \mathbb{Z}[q]$ a polynomial. The triple $(X, \mathcal{C}_n, f)$ has the *cyclic sieving phenomenon* [RSW04] if for all m, the number of elements of X fixed by c^m is $f(\zeta^m)$, where ζ is any primitive n^{th} root of unity. D. White [Whi07] discovered a cyclic sieving for $2 \times n$ standard Young tableaux. For this, he used a q-analogue of the hook-length formula (that is, a q-analogue of the Catalan numbers) and a group action by jeu de taquin promotion. B. Rhoades [Rho10, Theorem 1.3] generalized this result from $\mathrm{SYT}(2 \times n)$ to $\mathrm{SYT}(m \times n)$. Our main result is a generalization of D. White's result in another direction, from $\mathrm{SYT}(2 \times n) = \mathrm{Inc}_0(2 \times n)$ to $\mathrm{Inc}_k(2 \times n)$.

We first define *K-promotion* for increasing tableaux. Define the *SE-neighbors* of a box to be the (at most two) boxes immediately below it or right of it. Let T be an increasing tableau with maximum entry M. Delete the entry 1 from T, leaving an empty box. Repeatedly perform the following operation simultaneously on all empty boxes until no empty box has a SE-neighbor: Label each empty box by the minimal label of its SE-neighbors and then remove that label from the SE-neighbor(s) in which it appears. If an empty box has no SE-neighbors, it remains unchanged. We illustrate the local changes in Figure 1.

$$\begin{array}{|c|c|}\hline & i \\ \hline j & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \mapsto \begin{array}{|c|c|}\hline i & \\ \hline j & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline & j \\ \hline i & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \mapsto \begin{array}{|c|c|}\hline i & j \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline & i \\ \hline i & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \mapsto \begin{array}{|c|c|}\hline i & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad \begin{array}{cc}\cline{2-2} \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} \\ \hline \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{i} \\ \hline\end{array} \mapsto \begin{array}{cc}\cline{2-2} \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{i} \\ \hline \multicolumn{1}{|c|}{i} & \multicolumn{1}{c|}{} \\ \hline\end{array}$$

Fig. 1: Local changes during K-promotion for $i < j$.

Notice that the number of empty boxes may change during this process. Finally we obtain the K-promotion $\mathcal{P}(T)$ by labeling all empty boxes by $M + 1$ and then subtracting one from every label. Figure 2 shows a full example of K-promotion.

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 4 & 5 \\ \hline\end{array} \mapsto \begin{array}{|c|c|c|}\hline & 2 & 4 \\ \hline 3 & 4 & 5 \\ \hline\end{array} \mapsto \begin{array}{|c|c|c|}\hline 2 & & 4 \\ \hline 3 & 4 & 5 \\ \hline\end{array} \mapsto \begin{array}{|c|c|c|}\hline 2 & 4 & \\ \hline 3 & & 5 \\ \hline\end{array} \mapsto \begin{array}{|c|c|c|}\hline 2 & 4 & 5 \\ \hline 3 & 5 & \\ \hline\end{array} \mapsto \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 4 & 5 \\ \hline\end{array}$$

Fig. 2: K-promotion.

Our definition of K-promotion is analogous to that of ordinary promotion, but uses the K-jeu de taquin of H. Thomas–A. Yong [TY09] in place of ordinary jeu de taquin. (The 'K' reflects their original development of K-jeu de taquin in application to K-theoretic Schubert calculus.) Observe that on standard Young tableaux, promotion and K-promotion coincide.

We will need:

Theorem 1.2 *For all n and k, there is an action of the cyclic group $\mathcal{C}_{2n-k}$ on $T \in \mathrm{Inc}_k(2 \times n)$, where a generator acts by K-promotion.*

In the case $k = 0$, Theorem 1.2 is implicit in work of M.-P. Schützenberger (cf. [Hai92, Sta09]). We provide two combinatorial proofs of Theorem 1.2, which we believe provide different insights. Finally we construct the following cyclic sieving.

Theorem 1.3 *For all n and k, the triple $\left(\mathrm{Inc}_k(2 \times n), \mathcal{C}_{2n-k}, f\right)$ has the cyclic sieving phenomenon, where*

$$f(q) := \frac{\begin{bmatrix} n-1 \\ k \end{bmatrix}_q \begin{bmatrix} 2n-k \\ n-k-1 \end{bmatrix}_q}{[n-k]_q}$$

is the q-enumerator from Theorem 1.1.

Our proof of Theorem 1.3 is elementary. In contrast, all proofs [Rho10, Pur12] of B. Rhoades' theorem for standard Young tableaux use representation theory or geometry. (Also [PPR09], where the authors give new proofs of the 2- and 3-row cases of B. Rhoades' result, uses representation theory.) It is natural to ask also for such proofs of Theorem 1.3. For $k > 0$, Theorem 1.2 does not generalize in the obvious way to tableaux of more than 3 rows. We do not know a common generalization of our Theorem 1.3 and B. Rhoades' theorem.

This note is an extended abstract of [Pec12]. Here we omit or sketch most of the proofs. The organization is as follows. In Section 2, we prove Theorem 1.1. We include an additional bijection (to be used in Section 4) between $\mathrm{Inc}_k(2 \times n)$ and certain noncrossing partitions that we interpret as generalized noncrossing matchings. In Section 3, we develop a strengthening of Theorem 1.2 through combinatorics of small Schröder paths. We also provide a counterexample to the analogous statement for 4-row increasing tableaux. In Section 4, we use noncrossing partitions to give a second proof of Theorem 1.2 and to prove Theorem 1.3.

2 Bijections and Enumeration

Proposition 2.1 *There is an explicit bijection between $\mathrm{Inc}_k(2 \times n)$ and $\mathrm{SYT}(n-k, n-k, 1^k)$.*

Proof: Let $T \in \mathrm{Inc}_k(2 \times n)$. The following algorithm produces a corresponding $S \in \mathrm{SYT}(n-k, n-k, 1^k)$. Observe that every value in $\{1, \ldots, 2n-k\}$ appears in T either once or twice. Let A be the set of numbers that appear twice. Let B be the set of numbers that appear in the second row immediately right of an element of A. Note $|A| = |B| = k$.

Let T' be the tableau of shape $(n-k, n-k)$ formed by deleting all elements of A from the first row of T and all elements of B from the second. The standard Young tableau S is given by appending B to the first column. An example is shown in Figure 3.

This algorithm is reversible. Given the standard Young tableau S of shape $(n-k, n-k, 1^k)$, let B be the set of entries below the first two rows. By inserting B into the second row of S while maintaining increasingness, we reconstruct the second row of T. Let A be the set of elements immediately left of an element of B in this reconstructed row. By inserting A into the first row of S while maintaining increasingness, we reconstruct the first row of T. □

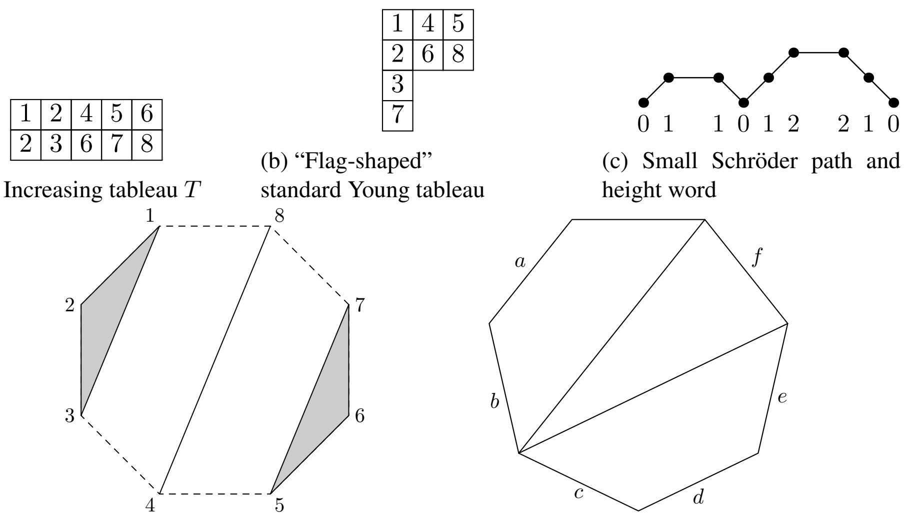
Corollary 2.2 *For all n and k,*

$$\sum_{T \in \mathrm{Inc}_k(2 \times n)} q^{\mathrm{maj}(T)} = q^{n+\frac{1}{2}(k^2+k)} \frac{\begin{bmatrix} n-1 \\ k \end{bmatrix}_q \begin{bmatrix} 2n-k \\ n-k-1 \end{bmatrix}_q}{[n-k]_q}.$$

$\square$

Proof of Theorem 1.1: The bijection between $\mathrm{Inc}_k(2 \times n)$ and $\mathrm{SYT}(n-k, n-k, 1^k)$ is given by Proposition 2.1. The q-enumeration (1) is exactly Corollary 2.2.

We now give a bijection between $\mathrm{Inc}_k(2 \times n)$ and small Schröder paths with k horizontal steps. Let $T \in \mathrm{Inc}_k(2 \times n)$. For each integer j from 1 to $2n-k$, we create one segment of a small Schröder path P_T. If j appears only in the first row, then the j^{th} segment of P_T is an upstep. If j appears only in the second row of T, the j^{th} segment of P_T is a downstep. If j appears in both rows of T, the j^{th} segment of P_T is horizontal. It is clear that the tableau T can be reconstructed from the small Schröder path P_T, so this operation gives a bijection. Thus increasing tableaux of shape (n, n) are counted by small Schröder numbers. $\square$



(a) Increasing tableau T

(b) "Flag-shaped" standard Young tableau

(c) Small Schröder path and its height word

(d) Noncrossing partition

(e) Polygon dissection

Fig. 3: A rectangular increasing tableau $T \in \mathrm{Inc}_2(5,5)$ with its corresponding standard Young tableau of shape $(3,3,1,1)$, small Schröder path, noncrossing partition of $\{1,\ldots,8\}$ with all blocks of size at least two, and heptagon dissection.

The following bijection will play an important role in our proof of Theorem 1.3 in Section 4. A partition of $\{1,\ldots,N\}$ is *noncrossing* if the convex hulls of the blocks are pairwise disjoint when the values $1,\ldots,N$ are equally spaced around a circle with 1 in the upper left and values increasing counterclockwise (cf. Figure 3(D)).

Proposition 2.3 *There is an explicit bijection between* $\mathrm{Inc}_k(2 \times n)$ *and noncrossing partitions of* $2n - k$ *into* $n - k$ *blocks all of size at least 2.*

Proof: Let $T \in \mathrm{Inc}_k(2 \times n)$. For each i in the second row of T, let s_i be the largest number in the first row that is less than i and that is not s_j for some $j < i$. Form a partition of $2n - k$ by declaring, for every i, that i and s_i are in the same block. We see this partition has $n - k$ blocks by observing that the largest elements of the blocks are precisely the numbers in the second row of T that do not also appear in the first row. Clearly there are no singleton blocks.

If the partition were *not* noncrossing, there would exist some elements $a < b < c < d$ with a, c in a block B and b, d in a distinct block B'. Observe that b must appear in the first row of T and c must appear in the second row of T (not necessarily exclusively). We may assume c to be the least element of B that is greater than b. We may then assume b to be the greatest element of B' that is less than c. Now consider s_c, which must exist since c appears in the second row of T. By definition, s_c is the largest number in the first row that is less than c and that is not s_j for some $j < c$. By assumption, b appears in the first row, is less than c, and is not s_j for any $j < c$; hence $s_c \geq b$. Since however b and c lie in distinct blocks, $s_c \neq b$, whence $b < s_c < c$. This is impossible, since we took c to be the least element of B greater than b. Thus the partition is necessarily noncrossing.

To reconstruct the increasing tableau, read the partition from 1 to $2n - k$. Place the smallest elements of blocks in only the first row, place the largest elements of blocks in only the second row, and place intermediate elements in both rows. $\square$

The set $\mathrm{Inc}_k(2 \times n)$ is also in bijection with $(n + 2)$-gon dissections by $n - k - 1$ diagonals. We do not describe this bijection, as it is well known (cf. [Sta96]) and will not be used except in Section 4 for comparison with previous results. The existence of a connection between increasing tableaux and polygon dissections was first suggested in [TY11]. An example of all these bijections is shown in Figure 3.

Remark 2.1 A *noncrossing matching* is a noncrossing partition with all blocks of size two. Like Dyck paths, polygon triangulations, and 2-row rectangular standard Young tableaux, noncrossing matchings are enumerated by the Catalan numbers. Since increasing tableaux were developed as a K-theoretic analogue of standard Young tableaux, it is tempting also to regard small Schröder paths, polygon dissections, and noncrossing partitions without singletons as K-theory analogues of Dyck paths, polygon triangulations, and noncrossing matchings, respectively. In particular, by analogy with [PPR09], it is tempting to think of noncrossing partitions without singletons as "K-webs" for $\mathfrak{sl}_2$, although their representation-theoretic significance is unknown.

3 K-Promotion and K-Evacuation

In this section, we prove Theorem 1.2. Let $\max(T)$ denote the largest entry in a tableau T. For a rectangular tableau T, we write $\mathtt{rot}(T)$ for the tableau formed by rotating 180 degrees and reversing the alphabet, so that label x becomes $\max(T) + 1 - x$. We define *K-evacuation* $\mathcal{E}$ as in [TY09, §4] by analogy with evacuation for standard Young tableaux, using K-jeu de taquin in place of ordinary jeu de taquin. Define *dual K-evacuation* $\mathcal{E}^*$ by $\mathcal{E}^* := \mathtt{rot} \circ \mathcal{E} \circ \mathtt{rot}$. (This definition of $\mathcal{E}^*$ only makes sense for rectangular tableaux. For a tableau T of general shape λ, in place of applying $\mathtt{rot}$, one should dualize λ (thought of as a poset) and reverse the alphabet. We will not make any essential use of this more general definition.)

Towards Theorem 1.2, we first develop basic combinatorics of the above operators that are well-known in the standard Young tableau case (cf. [Sta09]). From these results, we observe that Theorem 1.2 follows from the claim that $\mathtt{rot}(T) = \mathcal{E}(T)$ for every $T \in \mathrm{Inc}_k(2 \times n)$. We first saw this approach in [Whi10] for the standard Young tableau case, although similar ideas appear in [Hai92, Sta09, . . .]; we are not sure where it first appeared.

Finally, beginning at Lemma 3.3, we prove that for $T \in \mathrm{Inc}_k(2 \times n)$, $\mathtt{rot}(T) = \mathcal{E}(T)$. Here the situation is more subtle than in the standard case. (For example, we will show that the claim is not generally true for T a rectangular increasing tableau with more than 3 rows.) We proceed by careful analysis of how $\mathtt{rot}, \mathcal{E}, \mathcal{E}^*$, and $\mathcal{P}$ act on the corresponding small Schröder paths.

Remark 3.1 *It is not hard to see that K-promotion is reversible, and hence permutes the set of increasing tableaux.*

Lemma 3.1 *K-evacuation and dual K-evacuation are involutions, $\mathcal{P}\circ\mathcal{E} = \mathcal{E}\circ\mathcal{P}^{-1}$, and for any increasing tableau T, $(\mathcal{E}^* \circ \mathcal{E})(T) = \mathcal{P}^{\max(T)}(T)$.*

Before proving Lemma 3.1, we briefly recall the *K-theory growth diagrams* of [TY09, §2, 4], which extend the standard Young tableau growth diagrams of S. Fomin (cf. [Sta99, Appendix 1]). We will write $[T]_j$ for the subtableau of T formed by deleting all entries $> j$. For $T \in \mathrm{Inc}_k(\lambda)$, consider the sequence of Young diagrams $(\mathrm{shape}([T]_j))_{0 \le j \le |\lambda|-k}$. Note that this sequence of diagrams uniquely encodes T. We draw this sequence of Young diagrams horizontally from left to right. Below this sequence, we draw, in successive rows, the sequences of Young diagrams associated to $\mathcal{P}^i(T)$ for $1 \le i \le |\lambda|-k$. Hence each row encodes the K-promotion of the row above it. We offset each row one space to the right. We will refer to this entire array as the *K-theory growth diagram for T*. (There are other K-theory growth diagrams for T that one might consider, but this is the only one we will need.) Figure 4 shows an example.

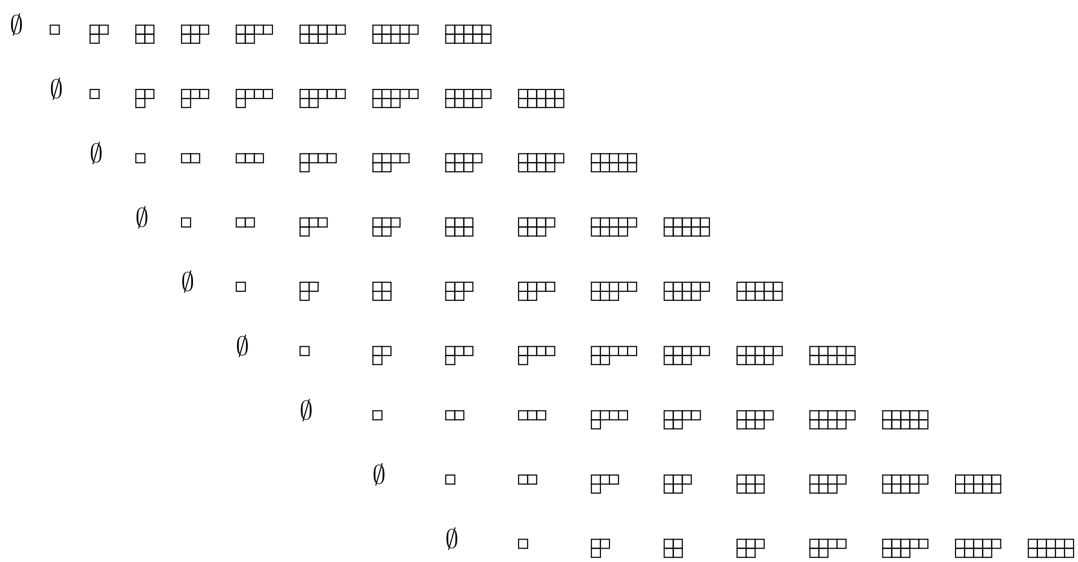


Fig. 4: The K-theory growth diagram for the tableau T of Figure 3(A).

We will write YD_{ij} for the Young diagram $\mathrm{shape}([\mathcal{P}^{i-1}(T)]_{j-i})$. This indexing is nothing more than imposing "matrix-style" or "English" coordinates on the K-theory growth diagram. For example in Figure 4, YD_{58} denotes $\boxplus$, the Young diagram in the fifth row from the top and the eighth column from the left.

Remark 3.2 *[TY09, Proposition 2.2]* In any 2×2 square $\begin{matrix} \lambda & \mu \\ \nu & \xi \end{matrix}$ of Young diagrams in a K-theory growth diagram, ξ is uniquely and explicitly determined by λ, μ and ν. Similarly λ is uniquely and explicitly determined by μ, ν and ξ. Furthermore these rules are symmetric, in the sense that if $\begin{matrix} \lambda & \mu \\ \nu & \xi \end{matrix}$ and $\begin{matrix} \xi & \mu \\ \nu & \rho \end{matrix}$ are both 2×2 squares of Young diagrams in K-theory growth diagrams, then $\lambda = \rho$.

Proof of Lemma 3.1: Fix a tableau $T \in \mathrm{Inc}_k(\lambda)$. All of these facts are proven as in the standard case (cf. [Sta09, §5]), except one uses K-theory growth diagrams instead of ordinary growth diagrams. The proof that K-evacuation is an involution appears in greater detail as [TY09, Theorem 4.1]. For rectangular shapes, the fact that dual K-evacuation is an involution follows from the fact that K-evacuation is, since $\mathcal{E}^* = \mathtt{rot} \circ \mathcal{E} \circ \mathtt{rot}$.

Essentially by definition, the central column (the column containing the rightmost $\emptyset$) of the K-theory growth diagram for T encodes the K-evacuation of the first row as well as the dual K-evacuation of the last row. The first row encodes T and the last row encodes $\mathcal{P}^{|\lambda|-k}(T)$. Hence $\mathcal{E}(T) = \mathcal{E}^*(\mathcal{P}^{|\lambda|-k}(T))$.

By the symmetry mentioned in Remark 3.2, one observes that the first row encodes the K-evacuation of the central column and that the last row encodes the dual K-evacuation of the central column. This yields $\mathcal{E}(\mathcal{E}(T)) = T$ and $\mathcal{E}^*(\mathcal{E}^*(\mathcal{P}^{|\lambda|-k}(T))) = \mathcal{P}^{|\lambda|-k}(T)$, showing that K-evacuation and dual K-evacuation are involutions. Combining the above observations, yields $(\mathcal{E}^* \circ \mathcal{E})(T) = \mathcal{P}^{|\lambda|-k}(T)$.

Finally to show $\mathcal{P} \circ \mathcal{E} = \mathcal{E} \circ \mathcal{P}^{-1}$, it is easiest to append an extra $\emptyset$ to the lower-right of the diagonal line of $\emptyset$s that appears in the K-theory growth diagram. This extra $\emptyset$ lies in the column just right of the central one. This column now encodes the K-evacuation of the second row. Hence by the symmetry mentioned in Remark 3.2, the K-promotion of this column is encoded by the central column. Thus if $S = \mathcal{P}(T)$, the central column encodes $\mathcal{P}(\mathcal{E}(S))$. But certainly $\mathcal{P}^{-1}(S) = T$ is encoded by the first row, and we have already observed that the central column encodes $\mathcal{E}(T)$. Therefore $\mathcal{P}(\mathcal{E}(S)) = \mathcal{E}(\mathcal{P}^{-1}(S))$. $\square$

Let $\mathrm{er}(T)$ be the least positive integer such that $(\mathcal{E}^* \circ \mathcal{E})^{\mathrm{er}(T)}(T) = T$. We call this number the *evacuation rank* of T. Similarly we define the *promotion rank* $\mathrm{pr}(T)$ to be the least positive integer such that $\mathcal{P}^{\mathrm{pr}(T)}(T) = T$.

Corollary 3.2 *Let T be a increasing tableau. Then $\mathrm{er}(T)$ divides $\mathrm{pr}(T)$, $\mathrm{pr}(T)$ divides $\max(T) \cdot \mathrm{er}(T)$, and the following are equivalent:*

(a) $\mathcal{E}(T) = \mathcal{E}^*(T)$,

(b) $\mathrm{er}(T) = 1$,

(c) $\mathrm{pr}(T)$ *divides* $\max(T)$.

Moreover if T is rectangular and $\mathcal{E}(T) = \mathtt{rot}(T)$, then $\mathcal{E}(T) = \mathcal{E}^*(T)$. $\square$

Thus to prove Theorem 1.2, it suffices to show that $\mathcal{E}(T) = \texttt{rot}(T)$ for every $T \in \mathrm{Inc}_k(2 \times n)$. We use the bijection between $\mathrm{Inc}_k(2 \times n)$ and small Schröder paths from Theorem 1.1. These paths are themselves in bijection with the sequence of their node heights, which we call the *height word*. Figure 3(C) shows an example. For $T \in \mathrm{Inc}_k(2 \times n)$, we write P_T for the corresponding small Schröder path and S_T for the corresponding height word.

Lemma 3.3 *For $T \in \mathrm{Inc}_k(2 \times n)$, the i^{th} letter of the height word S_T is the difference between the lengths of the first and second rows of the Young diagram* $\mathrm{shape}([T]_{i-1})$. □

Lemma 3.4 *Let $T \in \mathrm{Inc}_k(2 \times n)$. Then $P_{\texttt{rot}(T)}$ is the reflection of P_T across a vertical line and $S_{\texttt{rot}(T)}$ is the word formed by reversing S_T.* □

Lemma 3.5 *Let $T \in \mathrm{Inc}_k(2 \times n)$ and $M = 2n - k$. Let x_i denote the $(M + 2 - i)^{th}$ letter of the height word $S_{\mathcal{P}^{i-1}(T)}$. Then $S_{\mathcal{E}(T)} = x_{M+1}x_M \dots x_1$.*

Proof: By consideration of the K-theory growth diagram for T. □

Lemma 3.6 *Let $T \in \mathrm{Inc}_k(2 \times n)$.*

(a) The word S_T may be written in exactly one way as $0w_1 0w_3$ or $0w_1 1w_2 0w_3$, where w_1 is a sequence of strictly positive integers that ends in 1 and contains no consecutive 1s, w_2 is a (possibly empty) sequence of strictly positive integers, and w_3 is a (possibly empty) sequence of nonnegative integers.

(b) Let w_1^- be the sequence formed by decrementing each letter of w_1 by 1. Similarly, let w_3^+ be formed by incrementing each letter of w_3 by 1.

If S_T is of the form $0w_1 0w_3$, then $S_{\mathcal{P}(T)} = w_1^- 1w_3^+ 0$. If S_T is of the form $0w_1 1w_2 0w_3$, then $S_{\mathcal{P}(T)} = w_1^- 1w_2 1w_3^+ 0$. □

For $T \in \mathrm{Inc}_k(2 \times n)$, take the first $2n - k + 1$ columns of the K-theory growth diagram for T. Replace each Young diagram in the resulting array by the difference between the lengths of its first and second rows. Figure 5 shows an example. We write a_{ij} for the number corresponding to the Young diagram YD_{ij}. By Lemma 3.3, we see that the i^{th} row of this array of nonnegative integers is exactly the first $2n - k + 2 - i$ letters of $S_{\mathcal{P}^{i-1}(T)}$. Therefore we will refer to this array as the *height growth diagram for* T, and denote it by $\mathrm{hgd}(T)$. Observe that the rightmost column of $\mathrm{hgd}(T)$ corresponds to the central column of the K-theory growth diagram for T.

Lemma 3.7 *In $\mathrm{hgd}(T)$ for $T \in \mathrm{Inc}_k(2 \times n)$, we have for all j that $a_{1j} = a_{j,2n-k+1}$.*

Proof: By induction on the length of the height word. □

Corollary 3.8 *In the notation of Lemma 3.5, $S_T = x_1 x_2 \dots x_{M+1}$.* □

Proposition 3.9 *Let $T \in \mathrm{Inc}_k(2 \times n)$. Then $\mathcal{E}(T) = \texttt{rot}(T)$.*

Proof: By Corollary 3.8, $S_T = x_1 x_2 \dots x_{2n-k+1}$. Hence $S_{\texttt{rot}(T)} = x_{2n-k+1} x_{2n-k} \dots x_1$, by Lemma 3.4. However Lemma 3.5 says that also $S_{\mathcal{E}(T)} = x_{2n-k+1} x_{2n-k} \dots x_1$. By the bijective correspondence between tableaux and height words, this yields $\mathcal{E}(T) = \texttt{rot}(T)$. □

$$
\begin{array}{ccccccccc}
0 & 1 & 1 & 0 & 1 & 2 & 2 & 1 & 0 \\
 & 0 & 1 & 1 & 2 & 3 & 3 & 2 & 1 \\
 & & 0 & 1 & 2 & 3 & 3 & 2 & 1 \\
 & & & 0 & 1 & 2 & 2 & 1 & 0 \\
 & & & & 0 & 1 & 1 & 0 & 1 \\
 & & & & & 0 & 1 & 1 & 2 \\
 & & & & & & 0 & 1 & 2 \\
 & & & & & & & 0 & 1 \\
 & & & & & & & & 0
\end{array}
$$

Fig. 5: The height growth diagram $\mathrm{hgd}(T)$ for the tableau T shown in Figure 3(A). The i^{th} row shows the first $10 - i$ letters of $S_{\mathcal{P}^{i-1}(T)}$. Lemma 3.7 says that row 1 is the same as column 9, read from top to bottom.

This completes our first proof of Theorem 1.2. We will obtain an alternate proof in Section 4. We now show a counterexample to the obvious generalization of Theorem 1.2 to increasing tableaux of more than two rows.

Example 3.10 *If T is the increasing tableau* $\begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 7 \\ \hline 3 & \underline{5} & 6 & 8 \\ \hline 5 & \underline{7} & 8 & 10 \\ \hline 7 & 9 & 10 & 11 \\ \hline \end{array}$, *then* $\mathcal{P}^{11}(T) = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 7 \\ \hline 3 & \underline{4} & 6 & 8 \\ \hline 5 & \underline{6} & 8 & 10 \\ \hline 7 & 9 & 10 & 11 \\ \hline \end{array}$. *(The underscores mark entries that differ between the two tableaux.) It can be verified that the promotion rank of this tableau is 33.*

Computer checks of small examples (including all with at most seven columns) did not identify such a counterexample for T a 3-row rectangular increasing tableau.

4 Cyclic Sieving

Proof of Theorem 1.3: Recall we defined

$$f(q) := \frac{\begin{bmatrix} n-1 \\ k \end{bmatrix}_q \begin{bmatrix} 2n-k \\ n-k-1 \end{bmatrix}_q}{[n-k]_q}$$

to be the q-enumerator for $\mathrm{Inc}_k(2 \times n)$ obtained in Theorem 1.1. Our strategy (modeled throughout on [RSW04, §7]) is to explicitly evaluate f at roots of unity and compare the result with a count of increasing tableaux. To count tableaux, we use the bijection with noncrossing partitions given in Proposition 2.3. We will find that the symmetries of these partitions more transparently encode the promotion ranks of the corresponding tableaux.

Lemma 4.1 *Let ζ be any primitive d^{th} root of unity, for d dividing $2n-k$. Then*

$$f(\zeta) = \begin{cases} \frac{(\frac{2n-k}{d})!}{(\frac{k}{d})!(\frac{n-k}{d})!(\frac{n-k}{d}-1)!\frac{n}{d}}, & \text{if } d|n \\ \frac{(\frac{2n-k}{d})!}{(\frac{k+2}{d}-1)!(\frac{n-k-1}{d})!(\frac{n-k-1}{d})!\frac{n+1}{d}}, & \text{if } d|n+1 \\ 0, & \text{otherwise.} \end{cases}$$

Proof: By explicit evaluation, as in [RSW04, §7]. □

We will write π for the bijection of Proposition 2.3 from $\mathrm{Inc}_k(2\times n)$ to noncrossing partitions of $2n-k$ into $n-k$ blocks all of size at least 2. In a noncrossing partition, there is at most one block whose convex hull contains the center of the disk; we call such a block the *central block*. For Π a noncrossing partition of N, we write $\mathcal{R}(\Pi)$ for the noncrossing partition given by rotating Π clockwise by $2\pi/N$.

Lemma 4.2 *For any $T \in \mathrm{Inc}_k(2 \times n)$, $\pi(\mathcal{P}(T)) = \mathcal{R}(\pi(T))$.* □

It remains now to count noncrossing partitions of $2n-k$ into $n-k$ blocks all of size at least 2 that are invariant under rotation by $2\pi/d$, and to show that we obtain the formula of Lemma 4.1. It is easy to show for such a partition Π that $d|n+1$ if and only if Π has a central block and that $d|n$ if and only if Π has no central block.

Arrange the numbers $1, 2, \dots, n, -1, \dots, -n$ counterclockwise at equally spaced points around a circle. Consider a partition of these points such that, for every block B, the set formed by negating all elements of B is also a block. If the convex hulls of the blocks are pairwise nonintersecting, we call such a partition a *noncrossing B_n-partition* or *type-B noncrossing partition* (cf. [Rei97]). There is an obvious bijection between noncrossing partitions of $2n-k$ that are invariant under rotation by $2\pi/d$ and noncrossing $B_{(2n-k)/d}$-partitions. The needed enumerations of type-B noncrossing partitions may be obtained from work of C. Athanasiadis, V. Reiner, and C. Savvidou [AR04, AS12]. □

Lemma 4.2 yields a second proof of Theorem 1.2. We observe that under the reformulation of Lemma 4.2, Theorem 1.3 bears a striking similarity to Theorem 7.2 of [RSW04] which gives a cyclic sieving on the set of *all* noncrossing partitions of $2n-k$ into $n-k$ parts with respect to the same cyclic group action.

Additionally, under the correspondence mentioned in Section 2 between $\mathrm{Inc}_k(2 \times n)$ and dissections of an $(n+2)$-gon with $n-k-1$ diagonals, Theorem 1.3 bears a strong resemblance to Theorem 7.1 of [RSW04], which gives a cyclic sieving on the same set with the same q-enumerator, but with respect to an action by $\mathcal{C}_{n+2}$ instead of $\mathcal{C}_{2n-k}$. S.-P. Eu–T.-S. Fu [EF08] reinterpret the $\mathcal{C}_{n+2}$-action as the action of a Coxeter element on the k-faces of an associahedron. We do not know such an interpretation of our action by $\mathcal{C}_{2n-k}$. In [RSW04], the authors note many similarities between their Theorems 7.1 and 7.2 and ask for a unified proof. It would be very satisfying if such a proof could also account for our Theorem 1.3.

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Schubert polynomials and k-Schur functions (Extended abstract)

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Abstract. The main purpose of this paper is to show that the multiplication of a Schubert polynomial of finite type A by a Schur function can be understood from the multiplication in the space of dual k-Schur functions. Using earlier work by the second author, we encode both problems by means of quasisymmetric functions. On the Schubert vs. Schur side, we study the r-Bruhat order given by Bergeron-Sottile, along with certain operators associated to this order. On the other side, we connect this poset with a graph on dual k-Schur functions given by studying the affine grassmannian order of Lam-Lapointe-Morse-Shimozono. Also, we define operators associated to the graph on dual k-Schur functions which are analogous to the ones given for the Schubert vs. Schur problem.

Résumé. Le but principal de cet article est de montrer que la multiplication d'un polynôme de Schubert de type fini A par une fonction de Schur peut être comprise à partir de la multiplication dans l'espace dual des fonctions k-Schur. Les travaux antérieurs par le second auteur, nous permet de coder ces deux problèmes au moyen de fonctions quasisymétriques. Du côté Schubert vs Schur, nous étudions l'ordre partiel r-Bruhat donné par Bergeron-Sottile, ainsi que certains opérateurs associés à cet ordre. Nous donnons une relation entre l'ordre r-Bruhat et le graphe de Bruhat sur les fonctions k-Schur dualles données par l'étude de l'ordre affine grassmannienne de Lam-Lapointe-Morse-Shimozono. En outre, nous définissons des opérateurs associés a ce graphe qui sont analogues à ceux donnés pour le problème Schubert vs Schur.

Keywords: Schubert polynomials, k-Schur functions, affine grassmannian, r-Bruhat order, strong order.

1 Introduction

A fundamental problem in algebraic combinatorics is to find combinatorial rules for certain properties of a given combinatorial Hopf algebra. The problem of providing a combinatorial rule for the structure constants of a particular basis is an instance of this situation. The classical example is the Littlewood-Richardson rule which describes the multiplication and comultiplication of Schur functions within the space of symmetric functions.

Providing a rule for this kind of problems is in general very hard and many such problems are still unsolved. In particular, this paper will consider: the multiplication of Schubert polynomials, and the multiplication and comultiplication of k-Schur functions.

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Schubert polynomials are known to multiply positively since their structure constants enumerate flags in suitable triple intersections of Schubert varieties. However, there is no positive combinatorial rule to construct these constants in general. Nevertheless, since Schur polynomials correspond to grassmannian varieties which are a special class of flag varieties, we have that the Littlewood-Richardson rule is a special case of this particular problem. Even if we consider a slightly larger class of Schubert polynomials, namely, multiplication of a Schubert polynomial by a Schur function, we find that for several years there was no solution for finding a positive rule for these structure constants. Fortunately, in [6] new identities were deduced, more tools were developed and the use of techniques along the way of [2, 5, 7–9] gave as a result a combinatorial rule for this problem [3], which we will refer later as Schubert vs. Schur. Also in [3], using the work of [12], we deduce, independently of [11], a combinatorial proof that the Gromov-Witten invariants are positive.

Let us turn our attention now to k-Schur functions and their duals. In [14], one definition is shown to be related to the homology of the affine grassmannian of the affine coxeter group $\tilde{A}_{k+1}$. More precisely, the k-Schur functions are shown to be the Schubert polynomials for the affine grassmannian and, as such, the structure constants of their multiplication must be positive integers. The space of k-Schur functions span a graded Hopf algebra, and its graded dual describes the cohomology of the affine grassmannian. Thus, the comultiplication structure is also given by positive integer constants. Also, the structure constants of k-Schur functions include, as a special case, the structure of the small quantum cohomology and in particular, as mentioned above, the Gromov-Witten invariants [18].

In a series of two papers we plan to give a positive rule (along the lines of [3]) for the multiplication of dual k-Schur with a Schur function and relate this to the Schubert vs Schur problem. This is done by an in-depth study of the affine strong Bruhat graph. In order to achieve this we need to adapt the tools we have in [3, 5, 7–9] and create new ones. In this paper we start our study the strong Bruhat graph restricted to affine grassmannian permutations (see [15]). Given two such permutations u, v let $K_{[u,v]}$ be the quasisymmetric function associated to them, constructed as in [5]. The coefficient $d^v_{u,\lambda}$ of a Schur function S_λ in $K_{[u,v]}$ is the same as the coefficient of the dual k-Schur $S^{*(k)}_v$ in the product $S_\lambda S^{*(k)}_u$. In this way we recover certain structure constants of the multiplication of dual k-Schur functions since when $\lambda \subseteq (c^r)$ and $c+r = k+1$ we have that $S_\lambda = S^{(k)*}_w$ for some w affine grassmannian. We also consider an explicit combinatorial embedding of the Schubert vs. Schur problem into the dual k-Schur problem. This is done by inclusion of the chains of the grassmannian-Bruhat order into the affine strong Bruhat graph. We remark that in [13], Knutson, Lam and Speyer show that the Schubert vs. Schur problem reduces geometrically to the dual k-Schur problem. Here we focus on the positive combinatorial aspect of the problems.

The paper is organized as follows. In Sections 2 and 3 we recall some background about Schubert polynomials and k-Schur functions, respectively. In Section 4 we study the affine strong Bruhat graph and introduce the main relations satisfied by saturated chains in this order. Also,we introduce the quasi-symmetric function $K_{[u,v]}$. Finally, Section 5 is dedicated to the inclusion of the chains of the grassmannian-Bruhat order.

2 Schubert Polynomials

We recall a few results from [5, 7–9]. Let $u \in \mathcal{S}_\infty := \bigcup_{n\geq 0} \mathcal{S}_n$ be an infinite permutation where all but a finite number of positive integers are fixed. Non-affine Schubert polynomials $\mathfrak{S}_u$ are indexed by such permutations [19,20]. These polynomials form a homogenuous basis of the polynomial ring $\mathbb{Z}[x_1, x_2, \ldots]$

in countably many variables. The coefficients $c_{u,v}^{w}$ in $\mathfrak{S}_u\mathfrak{S}_v = \sum_v c_{u,v}^{w}\mathfrak{S}_w$, are known to be positive. As shown in example 6.2 of [5] (see also [9]), we can encode some of the coefficients above with a quasisymmetric function as follows. Let $\ell(w)$ be the length of a permutation $w \in \mathcal{S}_\infty$. We define the r-Bruhat order $<_r$ by its covers. Given permutations $u, w \in \mathcal{S}_\infty$, we say that $u \lessdot_r w$ if $\ell(u) + 1 = \ell(w)$ and $u^{-1}w = (i,j)$, where (i,j) is a reflection with $i \le r < j$. When $u \lessdot_r w$, we write $wu^{-1} = (a,b)$ with $a < b$ and label the cover $u \lessdot_r w$ in the r-Bruhat order with the integer b.

We enumerate chains in the r-Bruhat order according to the descents in their sequence of labels of the edges. More precisely, we use the *descent Pieri operator*

$$x.\mathbf{H}_k := \sum_{\omega} \mathrm{end}(\omega), \tag{2.1}$$

where the sum is over all chains ω of length k in the r-Bruhat order starting at $x \in \mathcal{S}_\infty$, $\omega : x \xrightarrow{b_1} x_1 \xrightarrow{b_2} \cdots \xrightarrow{b_k} x_k =: \mathrm{end}(\omega)$, with no descents, that is $b_1 \le b_2 \le \cdots \le b_k$. Let $\langle \cdot, \cdot \rangle$ be the bilinear form on $\mathbb{Z}\mathcal{S}_\infty$ induced by the Kronecker delta function on the elements of $\mathcal{S}_\infty$. Given $u \le_r w$, let $n = \ell(w) - \ell(u)$ be the rank of the interval $[u,w]_r$ and let

$$K_{[u,w]_r} = \sum_{\alpha \models n} \langle u.\mathbf{H}_{\alpha_1} ... \mathbf{H}_{\alpha_k}, w \rangle M_\alpha \tag{2.2}$$

summing over all compositions $\alpha = (\alpha_1, \dots, \alpha_k)$ of n, where M_α is the monomial quasisymmetric function indexed by α (see [1, 5]). Now, given a saturated chain ω in the interval $[u,w]_r$ with labels $b_1, b_2, \dots, b_n$, we let $D(\omega) = (d_1, d_2, \dots, d_s)$ denote the unique composition of n such that $b_i > b_{i+1}$ exactly in position $i \in \{d_1, d_1 + d_2, \dots, d_1 + d_2 + \cdots + d_{s-1}\}$. The chain ω contributes to the coefficient of M_α if and only if $\alpha \le D(\omega)$ under refinement. We thus have

$$K_{[u,w]_r} = \sum_{\omega \in [u,w]_r} F_{D(\omega)}. \tag{2.3}$$

where F_β denotes the fundamental quasisymmetric function for a composition β.

The descent Pieri operators on this labelled poset are symmetric as $\mathbf{H}_m$ models the action of the Schur polynomial $h_m(x_1, \dots, x_r)$ on the basis of Schubert classes (indexed by $\mathcal{S}_\infty$) in the cohomology of the flag manifold $SL(n, \mathbb{C})/B$. The quasisymmetric function $K_{[u,w]_r}$ is then a symmetric function and we can expand it in terms of Schur functions S_λ.

Proposition 2.1 ([9])

$$K_{[u,w]_r} = \sum_{\lambda} c_{u,(\lambda,r)}^{w} S_\lambda \tag{2.4}$$

where $c_{u,(\lambda,r)}^{w}$ *is the coefficient of the Schubert polynomial* $\mathfrak{S}_w$ *in the product* $\mathfrak{S}_u \cdot S_\lambda(x_1, \dots, x_r)$.

Geometry shows that these coefficients $c_{u,(\lambda,k)}^{w}$ are non-negative. To our knowledge, the work in [3] is the first combinatorial proof of this fact.

Let us recall the combinatorial analysis in [8] to study chains in the r-Bruhat order. By definition, a saturated chain in $[u,w]_r$ of the form $\omega : u = u_0 \xrightarrow{b_1} u_1 \xrightarrow{b_2} \cdots \xrightarrow{b_n} u_n = w$, is completely characterized by the sequence of transpositions $(a_1, b_1), (a_2, b_2), \dots (a_n, b_n)$ where $(a_i, b_i)u_{i-1} = u_i$.

Let $\mathbf{u}_{ab}\colon\ \mathbb{Z}\mathcal{S}_\infty \longrightarrow \mathbb{Z}\mathcal{S}_\infty$ denote the operator such that $\mathbf{u}_{ab}u = (a\ b)u$ if $u \lessdot_r (a,b)u$, and $\mathbf{u}_{ab}u = 0$ otherwise. We have shown in [8] that these operators satisfy the following relations:

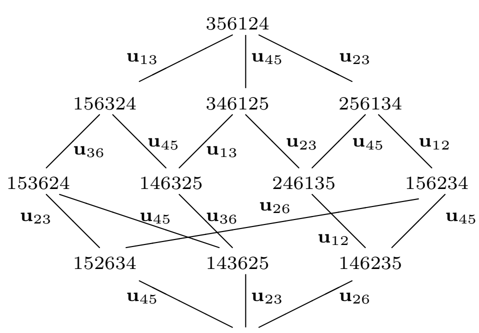
$$
\begin{array}{llcll}
(1) & \mathbf{u}_{bc}\mathbf{u}_{cd}\mathbf{u}_{ac} & \equiv & \mathbf{u}_{bd}\mathbf{u}_{ab}\mathbf{u}_{bc}, & \text{if } a<b<c<d,\\
(2) & \mathbf{u}_{ac}\mathbf{u}_{cd}\mathbf{u}_{bc} & \equiv & \mathbf{u}_{bc}\mathbf{u}_{ab}\mathbf{u}_{bd}, & \text{if } a<b<c<d,\\
(3) & \mathbf{u}_{ab}\mathbf{u}_{cd} & \equiv & \mathbf{u}_{cd}\mathbf{u}_{ab}, & \text{if } b<c \text{ or } a<c<d<b,\\
(4) & \mathbf{u}_{ac}\mathbf{u}_{bd} & \equiv & \mathbf{u}_{bd}\mathbf{u}_{ac} \ \equiv\ \mathbf{0}, & \text{if } a\le b<c\le d,\\
(5) & \mathbf{u}_{bc}\mathbf{u}_{ab}\mathbf{u}_{bc} & \equiv & \mathbf{u}_{ab}\mathbf{u}_{bc}\mathbf{u}_{ab} \ \equiv\ \mathbf{0}, & \text{if } a<b<c.
\end{array}
\tag{2.5}
$$

The $\mathbf{0}$ in relations (4) and (5) means that no chain in any r-Bruhat order can contain such a sequence of transpositions. On the other hand, relations (1), (2) and (3) are complete and transitively connect any two chains in a given interval $[u,w]_r$. It is also important to notice that the relations are independent of r. This is a fact noticed in [6]: a nonempty interval $[u,w]_r$ in the r-Bruhat order is isomorphic to a nonempty interval $[x,y]_{r'}$ in a r'-Bruhat order as long as $wu^{-1} = yx^{-1}$. This implies several identities among the structure constants.

When we write a sequence of operators $\mathbf{u}_{a_nb_n}\cdots\mathbf{u}_{a_2b_2}\mathbf{u}_{a_1b_1}$, if nonzero, it corresponds to a unique chain in some nonempty interval $[u,w]_r$ for some r and $w^{-1}u = (a_n,b_n)\cdots(a_1,b_1)$. To compute the quasisymmetric function $K_{[u,w]_r}$ as in equation (2.3), it suffices to generate one chain in $[u,w]_r$ and we can obtain the other ones using relations (1), (2) and (3) above.

Given any $\zeta \in \mathcal{S}_\infty$ we produce a chain in a nonempty interval $[u,w]_r$ as follows. Let $up(\zeta) = \{a : \zeta^{-1}(a) < a\}$. This is a finite set and we can set $r = |up(\zeta)|$. To construct w, we sort the elements in $up(\zeta) = \{i_1 < i_2 < \cdots < i_r\}$ and its complement $up^c(\zeta) = \mathbb{Z}_{>0} \setminus up(\zeta) = \{j_1 < j_2 < \ldots\}$. Next, we put $w = [i_1, i_2, \ldots, i_r, j_1, j_2, \ldots] \in \mathcal{S}_\infty$ and then we let $u = \zeta^{-1}w$. Notice that u, w and r constructed this way depend on ζ. From [6,8], we have that $[u,w]_r$ is non-empty and now we want to construct a chain in $[u,w]_r$. This is done recursively as follows: let $a_1 = u(i_1)$ where $i_1 = \max\{i \le r : u(i) < w(i)\}$ and $b_1 = u(j_1)$ where $j_1 = \min\{j > r : u(j) > u(i_1) \ge w(j)\}$ then $\mathbf{u}_{a_nb_n}\cdots\mathbf{u}_{a_2b_2}\mathbf{u}_{a_1b_1}$ is a chain in $[u,w]_r$ for any chain $\mathbf{u}_{a_nb_n}\cdots\mathbf{u}_{a_2b_2}$ in $[(a_1,b_1)u, w]_r$.

Example 2.2 Consider $\zeta = [3,6,2,5,4,1,...]$ where all other values are fixed. We have that $up(\zeta) = \{3,5,6\}$ and $up^c(\zeta) = \{1,2,4,...\}$. In this case, $r = 3$, $w = [3,5,6,1,2,4,...]$ and $u = [1,4,2,6,3,5,...]$. The recursive procedure above produces the chain $\mathbf{u}_{23}\mathbf{u}_{12}\mathbf{u}_{45}\mathbf{u}_{26}$ in $[u,v]_3$. We get all other chains by using the relations (2.5): $\mathbf{u}_{23}\mathbf{u}_{12}\mathbf{u}_{45}\mathbf{u}_{26}$, $\mathbf{u}_{23}\mathbf{u}_{12}\mathbf{u}_{26}\mathbf{u}_{45}$, $\mathbf{u}_{23}\mathbf{u}_{45}\mathbf{u}_{12}\mathbf{u}_{26}$, $\mathbf{u}_{45}\mathbf{u}_{23}\mathbf{u}_{12}\mathbf{u}_{26}$, $\mathbf{u}_{45}\mathbf{u}_{13}\mathbf{u}_{36}\mathbf{u}_{23}$, $\mathbf{u}_{13}\mathbf{u}_{45}\mathbf{u}_{36}\mathbf{u}_{23}$, $\mathbf{u}_{13}\mathbf{u}_{36}\mathbf{u}_{45}\mathbf{u}_{23}$, $\mathbf{u}_{13}\mathbf{u}_{36}\mathbf{u}_{23}\mathbf{u}_{45}$. The interval and the quasisymmetric function obtained in this case is



$$
\begin{aligned}
K_{[u,w]_3} &= F_{13} + 2F_{121} + 2F_{22} + F_{112}\\
&+ F_{31} + F_{211}\\
&= S_{31} + S_{22} + S_{211}.
\end{aligned}
$$

Notice that the functions $K_{[u,w]_r}$ encode the nonzero connected components of the given interval under the relations (2.5). In Section 5 we will show that the connected components of the chains for the r-Bruhat order where r is arbitrary, embed as a connected component of the corresponding theory for the 0-grassmannian in the affine strong Bruhat graph governing the multiplication of dual k-Schur functions.

3 k-Schur Functions and affine Grassmannians.

The k-Schur functions were originally defined combinatorially in terms of k-atoms, and conjecturally provide a positive decomposition of the Macdonald polynomials [16]. These functions have several definitions and it is conjectural that they are equivalent (see [15]). In this paper we will adopt the definition given by the k-Pieri rule and k-tableaus (see [15, 17]) since this gives us a relation with the homology and cohomology of the affine grassmannians and therefore, we get positivity in their structure constants.

The affine symmetric group W is generated by reflections s_i for $i \in \{0, 1, \ldots, k\}$, subject to the relations: $s_i^2 = 1$; $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$; $s_i s_j = s_j s_i$ if $i - j \neq \pm 1$, where $i - j$ and $i + 1$ are understood to be taken modulo $k + 1$. Let $w \in W$ and denote its length by $\ell(w)$, given by the minimal number of generators needed to write a reduced expression for w. We let W_0 denote the parabolic subgroup obtained from W by removing the generator s_0. This is naturally isomorphic to the symmetric group $\mathcal{S}_{k+1}$. For more details on affine symmetric group see [10].

Let $u \in W$ be an affine permutation. This permutation can be represented using window notation. That is, u can be seen as a bijection from $\mathbb{Z}$ to $\mathbb{Z}$, so that if u_i is the image of the integer i under u, then it can be seen as a sequence:

$$u = \cdots | u_{-k} \ \cdots \ u_{-1} \ u_0 \underbrace{| u_1 \ u_2 \ \cdots \ u_{k+1} |}_{\text{main window}} u_{k+2} \ u_{k+3} \ \cdots \ u_{2k+2} | \cdots$$

Moreover, u satisfies the property that $u_{i+k+1} = u_i + k + 1$ for all i, and the sum of the entries in the main window $u_1 + u_2 + \cdots + u_{k+1} = \binom{k+2}{2}$. Notice that in view of the first property, u is completely determined by the entries in the main window. In this notation, the generator $u = s_i$ is the permutation such that $u_{i+m(k+1)} = i + 1 + m(k+1)$ and $u_{i+1+m(k+1)} = i + m(k+1)$ for all m, and $u_j = j$ for all other values. The multiplication uw of permutations u, w in W is the usual composition given by $(uw)_i = u_{w_i}$. In view of this, the parabolic subgroup W_0 corresponds to the $u \in W$ such that the numbers $\{1, 2, \ldots, k+1\}$ appear in the main window. We will put $\bar{i} = -i$ and by convention, we consider 0 to be negative.

Now, let W^0 denote the set of minimal length coset representatives of W/W_0. In this paper we take right coset representatives, although left coset representatives could be taken also. The set of permutations in W^0 are the *affine grassmannian permutations* of W, or 0-grassmannians for short.

In this paper, any k-Schur function $S_u^{(k)}$ will be indexed by some $u \in W^0$, although k-bounded partitions or $k+1$-cores could be used instead of elements in W^0. A permutation $u \in W$ is 0-grassmannian if the numbers $1, 2, \ldots, k+1$ appear from left to right in the sequence u.

3.1 k-Schur functions.

Given $u \in W$, we say that $u \lessdot_w us_i$ is a cover for the weak order if $\ell(us_i) = \ell(u) + 1$ and we label this cover by i. The weak order on W is the transitive closure of these covers. The Pieri rule for k-Schur functions is described by certain chains in the weak order of W restricted to W^0 (see [14, 15, 17]). On the other hand, this same rule is satisfied by the Schubert grassmannian for the affine symmetric group [14].

Here, we describe the Pieri rule as follows. A saturated chain ω of length m in the weak order with end point $\mathrm{end}(\omega)$, gives us a sequence of labels $(i_1, i_2, \ldots, i_m)$. We say that the sequence $(i_1, i_2, \ldots, i_m)$ is cyclically increasing if $i_1, i_2, \ldots, i_m$ lies clockwise on a clock with hours $0, 1, \ldots, k$ and $\min\big\{j : 0 \leq j \leq k;\ j \notin \{i_1, i_2, \ldots, i_m\}\big\}$ lies between i_m and i_1. In particular we must have $1 \leq m \leq k$. Now, to express the Pieri rule, we first remark that for $1 \leq m \leq k$, the homogeneous symmetric function h_m corresponds to the k-Schur $S^{(k)}_{v(m)}$ where $v(m)$ is a 0-grassmannian whose main window is given by $|2\ \cdots\ m\ \bar{0}\ m+1\ \cdots\ k\ k+2|$. Then,

$$S^{(k)}_u h_m := \sum_{\omega \text{ cyclically increasing}} S^{(k)}_{\mathrm{end}(\omega)}, \tag{3.1}$$

where ω has length exactly m.
Iterating equation (3.1) one can easily see that

$$h_\lambda = \sum_u \mathrm{K}_{\lambda,u} S^{(k)}_u \tag{3.2}$$

is a triangular relation [17]. One way to define k-Schur functions is to start with equation (3.1) as a rule, and define them as follows. The *k-Schur functions* are the unique symmetric funtions $S^{(k)}_u$ obtained by inverting the matrix $[\mathrm{K}_{\lambda,u}]$ from (3.2) above.

It is clear that we can define a Pieri operator like equation (2.1) using the notion of a cyclically increasing chain. Using equation (2.2), this allows us to define a function $K_{[u,w]_w}$ for any interval in the weak order of W.

Example 3.1 Let $k = 2$ and $u = |\bar{0}\ 2\ 4|$. We consider the interval $[u, w]_w$ in the weak order where $w = |\bar{3}\ 4\ 5|$. This interval is a single chain $u = |\bar{0}\ 2\ 4| \xrightarrow{1} |2\ \bar{0}\ 4| \xrightarrow{2} |2\ 4\ \bar{0}| \xrightarrow{0} |\bar{3}\ 4\ 5| = w$. In this case, we have that $\langle u.\mathbf{H}_1\mathbf{H}_1\mathbf{H}_1, w\rangle = \langle u.\mathbf{H}_2\mathbf{H}_1, w\rangle = \langle u.\mathbf{H}_1\mathbf{H}_2, w\rangle = 1$ are the only nonzero entries in (2.2) and we get $K_{[u,w]_w} = M_{111} + M_{21} + M_{12} = F_{12} + F_{21} - F_{111} = S_{21} - S_{111}$.

This small example shows some of the behavior of the (quasi)symmetric function $K_{[u,w]_w}$ for the weak order of W. In general, it is not F-positive nor Schur positive. Although, these functions contain some information about the structure constants, it is not enough to fully understand them combinatorially, in particular, these functions lack some of the properties needed to use the theory developed in [2]. These functions were first defined in [5] in terms of the M-basis, but the definition given there in terms of the F-basis is wrong. Later on, Postnikov rediscovered them in [22] with more combinatorics involved, even though their combinatorial expansion in terms of Schur functions is still open.

3.2 Dual k-Schur functions.

Let $\Lambda = \mathbb{Z}[h_1, h_2, \ldots]$ be the Hopf algebra of symmetric functions (see [21]). The space of k-Schur functions $\Lambda_{(k)}$ can be seen as a Hopf subalgebra of Λ spanned by $\mathbb{Z}[h_1, h_2, \ldots, h_k]$ where h_i is the homogeneous symmetric function of degree i. The space Λ is a self dual Hopf algebra where the Schur functions S_λ form a self dual basis under the pairing $\langle h_\lambda, m_\mu\rangle = \delta_{\lambda,\mu}$ where the m_λ denote the monomial symmetric functions. Then, we have the inclusion $\Lambda_{(k)} \hookrightarrow \Lambda$, which turns into a projection $\Lambda \twoheadrightarrow \Lambda^{(k)}$ when passing to the dual space, where $\Lambda^{(k)} = \Lambda^*_{(k)}$ is the graded dual of $\Lambda_{(k)}$. It can be checked that the kernel of this projection is the linear span of $\{m_\lambda : \lambda_1 > k\}$, hence $\Lambda^{(k)} \cong \Lambda\big/\langle m_\lambda : \lambda_1 > k\rangle$.

The graded dual basis to $S_u^{(k)}$ will be denoted here by $\mathfrak{S}_u^{(k)} = S_u^{(k)*}$ which are also known as the affine Stanley symmetric functions. The multiplication of the dual k-Schur $\mathfrak{S}_u^{(k)}$ is described in terms of the affine Bruhat graph.

4 Affine Bruhat Graph

Let $t_{a,b}$ be the transposition in W such that for all $m \in \mathbb{Z}$, permutes $a + m(k+1)$ and $b + m(k+1)$ where $b - a \leq k$. The *affine Bruhat order* is given by its covering relation. Namely, for $u \in W$, we have $u \lessdot ut_{a,b}$ is a cover in the affine Bruhat order if $\ell(ut_{a,b}) = \ell(u) + 1$.

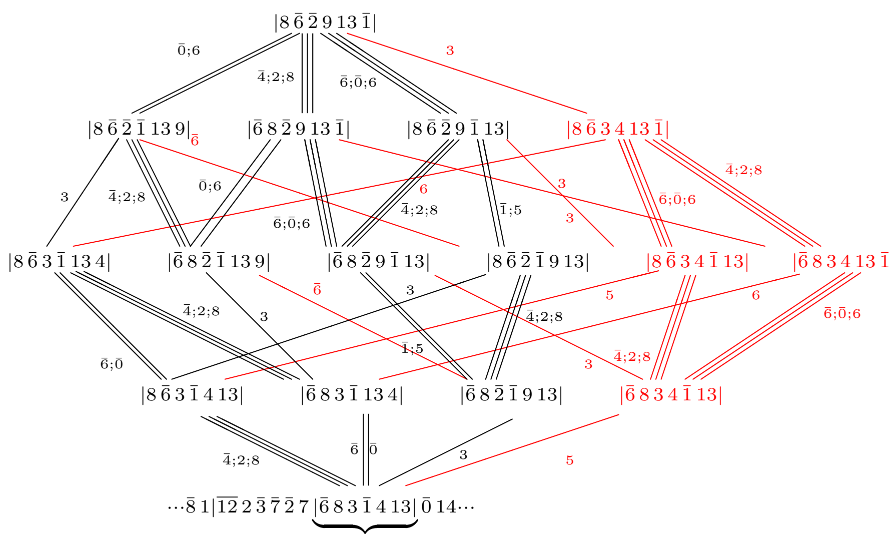
Proposition 4.1 (see [10]) *For $u \in W$ and $b - a \leq k$, we have that $u \lessdot ut_{a,b}$ is a cover in the* Bruhat order *if and only if $u(a) < u(b)$ and for all $a < i < b$ we have $u(i) < u(a)$ or $u(i) > u(b)$.*

The affine 0-Bruhat order $\lessdot_0$ arises as a suborder of the Bruhat order. For $u \in W$, a covering $u \lessdot_0 ut_{a,b}$ is encoded by transposition $t_{a,b}$ satisfying proposition 4.1 and also $u(a) \leq 0 < u(b)$. A transposition $t_{a',b'}$ satisfying the same conditions as $t_{a,b}$ gives the same affine Bruhat covering relation as long as $a' \equiv a$, $b' \equiv b$ modulo $k+1$. In view of this, we introduce a multigraph instead of a graph for the affine 0-Bruhat order, since we want to keep track of the distinct a, b such that $u \lessdot_0 ut_{a,b}$ is an affine 0-Bruhat covering for a given u.

We then define the following operators in a similar way to the ones defined in Section 2. For any $b - a \leq k+1$, let $\mathbf{t}_{ab}\colon\ \mathbb{Z}W \longrightarrow \quad \mathbb{Z}W$ be the operator on the right such that $u\mathbf{t}_{ab} = ut_{a,b}$ if $u \lessdot ut_{a,b}$ and $u(a) \leq 0 < u(b)$, and $u\mathbf{t}_{ab} = 0$ otherwise. Remark now that if $u\mathbf{t}_{ab} \neq 0$, then $u\mathbf{t}_{ab} = u\mathbf{t}_{a',b'} \neq 0$ for only finitely many values of m with $a' = a + m(k+1)$ and $b' = b + m(k+1)$.

The *affine 0-Bruhat graph* is the directed multigraph with vertices W and a labeled edge $u \xrightarrow{b} u\mathbf{t}_{ab}$ for every $u\mathbf{t}_{a,b} \neq 0$. We denote by $[u, w]$ the *set of paths* from u to w. Remark that all such paths will have the same length, namely $\ell(w) - \ell(u)$.

Example 4.2 We give below the interval $[|\bar{6}\,8\,3\,\bar{1}\,4\,13|, |8\,\bar{6}\,\bar{2}\,9\,13\,\bar{1}|]$ in the affine 0-Bruhat graph:



In this example we see that there are three arrows from $u = |\bar{6}\,8\,3\,\bar{1}\,4\,13|$ to $w = |8\,\bar{6}\,3\,\bar{1}\,4\,13|$, given by $u\mathbf{t}_{\bar{5}\bar{4}} = u\mathbf{t}_{12} = u\mathbf{t}_{78} = w$ and labeled $\bar{4}, 2, 8$, respectively. Also, we have that $u\mathbf{t}_{\overline{11}\,\overline{10}} = 0$. The shaded area of the graph represents the embedding of the interval in Example 2.2 as explain in the next sections.

For $u \in W^0$ such that $u\mathbf{t}_{ab} = w$, we have that $w \in W^0$ (see [15, Prop. 2.6]). In view of this nice behaviour we will restrict the affine 0-Bruhat graph to permutations in W^0.

4.1 Multiplication dual k-Schur.

For dual k-Schur functions $\mathfrak{S}_u^{(k)}$, the analogue of the Pieri formula (3.1) is given by

$$\mathfrak{S}_u^{(k)} h_m := \sum_{\substack{u\mathbf{t}_{a_1b_1}\cdots\mathbf{t}_{a_mb_m}\neq 0 \\ b_1<b_2<\ldots<b_m}} \mathfrak{S}^{(k)}_{u\mathbf{t}_{a_1b_1}\cdots\mathbf{t}_{a_mb_m}}, \tag{4.1}$$

where the sum is over all increasing paths $b_1 < b_2 < \cdots < b_m$ starting at u [15].

Since the Pieri formula is encoded by increasing chains in the affine 0-Bruhat graph restricted to W^0, we can define Pieri operators similar to equation (2.1) using increasing chains. This allows us to define the functions $K_{[u,w]}$ for any interval in the affine 0-Bruhat graph restricted to W^0. In contrast with the weak order, where we had cyclically increasing chains, any chain $\omega \in [u,w]$ has a well defined notion of descent. More precisely, for $\omega = \mathbf{t}_{a_1b_1}\mathbf{t}_{a_2b_2}\cdots\mathbf{t}_{a_mb_m}$ we have $D(\omega) = (d_1, d_2, \ldots, d_s)$ denotes the unique composition of n such that $b_i > b_{i+1}$ exactly in position $i \in \{d_1, d_1+d_2, \ldots, d_1+d_2+\cdots+d_{s-1}\}$. As in equation (2.3) we have

$$K_{[u,w]} \;=\; \sum_{\omega\in[u,v]} F_{D(\omega)} \tag{4.2}$$

and in this case $K_{[u,w]}$ is F-positive. Following [5], we have

Theorem 4.3

$$K_{[u,w]} \;=\; \sum_{\lambda} c^w_{u,\lambda}\, S_\lambda \tag{4.3}$$

where $c^w_{u,\lambda}$ is the coefficient of the dual k-Schur function $\mathfrak{S}_w^{(k)}$ in the product $\mathfrak{S}_u^{(k)} \cdot S_\lambda$.

Example 4.4 Considering the interval $[u,w] = [|\bar{6}\,8\,3\,\bar{1}\,4\,13|, |8\,\bar{6}\,\bar{2}\,9\,13\,\bar{1}|]$ we have in example 4.2. The total number of chains is 240. In this case $K_{[u,w]} = 9F_{1111} + 30F_{112} + 51F_{121} + 30F_{13} + 30F_{211} + 51F_{22} + 30F_{31} + 9F_4$, is symmetric and the expansion in term of Schur functions is positive $K_{[u,w]} = 9S_4 + 30S_{31} + 21S_{22} + 30S_{211} + 9S_{1111}$. The reader is encouraged to use SAGE and see that the coefficients are indeed the structure constants we claim in Theorem 4.3.

4.2 Relations of the operators $\mathbf{t}_{ab}$.

The purpose of this section is to understand some of the relations satisfied by the $\mathbf{t}_{ab}$ operators restricted to W^0, similar to the work done with Schubert polynomials in [3, 8]. The main theorem of this section presents the needed relations among these operators.

These relations depend on the following data. For $\mathbf{t}_{ab}$ we need to consider $a, b, \overline{a}, \overline{b}$ where $\overline{a}$ and $\overline{b}$ are the residue modulo $k+1$ of a and b respectively. Remark that $\overline{a} \neq \overline{b}$ since $b - a < k+1$. For $u \in W^0$ we have that, if non-zero, $u\mathbf{t}_{ab}$ and $u\mathbf{t}_{ab}\mathbf{t}_{cd}$ are both in W^0. The different relations satisfied by the operators $\mathbf{t}_{ab}$ and $\mathbf{t}_{cd}$ depend on the relation among $\overline{a}, \overline{b}, \overline{c}, \overline{d}$. We present some of them next.

(A) $\mathbf{t}_{ab}\mathbf{t}_{cd} \equiv \mathbf{t}_{cd}\mathbf{t}_{ab}$ if $\overline{a}, \overline{b}, \overline{c}, \overline{d}$ are distinct.

(B1) $\mathbf{t}_{ab}\mathbf{t}_{cd} \equiv \mathbf{t}_{cd}\mathbf{t}_{ab} \equiv 0$ if $(a < c < b < d)$ or $(b = c$ and $d - a > k + 1)$.

(B2) $\mathbf{t}_{ab}\mathbf{t}_{cd} \equiv 0$ if $(\overline{a} = \overline{c}$ and $b \leq d)$ or $(\overline{b} = \overline{d}$ and $c \leq a)$.

There are more possible zeros than what we present in (B), but we will satisfy ourselves with these ones for now. It will be more important to identify them in the second part of this work. Now if the numbers a, b, c, d are not distinct, then we must have $b = c$ or $d = a$. If $b = c$, then $d - a \leq k + 1$ in view of (B). Similarly if $d = a$ then $b - c \leq k + 1$.

(C1) $\mathbf{t}_{ab}\mathbf{t}_{bd} = \mathbf{t}_{ab}\mathbf{t}_{b-k-1,a}$ if $d - a = k + 1$,

(C2) $\mathbf{t}_{ab}\mathbf{t}_{bd}$ and $\mathbf{t}_{bd}\mathbf{t}_{ab}$ if $d - a < k + 1$.

Now we look at the cases $\mathbf{t}_{ab}\mathbf{t}_{cd}$ where a, b, c, d are distinct but some equalities occur between $\overline{a}, \overline{b}$ and $\overline{c}, \overline{d}$. By symmetry of the relation we will assume that $b < d$ which (excluding (B)) implies that $a < b < c < d$.

(D) $\mathbf{t}_{ab}\mathbf{t}_{cd} = \mathbf{t}_{d-k-1,c}\mathbf{t}_{b-k-1,a}$ if $\overline{b} = \overline{c},\ \overline{d} = \overline{a}$ and $(b - a) + (d - c) = k + 1$.

All the relations above are *local*. This means that if $\mathbf{t}_{ab}\mathbf{t}_{cd} = \mathbf{t}_{c'd'}\mathbf{t}_{a'b'}$, then $|a' - a|$, $|b' - b|$, $|c' - c|$ and $|d' - d|$ are strictly less than $k+1$. For example in (D) we have $|b-k-1-a|$, $|a-b|$, $|d-k-1-c|$ and $|c - d|$ which are strictly less than $k + 1$.

We now consider some more relations of length three:

(E1) $\mathbf{t}_{bc}\mathbf{t}_{cd}\mathbf{t}_{ac} \equiv \mathbf{t}_{bd}\mathbf{t}_{ab}\mathbf{t}_{bc}$ if $a < b < c < d$,

(E2) $\mathbf{t}_{ac}\mathbf{t}_{cd}\mathbf{t}_{bc} \equiv \mathbf{t}_{bc}\mathbf{t}_{ab}\mathbf{t}_{bd}$ if $a < b < c < d$.

also we have

(F) $\mathbf{t}_{bc}\mathbf{t}_{ab}\mathbf{t}_{bc} \equiv \mathbf{t}_{ab}\mathbf{t}_{bc}\mathbf{t}_{ab} \equiv \mathbf{0}$ if $a < b < c$ and $c - a < k + 1$.

Theorem 4.5 *The relations (A)–(F) above describe relations between* $\mathbf{t}$*-operators in the Strong Bruhat graph. (The proof is done case by case.)*

Remark 4.6 If we consider the permutation u we can derive more relations of length 2. Let $r = (b - a) + (d - c)$:

(X1) $u\mathbf{t}_{ab}\mathbf{t}_{cd} = u\mathbf{t}_{d,c+r}\mathbf{t}_{b-r,a}$ if $r < k + 1,\ \overline{d} = \overline{a},\ u(c) \leq 0$ and $u(d) \leq 0$,

(X2) $u\mathbf{t}_{ab}\mathbf{t}_{cd} = u\mathbf{t}_{cd}\mathbf{t}_{b-r,b}$ if $r < k + 1,\ \overline{d} = \overline{a}$ and $u(d) > 0$,

(X3) $u\mathbf{t}_{ab}\mathbf{t}_{cd} = u\mathbf{t}_{d-r,d}\mathbf{t}_{ab}$ if $r < k + 1,\ \overline{b} = \overline{c}$ and $u(a + r) \leq 0$,

(X4) $u\mathbf{t}_{ab}\mathbf{t}_{cd} = u\mathbf{t}_{d-r,c}\mathbf{t}_{b,a+r}$ if $r < k + 1,\ \overline{b} = \overline{c},\ u(b) > 0$ and $u(a + r) > 0$,

(X5) $u\mathbf{t}_{ab}\mathbf{t}_{cd} = u\mathbf{t}_{cd}\mathbf{t}_{a,b+c-d}$ if $\overline{b} = \overline{d},\ b - a > d - c$ and $u(d - b + a) > 0$,

(X6) $u\mathbf{t}_{ab}\mathbf{t}_{cd} = u\mathbf{t}_{c,d-b+a}\mathbf{t}_{a,b}$ if $\overline{b} = \overline{d},\ b - a < d - c$ and $u(a) \leq 0$.

In the (X) relations, the conditions we impose on u are minimal to assure that both sides of the equality are non-zero. These conditions are not given by the definition of the operators $\mathbf{t}_{ab}$. For example in (X1), the left hand side is non-zero regardless of the value of $u(d)$ but to guarantee that the right hand side is non-zero, we must have $u(d) \leq 0$. This shows that as operators $\mathbf{t}_{ab}\mathbf{t}_{cd} \neq \mathbf{t}_{d,c+r}\mathbf{t}_{b-r,a}$.

5 Schubert vs Schur Imbedded Inside Dual k-Schur

When comparing the relations (2.5) and the ones given in Section 4.2 we see that it may be possible to find a homomorphism from the Schubert vs Schur operators $\mathbf{u}_{ab}$ to the Dual k-Schur operators $\mathbf{t}_{a'b'}$. Such a homomorphism vanishes on many chains and this is the expected behavior.

Example 5.1 If we compare Example 2.2 and Example 4.2, the map $\mathbf{u}_{ab} \mapsto \mathbf{t}_{a-3,b-3}$ is a homomorphism that preserves all the chains from the first interval to the second one.

Now, given a non-empty interval $[x, y]_r$ in the r-Bruhat order, we want to find integers k, s and an explicit interval $[u, v]$ in the strong 0-Bruhat graph such that the homomorphism $\mathbf{u}_{ab} \mapsto \mathbf{t}_{a-s,b-s}$ maps the non-zero chains of $[x, y]_r$ to non-zero chains of $[u, v]$. In fact, we only need to assume that we have a non-zero operator $\mathbf{u}_{a_n b_n} \cdots \mathbf{u}_{a_1 b_1}$ and obtain the other ones using the corresponding relations. Then, the interval $[x, y]_r$ is isomorphic to the one described in Section 2.
For this purpose, let $\zeta = (a_n, b_n) \cdots (a_1, b_1)$, $up(\zeta) = \{i_1 < i_2 < \cdots < i_r\}$ and $up^c(\zeta) = \{j_1 < j_2 < \cdots\}$, then $r = |up(\zeta)|$. As in Section 2 we have that $[x, y]_r$ is nonempty for $y = [i_1, i_2, \ldots, i_r, j_1, j_2, \ldots]$ and $x = \zeta^{-1} y$.

Let k be such that $\alpha = x(\alpha) = y(\alpha)$ for all $\alpha > k + 1$. Such a k exists since x and y have finitely many non-fixed points. Put $x_\alpha = x(\alpha)$ and take the permutation $[x_1, x_2, \ldots, x_{k+1}]$. Now, we consider the positions $\alpha_1 < \cdots < \alpha_\ell < r < \beta_1 < \cdots < \beta_t < k + 1$ for which there are descents before and after r. In other words, where $x_{\alpha_i} > x_{\alpha_{i+1}}$ and $x_{\beta_j} > x_{\beta_{j+1}}$ for $1 \le i \le \ell - 1$ and $1 \le j \le t - 1$. This defines segments $1, 2, \ldots, \alpha_1; \quad \cdots \quad \alpha_\ell + 1, \ldots, r; \quad r + 1, \ldots, \beta_1; \quad \cdots \quad \beta_t + 1, \ldots, k + 1$. We want to construct a 0-grassmannian in the $k + 1$-affine permutation group W with this information such that in some adjacent $k + 1$ positions we have a permutation that has the same patterns as x^{-1}. The reason we want to look at the inverse permutation x^{-1} is because the $\mathbf{u}$ operators act on the left whereas the $\mathbf{t}$ operators act on the right.
For this purpose, we first place the values $1, 2, \ldots, k + 1$ on the $\mathbb{Z}$-axis as follows.

$$\begin{array}{rll}
1, 2, \ldots, k - \beta_t + 1 & \text{in positions} & x_{\beta_t+1} - t(k+1), \ldots, x_{k+1} - t(k+1) \\
& \cdots & \\
k - \beta_1 + 2, \ldots, k - r + 1 & \text{in positions} & x_{r+1}, \ldots, x_{\beta_1} \\
k - r + 2, \ldots, k - \alpha_\ell + 1 & \text{in positions} & x_{\alpha_\ell+1} + (k+1), \ldots, x_r + (k+1) \\
& \cdots & \\
k - \alpha_1 + 2, \ldots, k + 1 & \text{in positions} & x_1 + (\ell+1)(k+1), \ldots, x_{\alpha_1} + (\ell+1)(k+1)
\end{array}$$

This construction places the values $1, 2, \ldots, k + 1$ on the $\mathbb{Z}$-axis from left to right in distinct positions modulo $k + 1$. We build a permutation u' of $\mathbb{Z}$ defining it with the relation $u'_{i+m(k+1)} = u'_i + m(k + 1)$. This may not be a permutation in W as the sum $u'_1 + u'_2 + \cdots + u'_{k+1}$ may not be $\binom{k+2}{2}$, but a simple shift gives us the desired result, as shown in the next lemma (proof ommited) which will be followed by an example to make this construction clearer.

Lemma 5.2 *Any permutation u' of $\mathbb{Z}$ such that $u'_{i+m(k+1)} = u'_i + m(k+1)$ and the values $1, 2, \ldots, k+1$ are in distinct positions modulo $k + 1$ satisfies $u'_1 + u'_2 + \cdots + u'_{k+1} = \binom{k+2}{2} - s(k + 1)$ for some s.*

Notice that each time we shift the values of u' by 1, like $v_i = u'_{i+1}$ we get that $v_1 + v_2 + \cdots + v_{k+1} = u'_1 + u'_2 + \cdots u'_{k+1} + (k + 1) = \binom{k+2}{2} + (1 - s)(k + 1)$. Hence, if u' is as above and if the entries

$1, 2, \ldots, k+1$ appear from left to right in u', then by defining the permutation u by $u_i = u'_{i+s}$, we get a 0-affine permutation in W^0.

Example 5.3 Let us take the permutation from Example 2.2. Let $\zeta = [3, 6, 2, 5, 4, 1, ...]$ where all other values are fixed. We can choose $k+1 = 6$. We have that $up(\zeta) = \{3, 5, 6\}$ and $up^c(\zeta) = \{1, 2, 4, ...\}$. In this case, $r = 3$, $y = [3, 5, 6, 1, 2, 4, ...]$ and $x = [1, 4, 2, 6, 3, 5, ...]$. The descents in the permutation x are in positions $\alpha = 2$ and $\beta = 4$ so that $\ell = t = 1$ and $\alpha < r < \beta$. With the procedure above, we get $1 = u'(x_5 - 6) = u'(-3),\ 2 = u'(x_6 - 6) = u'(-1);\ 3 = u'(x_4) = u'(6);\ 4 = u'(x_3 + 6) = u'(8);\ 5 = u'(x_1 + 12) = u'(13),\ 6 = u'(x_2 + 12) = u'(16)$. Once we determine the values in the positions above, all other values of u' are determined as follows

$$u' = \cdots |\overline{13}\,\bar{8}\,1\,\overline{12}\,2\,\bar{3}\,\underbrace{|\bar{7}\,\bar{2}\,7\,\bar{6}\,8\,3|}_{\text{main}}\,\bar{1}\,4\,13\,\bar{0}\,14\,9|5\,10\,19\,6\,20| \cdots$$

the sum of the entries in the main window of u' is $3 = \binom{7}{2} - 3(6)$, hence $s = 3$. We see that the entries of u' in the main window $[\bar{7}\,\bar{2}\,7\,\bar{6}\,8\,3]$ are in the same relative order as $x^{-1} = [1\,3\,5\,2\,6\,4]$. We also see that the smallest $r = 3$ entries of the main window of u' are ≤ 0 and the remaining ones are positive. Now we get u by shifting the positions of u' by s:

$$u = \cdots \overline{13}\,\bar{8}\,1|\overline{12}\,2\,\bar{3}\,\bar{7}\,\bar{2}\,7\,\underbrace{|\bar{6}\,8\,3\,\bar{1}\,4\,13|}_{\text{main}}\,\bar{0}\,14\,9\,5\,10\,19|6\,20 \cdots$$

We remark that by construction, the entries $[u_{1-s}, u_{2-s}, \ldots, u_{k+1-s}]$ are the same as $[u'_1, u'_2, \ldots, u'_{k+1}]$ which in turn are in the same relative order as in x^{-1}. Therefore, from the previous paragraph we see that the smallest r entries in $[u_{1-s'}, u_{2-s'}, \ldots, u_{k+1-s'}]$ are ≤ 0 and the other entries in that window are positive. This implies that if x is covered by a non-zero permutation given by $\mathbf{u}_{ab}x$ where $x_a^{-1} \leq r < x_b^{-1}$, then we have $u\mathbf{t}_{a-s,b-s}$ is a cover in the 0-Bruhat graph. Recursively, we get that

Theorem 5.4 *Let $[x, y]_r$ be a non-empty interval $[x, y]_r$ in the r-Bruhat order and let u and s be as above. For any maximal chain $\mathbf{u}_{a_n b_n} \cdots \mathbf{u}_{a_1 b_1}$ in the interval $[x, y]_r$ we have that the chain $\mathbf{t}_{a_1-s, b_1-s} \cdots \mathbf{t}_{a_n-s, b_n-s}$ is a non-zero maximal chain in the 0-affine Bruhat graph in $[u, u\mathbf{t}_{a_1-s,b_1-s} \cdots \mathbf{t}_{a_n-s,b_n-s}]$.*

This theorem shows our main claim, namely the fact that the Schubert vs Schur problem is imbedded in the dual k-Schur problem. In the second part of our program [4] we will construct dual Knuth operators on the intervals $[u, w]$. Under the morphism above, connected components of certain dual equivalent graphs obtained in [3] are mapped to connected components of the dual equivalent graph of $[u, w]$. This shows in a stronger sense the imbedding above and explains the difficulty of the two problems. This allows us to conclude that solving the dual k-Schur problem is harder than the problem of Schubert vs Schur.

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Gog, Magog and Schützenberger II: left trapezoids

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Abstract. We are interested in finding an explicit bijection between two families of combinatorial objects: Gog and Magog triangles. These two families are particular classes of Gelfand-Tsetlin triangles and are respectively in bijection with alternating sign matrices (ASM) and totally symmetric self complementary plane partitions (TSSCPP). For this purpose, we introduce left Gog and GOGAm trapezoids. We conjecture that these two families of trapezoids are equienumerated and we give an explicit bijection between the trapezoids with one or two diagonals.

Résumé. Nous nous intéressons ici à trouver une bijection explicite entre deux familles d'objets combinatoires: les triangles Gog et Magog. Ces deux familles d'objets sont des classes particulières des triangles de Gelfand-Tsetlin et sont respectivement en bijection avec les matrices à signes alternants (ASMs) et les partitions planes totalement symétriques auto-complémentaires (TSSCPPs). Pour ce faire, nous introduisons les Gog et les GOGAm trapèzes gauches. Nous conjecturons que ces deux familles de trapèzes sont équipotents et nous donnons une bijection explicite entre ces trapèzes à une et deux lignes.

Keywords: Gog, Magog triangles and trapezoids, Schützenberger Involution, alternating sign matrices, totally symmetric self complementary plane partitions

1 Introduction

This paper is a sequel to [1], to which we refer for more on the background of the Gog-Magog problem (see also [2] and [3] for a thorough discussion). It is a well known open problem in bijective combinatorics to find a bijection between alternating sign matrices and totally symmetric self complementary plane partitions. One can reformulate the problem using so-called Gog and Magog triangles, which are particular species of Gelfand-Tsetlin triangles. In particular, Gog triangles are in simple bijection with alternating sign matrices of the same size, while Magog triangles are in bijection with totally symmetric self complementary plane partitions. In [4], Mills, Robbins and Rumsey introduced trapezoids in this problem by cutting out k diagonals on the right (with the conventions used in the present paper) of a triangle of size n, and conjectured that Gog and Magog trapezoids of the same size are equienumerated. Zeilberger [7] proved this conjecture, but no explicit bijection is known, except for $k = 1$ (which is a relatively easy problem) and for $k = 2$, this bijection being the main result of [1]. In this last paper a new class of triangles and trapezoids was introduced, called GOGAm triangles (or trapezoids), which are in bijection with the Magog triangles by the Schützenberger involution acting on Gelfand-Tsetlin triangles.

In this paper we introduce a new class of trapezoids by cutting diagonals of Gog and GOGAm triangles on the left instead of the right. We conjecture that the left Gog and GOGAm trapezoids of the same shape are equienumerated, and give a bijective proof of this for trapezoids composed of one or two diagonals. Furthermore we show that our bijection is compatible with the previous bijection between right trapezoids. It turns out that the bijection we obtain for left trapezoids is much simpler than the one of [1] for right trapezoids.Finally we can also consider rectangles (intersections of left and right trapezoids). For such rectangles we also conjecture that Gog and GOGAm are equienumerated.

Our results are presented in this paper as follows. In section 2 we give some elementary definitions about Gelfand-Tsetlin triangles, Gog and GOGAm triangles, and then define left and right Gog and GOGAm trapezoids and describe their minimal completion. Section 3 is devoted to the formulation of a conjecture on the existence of a bijection between Gog and GOGAm trapezoids of the same size. We end this paper by section 4 where we give a bijection between $(n, 2)$ left Gog and GOGAm trapezoids and we show how its work on an example. Finally, we consider another combinatorial object; rectangles.

2 Basic definitions

We start by giving definitions of our main objects of study. We refer to [1] for more details.

2.1 Gelfand-Tsetlin triangles

Definition 2.1 *A Gelfand-Tsetlin triangle of size n is a triangular array $X = (X_{i,j})_{n\geqslant i\geqslant j\geqslant 1}$ of positive integers*

$$\begin{array}{ccccccccccc}
X_{n,1} & & X_{n,2} & & \cdots & & \cdots & & X_{n,n-1} & & X_{n,n} \\
& X_{n-1,1} & & X_{n-1,2} & & \cdots & & \iddots & & X_{n-1,n-1} & \\
& & \ddots & & & & \iddots & & \iddots & & \\
& & & \ddots & & \iddots & & \iddots & & & \\
& & & & X_{2,1} & & X_{2,2} & & & & \\
& & & & & X_{1,1} & & & & &
\end{array} \tag{1}$$

such that

$$X_{i+1,j} \leqslant X_{i,j} \leqslant X_{i+1,j+1} \quad \text{for } n-1 \geqslant i \geqslant j \geqslant 1. \tag{2}$$

The set of all Gelfand-Tsetlin triangles of size n is a poset for the order such that $X \leqslant Y$ if and only if $X_{ij} \leqslant Y_{ij}$ for all i, j. It is also a lattice for this order, the infimum and supremum being taken entrywise: $\max(X, Y)_{ij} = \max(X_{ij}, Y_{ij})$.

2.2 Gog triangles and trapezoids

Definition 2.2 *A Gog triangle of size n is a Gelfand-Tsetlin triangle such that*

1. *its rows are strictly increasing;*

$$X_{i,j} < X_{i,j+1}, \qquad j < i \leqslant n-1 \tag{3}$$

2. *and such that*

$$X_{n,j} = j, \qquad 1 \leqslant j \leqslant n. \tag{4}$$

Here is an example of Gog triangle of size $n = 5$.

$$\begin{array}{ccccccccc}
1 & & 2 & & 3 & & 4 & & 5 \\
& 1 & & 3 & & 4 & & 5 & \\
& & 1 & & 4 & & 5 & & \\
& & & 2 & & 4 & & & \\
& & & & 3 & & & &
\end{array} \tag{5}$$

It is immediate to check that the set of Gog triangles of size n is a sublattice of the Gelfand-Tsetlin triangles.

Definition 2.3 *A (n, k) right Gog trapezoid (for $k \leqslant n$) is an array of positive integers $X = (X_{i,j})_{n \geqslant i \geqslant j \geqslant 1; i-j \leqslant k-1}$ formed from the k rightmost SW-NE diagonals of some Gog triangle of size n.*

Below is a $(5, 2)$ right Gog trapezoid.

$$\begin{array}{cccccc}
& & & 4 & & 5 \\
& & 4 & & 5 & \\
& 3 & & 4 & & \\
1 & & 3 & & & \\
& 2 & & & &
\end{array} \tag{6}$$

Definition 2.4 *A (n, k) left Gog trapezoid (for $k \leqslant n$) is an array of positive integers $X = (X_{i,j})_{n \geqslant i \geqslant j \geqslant 1; k \geqslant j}$ formed from the k leftmost NW-SE diagonals of a Gog triangle of size n.*

A more direct way of checking that a left Gelfand-Tsetlin trapezoid is a left Gog trapezoid is to verify that its rows are strictly increasing and that its SW-NE diagonals are bounded by $1, 2, \ldots, n$ as it is shown in the figure below which represents a $(5, 2)$ left Gog trapezoid.

$$\begin{array}{cccccccccccccc}
&&&\color{blue}{1}&&&&\color{blue}{2}&&&&&& \\
&\color{blue}{\leqslant}&&&&\color{blue}{\leqslant}&&&&&&&& \\
1&&&&2&&&&&\color{blue}{3}&&&& \\
&&&&&&&\color{blue}{\leqslant}&&&&&& \\
&&1&&&&3&&&&&\color{blue}{4}&& \\
&&&&&&&&&\color{blue}{\leqslant}&&&& \\
&&&&2&&&&3&&&&&\color{blue}{5} \\
&&&&&&&&&&&\color{blue}{\leqslant}&& \\
&&&&&&2&&&&4&&& \\
&&&&&&&&4&&&&&
\end{array} \tag{7}$$

There is a simple involution $X \to \tilde{X}$ on Gog triangles given by

$$\tilde{X}_{i,j} = n + 1 - X_{i,i+1-j} \tag{8}$$

which exchanges left and right trapezoids of the same size. This involution corresponds to a vertical symmetry of associated ASMs.

2.2.1 Minimal completion

Since the set of Gog triangles is a lattice, given a left (resp. a right) Gog trapezoid, there exists a smallest Gog triangle from which it can be extracted. We call this Gog triangle the *canonical completion* of the left (resp. the right) Gog trapezoid. Their explicit value is computed in the next Proposition.

Proposition 2.5

1. *Let X be a (n, k) right Gog trapezoid, then its canonical completion satisfies*

$$X_{ij} = j \quad \textit{for} \quad i \geqslant j + k. \tag{9}$$

2. *Let X be a (n, k) left Gog trapezoid, then its canonical completion satisfies*

$$X_{i,j} = \max(X_{i,k} + j - k, X_{i-1,k} + j - k - 1, \ldots, X_{i-j+k,k}) \quad \textit{for} \quad j \geqslant k. \tag{10}$$

Proof: The first case (right trapezoids) is trivial, the formula for the second case (left trapezoids) is easily proved by induction on $j - k$. □

For example, the completion of the $(5, 2)$ left Gog trapezoid in (7)

$$\begin{array}{ccccccccc} 1 & & 2 & & 3 & & 4 & & 5 \\ & 1 & & 3 & & 4 & & 5 & \\ & & 2 & & 3 & & 4 & & \\ & & & 2 & & 4 & & & \\ & & & & 4 & & & & \end{array} \tag{11}$$

Remark that the supplementary entries of the canonical completion of a left Gog trapezoid depend only on its rightmost NW-SE diagonal.

The right trapezoids defined above coincide (modulo easy reindexations) with those of Mills, Robbins, Rumsey [4], and Zeilberger [7]. They are in obvious bijection with the ones in [1] (actually the Gog trapezoids of [1] are the canonical completions of the right Gog trapezoids defined above).

2.3 GOGAm triangles and trapezoids

Definition 2.6 *A GOGAm triangle of size n is a Gelfand-Tsetlin triangle such that $X_{nn} \leqslant n$ and, for all $1 \leqslant k \leqslant n - 1$, and all $n = j_0 > j_1 > j_2 \ldots > j_{n-k} \geqslant 1$, one has*

$$\left(\sum_{i=0}^{n-k-1} X_{j_i+i,j_i} - X_{j_{i+1}+i,j_{i+1}}\right) + X_{j_{n-k}+n-k,j_{n-k}} \leqslant k \tag{12}$$

It is shown in [1] that GOGAm triangles are exactly the Gelfand-Tsetlin triangles obtained by applying the Schützenberger involution to Magog triangles. It follows that the problem of finding an explicit bijection between Gog and Magog triangles can be reduced to that of finding an explicit bijection between Gog and GOGAm triangles. In the sequel, Magog triangles will not be considered anymore.

Definition 2.7 *A (n,k) right GOGAm trapezoid (for $k \leqslant n$) is an array of positive integers $X = (x_{i,j})_{n \geqslant i \geqslant j \geqslant 1; i-j \leqslant k-1}$ formed from the k rightmost SW-NE diagonals of a GOGAm triangle of size n.*

Below is a $(5,2)$ right GOGAm trapezoid.

$$\begin{array}{ccccccccc} & & & 2 & & 4 & \\ & & 2 & & 4 & & \\ & 2 & & 4 & & & \\ 1 & & 4 & & & & \\ & 3 & & & & & \end{array} \tag{13}$$

Definition 2.8 *A (n,k) left GOGAm trapezoid (for $k \leqslant n$) is an array of positive integers $X = (x_{i,j})_{n \geqslant i \geqslant j \geqslant 1; k \geqslant j}$ formed from the k leftmost NW-SE diagonals of a GOGAm trapezoid of size n.*

Below is a $(5,2)$ left GOGAm trapezoid.

$$\begin{array}{cccccc} 1 & & 1 & & & \\ & 1 & & 2 & & \\ & & 1 & & 2 & \\ & & & 2 & & 3 \\ & & & & 3 & \end{array} \tag{14}$$

2.3.1 Minimal completion

The set of GOGAm triangles is not a sublattice of the Gelfand-Tsetlin triangles, nevertheless, given a right (resp. a left) GOGAm trapezoid, we shall see that there exists a smallest GOGAm triangle which extends it. We call this GOGAm triangle the *canonical completion* of the left (resp. the right) GOGAm trapezoid.

Proposition 2.9

1. *Let X be a (n,k) right GOGAm trapezoid, then its canonical completion is given by*

$$X_{ij} = 1 \quad \text{for } n \geqslant i \geqslant j + k. \tag{15}$$

2. *Let X be a (n,k) left GOGAm trapezoid, then its canonical completion is given by*

$$X_{i,j} = X_{i-j+k,k} \quad \text{for } n \geqslant i \geqslant j \geqslant k \tag{16}$$

in other words, the added entries are constant on SW-NE diagonals

Proof: In both cases, the completion above is the smallest Gelfand-Tsetlin triangle containing the trapezoid, therefore it is enough to check that if X is a (n,k) right or left GOGAm trapezoid, then its completion, as indicated in the proposition 2.9, is a GOGAm triangle. The claim follows from the following lemma.

Lemma 2.10 *Let X be a GOGAm triangle.*

i) *The triangle obtained from X by replacing the entries on the upper left triangle $(X_{ij}, n \geqslant i \geqslant j+k)$ by 1 is a GOGAm triangle.*

ii) *Let $n \geqslant m \geqslant k \geqslant 1$. If X is constant on each partial SW-NE diagonal $(X_{i+l,k+l}; n-i \geqslant l \geqslant 0)$ for $i \geqslant m+1$ then the triangle obtained from X by replacing the entries $(X_{m+l,k+l}; n-m \geqslant l \geqslant 1)$ by $X_{m,k}$ is a GOGAm triangle.*

Proof: It is easily seen that the above replacements give a Gelfand-Tsetlin triangle. Both proofs then follow by inspection of the formula (12), which shows that, upon making the above replacements, the quantity on the left cannot increase. $\square$

End of proof of Proposition 2.9. The case of right GOGAm triangles is dealt with by part **i)** of the preceding Lemma. The case of left trapezoids follows by replacing successively the SW-NE partial diagonals as in part **ii)** of the Lemma. $\square$

For example, the completion of the $(5,2)$ left GOGAm trapezoid in (14) is as follows.

$$\begin{array}{ccccccccc}
1 & & 1 & & 2 & & 2 & & 3 \\
& 1 & & 2 & & 2 & & 3 & \\
& & 1 & & 2 & & 3 & & \\
& & & 2 & & 3 & & & \\
& & & & 3 & & & &
\end{array} \tag{17}$$

3 Results and conjectures

Theorem 3.1 (Zeilberger [7]) *For all $k \leqslant n$, the (n,k) right Gog and GOGAm trapezoids are equienumerated*

Actually Zeilberger proves this theorem for Gog and Magog trapezoids, but composing by the Schützenberger involution yields the above result. In [1] a bijective proof is given for $(n,1)$ and $(n,2)$ right trapezoids.

Conjecture 3.2 *For all $k \leqslant n$, the (n,k) left Gog and GOGAm trapezoids are equienumerated.*

In the next section we will give a bijective proof of this conjecture for $(n, 1)$ and $(n, 2)$ trapezoids.

Remark that the right and left Gog trapezoids of shape (n, k) are equienumerated (in fact a simple bijection between them was given above).

If we consider left GOGAm trapezoids as GOGAm triangles, using the canonical completion, then we can take their image by the Schützenberger involution and obtain a subset of the Magog triangles, for each (n, k). It seems however that this subset does not have a simple direct characterization. This shows that GOGAm triangles and trapezoids are a useful tool in the bijection problem between Gog and Magog triangles.

4 Bijections between Gog and GOGAm left trapezoids

4.1 $(n, 1)$ *left trapezoids*

The sets of $(n, 1)$ left Gog trapezoids and of $(n, 1)$ left GOGAm trapezoids coincide with the set of sequences $X_{n,1}, \ldots, X_{1,1}$ satisfying $X_{j,1} \leqslant n - j + 1$ (note that these sets are counted by Catalan numbers). Therefore the identity map provides a trivial bijection between these two sets.

4.2 $(n, 2)$ *left trapezoids*

In order to treat the $(n, 2)$ left trapezoids we will recall some definitions from [1].

4.2.1 *Inversions*

Definition 4.1 *An inversion in a Gelfand-Tsetlin triangle is a pair (i, j) such that $X_{i,j} = X_{i+1,j}$.*

For example, the Gog triangle in (18) contains three inversions, $(2, 2)$, $(3, 1)$, $(4, 1)$, the respective equalities being in red on this picture.

$$\begin{array}{ccccccccc}
1 & & 2 & & 3 & & 4 & & 5 \\
 & 1 & & 3 & & 4 & & 5 & \\
 & & 1 & & 4 & & 5 & & \\
 & & & 2 & & 4 & & & \\
 & & & & 3 & & & &
\end{array} \tag{18}$$

Definition 4.2 *Let $X = (X_{i,j})_{n \geqslant i \geqslant j \geqslant 1}$ be a Gog triangle and let (i, j) be such that $1 \leqslant j \leqslant i \leqslant n$.*

An inversion (k, l) covers (i, j) if $i = k + p$ and $j = l + p$ for some p with $1 \leqslant p \leqslant n - k$.

The entries (i, j) covered by an inversion are depicted with " + " on the following picture.

$$\begin{array}{ccccccccc}
\circ & & + & & + & & \circ & & + \\
 & \circ & & + & & \circ & & + & \\
 & & \circ & & \circ & & + & & \\
 & & & \circ & & \circ & & & \\
 & & & & \circ & & & &
\end{array} \tag{19}$$

4.2.2 Standard procedure

The basic idea for our bijection is that for any inversion in the Gog triangle we should subtract 1 to the entries covered by this inversion, scanning the inversions along the successive NW-SE diagonals, starting from the rightmost diagonal, and scanning each diagonals from NW to SE. We call this the *standard procedure*. This procedure does not always yield a Gelfand-Tsetlin triangle, but one can check that it does so if one starts from a Gog triangle corresponding to a permutation matrix in the correspondance between alternating sign matrices and Gog triangles. Actually the triangle obtained is also a GOGAm triangle.

Although we will not use it below, it is informative to make the following remark.

Proposition 4.3 *Let X be the canonical completion of a left (n,k) Gog trapezoid. The triangle Y obtained by applying the standard procedure to the $n-k+1$ rightmost NW-SE diagonals of X is a Gelfand-Tsetlin triangle such that $Y_{i+l,k+l} = X_{i,k}$ for $n-i \geqslant l \geqslant 1$.*

Proof: The Proposition is proved easily by induction on the number $n-k+1$. □

For example, applied to the $(5,2)$ left Gog trapezoid in (7), this yields

$$\begin{array}{ccccccccc} 1 & & 2 & & 3 & & 3 & & 4 \\ & 1 & & 3 & & 3 & & 4 & \\ & & 2 & & 3 & & 4 & & \\ & & & 2 & & 4 & & & \\ & & & & 4 & & & & \end{array} \tag{20}$$

Like in [1] the bijection between left Gog and GOGAm trapezoids will be obtained by a modification of the Standard Procedure.

4.2.3 Characterization of $(n,2)$ GOGAm trapezoids

The family of inequalities (12) simplifies in the case of $(n,2)$ GOGAm trapezoids, indeed if we identify such a trapezoid with its canonical completion, then most of the terms in the left hand side are zero, so that these inequalities reduce to

$$X_{i,2} \leqslant n-i+2 \tag{21}$$

$$X_{i,2} - X_{i-1,1} + X_{i,1} \leqslant n-i+1 \tag{22}$$

Remark that, since $-X_{i-1,1} + X_{i,1} \leqslant 0$, the inequality (22) follows from (21) unless $X_{i-1,1} = X_{i,1}$.

4.2.4 From Gog to GOGAm

Let X be a $(n,2)$ left Gog trapezoid. We shall construct a $(n,2)$ left GOGAm trapezoid Y by scanning the inversions in the leftmost NW-SE diagonal of X, starting from NW. Let us denote by $n > i_1 > \ldots > i_k \geqslant 1$ these inversions, so that $X_{i,1} = X_{i+1,1}$ if and only if $i \in \{i_1, \ldots, i_k\}$. We also put $i_0 = n$. We will construct a sequence of $(n,2)$ left Gelfand-Tsetlin trapezoids $X = Y^{(0)}, Y^{(1)}, Y^{(2)}, \ldots, Y^{(k)} = Y$.

Let us assume that we have constructed the trapezoids up to $Y^{(l)}$, that $Y^{(l)} \leqslant X$, that $Y_{ij}^{(l)} = X_{ij}$ for $i \leqslant i_l$, and that inequalities (21) and (22) are satisfied by $Y^{(l)}$ for $i \geqslant i_l + 1$. This is the case for $l = 0$.

Let m be the largest integer such that $Y_{m,2}^{(l)} = Y_{i_{l+1}+1,2}^{(l)}$. We put

$$\begin{array}{lcllll} Y_{i,1}^{(l+1)} & = & Y_{i,1}^{(l)} & \text{for} & n \geqslant i \geqslant m & \text{and} \quad i_{l+1} \geqslant i \\ Y_{i,1}^{(l+1)} & = & Y_{i+1,1}^{(l)} & \text{for} & m-1 \geqslant i > i_{l+1} & \\ Y_{i,2}^{(l+1)} & = & Y_{i,2}^{(l)} & \text{for} & n \geqslant i \geqslant m+1 & \text{and} \quad i_{l+1} \geqslant i \\ Y_{i,2}^{(l+1)} & = & Y_{i,2}^{(l)} - 1 & \text{for} & m \geqslant i \geqslant i_{l+1}+1. & \end{array}$$

From the definition of m, and the fact that X is a Gog trapezoid, we see that this new triangle is a Gelfand-Tsetlin triangle, that $Y^{(l+1)} \leqslant X$, and that $Y_{ij}^{(l+1)} = X_{ij}$ for $i \leqslant i_{l+1}$. Let us now check that the trapezoid $Y^{(l+1)}$ satisfies the inequalities (21) and (22) for $i \geqslant i_{l+1} + 1$. The first series of inequalities, for $i \geqslant i_{l+1}+1$, follow from the fact that $Y^{(l)} \leqslant X$. For the second series, they are satisfied for $i \geqslant m+1$ since this is the case for $Y^{(l)}$. For $m \geqslant i \geqslant i_{l+1} + 1$, observe that

$$Y_{i,2}^{(l+1)} - Y_{i+1,1}^{(l+1)} + Y_{i+1,1}^{(l+1)} \leqslant Y_{i,2}^{(l+1)} = Y_{m,2}^{(l+1)} = Y_{m,2}^{(k)} - 1 \leqslant n - m + 1$$

by (21) for $Y(l)$, from which (22) follows.

This proves that $Y^{(l+1)}$ again satisfies the induction hypothesis. Finally $Y = Y^{(k)}$ is a GOGAm triangle: indeed inequalities (21) follow again from $Y^{(l+1)} \leqslant X$, and (22) for $i \leqslant i_k$ follow from the fact that there are no inversions in this range. It follows that the above algorithm provides a map from $(n, 2)$ left Gog trapezoids to $(n, 2)$ left GOGAm trapezoids.Observe that the number of inversions in the leftmost diagonal of Y is the same as for X, but the positions of these inversions are not the same in general.

4.2.5 Inverse map

We now describe the inverse map, from GOGAm left trapezoids to Gog left trapezoids.

We start from an $(n, 2)$ GOGAm left trapezoid Y, and construct a sequence
$Y = Y^{(k)}, Y^{(k-1)}, Y^{(k-2)}, \ldots, Y^{(0)} = X$ of intermediate Gelfand-Tsetlin trapezoids.
Let $n - 1 \geqslant \iota_1 > \iota_2 \ldots > \iota_k \geqslant 1$ be the inversions of the leftmost diagonal of Y, and let $\iota_{k+1} = 0$. Assume that $Y^{(l)}$ has been constructed and that $Y_{ij}^{(l)} = Y_{ij}$ for $i - j \geqslant \iota_{l+1}$. This is the case for $l = k$.

Let p be the smallest integer such that $Y_{\iota_l+1,2}^{(l)} = Y_{p,2}^{(l)}$. We put

$$\begin{array}{lcllll} Y_{i,1}^{(l-1)} & = & Y_{i,1}^{(l)} & \text{for} & n \geqslant i \geqslant \iota_l + 1 & \text{and} \quad p \geqslant i \\ Y_{i,1}^{(l-1)} & = & Y_{i-1,1}^{(l)} & \text{for} & \iota_l \geqslant i \geqslant p & \\ Y_{i,2}^{(l-1)} & = & Y_{i,2}^{(l)} & \text{for} & n \geqslant i \geqslant \iota_l + 2 & \text{and} \quad p-1 \geqslant i \\ Y_{i,2}^{(l-1)} & = & X_{i,2}^{(l)} - 1 & \text{for} & \iota_l + 1 \geqslant i \geqslant p. & \end{array}$$

It is immediate to check that if X is an $(n, 2)$ left Gog trapezoid, and Y is its image by the first algorithm then the above algorithm applied to Y yields X back, actually the sequence $Y^{(l)}$ is the same. Therefore in order to prove the bijection we only need to show that if Y is a $(n, 2)$ left GOGAm trapezoid then the algorithm is well defined and X is a Gog left trapezoid. This is a bit cumbersome, but not difficult, and very similar to the opposite case, so we leave this task to the reader.

4.2.6 A statistic

Observe that in our bijection the value of the bottom entry $X_{1,1}$ is unchanged when we go from Gog to GOGAm trapezoids. The same was true of the bijection in [1] for right trapezoids. Actually we make the following conjecture, which extends Conjecture 3.2 above.

Conjecture 4.4 *For each n, k, l the (n, k) left Gog and GOGAm trapezoids with bottom entry $X_{1,1} = l$ are equienumerated.*

4.3 An example

In this section we work out an example of the algorithm from the Gog trapezoid X to the GOGAm trapezoid Y by showing the successive trapezoids $Y^{(k)}$. At each step we indicate the inversion in green, as well as the entry covered by this inversion in shaded green, and the values of the parameters i_l, p. The algorithm also runs backwards to yield the GOGAm→Gog bijection.

$$\begin{array}{cccccccc}
1 & & 2 & & & & & \\
 & 1 & & 2 & & & & \\
 & & 1 & & 4 & & & \\
 & & & 2 & & 4 & & \\
 & & & & 3 & & 4 & \\
 & & & & & 3 & & 4 \\
 & & & & & & 3 &
\end{array}$$

$X = Y^{(0)}$

$i_1 = 6$

$m = 7$

$$\begin{array}{cccccccc}
1 & & 1 & & & & & \\
 & 1 & & 2 & & & & \\
 & & 1 & & 4 & & & \\
 & & & 2 & & 4 & & \\
 & & & & 3 & & 4 & \\
 & & & & & 3 & & 4 \\
 & & & & & & 3 &
\end{array}$$

$Y^{(2)}$

$i_2 = 5$

$m = 6$

$$
\begin{array}{ccccccccc}
1 & & 1 & & & & & & Y^{(2)} \\
& 1 & & 1 & & & & & \\
& & 1 & & 4 & & & & i_3 = 2 \\
& & & 2 & & 4 & & & m = 5 \\
& & & & 3 & & 4 & & \\
& & & & & 3 & & 4 & \\
& & & & & & 3 & &
\end{array}
$$

$$
\begin{array}{ccccccccc}
1 & & 1 & & & & & & Y^{(3)} \\
& 1 & & 1 & & & & & \\
& & 1 & & 3 & & & & i_4 = 1 \\
& & & 1 & & 3 & & & m = 2 \\
& & & & 2 & & 3 & & \\
& & & & & 3 & & 4 & \\
& & & & & & 3 & &
\end{array}
$$

$$
\begin{array}{ccccccccc}
1 & & 1 & & & & & & \\
& 1 & & 1 & & & & & \\
& & 1 & & 3 & & & & \\
& & & 1 & & 3 & & & Y^{(4)} = Y \\
& & & & 2 & & 3 & & \\
& & & & & 3 & & 3 & \\
& & & & & & 3 & &
\end{array}
$$

4.4 Rectangles

Definition 4.5 *For (n, k, l) satisfying $k + l \leqslant n + 1$, a (n, k, l) Gog rectangle is an array of positive integers $X = (x_{i,j})_{n \geqslant i \geqslant j \geqslant 1; k \geqslant j; j+l \geqslant i+1}$ formed from the intersection of the k leftmost NW-SE diagonals and the l rightmost SW-NE diagonals of a Gog triangle of size n.*

Definition 4.6 *For (n, k, l) satisfying $k + l \leqslant n + 1$, a (n, k, l) GOGAm rectangle is an array of positive integers $X = (x_{i,j})_{n \geqslant i \geqslant j \geqslant 1; k \geqslant j; j+l \geqslant i+1}$ formed from the intersection of the k leftmost NW-SE diagonals and the l rightmost SW-NE diagonals of a GOGAm triangle of size n.*

Similarly to the case of trapezoids, one can check that a (n, k, l) Gog (resp. GOGAm) rectangle has a canonical (i.e. minimal) completion as a (n, k) left trapezoid, or as a (n, l) right trapezoid, and finally as a triangle of size n.

Conjecture 4.7 *For any (n, k, l) satisfying $k + l \leqslant n + 1$ the (n, k, l) Gog and GOGAm rectangles are equienumerated.*

As in the case of trapezoids, there is also a refined version of the conjecture with the statistic X_{11} preserved.

One can check, using standard completions, that our bijections, in [1] and in the present paper, restrict to bijections for rectangles of size $(n, k, 2)$ or $(n, 2, l)$. Furthermore, in the case of $(n, 2, 2)$ rectangles, the bijections coming from left and right trapezoids are the same.

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This work is based on computer exploration and the authors use the open-source mathematical software Sage [6] and one of its extention, Sage-Combinat [5]

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The explicit molecular expansion of the combinatorial logarithm

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Abstract. Just as the power series of $\log(1+X)$ is the analytical substitutional inverse of the series of $\exp(X)-1$, the (virtual) combinatorial species, $\mathrm{Lg}(1+X)$, is the combinatorial substitutional inverse of the combinatorial species, $E(X)-1$, of non-empty finite sets. This *combinatorial logarithm*, $\mathrm{Lg}(1+X)$, has been introduced by A. Joyal in 1986 by making use of an iterative scheme. Given a species $F(X)$ (with $F(0)=1$), one of its main applications is to express the species, $F^{\mathrm{c}}(X)$, of *connected* F-structures through the formula $F^{\mathrm{c}} = \mathrm{Lg}(F) = \mathrm{Lg}(1+F_+)$ where F_+ denotes the species of non-empty F-structures. Since its creation, equivalent descriptions of the combinatorial logarithm have been given by other combinatorialists (G. L., I. Gessel, J. Li), but its exact decomposition into irreducible components (molecular expansion) remained unclear. The main goal of the present work is to fill this gap by computing explicitly the molecular expansion of the combinatorial logarithm and of $-\mathrm{Lg}(1-X)$, a " cousin " of the tensorial species, $\mathrm{Lie}(X)$, of free Lie algebras.

Résumé. Tout comme la série de puissances de $\log(1+X)$ est l'inverse substitutionnel analytique de la série de $\exp(X)-1$, l'espèce de structures (virtuelle) $\mathrm{Lg}(1+X)$, est l'inverse substitutionnel combinatoire de l'espèce, $E(X)-1$, des ensembles finis non vides. Ce *logarithme combinatoire*, $\mathrm{Lg}(1+X)$, a été introduit par A. Joyal en 1986 en faisant appel à un schéma itératif. Étant donnée une espèce $F(X)$ (telle que $F(0)=1$), l'une de ses principales applications est d'exprimer l'espèce, $F^{\mathrm{c}}(X)$, des F-structures *connexes* par la formule $F^{\mathrm{c}} = \mathrm{Lg}(F) = \mathrm{Lg}(1+F_+)$ où F_+ désigne l'espèce des F-structures non vides. Depuis sa création, des descriptions équivalentes du logarithme combinatoire ont été formulées par d'autres combinatoriciens (G. L., I, Gessel, J. Li), mais sa décomposition exacte en composantes irréductibles (développement moléculaire) est demeurée obscure. Le but principal du présent travail est de combler cette lacune en calculant explicitement le développement moléculaire du logarithme combinatoire et de $-\mathrm{Lg}(1-X)$, un " cousin " de l'espèce tensorielle, $\mathrm{Lie}(X)$, des algèbres de Lie libres.

Keywords: combinatorial species, combinatorial logarithm, molecular expansion, generating functions

1 Introduction

1.1 Counting connected structures

Since simple graphs are assemblies of connected simple graphs, it is well known that the exponential generating series, $G(x)$, which counts simple graphs, satisfies $G(x)=\exp(G^{\mathrm{c}}(x))$, where $G^{\mathrm{c}}(x)$ is the exponential generating series of connected simple graphs. Now, taking the (analytical) logarithm of both sides of this equation gives,

$$G^{\mathrm{c}}(x) = \log(G(x)) = \log \sum_{n \geq 0} 2^{n(n-1)/2} x^n / n!, \tag{1.1}$$

from which connected simple graphs can be counted exactly, recursively or asymptotically. More generally, the analogous formula,

$$F^{\mathrm{c}}(x) = \log(F(x)), \tag{1.2}$$

holds for the exponential generating series of any species of structures, F and F^{c}, for which F-structures are assemblies of F^{c}-structures. That is, for which the combinatorial equation,

$$F(X) = E \circ F^{\mathrm{c}}(X), \tag{1.3}$$

holds, where $\circ$ denotes the substitution of species, E is the species of finite sets (E stands for *ensembles*, in French) and X is the species of singletons (that is, one-element sets). Taking the cycle index series of both members of (1.3) yields, $Z_F = Z_E \circ Z_{F^{\mathrm{c}}}$, where $\circ$ now denotes the classical plethystic substitution. Since $Z_E = \exp \sum \frac{1}{k} p_k$, this can be written explicitly as[(i)]

$$Z_F(p_1, p_2, p_3, \dots) = \exp \sum_{k \geq 1} \frac{1}{k} Z_{F^{\mathrm{c}}}(p_k, p_{2k}, p_{3k}, \cdots). \tag{1.4}$$

Taking the logarithm of both sides of (1.4) and using Möbius inversion gives (see [BLL98]) the following refinement of (1.2),

$$Z_{F^{\mathrm{c}}}(p_1, p_2, p_3, \dots) = \sum_{k \geq 1} \frac{\mu(k)}{k} \log Z_F(p_k, p_{2k}, p_{3k}, \cdots), \tag{1.5}$$

where μ denotes the classical Möbius function. As a consequence, the (ordinary) generating series, $\widetilde{F^{\mathrm{c}}}(x)$, which counts *unlabeled* F^{c}-structures is obtained via the substitutions, $p_i := x^i, i = 1, 2, \dots$, in (1.5). All of this is classical in Pólya theory in the context of combinatorial species. For an introduction to species, the reader can consult the basic paper of A. Joyal [Joy81] or the book [BLL98] by Bergeron, Labelle, and Leroux.

1.2 *Solving the combinatorial equation* $F = E \circ F^{\mathrm{c}}$ *for the species* F^{c}

In 1986, Joyal [Joy86] went a step further by solving the combinatorial equation (1.3) for the species F^{c} in terms of the species F, thereby refining simultaneously both (1.2) and (1.5). He proceeded along the following lines. Let 1 denote the species of *the* empty set. We have $E = 1 + E_+$, where E_+ is the species of non empty finite sets and $F = 1 + F_+$, where F_+ is the species of F-structures on non empty finite sets[(ii)]. Combinatorial equation (1.3) is then equivalent to

$$F_+(X) = E_+ \circ F^{\mathrm{c}}(X). \tag{1.6}$$

[(i)] The variables p_k stand for the power sums and are often denoted by x_k in the theory of combinatorial species.
[(ii)] Since an empty assembly of F^{c}-structures *is* the empty set.

Now, $E_+ = X + E_{\geq 2}$, where $E_{\geq 2}$ is the species of finite sets having at least 2 elements. By his implicit species theorem (see [Joy81]), Joyal concluded that E_+ has a substitutional inverse, $E_+^{<-1>}$, in the realm of *virtual species* (that is, formal differences of species). By adapting the Newton interpolation formula to species (using a special " difference operator " δ), he also gave the following combinatorial formula for this substitutional inverse,

$$E_+^{<-1>}(X) = \sum_{n\geq 0}(-1)^n W_n(X) = \sum_{n:\,\mathrm{even}} W_n(X) - \sum_{n:\,\mathrm{odd}} W_n(X), \tag{1.7}$$

where $W_n = W_n(X)$ are species defined by the recursive scheme,

$$W_0 = X, \qquad W_n = \delta W_{n-1} = W_{n-1} \circ E_+ - W_{n-1}, \quad n \geq 1. \tag{1.8}$$

Hence, a W_n-structure, on a finite set U, is a strictly increasing sequence, $\hat{0} = R_0 < R_1 < \cdots < R_n = \hat{1}$, in the lattice of equivalence relations on U, where $\hat{0}$ and $\hat{1}$ respectively denote the finest and the coarsest equivalence relation on U. Applying $E_+^{<-1>}$ to (1.6) and using (1.7) finally gives,

$$F^{\mathrm{c}}(X) = E_+^{<-1>} \circ F_+(X) = \sum_{n\geq 0}(-1)^n W_n(F_+(X)). \tag{1.9}$$

In the present paper, we use the notation, $\mathrm{Lg}(1+X)$, to denote $E_+^{<-1>}(X)$ and call it the *combinatorial logarithm*[(iii)], by analogy with the fact that, in analysis, the power series of $\log(1+X)$ is the substitutional inverse of that of $\exp(X) - 1$. Summarizing, we have,

$$\mathrm{Lg}(1+X) \underset{\mathrm{def}}{=} E_+^{<-1>}(X) \quad \text{and} \quad F^{\mathrm{c}} = \mathrm{Lg}(1+F_+) = \mathrm{Lg}(F). \tag{1.10}$$

Note that (1.10) associates a virtual species F^{c} to *any* species F for which $F(0) = 1$, even in the case where F does not possess connected structures. For this reason, $\mathrm{Lg}(1+X) = (1+X)^{\mathrm{c}}$ is sometimes called, by abuse of language, the virtual species of " connected " $(1+X)$-structures. Although very useful and conceptually elegant, it turns out that the species $\sum_{n:\,\mathrm{even}} W_n$ and $\sum_{n:\,\mathrm{odd}} W_n$ in Joyal's expression for the combinatorial logarithm have plenty of subspecies in common. That is,

$$\mathrm{Lg}(1+X) = \sum_{n:\,\mathrm{even}} W_n - \sum_{n:\,\mathrm{odd}} W_n, \tag{1.11}$$

is *not* a completely reduced expression as a difference of species. Other equivalent $-$ but still *not* completely reduced $-$ expressions for the combinatorial logarithm have been given using special classes of graphs. For example, Gessel and Li in [GL11], found formula (1.12a), where $\mathcal{Q}^{\mathrm{c}}$ is the species of *connected co-point-determining graphs* and $\mathcal{P}^{\mathrm{c}}_{\geq 2}$ is that of *connected point-determining graphs having at least two vertices*. Later, J. Li [Li12] found the further reduced formula (1.12b), where $\mathcal{T}^{\mathrm{c}}$ is the species of *connected co-point-determining cographs* and $\mathcal{S}^{\mathrm{c}}_{\geq 2}$ is that of *connected point-determining cographs having at least two vertices*.

$$a) \quad \mathrm{Lg}(1+X) = \mathcal{Q}^{\mathrm{c}} - \mathcal{P}^{\mathrm{c}}_{\geq 2}, \qquad b) \quad \mathrm{Lg}(1+X) = \mathcal{T}^{\mathrm{c}} - \mathcal{S}^{\mathrm{c}}_{\geq 2}. \tag{1.12}$$

Our main goal is to give a completely reduced expression for the combinatorial logarithm. In Section 2, we describe the irreducible components of $F^{\mathrm{c}} = \mathrm{Lg}(F)$ and, in particular, of $\mathrm{Lg}(1+X)$ and $-\mathrm{Lg}(1-X)$, together with their exact multiplicities. Section 3 contains a compact table for the combinatorial logarithm up to degree 10.

(iii) Joyal uses the notation $\log(1+X)$ for the combinatorial logarithm, but we use it for the *analytical* logarithm in the present text.

2 Explicit molecular expansions

2.1 Molecular expansions in general

We first recall the general notions of molecular and atomic species. A species M is *molecular* if $M \neq 0$ and any two M-structures are isomorphic. Equivalently, M is irreducible under the combinatorial sum. A molecular species A is *atomic* if $A \neq 1$ and is irreducible over the combinatorial product. Y. N. Yeh proved in [Yeh86] that every molecular species can be written in a unique way (up to isomorphism) as a commutative finite product of atomic species. The sets $\mathbf{M}$ of all molecular species and $\mathbf{A}$ of all atomic species (up to isomorphism) are countable and we have, up to degree three,

$$\mathbf{M} = \{1, X, E_2, X^2, E_3, C_3, XE_2, X^3, \ldots\}, \quad \mathbf{A} = \{X, E_2, E_3, C_3, \ldots\}, \tag{2.1}$$

where X^n is the species of linear orderings of length n, C_n is the species of oriented n-cycles, and E_n is the species of n-sets. Note that $\mathbf{M}$ is the free commutative monoid (under combinatorial product) generated by $\mathbf{A}$. Moreover, each molecular species, M, is completely determined by the stabilizer $H = \mathrm{Stab}(s) \leq S_n$ of anyone of its structures, say s on $[n]$, where n is the degree of M and $[n] = \{1, 2, \ldots, n\}$. We write $M = X^n/H =$ linear orderings of length n modulo H. In particular, we have $X^n = X^n/\{1\}, \quad E_n = X^n/S_n, \quad C_n = X^n/\langle(1\,2\,\ldots\,n)\rangle$. Two molecular species, X^n/H and X^m/K, are isomorphic (and we write, $X^n/H = X^m/K$) if and only if $n = m$ and H and K are conjugate in S_n. Let now F be any species, not necessarily molecular. Then, one can always write F as a (countable) linear combination with nonnegative integer coefficients of molecular species,

$$F = \sum_{M \in \mathbf{M}} f_M M \in \mathbb{N}[[\mathbf{A}]], \tag{2.2}$$

where f_M denotes the number of subspecies of F that are isomorphic to M. The coefficient f_M is called the *multiplicity* of M in F. Summation (2.2) is unique and is called the *molecular expansion* of F. This expansion is very strong since it is a common refinement of the classical generating series, $F(x), \widetilde{F}(x), Z_F(p_1, p_2, p_3, \ldots)$, associated to the species F. For an example of molecular expansion, consider the well-known species $T = T(X)$ of rooted trees, defined by the combinatorial functional equation $T = XE(T)$. Up to degree 6, we have (see the book [BLL98] by Bergeron, Labelle, and Leroux, for example),

$$\begin{aligned} T = {} & X + X^2 + XE_2 + X^3 + XE_3 + 2X^4 + X^2E_2 + XE_4 + 3X^3E_2 + X\cdot(E_2 \circ X^2) + 3X^5 \\ & + X^2E_3 + X^2E_4 + 6X^4E_2 + 2X^2\cdot(E_2 \circ X^2) + 3X^3E_3 + X^2{E_2}^2 + XE_5 + 6X^6 + \cdots . \end{aligned} \tag{2.3}$$

Note that the species, $E_2 \circ X^2$, which occurs in this expansion is atomic. The usual combinatorial operations (sum, product, composition, differentiation, etc), as well as molecular expansions, have been extended by Joyal and Yeh ([Joy85], [Yeh86]) to *virtual species*, that is formal differences

$$\Phi = F - G, \tag{2.4}$$

of (ordinary) species F and G. The molecular expansion of Φ is defined by,

$$\Phi = \sum_{M \in \mathbf{M}} \phi_M M = \sum_{M \in \mathbf{M}} (f_M - g_M) M \in \mathbb{Z}[[\mathbf{A}]], \tag{2.5}$$

where $\sum_{M\in\mathbf{M}} f_M M$ and $\sum_{M\in\mathbf{M}} g_M M$ are the molecular expansions of F and G, respectively[(iv)]. Every virtual species, Φ, can be represented in the form (2.4) in an infinite number of way, just as $-5 = 0-5 = 1-6 = 2-7 = \cdots$, in the context of the ring, $\mathbb{Z}$, of integers. The less F and G have subspecies in common, the more representation (2.5) is said to be *reduced*. It is always possible to canonically choose F and G in such a way that they have *no* subspecies in common (i.e., are " stranger " species, to use a terminology taken from the theory of signed measures). The corresponding representation is denoted,

$$\Phi = \Phi^+ - \Phi^-, \tag{2.6}$$

and is called the (*completely*) *reduced form* of the virtual species Φ. The species Φ^+ (resp., Φ^-) is called the *positive* (resp., *negative*) part of Φ and the coefficients ϕ_M in (2.5) satisfy $\phi_M > 0$ if M appears in Φ^+ and $\phi_M < 0$ if M appears in Φ^- (otherwise, $\phi_M = 0$). Note that Φ^+ and Φ^- are characterized by the fact that no molecular species appears in both of their molecular expansions.

2.2 The explicit expansions of $\mathrm{Lg}(1+X)$ and of $-\mathrm{Lg}(1-X)$

Let us start by taking a closer look at the Joyal species, W_n, defined by (1.8). It is easy to see that,

$$W_n = \sum_{0\le k\le n} (-1)^k \binom{n}{k} E_+^{<n-k>}, \tag{2.7}$$

where $E_+^{<i>}$ denotes the i-fold iterate of E_+ under $\circ$ (with $E_+^{<0>} = X$). Using the expansion formulas,

$$E_+ = E - 1 = X + E_2 + E_3 + \cdots, \quad E(mA + nB + \cdots) = E(A)^m E(B)^n \cdots, \tag{2.8}$$

massive simplification and cancellation of terms occur in (2.7) and (1.11), and we have, up to degree 6, the molecular expansions,

$$\begin{aligned}\mathrm{Lg}(1+X)^+ = {} & X + XE_2 + XE_3 + E_2\circ E_2 + X^3E_2 + XE_4 + E_2E_3 + X^3E_3 \\ & + 2\,X^2E_2^2 + XE_5 + E_2E_4 + E_3\circ E_2 + E_2\circ E_3 + \cdots,\end{aligned} \tag{2.9}$$

$$\begin{aligned}\mathrm{Lg}(1+X)^- = {} & E_2 + E_3 + X^2E_2 + E_4 + X^2E_3 + XE_2^2 + E_5 + X^4E_2 + X^2E_4 \\ & + 2\,XE_2E_3 + E_2\cdot(E_2\circ E_2) + E_6 + E_2\circ(XE_2) + \cdots,\end{aligned} \tag{2.10}$$

Of course, the molecular species that appear in $\mathrm{Lg}(1+X)^+$ and $\mathrm{Lg}(1+X)^-$ are all *set-like* (that is, they are build from the E_n's using only products and substitutions), but their exact nature and multiplicities are far from being obvious. In fact, these " surviving " molecular species are *strictly* included in the class of set-like molecular species, since, for example, the set-like molecular species, $X\cdot(E_2\circ E_2)$, neither appears in (2.9) nor in (2.10). We will exhibit their exact form and describe their multiplicities explicitly, using special kinds of integer partitions together with an arithmetical function related to the Möbius function. To do so, we will need to work in the more general setting, $\mathbb{Q}[[\mathbf{A}]]$, of *rational species*, that is, of countably summable linear combinations of molecular species with rational coefficients[(v)]. All usual combinatorial

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[(iv)] As a ring, $\mathbb{Z}[[\mathbf{A}]]$ is the completion (under countable summability) of the family, $\mathbf{B}(S_n)_{n\ge 0}$, of the Burnside rings of virtual set-like representations of the symmetric groups, $S_n, n \ge 0$.

[(v)] More general settings are also possible, for example, $\mathbb{C}[[\mathbf{A}]]$, but $\mathbb{Q}[[\mathbf{A}]]$ is sufficient here.

operations have been extended to this ring by A. Joyal in [Joy85] and Y.-N. Yeh in [Yeh86]. Since the classical ring, $\mathbb{Q}[[X]]$, of power series in X " sits " in $\mathbb{Q}[[\mathbf{A}]]$, both kinds (analytical and combinatorial) of exponentials and logarithms are special cases of rational species, with molecular expansions:

$$\begin{aligned} \exp(X) &= \sum_{n\geq 0} \frac{1}{n!} X^n \in \mathbb{Q}[[\mathbf{A}]], \quad \log(1+X) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} X^n \in \mathbb{Q}[[\mathbf{A}]]. \\ E(X) &= \sum_{n\geq 0} E_n(X) \in \mathbb{N}[[\mathbf{A}]], \quad \mathrm{Lg}(1+X) = \sum_{M} \omega_M M(X) \in \mathbb{Z}[[\mathbf{A}]], \end{aligned} \tag{2.11}$$

where the coefficients $\omega_M \in \mathbb{Z}$ are to be determined explicitely. We need to introduce some preliminary definitions, lemmas and notation. For technical reasons, in this paper, a partition of an integer $n \geq 0$ will be a weakly *increasing*(vi) sequence $\lambda = (\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_r)$ of positive integers such that $\lambda_1 + \lambda_2 + \cdots + \lambda_r = n$. We write $\lambda \vdash n$. The number λ_i is called the i-th *part* of λ and $r = \#\lambda$, is the number of parts of λ. As usual, $m_j = m_j(\lambda), j = 1, \ldots, n$ denotes the *multiplicity* of part j in λ; that is, $m_j(\lambda) = \mathrm{card}\{i : \lambda_i = j\}$. The expression $1^{m_1}2^{m_2}3^{m_3}\cdots n^{m_n}$ is called the *type* of λ and $n = |\lambda|$ is called the *size* of λ. It turns out to be useful to freely use the abuse of notation of identifying a partition with its type; so that,

$$\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r) = 1^{m_1}2^{m_2}\cdots n^{m_n}, \quad m_j = m_j(\lambda), \quad j = 1, \ldots, n. \tag{2.12}$$

Moreover, if $\lambda \vdash n$ and $d \geq 1$, we denote by λ^k the partition of kn defined by

$$\lambda^k = 1^{km_1}2^{km_2}\cdots n^{km_n} \vdash kn, \quad m_j = m_j(\lambda), \quad j = 1, \ldots, n. \tag{2.13}$$

So that $\lambda^k \vdash kn$ have the same parts as λ, but each individual part of λ occurs k times in λ^k. For example, $(1,1,1,4,4,4,4,4,4,5,5,5) = (1,4,4,5)^3 \vdash 42$ and $(2,4,4,7) = (2,4,4,7)^1 \vdash 17$.

Definition 2.1 *A partition $\lambda \vdash n$ is called,*
• primary*, if* $\gcd(m_1(\lambda), m_2(\lambda), \ldots, m_n(\lambda)) = 1$; • fat*, if it has a part $\lambda_i > 1$. Equivalently, $\lambda \neq 1^n$;*
• non-repeating*, if $\lambda \neq m^k$ with $m \geq 1, k > 1$.*

Lemma 2.2 *Every non-empty partition λ can be written in the form $\lambda = \tau^k$, where τ is a primary partition. Moreover, λ is fat (resp., non-repeating) if and only if τ is fat (resp., non-repeating).* □

Borrowing notational conventions from the theory of symmetric functions, we now associate a set-like molecular species, $E_\lambda = E_\lambda(X)$, of degree n, to every partition $\lambda \vdash n$, by the combinatorial products,

$$E_\lambda \underset{\mathrm{def}}{=} E_{\lambda_1}E_{\lambda_2}\cdots E_{\lambda_k} = X^{m_1}E_2^{m_2}\cdots E_n^{m_n}, \quad (E_1 = X). \tag{2.14}$$

Note that, $E_\lambda = X^n/S_\lambda = X^n/S_{\lambda_1,\lambda_2,\ldots,\lambda_k}$, where S_λ denotes the Young subgroup of S_n of type λ. Also, if $\alpha \vdash m, \beta \vdash n$, then $E_\alpha \circ E_\beta = X^m/S_\alpha \circ X^n/S_\beta = X^{mn}/S_\alpha \wr S_\beta$, where $\wr$ is the wreath product.

Lemma 2.3 *Every molecular species, $M \neq 1$, can be written canonically in exactly one of the two forms,*

$$M = P^k \quad \textit{or} \quad M = (E_\tau \circ Q)^k, \tag{2.15}$$

where τ is a primary fat partition, P, Q are molecular and $k \geq 1$.

(vi) Instead of weakly *decreasing*, contrarily to the usual practice in the theory of partitions.

Proof: Let $M = A_1^{\alpha_1} A_2^{\alpha_2} \cdots$ be the atomic factorization of M, take $k = \gcd(\alpha_1, \alpha_2, \ldots)$ and consider the molecular species $P = A_1^{\alpha_1/k} A_2^{\alpha_2/k} \cdots$. Clearly, we canonically have $M = P^k$. Now, if P is not of the form $E_\tau \circ Q$, with $|\tau| > 1$, then we are done. On the contrary, if $P = E_\tau \circ Q$, with $|\tau| > 1$, then τ must be fat since if $\tau = 1^s, s > 1$, then $M = P^k = (E_{1^s} \circ Q)^k = (Q^s)^k = Q^{sk}$, contradicting the fact that $k = \gcd(\alpha_1, \alpha_2, \ldots)$, since $sk > k$. Moreover, τ must be primary, since if $\tau = \phi^s, s > 1$, then $M = P^k = (E_{\phi^s} \circ Q)^k = ((E_\phi \circ Q)^s)^k = (E_\phi \circ Q)^{sk}$, which is again a contradiction. $\square$

Finally, we need the following special function π defined on the set, $\mathbb{P}^+$, of non empty partitions and the " Möbius-like " arithmetical function, ν, defined on the set, $\mathbb{N}^+$, of positive integers.

Definition 2.4 *The function, $\pi : \mathbb{P}^+ \to \mathbb{Q}$, is defined, for $\lambda = 1^{m_1} 2^{m_2} \cdots n^{m_n} \vdash n$ by,*

$$\pi(\lambda) = \frac{(-1)^{\#\lambda - 1}}{\#\lambda} \binom{\#\lambda}{m_1, \ldots, m_n}. \tag{2.16}$$

In particular, for $\lambda = 1^n$, $\pi(1^n)$ reduces to the number-theoretic multiplicative function,

$$\theta : \mathbb{N}^+ \to \mathbb{Q},\ n \mapsto \frac{(-1)^{n-1}}{n}. \tag{2.17}$$

Definition 2.5 *The function, $\nu : \mathbb{N}^+ \to \mathbb{Q}$, is the inverse of θ under the Dirichlet $(\star)$ convolution*[vii]*. Explicitly,*

$$\nu(n) = \begin{cases} \frac{1}{2}\mu(i)/i, & if\ \ n = 2^k i,\ i\ odd,\ k \geq 1, \\ \mu(i)/i, & otherwise. \end{cases} \tag{2.18}$$

We are now ready to state and prove our main result from which each individual coefficient of the molecular expansion of the *combinatorial* logarithm, $\mathrm{Lg}(F)$, of a species, F, can be computed from the coefficients of the molecular expansion of its *analytical* logarithm, $\log(F)$. This analytical logarithm is very easy to expand, since we have, in view of (2.11),

$$\log(F) = \log(1 + F_+) = \sum_{n \geq 1} \frac{(-1)^n}{n} (F_+)^n. \tag{2.19}$$

The expansions of $\mathrm{Lg}(1 + X)$ and $-\mathrm{Lg}(1 - X)$ will then follow as special cases (Corollaries 2.7 – 2.8).

Theorem 2.6 *Consider a species, $F = 1 + F_+$, with molecular expansion $F = 1 + \sum_{M \neq 1} f_M M$, together with the molecular expansions of its two kinds of logarithms,*

$$\mathrm{Lg}(F) = \sum_{M \neq 1} g_M M, \qquad \log(F) = \sum_{M \neq 1} h_M M. \tag{2.20}$$

Then the coefficients g_M can be computed from the coefficients h_M via the recursive scheme,

$$g_M = h_M - \sum_{\substack{E_\lambda \circ N = M \\ |\lambda| > 1}} \pi(\lambda) g_N. \tag{2.21}$$

(vii) $(f \star g)(n) = \sum_{d|n} f(d) g(n/d)$.

More precisely, if M is written in the canonical form (2.15)*, then*

$$g_M = \begin{cases} \nu(k) \star h_{P^k}, & if\ M = P^k, \\ \nu(k) \star \left(h_{(E_\tau \circ Q)^k} - \pi(\tau^k) g_Q\right), & if\ M = (E_\tau \circ Q)^k, \end{cases} \tag{2.22}$$

where $(\star)$ *denotes Dirichlet convolution.*

Proof: Consider the special rational species, $\widehat{X}$, of *pseudo-singletons*, that we introduced in [Lab90] as the *analytical* logarithm of the species, E, of finite sets. Expanding, we have explicitly (see [Lab08], for more detail),

$$\begin{aligned} \widehat{X} \underset{\text{def}}{=} \log(E) = \log(1 + E_+) &= \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} (E_1 + E_2 + E_3 + \cdots)^k \\ &= P_1 + \frac{1}{2} P_2 + \frac{1}{3} P_3 + \cdots + \frac{1}{n} P_n + \cdots \in \mathbb{Q}[[\mathbf{A}]], \end{aligned} \tag{2.23}$$

where $P_n = P_n(X)$ are *virtual* species that are " combinatorial liftings " of the classical power sums symmetric functions[(viii)], p_n, and can be computed by the " Newton like " combinatorial recursive scheme,

$$P_1 = X, \qquad P_n = nE_n - E_1 P_{n-1} - E_2 P_{n-2} - \cdots - E_{n-1} P_1, \quad n \geq 2. \tag{2.24}$$

Taking *analytical* exponential, $\exp$, of (2.23) gives alternate expressions for the species of finite sets:

$$E = \exp(\widehat{X}) = e^{\widehat{X}} = \exp\left(\sum_{n \geq 1} \frac{1}{n} P_n\right). \tag{2.25}$$

Expanding (2.23) and using (2.16), we get the molecular expansion,

$$\frac{1}{n} P_n = \sum_{\lambda \vdash n} \pi(\lambda) E_\lambda. \tag{2.26}$$

Now, a basic property of P_n is that it behaves *linearly* (see [Lab08]) under substitution[(ix)]:

$$P_n \circ (aA + bB + \cdots) = aP_n \circ A + bP_n \circ B + \cdots, \quad a, b, \ldots \in \mathbb{Q}, \quad A, B, \ldots \in \mathbb{Q}[[\mathbf{A}]]. \tag{2.27}$$

Of course, such a linear behavior is *far* from being true in general. In particular, it is far from being true for E_λ. Nevertheless, thanks to $(2.26) - (2.27)$, we have,

$$\left(\sum_{\lambda \vdash n} \pi(\lambda) E_\lambda\right) \circ (aA + bB + \cdots) = \sum_{\lambda \vdash n} \pi(\lambda)(aE_\lambda \circ A + bE_\lambda \circ B + \cdots). \tag{2.28}$$

(viii) In fact, at the level of linear representations, the cycle index series of P_n is p_n. That is, $Z_{P_n} = p_n$.

(ix) They even behave *plethystic linearly* under substitution: for weight variables, $s, t, \ldots$, we have, $P_n(asF + btG + \cdots) = as^n P_n(F) + bt^n P_n(G) + \cdots$.

This last equation is *crucial* in the next steps of the present proof. By (2.25), we have,

$$\exp\left(\left(\sum_n \frac{1}{k}P_k\right)\circ\left(\sum_{N\neq 1} g_N N\right)\right) = E(\sum_{N\neq 1} g_N N) = E(\mathrm{Lg}(F)) = F = 1 + \sum_{M\neq 1} f_M M. \quad (2.29)$$

Taking log, using (2.26) and linearity property (2.28), gives,

$$\sum_{\substack{k,N\\ \lambda\vdash k}} \pi(\lambda) g_N E_\lambda \circ N = \log(F) = \sum_{M\neq 1} h_M M. \quad (2.30)$$

Extracting the coefficient of M on the leftmost and rightmost sides of (2.30), we can write,

$$\sum_{E_\lambda\circ N = M} \pi(\lambda) g_N = h_M, \quad (2.31)$$

which is equivalent to the recursive scheme (2.21), since $\pi(1) = 1$ and $E_1 \circ N = X \circ N = N$. Finally, consider the canonical form (2.15) of M. If $M = P^k$, then $E_\lambda \circ N = M$, with $|\lambda| > 1$, if and only if $\lambda = 1^d, N = P^{k/d}$, with $1 < d|k$. So that, (2.21) takes the form,

$$g_{P^k} = h_{P^k} - \sum_{1<d|k} \pi(1^d) g_{P^{k/d}} = h_{P^k} + \pi(1) g_{P^k} - \theta(k) \star g_{P^k}, \quad (2.32)$$

which reduces to $\theta(k) \star g_{P^k} = h_{P^k}$. This is equivalent to $g_{P^k} = \nu(k) \star h_{P^k}$. On the other hand, if $M = (E_\tau \circ Q)^k$, where τ is a primary fat partition, then $E_\lambda \circ N = M$, with $|\lambda| > 1$, if and only if $\lambda = 1^d, N = (E_\tau \circ Q)^{k/d}$, with $1 < d|k$, **or** $\lambda = \tau^k, N = Q$. This time, (2.21) takes the form,

$$\begin{aligned} g_{(E_\tau\circ Q)^k} &= h_{(E_\tau\circ Q)^k} - \pi(\tau^k) g_Q - \sum_{1<d|k} \pi(1^d) g_{(E_\tau\circ Q)^{k/d}} \\ &= h_{(E_\tau\circ Q)^k} - \pi(\tau^k) g_Q + g_{(E_\tau\circ Q)^k} - \theta(k) \star g_{(E_\tau\circ Q)^k}. \end{aligned} \quad (2.33)$$

This reduces to $\theta(k) \star g_{(E_\tau\circ Q)^k} = h_{(E_\tau\circ Q)^k} - \pi(\tau^k) g_Q$, which proves (2.22). □

Corollary 2.7 *The molecular expansion of the combinatorial logarithm is explicitly given by,*

$$\mathrm{Lg}(1+X) = \sum_M \omega_M M, \quad (2.34)$$

where each molecular component, M, is of the form of a finite composition,

$$M = E_{\phi^{(1)}} \circ E_{\phi^{(2)}} \circ \cdots \circ E_{\phi^{(s)}}, \quad s \geq 0, \quad (2.35)$$

in which each $\phi^{(i)}$ is a non-repeating fat *partition. The coefficients, $\omega_M \in \mathbb{Z}\backslash\{0\}$, and their sign, $\mathrm{sgn}(\omega_M)$, are given by,*

$$\omega_M = c(\phi^{(1)}) \cdots c(\phi^{(s)}), \qquad \mathrm{sgn}(\omega_M) = (-1)^{\#\phi^{(1)} + \cdots + \#\phi^{(s)}}, \quad (2.36)$$

where each factor, for non-repeating fat $\phi = \tau^k$ with primary non-repeating fat τ, is given by,

$$c(\phi) = c(\tau^k) = -\nu(k) \star \pi(\tau^k) = -\sum_{d|k} \nu(k/d) \pi(\tau^d). \quad (2.37)$$

Proof: Take $F = 1 + X$ in Theorem 2.6. Then $g_M = \omega_M$ and, from (2.11), $h_M = \theta(k) = \frac{(-1)^{k-1}}{k}$, if $M = X^k$ and 0, otherwise. Hence, by (2.22), $\omega_{X^k} = \nu(k) \star \theta(k) = 1$, if $k = 1$; 0, otherwise. If $M = (E_\tau \circ Q)^k = E_{\tau^k} \circ Q$, with τ primary fat, then, by (2.22), $\omega_{E_{\tau^k} \circ Q} = \nu(k) \star (0 - \pi(\tau^k)\omega_Q) = -\nu(k) \star \pi(\tau^k)\omega_Q = c(\tau^k)\omega_Q$. In particular, if $\tau^k = m^k$, with $1 < m \in \mathbb{N}$, then $\omega_{E_{m^k} \circ Q} = \nu(k) \star (0 - \pi(m^k)\omega_Q) = -\nu(k) \star \theta(k)\omega_Q = -\omega_Q$ if $k = 1$; 0, otherwise. Summarizing, let $\phi = \tau^k$, be fat, then, $\omega_{E_\phi \circ Q} = c(\phi)\omega_Q$, where $c(\phi)$ is defined by (2.37). Moreover, if $\phi = m^k$, with $k > 1$, i.e., when ϕ is repeating, then $\omega_{E_\phi \circ Q} = 0$. This means that the molecular species that can contribute to $\mathrm{Lg}(1 + X)$ are of the form X or of the form $E_\phi \circ Q$, where ϕ is non-repeating fat and Q also contribute to $\mathrm{Lg}(1 + X)$. This implies that formula (2.36) for ω_M holds, since, in this case, $\omega_{E_\phi \circ Q} = c(\phi)\omega_Q$. The sign of ω_M follows from the fact that the leading term in (2.37) corresponds to $d = k$ and its sign is $(-1)^{\#\phi}$. $\square$

The virtual species, $-\mathrm{Lg}(1-X) = \mathrm{Lg}\frac{1}{(1-X)}$, of " connected " linear orders, is a " cousin" of the tensorial species, $\mathrm{Lie}(X)$, of free Lie algebras (see A. Joyal in [Joy86] and C. Reutenauer in [Reu86]).

Corollary 2.8 *The following molecular expansion holds,*

$$-\mathrm{Lg}(1 - X) = \sum_M \ell_M M, \tag{2.38}$$

where each molecular component, M, is of the form of a finite composition,

$$M = E_{\phi^{(1)}} \circ E_{\phi^{(2)}} \circ \cdots \circ E_{\phi^{(s)}} \circ X^{2^j}, \quad s \geq 0, \quad j \geq 0, \tag{2.39}$$

in which each $\phi^{(i)}$ is a non-repeating fat *partition. The coefficients, $\ell_M \in \mathbb{Z}\backslash\{0\}$, and their sign,* $\mathrm{sgn}(\ell_M)$, *are given by,*

$$\ell_M = c(\phi^{(1)}) \cdots c(\phi^{(s)}), \qquad \mathrm{sgn}(\ell_M) = (-1)^{\sum_{i=1}^s \#\phi^{(i)}}. \tag{2.40}$$

Proof: Take $F = 1 - X$ in Theorem 2.6 and argue as in the proof of Corollary 2.7. A simpler proof is to use $\frac{1}{1-X} = (1 + X)(1 + X^2)(1 + X^4) \cdots (1 + X^{2^j}) \cdots$ and apply Corollary 2.7, to obtain,

$$-\mathrm{Lg}(1 - X) = \mathrm{Lg}\left(\frac{1}{1 - X}\right) = \sum_{j \geq 0} \mathrm{Lg}(1 + X^{2^j}) = \sum_{M \neq 1, j \geq 0} \omega_M M(X^{2^j}). \tag{2.41}$$

$\square$

Of course, from (1.5), we have the underlying cycle index series,

$$Z_{\mathrm{Lg}(1+X)} = \sum_{k \geq 1} \frac{\mu(k)}{k} \log(1 + p_k), \qquad Z_{-\mathrm{Lg}(1-X)} = -\sum_{k \geq 1} \frac{\mu(k)}{k} \log(1 - p_k). \tag{2.42}$$

Many other applications of Theorem 2.6 are possible. But, due to lack of space, we conclude with a Table, obtained using Maple, which gives the explicit molecular expansion of the combinatorial logarithm up to degree 10. Much more extended tables are easily obtained.

3 Compact table for the combinatorial logarithm up to degree 10

Note that for $s = 0$ in (2.35), the sequence, $\phi^{(1)}, \dots, \phi^{(s)}$, of non-repeating fat partitions is empty. So that the corresponding s-fold composition is a 0-fold composition, hence is equal to X, which is the neutral element under composition. Moreover, the corresponding product (2.36) being empty, is equal to 1. This is coherent with the fact that the molecular expansion of $\mathrm{Lg}(1+X)$ starts with X (see (2.9)). Moreover, if $M = E_{\phi^{(1)}} \circ \cdots \circ E_{\phi^{(s)}}$, is a molecular component in (2.34) then, for any permutation, $\sigma \in S_s$, $M^\sigma = E_{\phi^{(\sigma(1))}} \circ \cdots \circ E_{\phi^{(\sigma(s))}}$, is also a molecular component and the coefficients are equal: $\omega_M = \omega_{M^\sigma}$. Table 1, below[(x)], gives the molecular expansion (2.34) up to degree 10 and uses this fact to compact its size. The following convention is used, for $s > 1$ and $M = E_{\phi^{(1)}} \circ \cdots \circ E_{\phi^{(s)}}$:

$$\overline{M} \underset{\mathrm{def}}{=} \sum_{N \in \Lambda} N, \qquad \Lambda = \{M^\sigma : \sigma \in S_s\}. \tag{3.1}$$

Acknowledgements. The authors would like to thank Jérôme Tremblay for his LaTeXand Maple help.

n	Compact form for the terms of degree n in the combinatorial logarithm $\mathrm{Lg}(1+X)$
0	0
1	X
2	$-E_2$
3	$-E_3 + E_{1,2}$
4	$-E_4 + E_{1,3} - E_{1^2,2} + E_2 \circ E_2$
5	$-E_5 + E_{1,4} + E_{2,3} - E_{1^2,3} - E_{1,2^2} + E_{1^3,2}$
6	$-E_6 + E_{1,5} + E_{2,4} - E_{1^2,4} - 2E_{1,2,3} + E_{1^3,3} + 2E_{1^2,2^2} - E_{1^4,2} + \overline{E_2 \circ E_3} - \overline{E_2 \circ E_{1,2}}$
7	$-E_7 + E_{1,6} + E_{2,5} + E_{3,4} - E_{1^2,5} - 2E_{1,2,4} - E_{1,3^2} - E_{2^2,3} + E_{1^3,4} + 3E_{1^2,2,3} + E_{1,2^3}$ $-E_{1^4,3} - 2E_{1^3,2^2} + E_{1^5,2}$
8	$-E_8 + E_{1,7} + E_{2,6} + E_{3,5} - E_{1^2,6} - 2E_{1,2,5} - 2E_{1,3,4} - E_{2^2,4} - E_{2,3^2} + E_{1^3,5} + 3E_{1^2,2,4}$ $+2E_{1^2,3^2} + 3E_{1,2^2,3} - E_{1^4,4} - 4E_{1^3,2,3} - 2E_{1^2,2^3} + E_{1^5,3} + 2E_{1^4,2^2} - E_{1^6,2} + \overline{E_2 \circ E_4}$ $-\overline{E_2 \circ E_{1,3}} + \overline{E_2 \circ E_{1^2,2}} - E_2 \circ E_2 \circ E_2$
9	$-E_9 + E_{1,8} + E_{2,7} + E_{3,6} + E_{4,5} - E_{1^2,7} - 2E_{1,2,6} - 2E_{1,3,5} - E_{1,4^2} - E_{2^2,5} - 2E_{2,3,4}$ $+E_{1^3,6} + 3E_{1^2,2,5} + 3E_{1^2,3,4} + 3E_{1,2^2,4} + 3E_{1,2,3^2} + E_{2^3,3} - E_{1^4,5} - 4E_{1^3,2,4} - 2E_{1^3,3^2}$ $-6E_{1^2,2^2,3} - E_{1,2^4} + E_{1^5,4} + 5E_{1^4,2,3} + 3E_{1^3,2^3} - E_{1^6,3} - 3E_{1^5,2^2} + E_{1^7,2} + E_3 \circ E_3$ $-\overline{E_3 \circ E_{1,2}} + E_{1,2} \circ E_{1,2}$
10	$-E_{10} + E_{1,9} + E_{2,8} + E_{3,7} + E_{4,6} - E_{1^2,8} - 2E_{1,2,7} - 2E_{1,3,6} - 2E_{1,4,5} - E_{2^2,6} - 2E_{2,3,5}$ $-E_{2,4^2} - E_{3^2,4} + E_{1^3,7} + 3E_{1^2,2,6} + 3E_{1^2,3,5} + 2E_{1^2,4^2} + 3E_{1,2^2,5} + 6E_{1,2,3,4} + E_{1,3^3}$ $+E_{2^3,4} + 2E_{2^2,3^2} - E_{1^4,6} - 4E_{1^3,2,5} - 4E_{1^3,3,4} - 6E_{1^2,2^2,4} - 6E_{1^2,2,3^2} - 4E_{1,2^3,3} + E_{1^5,5}$ $+5E_{1^4,2,4} + 2E_{1^4,3^2} + 10E_{1^3,2^2,3} + 2E_{1^2,2^4} - E_{1^6,4} - 6E_{1^5,2,3} - 5E_{1^4,2^3} + E_{1^7,3} + 4E_{1^6,2^2}$ $-E_{1^8,2} + \overline{E_2 \circ E_5} - \overline{E_2 \circ E_{1,4}} - \overline{E_2 \circ E_{2,3}} + \overline{E_2 \circ E_{1^2,3}} + \overline{E_2 \circ E_{1,2^2}} - \overline{E_2 \circ E_{1^3,2}}$

Tab. 1: Compact form for the terms of degree n in the combinatorial logarithm, for $0 \leq n \leq 10$.

(x) Made using the Maple package `combinat` together with the `define('linear')` command.

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On the Spectra of Simplicial Rook Graphs

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Abstract. The *simplicial rook graph* $SR(d, n)$ is the graph whose vertices are the lattice points in the nth dilate of the standard simplex in $\mathbb{R}^d$, with two vertices adjacent if they differ in exactly two coordinates. We prove that the adjacency and Laplacian matrices of $SR(3, n)$ have integral spectra for every n. We conjecture that $SR(d, n)$ is integral for all d and n, and give a geometric construction of almost all eigenvectors in terms of characteristic vectors of lattice permutohedra. For $n \leq \binom{d}{2}$, we give an explicit construction of smallest-weight eigenvectors in terms of rook placements on Ferrers diagrams. The number of these eigenvectors appears to satisfy a Mahonian distribution.

Resumé. Le *graphe des tours simplicials* $SR(d, n)$ est le graphe dont les sommets sont les points du réseau dans le $n$ième dilation du simplexe standard dans $\mathbb{R}^d$; deux sommets sont adjacents s'ils différent dans exactement deux coordonnées. Nous montrons que tous les valeurs propres des matrices d'adjacence et laplacienne de $SR(3, n)$ sont entiers, pour tous les n. Nous conjecturons que les valeurs propres sont entiers pour tous d et n, et donnons une construction géometrique de presque tous les vecteurs propres en termes des vecteurs caractéristiques de permutoèdres treillis. Pour $n \leq \binom{d}{2}$, nous donnons une construction explicite des vecteurs propres de plus petits poids en termes des placements des tours sur diagrammes de Ferrers. Le nombre de ces vecteurs propres semble satisfaire une distribution Mahonian.

Keywords: simplicial rook graph, adjacency matrix, Laplacian matrix, spectral graph theory

1 Introduction

Let d and n be nonnegative integers. The *simplicial rook graph* $SR(d, n)$ is the graph with vertices

$$V(d, n) := \left\{ x = (x_1, \dots, x_d) \colon 0 \leq x_i \leq n,\ \sum_{i=1}^{d} x_i = n \right\}$$

with two vertices adjacent if they agree in all but two coordinates. This graph has $N = \binom{n+d-1}{d-1}$ vertices and is regular of degree $\delta = (d-1)n$. Geometrically, let Δ^{d-1} denote the standard simplex in $\mathbb{R}^d$ (i.e., the convex hull of the standard basis vectors $\mathbf{e}_1, \dots, \mathbf{e}_d$) and let $n\Delta^{d-1}$ denote its n^{th} dilate (i.e., the convex hull of $n\mathbf{e}_1, \dots, n\mathbf{e}_d$). Then $V(d, n)$ is the set of lattice points in $n\Delta^{d-1}$, with two points adjacent if their difference is a multiple of $\mathbf{e}_i - \mathbf{e}_j$ for some i, j. Thus the independence number of $SR(d, n)$ is

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the maximum number of nonattacking rooks that can be placed on a simplicial chessboard with $n+1$ "squares" on each side. For $d=3$, this independence number is $\lfloor (2n+3)/3 \rfloor$ Blackburn et al. (2011); Nivasch and Lev (2005).

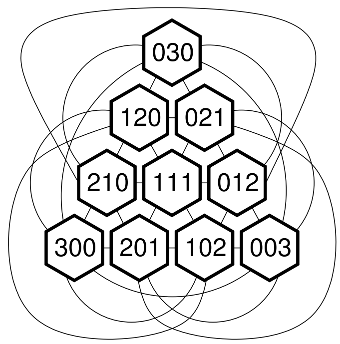

Fig. 1: The graph $SR(3,3)$.

As far as we can tell, the class of simplicial rook graphs has not been studied before. For some small values of the parameters, $SR(d,n)$ is a well-known graph: $SR(2,n)$ and $SR(d,1)$ are complete of orders $n+1$ and d respectively; $SR(3,2)$ is isomorphic to the octahedron; and $SR(d,2)$ is isomorphic to the Johnson graph $J(d+1,2)$. On the other hand, simplicial rook graphs are not in general strongly regular or distance-regular, nor are they line graphs or noncomplete extended p-sums (in the sense of (Cvetković et al., 1988, p. 55)). They are also not to be confused with the *simplicial grid graph*, in which two vertices are adjacent only if their difference vector is exactly $\mathbf{e}_i - \mathbf{e}_j$ (as opposed to some scalar multiple) nor with the *triangular graph* T_n, which is the line graph of K_n (Brouwer and Haemers, 2012, p.23), (Godsil and Royle, 2001, §10.1).

Let G be a simple graph on vertices $[n] = \{1, \ldots, n\}$. The *adjacency matrix* $A = A(G)$ is the $n \times n$ symmetric matrix whose (i,j) entry is 1 if ij is an edge, 0 otherwise. The *Laplacian matrix* is $L = L(G) = D - A$, where D is the diagonal matrix whose (i,i) entry is the degree of vertex i. The graph G is said to be *integral* (resp. *Laplacian integral*) if all eigenvalues of A (resp. L) are integers. If G is regular of degree δ, then these conditions are equivalent, since every eigenvector of A with eigenvalue λ is an eigenvector of L with eigenvalue $\delta - \lambda$.

We can now state our main theorem.

Theorem 1.1 *For every $n \geq 1$, the simplicial rook graph $SR(3,n)$ is integral and Laplacian integral, with eigenvalues as follows:*

If $n = 2m+1$ is odd:

Eigenvalue of A	**Eigenvalue of L**	**Multiplicity**	**Eigenvector**
-3	$4m+5 = 2n+3$	$\binom{2m}{2}$	$\mathbf{H}_{a,b,c}$
$-2, -1, \ldots, m-3$	$3m+5 \ldots, 4m+4$	3	$\mathbf{P}_k$
$m-1$	$3m+3$	2	$\mathbf{R}$
$m, \ldots, 2m-1 = n-2$	$2m+3 \ldots, 3m+2$	3	$\mathbf{Q}_k$
$4m+2 = 2n$	0	1	$\mathbf{J}$

If $n = 2m$ is even:

Eigenvalue of A	**Eigenvalue of L**	**Multiplicity**	**Eigenvector**
-3	$4m+3=2n+3$	$\binom{2m-1}{2}$	$\mathbf{H}_{a,b,c}$
$-2,-1,\dots,m-4$	$3m+4,\dots,4m+2$	3	$\mathbf{P}_k$
$m-3$	$3m+3$	2	$\mathbf{R}$
$m-1,\dots,2m-2=n-2$	$2m+2,\dots,3m+1$	3	$\mathbf{Q}_k$
$4m=2n$	0	1	$\mathbf{J}$

Integrality and Laplacian integrality typically arise from tightly controlled combinatorial structure in special families of graphs, including complete graphs, complete bipartite graphs and hypercubes (classical; see, e.g., (Stanley, 1999, §5.6)), Johnson graphs Krebs and Shaheen (2008), Kneser graphs Lovász (1979) and threshold graphs Merris (1994). (General references on graph eigenvalues and related topics include Balińska et al. (2002); Brouwer and Haemers (2012); Cvetković et al. (1988); Godsil and Royle (2001).) For simplicial rook graphs, lattice geometry provides this combinatorial structure. To prove Theorem 1.1, we construct a basis of $\mathbb{R}^{\binom{n+2}{2}}$ consisting of eigenvectors of $A(SR(3,n))$, as indicated in the tables above. The basis vectors $\mathbf{H}_{a,b,c}$ for the largest eigenspace are signed characteristic vectors for hexagons centered at lattice points in the interior of $n\Delta^3$ (see Figure 2). The other eigenvectors $\mathbf{P}_k, \mathbf{R}, \mathbf{Q}_k$ can be expressed as certain sums of characteristic vectors of lattice lines.

Theorem 1.1, together with Kirchhoff's matrix-tree theorem (Godsil and Royle, 2001, Lemma 13.2.4) implies the following formula for the number of spanning trees of $SR(d,n)$.

Corollary 1.2 *The number of spanning trees of $SR(3,n)$ is*

$$\begin{cases} \dfrac{32(2n+3)^{\binom{n-1}{2}} \prod_{a=n+2}^{2n+2} a^3}{3(n+1)^2(n+2)(3n+5)^3} & \text{if } n \text{ is odd,} \\[2ex] \dfrac{32(2n+3)^{\binom{n-1}{2}} \prod_{a=n+2}^{2n+2} a^3}{3(n+1)(n+2)^2(3n+4)^3} & \text{if } n \text{ is even.} \end{cases}$$

Based on experimental evidence gathered using Sage Stein et al. (2012), we make the following conjecture:

Conjecture 1.3 *The graph $SR(d,n)$ is integral for all d and n.*

We discuss the case of arbitrary d in Section 3. The construction of hexagon vectors generalizes as follows: for each permutohedron whose vertices are lattice points in $n\Delta^{d-1}$, its signed characteristic vector is an eigenvector of eigenvalue $-\binom{d}{2}$ (Proposition 3.1). This is in fact the smallest eigenvalue of $SR(d,n)$ when $n \geq \binom{d}{2}$. Moreover, these eigenvectors are linearly independent and, for fixed d, account for "almost all" of the spectrum as $n \to \infty$, in the sense that

$$\lim_{n\to\infty} \frac{\dim\text{(span of permutohedron eigenvectors)}}{|V(d,n)|} = 1.$$

When $n < \binom{d}{2}$, the simplex $n\Delta^{d-1}$ is too small to contain any lattice permutohedra. On the other hand, the signed characteristic vectors of *partial permutohedra* (i.e., intersections of lattice permutohedra with

$SR(d,n)$) are eigenvectors with eigenvalue $-n$. Experimental evidence indicates that this is in fact the smallest eigenvalue of $A(d,n)$, and that these partial permutohedra form a basis for the corresponding eigenspace. Unexpectedly, its dimension appears to be the *Mahonian number* $M(d,n)$ of permutations in $\mathfrak{S}_d$ with exactly n inversions (sequence #A008302 in Sloane (2012)). We construct a family of eigenvectors by placing rooks (ordinary rooks, not simplicial rooks!) on Ferrers boards.

The reader is referred to Martin and Wagner (2012) for the full version of this article, including proofs of all results. The authors thank Cristi Stoica for bringing their attention to references Nivasch and Lev (2005) and Blackburn et al. (2011), and Noam Elkies and several other members of the MathOverflow community for a stimulating discussion. The open-source software package Sage Stein et al. (2012) was a valuable tool in carrying out this research.

2 Proof of the Main Theorem

We begin by reviewing some basic algebraic graph theory; for a general reference, see, e.g., Godsil and Royle (2001). Let $G = (V, E)$ be a simple undirected graph with N vertices. The *adjacency matrix* $A(G)$ is the $N \times N$ matrix whose (i, j) entry is 1 if vertices i and j are adjacent, 0 otherwise. The *Laplacian matrix* is $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees. These are both real symmetric matrices, so they are diagonalizable, with real eigenvalues, and eigenspaces with different eigenvalues are orthogonal (Godsil and Royle, 2001, §8.4).

Proposition 2.1 *The graph $SR(d,n)$ has $N = \binom{n+d-1}{d-1}$ vertices and is regular of degree $(d-1)n$. In particular, its adjacency and Laplacian matrices have the same eigenvectors.*

Proof: Counting vertices is the classic "stars-and-bars" problem (with n stars and $d-1$ bars). For each $x \in V(d,n)$ and each pair of coordinates i, j, there are $x_i + x_j$ other vertices that agree with x in all coordinates but i and j. Therefore, the degree of x is $\sum_{1 \leq i < j \leq n}(x_i + x_j) = (d-1)\sum_{i=1}^{n} x_i = (d-1)n$. $\square$

In the rest of this section, we focus exclusively on the case $d = 3$, and regard n as fixed. We fix $N := \binom{n+2}{2}$, the number of vertices of $SR(3,n)$, and abbreviate $A = A(3,n)$. The matrix A acts on the vector space $\mathbb{R}^N$ with standard basis $\{\mathbf{e}_{ijk} \colon (i,j,k) \in V(3,n)\}$. We will sometimes consider the standard basis vectors as ordered lexicographically, for the purpose of showing that a collection of vectors is linearly independent.

2.1 Hexagon vectors

Let $(a,b,c) \in V(3,n)$ with $a,b,c > 0$. The corresponding *hexagon vector* is defined as

$$\mathbf{H}_{a,b,c} := \mathbf{e}_{a-1,b,c+1} - \mathbf{e}_{a,b-1,c+1} + \mathbf{e}_{a+1,b-1,c} - \mathbf{e}_{a+1,b,c-1} + \mathbf{e}_{a,b+1,c-1} - \mathbf{e}_{a-1,b+1,c}.$$

Geometrically, this is the characteristic vector, with alternating signs, of a regular lattice hexagon centered at the lattice point (a,b,c) in the interior of $n\Delta^2$ (see Figure 2).

It is not hard to check that the vectors $\mathbf{H}_{a,b,c}$ are linearly independent, and each is an eigenvector of $A(d,n)$ with eigenvalue -3. The number of possible "centers" (a,b,c) is $\binom{n-2}{3}$, so there are still $3n$ eigenvectors to determine (since $3n$ is the number of vertices of $SR(d,n)$ with at least one coordinate zero).

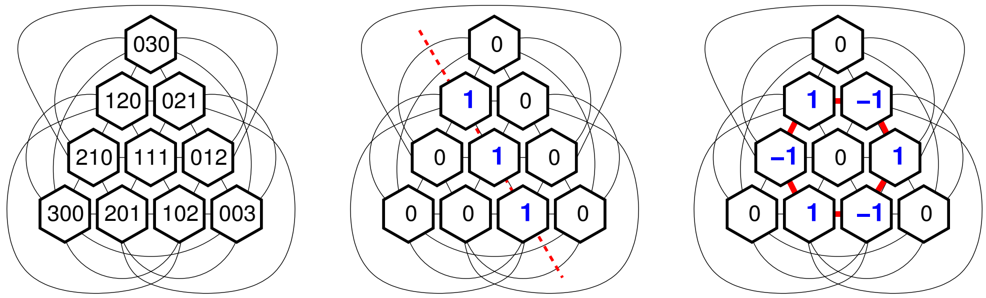

Fig. 2: (left) The graph $SR(3,3)$. (center) The vector $\mathbf{X}_1$ and the lattice line it supports. (right) $\mathbf{H}_{1,1,1}$.

Define

$$\mathbf{X}_i := \sum_{j+k=n-i} \mathbf{e}_{ijk}, \quad \mathbf{Y}_j := \sum_{i+k=n-j} \mathbf{e}_{ijk}, \quad \mathbf{Z}_k := \sum_{i+j=n-k} \mathbf{e}_{ijk}.$$

These vectors $\mathbf{X}_i, \mathbf{Y}_j, \mathbf{Z}_k$ are the characteristic vectors of lattice lines in $n\Delta^2$; see Figure 2. It can be checked that they span a vector space W of dimension $3n$, and that each one is orthogonal to every hexagon eigenvector. Therefore W is the span of all the other eigenvectors. Moreover, the symmetric group $\mathfrak{S}_3$ acts on $SR(3,n)$ (hence on each of its eigenspaces) by permuting the coordinates of vertices.

Theorem 2.2 *The eigenvectors of $A(d,n)$ are as follows.*

- *Let $\mathbf{J} = \sum_{i=0}^{n} \mathbf{X}_i = \sum_{i=0}^{n} \mathbf{Y}_i = \sum_{i=0}^{n} \mathbf{Z}_i$. Then $\mathbf{J}$ is an eigenvector with eigenvalue $2n$.*
- *Let $m = \lfloor n/2 \rfloor$ and $\mathbf{R} := \mathbf{X}_m - \mathbf{Y}_m - \mathbf{X}_{m+1} + \mathbf{Y}_{m+1}$. Then the $\mathfrak{S}_3$-orbit of $\mathbf{R}$ is an eigenspace with dimension 2 and eigenvalue $(n-6)/2$ if n is even, or $(n-3)/2$ if n is odd.*
- *For each k with $0 \le k \le \lfloor \frac{n-3}{2} \rfloor$, let*

$$\mathbf{P}_k := -(n-2k-1)(n-2k-2)\mathbf{Z}_{n-k} + \sum_{i=k+1}^{n-k-1} \Big[2(i-k-1)\mathbf{Z}_i + (2i-n)(\mathbf{X}_i + \mathbf{Y}_i) \Big].$$

  Then the $\mathfrak{S}_3$-orbit of $\mathbf{P}_k$ is an eigenspace with dimension 3 and eigenvalue $k-2$.
- *For each k with $0 \le k \le \lfloor \frac{n-2}{2} \rfloor$, let*

$$\mathbf{Q}_k = (n-2k+1)(n-2k+2)\mathbf{Z}_k + \sum_{j=k}^{n-k} \Big[(2j-n)(\mathbf{X}_j + \mathbf{Y}_j) - 2(n-j-k+1)\mathbf{Z}_j \Big].$$

  Then the $\mathfrak{S}_3$-orbit of $\mathbf{P}_k$ is an eigenspace with dimension 3 and eigenvalue $n-k-2$.

We omit the proof, which is a more or less direct calculation, requiring the action of $A(d,n)$ on the vectors $\mathbf{X}_i, \mathbf{Y}_j, \mathbf{Z}_k$ and several summation identities.

3 Simplicial rook graphs in arbitrary dimension

We now consider the graph $SR(d,n)$ for arbitrary d and n, with adjacency matrix $A = A(d,n)$. Recall that $SR(d,n)$ has $N := \binom{n+d-1}{d-1}$ vertices and is regular of degree $(d-1)n$. If two vertices $a = (a_1,\dots,a_d), b = (b_1,\dots,b_d) \in V(d,n)$ differ only in their i^{th} and j^{th} positions (and are therefore adjacent), we write $a \underset{i,j}{\sim} b$.

Let $\mathfrak{S}_d$ be the symmetric group of order d, and let $\mathfrak{A}_d \subset \mathfrak{S}_d$ be the alternating subgroup. Let ε be the sign function

$$\varepsilon(\sigma) = \begin{cases} 1 & \text{for } \sigma \in \mathfrak{A}_d, \\ -1 & \text{for } \sigma \notin \mathfrak{A}_d. \end{cases}$$

Let $\tau_{ij} \in \mathfrak{S}_d$ denote the transposition of i and j. Note that $\mathfrak{S}_d = \mathfrak{A}_d \cup \mathfrak{A}_d\tau_{ij}$ for each i,j.

In analogy to the vectors $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ used in the $d = 3$ case, define

$$\mathbf{X}_\alpha^{(i,j)} = \mathbf{e}_\alpha + \sum_{\beta\colon\, \beta \underset{i,j}{\sim} \alpha} \mathbf{e}_\beta. \tag{3.1}$$

That is, $\mathbf{X}_\alpha^{(i,j)}$ is the characteristic vector of the lattice line through α in direction $\mathbf{e}_i - \mathbf{e}_j$. In particular, if $\alpha \underset{i,j}{\sim} \beta$, then $\mathbf{X}_\alpha^{(i,j)} = \mathbf{X}_\beta^{(i,j)}$. Moreover, the column of A indexed by α is

$$A\mathbf{e}_\alpha = -\binom{d}{2}\mathbf{e}_\alpha + \sum_{1 \le i < j \le d} \mathbf{X}_\alpha^{(i,j)}. \tag{3.2}$$

since e_α itself appears in each summand $\mathbf{X}_\alpha^{(i,j)}$.

3.1 Permutohedron vectors

We now generalize the construction of hexagon vectors to arbitrary dimension. The idea is that for each point p in the interior of $n\Delta^{d-1}$ and sufficiently far away from its boundary, there is a lattice permutohedron centered at p, all of whose points are vertices of $SR(d,n)$ (see Figure 3), and the signed characteristic vector of this permutohedron is an eigenvector of $A(d,n)$.

Specifically, let $w = ((1-d)/2, (3-d)/2, \dots, (d-3)/2, (d-1)/2) \in \mathbb{R}^d$. Let $p \in \mathbb{Z}^d$ (if d is odd) or $(\mathbb{Z} + \frac{1}{2})^d$ (if d is even). Then

$$\mathbf{H}_p = \sum_{\sigma \in \mathfrak{S}_d} \varepsilon(\sigma)\mathbf{e}_{p+\sigma(w)}$$

is the signed characteristic vector of the smallest lattice permutohedron with center p; we call $\mathbf{H}_p$ a *permutohedron vector*.

Proposition 3.1 *Fix $d,n \in \mathbb{N}$, and let $p, w, \mathbf{H}_p$ be as above.*

1. *If $\{p + \sigma(w) \colon \sigma \in \mathfrak{S}_d\}$ are distinct vertices of $SR(d,n)$, then $\mathbf{H}_p$ is an eigenvector of $A(d,n)$ with eigenvalue $-\binom{d}{2}$.*

2. *The set of all such eigenvectors $\mathbf{H}_p$ is linearly independent, and its cardinality is $\binom{n - \frac{(d-1)(d-2)}{2}}{d-1}$.*

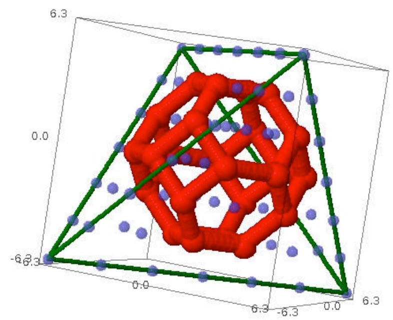

Fig. 3: A permutohedron vector ($n = 6$, $d = 4$).

This result says that we can construct a large eigenspace by fitting many congruent permutohedra into the dilated simplex. In fact, the permutohedron eigenvectors account for "almost all" of the eigenvectors in the following sense: if $\mathcal{H}_{d,n} \subseteq \mathbb{R}^N$ is the linear span of the eigenvectors constructed in Prop. 3.1, then for each fixed d, we have

$$\lim_{n\to\infty} \frac{\dim \mathcal{H}_{d,n}}{|V(d,n)|} = \lim_{n\to\infty} \frac{\binom{n-\frac{(d-1)(d-2)}{2}}{d-1}}{\binom{n+d-1}{d-1}} = 1. \tag{3.3}$$

The characteristic vectors of lattice lines in $\mathbb{R}^d$ can be shown to be orthogonal to $\mathcal{H}_{d,n}$. We conjecture that those characteristic vectors in fact span the orthogonal complement. We have verified this statement computationally for $d = 4$ and $n \leq 11$, and for $d = 5$ and $n = 7, 8, 9$. We do not have a proof of the general statement; part of the difficulty is that it is not clear what subset of the $\mathbf{X}_\alpha^{(i,j)}$ ought to form a basis (in contrast to the case $d = 3$).

Proposition 3.2 *Suppose that $d \geq 1$ and $n \geq \binom{d}{2}$. Then the smallest eigenvalue of $SR(d,n)$ is $-\binom{d}{2}$.*

We omit the short proof, whose main idea was suggested to the authors by Noam Elkies. The smallest eigenvalue is significant in spectral graph theory; for instance, it is related to the independence number (Godsil and Royle, 2001, Lemma 9.6.2).

3.2 The small-n case and Mahonian numbers

When $n < \binom{d}{2}$, there are no permutohedron vectors — the simplex $n\Delta^{d-1}$ is too small to contain any lattice permutohedra.

Experimental evidence indicates that the smallest eigenvalue of $SR(d,n)$ is $-n$, and moreover that the multiplicity of this eigenvalue equals the number $M(d,n)$ of permutations in $\mathfrak{S}_d$ with exactly n inversions. The numbers $M(d,n)$ are well known in combinatorics as the *Mahonian numbers*, or as the coefficients of the q-factorial polynomials; see (Sloane, 2012, sequence #A008302). In the rest of this section, we construct $M(d,n)$ linearly independent eigenvectors of eigenvalue $-n$; however, we do not know how to rule out the possibility of additional eigenvectors of equal or smaller eigenvalue

We review some basics of rook theory; for a general reference, see, e.g., Butler et al. (2012). For a sequence of positive integers $c = (c_1, \dots, c_d)$, the *skyline board* $\mathrm{Sky}(c)$ consists of a sequence of d columns, with the i^{th} column containing c_i squares. A *rook placement* on $\mathrm{Sky}(c)$ consists of a choice of one square in each column. A rook placement is *proper* if all d squares belong to different rows.

An *inversion* of a permutation $\pi = (\pi_1, \dots, \pi_d) \in \mathfrak{S}_d$ is a pair i, j such that $i < j$ and $\pi_i > \pi_j$. Let $\mathfrak{S}_{d,n}$ denote the set of permutations of $[d]$ with exactly n inversions.

Definition 3.3 *Let $\pi \in \mathfrak{S}_{d,n}$. The* inversion word *of π is $a = a(\pi) = (a_1, \dots, a_d)$, where*

$$a_i = \#\{j \in [d] \colon i < j \text{ and } \pi_i > \pi_j\}.$$

*Note that a is a weak composition of n with d parts, hence a vertex of $SR(d,n)$. A permutation $\sigma \in \mathfrak{S}_{d,n}$ is π-*admissible *if σ is a proper skyline rook placement on $\mathrm{Sky}(a_1+1, \dots, a_d+d)$; that is, if*

$$x(\sigma) = a(\pi) + \mathbf{w} - \sigma(\mathbf{w}) = a(\pi) + \mathrm{id} - \sigma$$

is a lattice point in $n\Delta^{d-1}$. Note that the coordinates of $x(\sigma)$ sum to n, so admissibility means that its coordinates are all nonnegative. The set of all π-admissible permutations is denoted $\mathrm{Adm}(\pi)$; that is,

$$\mathrm{Adm}(\pi) = \{\sigma \in \mathfrak{S}_d \colon a_i - \sigma_i + i \geq 0 \quad \forall i = 1, \dots, d\}.$$

The corresponding partial permutohedron *is*

$$\mathrm{Parp}(\pi) = \{x(\sigma) \colon \sigma \in \mathrm{Adm}(\pi)\}.$$

That is, $\mathrm{Parp}(\pi)$ is the set of permutations corresponding to lattice points in the intersection of $n\Delta^{d-1}$ with the standard permutohedron centered at $a(\pi) + \mathbf{w}$. The partial permutohedron vector *is the signed characteristic vector of $\mathrm{Parp}(\pi)$, that is,*

$$\mathbf{F}_\pi = \sum_{\sigma \in \mathrm{Parp}(\pi)} \varepsilon(\sigma) \mathbf{e}_{x(\sigma)}.$$

Example 3.4 *Let $d = 4$ and $\pi = 3142 \in \mathfrak{S}_d$. Then π has $n = 3$ inversions, namely 12, 14, 34. Its inversion word is accordingly $a = (2,0,1,0)$. The π-admissible permutations are the proper skyline rook placements on $\mathrm{Sky}(2+1, 0+2, 1+3, 0+4) = \mathrm{Sky}(3,2,4,4)$, namely 1234, 1243, 2134, 2143, 3124, 3142, 3214, 3241 (see Figure 4). The corresponding lattice points $x(\sigma)$ can be read off from the rook placements by counting the number of empty squares above each rook, obtaining respectively 2010, 2001, 1110, 1101, 0120, 0102, 0030, 0003; these are the neighbors of a in $\mathrm{Parp}(\pi)$. Thus $\mathbf{F}_\pi = \mathbf{e}_{2010} - \mathbf{e}_{2001} - \mathbf{e}_{1110} + \mathbf{e}_{1101} - \mathbf{e}_{0120} + \mathbf{e}_{0102} + \mathbf{e}_{0030} - \mathbf{e}_{0003}$; see Figure 5.*

Theorem 3.5 *Let $\pi \in \mathfrak{S}_{d,n}$ and $A = A(d,n)$. Then $\mathbf{F}_\pi$ is an eigenvector of A with eigenvalue $-n$. Moreover, for every pair d, n with $n < \binom{d}{2}$, the set $\{\mathbf{F}_\pi \colon \pi \in \mathfrak{S}_{d,n}\}$ is linearly independent. In particular, the dimension of the $(-n)$-eigenspace of A is at least the Mahonian number $M(d,n)$.*

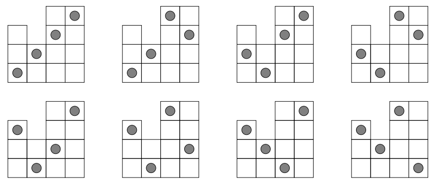

Fig. 4: Rook placements on the skyline board $\mathrm{Sky}(3, 2, 4, 4)$.

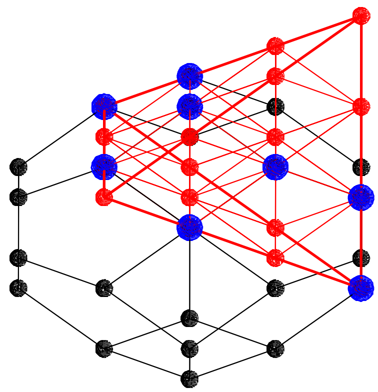

Fig. 5: The partial permutohedron $\mathrm{Parp}(3142)$ in $SR(4, 3)$.

Proof: We include the proof in order to illustrate the connections to (non-simplicial) rook theory. First, linear independence follows from the observation that the lexicographically leading term of $\mathbf{F}_\pi$ is $\mathbf{e}_{a(\pi)}$, and these terms are different for all $\pi \in \mathfrak{S}_{d,n}$.

Second, let $\sigma \in \mathrm{Adm}(\pi)$. Then the coefficient of $\mathbf{e}_{x(\sigma)}$ in $\mathbf{F}_\pi$ is $\varepsilon(\sigma) \in \{1, -1\}$. We will show that the coefficient of $\mathbf{e}_{x(\sigma)}$ in $A\mathbf{F}_\pi$ is $-n\varepsilon(\sigma)$, i.e., that

$$\varepsilon(\sigma) \sum_{\rho} \varepsilon(\rho) = -n, \tag{3.4}$$

the sum over all ρ such that $\rho \sim \sigma$ and $\rho \in \mathrm{Parp}(\pi)$. (Here and subsequently, $\sim$ denotes adjacency in $SR(d, n)$.) Each such rook placement ρ is obtained by multiplying σ by the transposition $(i\ j)$, that is, by choosing a rook at (i, σ_i), choosing a second rook at (j, σ_j) with $\sigma_j > \sigma_i$, and replacing these two rooks with rooks in positions (i, σ_j) and (j, σ_i). For each choice of i, there are $(a_i + i) - \sigma_i$ possible j's, and $\sum_i (a_i + i - \sigma_i) = n$. Moreover, the sign of each such ρ is opposite to that of σ, proving (3.4).

Third, let $y = (y_1, \ldots, y_d) \in V(d, n) \setminus \mathrm{Parp}(\pi)$. Then the coefficient of $e_{x(\sigma)}$ in $\mathbf{F}_\pi$ is 0. We show that the coefficient of $\mathbf{e}_{x(\sigma)}$ in $A\mathbf{F}_\pi$ is also 0, i.e., that

$$\sum_{\sigma \in N} \varepsilon(\sigma) = 0. \tag{3.5}$$

where $N = \{\rho : x(\rho) \sim y\} \cap \mathrm{Parp}(\pi)$. In order to prove this, we construct a sign-reversing involution on N. Let $a = a(\pi)$ and let $b = (b_1, \dots, b_d) = (a_1 + 1 - y_1, a_2 + 2 - y_2, \dots, a_d + d - y_d)$. Note that $b_i \leq a_i + i$ for every i; therefore, we can regard b as a rook placement on $\mathrm{Sky}(a_1 + 1, \dots, a_d + d)$. (It is possible that $b_i \leq 0$ for one or more i; we will consider that case shortly.) To say that $y \notin \mathbf{F}_\pi$ is to say that b is not a proper π-skyline rook placement; on the other hand, we have $\sum b_i = \binom{d+1}{2}$ (as would be the case if b were proper). Hence the elements of N are the proper π-skyline rook skyline placements obtained from b by moving one rook up and one other rook down, necessarily by the same number of squares. Let $b(i \uparrow q, j \downarrow r)$ denote the rook placement obtained by moving the i^{th} rook up to row q and the j^{th} rook down to row r.

We now consider the various possible ways in which b can fail to be proper.

Case 1: $b_i \leq 0$ for two or more i. In this case $N = \emptyset$, because moving only one rook up cannot produce a proper π-skyline rook placement.

Case 2: $b_i \leq 0$ for exactly one i. The other rooks in b cannot all be at different heights, because that would imply that $\sum b_i \leq 0 + (2 + \cdots + d) < \binom{d+1}{2}$. Therefore, either $N = \emptyset$, or else $b_j = b_k$ for some j, k and there are rooks at all heights except q and r for some $q, r < b_j = b_k$.

Then $b(i \uparrow q, j \downarrow r)$ is proper if and only if $b(i \uparrow q, k \downarrow r)$ is proper, and likewise $b(i \uparrow r, j \downarrow q)$ is proper if and only if $b(i \uparrow r, k \downarrow q)$ is proper. Each of these pairs is related by the transposition $(j\ k)$, so we have the desired sign-reversing involution on N.

Case 3: $b_i \geq 1$ for all i. Then the reason that b is not proper must be that some row has no rooks and some row has more than one rook. There are several subcases:

Case 3a: For some $q \neq r$, there are two rooks at height q, no rooks at height r, and one rook at every other height. But this is impossible because then $\sum b_i = \binom{d+1}{2} + q - r \neq \binom{d+1}{2}$.

Case 3b: There are four or more rooks at height q, or three at height q and two or more at height r. In both cases $N = \emptyset$.

Case 3c: We have $b_i = b_j = b_k$; no rooks at heights q or r for some $q < r$; and one rook at every other height. Then

$$N \subseteq \left\{ \begin{array}{lll} b(i \uparrow r, j \downarrow q), & b(j \uparrow r, i \downarrow q), & b(k \uparrow r, i \downarrow q), \\ b(i \uparrow r, k \downarrow q), & b(j \uparrow r, k \downarrow q), & b(k \uparrow r, j \downarrow q). \end{array} \right\}$$

For each column of the table above, its two rook placements are related by a transposition (e.g., $(j\ k)$ for the first column) and either both or neither of those rook placements are proper (e.g., for the first column, depending on whether or not $b_i \leq r$). Therefore, we have the desired sign-reversing involution on N.

Case 3d: We have $b_i = b_j = q$; $b_k = b_\ell = r$, and one rook at every other height except heights s and t. Now the desired sign-reversing involution on N is toggling the rook that gets moved down; for instance, $b(j \uparrow s, k \downarrow t)$ is proper if and only if $b(j \uparrow s, \ell \downarrow t)$ is proper.

This completes the proof of (3.5), which together with (3.4) completes the proof that $\mathbf{F}_\pi$ is an eigenvector of $A(d, n)$ with eigenvalue $-n$. $\square$

Conjecture 3.6 *If $n \leq \binom{d}{2}$, then in fact $\tau(SR(d, n)) = -n$, and the dimension of the corresponding eigenspace is the Mahonian number $M(d, n)$.*

We have verified this conjecture, using Sage, for all $d \leq 6$. It is not clear in general how to rule out the possibility of a smaller eigenvalue, or of additional $(-n)$-eigenvectors linearly independent of the $\mathbf{F}_\pi$.

The proof of Theorem 3.5 implies that every partial permutohedron $\mathrm{Parp}(\pi)$ induces an n-regular subgraph of $SR(d, n)$. Another experimental observation is the following:

Conjecture 3.7 *For every $\pi \in \mathfrak{S}_{d,n}$, the induced subgraph $SR(d, n)|_{\mathrm{Parp}(\pi)}$ is Laplacian integral.*

We have verified this conjecture, using Sage, for all permutations of length $d \leq 6$. We do not know what the eigenvalues are, but these graphs are not in general strongly regular (as evidenced by the observation that they have more than 3 distinct eigenvalues).

4 Corollaries, alternate methods, and further directions

4.1 The independence number

The independence number of $SR(d, n)$ can be interpreted as the maximum number of nonattacking "rooks" that can be placed on a simplicial chessboard of side length $n + 1$. By (Godsil and Royle, 2001, Lemma 9.6.2), the independence number $\alpha(G)$ of a δ-regular graph G on N vertices is at most $-\tau N/(\delta - \tau)$, where τ is the smallest eigenvalue of $A(G)$. For $d = 3$ and $n \geq 3$, we have $\tau = -3$, which implies that the independence number $\alpha(SR(d, n))$ is at most $3(n + 2)(n + 1)/(4n + 6)$. This is of course a weaker result (except for a few small values of n) than the exact value $\lfloor (3n + 3)/2 \rfloor$ obtained in Nivasch and Lev (2005) and Blackburn et al. (2011).

Question 4.1 *What is the independence number of $SR(d, n)$? That is, how many nonattacking rooks can be placed on a simplicial chessboard?*

Proposition 3.2 implies the upper bound

$$\alpha(SR(d, n)) \leq \frac{d(d + 1)}{(2n + d)(d - 1)} \binom{n + d - 1}{d - 1}$$

for $n \geq \binom{d}{2}$, but this bound is not sharp (for example, the bound for $SR(4, 6)$ is $\alpha \leq 21$, but computation indicates that $\alpha = 16$).

The theory of interlacing and equitable partitions Haemers (1995), (Godsil and Royle, 2001, chapter 9) may be useful in describing the spectrum of $SR(d, n)$. Briefly, given a graph G, one constructs a square matrix P whose columns and rows correspond to orbits of vertices under the action of the automorphism group of G; under suitable conditions, every eigenvalue of P is also an eigenvalue of $A(G)$. When $G = SR(n, d)$, the spectrum of $P(G)$ appears to be a proper subset of that of $A(G)$; on the other hand, in all cases we have checked computationally ($d = 4, n \leq 30$; $d = 5, n \leq 25$), the matrices $P(SR(n, d))$ have integral spectra, which is consistent with Conjecture 1.3.

Question 4.2 *Is $SR(d, n)$ determined up to isomorphism by its spectrum?*

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Interpolation, box splines, and lattice points in zonotopes

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Abstract. Given a finite list of vectors $X \subseteq \mathbb{R}^d$, one can define the box spline B_X. Box splines are piecewise polynomial functions that are used in approximation theory. They are also interesting from a combinatorial point of view and many of their properties solely depend on the structure of the matroid defined by the list X. The support of the box spline is the zonotope $Z(X)$. We show that if the list X is totally unimodular, any real-valued function defined on the set of lattice points in the interior of $Z(X)$ can be extended to a function on $Z(X)$ of the form $p(D)B_X$ in a unique way, where $p(D)$ is a differential operator that is contained in the so-called internal $\mathcal{P}$-space. This was conjectured by Olga Holtz and Amos Ron. We also point out connections between this interpolation problem and matroid theory, including a deletion-contraction decomposition.

Résumé. Étant donné une liste finie de vecteurs $X \subseteq \mathbb{R}^d$, on peut définir la box spline B_X. Les box splines sont des fonctions continues par morceaux qui sont utilisées en théorie de l'approximation. Elles sont aussi intéressantes d'un point de vue combinatoire et beaucoup de leurs propriétés dépendent uniquement de la structure du matroïde défini par la liste X. Le support de la box spline est le zonotope $Z(X)$. Si la liste X est totalement unimodulaire, nous démontrons que toute fonction à valeurs réelles définie sur l'ensemble des points du réseau à l'intérieur de $Z(X)$ peut être étendue à une fonction sur $Z(X)$ de la forme $p(D)B_X$ de manière unique, où $p(D)$ est un opérateur différentiel qui est contenu dans l'espace appelé $\mathcal{P}$-interne. Cela a été conjecturé par Olga Holtz et Amos Ron. Nous indiquons aussi des relations entre ce problème d'interpolation et la théorie des matroïdes, en plus d'une décomposition suppressions-contractions.

Keywords: matroid, zonotope, lattice points, interpolation, box spline

1 Introduction

Given a set $\Theta = \{u_1, \ldots, u_k\}$ of k distinct points on the real line and a function $f : \Theta \to \mathbb{R}$, it is well-known that there exists a unique polynomial p_f in the space of univariate polynomials of degree at most $k-1$ s. t. $p_f(u_i) = f(u_i)$ for $i = 1, \ldots, k$.

If Θ is contained in $\mathbb{R}^d$ for an integer $d \geq 2$, the situation becomes more difficult. Not all of the properties of the univariate case can be preserved simultaneously. The minimal number m_Θ s. t. for every

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$f : \Theta \to \mathbb{R}$ there exists a polynomial $p_f \in \mathbb{R}[x_1, \ldots, x_d]$ of total degree at most m_Θ that satisfies $p_f(u_i) = f(u_i)$ depends on the geometric configuration of the points in Θ. Furthermore, the interpolating polynomial p_f of degree at most m_Θ is in general not uniquely determined. This is only possible if the dimension of the space of polynomials of degree at most m_Θ happens to be equal to k.

Uniqueness is possible if we choose the interpolating polynomials from a special space. Carl de Boor and Amos Ron introduced the *least solution* to the polynomial interpolation problem. For an arbitrary finite point set $\Theta \subseteq \mathbb{R}^d$, they construct a space of multivariate polynomials $\Pi(\Theta)$ that has dimension $|\Theta|$ and that contains a unique polynomial interpolating polynomial p_f for every function $f : \Theta \to \mathbb{R}$ [12, 13].

In this paper, we construct a space that contains unique interpolating functions for the special case where Θ is the set of lattice points in the interior of a zonotope. The space is of a very special nature: it is obtained by applying certain differential operators to the box spline. This is interesting because it connects various algebraic and combinatorial structures with interpolation and approximation theory. More information on multivariate polynomial interpolation can be found in the survey paper [17].

We use the following setup: U denotes a d-dimensional real vector space and $\Lambda \subseteq U$ a lattice. Let $X = (x_1, \ldots, x_N) \subseteq \Lambda$ be a finite list of vectors that spans U. We assume that X is totally unimodular with respect to Λ, *i. e.* every basis for U that can be selected from X is also a lattice basis. The symmetric algebra over U is denoted by $\mathrm{Sym}(U)$. We fix a basis $s_1, \ldots, s_d$ for the lattice. This makes it possible to identify Λ with $\mathbb{Z}^d$, U with $\mathbb{R}^d$, $\mathrm{Sym}(U)$ with the polynomial ring $\mathbb{R}[s_1, \ldots, s_d]$, and X with a $(d \times N)$ matrix. Then X is totally unimodular if and only if every non-singular square submatrix of this matrix has determinant 1 or -1. A base-free setup is however more convenient when working with quotient vector spaces.

The *zonotope* $Z(X)$ is defined as

$$Z(X) := \left\{ \sum_{i=1}^{N} \lambda_i x_i : 0 \le \lambda_i \le 1 \right\}. \tag{1}$$

We denote its set of interior lattice points by $\mathcal{Z}_-(X) := \mathrm{int}(Z(X)) \cap \Lambda$. The *box spline* $B_X : U \to \mathbb{R}$ is a piecewise polynomial function that is supported on the zonotope $Z(X)$. It is defined by

$$B_X(u) := \frac{1}{\sqrt{\det(XX^T)}} \mathrm{vol}_{N-d} \left\{ (\lambda_1, \ldots, \lambda_N) \in [0,1]^N : \sum_{i=1}^{N} \lambda_i x_i = u \right\}. \tag{2}$$

For examples, see Figure 1 and Example 10. A good reference for box splines and their applications in approximation theory is [11]. Our terminology is closer to [14, Chapter 7], where splines are studied from an algebraic point of view.

A vector $u \in U$ defines a linear form $p_x \in \mathrm{Sym}(U)$. For a sublist $Y \subseteq X$, we define $p_Y := \prod_{y \in Y} p_y$. For example, if $Y = ((1,0),(1,2))$, then $p_Y = s_1^2 + 2s_1 s_2$. Now we define the

$$\textit{central } \mathcal{P}\textit{-space } \mathcal{P}(X) := \mathrm{span}\{p_Y : \mathrm{rk}(X \setminus Y) = \mathrm{rk}(X)\} \tag{3}$$

$$\text{and the } \textit{internal } \mathcal{P}\textit{-space } \mathcal{P}_-(X) := \bigcap_{x \in X} \mathcal{P}(X \setminus x). \tag{4}$$

The space $\mathcal{P}_-(X)$ was introduced in [18] where it was also shown that the dimension of this space is equal to $|\mathcal{Z}_-(X)|$. The space $\mathcal{P}(X)$ first appeared in approximation theory [1, 10, 16]. Later, spaces of this type and generalisations were also studied by authors in other fields, *e. g.* [2, 4, 19, 22, 24, 26, 29].

We will let the elements of $\mathcal{P}_-(X)$ act as differential operators on the box spline. For $p \in \mathcal{P}_-(X) \subseteq \mathrm{Sym}(U) \cong \mathbb{R}[s_1, \ldots, s_r]$, we write $p(D)$ to denote the differential operator obtained from p by replacing the variable s_i by $\frac{\partial}{\partial s_i}$.

The following proposition ensures that the box spline is sufficiently smooth so that the derivatives that appear in the Main Theorem actually exist.

Proposition 1. *Let $X \subseteq \Lambda \subseteq U \cong \mathbb{R}^d$ be a list of vectors that is totally unimodular and let $p \in \mathcal{P}_-(X)$. Then $p(D)B_X$ is a continuous function.*

Now we are ready to state the Main Theorem. It was conjectured by Olga Holtz and Amos Ron [18, Conjecture 1.8].

Theorem 2 (Main Theorem)**.** *Let $X \subseteq \Lambda \subseteq U \cong \mathbb{R}^d$ be a list of vectors that is totally unimodular. Let f be a real valued function on $\mathcal{Z}_-(X)$, the set of interior lattice points of the zonotope defined by X.*

Then there exists a unique polynomial $p \in \mathcal{P}_-(X) \subseteq \mathbb{R}[s_1, \ldots, s_d]$, s. t. $p(D)B_X$ equals f on $\mathcal{Z}_-(X)$.

Here, $p(D)$ denotes the differential operator obtained from p by replacing the variable s_i by $\frac{\partial}{\partial s_i}$ and B_X denotes to the box spline defined by X.

Remark 3. Total unimodularity of the list X is a crucial requirement in Theorem 2. Namely, the dimension of $\mathcal{P}_-(X)$ and $|\mathcal{Z}_-(X)|$ agree if and only if X is totally unimodular. Note that if one vector in X is multiplied by an integer $\lambda \geq 2$, $|\mathcal{Z}_-(X)|$ increases while $\mathcal{P}_-(X)$ stays the same.

Total unimodularity also enables us to make a simple deletion-contraction proof: it implies that Λ/x is a lattice for all $x \in X$. In general, quotients of lattices may contain torsion elements.

Remark 4. We have mentioned above that $\dim(\mathcal{P}_-(X)) = |\mathcal{Z}_-(X)|$ holds. This is a consequence of a deep connection between the spaces $\mathcal{P}_-(X)$ and $\mathcal{P}(X)$ and matroid theory. The Hilbert series of these two spaces are evaluations of the Tutte polynomial of the matroid defined by X [2]. One can deduce that the Hilbert series of the internal space is equal to the h-polynomial of the broken-circuit complex [5] of the matroid $M^*(X)$ that is dual to the matroid defined by X and the Hilbert series of the central space equals the h-polynomial of the matroid complex of $M^*(X)$. The Ehrhart polynomial of a zonotope that is defined by a totally unimodular matrix is also an evaluation of the Tutte polynomial (see *e. g.* [30]). In summary, for a totally unimodular matrix X

$$\dim \mathcal{P}_-(X) = |\mathcal{Z}_-(X)| = \mathfrak{T}_X(0,1) \text{ and } \dim \mathcal{P}(X) = \mathrm{vol}(Z(X)) = \mathfrak{T}_X(1,1) \tag{5}$$

holds, where $\mathfrak{T}_X$ denotes the Tutte polynomial of the matroid defined by X.

It is also interesting to know that the Ehrhart polynomial of an arbitrary zonotope defined by an integer matrix is an evaluation of the arithmetic Tutte polynomial [6, 7].

The full-length version of this extended abstract that includes all proof is available on the arXiv [23]. Various related papers have been presented at FPSAC in recent years (*e. g.* [2, 21, 27]).

Organisation of this extended abstract. In Section 2 we will discuss some basic properties of splines. Section 3 is devoted to the one-dimensional case. In Section 4 we will recall the wall-crossing formula for splines that can be used to prove Proposition 1. In Section 5 we will define deletion and contraction and in Section 6 we will state a deletion-contraction decomposition of our interpolation problem that implies the Main Theorem.

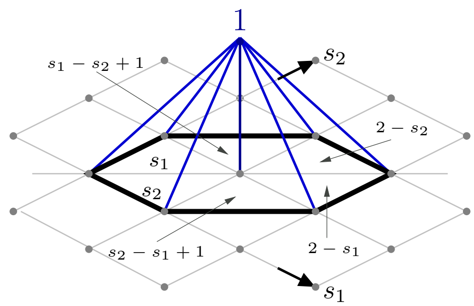

Figure 1: A very simple two-dimensional example. Here, $X = ((1,0),(0,1),(1,1))$, $\mathcal{P}_-(X) = \mathbb{R}$, and there is only one interior lattice point in $Z(X)$.

2 Splines

In this section we will introduce the multivariate spline and discuss some basic properties of splines. Proofs of the results that we mention here can be found in [14, Chapter 7] and some also in [11].

If the convex hull of the vectors in X does not contain 0, we define the *multivariate spline* (or truncated power) $T_X : U \to \mathbb{R}$ by

$$T_X(u) := \frac{1}{\sqrt{\det(XX^T)}} \operatorname{vol}_{N-d}\{(\lambda_1,\ldots,\lambda_N) \in \mathbb{R}^N_{\geq 0} : \sum_{i=1}^N \lambda_i x_i = u\}. \tag{6}$$

The support of T_X is the *cone* $\operatorname{cone}(X) := \left\{\sum_{i=1}^N \lambda_i x_i : \lambda_i \geq 0\right\}$.

Sometimes it is useful to think of the two splines B_X and T_X as distributions. In particular, one can then define the splines for lists $X \subseteq U$ that do not span U.

Remark 5. Let $X \subseteq U \cong \mathbb{R}^r$ be a finite list of vectors. The multivariate spline T_X and the box spline B_X are distributions that are characterised by the formulae

$$\int_U \varphi(u) B_X(u)\,\mathrm{d}u = \int_0^1 \cdots \int_0^1 \varphi\left(\sum_{i=1}^N \lambda_i x_i\right) \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_N \tag{7}$$

$$\text{and } \int_U \varphi(u) T_X(u)\,\mathrm{d}u = \int_0^\infty \cdots \int_0^\infty \varphi\left(\sum_{i=1}^N \lambda_i x_i\right) \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_N. \tag{8}$$

where φ denotes a test function.

Remark 6. Convolutions of splines are again splines. In particular,

$$T_X = T_{x_1} * \cdots * T_{x_N} \text{ and } B_X = B_{x_1} * \cdots * B_{x_n}. \tag{9}$$

For $x \in X$, differentiation of the two splines in direction x is particularly easy:

$$D_x T_X = T_{X \setminus x} \tag{10}$$

$$\text{and } D_x B_X = \nabla_x B_{X\setminus x} := B_{X\setminus x} - B_{X\setminus x}(\cdot - x). \tag{11}$$

Remark 7. For a basis $C \subseteq U$,

$$B_C = \frac{\chi_{Z(C)}}{|\det(C)|} \text{ and } T_C = \frac{\chi_{\mathrm{cone}(C)}}{|\det(C)|}, \tag{12}$$

where $\chi_A : U \to \{0, 1\}$ denotes the indicator function of the set $A \subseteq U$. In conjunction with (9), (12) provides a simple recursive method to calculate the splines.

Remark 8. The box spline can easily be obtained from the multivariate spline. Namely,

$$B_X(u) = \sum_{S \subseteq X} (-1)^{|S|} T_X\left(u - a_S\right), \tag{13}$$

where $a_S := \sum_{a \in S} a$.

3 Cardinal B-splines

In this section we will discuss the one-dimensional case of Theorem 2. This case can be used as base case for the inductive proof of the Main Theorem.

Let $X_N := (\underbrace{1, \ldots, 1}_{N \text{ times}}) \subseteq \mathbb{Z} \subseteq \mathbb{R}^1$. WLOG every totally unimodular list of vectors in $\mathbb{R}^1$ can be written in this way.

One can easily calculate the corresponding box splines (cf. Remark 7):

$$B_{X_{N+1}}(u) = \int_0^1 B_{X_N}(u - \tau)\,\mathrm{d}\tau = \sum_{j=0}^{N+1} \frac{(-1)^j}{N!} \binom{N+1}{j} (u - j)_+^N, \tag{14}$$

where $(u - j)_+^N := \max(u - j, 0)^N$. The functions $B_{X_{N+1}}$ are called *cardinal B-splines* in the literature (*e. g.* [9]).

Note that $\mathcal{Z}_-(X_{N+1}) = \{1, 2, \ldots, N\}$,

$$\mathcal{P}_{X_{N+1}} = \mathrm{span}\{1, s, \ldots, s^N\}, \text{ and } \mathcal{P}_-(X_{N+1}) = \mathrm{span}\{1, s, \ldots, s^{N-1}\}.$$

Hence, in the one-dimensional case, Theorem 2 is equivalent to the following proposition.

Proposition 9. *Let $N \in \mathbb{N}$. For every function $f : \{1, \ldots, N\} \to \mathbb{R}$, there exist uniquely determined numbers $\lambda_1, \ldots, \lambda_N \in \mathbb{R}$ s. t.*

$$\sum_{i=1}^{N} \lambda_i D_x^{i-1} B_{X_{N+1}}(j) = f(j) \text{ for } j = 1, \ldots, N. \tag{15}$$

For $N \in \mathbb{N}$, we consider the matrix $(N \times N)$-matrix M^N whose entries are given by

$$m_{ij}^N = D_x^{i-1} B_{X_{N+1}}(j). \tag{16}$$

Proposition 9 is equivalent to M^N having full rank. Here are a few simple examples (see also Figure 2).

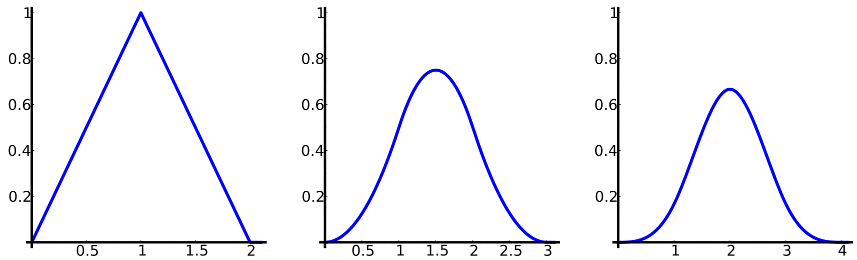

Figure 2: The cardinal B-splines B_2, B_3, and B_4.

Example 10.

$$B_{X_2}(s) = s - 2(s-1)_+ + (s-2)_+ \tag{17}$$

$$B_{X_3}(s) = \frac{1}{2}\left(s^2 - 3(s-1)_+^2 + 3(s-2)_+^2 - (s-3)_+^2\right) \tag{18}$$

$$B_{X_4}(s) = \frac{1}{6}\left(s^3 - 4(s-1)_+^3 + 6(s-2)_+^3 - 4(s-3)_+^3 + (s-4)_+^3\right) \tag{19}$$

$$M^2 = \begin{pmatrix} 1 \end{pmatrix} \qquad M^3 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & -1 \end{pmatrix} \qquad M^4 = \begin{pmatrix} \frac{1}{6} & \frac{4}{6} & \frac{1}{6} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ 1 & -2 & 1 \end{pmatrix}$$

4 Smoothness and wall-crossing

In this section we will discuss the proof of Proposition 1. First, we will mention some results on the structure of the multivariate spline T_X that can be used in the proof. The Wall-Crossing Theorem describes the behaviour of T_X when we pass from one region of polynomiality to another.

Definition 11. A *tope* is a connected component of the complement of

$$\mathcal{H}_X := \{\mathrm{span}(Y) : Y \subseteq X,\, \mathrm{rk}(Y) = \mathrm{rk}(X) - 1\} \subseteq U \tag{20}$$

The following theorem is a consequence of Lemma 3.3 and Proposition 3.7 in [15].

Theorem 12. *Let $X \subseteq U \cong \mathbb{R}^d$ be a list of vectors N that spans U and whose convex hull does not contain 0.*

Then T_X agrees with a homogeneous polynomial f^τ of degree $N-d$ on every tope τ.

Given a hyperplane H and a tope τ which does not intersect H (but its closure may do so), we partition $X \setminus H$ into two sets A_H^τ and B_H^τ. The set A_H^τ contains the vectors that lie on the same side of H as τ and B_H^τ contains the vectors that lie on the other side. Note that the convex hull of $(A_H^\tau, -B_H^\tau)$ does not contain 0. Hence, we can define the multivariate spline

$$T_{X\setminus H}^\tau := (-1)^{|B_H^\tau|} T_{(A_H^\tau, -B_H^\tau)}. \tag{21}$$

Now we are ready to state the wall-crossing formula as in [15, Theorem 4.10]. Related results are in [8, 28].

Theorem 13 (Wall-crossing for multivariate splines)**.** *Let τ_1 and τ_2 be two topes whose closures have an $r-1$ dimensional intersection τ_{12} that spans a hyperplane H. Then there exists a uniquely determined distribution $f^{\tau_{12}}$ that is supported on H s. t. the difference of the local pieces of T_X in τ_1 and τ_2 is equal to the polynomial*

$$T_X^{\tau_1} - T_X^{\tau_2} = (T_{X\setminus H}^{\tau_1} - T_{X\setminus H}^{-\tau_1}) * f^{\tau_{12}}. \tag{22}$$

Proof of Proposition 1 (sketch): Let $u \in U$. If $u \in U \setminus \mathcal{H}_X$, there is nothing to prove: by Theorem 12, T_X is polynomial in a neighbourhood of u and hence smooth. If $u \in \mathcal{H}_X$, u is contained in the closure of at least two topes. We have to show that the derivatives of the polynomial pieces in the topes agree on u. This can be done using the wall-crossing formula. □

Remark 14. Holtz and Ron conjectured that $\mathcal{P}_-(X)$ is spanned by polynomials p_Y where Y runs over all sublists of X s. t. $X \setminus (Y \cup x)$ has full rank for all $x \in X$ [18, Conjecture 6.1]. By formula (11), this would have implied Proposition 1. However, this conjecture has recently been disproved [3].

5 Deletion and contraction

In this section we will introduce the operations deletion and contraction which will can be used in an inductive proof of the Main Theorem. We will also state two lemmas about deletion and contraction for box splines and zonotopes.

Let $x \in X$. We call the list $X \setminus x$ the *deletion* of x. The image of $X \setminus x$ under the canonical projection $\pi_x : U \to U/\operatorname{span}(x) =: U/x$ is called the *contraction* of x. It is denoted by X/x.

The projection π_x induces a map $\mathrm{Sym}(\pi_x) : \mathrm{Sym}(U) \to \mathrm{Sym}(U/x)$. If we identify $\mathrm{Sym}(U)$ with the polynomial ring $\mathbb{R}[s_1, \ldots, s_r]$ and $x = s_r$, then $\mathrm{Sym}(\pi_x)$ is the map from $\mathbb{R}[s_1, \ldots, s_r]$ to $\mathbb{R}[s_1, \ldots, s_{r-1}]$ that sends s_r to zero and $s_1, \ldots, s_{r-1}$ to themselves. The space $\mathcal{P}(X/x)$ is contained in the symmetric algebra $\mathrm{Sym}(U/x)$.

Since X is totally unimodular, $\Lambda/x \subseteq U/x$ is a lattice for every $x \in X$ and X/x is totally unimodular with respect to this lattice.

The following two lemmas describe the behaviour zonotopes and box splines under deletion and contraction. See Figure 3 for an illustration of the first lemma.

Lemma 15. *The following map is a bijection:*

$$\mathcal{Z}_-(X) \setminus \mathcal{Z}_-(X \setminus x) \to \mathcal{Z}_-(X/x) \tag{23}$$

$$z \mapsto \bar{z}. \tag{24}$$

Lemma 16. *Let $x \in X$, $u \in U$, and $\bar{u} = u + \operatorname{span}(x)$ the coset of u in X/x. Then*

$$B_{X/x}(\bar{u}) = \int_{\mathbb{R}} B_{X\setminus x}(u + \tau x)\,\mathrm{d}\tau = \sum_{\lambda \in \mathbb{Z}} B_X(u + \lambda x). \tag{25}$$

6 Exact sequences

In this section we will state a result involving deletion and contraction that implies the Main Theorem. We start with a simple observation.

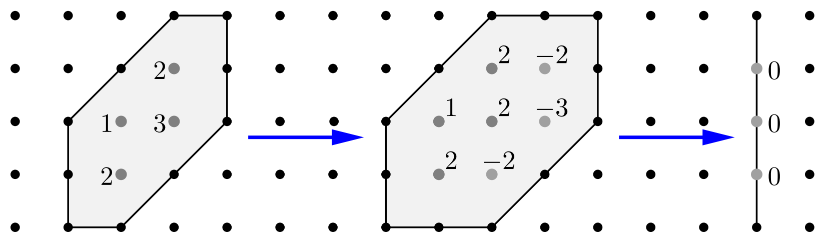

Figure 3: Deletion and contraction for a zonotope and a function defined on the interior lattice points of the zonotope.

Remark 17. If X contains a *coloop*, *i. e.* an element x s. t. $\operatorname{rk}(X \setminus x) < \operatorname{rk}(X)$, then $\mathcal{Z}_-(X) = \emptyset$ and $\mathcal{P}_-(X) = \{0\}$. Hence, Theorem 2 is trivially satisfied.

We will consider the set $\Xi(X) := \{f : \Lambda \to \mathbb{R} : \operatorname{supp}(f) \subseteq \mathcal{Z}_-(X)\}$ and the map

$$\begin{aligned} \gamma_X : \mathcal{P}_-(X) &\to \Xi(X) \\ p &\mapsto \big[\, \Lambda \ni z \mapsto p(D)B_X(z) \,\big] \end{aligned} \tag{26}$$

Note that the Main Theorem is equivalent to γ_X being an isomorphism.

Proposition 18. *Let $d \geq 2$ and let $\Lambda \subseteq U \cong \mathbb{R}^d$ be a lattice. Let $X \subseteq \Lambda$ be a finite list of vectors that spans U and that is totally unimodular with respect to Λ. Let $x \in X$ be a non-zero element s. t. $\operatorname{rk}(X \setminus x) = \operatorname{rk}(X)$.*

Then the following diagram of real vector spaces is commutative, the rows are exact and the vertical maps are isomorphisms:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{P}_-(X \setminus x) \arrow[r, "\cdot p_x"] \arrow[d, "\gamma_{X\setminus x}"] & \mathcal{P}_-(X) \arrow[r, "\operatorname{Sym}(\pi_x)"] \arrow[d, "\gamma_X"] & \mathcal{P}_-(X/x) \arrow[r] \arrow[d, "\gamma_{X/x}"] & 0 \\ 0 \arrow[r] & \Xi(X \setminus x) \arrow[r, "\nabla_x"] & \Xi(X) \arrow[r, "\Sigma_x"] & \Xi(X/x) \arrow[r] & 0 \end{tikzcd} \tag{27}$$

$$\textit{where } \nabla_x(f)(z) := f(z) - f(z - x), \tag{28}$$

$$\Sigma_x(f)(\bar{z}) := \sum_{x \in \bar{z} \cap \Lambda} f(x) = \sum_{\lambda \in \mathbb{Z}} f(\lambda x + z) \textit{ for some } z \in \bar{z}. \tag{29}$$

7 Outlook

In a forthcoming paper the Main Theorem will be made more explicit. Namely, in [25] polynomials $f_z \in \mathcal{P}_-(X)$ will be constructed for all $z \in \mathcal{Z}_-(X)$ s. t. $f_z(D)B_X$ equals one on z and vanishes elsewhere on $\mathcal{Z}_-(X)$. The construction of these polynomials involves Todd operators.

A variant of the Khovanskii-Pukhlikov formula [20] that relates the volume and the number of integer points in a smooth polytope is obtained as a corollary.

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Renormalization group-like proof of the universality of the Tutte polynomial for matroids

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Abstract In this paper we give a new proof of the universality of the Tutte polynomial for matroids. This proof uses appropriate characters of Hopf algebra of matroids, algebra introduced by Schmitt (1994). We show that these Hopf algebra characters are solutions of some differential equations which are of the same type as the differential equations used to describe the renormalization group flow in quantum field theory. This approach allows us to also prove, in a different way, a matroid Tutte polynomial convolution formula published by Kook, Reiner and Stanton (1999). This FPSAC contribution is an extended abstract.

Résumé. Dans cet article, nous donnons une nouvelle preuve de l'universalité du polynôme de Tutte pour les matroïdes. Cette preuve utilise des caractères appropriés de l'algèbre de Hopf des matroïdes introduite par Schmitt (1994). Nous montrons que ces caractères algèbre de Hopf sont des solutions de des équations différentielles du même type que les équations différentielles utilisées pour décrire le flux du groupe de renormalisation en théorie quantique de champs. Cette approche nous permet aussi de démontrer, d'une manière différente, une formule de convolution du polynôme de Tutte des matroïdes, formule publiée par Kook, Reiner et Stanton (1999). Cette contribution FPSAC est un résumé étendu.

Keywords: Tutte polynomial for matroids, Hopf algebras for matroids, Hopf algebra characters, matroid recipe theorem, Combinatorial Physics

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1 Introduction

The interplay between algebraic combinatorics and quantum field theory (QFT) has become more and more present within the spectrum of Combinatorial Physics (spectrum represented by many other subjects, such as the combinatorics of quantum mechanics, of statistical physics or of integrable systems - see, for example, Blasiak and Flajolet (2011), Blasiak et al. (2010), the review article Di Francesco (2012), the Habilitation Tanasa (2012) and references within).

One of the most known results lying at this frontier between algebraic combinatorics and QFT is the celebrated Hopf algebra Connes and Kreimer (2000), describing the combinatorics of renormalization in QFT. It is worth emphasizing that the coproduct of this type of Hopf algebra is based on a selection/contraction rule (one has on the coproduct left hand side (lhs) some selection of a subpart of the entity the coproduct acts on, while on the coproduct right hand side (rhs) one has the result of the *contraction* of the selected subpart). This type of rule (appearing in other situations in Mathematical Physics, see also Tanasa and Vignes-Tourneret (2008); Tanasa and Kreimer (2012); Markopoulou (2003); Tanasa (2010)) is manifestly distinct from the selection/deletion one, largely studied in algebraic combinatorics (see, for example, Duchamp et al. (2011) and references within).

In this paper, we use characters of the matroid Hopf algebra introduced in Schmitt (1994) to prove the universality property of the Tutte polynomial for matroids. We use a Combinatorial Physics approach, namely we use a renormalization group-like differential equation to prove the respective recipe theorem. Our method also allows to give a new proof of a matroid Tutte polynomial convolution formula given in Kook et al. (1999). This approach generalizes the one given in Krajewski and Martinetti (2011) for the universality of the Tutte polynomial for graphs. Moreover, the demonstrations we give here allow us to also have proofs of the graph results conjectured in Krajewski and Martinetti (2011).

The paper is structured as follows. In the following section we briefly present the renormalization group flow equation and show that equations of such type have already been successfully used in combinatorics. The third section defines matroids, as well the Tutte polynomial for matroids and the matroid Hopf algebra defined in Schmitt (1994). The following section defines some particular infinitesimal characters of this Hopf algebra as well as their exponential - proven, later on, to be non-trivially related to the Tutte polynomial for matroids. The fifth section uses all this tools to give a new proof of the matroid Tutte polynomial convolution formula given in Kook et al. (1999) and of the recipe theorem for the Tutte polynomial of matroids.

2 Renormalization group in quantum field theory - a glimpse

A QFT model (for a general introduction to QFT and not just an introduction, see for example the book Zinn-Justin (2002)) is defined by means of a functional integral of the exponential of an *action* S which, from a mathematical point of view, is a functional of the *fields* of the model. For the Φ^4 scalar model - the simplest QFT model - there is only one type of field, which we denote by $\Phi(x)$. From a mathematical point of view, for an Euclidean QFT scalar model, the field $\Phi(x)$ is a function, $\Phi : \mathbb{R}^D \to \mathbb{K}$, where D is usually taken equal to 4 (the dimension of the space) and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ (real, respectively complex fields).

The quantities computed in QFT are generally divergent. One thus has to consider a real, positive, *cut-off* Λ - the flowing parameter. This leads to a family of cut-off dependent actions, family denoted by S_Λ. The derivation $\Lambda\frac{\partial S_\Lambda}{\partial\Lambda}$ gives the *renormalization group equation*.

The quadratic part of the action - the *propagator* of the model - can be written in the following way

$$C_{\Lambda,\Lambda_0}(p,q) = \delta(p-q)\int_{\frac{1}{\Lambda_0}}^{\frac{1}{\Lambda}} d\alpha e^{-\alpha p^2}, \tag{2.1}$$

with p and q living in the Fourier transformed space $\mathbb{R}^D$ and Λ_0 a second real, positive cut-off. In perturbative QFT, one has to consider *Feynman graphs*, and to associate to each such a graph a *Feynman integral* (further related to quantities actually measured in physical experiments). The contribution of an edge of such a Feynman graph to its associated Feynman integral is given by an integral such as (2.1).

One can then get (see Polchinski (1984)) the Polchinski flow equation

$$\Lambda\frac{\partial S_\Lambda}{\partial\Lambda} = \int_{\mathbb{R}^{2D}} \frac{1}{2} d^D p d^D q \Lambda \frac{\partial C_{\Lambda,\Lambda_0}}{\partial\Lambda}\left(\frac{\delta^2 S}{\delta\tilde{\Phi}(p)\delta\tilde{\Phi}(q)} - \frac{\delta S}{\delta\tilde{\Phi}(p)}\frac{\delta S}{\delta\tilde{\Phi}(q)}\right), \tag{2.2}$$

where $\tilde{\Phi}$ represents the Fourier transform of the function Φ. The first term in the right hand side (rhs) of the equation above corresponds to the derivation of a propagator associated to a bridge in the respective Feynman graph. The second term corresponds to an edge which is not a bridge and is part of some circuit in the graph. One can see this diagrammatically in Fig. 1.

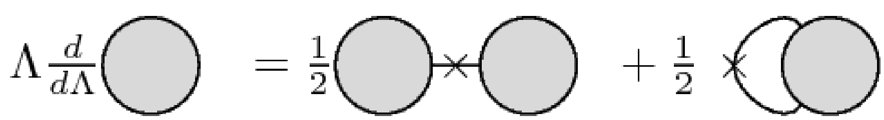

Fig. 1: Diagrammatic representation of the flow equation.

This equation can then be used to prove perturbative renormalizability in QFT. Let us also stress here, that an equation of this type is also used to prove a result of E. M. Wright which expresses the generating function of connected graphs under certain conditions (fixed excess). To get this generating functional (see, for example, Proposition $II.6$ the book Flajolet and Sedgewick (2008)), one needs to consider contributions of two types of edges (first contribution when the edge is a bridge and a second one when not - see again Fig. 1).

As already announced in the Introduction, we will use such an equation to prove the universality of the matroid Tutte polynomial (see section 5).

3 Matroids: the Tutte polynomial and the Hopf algebra

In this section we recall the definition and some properties of the Tutte polynomial for matroids as well as of the matroid Hopf algebra defined in Schmitt (1994).

Following the book Oxley (1992), one has the following definitions:

Definition 3.1 *A* **matroid** *M is a pair* $(E,\mathcal{I})$ *consisting of a finite set E and a collection of subsets of E satisfying:*

(I1) $\mathcal{I}$ *is non-empty.*

(I2) Every subset of every member of $\mathcal{I}$ is also in $\mathcal{I}$.

(I3) If X and Y are in $\mathcal{I}$ and $|X| = |Y| + 1$, then there is an element x in $X - Y$ such that $Y \cup \{x\}$ is in $\mathcal{I}$.

The set E is the ground set of the matroid and the members of $\mathcal{I}$ are the independent sets of the matroid.

One can associate matroid to graphs - graphic matroids. Nevertheless, not all matroids are graphic matroids.

Let E be an $n-$element set and let $\mathcal{I}$ be the collection of subsets of E with at most r elements, $0 \leq r \leq n$. One can check that $(E, \mathcal{I})$ is a matroid; it is called the **uniform matroid** $U_{r,n}$.

Remark 3.2 *If one takes $n = 1$, there are only two matroids, namely $U_{0,1}$ and $U_{1,1}$ and both of these matroids are graphic matroids. The graphs these two matroids correspond to are the graphs with one edge of Fig. 2 and Fig. 3. In the first case, the edge is a loop (in graph theoretical terminology) or a tadpole*

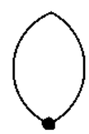

Fig. 2: The graph corresponding to the matroid $U_{0,1}$.

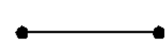

Fig. 3: The graph corresponding to the matroid $U_{1,1}$.

(in QFT language). In the second case, the edge represents a bridge (in graph theoretical terminology) or a 1-particle-reducible line (in QFT terminology) - the number of connected components of the graphs increases by 1 if one deletes the respective edge. In matroid terminology, these two particular cases correspond to a loop and respectively to a coloop (see Definitions 3.6 and respectively 3.7 below).

Definition 3.3 *The collection of maximal independent sets of a matroid are called bases. The collection of minimal dependent sets of a matroid are called circuits.*

Let $M = (E, \mathcal{I})$ be a matroid and let $\mathcal{B} = \{\mathcal{B}\}$ be the collection of bases of M. Let $\mathcal{B}^\star = \{E - B : B \in \mathcal{B}\}$. Then $\mathcal{B}^\star$ is the collection of bases of a matroid $M^\star$ on E. The matroid M^* is called the dual of M.

Definition 3.4 *Let $M = (E, \mathcal{I})$ be a matroid. The* **rank** *$r(A)$ of $A \subset E$ is defined as the cardinal of a maximal independent set in A.*

$$r(A) = max\{|B| \text{ s.t. } B \in \mathcal{I}, B \subset A\} . \tag{3.1}$$

Definition 3.5 *Let $M = (E, \mathcal{I})$ be a matroid with a ground set E. The* **nullity** *function is given by*

$$n(M) = |E| - r(M) . \tag{3.2}$$

Definition 3.6 *Let $M = (E, \mathcal{I})$ be a matroid. The element $e \in E$. is a* ***loop*** *iff $\{e\}$ is the circuit.*

Definition 3.7 *Let $M = (E, \mathcal{I})$ be a matroid. The element $e \in E$ is a* **coloop** *iff, for any basis B, $e \in B$.*

Let us now define two basic operations on matroids. Let M be a matroid $(E, \mathcal{I})$ and T be a subset of E. Let $\mathcal{I}' = \{I \subseteq E - T : I \in \mathcal{I}\}$. One can check that $(E - T, \mathcal{I}')$ is a matroid. We denote this matroid by $M \backslash T$ - the **deletion** of T from M. The **contraction** of T from M, M/T, is given by the formula: $M/T = (M^\star \backslash T)^\star$.

Let us also recall the following results:

Lemma 3.8 *Let M be a matroid $(E, \mathcal{I})$ and T be a subset of E. One has*

$$M|_T = M\backslash_{E-T} \; . \tag{3.3}$$

Lemma 3.9 *If e is a coloop of a matroid $M = (E, \mathcal{I})$, then $M/e = M \backslash e$.*

Lemma 3.10 *Let $M = (E, \mathcal{I})$ be a matroid and $T \subseteq E$, then, for all $X \subseteq E - T$,*

$$r_{M/T}(X) = r_M(X \cup T) - r_M(T) \; . \tag{3.4}$$

Let us now define the Tutte polynomial for matroids:

Definition 3.11 *Let $M = (E, \mathcal{I})$ be a matroid. The* **Tutte polynomial** *is given by the following formula:*

$$T_M(x, y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{n(A)}. \tag{3.5}$$

The sum is computed over all subset of the matroid's ground set.

Definition 3.12 *Let ψ be the matroid duality map, that is a map associating to any matroid M its dual, $\psi(M) = M^\star$.*

It is worth stressing here that one can define the dual of any matroid; this is not the case for graphs, where only the dual of planar graph can be defined.

Let us recall, from Brylawsky and Oxley (1992) that

$$T_M(x, y) = T_{M^\star}(y, x). \tag{3.6}$$

In Schmitt (1994), as a particularization of a more general construction of incidence Hopf algebras, the following result was proved:

Proposition 3.13 *If $\mathcal{M}$ is a minor-closed family of matroids then $k(\widetilde{\mathcal{M}})$ is a coalgebra, with coproduct Δ and counit ϵ determined by*

$$\Delta(M) = \sum_{A \subseteq E} M|A \otimes M/A \tag{3.7}$$

and $\epsilon(M) = \begin{cases} 1, & \text{if } E = \emptyset, \\ 0 & \text{otherwise}, \end{cases}$ *for all $M = (E, \mathcal{I}) \in \mathcal{M}$. If, furthermore, the family $\mathcal{M}$ is closed under formation of direct sums, then $k(\widetilde{\mathcal{M}})$ is a Hopf algebra, with product induced by direct sum.*

We refer to this Hopf algebra as the matroid Hopf algebra. We follow Crapo and Schmitt (2005) and, by a slight abuse of notation, we denote in the same way a matroid and its isomorphic class, since the distinction will be clear from the context (as it is already in Proposition 3.13).

We denote the unit of this Hopf algebra by $\mathbf{1}$ (the empty matroid, or $U_{0,0}$).

4 Characters of matroid Hopf algebra

Let us give the following definitions:

Definition 4.1 *Let f, g be two mappings in $Hom(\mathcal{M}, \mathcal{M})$. The convolution product of f and g is given by the following formula*

$$f * g = m \circ (f \otimes g) \circ \Delta, \tag{4.1}$$

where m denotes the Hopf algebra multiplication, given here by direct sum (see above).

Definition 4.2 *A matroid Hopf algebra* **character** *f is an algebra morphism from the matroid Hopf algebra into a fixed commutative ring $\mathbb{K}$, such that*

$$f(M_1 \oplus M_2) = f(M_1)f(M_2), \quad f(\mathbf{1}) = 1_{\mathbb{K}}. \tag{4.2}$$

Definition 4.3 *A matroid Hopf algebra* **infinitesimal character** *g is an algebra morphism from the matroid Hopf algebra into a fixed commutative ring $\mathbb{K}$, such that*

$$g(M_1 \oplus M_2) = f(M_1)\epsilon(M_2) + \epsilon(M_1)g(M_2). \tag{4.3}$$

Since we work in a Hopf algebra where the non-trivial part of the coproduct is nilpotent, we can also define an exponential map by the following expression

$$\exp_*(\delta) = \epsilon + \delta + \frac{1}{2}\delta * \delta + \dots \tag{4.4}$$

where δ is an infinitesimal character.

As already stated above (see Remark 3.2), there are only two matroids with unit cardinal ground set, $U_{0,1}$ and $U_{1,1}$. We now define two maps δ_{loop} and δ_{coloop}.

$$\delta_{\text{loop}}(M) = \begin{cases} 1_{\mathbb{K}} \text{ if } M = U_{0,1}, \\ 0_{\mathbb{K}} \text{ otherwise }. \end{cases} \tag{4.5}$$

$$\delta_{\text{coloop}}(M) = \begin{cases} 1_{\mathbb{K}} \text{ if } M = U_{1,1}, \\ 0_{\mathbb{K}} \text{ otherwise }. \end{cases} \tag{4.6}$$

One can directly check that these maps are *infinitesimal characters* of the matroid Hopf algebra defined above.

One then has:

Lemma 4.4 *Let $M = (E, \mathcal{I})$ be a matroid. One has*

$$exp_*\{a\delta_{\text{coloop}} + b\delta_{\text{loop}}\}(M) = a^{r(M)}b^{n(M)}. \tag{4.7}$$

Proof: Using the definition (4.4), the lhs of the identity (4.7) above writes:

$$\left(\sum_{k=0}^{\infty}\frac{(a\delta_{\text{coloop}}+b\delta_{\text{loop}})^k}{k!}\right)(M). \tag{4.8}$$

All the terms in the sum above vanish, except the one for whom k is equal to $|E|$. Using the definition (4.1) of the convolution product, this term writes

$$\frac{1}{k!}\left(\sum_{i=0}^{k}a^{k-i}b^i\sum_{\substack{i_1,\dots,i_n\\ j_1,\dots,j_m\\ i_1+\dots+i_n=k-i\\ j_1+\dots+j_m=i}}\delta_{\text{coloop}}^{\otimes(i_1)}\otimes\delta_{\text{loop}}^{\otimes(j_1)}\otimes\cdots\otimes\delta_{\text{coloop}}^{\otimes(i_n)}\otimes\delta_{\text{loop}}^{\otimes(j_m)}\right)\left(\sum_{(i)}M^{(1)}\otimes\cdots\otimes M^{(k)}\right), \tag{4.9}$$

where we have used the notation $\Delta^{(k-1)}(M)=\sum_{(i)}M^{(1)}\otimes\cdots\otimes M^{(k)}$. Using the definitions (4.5) and respectively (4.6) of the infinitesimal characters δ_{loop} and respectively δ_{coloop}, implies that the submatroids $M^{(j)}$ $(j=1,\dots,k)$ are equal to $U_{0,1}$ or $U_{1,1}$.

Using the definition of the nullity and of the rank of a matroid concludes the proof. □

We now define the following map:

$$\alpha(x,y,s,M):=\exp_* s\{\delta_{\text{coloop}}+(y-1)\delta_{\text{loop}}\}*\exp_* s\{(x-1)\delta_{\text{coloop}}+\delta_{\text{loop}}\}(M). \tag{4.10}$$

One then has:

Proposition 4.5 *The map* (4.10) *is a character.*

Proof: The proof can be done by a direct check. On a more general basis, this is a consequence of the fact that δ_{loop} and δ_{coloop} are infinitesimal characters and the space of infinitesimal characters is a vector space; thus $s\{\delta_{\text{coloop}}+(y-1)\delta_{\text{loop}}\}$ and $s\{(x-1)\delta_{\text{coloop}}+\delta_{\text{loop}}\}$ are infinitesimal characters. Since, $\exp_*(h)$ is a character when h is an infinitesimal character and since the convolution of two characters is a character, one gets that α is a character. □

Let us define a map

$$\varphi_{a,b}(M)\longmapsto a^{r(M)}b^{n(M)}M\ . \tag{4.11}$$

Lemma 4.6 *The map $\varphi_{a,b}$ is a bialgebra automorphism.*

Proof: One can directly check that the map $\varphi_{a,b}$ is an algebra automorphism. Let us now check that this map is also a coalgebra automorphism. Using Lemma 3.8 and Lemma 3.10,

$$r(M|_T)+r(M/T)=r(M). \tag{4.12}$$

Thus, using the definitions of the map $\varphi_{a,b}$ of the matroid coproduct, one has:

$$\Delta\circ\varphi_{a,b}(M)=\sum_{T\subseteq E}(a^{r(M|_T)}b^{n(M|_T)}M|_T)\otimes(a^{r(M/_T)}b^{n(M/_T)}M/_T). \tag{4.13}$$

Using again the definition of the map $\varphi_{a,b}$ leads to

$$\Delta \circ \varphi_{a,b}(M) = (\varphi_{a,b} \otimes \varphi_{a,b}) \circ \Delta(M), \tag{4.14}$$

which concludes the proof. □

5 Proof of the universality of the Tutte polynomial for matroids

Let $M = (E, \mathcal{I})$ be a matroid. One has:

$$\begin{aligned}\alpha(x, y, s, M) \;&=\; \exp_*\left(s(\delta_{\text{coloop}} + (y-1)\delta_{\text{loop}})\right) * \exp_*\left(s(-\delta_{\text{coloop}} + \delta_{\text{loop}})\right) \\ &* \;\exp_*\left(s(\delta_{\text{coloop}} - \delta_{\text{loop}})\right) * \exp_*\left(s((x-1)\delta_{\text{coloop}} + \delta_{\text{loop}})\right).\end{aligned} \tag{5.1}$$

Proposition 5.1 *Let $M = (E, \mathcal{I})$ be a matroid. The character α is related to the Tutte polynomial of matroids by the following identity:*

$$\alpha(x, y, s, M) = s^{|E|} T_M(x, y)\,. \tag{5.2}$$

Proof: Using the definition (4.1) of the convolution product in the definition (4.10) of the character α, one has the following identity:

$$\alpha(x, y, s, M) = \sum_{A \subseteq E} \exp_* s\{\delta_{\text{coloop}} + (y-1)\delta_{\text{loop}}\}(M|_A)\; \exp_* s\{(x-1)\delta_{\text{coloop}} + \delta_{\text{loop}}\}(M/A). \tag{5.3}$$

We can now apply Lemma 4.4 on each of the two terms in the rhs of equation (5.3) above. This leads to the result. □

Using (3.6) and the Proposition 5.1, one has:

Corollary 5.2 *One has:*

$$\alpha(x, y, s, M) = \alpha(y, x, s, M^\star). \tag{5.4}$$

This allows to give a different proof of a matroid Tutte polynomial convolution identity, which was shown in Kook et al. (1999). One has:

Corollary 5.3 *(Theorem 1 of Kook et al. (1999)) The Tutte polynomial satisfies*

$$T_M(x, y) = \sum_{A \subset E} T_{M|A}(0, y) T_{M/A}(x, 0). \tag{5.5}$$

Proof: Taking $s = 1$, this is as a direct consequence of identity (5.1), and of Proposition 5.1. □

Let us now define:

$$[f, g]_* := f * g - g * f. \tag{5.6}$$

Using the definition (4.10) of the Hopf algebra character α, one can directly prove the following result:

Proposition 5.4 *The character α is the solution of the differential equation:*

$$\frac{d\alpha}{ds} = x\alpha * \delta_{\text{coloop}} + y\delta_{\text{loop}} * \alpha + [\delta_{\text{coloop}}, \alpha]_* - [\delta_{\text{loop}}, \alpha]_* \ . \tag{5.7}$$

It is the fact that the matroid Tutte polynomial is a solution of the differential equation (5.7) that will be used now to prove the universality of the matroid Tutte polynomial. In order to do that, we take a four-variable matroid polynomial $Q_M(x, y, a, b)$ satisfying a multiplicative law and which has the following properties:

- if e is a coloop, then
$$Q_M(x, y, a, b) = xQ_{M\backslash e}(x, y, a, b) \ , \tag{5.8}$$
- if e is a loop, then
$$Q_M(x, y, a, b) = yQ_{M/e}(x, y, a, b) \tag{5.9}$$
- if e is a nonseparating point, then
$$Q_M(x, y, a, b) = aQ_{M\backslash e}(x, y, a, b) + bQ_{M/e}(x, y, a, b). \tag{5.10}$$

Remark 5.5 *Note that, when one deals with the same problem in the case of graphs, a supplementary multiplicative condition for the case of one-point joint of two graphs (i. e. identifying a vertex of the first graph and a vertex of the second graph into a single vertex of the resulting graph) is required (see, for example, Ellis-Monaghan and Merino (2010) or Sokal (2005)).*

We now define the map:

$$\beta(x, y, a, b, s, M) := s^{|E|} Q_M(x, y, a, b). \tag{5.11}$$

One then directly check (using the definition (5.11) above and the multiplicative property of the polynomial Q) that this map is again a matroid Hopf algebra character.

Proposition 5.6 *The character* (5.11) *satisfies the following differential equation:*

$$\frac{d\beta}{ds}(M) = \left(x\beta * \delta_{\text{coloop}} + y\delta_{\text{loop}} * \beta + b[\delta_{\text{coloop}}, \beta]_* - a[\delta_{\text{loop}}, \beta]_*\right)(M). \tag{5.12}$$

Proof: Applying the definition (4.1) of the convolution product, the rhs of equation (5.12) above writes

$$\begin{aligned} &= (x-b)\sum_{A\subseteq E} \beta(M|_A)\delta_{\text{coloop}}(M/_A) + (y-a)\sum_{A\subseteq E} \delta_{\text{loop}}(M|_A)\beta(M/_A) \\ &+ b\sum_{A\subseteq E} \delta_{\text{coloop}}(M|_A)\beta(M/_A) + a\sum_{A\subseteq E} \beta(M|_A)\delta_{\text{loop}}(M/_A). \end{aligned} \tag{5.13}$$

Using the definitions (4.5) and respectively (4.6) of the infinitesimal characters δ_{loop} and respectively δ_{coloop}, constraints the sums on the subsets A above. The rhs of (5.12) becomes:

$$\begin{aligned} &(x-b)\textstyle\sum_{A, M/_A = U_{1,1}} \beta(M|_A) + (y-a)\sum_{A, M|_A = U_{0,1}} \beta(M/_A) \\ &\quad + b\textstyle\sum_{A, M|_A = U_{1,1}} \beta(M/_A) + a\sum_{A, M/_A = U_{0,1}} \beta(M|_A) \end{aligned} \tag{5.14}$$

We now apply the definition of the Hopf algebra character β; one obtains:

$$s^{|E|-1}[(x-b)\sum_{A,M/_A=U_{1,1}} Q(x,y,a,b,M|_A)+(y-a)\sum_{A,M|_A=U_{0,1}} Q(x,y,a,b,M/_A) \\ +b\sum_{A,M|_A=U_{1,1}} Q(x,y,a,b,M/_A)+a\sum_{A,M/_A=U_{0,1}} Q(x,y,a,b,M|_A)]. \tag{5.15}$$

We can now directly analyze the four particular cases $M/_A = U_{1,1}$, $M/_A = U_{0,1}$, $M|_A = U_{1,1}$ and $M|_A = U_{0,1}$:

- If $M/_A = U_{1,1}$, we can denote the ground set of $M/_A$ by $\{e\}$. Note that e is a coloop. From the Lemma 3.8, one has $M|_A = M\backslash_{E-A} = M\backslash e$. One then has $Q(x,y,a,b,M) = xQ(x,y,a,b,M|_A)$.
- If $M|_A = U_{0,1}$, then $A=\{e\}$ and e is a loop of M. Thus, one has $Q(x,y,a,b,M) = yQ(x,y,a,b,M/_A)$
- If $M|_A = U_{1,1}$, then $A=\{e\}$. One has to distinguish between two subcases:
  - e is a coloop of M. Then, by Lemma 3.9, $M/e = M\backslash e$. Thus, one has $Q(x,y,a,b,M) = xQ(x,y,a,b,M|_A)$.
  - e is a nonseparating point of M.
- If $M/_A = U_{0,1}$, one can denote the ground set of $M/_A$ by $\{e\}$. There are again two subcases to be considered:
  - e is a loop of M, one has that $M|_A = M\backslash_{(E-A)} = M\backslash_{\{e\}} = M/e$. Then one has $Q(x,y,a,b,M) = yQ(x,y,a,b,M|_A)$.
  - e is a nonseparating point of M, then one has $M|_A = M\backslash_{(E-A)} = M\backslash_{\{e\}}$

We now insert all of this in equation (5.15); this leads to three types of sums over some element e of the ground set E, e being a loop, a coloop or a nonseparating point:

$$s^{|E|-1}[\sum_{e\in E:e \text{ is a coloop}} Q(x,y,a,b,M)+\sum_{e\in E:e \text{ is a loop}} Q(x,y,a,b,M)+\sum_{e\in E:e \text{ is a regular element}} Q(x,y,a,b,M)] \tag{5.16}$$

This rewrites as

$$|E|s^{|E|-1}Q(x,y,a,b,M) = \frac{d\beta}{ds}(M), \tag{5.17}$$

which completes the proof. □

We can now state the main result of this paper, the recipe theorem specifying how to recover the matroid polynomial Q as an evaluation of the Tutte polynomial T_M:

Theorem 5.7 *One has:*

$$Q(x,y,a,b,M) = a^{n(M)}b^{r(M)}T_M(\frac{x}{b},\frac{y}{a}). \tag{5.18}$$

Proof: The proof is a direct consequence of Propositions 5.1, 5.4 and 5.6 and of Lemma 4.6. This comes from the fact that one can apply the automorphism ϕ defined in (4.11) to the differential equation (5.12). One then obtains the differential equation (5.7) with modified parameters x/b and y/a. Finally, the solution of this differential equation is (trivially) related to the matroid Tutte polynomial T_M (see Proposition 5.1) and this concludes the proof. □

6 Conclusions

We have thus proved in this paper the universality of the matroid Tutte polynomial using differential equations of the same type as the Polchinski flow equation used in renormalization proofs in QFT. This analogy comes from the fact we differentiate with respect to two distinct type of edges of the graphs (see section 2). The role of these two types of graph edges is played in the matroid case studied here by the loop and the coloop type of elements in the matroid ground set.

As already announced in the Introduction, the matroid proofs given in this paper allow to prove the corresponding results for graphs. These graphs results were already conjectured in Krajewski and Martinetti (2011).

Let us end this paper by indicating as a possible direction for future work the investigation of the existence of a polynomial realization of matroid Hopf algebras. This objective appears as particularly interesting because polynomial realizations of Hopf algebras substantially simplify the coproduct coassociativity check (see, for example, the online version of the talk Thibon (2012)). Such an example of polynomial realizations for the Hopf algebra of trees Connes and Kreimer (1998) (amongst other combinatorial Hopf algebras) was given in Foissy et al. (2010). Let us also mention that the task of finding polynomial realizations for matroid Hopf algebras appears to us to be close to the one of finding polynomial realizations for graph Hopf algebras, since both these algebraic combinatorial structures are based on the selection/contraction rule stated in the Introduction.

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On some generalized q-Eulerian polynomials

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Abstract. The (q,r)-Eulerian polynomials are the $(\mathrm{maj}-\mathrm{exc}, \mathrm{fix}, \mathrm{exc})$ enumerative polynomials of permutations. Using Shareshian and Wachs' exponential generating function of these Eulerian polynomials, Chung and Graham proved two symmetrical q-Eulerian identities and asked for bijective proofs. We provide such proofs using Foata and Han's three-variable statistic $(\mathrm{inv}-\mathrm{lec}, \mathrm{pix}, \mathrm{lec})$. We also prove a new recurrence formula for the (q,r)-Eulerian polynomials and study a q-analogue of Chung and Graham's restricted Eulerian polynomials. In particular, we obtain a symmetrical identity for these restricted q-Eulerian polynomials with a combinatorial proof.

Résumé. Les (q,r)-polynômes Eulériens sont les polynomômes énumératives des permutations par rapport au poids $(\mathrm{maj}-\mathrm{exc}, \mathrm{fix}, \mathrm{exc})$. En utilisant la fonction génératrice de ces polynômes Eulériens due à Shareshien et Wachs, Chung et Graham ont démontré deux identités symétriques q-Eulériennes et demandé des preuves bijectives. Nous donnons de telles preuves en utilisant les statistiques trivariées $(\mathrm{inv}-\mathrm{lec}, \mathrm{pix}, \mathrm{lec})$ de Foata et Han. Nous démontrons aussi une nouvelle récurrence pour ces (q,r)-polynômes Eulériens et étudions un q-analogue des polynômes Eulériens restreints de Chung et Graham. En particulier, nous obtenons une identité symétrique pour ces q-polynômes Eulériens restreints avec une preuve combinatoire.

Keywords: Eulerian numbers; symmetrical Eulerian identities; hook factorization; descents; admissible inversions

1 Introduction

The *Eulerian polynomials* $A_n(t) := \sum_{k=0}^{n} A_{n,k}t^k$ are defined by the exponential generating function

$$\sum_{n\geq 0} A_n(t)\frac{z^n}{n!} = \frac{(1-t)e^z}{e^{zt}-te^z}. \tag{1}$$

The coefficients $A_{n,k}$ are called *Eulerian numbers*. The Eulerian numbers arise in a variety of contexts in mathematics. Let $\mathfrak{S}_n$ denote the set of permutations of $[n] := \{1,2,\dots,n\}$. For each $\pi \in \mathfrak{S}_n$, a value i, $1 \leq i \leq n-1$, is an *excedance* (resp. *descent*) of π if $\pi(i) > i$ (resp. $\pi(i) > \pi(i+1)$). Denote by $\mathrm{exc}(\pi)$ and $\mathrm{des}(\pi)$ the number of excedances and descents of π, respectively. It is well-known that the Eulerian number $A_{n,k}$ counts permutations in $\mathfrak{S}_n$ with k descents (or k excedances), that is $A_n(t) = \sum_{\pi\in\mathfrak{S}_n} t^{\mathrm{des}\,\pi} = \sum_{\pi\in\mathfrak{S}_n} t^{\mathrm{exc}\,\pi}$. The reader is referred to [7] for some leisurely historical introductions of Eulerian polynomials and Eulerian numbers.

Several q-analogs of Eulerian polynomials with combinatorial meanings have been studied in the literature (see [2, 6, 16, 20]). Recall that the *major index*, $\mathrm{maj}(\pi)$, of a permutation $\pi \in \mathfrak{S}_n$ is the sum of all the descents of π, i.e., $\mathrm{maj}(\pi) := \sum_{\pi(i)>\pi(i+1)} i$. An element $i \in [n]$ is a *fixed point* of $\pi \in \mathfrak{S}_n$ if $\pi(i) = i$ and we denote by $\mathrm{fix}(\pi)$ the number of fixed points of π. Define the (q,r)*-Eulerian polynomials* $A_n(t,r,q)$ by the following extension of (1):

$$\sum_{n\geq 0} A_n(t,r,q)\frac{z^n}{(q;q)_n} = \frac{(1-t)e(rz;q)}{e(tz;q)-te(z;q)}, \tag{2}$$

where $(q;q)_n := \prod_{i=1}^n (1-q^i)$ and $e(z;q)$ is the q-exponential function defined by $e(z;q) := \sum_{n\geq 0} \frac{z^n}{(q;q)_n}$. The following interpretation for $A_n(t,r,q)$ was given by Shareshian and Wachs [16, 18]:

$$A_n(t,r,q) := \sum_{\pi\in\mathfrak{S}_n} t^{\mathrm{exc}\,\pi} r^{\mathrm{fix}\,\pi} q^{(\mathrm{maj}-\mathrm{exc})\,\pi}. \tag{3}$$

These polynomials have attracted the attention of several authors (cf.[8, 9, 10, 11, 13, 14, 17]).

Let $A_n(t,q) = A_n(t,1,q)$. Define the *q-Eulerian numbers* $A_{n,k}(q)$ and the *fixed point q-Eulerian numbers* $A_{n,k}^{(j)}(q)$:

$$A_n(t,q) = \sum_k A_{n,k}(q)t^k \quad \text{and} \quad A_n(t,r,q) = \sum_{j,k} A_{n,k}^{(j)}(q) r^j t^k.$$

By (3), we have the following interpretations

$$A_{n,k}(q) = \sum_{\substack{\pi\in\mathfrak{S}_n\\ \mathrm{exc}\,\pi=k}} q^{(\mathrm{maj}-\mathrm{exc})\pi} \quad \text{and} \quad A_{n,k}^{(j)}(q) = \sum_{\substack{\pi\in\mathfrak{S}_n\\ \mathrm{exc}\,\pi=k\\ \mathrm{fix}\,\pi=j}} q^{(\mathrm{maj}-\mathrm{exc})\pi}. \tag{4}$$

Recall that the q-binomial coefficients $\begin{bmatrix} n\\ k\end{bmatrix}_q$ are defined by $\begin{bmatrix} n\\ k\end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k}$ for $0 \leq k \leq n$, and $\begin{bmatrix} n\\ k\end{bmatrix}_q = 0$ if $k < 0$ or $k > n$.

Answering a question of Chung et al. [5], Han et al. [13] found and proved the following symmetrical q-Eulerian identity:

$$\sum_{k\geq 1} \begin{bmatrix} a+b\\ k\end{bmatrix}_q A_{k,a-1}(q) = \sum_{k\geq 1} \begin{bmatrix} a+b\\ k\end{bmatrix}_q A_{k,b-1}(q), \tag{5}$$

where a,b are integers with $a,b \geq 1$. Besides a generating function proof using (2), a bijective proof of (5) was also given in [13]. Recently, through analytical arguments, Chung and Graham [4] derived from (2) the following two further symmetrical q-Eulerian identities:

$$\sum_{k\geq 1} (-1)^k \begin{bmatrix} a+b\\ k\end{bmatrix}_q q^{\binom{a+b-k}{2}} A_{k,a}(q) = \sum_{k\geq 1} (-1)^k \begin{bmatrix} a+b\\ k\end{bmatrix}_q q^{\binom{a+b-k}{2}} A_{k,b}(q), \tag{6}$$

$$\sum_{k\geq 1} \begin{bmatrix} a+b+j+1\\ k\end{bmatrix}_q A_{k,a}^{(j)}(q) = \sum_{k\geq 1} \begin{bmatrix} a+b+j+1\\ k\end{bmatrix}_q A_{k,b}^{(j)}(q), \tag{7}$$

where a,b,j are integers with $a,b \geq 1$ and $j \geq 0$, and asked for bijective proofs. Our first aim is to provide such proofs using another interpretation of $A_n(t,r,q)$ introduced by Foata and Han [9], which was already shown to be successful in the bijective proof of (5) in [13].

Next, for $1 \leq j \leq n$, we shall define the *restricted q-Eulerian polynomial* $B_n^{(j)}(t,q)$ by the exponential generating function:

$$\sum_{n\geq j} B_n^{(j)}(t,q)\frac{z^{n-1}}{(q;q)_{n-1}} = \left(\frac{A_{j-1}(t,q)(qz)^{j-1}}{(q;q)_{j-1}}\right)\frac{e(tz;q)-te(tz;q)}{e(tz;q)-te(z;q)}. \tag{8}$$

and the *restricted q-Eulerian number* $B_{n,k}^{(j)}(q)$ by $B_n^{(j)}(t,q) = \sum_k B_{n,k}^{(j)}(q)t^k$. We find the following generalized symmetrical identity for the restricted q-Eulerian polynomials.

Theorem 1 *Let a, b, j be integers with $a, b \geq 1$ and $j \geq 2$. Then*

$$\sum_{k\geq 1}\begin{bmatrix} a+b+1 \\ k-1 \end{bmatrix}_q B_{k,a}^{(j)}(q) = \sum_{k\geq 1}\begin{bmatrix} a+b+1 \\ k-1 \end{bmatrix}_q B_{k,b}^{(j)}(q). \tag{9}$$

When $q = 1$, the above identity was proved by Chung and Graham [4], who also asked for a bijective proof. We shall give a bijective proof and an analytical proof of (9), the latter leads to a new recurrence formula for $A_n(t,r,q)$.

Theorem 2 *The (q,r)-Eulerian polynomials satisfy the following recurrence formula:*

$$A_{n+1}(t,r,q) = rA_n(t,r,q) + tA_n(t,q) + t\sum_{j=1}^{n-1}\begin{bmatrix} n \\ j \end{bmatrix}_q q^j A_j(t,r,q)A_{n-j}(t,q) \tag{10}$$

for $n \geq 1$ and $A_1(t,r,q) = r$.

This paper is organized as follows. In section 2, we review some preliminaries about the three-variable statistic $(\mathrm{inv}, \mathrm{pix}, \mathrm{lec})$ and give the bijective proofs of (6) and (7). In section 3, we first prove Theorem 2 and then define a new statistic called "rix", which together with descents and *admissible inversions* (a statistic on permutations which appears in the context of poset topology [16]) gives another interpretation of $A_n(t,r,q)$. In section 4, we give two combinatorial interpretations of $B_{n,k}^{(j)}(q)$ and two proofs of Theorem 1.

2 Bijective proofs of (6) and (7)

2.1 Preliminaries

A word $w = w_1w_2\ldots w_m$ on $\mathbb{N}$ is called a *hook* if $w_1 > w_2$ and either $m = 2$, or $m \geq 3$ and $w_2 < w_3 < \ldots < w_m$. As shown in [12], each permutation $\pi = \pi_1\pi_2\ldots\pi_n$ admits a unique factorization, called its *hook factorization*, $p\tau_1\tau_2...\tau_r$, where p is an increasing word and each factor $\tau_1, \tau_2, \ldots, \tau_k$ is a hook. To derive the hook factorization of a permutation, one can start from the right and factor out each hook step by step. Denote by $\mathrm{inv}(w)$ the numbers of *inversions* of a word $w = w_1w_2\ldots w_m$, i.e., the number of pairs (w_i, w_j) such that $i < j$ and $w_i > w_j$. Then we define

$$\mathrm{lec}(\pi) := \sum_{1\leq i\leq k}\mathrm{inv}(\tau_i) \quad \text{and} \quad \mathrm{pix}(\pi) := \text{length of the factor } p.$$

For example, the hook factorization of $\pi = 1\ 3\ 4\ 14\ 12\ 2\ 5\ 11\ 15\ 8\ 6\ 7\ 13\ 9\ 10$ is

$$1\ 3\ 4\ 14\,|\,12\ 2\ 5\ 11\ 15\,|\,8\ 6\ 7\,|\,13\ 9\ 10.$$

Hence $p = 1\ 3\ 4\ 14$, $\tau_1 = 12\ 2\ 5\ 11\ 15$, $\tau_2 = 8\ 6\ 7$, $\tau_3 = 13\ 9\ 10$, $\mathrm{pix}(\pi) = 4$ and

$$\mathrm{lec}(\pi) = \mathrm{inv}(12\ 2\ 5\ 11\ 15) + \mathrm{inv}(8\ 6\ 7) + \mathrm{inv}(13\ 9\ 10) = 7.$$

Let $\mathcal{A}_0, \mathcal{A}_1, ..., \mathcal{A}_r$ be a series of sets on $\mathbb{N}$. Denote by $\mathrm{inv}(\mathcal{A}_0, \mathcal{A}_1, ..., \mathcal{A}_r)$ the number of pairs (k,l) such that $k \in \mathcal{A}_i$, $l \in \mathcal{A}_j$, $k > l$ and $i < j$. We usually write $\mathrm{cont}(\mathcal{A})$ to denote the set of all letters in a word $\mathcal{A}$. So we have $(\mathrm{inv} - \mathrm{lec})\pi = \mathrm{inv}(\mathrm{cont}(p), \mathrm{cont}(\tau_1), \dots, \mathrm{cont}(\tau_r))$ if $p\tau_1\tau_2...\tau_r$ is the hook factorization of π.

From Foata and Han [9, Theorem 1.4], we derive the following combinatorial interpretations of the (q,r)-Eulerian polynomials

$$A_n(t,r,q) = \sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{lec}\pi} r^{\mathrm{pix}\pi} q^{(\mathrm{inv}-\mathrm{lec})\pi}. \tag{11}$$

Therefore

$$A_{n,k}(q) = \sum_{\substack{\pi \in \mathfrak{S}_n \\ \mathrm{lec}\pi = k}} q^{(\mathrm{inv}-\mathrm{lec})\pi} \quad \text{and} \quad A_{n,k}^{(j)}(q) = \sum_{\substack{\pi \in \mathfrak{S}_n \\ \mathrm{lec}\pi = k \\ \mathrm{pix}\pi = j}} q^{(\mathrm{inv}-\mathrm{lec})\pi}. \tag{12}$$

It is known [19, Proposition 1.3.17] that the q-binomial coefficient has the interpretation

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{(\mathcal{A},\mathcal{B})} q^{\mathrm{inv}(\mathcal{A},\mathcal{B})}, \tag{13}$$

where the sum is over all ordered partitions $(\mathcal{A},\mathcal{B})$ of $[n]$ such that $|\mathcal{A}| = k$.

We will give bijective proofs of (6) and (7) using the interpretations in (12) and (13).

Remark 1 *In [9], a bijection on $\mathfrak{S}_n$ that carries the triplet (fix, exc, maj) to (pix, lec, inv) was constructed without being specified. This bijection consists of two steps. The first step (see [9, section 6]) uses the word analogue of* Kim-Zeng's decomposition *and an updated version of* Gessel-Reutenauer standardization *to construct a bijection on $\mathfrak{S}_n$ that transforms the triplet (fix, exc, maj) to $(pix, lec, imaj)$, where $imaj(\pi) := maj(\pi^{-1})$ for each permutation π. The second step (see [9, section 7]) uses* Foata's second fundamental transformation *to carry the triplet $(pix, lec, imaj)$ to (pix, lec, inv). In view of this bijection, one can construct bijective proofs of* (5), (6) *and* (7) *using the original interpretations in* (4)*, through the bijective proof of* (5) *in [13] and our bijective proofs,.*

To construct our bijective proofs, we need two elementary transformations from [13] that we recall now. Let τ be a hook with $\mathrm{inv}(\tau) = k$ and $\mathrm{cont}(\tau) = \{x_1, \dots, x_m\}$, where $x_1 < \dots < x_m$. Define

$$d(\tau) = x_{m-k+1}x_1 \dots x_{m-k}x_{m-k+2} \dots x_m. \tag{14}$$

Clearly, $d(\tau)$ is the unique hook with $\mathrm{cont}(d(\tau)) = \mathrm{cont}(\tau)$ and satisfying $\mathrm{inv}(d(\tau)) = m - k = |\mathrm{cont}(\tau)| - \mathrm{inv}(\tau)$. Let τ be a hook or an increasing word with $\mathrm{inv}(\tau) = k$ and $\mathrm{cont}(\tau) = \{x_1, \dots, x_m\}$, where $x_1 < \dots < x_m$. Define

$$d'(\tau) = x_{m-k}x_1 \dots x_{m-k-1}x_{m-k+1} \dots x_m. \tag{15}$$

It is not difficult to see that, $d'(\tau)$ is the unique hook (when $k < m-1$) or increasing word (when $k = m-1$) with $\mathrm{cont}(d(\tau)) = \mathrm{cont}(\tau)$ and satisfying $\mathrm{inv}(d(\tau)) = m-k-1 = |\mathrm{cont}(\tau)| - 1 - \mathrm{inv}(\tau)$.

2.2 Bijective proof of (6)

Let $\mathfrak{S}_n(k) = \{\pi \in \mathfrak{S}_n : \text{pix}(\pi) = k\}$ and $\mathcal{D}_n = \mathfrak{S}_n(0)$. We first notice that the left-hand side of (6) has the following interpretation:

$$\sum_{\substack{\pi \in \mathcal{D}_n \\ \text{lec}\,\pi = a}} q^{(\text{inv}-\text{lec})\pi} = \sum_{k \geq 1} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}} A_{k,a}(q). \tag{16}$$

This interpretation follows immediately from [18, Corollary 4.4] and (11). Note that one can also give a direct combinatorial proof similarly as in [21].

Now, by (16), the symmetrical identity (6) is equivalent to the $j = 0$ case of the following Lemma.

Lemma 1 *For $0 \leq j \leq n$, there is an involution $\mathbf{v} \mapsto \mathbf{u}$ on $\mathfrak{S}_n(j)$ satisfying*

$$\textit{lec}(\mathbf{u}) = n - j - \textit{lec}(\mathbf{v}) \quad \textit{and} \quad (\textit{inv} - \textit{lec})\mathbf{u} = (\textit{inv} - \textit{lec})\mathbf{v}.$$

Proof: Let $\mathbf{v} = p\tau_1\tau_2 \ldots \tau_r$ be the hook factorization of $\mathbf{v} \in \mathfrak{S}_n(j)$, where p is an increasing word and each factor $\tau_1, \tau_2, \ldots, \tau_r$ is a hook. We define $\mathbf{u} = pd(\tau_1) \ldots d(\tau_r)$, where d is defined in (14). It is easy to check that this mapping is an involution on $\mathfrak{S}_n(j)$ with the desired properties. □

By (12), Lemma 1 gives a simple bijective proof of the following known [4, 18] symmetric property of the fixed point q-Eulerian numbers.

Corollary 1 *For $n, k, j \geq 0$,*

$$A_{n,k}^{(j)}(q) = A_{n,n-j-k}^{(j)}(q). \tag{17}$$

2.3 Bijective proof of (7)

Recall [13] that, for a fixed positive integer n, a *two-pix-permutation of* $[n]$ is a sequence of words

$$\mathbf{v} = (p_1, \tau_1, \tau_2, \ldots, \tau_{r-1}, \tau_r, p_2) \tag{18}$$

satisfying the following conditions:

(C1) p_1 and p_2 are two increasing words, possibly empty;

(C2) $\tau_1, \ldots, \tau_r$ are hooks for some positive integer r;

(C3) The concatenation $p_1\tau_1\tau_2 \ldots \tau_{r-1}\tau_r p_2$ of all components of $\mathbf{v}$ is a permutation of $[n]$.

We also extend the two statistics to the two-pix-permutations by

$$\text{lec}(\mathbf{v}) = \sum_{1 \leq i \leq r} \text{inv}(\tau_i) \quad \text{and} \quad \text{inv}(\mathbf{v}) = \text{inv}(p_1\tau_1\tau_2 \ldots \tau_{r-1}\tau_r p_2).$$

It follows that

$$(\text{inv} - \text{lec})\mathbf{v} = \text{inv}(\text{cont}(p_1), \text{cont}(\tau_1), \text{cont}(\tau_2), \ldots, \text{cont}(\tau_r), \text{cont}(p_2)). \tag{19}$$

Let $\mathcal{W}_n(j)$ denote the set of all two-pix-permutations with $|p_1| = j$.

Lemma 2 *Let a, j be fixed nonnegative integers. Then*

$$\sum_{\substack{\mathbf{v}\in\mathcal{W}_n(j)\\ lec\mathbf{v}=a}} q^{(inv-lec)\mathbf{v}} = \sum_{k\geq 1} \begin{bmatrix} n \\ k \end{bmatrix}_q A_{k,a}^{(j)}(q). \tag{20}$$

Proof: By the hook factorization, the two-pix-permutation in (18) is in bijection with the pair (σ, p_2), where $\sigma = p_1\tau_1\tau_2\ldots\tau_{r-1}\tau_r$ is a permutation on $[n]\setminus \mathrm{cont}(p_2)$ and p_2 is an increasing word. Thus, by (12), (13) and (19), the generating function of all two-pix-permutations $\mathbf{v}$ of $[n]$ with $|p_1| = j$ such that $\mathrm{lec}(\mathbf{v}) = a$ and $|p_2| = n-k$ with respect to the weight $q^{(\mathrm{inv}-\mathrm{lec})\mathbf{v}}$ is $\begin{bmatrix} n \\ n-k \end{bmatrix}_q A_{k,a}^{(j)}(q)$. □

Lemma 3 *Let j be a fixed nonnegative integer. Then there is an involution $\mathbf{v}\mapsto\mathbf{u}$ on $\mathcal{W}_n(j)$ satisfying*

$$lec(\mathbf{v}) = n-j-1-lec(\mathbf{u}), \quad \textit{and} \quad (inv-lec)\mathbf{v} = (inv-lec)\mathbf{u}.$$

Proof: We give an explicit construction of the bijection using the involutions d and d' defined in (14) and (15).

Let $\mathbf{v} = (p_1, \tau_1, \tau_2, \ldots, \tau_{r-1}, \tau_r, p_2)$ be a two-pix-permutation of $[n]$ with $|p_1| = j$. If $p_2 \neq \emptyset$, then

$$\mathbf{u} = (p_1, d(\tau_1), d(\tau_2), \ldots, d(\tau_{r-1}), d(\tau_r), d'(p_2)),$$

otherwise,

$$\mathbf{u} = (p_1, d(\tau_1), d(\tau_2), \ldots, d(\tau_{r-1}), d'(\tau_r)).$$

As d and d' are involutions, this mapping is an involution on $\mathcal{W}_n(j)$.

Since we have $\mathrm{lec}(d(\tau_i)) = |\mathrm{cont}(\tau_i)| - \mathrm{lec}(\tau_i)$ for $1\leq i\leq r$ and $\mathrm{lec}(d'(p_2)) = |\mathrm{cont}(p_2)| - 1$ in the case $p_2\neq\emptyset$, it follows that $\mathrm{lec}(\mathbf{u}) = \sum_{i=1}^r |\mathrm{cont}(\tau_i)| + |\mathrm{cont}(p_2)| - 1 - \mathrm{lec}(\mathbf{v}) = n-j-1-\mathrm{lec}(\mathbf{v})$. The above identity is also valid when $p_2 = \emptyset$.

Finally it follows from (19) that $(\mathrm{inv}-\mathrm{lec})\mathbf{u} = (\mathrm{inv}-\mathrm{lec})\mathbf{v}$. This finishes the proof of the lemma. □

Combining Lemmas 2 and 3 we obtain a bijective proof of (7).

3 A new recurrence formula for the (q, r)-Eulerian polynomials

The *Eulerian differential operator* δ_x involved here is defined by

$$\delta_x(f(x)) := \frac{f(x)-f(qx)}{x},$$

for any $f(x)\in\mathbb{Q}[q][[x]]$ in the ring of formal power series in x over $\mathbb{Q}[q]$ (instead of the traditional $(f(x)-f(qx))/((1-q)x)$, see [3]). A proof of Theorem 2 can be obtained by applying δ_z to both sides of (2), which is straightforward and is omitted.

Remark 2 *A different recurrence formula for $A_n(t,r,q)$ was obtained in [18, Corollary 4.3]. Eq.* (10) *is similar to two recurrence formulas in the literature: one for the (inv, des)-q-Eulerian polynomials in [15, Corollary 2.22] (see also [3]) and the other one for the (maj, des)-q-Eulerian polynomials in [15, Corollary 3.6].*

We shall give another interpretation of $A_n(t,r,q)$ in the following.

Let $\pi \in \mathfrak{S}_n$. Recall that an *inversion* of π is a pair $(\pi(i),\pi(j))$ such that $1 \le i < j \le n$ and $\pi(i) > \pi(j)$. An *admissible inversion* of π is an inversion $(\pi(i),\pi(j))$ that satisfies either

- $1 < i$ and $\pi(i-1) < \pi(i)$ or
- there is some l such that $i < l < j$ and $\pi(i) < \pi(l)$.

We write $\mathrm{ai}(\pi)$ the number of admissible inversions of π. Define the statistic $\mathrm{aid}(\pi) := \mathrm{ai}(\pi) + \mathrm{des}(\pi)$. For example, if $\pi = 42153$ then there are 5 inversions, but only $(4,3)$ and $(5,3)$ are admissible. So $\mathrm{inv}(\pi) = 5$, $\mathrm{ai}(\pi) = 2$ and $\mathrm{aid}(\pi) = 2 + 3 = 5$. The statistics ai and aid were first studied by Shareshian and Wachs [16] in the context of Poset Topology. Here we follow the definitions in [14]. The curious result that the pairs $(\mathrm{aid}, \mathrm{des})$ and $(\mathrm{maj}, \mathrm{exc})$ are equidistributed on $\mathfrak{S}_n$ was proved in [14] using techniques of Rees products and lexicographic shellability.

Let $\mathcal{W}$ be the set of all the words on $\mathbb{N}$. We define a new statistic, denoted by "rix", on $\mathcal{W}$ recursively. Let $W = w_1 w_2 \cdots w_n$ be a word in $\mathcal{W}$ and w_i be the rightmost maximum element of W. We define $\mathrm{rix}(W)$ by (with convention that $\mathrm{rix}(\emptyset) = 0$)

$$\mathrm{rix}(W) := \begin{cases} 0, & \text{if } i = 1 \neq n, \\ 1 + \mathrm{rix}(w_1 \cdots w_{n-1}), & \text{if } i = n, \\ \mathrm{rix}(w_{i+1} w_{i+2} \cdots w_n), & \text{if } 1 < i < n. \end{cases}$$

For example, we have $\mathrm{rix}(1\,5\,2\,4\,3\,3\,5) = 1 + \mathrm{rix}(1\,5\,2\,4\,3\,3) = 1 + \mathrm{rix}(2\,4\,3\,3) = 1 + \mathrm{rix}(3\,3) = 2 + \mathrm{rix}(3) = 3$. As every permutation can be viewed as a word on $\mathbb{N}$, this statistic is well-defined on permutations.

For $1 \le j \le n$, we write $\mathfrak{S}_n^{(j)}$ the set of permutations $\pi \in \mathfrak{S}_n$ with $\pi(j) = n$. We define

$$B_n(t,r,q) := \sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{des}\,\pi} r^{\mathrm{rix}\,\pi} q^{\mathrm{ai}\,\pi}$$

and its restricted version by

$$B_n^{(j)}(t,r,q) := \sum_{\pi \in \mathfrak{S}_n^{(j)}} t^{\mathrm{des}\,\pi} r^{\mathrm{rix}\,\pi} q^{\mathrm{ai}\,\pi}. \tag{21}$$

Theorem 3 *We have the following interpretation for (q,r)-Eulerian polynomials:*

$$A_n(t,r,q) = \sum_{\pi \in \mathfrak{S}_n} t^{\mathit{des}\,\pi} r^{\mathit{rix}\,\pi} q^{\mathit{ai}\,\pi}. \tag{22}$$

Proof: We will show that $B_n(t,r,q)$ satisfies the same recurrence formula and initial condition as $A_n(t,r,q)$. For $n \ge 1$, it is clear from the definition of $B_n(t,r,q)$ that

$$B_{n+1}(t,r,q) = \sum_{1 \le j \le n+1} B_{n+1}^{(j)}(t,r,q). \tag{23}$$

It is easy to see that

$$B_{n+1}^{(1)}(t,r,q) = tB_n(t,1,q) \quad \text{and} \quad B_{n+1}^{(n+1)}(t,r,q) = rB_n(t,r,q). \tag{24}$$

We then consider $B_{n+1}^{(j)}(t,r,q)$ for the case of $1 < j < n+1$.

For a set X, we denote by $\binom{X}{m}$ the m-element subsets of X and $\mathfrak{S}_X$ the set of permutations of X. Let $\mathcal{W}(n,j)$ be the set of all triples (W,π_1,π_2) such that $W \in \binom{[n]}{j}$ and $\pi_1 \in \mathfrak{S}_W, \pi_2 \in \mathfrak{S}_{[n]\setminus W}$. It is not difficult to see that the mapping $\pi \mapsto (W,\pi_1,\pi_2)$ defined by

- $W = \{\pi(i) : 1 \le i \le j-1\}$,
- $\pi_1 = \pi(1)\pi(2)\cdots\pi(j-1)$ and $\pi_2 = \pi(j+1)\pi(j+2)\cdots\pi(n)$

is a bijection between $\mathfrak{S}_n^{(j)}$ and $\mathcal{W}(n-1,j-1)$ and satisfies

$$\mathrm{des}(\pi) = \mathrm{des}(\pi_1) + \mathrm{des}(\pi_2) + 1, \quad \mathrm{rix}(\pi) = \mathrm{rix}(\pi_2)$$

and

$$\mathrm{ai}(\pi) = \mathrm{ai}(\pi_1) + \mathrm{ai}(\pi_2) + \mathrm{inv}(W,[n-1]\setminus W) + n - j.$$

Thus, for $1 < j < n+1$, we have

$$\begin{aligned}
B_{n+1}^{(j)}(t,r,q) &= \sum_{\pi\in\mathfrak{S}_{n+1}^{(j)}} t^{\mathrm{des}\pi} r^{\mathrm{rix}\pi} q^{\mathrm{ai}\pi} \\
&= tq^{n+1-j} \sum_{(W,\pi_1,\pi_2)\in\mathcal{W}(n,j-1)} q^{\mathrm{inv}(W,[n]\setminus W)} q^{\mathrm{ai}(\pi_1)} t^{\mathrm{des}(\pi_1)} r^{\mathrm{rix}(\pi_2)} q^{\mathrm{ai}(\pi_2)} t^{\mathrm{des}(\pi_2)} \\
&= tq^{n+1-j} \sum_{W\in\binom{[n]}{j-1}} q^{\mathrm{inv}(W,[n]\setminus W)} \sum_{\pi\in\mathfrak{S}_W} q^{\mathrm{ai}(\pi_1)} t^{\mathrm{des}(\pi_1)} \sum_{\pi_2\in\mathfrak{S}_{[n]\setminus W}} r^{\mathrm{rix}(\pi_2)} q^{\mathrm{ai}(\pi_2)} t^{\mathrm{des}(\pi_2)} \\
&= tq^{n+1-j} \begin{bmatrix} n \\ j-1 \end{bmatrix}_q B_{j-1}(t,1,q) B_{n+1-j}(t,r,q),
\end{aligned} \tag{25}$$

where we apply (13) to the last equality. Substituting (24) and (25) into (23) we obtain

$$B_{n+1}(t,r,q) = rB_n(t,r,q) + tB_n(t,1,q) + t\sum_{j=1}^{n-1} \begin{bmatrix} n \\ j \end{bmatrix}_q q^j B_j(t,r,q) B_{n-j}(t,1,q).$$

By Theorem 2, $B_n(t,r,q)$ and $A_n(t,r,q)$ satisfy the same recurrence formula and initial condition, thus $B_n(t,r,q) = A_n(t,r,q)$. This finishes the proof of the theorem. $\square$

Corollary 2 *The three triplets* (rix, des, aid), (fix, exc, maj) *and* (pix, lec, inv) *are equidistributed on* $\mathfrak{S}_n$.

Remark 3 *At the* Permutation Patterns 2012 *conference, Alexander Burstein [1] gave a direct bijection on* $\mathfrak{S}_n$ *that transforms the triple* (rix, des, aid) *to* (pix, lec, inv). *The new statistic "rix" was introduced independently therein under the name "aix". Actually, the definitions of both are slightly different, but they are the same up to an easy transformation. It would be very interesting to find a similar bijective proof of the equidistribution of* (rix, des, aid) *and* (fix, exc, maj).

4 A symmetrical identity for restricted q-Eulerian polynomials

4.1 An interpretation of $B_{n,k}^{(j)}(q)$ and a proof of Theorem 1

It follows from (2) and (8) that $B_{1,0}^{(1)}(q) = 1$ and $B_{n,k}^{(1)}(q) = A_{n-1,k-1}(q)$ for $n \geq 2$. For $2 \leq j \leq n$, we have the following interpretation for $B_{n,k}^{(j)}(q)$, which shows that $B_{n,k}^{(j)}(q)$ is really a q-analogue of the *restricted Eulerian number* studied in [4] and defined to be the number of permutations in $\mathfrak{S}_n^{(j)}$ with k descents.

Lemma 4 *For* $2 \leq j \leq n$, $B_{n,k}^{(j)}(q) = \sum_{\substack{\pi \in \mathfrak{S}_n^{(j)} \\ des(\pi)=k}} q^{ai(\pi)+2j-n-1}$.

Proof: When $j \geq 2$, by the recurrence relation (25), one can compute without difficulty that the exponential generating function $\sum_{n \geq j} q^{2j-n-1} B_n^{(j)}(t,1,q) \frac{z^{n-1}}{(q;q)_{n-1}}$ is exactly the right side of (8) using (2) and (22), which would finish the proof of the lemma. □

Lemma 5 *For* $1 < j < n$, *we have*

$$B_{n,k}^{(j)}(q) = B_{n,n-1-k}^{(j)}(q).$$

Proof: We first construct an involution $f : \pi \mapsto \pi'$ on $\mathfrak{S}_n$ satisfying

$$\mathrm{ai}(\pi) = \mathrm{ai}(\pi') \quad \text{and} \quad \mathrm{des}(\pi) = n - 1 - \mathrm{des}(\pi'). \tag{26}$$

For $n = 1$, define $f(\mathrm{id}) = \mathrm{id}$. For $n \geq 2$, suppose that $\pi = \pi_1 \cdots \pi_n$ is a permutation of $\{\pi_1, \cdots, \pi_n\}$ and π_j is the maximum element in $\{\pi_1, \cdots, \pi_n\}$. We construct f recursively as follows

$$f(\pi) = \begin{cases} f(\pi_2\pi_3 \cdots \pi_n)\, \pi_1, & \text{if } j = 1, \\ \pi_n\, f(\pi_1\pi_2 \cdots \pi_{n-1}), & \text{if } j = n, \\ f(\pi_1\pi_2 \cdots \pi_{j-1})\, \pi_j\, f(\pi_{j+1}\pi_{j+2} \cdots \pi_n), & \text{otherwise.} \end{cases}$$

For example, if $\pi = 3\,2\,5\,7\,6\,4\,1$, then $f(\pi) = f(3\,2\,5)\,7\,f(6\,4\,1) = 5\,f(3\,2)7\,f(4\,1)\,6 = 5\,2\,3\,7\,1\,4\,6$. Clearly, $\mathrm{ai}(\pi) = 7 = \mathrm{ai}(\pi')$ and $\mathrm{des}(\pi) = 4 = 7 - 1 - \mathrm{des}(\pi')$. It is not difficult to see that f is an involution. We can show that f satisfies (26) by induction on n, which is routine and left to the reader.

For each $\pi = \pi_1 \cdots \pi_{j-1}\, n\, \pi_{j+1} \cdots \pi_n$ in $\mathfrak{S}_n^{(j)}$, we then define $g(\pi) = f(\pi_1 \cdots \pi_{j-1})\, n\, f(\pi_{j+1} \cdots \pi_n)$. As f is an involution, g is an involution on $\mathfrak{S}_n^{(j)}$. It follows from (26) that $\mathrm{ai}(g(\pi)) = \mathrm{ai}(\pi)$ and $\mathrm{des}(\pi) = n - 1 - \mathrm{des}(g(\pi))$, which completes the proof in view of Lemma 4. □

Remark 4 *Supposing that* $\pi = \pi_1 \cdots \pi_n$ *is a permutation of* $\{\pi_1, \cdots, \pi_n\}$ *and* π_j *is the maximum element in* $\{\pi_1, \cdots, \pi_n\}$, *we modify* f *to* f' *as follows:*

$$f'(\pi) = \begin{cases} f'(\pi_2\pi_3 \cdots \pi_n)\, \pi_1, & \textit{if } j = 1, \\ \pi, & \textit{if } j = n, \\ f'(\pi_1\pi_2 \cdots \pi_{j-1})\, \pi_j\, f'(\pi_{j+1}\pi_{j+2} \cdots \pi_n), & \textit{otherwise.} \end{cases}$$

The reader is invited to check that f' *would provide another bijective proof of Corollary 1 using* (des, rix, ai).

Through some similar analytical arguments as [4, Theorem 2] starting with the generating function (8) and using Lemma 4 and 5 we can get a proof of Theorem 1. The details are omitted.

4.2 Another interpretation of $B_{n,k}^{(j)}(q)$ and a bijective proof of Theorem 1

Let $\bar{\mathfrak{S}}_n^{(j)} := \{\pi \in \mathfrak{S}_n : \pi(j+1) = 1\}$ for $1 \le j < n$ and $\bar{\mathfrak{S}}_n^{(n)} := \{\pi'\square 1 : \pi' \in \mathfrak{S}_{[n]\setminus\{1\}}\}$. The "$\square$" in $\pi = \pi_1\pi_2\cdots\pi_{n-1}\square 1 \in \bar{\mathfrak{S}}_n^{(n)}$ means that the n-th position of π is empty and the hook factorization of π is defined to be $p\tau_1\cdots\tau_r\square 1$, where $p\tau_1\cdots\tau_r$ is the hook factorization of $\pi_1\cdots\pi_{n-1}$ and "$\square 1$" is viewed as a hook. We also define $\operatorname{lec}(\pi_1\pi_2\cdots\pi_{n-1}\square 1) = \sum_{i=1}^r \operatorname{lec}(\tau_i)$ and $\operatorname{inv}(\pi_1\pi_2\cdots\pi_{n-1}\square 1) = \operatorname{inv}(\pi_1\pi_2\cdots\pi_{n-1}1)$. For example, $\bar{\mathfrak{S}}_3^{(3)} = \{32\square 1, 23\square 1\}$ with $\operatorname{lec}(32\square 1) = 1, \operatorname{lec}(23\square 1) = 0$ and $\operatorname{inv}(32\square 1) = 3, \operatorname{inv}(23\square 1) = 2$.

Lemma 6 *Let $B_{n,k}^{(j)}(q)$ be defined by* (8). *Then* $B_{n,k}^{(j)}(q) = \sum_{\substack{\pi\in\bar{\mathfrak{S}}_n^{(j)}\\ lec(\pi)=k}} q^{(inv-lec)\pi}$.

Proof: Let $\bar{B}_n^{(j)}(t,q) := \sum_{\pi\in\bar{\mathfrak{S}}_n^{(j)}} q^{(\mathrm{inv}-\mathrm{lec})\pi} t^{\mathrm{lec}\,\pi}$. We recall that, to derive the hook factorization of a permutation, one can start from the right and factor out each hook step by step. Therefore, the hook factorization of $\pi = \pi_1\cdots\pi_{j-1}\pi_j 1\pi_{j+2}\cdots\pi_n$ in $\pi \in \bar{\mathfrak{S}}_n^{(j)}$ is $p\tau_1\cdots\tau_s\tau_1'\cdots\tau_r'$, where $p\tau_1\cdots\tau_s$ and $\tau_1'\cdots\tau_r'$ are hook factorizations of $\pi_1\cdots\pi_{j-1}$ and $\pi_j 1\pi_{j+2}\cdots\pi_n$, respectively. When $n > j$, it is not difficult to see that

$$\operatorname{lec}(\pi_j 1\pi_{j+2}\cdots\pi_n) = 1 + \operatorname{lec}(\pi_j\pi_{j+2}\cdots\pi_n)$$

and

$$(\operatorname{inv} - \operatorname{lec})(\pi_j 1\pi_{j+2}\cdots\pi_n) = (\operatorname{inv} - \operatorname{lec})(\pi_j\pi_{j+2}\cdots\pi_n).$$

Thus by (13), we have

$$\bar{B}_n^{(j)}(t,q) = A_{j-1}(t,q)q^{j-1}\begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_q tA_{n-j}(t,q) \tag{27}$$

for $n > j$. Clearly, $\bar{B}_j^{(j)}(t,q) = A_{j-1}(t,q)q^{j-1}$. So by (2), the exponential generating function $\sum_{n\ge j}\bar{B}_n^{(j)}(t,q)\frac{z^{n-1}}{(q;q)_{n-1}}$ is the right side of (8). This finishes the proof of the lemma. $\square$

Remark 5 *This interpretation can also be deduced directly from the interpretation in Lemma 4 using Burstein's bijection [1].*

For $X \subset [n]$ with $|X| = m$ and $1 \in X$, we can define $\bar{\mathfrak{S}}_X^{(j)}$ for $1 \le j \le m$ similarly as $\bar{\mathfrak{S}}_m^{(j)}$ like this:

$$\bar{\mathfrak{S}}_X^{(j)} := \{\pi \in \mathfrak{S}_X : \pi(j+1) = 1\} \text{ for } 1 \le j < m \text{ and } \bar{\mathfrak{S}}_X^{(m)} := \{\pi'\square 1 : \pi' \in \mathfrak{S}_{X\setminus\{1\}}\}.$$

For $1 \le j \le n$, we define a *j-restricted two-pix-permutation of* $[n]$ to be a pair $\mathbf{v} = (\pi, p_2)$ such that p_2 (possibly empty) is an increasing words on $[n]$ and $\pi \in \bar{\mathfrak{S}}_X^{(j)}$ with $X = [n] \setminus \{\operatorname{cont}(p_2)\}$. Similarly, we define $\operatorname{lec}(\mathbf{v}) = \operatorname{lec}(\pi)$ and $\operatorname{inv}(\mathbf{v}) = \operatorname{inv}(\pi) + \operatorname{inv}(\operatorname{cont}(\pi), \operatorname{cont}(p_2))$. Let $\mathcal{W}_n^{(j)}$ denote the set of all j-restricted two-pix-permutations of $[n]$.

Lemma 7 *Let a, j be positive integers. Then*

$$\sum_{\substack{\mathbf{v}\in\mathcal{W}_n^{(j)}\\ lec\mathbf{v}=a}} q^{(inv-lec)\mathbf{v}} = \sum_{k\ge 1}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q B_{k,a}^{(j)}(q). \tag{28}$$

Proof: It follows from Lemma 6 and some similar arguments as in the proof of Lemma 2. □

Lemma 8 *Let $2 \le j \le n$. Then there is an involution $\mathbf{v} \mapsto \mathbf{u}$ on $\mathcal{W}_n^{(j)}$ satisfying*

$$lec(\mathbf{v}) = n - 2 - lec(\mathbf{u}), \quad and \quad (inv - lec)\mathbf{v} = (inv - lec)\mathbf{u}. \tag{29}$$

Proof: Suppose $\mathbf{v} = (\pi, p_2) \in \mathcal{W}_n^{(j)}$ and $\pi = \tau_0\tau_1 \cdots \tau_r$ is the hook factorization of π such that τ_0 is a hook or an increasing word and τ_i $(1 \le i \le r)$ are hooks. We also assume that $p_2 = x_1 \cdots x_l$ if p_2 is not empty. Note that $1 \notin \text{cont}(\tau_0)$ since $j \neq 1$. We will use the involutions d and d' defined in (14) and (15). There are several cases to be considered:

(i) $\tau_r = \Box 1$. Then

$$\mathbf{u} = \begin{cases} (d'(\tau_0)d(\tau_1)\cdots d(\tau_{r-1})x_l 1 x_1 x_2 \cdots x_{l-1}, \emptyset), & \text{if } p_2 \neq \emptyset; \\ (d'(\tau_0)d(\tau_1)\cdots d(\tau_{r-1})\Box 1, \emptyset), & \text{otherwise.} \end{cases}$$

(ii) $\tau_r = y_s 1 y_1 \cdots y_{s-1}$. Then

$$\mathbf{u} = \begin{cases} (d'(\tau_0)d(\tau_1)\cdots d(\tau_{r-1})d(\tau_r)d'(p_2), \emptyset), & \text{if } p_2 \neq \emptyset; \\ (d'(\tau_0)d(\tau_1)\cdots d(\tau_{r-1})\Box 1, y_1 \cdots y_s), & \text{if } p_2 = \emptyset \text{ and } y_s > y_{s-1}; \\ (d'(\tau_0)d(\tau_1)\cdots d(\tau_{r-1})d'(\tau_r), \emptyset), & \text{otherwise.} \end{cases}$$

(iii) $1 \notin \text{cont}(\tau_r)$. Then

$$\mathbf{u} = \begin{cases} (d'(\tau_0)d(\tau_1)\cdots d(\tau_{r-1})d(\tau_r)d'(p_2), \emptyset), & \text{if } p_2 \neq \emptyset; \\ (d'(\tau_0)d(\tau_1)\cdots d(\tau_{r-1}), d'(\tau_r)), & \text{if } p_2 = \emptyset \text{ and } \text{lec}(\tau_r) = |\tau_r| - 1; \\ (d'(\tau_0)d(\tau_1)\cdots d(\tau_{r-1})d'(\tau_r), \emptyset), & \text{otherwise.} \end{cases}$$

First of all, one can check that $\mathbf{u} \in \mathcal{W}_n^{(j)}$. Secondly, as d, d' are involutions, the above mapping is an involution. Finally, this involution satisfies (29) in all cases. This completes the proof of the lemma. □

Combining Lemmas 7 and 8 we obtain a bijective proof of Theorem 1.

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On a Classification of Smooth Fano Polytopes

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Abstract. The d-dimensional simplicial, terminal, and reflexive polytopes with at least $3d-2$ vertices are classified. In particular, it turns out that all of them are smooth Fano polytopes. This improves previous results of Casagrande (2006) and Øbro (2008). Smooth Fano polytopes play a role in algebraic geometry and mathematical physics. This text is an extended abstract of Assarf et al. (2012).

Résumé. Nous classifions les polytopes simpliciaux, terminaux et réflexifs de dimension d avec au moins $3d-2$ sommets. En particulier, tous ces polytopes se trouvent être des polytopes de Fano lisses. Nous améliorons des résultats antérieurs de Casagrande (2006) et d'Øbro (2008). Les polytopes de Fano lisses apparaissent en géométrie algébrique et en physique mathématique. Ce texte est un résumé étendu de Assarf et al. (2012).

Keywords: toric Fano varieties, lattice polytopes, terminal polytopes, smooth polytopes

1 Introduction

A *lattice polytope* P is a convex polytope whose vertices lie in a lattice N contained in the vector space $\mathbb{R}^d$. Fixing a basis of N describes an isomorphism to $\mathbb{Z}^d$. Throughout this paper, we restrict our attention to the standard lattice $N = \mathbb{Z}^d$. A d-dimensional lattice polytope $P \subset \mathbb{R}^d$ is called *reflexive* if it contains the origin $\mathbf{0}$ as an interior point and its polar polytope is a lattice polytope in the dual lattice $M := \operatorname{Hom}(N, \mathbb{Z}) \cong \mathbb{Z}^d$. A lattice polytope P is *terminal* if $\mathbf{0}$ and the vertices are the only lattice points in $P \cap \mathbb{Z}^d$. It is *simplicial* if each face is a simplex. We say that P is a *smooth Fano polytope* if $P \subseteq \mathbb{R}^d$ is simplicial with $\mathbf{0}$ in the interior and the vertices of each facet form a lattice basis of $\mathbb{Z}^d$.

In algebraic geometry, reflexive polytopes correspond to *Gorenstein toric Fano varieties*. The toric variety X_P of a polytope P is determined by the face fan of P, that is, the fan spanned by all faces of P; see (Ewald, 1996) or (Cox et al., 2011) for details. The toric variety X_P is $\mathbb{Q}$*-factorial* (some multiple of a Weil divisor is Cartier) if and only if the polytope P is simplicial. In this case the *Picard number* of X equals $n-d$, where n is the number of vertices of P. The polytope P is smooth if and only if the variety X_P is a manifold (that is, it has no singularities). Note that the notions detailed above are not entirely standardized in the literature. For example, our definitions agree with (Nill, 2005), but disagree with (Kreuzer and Nill, 2009).

Our main result is a classification of those simplicial, terminal, and reflexive lattice polytopes with at least $3d-2$ vertices. We show that such a polytope is lattice equivalent to a direct sum of del Pezzo polytopes,

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pseudo del Pezzo polytopes, or a (possibly skew) bipyramid over (pseudo) del Pezzo polytopes. In particular, a simplicial, terminal, and reflexive polytope with at least $3d-2$ vertices is necessarily smooth Fano. The precise statement can be found in Theorem 2 below.

This extends results of Casagrande who proved that the number of a d-dimensional simplicial, terminal, and reflexive lattice polytopes does not exceed $3d$; she also showed that, up to lattice equivalence, only one type exists which attains this bound (and the dimension d is even) (Casagrande, 2006). Moreover, our result also extends Øbro's classification of all polytopes of the named kind with $3d-1$ vertices (Øbro, 2008). Our proof employs techniques similar to those used by (Øbro, 2008) and (Nill and Øbro, 2010), but requires more organization since a greater variety of possibilities occurs. Translated into the language of toric varieties our main result establishes that any d-dimensional terminal $\mathbb{Q}$-factorial Gorenstein toric Fano variety with Picard number at least $2d-2$ decomposes as a (possibly trivial) toric fiber bundle with known fiber and base space; the precise statement is Corollary 4. As a key benefit of our systematic approach a certain general pattern emerges, and we state this as Conjecture 3 below. Like our main result this conjecture also admits a direct translation to toric varieties.

The interest in structural results of this type originates in applications of algebraic geometry to mathematical physics. For instance, (Batyrev and Borisov, 1996) use reflexive polytopes to construct pairs of mirror symmetric Calabi-Yau manifolds. Up to unimodular equivalence, there exists only a finite number of such polytopes in each dimension, and they have been classified up to dimension 4, see (Batyrev, 1991), (Kreuzer and Skarke, 1997, 2002). Smooth reflexive polytopes have been classified up to dimension 8 by (Øbro, 2007); see (Brown and Kasprzyk, 2009–2012) for data. By enhancing Øbro's implementation within the `polymake` framework (Gawrilow and Joswig, 2000) this classification was extended to dimension 9 (Lorenz and Paffenholz, 2008); from that site the data is available in `polymake` format.

In this extended abstract we will only summarize the essential ideas for the proofs. In addition, we will detail the 6-dimensional case. For full proofs we refer to the paper (Assarf et al., 2012).

2 Lattice Polytopes

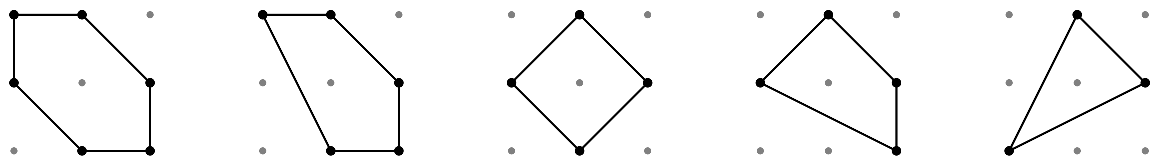
A polytope $P \subset \mathbb{R}^d$ is a *lattice polytope* if its *vertex set* $\mathrm{Vert}(P)$ is contained in $\mathbb{Z}^d$ (more generally, contained in some lattice $N \subseteq \mathbb{R}^d$). See (Ewald, 1996) for background on lattice polytopes. P is called *reflexive*, if P contains the origin in its interior and its dual P^* is a lattice polytope in the dual lattice. P is *terminal* if $P \cap N = \mathrm{Vert}(P) \cup \{\mathbf{0}\}$. More generally, P is *canonical* if the origin is the only interior lattice point in P. Two lattice polytopes are *lattice equivalent* if one can be mapped to the other by a transformation in $\mathrm{GL}_d\mathbb{Z}$ followed by a lattice translation.

We start out with listing all possible types of 2-dimensional terminal and reflexive lattice polytopes in Figure 1. Up to lattice equivalence five cases occur which we denote as P_6, P_5, P_{4a}, P_{4b}, and P_3, respectively; one hexagon, one pentagon, two quadrangles, and a triangle; see (Ewald, 1996, Thm. 8.2). All of them are *smooth Fano* polytopes, that is, the origin lies in the interior and the vertex set of each facet forms a lattice basis. The only 1-dimensional reflexive polytope is the interval $[-1, 1]$.

Let $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ be polytopes with the origin in their respective relative interiors. The polytope

$$P \oplus Q \;=\; \mathrm{conv}(P \cup Q) \quad \subset \; \mathbb{R}^{d+e}$$

is the *direct sum* of P and Q. This construction also goes by the name "linear join" of P and Q. Clearly, forming direct sums is commutative and associative. Notice that the polar polytope $(P \oplus Q)^* = P^* \times Q^*$ is



(a) P_6 (b) P_5 (c) P_{4a} (d) P_{4b} (e) P_3

Figure 1: The 2-dimensional reflexive and terminal lattice polytopes

the direct product. An important special case is the *proper bipyramid* $[-1,1] \oplus Q$ over Q. More generally, we consider the possibly *skew* bipyramids

$$\mathrm{BP}(Q, v, w) \ := \ \mathrm{conv}((\{0\} \times Q) \cup \{w, v - w\}),$$

where $v \in Q \cap \mathbb{Z}^e$ is a lattice point in Q and w is orthogonal to the affine hull of Q with $|w| = 1$. In particular, choosing $v = 0$ recovers the proper bipyramid. The relevance of these constructions for simplicial, terminal, and reflexive polytopes stems from the following lemma; see also (Ewald, 1996, §V.7.7) and Figure 2 below. The reader can find the simple proof in (Assarf et al., 2012, Lemmas 2,3,4).

Lemma 1 *Let $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ both be lattice polytopes. Then the direct sum $P \oplus Q \subset \mathbb{R}^{d+e}$ is simplicial, terminal, or reflexive if and only if P and Q are.*

In particular, this applies to the case that $P = [-1,1] \oplus Q$ is a proper bipyramid over a $(d-1)$-dimensional lattice polytope Q. More generally, P is a simplicial, terminal, or reflexive skew bipyramid if and only if Q has the corresponding property.

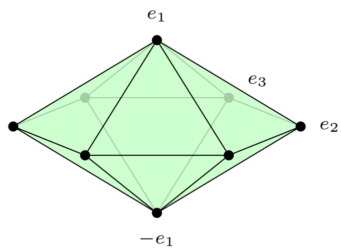
The latter case of the lemma occurs frequently in the classification. Let $e_1, e_2, \ldots, e_d$ be the standard basis of $\mathbb{Z}^d$ in $\mathbb{R}^d$. Here and throughout we abbreviate $\mathbf{1} = (1, 1, \ldots, 1)$. For even d the d-polytopes

$$\mathrm{DP}(d) \ = \ \mathrm{conv}\{\pm e_1, \pm e_2, \ldots, \pm e_d, \pm \mathbf{1}\} \quad \subset \mathbb{R}^d$$

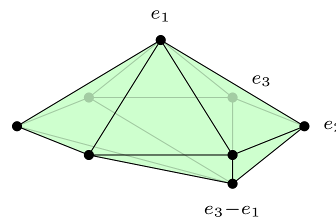
with $2d+2$ vertices form a 1-parameter family of smooth Fano polytopes; see (Ewald, 1996, §V.8.3). They are usually called *del Pezzo* polytopes. If $-\mathbf{1}$ is not a vertex the resulting polytopes are sometimes called *pseudo del Pezzo*. Notice that the 2-dimensional del Pezzo polytope $\mathrm{DP}(2)$ is lattice equivalent to the hexagon P_6 shown in Figure 1, and the 2-dimensional pseudo del Pezzo polytope is lattice equivalent to the pentagon P_5. While the definition of $\mathrm{DP}(d)$ also makes sense in odd dimensions, the polytopes obtained are not simplicial.

For *centrally symmetric* smooth Fano polytopes (Voskresenskiĭ and Klyachko, 1984) provide a classification result. They showed that every centrally symmetric smooth Fano polytope can be written as a sum of line segments and del Pezzo polytopes. This was later generalized to simplicial and reflexive pseudo-symmetric polytopes by (Ewald, 1988, 1996) in the smooth case, and by (Nill, 2006, Thm. 0.1) in the general case. A polytope is *pseudo-symmetric* if there exists a facet F, such that $-F = \{-v \mid v \in F\}$ is also a facet. They proved that any pseudo-symmetric simplicial and reflexive polytope is lattice equivalent to a direct sum of a (possibly trivial) cross polytope, del Pezzo polytopes, and pseudo del Pezzo polytopes.

A direct sum of d intervals $[-1,1] \oplus [-1,1] \oplus \cdots \oplus [-1,1]$ is the same as the regular cross polytope $\mathrm{conv}\{\pm e_1, \pm e_2, \ldots, \pm e_d\}$. The direct sum of several intervals with a polytope Q is the same as an iterated proper bipyramid over Q. Casagrande showed that any simplicial and reflexive d-polytope P has at most $3d$ vertices, and if it does have exactly $3d$ vertices then d is even, and P is a centrally symmetric smooth Fano



(a) Proper bipyramid over P_6



(b) Skew bipyramid over P_6

Figure 2: The smooth Fano 3-polytopes with $3d-1=8$ vertices. Combinatorially, both are bipyramids over P_6.

polytope (Casagrande, 2006, Thm. 3). Thus, in this case P is lattice equivalent to a direct sum of $\frac{d}{2}$ copies of $P_6 \cong \mathrm{DP}(2)$.

Øbro classified the simplicial, terminal, and reflexive d-polytopes with $3d-1$ vertices (Øbro, 2008). Up to lattice equivalence, there is the interval $[-1,1]$ in dimension 1 and the pentagon P_5 in dimension 2. Forming suitable direct sums and (skew) bipyramids gives more smooth Fano d-polytopes with $3d-1$ vertices via

$$P_5 \oplus P_6^{\oplus(\frac{d}{2}-1)} \quad \text{for even } d\text{, and} \quad \mathrm{BP}(P_6^{\oplus(\frac{d-1}{2})}, v, e_d) \quad \text{for odd } d \text{ and } v \in \mathbb{Z}^{d-1} \cap P_6^{\oplus(\frac{d-1}{2})} .$$

Note that, up to lattice isomorphism, there are only two choices for v, either 0, which gives a proper bipyramid, or some vertex, which results in a skew bipyramid. The 3-dimensional cases are shown in Figure 2. Up to lattice equivalence, these are the only d-dimensional simplicial, terminal, and reflexive polytopes with $3d-1$ vertices (Øbro, 2008, Thm. 1). It turns out that all these polytopes are smooth Fano. Our main result is the following classification, which is a summary of (Assarf et al., 2012, Thm. 7).

Theorem 2 *For even $d \geq 6$ there are three combinatorial types of d-dimensional simplicial, terminal, and reflexive polytopes with $3d-2$ vertices. These three cases split into eleven types up to lattice equivalence. For odd $d \geq 5$ there is only one combinatorial type that splits into five types up to lattice equivalence.*

For $d=1$ there is one combinatorial type, for $d=2$ there is one combinatorial type with two different lattice realizations, for $d=3$ there is one combinatorial type with 4 different lattice realizations, and, finally, for $d=4$ there are three combinatorial types with ten different lattice realizations; see (Batyrev, 1999).

We list the types explicitly. To this end we label the vertices of P_5 by $v_1, v_2, \ldots, v_5$ and those of P_6 with $w_1, w_2, \ldots, w_6$ in clockwise order. For P_5, let v_1 be the unique vertex such that $-v_1 \notin P_5$. For even $d \geq 4$ the three combinatorial types are

$$P_5^{\oplus 2} \oplus P_6^{\oplus(\frac{d}{2}-2)}, \qquad \mathrm{DP}(4) \oplus P_6^{\oplus(\frac{d}{2}-2)}, \qquad \text{and} \qquad \mathrm{BP}(\mathrm{BP}(P_6^{\oplus\frac{d-2}{2}}, x, a), y, b),$$

for a lattice point x of $P_6^{\oplus\frac{d-2}{2}}$, a lattice point y of $\mathrm{BP}(P_6^{\oplus\frac{d-2}{2}}, x, a)$ and transversal vectors a, b. The last case splits, up to lattice equivalence, into eight types if $d=4$ and nine if $d \geq 6$. The relevant choices of x, y are

$$(0,0), \quad (0,c), \quad (0,w_1), \quad (w_1,w_1), \quad (w_1,w_2), \quad (w_1,w_3), \quad (w_1,w_4), \quad \text{and} \quad (w_1,c)$$

for $d=4$, where all w_i are vertices of some copy of P_6; here c denotes one of the two apices of the bipyramid $\mathrm{BP}(P_6^{\oplus\frac{d-2}{2}}, x, a)$. For $d \geq 6$ we can additionally choose two vertices in different copies of P_6. It is a key step in our proof to recognize these (proper or skew) bipyramids. The fact that the group of lattice automorphisms of P_6, which is isomorphic to the dihedral group of order 12, acts sharply transitively on adjacent pairs of vertices then entails the classification up to lattice equivalence. For odd $d \geq 5$ the one combinatorial type is $\mathrm{BP}(P_5 \oplus P_6^{\oplus \frac{d-3}{2}}, x, a)$ for some lattice point $x \in P_5 \oplus P_6^{\oplus \frac{d-3}{2}}$. The five different lattice isomorphism types are realized by choosing x in $\{0, v_1, v_2, v_3, w_1\}$.

We do believe that the list of the classifications obtained so far follows a pattern.

Conjecture 3 *Let P be a d-dimensional smooth Fano polytope with n vertices such that $n \geq 3d - k$ for $k \leq \frac{d}{3}$. If d is even then P is lattice equivalent to $Q \oplus P_6^{\oplus(\frac{d-3k}{2})}$ where Q is a $3k$-dimensional smooth Fano polytope with $n - 3d + 9k \geq 8k$ vertices. If d is odd then P is lattice equivalent to $Q \oplus P_6^{\oplus(\frac{d-3k-1}{2})}$ where Q is a $(3k{+}1)$-dimensional smooth Fano polytope with $n - 3d + 9k - 3 \geq 8k - 3$ vertices.*

This conjecture is best possible in the following sense: The k-fold direct sum of skew bipyramids over P_6 yields a smooth Fano polytope of dimension $d = 3k$ with $8k = 3d - k$ vertices, but it has no copy of P_6 as a direct summand. However, it does contain $P_6^{\oplus k}$ as a subpolytope of dimension $2k = \frac{2}{3}d$.

If the conjecture above holds the full classification of the smooth Fano polytopes of dimension at most nine Lorenz and Paffenholz (2008) would automatically yield a complete description of all d-dimensional smooth Fano polytopes with at least $3d - 3$ vertices.

3 Toric Varieties

Reading a lattice point $a \in \mathbb{Z}^d$ as the exponent vector of the monomial $z^a = z_1^{a_1} z_2^{a_2} \ldots z_d^{a_d}$ in the Laurent polynomial ring $\mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, \ldots, z_d^{\pm d}]$ provides an isomorphism from the additive group of $\mathbb{Z}^d$ to the multiplicative group of Laurent monomials. This way the maximal spectrum X_σ of a lattice cone σ becomes an *affine toric variety*. If Σ is a fan of lattice cones, gluing the duals of the cones along common faces yields a *(projective) toric variety* X_Σ. This complex algebraic variety admits a natural action of the embedded dense torus corresponding to the (dual of) the trivial cone $\{\mathbf{0}\}$ which is contained in each cone of Σ. If $P \in \mathbb{R}^d$ is a lattice polytope containing the origin, then the *face fan*

$$\Sigma(P) \;=\; \{\mathrm{pos}(F) \;\mid F \text{ face of } P\}$$

is such a fan of lattice cones. We denote the associated toric variety by $X_P = X_{\Sigma(P)}$. The face fan of a polytope is isomorphic to the normal fan of its polar. Two lattice polytopes P and Q are lattice equivalent if and only if X_P and X_Q are isomorphic as toric varieties.

Let P be a full-dimensional lattice polytope containing the origin as an interior point. Then the toric variety X_P is smooth if and only if P is smooth in the sense of the definition given above, that is, the vertices of each facet of P are required form a lattice basis. A smooth compact projective toric variety X_P is a *toric Fano variety* if its anticanonical divisor is very ample. This holds if and only if P is a smooth Fano polytope; see (Ewald, 1996, §VII.8.5).

We now describe the toric varieties arising from the polytopes listed in our Theorem 2. For the list of two-dimensional toric Fano varieties we use the same notation as in Figure 1; see (Ewald, 1996, §VII.8.7). The toric variety X_{P_3} is the complex projective plane $\mathbb{P}_2$. The toric variety $X_{P_{4a}}$ is isomorphic to a direct product $\mathbb{P}_1 \times \mathbb{P}_1$ of lines, and $X_{P_{4b}}$ is the smooth *Hirzebruch surface* $\mathcal{H}_1$. The toric variety X_{P_5} is a blow-up of $\mathbb{P}_2$ at two points or, equivalently, a blow-up of $\mathbb{P}_1 \times \mathbb{P}_1$ at one torus invariant point. The toric varieties associated with the del Pezzo polytopes $\mathrm{DP}(d)$ are called *del Pezzo varieties*; notice that this notion also occurs with a different meaning in the literature. The toric variety X_{P_6} is a del Pezzo surface or, equivalently, a blow-up of $\mathbb{P}_2$ at three non-collinear torus invariant points.

Two polytope constructions play a role in our classification, direct sums and (skew) bipyramids. We want to translate them into the language of toric varieties. Let $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ both be full-dimensional lattice polytopes containing the origin. Then the toric variety $X_{P \oplus Q}$ is isomorphic to the direct product $X_P \times X_Q$. In particular, for $P = [-1, 1]$ we have that the toric variety

$$X_{[-1,1] \oplus Q} \;=\; \mathbb{P}_1 \times X_Q$$

over the regular bipyramid over Q is a direct product with the projective line $\mathbb{P}_1 \cong X_{[-1,1]}$. More generally, the toric variety of a skew bipyramid over Q is a toric fiber bundle with base space $\mathbb{P}_1$ and generic fiber X_Q; see (Ewald, 1996, §VI.6.7). An example is the smooth Hirzebruch surface $\mathcal{H}_1 \cong X_{P_{4b}}$, which is a (projective) line bundle over $\mathbb{P}_1$.

In order to translate Theorem 2 to toric varieties we need a few more definitions. For the sake of brevity we explain these in polytopal terms and refer to (Ewald, 1996) for the details. A toric variety X_P associated with a canonical lattice d-polytope P is $\mathbb{Q}$*-factorial* (or *quasi-smooth*) if P is simplicial; see (Ewald, 1996, §VI.3.9). In this case the *Picard number* equals $n - d$ where n is the number of vertices of P; see (Ewald, 1996, §VII.2.17). We call this toric variety a 2*-stage fiber bundle over* Z if X is a fiber bundle with base space Y such that Y itself is a fiber bundle with base space Z. The following is now a corollary of Theorem 2.

Corollary 4 *Let X be d-dimensional terminal $\mathbb{Q}$-factorial Gorenstein toric Fano variety with Picard number $2d-2$. We assume $d \geq 4$.*

If d is even, then X is isomorphic to

i. a 2-stage toric fiber bundle such that the base spaces of both stages are projective lines and the generic fiber of the second stage is the direct product of $\frac{d-2}{2}$ copies of the del Pezzo surface X_{P_6}, or

ii. the direct product of two copies of X_{P_5} and $\frac{d}{2} - 2$ copies of X_{P_6} or

iii. the direct product of the del Pezzo fourfold $X_{\mathrm{DP}(4)}$ and $\frac{d}{2} - 2$ copies of X_{P_6}.

If d is odd then X is isomorphic to

iv. a toric fiber bundle over a projective line with generic fiber isomorphic to the direct product of X_{P_5} and $\frac{d-3}{2}$ copies of X_{P_6}.

All fiber bundles in the preceding result may or may not be trivial. Classifying the polytopes in Theorem 2 up to lattice equivalence is tantamount to classifying the associated toric varieties up to toric isomorphism. As detailed above there is one type for $d = 1$, two types for $d = 2, 3$, ten for $d = 4$, five for any odd dimension $d \geq 5$ and eleven types for even dimensions $d \geq 6$. For $d = 6$ this is explained in detail in Section 5 below. In dimensions up to and including 4 this is known from work of Batyrev (1991, 1999).

4 Special Facets and η-Vectors

In this section we will describe our major technical tools. This follows the approach of Øbro (2008). Let $P \subset \mathbb{R}^d$ be a reflexive lattice d-polytope with vertex set $\mathrm{Vert}(P)$. In particular, the origin $\mathbf{0}$ is an interior point. We let $v_P \;:=\; \sum_{v \in \mathrm{Vert}(P)} v$ be the *vertex sum* of P. As P is a lattice polytope v_P is a lattice point.

$\mathrm{ecc}(P)$	2	1	1	0	0	0	0
η_1	d	d	d	d	d	d	d
η_0	d	d	$d-1$	d	d	$d-1$	$d-2$
η_{-1}	$d-2$	$d-3$	$d-1$	$d-3$	$d-4$	$d-2$	d
η_{-2}	0	1	0	0	2	1	0
η_{-3}	0	0	0	1	0	0	0

Table 1: List of possible η-vectors of simplicial, terminal, and reflexive d-polytopes with $3d-2$ vertices, where $\mathrm{ecc}(P)$ denotes the eccentricity of P. Marked with a gray background are the η-vectors, which do not occur.

Now, a facet F of P is called *special* if the *face cone* $\mathrm{pos}\, F$ spanned by F contains v_P. Since the fan $\Sigma(P)$ generated by the face cones is complete, a special facet always exists. However, it is not necessarily unique. For instance, if P is centrally symmetric, we have $v_P = \mathbf{0}$, and each facet is special.

Since P is reflexive, for each facet F of P the primitive outer facet normal vector u_F satisfies $\langle u_F, x\rangle \leq 1$ for all points $x \in P$ and the set $\{x \in \mathbb{R}^d \mid \langle u_F, x\rangle = 1\}$ is the affine hull of F. We define

$$H(F,k) \;:=\; \left\{x \in \mathbb{R}^d \;\middle|\; \langle u_F, x\rangle = k\right\}\,, \quad V(F,k) \;:=\; H(F,k) \cap \mathrm{Vert}(P)\,, \quad \text{and} \quad \eta_k^F \;:=\; |V(F,k)|$$

for any integer $k \leq 1$. The sequence of numbers $\eta^F = (\eta_1^F, \eta_0^F, \eta_{-1}^F, \dots)$ is the *η-vector* of P with respect to F (we usually omit F in the notation). We omit any trailing zeros so that η has finite length. We have

$$\mathrm{Vert}(P) \;=\; \bigcup_{k\leq 1} V(F,k) \;\subseteq\; \bigcup_{k \leq 1} H(F,k)\,.$$

Thus $\eta_1^F + \eta_0^F + \eta_{-1}^F + \cdots = |\,\mathrm{Vert}(P)|$ is the number of vertices of P. If a vertex v is contained in $V(F,k)$ we call the number k the *level* of v with respect to F. As P is simplicial we have $\eta_1 = d$ for any facet F. Furthermore, one can show that for any facet F any vertex on level 0 is contained in a facet adjacent to F. Looking at a special facet and evaluating

$$0 \;\leq\; \langle u_F, v_P\rangle \;=\; \langle u_F, \sum_{k\leq 1}\sum_{v\in V(F,k)} v\rangle \;=\; \sum_{k\leq 1}\sum_{v\in V(F,k)} \langle u_F, v\rangle \;=\; d + \sum_{k\leq -1}\sum_{v\in V(F,k)}\langle u_F, v\rangle \tag{1}$$

shows that there can only be at most d many vertices below level 0. Thus, P has at most $3d$ vertices, implying the upper bound of (Casagrande, 2006). This allows to deduce a list of potential η-vectors from (1). Now we assume that P has exactly $3d-2$ vertices. *A priori*, the potential cases are listed in Table 1. The maximum level of v_P is 2. Our classification shows that not all of the η-vectors listed actually occur. Some can be ruled out by a direct argument, some only *a posteriori*. Those that do not occur are marked in gray in the table.

Our overall proof strategy is as follows. It turns out that the level of v_P is the same for each special facet. Hence, this is an invariant of the polytope, which we call the *eccentricity* $\mathrm{ecc}(P)$. We look at the three possible cases separately. We choose a special facet F of P. As a refinement, we consider separate cases according to the η-vector of F. A key is the observation, that we can, up to lattice equivalence, restrict the possible choices for vertices in levels $1, 0$, and -1 of F. This is summarized in the proposition below; see (Assarf et al., 2012, Prop. 32). Given this initial distribution of the vertices we want to determine the remaining vertices. Sometimes this turns out to be quite difficult. In this cases we switch to a special neighboring facet with a different η-vector which is easier to analyze or already have been analyzed. With $\mathrm{opp}(F)$ we denote the set of all vertices which lie in a facet adjacent to F but which are not vertices of F itself.

Proposition 5 *Let P be a d-dimensional simplicial, terminal, and reflexive polytope such that F is a special facet. Up to lattice equivalence, we can assume that $F = \operatorname{conv}\{e_1, e_2, \dots, e_d\}$ and there is a map $\phi : \operatorname{Vert}(F) \to \operatorname{Vert}(F) \cup \{0\}$ such that:*

i. if $\eta_0^F = d$, then

$$\begin{aligned} V(F,0) &= \{\phi(e_1) - e_1,\, \phi(e_2) - e_2,\, \dots,\, \phi(e_d) - e_d\} \\ V(F,-1) &\subseteq \{-e_1,\, -e_2,\, \dots,\, -e_d\}\,. \end{aligned}$$

ii. If $\eta_0^F = d-1$ and $\operatorname{opp}(F) = V(F,0)$, then, for $a, b \in [d] \setminus \{1,2\}$ not necessarily distinct,

$$\begin{aligned} V(F,0) &= \{-e_1 - e_2 + e_a + e_b,\, \phi(e_3) - e_3,\, \dots,\, \phi(e_d) - e_d\} \\ V(F,-1) &\subseteq \{-e_1,\, -e_2,\, \dots,\, -e_d\} \,\cup\, \{-e_1 - e_2 + e_s \mid s \in [d]\} \end{aligned}$$

iii. If $\eta_0^F = d-1$ and $\operatorname{opp}(F) \neq V(F,0)$, then

$$\begin{aligned} V(F,0) &= \{\phi(e_2) - e_2,\, \phi(e_3) - e_3,\, \dots,\, \phi(e_d) - e_d\} \\ V(F,-1) &\subseteq \{-e_1,\, -e_2,\, \dots,\, -e_d\} \,\cup\, \{-2e_1 - e_r + e_s + e_t \mid r,s,t \in [d] \text{ pairwise distinct },\ r \neq 1\}\,. \end{aligned}$$

The first case above occurs in (Øbro, 2008). This result allows us to control most of the vertices of a simplicial, terminal, and reflexive polytope if η_0 is given. In this way an approach to the classification is by examining choices for the vertices on the levels k for $k \leq -2$.

5 The Classification Explained in Dimension Six

In this section we will explicitly list the 6-dimensional simplicial, terminal, and reflexive polytopes with exactly $3 \cdot 6 - 2 = 16$ vertices. This is the smallest even dimension in which all eleven types up to lattice equivalence arise. This list in dimension 6 is already subsumed in the classifications (Brown and Kasprzyk, 2009–2012) and (Lorenz and Paffenholz, 2008); and we will refer to the latter. Here we will organize the polytopes in a way such that it fits the line of arguments in (Assarf et al., 2012). Additional comments are meant to give the reader an idea about the organization of our proof.

Throughout let P be a d-dimensional simplicial, terminal, and reflexive polytope with $3d-2$ vertices such that F is a special facet. The vertex sum v_P lies on level 0, 1 or 2 with respect to F. Throughout we assume that d is even and $d \geq 4$. It turns out that each such polytope P contains a copy of the hexagon P_6 as a subpolytope, albeit not necessarily as a direct summand. So we normalized the examples in a way that P_6 always lies in in the coordinate subspace $\operatorname{lin}\{e_1, e_2\}$. This way the differences among the examples are particularly easy to spot.

5.1 Polytopes of Eccentricity 2

The classification becomes more involved the more symmetric P is. The most eccentric case occurs if the vertex sum lies on level 2, and this is the easiest. Table 1 tells us that there is only one kind of η-vector, namely $\eta^F = (d, d, d-2)$. What makes this case simpler than others is that we immediately have $\eta_0 = d$, which forces that the vertices on F form a lattice basis, and the vertices on level 0 can be determined (Øbro, 2008). In this case the partial description of the vertices in Proposition 5 is already good enough to get the full picture with little extra effort. It turns out that P is lattice equivalent to $P_5^{\oplus 2} \oplus P_6^{\oplus \frac{d}{2}-2}$ or to a skew bipyramid over a $(d-1)$-dimensional smooth Fano polytope with $3(d-1) - 1 = 3d - 4$ vertices.

Example 6 *For $d = 6$ the first case is $P \cong P_6 \oplus P_5 \oplus P_5$ such that $v_P = e_3 + e_5$. Here and in the examples below, we list the vertices sorted by level.*

$$e_1, e_2, e_3, e_4, e_5, e_6$$
$$\pm(e_1 - e_2), \pm(e_3 - e_4), \pm(e_5 - e_6)$$
$$-e_1, -e_2, -e_4, -e_6$$

In the database (Lorenz and Paffenholz, 2008) this occurs as `F.6D.6552`. The polytope has 24 special facets.

If the polytope P is not of the type above then, for $d = 6$, the polytope P is a double skew bipyramid over $P_6 \oplus P_6$. Four more cases arise depending on the relative positions of the apices of the two bipyramids. To form a skew bipyramid we need to pick a vertex of the base. Since the group of lattice automorphisms of P_6 acts transitively on the vertices, we may assume that the first skew bipyramid is $\mathrm{BP}(P_6^{\oplus 2}, e_1, e_5)$. The three distinct relative positions of two vertices of P_6 lead to the next three cases.

Example 7 *For $d = 6$ the second type is given by $\mathrm{BP}(\mathrm{BP}(P_6^{\oplus 2}, e_1, e_5), e_1, e_6)$ such that $v_P = 2e_1$.*

$$e_1, e_2, e_3, e_4, e_5, e_6$$
$$\pm(e_1 - e_2), \pm(e_3 - e_4), e_1 - e_5, e_1 - e_6$$
$$-e_1, -e_2, -e_3, -e_4$$

In the database this occurs as `F.6D.5346`. The polytope has 48 special facets.

Example 8 *For $d = 6$ the third type is given by $\mathrm{BP}(\mathrm{BP}(P_6^{\oplus 2}, e_1, e_5), e_2, e_6)$ such that $v_P = e_1 + e_2$.*

$$e_1, e_2, e_3, e_4, e_5, e_6$$
$$\pm(e_1 - e_2), \pm(e_3 - e_4), e_1 - e_5, e_2 - e_6$$
$$-e_1, -e_2, -e_3, -e_4$$

In the database this occurs as `F.6D.5680`. The polytope has 24 special facets.

Example 9 *For $d = 6$ the fourth type is given by $\mathrm{BP}(\mathrm{BP}(P_6^{\oplus 2}, e_1, e_5), e_3, e_6)$ such that $v_P = e_1 + e_3$.*

$$e_1, e_2, e_3, e_4, e_5, e_6$$
$$\pm(e_1 - e_2), \pm(e_3 - e_4), e_1 - e_5, e_3 - e_6$$
$$-e_1, -e_2, -e_3, -e_4$$

In the database this occurs as `F.6D.5553`. The polytope has 16 special facets.

The final case in this section differs from the above in that the base vertex of the second skew bipyramid is an apex of the first stage.

Example 10 *For $d = 6$ the fifth type is given by $\mathrm{BP}(\mathrm{BP}(P_6^{\oplus 2}, e_1, e_5), e_5, e_6)$ such that $v_P = e_1 + e_5$.*

$$e_1, e_2, e_3, e_4, e_5, e_6$$
$$\pm(e_1 - e_2), \pm(e_3 - e_4), e_1 - e_5, e_5 - e_6$$
$$-e_1, -e_2, -e_3, -e_4$$

In the database this occurs as `F.6D.5685`. The polytope has 24 special facets.

5.2 Polytopes of Eccentricity 1

If the vertex sum lies on level one, then the situation is still somewhat benign. Our proof strategy is to first consider polytopes P with a special facet that have η-vector $(d, d, d-3, 1)$. In (Assarf et al., 2012, Prop. 36) we show that in this case P, again, must be a skew bipyramid. Notice, however, that our classification shows *a posteriori* that this case does not occur. Table 1 then says that the only choice left is $\eta = (d, d-1, d-1)$. In this situation (Assarf et al., 2012, Prop. 39) shows that, once more, P is a double bipyramid.

In the first two cases the first stage is a proper bipyramid. For the second stage then the base vertex can either be in the base of the first stage or an apex.

Example 11 *For $d = 6$ the sixth type is given by* $\mathrm{BP}(\mathrm{BP}(P_6^{\oplus 2}, 0, e_5), e_1, e_6)$ *such that* $v_P = e_1$.

$$\begin{gathered} e_1, e_2, e_3, e_4, e_5, e_6 \\ \pm(e_1 - e_2), \pm(e_3 - e_4), e_1 - e_6 \\ -e_1, -e_2, -e_3, -e_4, -e_5 \end{gathered}$$

In the database this occurs as `F.6D.5711`*. The polytope has 48 special facets.*

Example 12 *For $d = 6$ the seventh type is given by* $\mathrm{BP}(\mathrm{BP}(P_6^{\oplus 2}, 0, e_5), e_5, e_6)$ *such that* $v_P = e_5$.

$$\begin{gathered} e_1, e_2, e_3, e_4, e_5, e_6 \\ \pm(e_1 - e_2), \pm(e_3 - e_4), e_5 - e_6 \\ -e_1, -e_2, -e_3, -e_4, -e_5 \end{gathered}$$

In the database this occurs as `F.6D.6558`*. The polytope has 72 special facets.*

For $v_P \in H(F, 1)$ there is only one choice of a double bipyramid where both stages are skew.

Example 13 *For $d = 6$ the eighth type is given by* $\mathrm{BP}(\mathrm{BP}(P_6^{\oplus 2}, e_2, e_5), e_1 - e_2, e_6)$ *such that* $v_P = e_1$.

$$\begin{gathered} e_1, e_2, e_3, e_4, e_5, e_6 \\ \pm(e_1 - e_2), \pm(e_3 - e_4), e_2 - e_5, e_1 - e_2 - e_6 \\ -e_1, -e_2, -e_3, -e_4 \end{gathered}$$

In the database this occurs as `F.6D.5702`*. The polytope has 48 special facets.*

5.3 Polytopes of Eccentricity 0

If the vertex sum of P is zero all facets are special. The easy subcase occurs when all η-vectors of P are of type $(d, d-2, d)$. We show that in this case P is centrally symmetric (Assarf et al., 2012, Prop. 40). Extending arguments of (Nill, 2006, Thm. 0.1) we show that such a polytope is lattice equivalent to a double proper bipyramid over $P_6^{\oplus \frac{d-2}{2}}$ or $\mathrm{DP}(4) \oplus P_6^{\oplus \frac{d}{2}-2}$.

Example 14 *If $d = 6$ the ninth type occurs for* $P \cong P_6 \oplus \mathrm{DP}(4)$.

$$\begin{gathered} e_1, e_2, e_3, e_4, e_5, e_6 \\ \pm(e_1 - e_2), \pm(e_3 + e_4 - e_5 - e_6) \\ -e_1, -e_2, -e_3, -e_4, -e_5, -e_6 \end{gathered}$$

In the database this occurs as `F.6D.3154`. *All 180 facets are special, and all of them have the same η-vector* $(6, 4, 6)$.

Example 15 *If $d = 6$ the tenth case is the direct sum of two hexagons P_6 and two line segments. In our notation, this means that $P \cong \mathrm{BP}(\mathrm{BP}(P_6^{\oplus 2}, 0, e_5), 0, e_6)$.*

$$\begin{gathered} e_1, e_2, e_3, e_4, e_5, e_6 \\ \pm(e_1 - e_2), \pm(e_3 - e_4) \\ -e_1, -e_2, -e_3, -e_4, -e_5, -e_6 \end{gathered}$$

In the database this occurs as `F.6D.6765`. *All 144 facets are special, and all of them have the same η-vector* $(6, 4, 6)$.

It remains to discuss the situation where $v_P = \mathbf{0}$ but P is not centrally symmetric. This is by far the most complicated case in our proof. It contributes to this complexity that we need to discuss four candidates of η-vectors. First, $\eta = (6, 6, 3, 0, 1)$ is excluded (Assarf et al., 2012, Prop. 4). Second, $\eta = (6, 6, 2, 2)$ is essentially reduced to a bipyramid (Assarf et al., 2012, Lem. 43) (but this case does not exist *a posteriori*). So this leaves two more η-vectors. Surprisingly, they lead to the same polytopes.

Example 16 *If $d = 6$ the final eleventh type occurs for $P \cong \mathrm{BP}(\mathrm{BP}(P_6^{\oplus 2}, e_1, e_5), -e_1, e_6)$. Up to lattice equivalence this is the only case in which $v_P = \mathbf{0}$ but P is not centrally symmetric.*

$$\begin{gathered} e_1, e_2, e_3, e_4, e_5, e_6 \\ \pm(e_1 - e_2), \pm(e_3 - e_4), e_1 - e_5 \\ -e_1, -e_2, -e_3, -e_4 \\ -e_1 - e_6 \end{gathered}$$

In the database this occurs as `F.6D.5713`. *All 144 facets are special, where 96 of them have the η-vector $(6, 5, 4, 1)$ and the other 48 the η-vector reads $(6, 4, 6)$. For instance, the facet which is induced by $\langle \mathbf{1}, x \rangle = 1$ has the η-vector $(6, 5, 4, 1)$, and the facet induced by $\langle \mathbf{1} - 2e_1 - 2e_6, x \rangle = 1$ has the η-vector $(6, 4, 6)$.*

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Counting smaller trees in the Tamari order

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Abstract. We introduce new combinatorial objects, the interval-posets, that encode intervals of the Tamari lattice. We then find a combinatorial interpretation of the bilinear form that appears in the functional equation of Tamari intervals described by Chapoton. Thus, we retrieve this functional equation and prove that the polynomial recursively computed from the bilinear form on each tree T counts the number of trees smaller than T in the Tamari order.

Résumé. Nous introduisons un nouvel objet, les intervalles-posets, pour encoder les intervalles de Tamari. Nous donnons ainsi une interprétation combinatoire à la forme bilinéaire qui apparaît dans l'équation fonctionnelle des intervalles de Tamari que donne Chapoton. De cette façon, nous retrouvons d'une nouvelle manière cette équation fonctionnelle et prouvons que le polynôme calculé récursivement à partir de la forme bilinéaire pour chaque arbre T compte le nombre d'arbres plus petits que T dans l'ordre de Tamari.

Keywords: binary trees, Tamari lattice, Tamari intervals

1 Introduction

The combinatorics of planar binary trees has already being linked with interesting algebraic properties. Loday and Ronco first introduced the **PBT** Hopf Algebra based on these objects [9]. It was re-built by Hivert, Novelli and Thibon [7] through the introduction of the sylvester monoid. The structure of **PBT** involves a very nice object which is linked to both algebra and classical algorithmics: the *Tamari lattice*.

This order on binary trees is based on the *right rotation* operation (see Figure 1), commonly used in sorting algorithms through binary search trees. The lattice itself first appeared in the context of the associahedron [8]. As its vertices are counted by Catalan numbers, the covering relations can be described by many combinatorial objects [10], two common ones being planar binary trees and Dyck paths. Recently Chapoton gave a formula for the number of intervals [4]:

$$I_n = \frac{2(4n+1)!}{(n+1)!(3n+2)!}, \tag{1}$$

where I_n is the number of intervals of the Tamari lattice of binary trees of size n. This formula was very recently generalized to a new set of lattices, the m-Tamari lattices [3].

It has been known since Björner and Wachs [2] that linear extensions of a certain labelling of binary trees correspond to intervals of the weak order on permutations. This was more explicitly described in [7] with sylvester classes. The elements of the basis **P** of **PBT** are defined as a sum on a sylvester class of elements of **FQSym**. The **PBT** algebra also admits two other bases **H** and **E** which actually correspond to respectively initial and final intervals of the Tamari order. They can be indexed by plane forests and, with a well chosen labelling, their linear extensions are intervals of the weak order on permutations corresponding to a union of sylvester classes. In this paper, we introduce a more general object, the Tamari interval-poset, which encodes a general interval of the Tamari lattice and whose linear extensions are exactly the corresponding sylvester classes (and so an interval of the weak order). This new object has nice combinatorial properties and allows to perform computations on Tamari intervals.

Thereby, we give a new proof of the formula of Chapoton (1). This proof is based on the study of a bilinear form that already appeared in [4] but was not explored yet. It leads to the definition of a new family of polynomials:

Definition 1.1 *Let T be a binary tree, the polynomial $\mathcal{B}_T(x)$ is recursively defined by*

$$\mathcal{B}_\emptyset := 1 \tag{2}$$

$$\mathcal{B}_T(x) := x\mathcal{B}_L(x)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1} \tag{3}$$

where L and R are respectively the left and right subtrees of T. We call $\mathcal{B}_T(x)$ the Tamari polynomial *of T and the* Tamari polynomials *are the set of all polynomials obtained by this process.*

This family of polynomials is yet unexplored in this context but a different computation made by Chapoton [5] on rooted trees seems to give a bivariate version. Our approach on Tamari interval-posets allows us to prove the following theorem:

Theorem 1.2 *Let T be a binary tree. Its Tamari polynomial $\mathcal{B}_T(x)$ counts the trees smaller than T in the Tamari order according to the number of nodes on their left border. In particular, $\mathcal{B}_T(1)$ is the number of trees smaller than T.*

Symmetrically, if $\tilde{\mathcal{B}}_T$ is defined by exchanging the role of left and right children in Definition 1.1, then it counts the number of trees greater than T according to the number of nodes on their right border.

This theorem will be proven in Section 3.2. In Section 2, we recall some definitions and properties of the Tamari lattice and introduce the notion of *interval-poset* to encode a Tamari interval. In Section 3, we show the implicit bilinear form that appears in the functional equation of the generating functions of Tamari intervals. We then explain how interval-posets can be used to give a combinatorial interpretation of this bilinear form and thereby give a new proof of the functional equation. Theorem 1.2 follows naturally. In Section 4, we give two independent contexts in which our problem can be generalized: flows of rooted trees and m-Tamari intervals.

2 Definitions of Tamari interval-posets

2.1 Binary trees and Tamari order

A binary tree is recursively defined by being either the empty tree ($\emptyset$) or a pair of binary trees, respectively called *left* and *right* subtrees, grafted on an internal node. If a tree T is composed of a root node x with A and B as respectively left and right subtrees, we write $T = x(A, B)$. The number of nodes of a tree T is called the size of T. The Tamari order is an order on trees of a given size using the *rotation* operation.

Definition 2.1 *Let y be a node of T with a non-empty left subtree x. The* right rotation *of T on y is a local rewriting which follows Figure 1, that is replacing $y(x(A,B),C)$ by $x(A,y(B,C))$ (note that A, B, or C might be empty).*

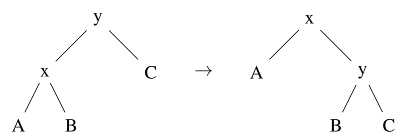


Figure 1: Right rotation on a binary tree.

The Tamari order is the transitive and reflexive closure of the right rotation: a tree T' is greater than a tree T if T' can be obtained by applying a sequence of right rotations on T. It is actually a lattice [8], see Figure 2 for some examples. One of the purposes of this article is to define combinatorial objects that correspond to Tamari intervals.

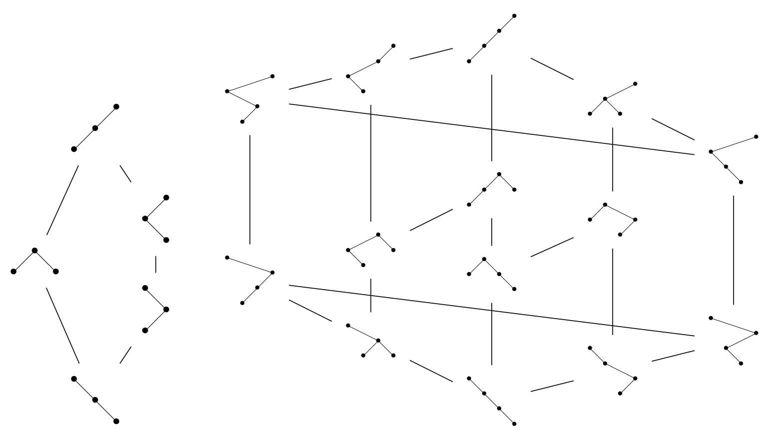
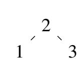


Figure 2: Tamari lattices of size 3 and 4.

The Tamari lattice is a quotient of the weak order on permutations [7]. To understand the relation between the two orders, we need the notion of *binary search tree*.

Definition 2.2 *A* binary search tree *is a labelled binary tree where for each node of label k, any label in his left (resp. right) subtree is lower than or equal to (resp. greater than) k.*

Figure 3 shows an example of a binary search tree. For a given binary tree T of size n, there is a unique labelling of T with $1,\dots,n$ such that T is a binary search tree. Such a labelled tree can then be seen as a poset. For example, the tree



is the poset where 1 and 3 are smaller than 2. We write $1 \prec 2$ and $3 \prec 2$. A linear extension of this poset is a permutation where if $a \prec b$ in the poset, then the number a is before b in the permutation. The linear extensions of the tree above are 132 and 312. The sets of permutations corresponding to the linear extensions of the binary trees of size n form a partition of $\mathfrak{S}_n$ and more precisely, each set is an interval of the right weak order on permutations called a *sylvester class* and the Tamari order is a lattice on these classes [7]. See Figures 3 for examples of sylvester classes.

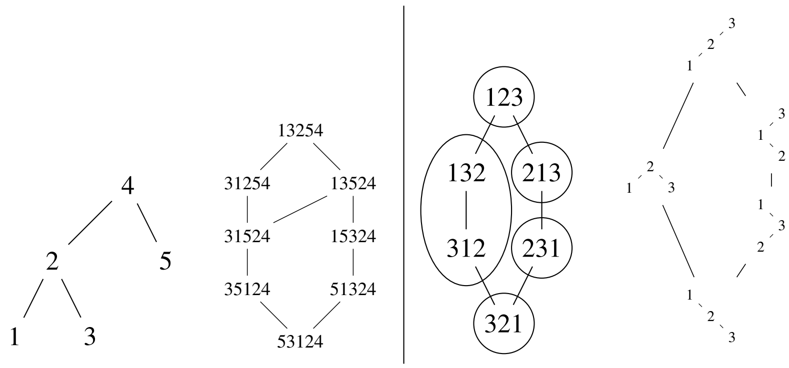


Figure 3: On the left: a binary search tree and its corresponding sylvester class, and on the right: the sylvester classes of the weak order of size 3, with the corresponding binary search trees.

2.2 Construction of interval-posets

We now introduce more general objects, called the *interval-poset*, that are in bijection with the intervals of the Tamari order. Let us first recall two bijections between binary search trees and forests of planar trees. A binary search tree T is a poset containing two kinds of relations: when a is in the left subtree of b, we have an increasing relation $a < b$ and $a \prec_T b$ and when b is in the right subtree of a, we have a decreasing relation $b > a$ and $b \prec_T a$. The two bijections we define consist in keeping only increasing (resp. decreasing) relations of the poset.

Definition 2.3 *The* increasing forest(i) *(noted $F_{\leq}$) of a binary search tree T is a forest poset on the nodes of T containing only increasing relations and such that:*

$$a \prec_{F_{\leq}(T)} b \Leftrightarrow a < b \text{ and } a \prec_T b. \tag{4}$$

It is equivalent to the following construction:

- *if a node labelled x has a left son labelled y in T then the node x has a son y in F;*
- *if a node labelled x has a right son labelled y in T then the node x has a brother y in F.*

(i) Note that what we call increasing means increasing labels from the leaf to the root and not from the root to the leaf as it is often the case.

In the same way, one can define the decreasing forest *(noted $F_{\geq}$) by switching the roles of the right and left son in the previous construction or, in terms of posets:*

$$b \prec_{F_{\geq}(F)} a \Leftrightarrow a < b \ \textit{and} \ b \prec_T a. \tag{5}$$

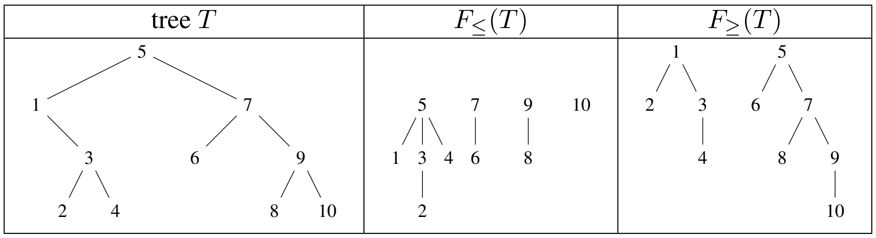


Figure 4: A tree with its corresponding increasing and decreasing forests.

In Figure 4, we can see a tree T with its decreasing and increasing forests. The linear extensions of the decreasing and increasing forests are actually initial and final intervals of the weak order.

Proposition 2.4 *The linear extensions of the increasing forest of a tree T is the union of the linear extensions of all trees lower than or equal to T (initial interval) and the linear extensions of the decreasing forest of T is the union of the linear extensions of all trees greater than or equal to T (final interval).*

Proof (sketch): We just need to recall that $\sigma \leq \mu$ in the weak order means that $coinv(\sigma) \subseteq coinv(\mu)$, where $coinv(\sigma) := \{(\sigma(i), \sigma(j)); i < j, \sigma(i) > \sigma(j)\}$. It is then easy to see that the linear extension with maximal (resp. minimal) number of co-inversions is the same for T than for $F_{\leq}$ (resp. $F_{\geq}$). Conversely, if the co-inversions of a permutation μ are included in the co-inversions of the maximal linear extension of a tree for the weak order, then μ is a linear extension of $F_{\leq}$. The same reasoning can be made for $F_{\geq}$. $\square$

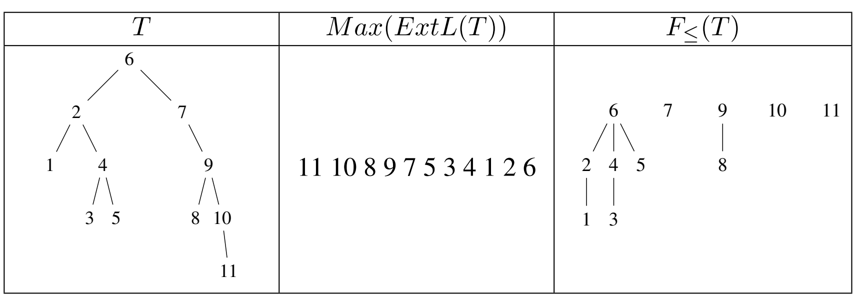


Figure 5: A tree with the maximum of its sylvester class and its increasing forest.

An example of this construction can be found in Figure 4 and another example of an increasing forest is given in Figure 5 with its maximal linear extension. If two trees T and T' are such that $T \leq T'$, then $F_{\geq}(T)$ and $F_{\leq}(T')$ share some linear extensions (by Proposition 2.4). More precisely, we have $ExtL(F_{\geq}(T)) \cap ExtL(F_{\leq}(T')) = [Min(ExtL(T)), Max(ExtL(T'))]$. This set corresponds exactly to the linear extensions of the trees of the interval $[T, T']$ in the Tamari order. It is then natural to construct a poset that would contain relations of both $F_{\geq}(T)$ and $F_{\leq}(T')$, see Figure 6 for an example. We give a characterization of these posets.

Definition 2.5 *An* interval-poset P *is a poset such that the following conditions hold:*

- $a \prec_P c$ *implies that for all* $a < b < c$*, we have* $b \prec_P c$*,*
- $c \prec_P a$ *implies that for all* $a < b < c$*, we have* $b \prec_P a$*.*

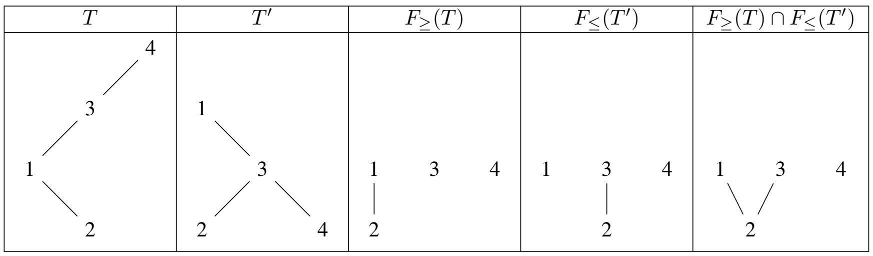


Figure 6: Two trees T and T', their decreasing and increasing forest and the interval-poset $[T, T']$. The linear extensions of the interval-poset correspond to the interval $[2134, 4231]$ of the weak order and 2134 (resp. 4231) is the minimal (resp. maximal) linear extension of T (resp. T').

Proposition 2.6 *The interval-posets are exactly the posets whose linear extensions correspond to Tamari intervals for the weak order.*

Indeed, it is easy to see that from an interval-poset, one can build $F_{\leq}$ (resp. $F_{\geq}$) by only considering the increasing relations (resp. decreasing relations). Conditions of Definition 2.5 are necessary and sufficient to obtain well-defined increasing (resp. decreasing) forests that correspond to proper binary search trees.

2.3 Combinatorial properties of interval-posets

Many operations on intervals can be easily done on interval-posets, all with trivial proofs.

Proposition 2.7 *(i) The intersection between two intervals I_1 and I_2 is given by the interval-poset I_3 containing all relations of I_1 and I_2. If I_3 is a valid poset (there is no cycle in the union of I_1 and I_2), then it is a valid interval-poset, otherwise the intersection is empty.*

(ii) An interval $I_1 := [T_1, T_1']$ is contained into an interval $I_2 := [T_2, T_2']$, i.e.*, $T_1 \geq T_2$ and $T_1' \leq T_2'$, if and only if all relations of the interval-poset I_1 are satisfied by the interval-poset I_2.*

(iii) If $I_1 := [T_1, T_1']$ is an interval, then $I_2 = [T_2, T_1']$, $T_2 \geq T_1$, if and only if all relations of the interval-poset I_1 are satisfied by I_2 and all new relations of I_2 are decreasing. Symmetrically, $I_3 = [T_1, T_3]$, $T_3 \leq T_1'$, if and only if all relations of the interval-poset I_1 are satisfied by I_3 and all new relations of I_3 are increasing.

3 Tamari polynomials

3.1 Bilinear form and enumeration

Let $\phi(y)$ be the generating function of Tamari intervals,

$$\phi(y) = 1 + y + 3y^2 + 13y^3 + 68y^4 + \dots\ . \tag{6}$$

where y counts the number of nodes in the trees or equivalently the number of vertices in the interval-posets. In [4], Chapoton gives a refined version of ϕ with a parameter x that counts the number of nodes on the left border of the smaller tree of the interval,

$$\Phi(x, y) = 1 + xy + (x + 2x^2)y^2 + (3x + 5x^2 + 5x^3)y^3 + \dots\ . \tag{7}$$

We know that an interval-poset I of $[T, T']$ is formed by two forest posets of respectively decreasing relations of T and increasing relations of T'. The number of nodes in the left border of T can then be seen as the number of trees in $F_{\geq}(T)$, *i.e.*, the poset formed by the decreasing relations of I. This way, one can interpret the refined generating function (7) directly on interval-posets. In [4, formula (6)], Chapoton gives a functional equation on Φ:[(ii)]

$$\Phi(x, y) = xy\Phi(x, y)\frac{x\Phi(x, y) - \Phi(1, y)}{x - 1} + 1. \tag{8}$$

The generating function Φ is then the solution of

$$\Phi = B(\Phi, \Phi) + 1 \tag{9}$$

where B is the bilinear form

$$B(f, g) = xyf(x, y)\frac{xg(x, y) - g(1, y)}{x - 1}. \tag{10}$$

By expanding (9), one obtains

$$\Phi = 1 + B(1, 1) + B(B(1, 1), 1) + B(1, B(1, 1)) + \dots \tag{11}$$

$$= \sum_T \mathcal{B}_T, \tag{12}$$

sum over all binary trees T, with $\mathcal{B}_T$ recursively defined by $\mathcal{B}_\emptyset := 1$ and $\mathcal{B}_T := B(\mathcal{B}_L, \mathcal{B}_R)$ where L and R are respectively the left and right children of T. Using a combinatorial interpretation of B, we actually prove that $\mathcal{B}_T$ counts the number of trees smaller than T in the Tamari order. We also obtain a new way of generating intervals and thus prove in a new way that the generating function of the interval satisfies the functional equation (8). Let us define an operation on interval-posets:

Definition 3.1 *Let I_1 and I_2 be two interval-posets of respective sizes k_1 and k_2. Then $\mathbb{B}(I_1, I_2)$ is the formal sum of all interval-posets of size $k_1 + k_2 + 1$ where,*

(ii) Our equation is slightly different from the one of [4, formula (6)]. Indeed, the definition of the degree of x differs by one and in our case Φ also counts the interval of size 0.

(i) the relations between vertices $1, \dots, k_1$ are exactly the ones from I_1,

(ii) the relations between $k_1+2, \dots, k_1+k_2+1$ are exactly the ones from I_2 shifted by k_1+1,

(iii) we have $i \prec k_1+1$ for all $i \leq k_1$,

(iv) there is no relation $k_1+1 \prec j$ for all $j > k_1+1$.

We call this operation the composition of intervals and extend it by bilinearity to all linear sums of intervals.

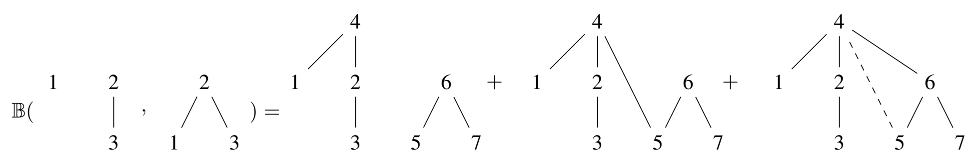


Figure 7: Composition of interval-posets: the three terms of the sum are obtained by adding respectively no, 1, and 2 decreasing relations between the second poset and the vertex 4. For the last term, two decreasing relations have been added: $5 \prec 4$ and $6 \prec 4$, the $5 \prec 4$ relation has been dashed as it is implicit through transitivity.

The sum we obtain by composing interval-posets actually corresponds to all possible ways of adding decreasing relations between the second poset and the new vertex k_1+1, as seen on Figure 7. Especially, there is no relations between vertices $1, \dots, k_1$ and $k_1+2, \dots, k_1+k_2+1$. Indeed, condition (iii) makes it impossible to have any relation $j \prec i$ with $i < k_1+1 < j$ as this would imply by Definition 2.5 that $k_1+1 \prec i$. And condition (iv) makes it impossible to have $i \prec j$ as this would imply $k_1+1 \prec j$.

Proposition 3.2 *Let I_1 and I_2 be two interval-posets. Let $\mathcal{P}$ be the linear function that associates with an interval-poset its monomial $x^{trees}y^{size}$ where the power of y is the number of vertices and the power of x the number of trees obtained by keeping only decreasing relations. Then*

$$\mathcal{P}(\mathbb{B}(I_1, I_2)) = B(\mathcal{P}(I_1), \mathcal{P}(I_2)). \tag{13}$$

As an example, in Figure 7, $\mathcal{P}(I_1) = \mathcal{P}(I_2) = x^2y^3$. And we have $\mathcal{P}(\mathbb{B}(I_1, I_2)) = x^5y^7 + x^4y^7 + x^3y^7 = B(x^2y^3, x^2y^3)$.

Proof: If I_1 and I_2 are two interval-posets of size respectively k_1 and k_2, we have by definition that all interval-posets of $\mathbb{B}(I_1, I_2)$ are of size k_1+k_2+1. Thus the power of y is the same in $B(\mathcal{P}(I_1), \mathcal{P}(I_2))$ and in $\mathcal{P}(\mathbb{B}(I_1, I_2))$ and we only have to consider the polynomial in x.

Let us assume that I_1 and I_2 contain respectively n and m trees formed by decreasing relations. The n trees of I_1 are kept unchanged on all terms of the result as no decreasing relation is added to the vertices $1, \dots, k_1$. Now, we call $v_1 < \dots < v_m$ the root vertices of the trees of I_2 shifted by k_1+1. By construction, $k_1+1 < v_1$, and this new vertex can either become a new root or a root to some of the previous trees. If we have $v_j \prec k_1+1$, by definition of an interval-poset, we also have $v_i \prec k_1+1$ for all $i < j$. The m trees of I_2 can then be replaced by either $m+1, m, \dots, 2$, or 1 trees, which mean the monomial x^m of $\mathcal{P}(I_2)$ becomes $x + x^2 + \dots + x^{m+1}$ in the composition. So,

$$\mathcal{P}(\mathbb{B}(I_1, I_2)) = y(x^n y^{k_1}) y^{k_2} x \frac{x^{m+1} - 1}{x - 1} \tag{14}$$

$$= B(\mathcal{P}(I_1), \mathcal{P}(I_2)). \tag{15}$$

□

To prove now that the generating function of the intervals is the solution of the bilinear equation (9), we only need the following proposition.

Proposition 3.3 *Let I be an interval-poset, then, there is exactly one pair of intervals I_1 and I_2 such that I appears in the composition $\mathbb{B}(I_1, I_2)$.*

Proof: Let I be an interval-poset of size n and let k be the vertex of I with maximal label such that $i \prec k$ for all $i < k$. The vertex 1 satisfies this property, so one can always find such a vertex. We prove that I only appears in the composition of I_1 by I_2, where I_1 is formed by the vertices and relations of $1, \dots, k-1$ and I_2 is formed by the re-normalized vertices and relations of $k+1, \dots, n$. Note that one or both of these intervals can be of size 0.

Conditions (i), (ii), and (iii) of Definition 3.1 are clearly satisfied by construction. If condition (iv) is not satisfied, it means that we have a relation $k \prec j$ with $j > k$. Then, by definition of an interval-poset, we also have $\ell \prec j$ for all $k < l < j$ and by definition of k, we have $i \prec k \prec j$ for all $i < k$, so for all $i < j$, we have $i \prec j$. This is not possible as k has been chosen to be maximal among vertices with this property.

This proves that I appears in the composition of I_1 by I_2. Now, if I appears in $\mathbb{B}(I'_1, I'_2)$, the vertex $k' = |I'_1| + 1$ is by definition the vertex where for all $i < k'$, we have $i \prec k'$ and for all $j > k'$, we have $k' \not\prec j$, this is exactly the definition of k. So $k' = k$ which makes $I'_1 = I_1$ and $I'_2 = I_2$. □

3.2 Main result

This composition operation on intervals is an analogue of the usual composition of binary trees that adds a root node to two given binary trees. In our case, a tree T is replaced by a sum of intervals $[T', T]$.

Proposition 3.4 *Let $T := k(T_1, T_2)$ be a binary tree and $S := \sum_{T' \leq T} [T', T]$. Then, if $S_1 := \sum_{T'_1 \leq T_1} [T'_1, T_1]$ and $S_2 := \sum_{T'_2 \leq T_2} [T'_2, T_2]$, we have $S = \mathbb{B}(S_1, S_2)$.*

With this new proposition, Theorem 1.2 would be fully proven by induction on the size of the tree. The initial case is trivial, and then if we assume that $\mathcal{P}(S_1) = \mathcal{B}_{T_1}(x)$ and $\mathcal{P}(S_2) = \mathcal{B}_{T_2}(x)$, Proposition 3.2 tells us that $\mathcal{P}(\mathbb{B}(S_1, S_2)) = B(\mathcal{B}_{T_1}, \mathcal{B}_{T_2})$.

Proof: Let T be a binary tree of size n. The initial interval $\mathcal{T} = [T_0, T]$, is given by the increasing bijection of Definition 2.3, it is a poset containing only increasing relations. By Proposition 2.7, (iii), the sum of all intervals $[T', T]$ is given by all possible ways of adding decreasing edges to the poset $\mathcal{T}$.

The increasing poset $\mathcal{T}$ can be formed recursively from the increasing posets $\mathcal{T}_1$ and $\mathcal{T}_2$ of the subtrees T_1 and T_2 as shown in Figure 8. The new vertex $k = |T_1| + 1$ is placed so that $i \prec k$ for all $i \in \mathcal{T}_1$ and the vertices of $\mathcal{T}_2$ are just shifted by k. Now, let I be an interval of the sum S, I contains the poset $\mathcal{T}$ and some extra decreasing relations. Let I_1 and I_2 be the subposets formed respectively by vertices $1, \dots, k-1$, and $k+1,\dots,n$. By construction, the posets I_1 and I_2 contain respectively the forest posets $\mathcal{T}_1$ and $\mathcal{T}_2$ and some extra decreasing relations. This means that I_1 appears in S_1 and I_2 appears in S_2. And we have that I appears in $\mathbb{B}(I_1, I_2)$. Indeed, conditions (i) and (ii) of Definition 3.1 are true by construction and conditions (iii) and (iv) are true because the increasing relations of I are exactly the ones of $\mathcal{T}$.

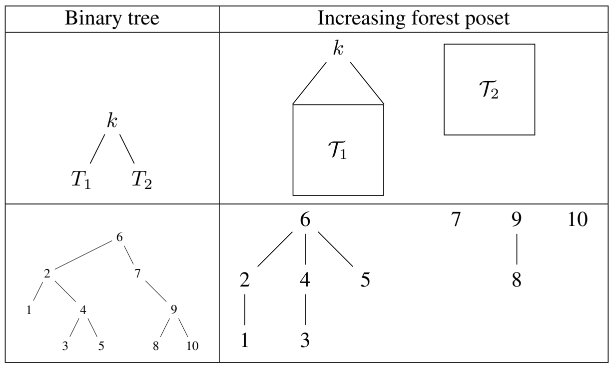


Figure 8: The recursive construction of $\mathcal{T}$ from $\mathcal{T}_1$ and $\mathcal{T}_2$.

Conversely, if I_1 and I_2 are two elements of respectively S_1 and S_2, their increasing relations are exactly the ones from respectively $\mathcal{T}_1$ and $\mathcal{T}_2$ which makes all interval-posets I of $\mathbb{B}(I_1, I_2)$ an element of S. Indeed, by definition of the composition, the increasing relations of I are exactly the ones of $\mathcal{T}$. $\square$

For a given tree T (with increasing poset $\mathcal{T}$), the coefficient of the monomial with maximal degree in x in $\mathcal{B}_T$ is always 1. It corresponds to the minimal tree of the Tamari order, or to the interval with no decreasing relations, *i.e.*, $\mathcal{T}$. The interval with the maximal number of decreasing relations corresponds to $[T, T]$. An example of $\mathcal{B}_T$ and of the computation of smaller trees is presented in Figure 9.

4 Final comments

4.1 Bivariate polynomials

In some very recent work [5], Chapoton computed some bivariate polynomials that seem to be similar to the ones we study. By computing the first polynomials of [5, formula (7)], one notices [6] that for $b = 1$ and $t = 1 - 1/x$ is equal to $\mathcal{B}_T(x)$, where T is a binary tree with no left subtree. The non planar rooted tree corresponding to T is the non planar version of the tree given by the decreasing bijection of Definition 4, *i.e.*, transforming left children of a node into its brothers.

A b parameter can be also be added to our formula. For an interval $[T', T]$, it is either the number of nodes in T' which have a right subtree, or in the interval-poset the number of nodes x with a relation $y \prec x$ and $y > x$. By a generalization of the linear function $\mathcal{P}$, one can associate a monomial in b, x, and y with each interval-poset. The bilinear form now reads:

$$B(f, g) = y\left(xbf\frac{xg - g_{x=1}}{x-1} - bxfg + xfg\right), \tag{16}$$

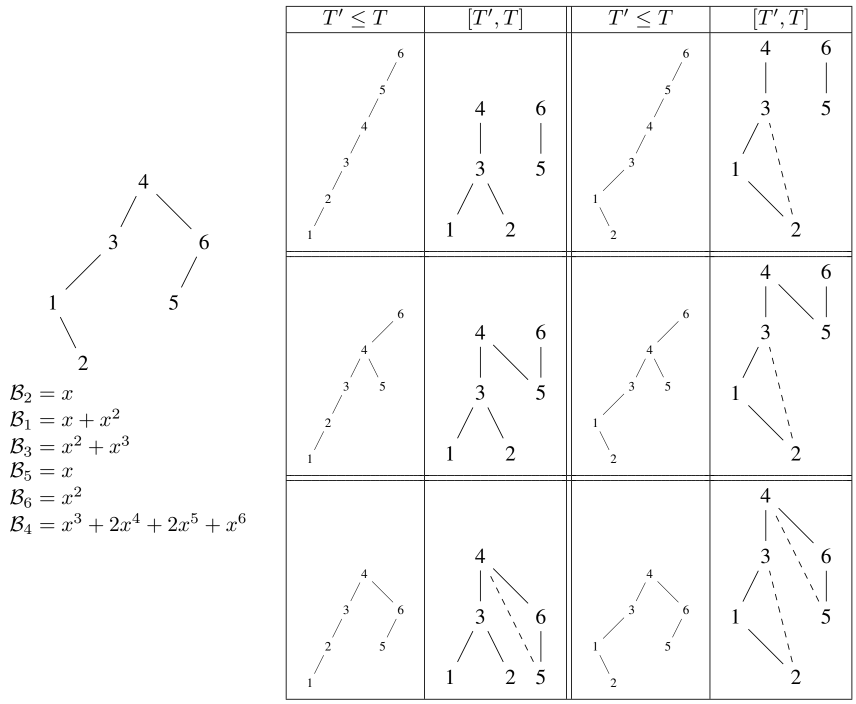


Figure 9: Example of the computation of $\mathcal{B}_T$ and list of all smaller trees with associated intervals

where f and g are polynomials in x, b, and y. Proposition 3.2 still holds, since a node with a decreasing relation is added in all terms of the composition but one. As an example, in Figure 7, one has $B(y^3x^2b, y^3x^2b) = y^7(x^5b^2 + x^4b^3 + x^3b^3)$.

With this definition of the parameter b, the bivariate polynomials $\mathcal{B}_T(x, b)$ where T has no left subtree seem to be exactly the ones computed by Chapoton in [5] when taken on $t = 1 - 1/x$. This correspondence and its meaning in terms of algebra and combinatorics should be explored in some future work.

4.2 m-Tamari

The Tamari lattice on binary trees can also be described in terms of Dyck paths. A Dyck path is a path on the grid formed by north and east steps, starting at $(0, 0)$ and ending at (n, n) and never going under the diagonal. One obtains a Dyck path from a binary tree by reading it in postfix order and writing a north step for each empty tree (also called leaf) and an east step for each node, and by ignoring the first leaf. As an example, the binary tree of Figure 9 gives the following path: $N, N, E, E, N, E, N, N, E, N, E, E$. The rotation consists in switching an east step e (immediately followed by a north step) with the shortest translated Dyck path starting right after e. One can now consider paths that end in (mn, n) and stay above the line $x = my$, called *m-ballot* paths and the same rotation operation will also give a lattice [1].

It is called the m-Tamari lattice, a formula counting the number of intervals was conjectured in [1] and was proven recently in [3]. The authors use a functional equation that is a direct generalization of (9). Let $\Phi_m(x, y)$ be the generating function of intervals of the m-Tamari lattice where y is the size n and x a statistic called number of contacts, then [3, formula (3)] reads

$$\Phi_m(x,y) = x + B_m(\Phi, \Phi, \ldots, \Phi), \tag{17}$$

where B_m is a m-linear form defined by

$$B_m(f_1, \ldots, f_m) := xyf_1\Delta(f_2\Delta(\ldots\Delta(f_m))\ldots), \tag{18}$$

$$\Delta(g) := \frac{g(x,y) - g(1,y)}{x-1}. \tag{19}$$

Expanding (8), we obtain a sum of m-ary trees. A process is described in [3] to associate a m-ballot path with a m-ary tree: the tree is read in prefix order, from the right to the left and each leaf (resp. node) is coded by an east (resp. north) step. Note that this process is not consistent with the classical bijection between Dyck path and binary trees: a different definition of the rotation is given which slightly changes the Tamari lattice and could be generalized to m-Tamari. However, by computer exploration, one notices that the analog of Theorem 1.2 seems to hold: for a given Dyck path, the polynomials obtained by the postfix and prefix tree interpretations of the path are equal. More generally, given a m-ballot path D, let T be the m-ary tree obtained by a prefix reading. Then the polynomial $\mathcal{B}_{m,T}$ of T where B_m is applied to the nodes and x to the leafs counts the number of m-ballot paths lower than D in the m-Tamari order. We shall prove this result in future work.

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PreLie-*decorated hypertrees*

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Abstract. Weighted hypertrees have been used by C. Jensen, J. McCammond, and J. Meier to compute some Euler characteristics in group theory. We link them to decorated hypertrees and 2-coloured rooted trees. After the enumeration of pointed and non-pointed types of decorated hypertrees, we compute the character for the action of the symmetric group on these hypertrees.

Résumé. Des hyperarbres pondérés ont été utilisés en théorie des groupes, par C. Jensen, J. McCammond et J. Meier pour calculer des caractéristiques d'Euler. Nous relions ces hyperarbres pondérés à des hyperarbres décorés, puis à des arbres enracinés 2-colorés. Après énumération des hyperarbres décorés pointés et non pointés, nous déterminons le caractère de l'action du groupe symétrique sur les hyperarbres.

Keywords: Enumerative combinatorics, Species, Hypertrees, Symmetric group action

1 Introduction

Hypergraphs are generalizations of graphs introduced by C. Berge in his book [Ber89] during the 1980's. Like graphs, they are defined by their vertices and edges, but the edges can contain more than two vertices. Hypertrees are hypergraphs in which there is one and only one walk between every pair of vertices. Several studies on hypertrees have been led such as the computation of the number of hypertrees on n vertices by L. Kalikow in [Kal99] and by W. D. Smith and D. Warme in [War98]. In the article [JMM07], C. Jensen, J. McCammond and J. Meier have used weighted hypertrees to compute the Euler characteristic of a subgroup of the automorphism group of a free product. The weight used was $(e-1)^{e-2}$ for an edge of cardinality e.

Thanks to operad theory, we also know a vector space of dimension $(n-1)^{n-2}$. This vector space is the component of arity $n-1$ of the PreLie operad introduced by F. Chapoton and M. Livernet in their paper [CL01]. A basis of this vector space is the set of rooted trees on $n-1$ vertices. These results lead to the following question: is there a combinatorial interpretation in terms of rooted trees of the weighted hypertrees used by C. Jensen, J. McCammond and J. Meier ?

In this paper, we present a combinatorial interpretation of weighted hypertrees used in [JMM07] in terms of hypertrees decorated by $\widehat{\mathrm{PreLie}}$, which are hypertrees in which vertices in every edges form a rooted tree. To enumerate them, we establish a bijection of species between the species of decorated hollow hypertrees and the species of 2-coloured rooted trees. In Section 2, we enumerate pointed and non-pointed versions of decorated hypertrees, which give back the result of C. Jensen, J. McCammond

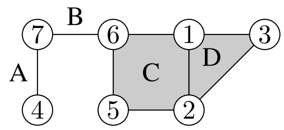

Figure 1: An example of hypergraph on $\{1, 2, 3, 4, 5, 6, 7\}$.

and J. Meier in [JMM07]. We then compute the action of the symmetric group on decorated hypertrees in Section 3, which is remarkable because the associated cycle index series is quite simple.

We use the language of species for which the book of F. Bergeron, G. Labelle and P. Leroux [BLL98] is a good reference. Note that we write *species* for *linear species*. This paper is an extended abstract of [Oge12]: we refer to the long version for a generalization of this construction and enumeration of decorated hypertrees to any species or linear species.

2 Description and relations of the decorated hypertrees

In this section, we introduce a type of *decorated hypertrees*, decorated by $\widehat{\mathrm{PreLie}}$ and give functional equations satisfied by these.

2.1 From hypergraphs to rooted and pointed hypertrees

We first recall the definition of hypergraphs and hypertrees.

Definition 2.1 *A* hypergraph (on a set V) *is an ordered pair* (V, E) *where* V *is a finite set and* E *is a collection of parts of* V *of cardinality at least two. The elements of* V *are called* vertices *and those of* E *are called* edges.

Definition 2.2 *Let* $H = (V, E)$ *be a hypergraph,* v *and* w *two vertices of* H*. A* walk from v to w in H *is an alternating sequence of vertices and edges* $(v = v_1, e_1, v_2, \dots, e_n, v_{n+1} = w)$ *where for all* i *in* $\{1, \dots, n+1\}$, $v_i \in V$, $e_i \in E$ *and for all* i *in* $\{1, \dots, n\}$, $\{v_i, v_{i+1}\} \subseteq e_i$.

Example 2.3 *On the hypergraph of the example of Figure 1, there are several walks from* 4 *to* 2*:* $(4, A, 7, B, 6, C, 2)$ *and* $(4, A, 7, B, 6, C, 1, D, 2)$.

In this article, we are interested in a special type of hypergraphs: hypertrees.

Definition 2.4 *A* hypertree *is a non empty hypergraph* H *such that, given any vertices* v *and* w *in* H*,*

- *there exists a walk from* v *to* w *in* H *with distinct edges* e_i*, i.e.* H *is* connected*,*
- *and this walk is unique, i.e.* H *has* no cycles.

The pair $H = (V, E)$ *is called* hypertree on V. *If* V *is the set* $\{1, \dots, n\}$*, then* H *is called a* hypertree on n vertices.

We now recall rooted and pointed variations of hypertrees.

Definition 2.5

- *A* rooted hypertree *is a hypertree* H *together with a vertex* s *of* H*. The hypertree* H *is said to be* rooted at s *and* s *is called the root of* H*.*

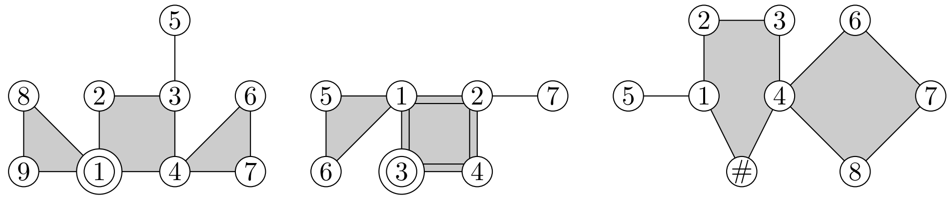

Figure 2: (Left) A hypertree on nine vertices, rooted at 1. (Middle) A hypertree on seven vertices, pointed at edge $\{1, 2, 3, 4\}$ and rooted at 3. (Right) A hollow hypertree on eight vertices. The hollow edge is the edge $\{1, 2, 3, 4\}$.

- *An* edge-pointed hypertree *is a hypertree H together with an edge e of H. The hypertree H is said to be* pointed at e.
- *An* edge-pointed rooted hypertree *is a hypertree H on at least two vertices, together with an edge a of H and a vertex v of a. The hypertree H is said to be* pointed at a and rooted at s.
- *A* hollow hypertree *on n vertices is a hypertree on $n+1$ vertices on the set $\{\#, 1, \ldots, n\}$, such that the vertex labelled by $\#$, called the gap, belongs to one and only one edge.*

2.2 *On the linear species* $\widehat{\mathrm{PreLie}}$ *and the operad* PreLie

An *operad* is a *species* with an operadic composition. It means that it is a functor F from the category of finite sets and bijections to the category of vector spaces, with a family of composition $\circ_i : F([n]) \times F([m]) \mapsto F([n+m-1])$, for all integers n and m, satisfying some axioms that we will not describe here, as they will not be needed in this paper.

The basic definitions on species can be found in the book [BLL98]. We can define the following operations on species:

Definition 2.6 *Let F and G be two species. We define the following operations on species:*

- $F'(I) = F(I \sqcup \{\bullet\})$, *(differentiation)*
- $(F+G)(I) = F(I) + G(I)$, *(addition)*
- $(F \times G)(I) = \sum_{I_1 \sqcup I_2 = I} F(I_1) \times G(I_2)$, *(product)*
- $(F \circ G)(I) = \bigoplus_{\pi \in \mathcal{P}(I)} F(\pi) \times \prod_{J \in \pi} G(J)$, *(substitution) where $\mathcal{P}(I)$ runs on the set of partitions of I.*

With each species F, we can associate a generating series of the dimension of $F([n])$ and a cycle index series, which gives the character for the action of the symmetric group $\mathfrak{S}_n$ on the set of vertices $[n]$ of the hypertree, such that operations on species can be translated in terms of operations on generating series and cycle index series.

The operad PreLie associates to any finite sets I the vector space whose basis is the set of rooted trees with vertex set I. This operad is described in the paper by F. Chapoton and M. Livernet [CL01]. In the article [Cha05], F. Chapoton proves that PreLie is anticyclic. It means that the action of the symmetric group $\mathfrak{S}_n$ on the vector space $\mathrm{PreLie}([n])$ obtained by permutations of vertices can be extended into an action of $\mathfrak{S}_{n+1}$ on $\mathrm{PreLie}([n])$. This implies the existence of a species $\widehat{\mathrm{PreLie}}$ whose differential is the species PreLie.

2.3 Definitions of decorated hypertrees

From a hypertree, we can define what we call a *decorated hypertree*. It consists in decorating the edges of the hypertree by the linear species $\widehat{\mathrm{PreLie}}$.

Definition 2.7 *Given a species $\mathcal{S}$, a* decorated (edge-pointed) hypertree *is obtained from a (edge-pointed) hypertree H by choosing for every edge e of H an element of $\widehat{\mathrm{PreLie}}(V_e)$, where V_e is the set of vertices in the edge e.*

The map which associates to a finite set I the set of decorated (resp. edge-pointed) hypertrees on I is a species, denoted by $\mathcal{H}_{\widehat{\mathrm{PreLie}}}$ (resp. $\mathcal{H}^a_{\widehat{\mathrm{PreLie}}}$).

We now give definitions for rooted or hollow versions of decorated hypertrees.

Definition 2.8 *A* decorated rooted *(resp.* edge-pointed rooted*, resp.* hollow*)* hypertree *is obtained from a rooted (resp. edge-pointed rooted, resp. hollow) hypertree H by choosing for every edge e of H an element of $\widehat{\mathrm{PreLie}}(V_e)$, where V_e is the set of vertices in the edge e.*

In rooted or hollow hypertrees, there is one distinguished vertex in every edge. Therefore, using the definition of the differential of a species, we obtain the following equivalent definition:

Definition 2.9 *Let us consider a rooted (resp. edge-pointed rooted, resp. hollow) hypertree H. Given an edge e of H, there is one vertex of e which is the nearest from the root (resp. the gap) of H in e: let us call it the* petiole p_e *of e. Then, a* decorated rooted *(resp.* rooted edge-pointed*, resp.* hollow*)* hypertree *is obtained from the hypertree H by choosing for every edge e of H an element in the vector space $\mathrm{PreLie}(V_e - \{p_e\})$, where the set V_e is the set of vertices of e. It means that edges of decorated hypertrees contain a vertex (or a gap) and a rooted tree.*

The map which associates to a finite set I the set of decorated rooted (resp. edge-pointed rooted, resp. hollow) hypertrees on I is a species, called the decorated rooted (resp. edge-pointed rooted, resp. hollow) hypertree species and denoted by $\mathcal{H}^p_{\widehat{\mathrm{PreLie}}}$ (resp. $\mathcal{H}^{pa}_{\widehat{\mathrm{PreLie}}}$, resp. $\mathcal{H}^c_{\widehat{\mathrm{PreLie}}}$).

2.4 Relations

2.4.1 Dissymetry principle

The reader may consult the book [BLL98, Chapter 2.3] for a deeper explanation on the dissymmetry principle. In a general way, a *dissymmetry principle* is the use of a natural center to obtain the expression of a non pointed species in terms of pointed species. An example of this principle is the use of the center of a tree to express unrooted trees in terms of rooted trees.

We will consider the following weight on any hypertree (pointed or not, rooted or not, hollow or not):

Definition 2.10 *The weight of a hypertree H with edge set E is given by:*

$$W_t(H) = t^{\#E-1}.$$

The expression of the hypertree species in terms of pointed and rooted hypertrees species is the following:

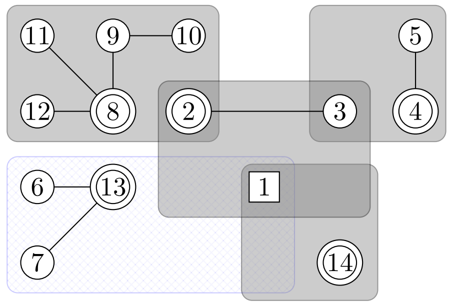

Figure 3: An example of an edge-pointed rooted hypertree with edges decorated by $\widehat{\mathrm{PreLie}}$. The root of the hypertree is in a white square, the roots of trees from the decoration are in two circles whereas the other vertices are in one circle. The grey rectangles are the non-pointed edges of the hypertrees and the blue dotted rectangle is the pointed one.

Proposition 2.11 ([Oge13]) *The species of hypertrees and the one of rooted hypertrees are related by:*

$$\mathcal{H} + \mathcal{H}^{pa} = \mathcal{H}^{p} + \mathcal{H}^{a}. \tag{1}$$

This bijection links hypertrees with k edges with hypertrees with k edges, and therefore it preserves the weight on hypertrees. Pointing a vertex or an edge of a hypertree and then decorating its edges is just the same as decorating its edges and then pointing a vertex or an edge. Therefore, the decoration of edges is compatible with the previous Proposition 2.11 and we obtain:

Proposition 2.12 (Dissymmetry principle for decorated hypertrees) *Given a species $\mathcal{S}$, the following relation holds:*

$$\mathcal{H}_{\widehat{\mathrm{PreLie}}} + \mathcal{H}^{pa}_{\widehat{\mathrm{PreLie}}} = \mathcal{H}^{p}_{\widehat{\mathrm{PreLie}}} + \mathcal{H}^{a}_{\widehat{\mathrm{PreLie}}}. \tag{2}$$

Thanks to this proposition, the study of decorated hypertrees is obtained with the study of pointed and rooted decorated hypertrees.

2.4.2 Functional equations

To study pointed and rooted decorated hypertrees, we decompose them into smaller pointed and rooted hypertrees to obtain relations between the different species. The previous species are related by the following proposition, where X is the species of singleton, which associates to every singleton s itself and the empty set otherwise, and Comm is the species of non-empty sets, which associates to every non-empty set S the set $\{S\}$ and the empty set otherwise:

Theorem 2.13 *The species $\mathcal{H}_{\widehat{\mathrm{PreLie}}}$, $\mathcal{H}^{p}_{\widehat{\mathrm{PreLie}}}$, $\mathcal{H}^{a}_{\widehat{\mathrm{PreLie}}}$, $\mathcal{H}^{pa}_{\widehat{\mathrm{PreLie}}}$ and $\mathcal{H}^{c}_{\widehat{\mathrm{PreLie}}}$ satisfy:*

$$\mathcal{H}^{p}_{\widehat{\mathrm{PreLie}}} = X \times \mathcal{H}'_{\widehat{\mathrm{PreLie}}}, \tag{3}$$

$$t\mathcal{H}^{p}_{\widehat{\mathrm{PreLie}}} = X + X \times \mathrm{Comm} \circ \left(t \times \mathcal{H}^{c}_{\widehat{\mathrm{PreLie}}}\right), \tag{4}$$

$$\mathcal{H}^{c}_{\widehat{\mathrm{PreLie}}} = \mathrm{PreLie} \circ t\mathcal{H}^{p}_{\widehat{\mathrm{PreLie}}}, \tag{5}$$

$$\mathcal{H}^{a}_{\widehat{\mathrm{PreLie}}} = \widehat{\mathrm{PreLie}} \circ t\mathcal{H}^{p}_{\widehat{\mathrm{PreLie}}}, \tag{6}$$

$$\mathcal{H}^{pa}_{\widehat{\mathrm{PreLie}}} = \mathcal{H}^{c}_{\widehat{\mathrm{PreLie}}} \times t\mathcal{H}^{p}_{\widehat{\mathrm{PreLie}}} = \frac{X}{t} \times \left(X(1+\mathrm{Comm}) \circ t\mathcal{H}^{c}_{\widehat{\mathrm{PreLie}}} \right). \tag{7}$$

Proof: If we multiply the series by t, the power of t corresponds to the number of edges in the associated hypertrees.

- The first relation is due to the general relation between a species and a pointed species.
- The second one is obtained from a decomposition of a rooted hypertree. If there is only one vertex, we keep the label of it and it gives X, the number of edges is 0. Otherwise, we separate the label of the root: it gives X. A hypertree with a gap contained in different edges remains. Separating these edges, we obtain a non-empty set of hollow hypertrees with edges decorated by $\widehat{\mathrm{PreLie}}$. There is the same number of edges in the set of hollow hypertrees as in the rooted hypertree. This operation is a bijection of species because a vertex alone is a rooted decorated hypertree and taking a non-empty forest of hollow decorated hypertrees and linking them by their gap on which we put a label gives a rooted decorated hypertree.
- The third relation is obtained by pointing the vertices in the hollow edge and breaking the edge: we obtain a non-empty forest of rooted decorated hypertrees. As we break an edge, there is one edge less in the forest of rooted hypertrees than in the hollow hypertree: it corresponds to a division by t. The set of roots is a rooted tree and induces this structure on the set of hypertrees: we obtain a rooted tree whose vertices are rooted decorated hypertrees. As this operation is reversible and does not depend on the labels of the hollow hypertree, this is a bijection of species.
- The fourth relation is obtained by pointing the vertices in the pointed edge and breaking it: we obtain a non-empty forest of at least two rooted decorated hypertrees. As we break an edge, there is one edge less in the forest of rooted hypertrees than in the edge-pointed hypertree: it corresponds to a division by t. The set of roots is a $\widehat{\mathrm{PreLie}}$-structure and induces this structure on the set of tree: we obtain a $\widehat{\mathrm{PreLie}}$-structure in which all elements are rooted decorated hypertrees. As this operation is reversible and does not depend on the labels of the hollow hypertree, this is a bijection of species.
- The last relation is obtained by separating the pointed edge from the other edges containing the root and putting a gap in the pointed edge instead of the root: the connected component of the pointed edge gives a decorated hollow hypertree and the connected component containing the root gives a rooted decorated hypertree. There is the same number of edges in the edge-pointed rooted hypertree as in the union of the hollow and the rooted hypertree.

□

Corollary 2.14 *Using the equations* (4) *and* (5) *of the previous proposition, we obtain:*

$$t\mathcal{H}^{p}_{\widehat{\mathrm{PreLie}}} = X + X \times \mathrm{Comm} \circ \left(t \times \mathrm{PreLie} \circ t\mathcal{H}^{p}_{\widehat{\mathrm{PreLie}}} \right) \tag{8}$$

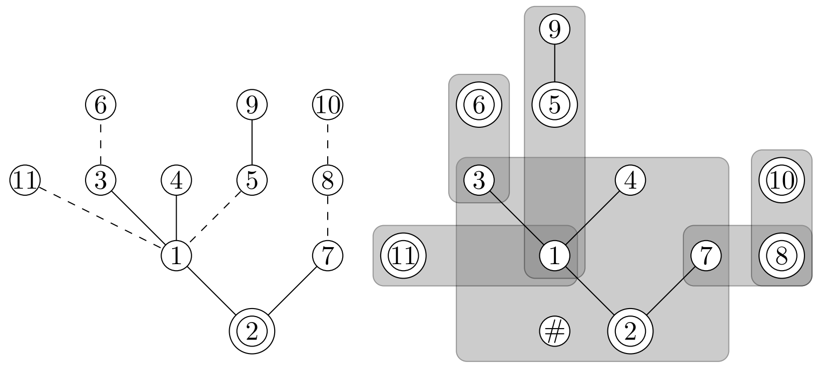

Figure 4: (Left) An example of 2-coloured rooted tree. The edges of E_1 are dashed whereas the edges of E_0 are plain. (Right) The hollow hypertree with edges decorated by $\widehat{\mathrm{PreLie}}$ associated to this 2-coloured rooted tree.

and

$$\mathcal{H}^c_{\widehat{\mathrm{PreLie}}} = \mathrm{PreLie} \circ \left(X + X \times \mathrm{Comm} \circ (t \times \mathcal{H}^c_{\widehat{\mathrm{PreLie}}}) \right). \tag{9}$$

These equations enables us to compute recursively the generating series and the cycle index series of all decorated hypertrees and rooted and pointed decorated hypertrees. Nevertheless, we can obtain closed formulas thanks to a bijection with 2-coloured rooted trees.

3 Computation of generating series and cycle index series of decorated hypertrees

3.1 Link with 2-coloured rooted trees and computation of generating series

We now draw the link between trees whose edges can be either plain (0) or dashed (1) and decorated hollow hypertrees.

Definition 3.1 *A* 2-coloured rooted tree *is a rooted tree* (V, E)*, where* V *is the set of vertices and* $E \subseteq V \times V$ *is the set of edges, together with a map* φ *from* E *to* $\{0, 1\}$*. It is equivalent to the data of a tree* (V, E) *and a decomposition* $E_0 \cup E_1$ *of* E*, with* $E_0 \cap E_1 = \emptyset$*. We consider the weight* $t^{\#E_1}$ *on this set.*

Theorem 3.2

1. *The species of hollow hypertrees decorated by* $\widehat{\mathrm{PreLie}}$ *is isomorphic to the species of 2-coloured rooted trees.*

2. *The species of rooted hypertrees decorated by* $\widehat{\mathrm{PreLie}}$ *is isomorphic to the species of 2-coloured rooted trees such that the edges adjacent to the root are all dashed, up to a coefficient* $\frac{1}{t}$ *to respect the weight.*

3. *The species of rooted edge-pointed hypertrees decorated by* $\widehat{\mathrm{PreLie}}$ *is isomorphic to the species of 2-coloured rooted trees such that all the edges adjacent to the root but one are dashed.*

This theorem enables us to compute explicitly the generating series and cycle index series for decorated hypertrees. Hence this is the key point to establish the link between $\widehat{\mathrm{PreLie}}$-decorated hypertrees and C. Jensen, J. McCammond and J. Meier's weighted hypertrees.

Proof:

1. A hollow hypertree with edges decorated by $\widehat{\mathrm{PreLie}}$ is a hollow hypertree in which, for all edges e, the vertices of e different from the gap or the petiole form a rooted tree.

   Let us consider a hollow hypertree H decorated by $\widehat{\mathrm{PreLie}}$ on vertex set V. We call E_0, the set of edges between elements of V in the rooted trees obtained from the decoration by $\widehat{\mathrm{PreLie}}$. The graph (V, E_0) is then a forest of trees obtained by deleting the edges of the hypertree H and forgetting the roots. For any edge e of H, we write r_e for the root of the rooted tree in e and p_e for the petiole of e. Let E_1 be the set of edges between r_e and p_e for all edges e of H. By definition of the sets, the intersection of E_0 with E_1 is empty. Moreover, to every path in H corresponds a path in $(V, E_0 \cup E_1)$. As H is a hypertree, the graph $(V, E_0 \cup E_1)$ is a tree T. We root that tree in the root r of the tree in the hollow edge of H: T is then a 2-coloured tree rooted in r.

   Conversely, let $T = (V, E_0 \cup E_1)$ be a 2-coloured rooted tree. The graph (V, E_0) is a forest of trees: we can root these trees in their closest vertex from the root. Let us call $T_1, \ldots, T_n$ this forest, where T_1 is the tree rooted in the root of T. For all i between 2 and n, there is one vertex of V linked with a vertex of T_i in T and closer than the root of T_i from the root of T: we call this vertex p_i. Then, we consider the hypergraph whose set of vertices is V, with edges containing the vertices of T_1 or the vertices of a T_i and p_i for all i between 2 and n. Adding the edges of every T_i, we obtain a hypergraph decorated by $\widehat{\mathrm{PreLie}}$. Moreover, paths in T and in the hypergraph are the same: the hypergraph is then a hypertree.

   As there is a bijection between the elements of E_1 and the edges of the hypertree different from the hollow one, the weight is preserved.

2. Let us consider a rooted hypertree H decorated by $\widehat{\mathrm{PreLie}}$ on vertex set V.

   If the cardinality of V is 1, then H is a 2-coloured rooted tree on one vertex. Otherwise, putting a gap in the edges containing the root r, deleting the root and separating the edges linked by the root, we obtain a forest of hollow hypertrees decorated by $\widehat{\mathrm{PreLie}}$. According to the first point, the species of hollow decorated hypertrees is in bijection with the species of 2-coloured rooted trees: the species of decorated rooted hypertrees is then in bijection with the species of a vertex r and a forest of 2-coloured trees. Linking the vertex r to the roots of the 2-coloured trees by dashed edges give the result. To preserve the weight, we must insert the factor $\frac{1}{t}$ in front of the species of 2-coloured trees.

3. A rooted edge-pointed hypertree is a rooted hypertree on at least two vertices with one edge adjacent to the root distinguished. We then distinguished one edge in the corresponding 2-coloured trees by changing it from dashed to plain. The weight is then preserved.

$\square$

Applying the results of Theorem 3.2, we obtain the following proposition, which establish the link between $\widehat{\mathrm{PreLie}}$-decorated hypertrees and J. McCammond and J. Meier's weighted hypertrees:

Corollary 3.3 *The generating series of the species of hollow hypertrees decorated by* $\widehat{\mathrm{PreLie}}$ *is given by:*

$$S^{c}_{\widehat{\mathrm{PreLie}}} = x + \sum_{n \geq 2} (tn+n)^{n-1} \frac{x^n}{n!}.$$

The generating series of the species of rooted hypertrees decorated by the linear species $\widehat{\mathrm{PreLie}}$ *is given by:*

$$S^{p}_{\widehat{\mathrm{PreLie}}} = \frac{x}{t} + \sum_{n \geq 2} n(tn+n-1)^{n-2} \frac{x^n}{n!}.$$

The generating series of the species of hypertrees decorated by $\widehat{\mathrm{PreLie}}$ *is given by:*

$$S_{\widehat{\mathrm{PreLie}}} = x + \sum_{n \geq 2} (tn+n-1)^{n-2} \frac{x^n}{n!}.$$

The generating series of the species of rooted edge-pointed hypertrees decorated by $\widehat{\mathrm{PreLie}}$ *is given by:*

$$S^{pa}_{\widehat{\mathrm{PreLie}}} = x + \sum_{n \geq 2} n(n+tn-1)^{n-3}(n-1)(1+2t) \frac{x^n}{n!}.$$

The generating series of the species of edge-pointed hypertrees decorated by $\widehat{\mathrm{PreLie}}$ *is given by:*

$$S^{a}_{\widehat{\mathrm{PreLie}}} = x + \sum_{n \geq 2} (n+tn-1)^{n-3}(n-1)(1+tn) \frac{x^n}{n!}.$$

We recover the result of C. Jensen, J. McCammond and J. Meier for weighted hypertrees and rooted weighted hypertrees in [JMM07, Theorem 3.9] and in [JMM07, Theorem 3.11]

Proof:

- According to Theorem 3.2, we have to count 2-coloured rooted trees on n vertices. There are n^{n-1} rooted trees on n vertices. To obtain 2-coloured rooted trees from rooted trees, we establish a bijection between the set of edge, of cardinality $n-1$, and the set $\{0, 1\}$, according to the set E_i to which the edge belongs.

- Let us count the number N_n^p of 2-coloured rooted trees such that the edges adjacent to the root are all dashed. There are n ways to choose the root. Cutting the root, we obtain a forest of j trees on $n-1$ vertices. We use the classical formula for the number of forests of j trees on $n-1$ vertices, which can be found in the book of M. Aigner and G. Ziegler [AZ04]. We obtain:

$$N_n^p = n \times \sum_{j=1}^{n-1} \binom{n-2}{j-1} \times ((n-1)(1+t))^{n-1-j} \times t^j.$$

  Re-indexing, we obtain the result.

- We apply the first equation of Proposition 2.13.

- We count in the same way as for the second point 2-coloured rooted trees such that all the edges adjacent to the root but one are dashed:

$$N_n^{pa} = n \times \sum_{j=1}^{n-1} \binom{n-2}{j-1} \times ((n-1)(1+t))^{n-1-j} \times j \times t^{j-1}.$$

- We apply the dissymetry principle of Proposition 2.12.

□

3.2 Computation of cycle index series of hypertrees decorated by $\widehat{\mathrm{PreLie}}$

We denote the cycle index series of usual species in the same way as the species itself. We compute the cycle index series of hypertrees decorated by $\widehat{\mathrm{PreLie}}$. We do not write the argument of the cycle index series $(t, p_1, p_2, \ldots)$ in this subsection.

We are now interested in the action of the symmetric group on the set of decorated hypertrees, which is given by cycle index series:

Proposition 3.4 *The cycle index series of hollow hypertrees decorated by* $\widehat{\mathrm{PreLie}}$ *is given by:*

$$Z^c_{\widehat{\mathrm{PreLie}}} = \frac{1}{1+t} \mathrm{PreLie} \circ (1+t)p_1. \tag{10}$$

Proof: By Theorem 3.2, the cycle index series of hollow hypertrees decorated by $\widehat{\mathrm{PreLie}}$ is given by the cycle index series of 2-coloured rooted trees. □

Let us define the following expressions, for λ a partition of an integer n, written $\lambda \vdash n$:

$$f_k(\lambda) = \sum_{l|k} l\lambda_l$$

and

$$P_k(\lambda) = \left(\left((1+t^k) f_k(\lambda) - 1\right)^{\lambda_k} - k\lambda_k(t^k+1) \times \left((1+t^k) f_k(\lambda) - 1\right)^{\lambda_k - 1} \right).$$

We obtain the following expression for the cycle index series of hypertrees decorated by $\widehat{\mathrm{PreLie}}$:

Proposition 3.5 *The cycle index series of rooted hypertrees decorated by* $\widehat{\mathrm{PreLie}}$ *is given by:*

$$Z^p_{\widehat{\mathrm{PreLie}}} = \sum_{n \geq 1} \sum_{\lambda \vdash n, \lambda_1 \neq 0} \lambda_1 (\lambda_1 t + \lambda_1 - 1)^{\lambda_1 - 2} \prod_{k \geq 2} P_k(\lambda) \times \frac{p_\lambda}{z_\lambda}. \tag{11}$$

The cycle index series of edge-pointed rooted hypertrees decorated by $\widehat{\mathrm{PreLie}}$ *is given by:*

$$Z^{pa}_{\widehat{\mathrm{PreLie}}} = \sum_{n \geq 1} \sum_{\lambda \vdash n, \lambda_1 \neq 0} \lambda_1 (\lambda_1 - 1)(2t+1)(\lambda_1 + \lambda_1 t - 1)^{\lambda_1 - 3} \prod_{k \geq 2} P_k(\lambda) \times \frac{p_\lambda}{z_\lambda}. \tag{12}$$

The cycle index series of edge-pointed hypertrees decorated by $\widehat{\mathrm{PreLie}}$ *is given by:*

$$Z^a_{\widehat{\mathrm{PreLie}}} = \sum_{n\geq 1} \sum_{\lambda\vdash n, \lambda_1\neq 0} (\lambda_1 - 1)(1+\lambda_1 t)(\lambda_1 + \lambda_1 t - 1)^{\lambda_1 - 3} \prod_{k\geq 2} P_k(\lambda) \times \frac{p_\lambda}{z_\lambda}. \tag{13}$$

The cycle index series of hypertrees decorated by $\widehat{\mathrm{PreLie}}$ *is given by:*

$$Z_{\widehat{\mathrm{PreLie}}} = \sum_{n\geq 1} \sum_{\lambda\vdash n, \lambda_1\neq 0} (\lambda_1 t + \lambda_1 - 1)^{\lambda_1 - 2} \prod_{k\geq 2} P_k(\lambda) \times \frac{p_\lambda}{z_\lambda}. \tag{14}$$

This proposition is proven in the paper [Oge12].

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Adinkras for Mathematicians

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Abstract. *Adinkras* are graphical tools created for the study of supersymmetry representations. Besides having inherent interest for physicists, the study of adinkras has already shown connections with coding theory and Clifford algebras. Furthermore, adinkras offer many natural and accessible mathematical problems of combinatorial nature. We present the foundations for a mathematical audience, make new connections to other fields (homological algebra, poset theory, and polytopes), and solve some of these problems. Original results include the enumeration of all hypercube adinkras through dimension 5, the enumeration of odd dashings of adinkras for any dimension, and a connection between rankings and the chromatic polynomial for certain graphs.

Résumé.

Les *adinkras* sont des dessins qui sont utilisés pour étudier les représentations des theories supersymmetriques. Outre leur intérêt en physique, les adinkras sont aussi utiles en connection avec la theory des codes et les algebres de Clifford. De plus, le adinkras offrent beaucoup de problèmes de nature combinatoires qui sont à la fois naturels et accessible. Nous présentons une introduction pour une audience de mathématiciens, présentons de nouvelles connections avec d'autre domaines (algèbres homologiques, ensembles partiellement ordonn és, polytopes), et resolvont certains problèmes. Parmi les résultats nouveaux, nous énumérons les adinkras de l'hypercube de dimension inférieur ou égal a 5, nous énumérons les *odd dashings* en toute dimension, et établissons une relation entre les *rankings* et le polynôme chromatique pour certains graphes.

Keywords: supersymmetry, representation theory, Clifford algebras, codes, topology

1 Introduction

In a series of papers, starting with Faux and Gates Jr (2005) and most recently Doran et al. (2011), different subsets of the "DFGHILM collaboration" (Doran, Faux, Gates, Hübsch, Iga, Landweber, Miller) have built and extended the machinery of *adinkras*. Following the ubiquitous spirit of visual diagrams in physics, adinkras are combinatorial objects that encode information about the representation theory of supersymmetry algebras. Adinkras have many intricate links with other fields such as graph theory, Clifford theory, and coding theory. Each of these connections provide many problems that can be compactly communicated to a (non-specialist) mathematician. In this paper, which is an extended abstract for a longer article (Zhang (2011)) and more recent results (mostly from Klein and Zhang), we extract and study some of these mathematical problems.

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In short, *adinkras* are *chromotopologies* (a class of edge-colored bipartite graphs) with two additional structures, a *dashing* condition on the edges and a *ranking* condition on the vertices. We redevelop the foundations in a self-contained manner in Sections 2, and an optional discussion of the relevant physics in Section 3.

Using this setup, we look at the two aforementioned conditions separately in Sections 5 and 6, making original connection with different areas of mathematics. In Section 5 we use homological algebra to study dashings; our main result is the enumeration of odd dashings for any chromotopology. In Section 6, we use the theory of posets to put a lattice structure on the set of all rankings of any bipartite graph (including chromotopologies) and count hypercube rankings up through dimension 5. We discuss the generalization of rankings in Section 7, including a useful definition of *discrete Lipschitz functions*, a formula for rankings involving the chromatic polynomial, and a connection with the theory of polytopes.

We wish that these purely combinatorial discussions will equip the readers with a visual model that allows them to appreciate (or to solve!) the original representation-theoretic problems in the physics literature. We revisit these questions in Section 8, where we give our concluding remarks.

2 Definitions

In this section, we deviate from the original literature, yielding slightly cleaner and more general mathematics, but the core ideas are the same.

For a graph G, we use $E(G)$ to denote the edges of G and $V(G)$ to denote the vertices of G. We assume basic notions of posets and lattices, as in Stanley (1997). In this paper, we think of each Hasse diagram for a poset as a directed graph, with $x \to y$ an edge if y covers x. Thus it makes sense to call the maximal elements (i.e. those x with no $y > x$) *sinks* and the minimal elements *sources*. A *ranked* poset (this is sometimes also called a *graded* poset, though there subtly different uses of that name so we avoid it) is a poset A equipped with a rank function $h\colon A \to \mathbb{Z}$ such that for all x covering y we have $h(x) = h(y) + 1$. There is a unique rank function h_0 among these such that 0 is the lowest value in the range of h_0, so it makes sense to define the *rank* of an element v as $h_0(v)$. The largest element in the range of h_0 is then the length of the longest chain in A; we call it the *height* of A.

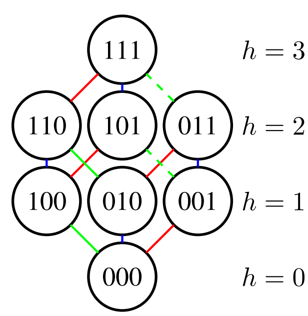

Fig. 1: An adinkra. We can take $\{000, 011, 101, 110\}$ to be either bosons or fermions.

An n-dimensional *adinkra topology*, or *topology* for short, is a finite, simple, connected, and bipartite graph A such that A is n-*regular* (every vertex has exactly n incident edges). We call the two sets in the bipartition of $V(A)$ *bosons* and *fermions*, though the actual choice is mostly arbitrary and we do not consider it part of the data. A *chromotopology* of dimension n is a topology A such that the edges are colored by n colors, which are elements of the set $[n] = \{1, 2, \ldots, n\}$ unless denoted otherwise, such that every vertex is incident to exactly one edge of each color, and for any distinct i and j, the edges in $E(A)$ with colors i and j form a disjoint union of 4-cycles.

An *adinkra* is a chromotopology A with two additional properties:

1. **ranked**: we give A the additional structure of a ranked poset, with rank function h_A (though we will usually just write h). In this paper, we will usually represent ranks via vertical placement, with higher h corresponding to being higher on the page.

2. **dashed**: we add an *odd dashing* A, which is a choice of making each edge of A either *dashed* or *solid*, such that every 2-colored 4-cycle contains an odd number of dashes.

An example of an adinkra is in Figure 1. Note that any chromotopology A can be ranked as follows: take one choice of bipartition of $V(A)$ into bosons and fermions. Assign the rank function h to take values 0 on all bosons and 1 on all fermions, forming a height-2 poset. Call such a ranked chromotopology a *valise* (see Figure 2). Thus, a chromotopology can be made into an adinkra if and only if it can be dashed. Call such chromotopologies *adinkraizable*.

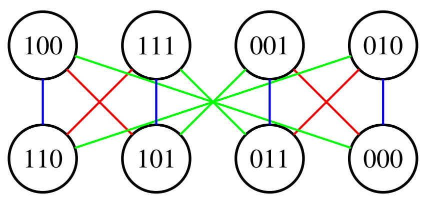

Fig. 2: A valise with topology I^3.

3 Motivation

We have neither the space nor the qualification to give a comprehensive review, so we encourage interested readers to explore the original physics literature. The reader is already equipped to understand most of the rest of the paper without needing to read this section.

The physical motivation for adinkras is to understand off-shell representations of the N-extended Poincaré superalgebra in the 1-dimensional worldline. There is no need to understand what all of these terms mean (the author certainly does not) to appreciate the rest of the discussion; we now give a simplified translation.

We consider the algebra $\mathfrak{po}^{1|N}$ generated by $N+1$ operators $Q_1, Q_2, \ldots, Q_N$ (the *supersymmetry generators*) and $H = i\partial_t$ (the *Hamiltonian*), such that $\{Q_I, Q_J\} = 2\delta_{IJ}H$ and $[Q_I, H] = 0$. Since H is basically a time derivative, it lowers the *engineering dimension* (physics units) of any function f by a single unit of time. Consider functions (equipped with engineering dimensions) $\{\phi_1, \ldots, \phi_m\}$ (the *bosonic fields* or *bosons*) and $\{\psi_1, \ldots, \psi_m\}$ (the *fermionic fields* or *fermions*), collectively called the *component fields*. We want to understand representations of $\mathfrak{po}^{1|N}$ acting on the infinite basis $\{H^k\phi_I, H^k\psi_J \mid k \in$

$\mathbb{N}; I, J \leq m\}$. This is a long-open problem and seems intractible, so we restrict our attention to one where the Q_I act as permutations (up to a scalar) on the basis: for any boson ϕ and any Q_I, $Q_I\phi = \pm(-iH)^s\psi$, where $s \in \{0, 1\}$, the sign, and the fermion ψ depends on ϕ and I. We enforce a similar requirement for fermions. We call the representations corresponding to these types of actions *adinkraic representations*. For each of these representations, we associate an *adinkra* via the following correspondence with the definitions in Section 2.

adinkras	representations
vertex bipartition	bosonic/fermionic bipartition
rank function (refines bipartition)	partition of component fields by engineering dimension
edge with color I	Q_I action without the sign or powers of $(-iH)$
dashing of an edge	sign in Q_I action
change of rank by an edge	powers of $(-iH)$ in Q_I action

To summarize: an adinkra encodes a representation of $\mathfrak{po}^{1|N}$. An adinkraic representation is a representation of $\mathfrak{po}^{1|N}$ that can be encoded into an adinkra.

When the poset structure of our adinkra A is a boolean lattice, we get what Doran et al. (2008a) calls the *exterior supermultiplet*, which coincides with the classical notion of the *superfield* introduced in Salam and Strathdee (1974). When A is a valise, we get Doran et al. (2008a)'s *Clifford supermultiplet*.

4 Topologies and Chromotopologies

In this section, we study chromotopologies and adinkraizable chromotopologies. Our approach is more general than the relevant sections of Doran et al. (2008a) and Doran et al. (2008b) and we obtain the main classification results of those papers as a special case, though all the main ideas, including the pleasant connections to codes and Clifford algebras, are already done in the original work.

We now give a quick review of codes (there are many references, including Huffman and Pless (2003)). An n-*bitstring* is a vector in $\mathbb{Z}_2^n$, which we usually write as $b_1 b_2 \cdots b_n$, $b_i \in \mathbb{Z}_2$. We distinguish two n-bitstrings $\overrightarrow{1_n} = 11\ldots1$ and $\overrightarrow{0_n} = 00\ldots0$, and when n is clear from context we suppress the subscript n. The number of 1's in a bitstring v is called the *weight* of the string, which we denote by $\mathrm{wt}(v)$. An (n, k)-*linear binary code*, or *code* for short, is a k-dimensional subspace of $\mathbb{Z}_2^n$. A code is *even* if all its bitstrings have weight divisible by 2 and *doubly even* if all its bitstrings have weight divisible by 4.

Define the n-dimensional *hypercube* to be the graph with 2^n vertices labeled by the n-bitstrings. If two vertices differ at the i-th bit i, color the edge between them by i. This graph is a chromotopology, so we call it the n-*cubical chromotopology* I_c^n. Our earlier example in Figure 1 had the chromotopology I_c^3. The hypercube is the main running example in this paper. We denote the underlying (colorless) graph of I_c^n as I^n.

For a code L, we now construct an edge-colored graph I_c^n/L, which which we call the *quotient* of I_c^n by L. Let $V(I_c^n/L)$ be the set of the equivalence classes of $\mathbb{Z}_2^n/L$ and define $p_L(v)$ as the image of v under the quotient $\mathbb{Z}_2^n/L$. Let there be an edge $p_L(v, w)$ in I_c^n/L with color i between $p_L(v)$ and $p_L(w)$ in I^n/L if there is at least one edge with color i of the form (v', w') in $\mathbb{Z}_2^n$, with $v' \in p_L^{-1}(v)$ and $w' \in p_L^{-1}(w)$. It can be checked that I_c^n/L is a n-regular graph.

Proposition 4.1 *The following hold for $A = I_c^n/L$, where L is a code.*

1. *A is a simple graph if and only if L has has no bitstrings of weight 1 or 2.*

2. *A can be ranked if and only if A is bipartite, which is true if and only if L is an even code.*

3. *A can be dashed if and only if L is a code with the following two conditions: first, all bitstrings must have weight 0 or $1 \pmod 4$; second, for any two bitstrings w_1 and w_2, we have $(w_1 \cdot w_2) + \mathrm{wt}(w_1)\,\mathrm{wt}(w_2) = 0 \pmod 2$, where the first term is the dot product in $\mathbb{Z}_2^n$.*

Proof idea: The first two parts are routine. The third can be proven by a translation of the dashing condition into relations in the multiplicative group of the signed monomials of the Clifford algebra $\mathbf{Cl}(n)$. □

These results give the following classifications, the second part being equivalent to a combination of (Doran et al., 2008a, Theorem 4.1) and (Doran et al., 2008b, Section 3.1):

Theorem 4.2 *Chromotopologies are in bijection with quotients I_c^n/L where L is an even code with no bitstring of weight 2. Adinkraizable chromotopologies are in bijection with such quotients where L is a doubly even code.*

Thanks to Theorem 4.2, we can assume that any chromotopology A we discuss comes from some (n,k)-code $L(A) = L$. If L is an (n,k)-code, we say that the corresponding A is an (n,k)-chromotopology. An $(n,0)$-chromotopology is exactly the n-cubical chromotopology, corresponding to the trivial code $\{\overrightarrow{0}\}$. The first non-cubical chromotopology, shown in Figure 3, is the result of quotienting the 4-cubical topology by the code $L = \{0000, 1111\}$, the smallest non-trivial doubly-even code. It has the topology of the bipartite graph $K_{4,4}$.

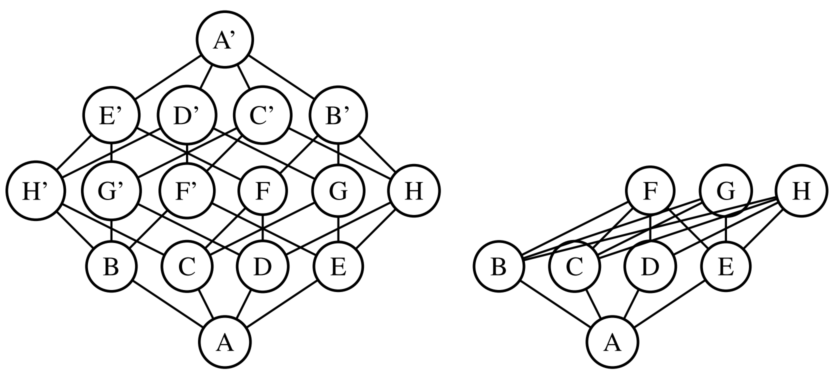

Fig. 3: The topologies I^4 and $I^4/\{0000, 1111\}$. Labels with the same letter are sent to the same vertex.

Before moving on, we introduce a helpful notion for later sections. Say that a color i *decomposes* a chromotopology A into A_0 and A_1, or $A = A_0 \amalg_i A_1$, if removing all edges with color i creates two disjoint chromotopologies A_0 and A_1, which are labeled and colored in a natural fashion. Whenever $A = A_0 \amalg_i A_1$, A_0 and A_1 are $(n-1,k)$ chromotopologies with isomorphic topologies. Our definition was inspired by observations in Doran et al. (2008b), where certain adinkras were called 1-*decomposable*. See Figure 4 for an example.

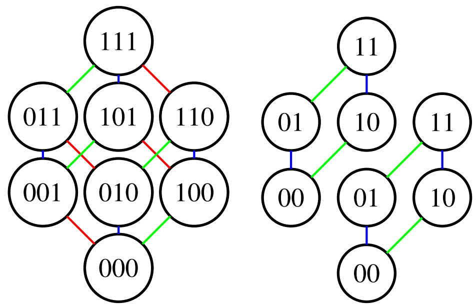

Fig. 4: The color 3 decomposes a ranked chromotopology A.

5 Dashing

Given an adinkraizable chromotopology A, define $o(A)$ to be the set of odd dashings of A. In this section, we introduce several seemingly unrelated ideas and combine them to count $|o(A)|$ for any adinkraizable chromotopology.

First, let an *even dashing* be a way to dash $E(A)$ such that every 2-colored 4-cycle contains an even number of dashed edges, and let $e(A)$ be the set of even dashings. The odd dashings form a torsor for the even dashings:

Lemma 5.1 *For any adinkraizable chromotopology A, we have $|o(A)| = |e(A)|$.*

Proof idea: Let $l = |E(A)|$. We may consider a dashing (with no parity constraints) of A as a vector in $\mathbb{Z}_2^l$, where each coordinate corresponds to an edge and is assigned 1 for a dashed edge and 0 for a solid edge. The obvious way to add dashings make all dashings form a vector space V of dimension l. Observe that $e(A)$ is a subspace of V, and that $o(A)$ is a coset in V of $e(A)$ and must then have the same cardinality as $e(A)$ given that at least one odd dashing exists. Since A is adinkraizable by definition, we are done. □

Dashings (of both sorts) behave extremely well under decompositions. In fact, if $A = A_0 \amalg_i A_1$, then each even (resp. odd) dashing of the induced graph of A_0 and each of the arbitrary choices of dashing the i-colored edges extends to exactly one even (resp. odd) dashing of A. Using this and an inductive argument, we obtain:

Proposition 5.2 *The number of even (or odd) dashings of I_c^n is*

$$|e(I_c^n)| = |o(I_c^n)| = 2^{2^n-1}.$$

We now borrow a concept from Douglas et al. (2010), which defines the *vertex switch* at a vertex v of a dashed chromotopology A as the operation that sends all dashed edges incident to v to solid edges, and vice-versa (this is in turn inspired by the theory of *two-graphs*). A vertex switch preserves odd dashings (in fact, parity in all 4-cycles), so the odd dashings of A can be split into orbits under vertex switches, which we will call the *labeled switching classes* (or *LSCs*) of A.

Proposition 5.3 *In an adinkraizable (n,k)-chromotopology, each LSC has $2^{2^{n-k}-1}$ dashings.*

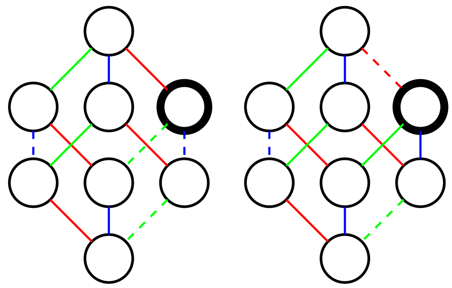

Fig. 5: Before and after a vertex switch at the outlined vertex.

We need a final observation. In Section 4, I_c^n plays the role of a universal cover, in the sense that its everything else comes from their quotients. We make this intuition rigorous with homological algebra. Over $\mathbb{Z}_2$, construct the following 2-dimensional complex $X(A)$ from a chromotopology A. Let C_0 be formal sums of elements of $V(A)$ and C_1 be formal sums of elements of $E(A)$. For each 2-colored 4-cycle C of A, create a 2-cell with C as its boundary as a generator in C_2, the boundary maps $\{d_i\colon C_i \to C_{i-1}\}$ are the natural choices (we do not worry about orientations since we are using $\mathbb{Z}_2$), giving homology groups $H_i = H_i(X(A))$.

Proposition 5.4 *Let A be an (n,k)-adinkraizable chromotopology with code L. Then $X(A) = X(I_c^n)/L$ as a quotient complex, where L acts freely on $X(I_c^n)$. We have that $X(I_c^n)$ is a simply-connected covering space of $X(A)$, with L the group of deck transformations.*

Finally, we combine all our ideas to generalize Theorem 5.2.

Theorem 5.5 *The number of even (or odd) dashings of an adinkraizable (n,k)-chromotopology A is*

$$|e(A)| = |o(A)| = 2^{2^{n-k}+k-1}.$$

Proof idea: The even dashings are exactly the orthogonal complement of the boundaries in C_1 (by the usual inner product), which works out to have $\mathbb{Z}_2$-dimension equalling $\dim(H_1) + \dim(C_0) - \dim(H_0)$. However, note that $\dim(C_0) - \dim(H_0) = 2^{n-k} - 1$, which is exactly the dimension of the vector space of the vertex switchings for a particular LSC from Proposition 5.3. Dividing, we get that the dimension of switching classes is precisely $\dim(H_1)$. By Proposition 5.4, $\pi_1(X(A)) = L$, the quotient group, which in this case is the vector space $\mathbb{Z}_2^k$. Since π_1 is abelian, $H_1 = \mathbb{Z}_2^k$ also. These dimensions basically complete the proof. $\square$

It is remarkable that this enumeration is dependent only on the dimension k of the code and not the code itself, a fact that was not obvious to us through elementary methods.

6 Ranking

Fix a chromotopology A. Call the set of all ranked chromotopologies with the same chromotopology as A the *rank family* $R(A)$ and the elements of $R(A)$ *rankings* of A. Figure 6 shows $R(I_c^2)$. In this section, we give some original structural results using the language of posets and lattices. Then, we count the rankings for I_c^n with $n \leq 5$ with the help of decomposition.

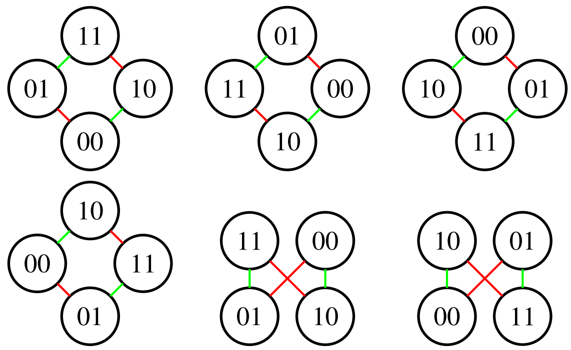

Fig. 6: The rank family of I^2.

The main structural theorem for rankings is the following theorem. Let $D(v, w)$ be the graph distance function between v and w:

Theorem 6.1 ((Doran et al., 2007, Theorem 4.1)) *Fix a chromotopology A. Let $S \subset V(A)$ and $h_S\colon S \to \mathbb{Z}$ be such that h_S takes only odd values on bosons and only even values on fermions, or vice-versa, and for every distinct s_1 and s_2 in S, we have $D(s_1, s_2) > |h_S(s_1) - h_S(s_2)|$. Then, there exists a unique ranking of A, corresponding to the rank function h, such that h agrees with h_S on S and A's set of sinks is exactly S.*

In other words, any ranking of A is determined by a set of sinks and their relative ranks (an analogous statement is true for sources). We can think of such a choice as the following: pick some nodes as sinks and "pin" them at acceptable relative ranks, and let the other nodes naturally "hang" down. Thus, Theorem 6.1 is also called the "Hanging Gardens" Theorem. Figure 7 shows an example.

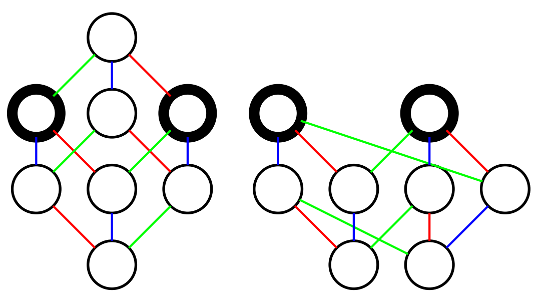

Fig. 7: Left: I^3. Right: Hanging Gardens on I^3 applied to the two outlined vertices.

In particular, note that we can pick the set of sinks to contain only a single element, which defines a unique ranking. Thus, for any vertex v of a chromotopology A, by Theorem 6.1 we can get a ranking A^v which "hangs" from its only sink v. We now discuss our original results.

We introduce two operators on $R(A)$. Given a ranking B in $R(A)$ (with rank function h) and a sink s, we define D_s, the *vertex lowering on* s, to change $h(s)$ to $h(s) - 2$ while keeping everything else in B

unchanged (visually, we have "flipped" s down two ranks and its edges with it). We define U_s, the *vertex raising on* s, to be the analogous operation for s a source. We call both of these operators *vertex flipping* operators. In Doran et al. (2007), it is shown that any two rankings with the same chromotopology A can be obtained from each other via a sequence of vertex-raising or vertex-lowering operations, so $R(A)$ has the structure of a connected graph. We can say more about $R(A)$.

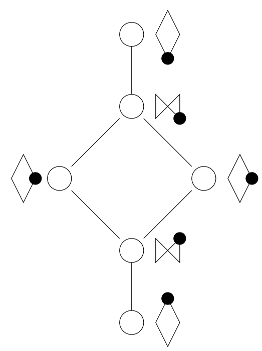

Fig. 8: The rank family poset for $P_v(I^2)$, where next to each node is a corresponding ranking. The rankings are presented as miniature posets, with the black dots corresponding to v.

Theorem 6.2 *For a chromotopology A and any vertex v of A, there exists a finite distributive symmetric ranked lattice $P_v(A)$ with vertex set $R(A)$. Each covering relation in $P_v(A)$ corresponds to vertex-flipping on some vertex $w \neq v$.*

Proof idea: Construct $P_v(A)$, as a ranked poset, in the following way: on the bottom rank 0 put A^v as the unique element. Once we finish constructing rank i, from any choice of element B on rank i and a source $w \in V(B)\backslash\{v\}$, apply U_w to obtain a ranking C and place it on rank $i+1$ such that C covers B. The lattice structure comes from constructing an auxiliary poset $E_v(A)$, showing $P_v(A)$ is the poset of order ideals of $E_v(A)$, and appealing to the fundamental theorem of finite distributive lattices. $\square$

The authors of Doran et al. (2007) noted that the rank family is reminiscent of a Verma module. Extend the U_s to act on formal sums in $\mathbb{R}[R(A)]$ and define $U(A)$ to be the algebra generated by all $U_{s\in A}$. The image of A^v under the action of $U(A)/U_v$ is $\mathbb{R}[R(A)]$, so we can consider A^v as a lowest-weight vector. If we allowed U_v we would get repetitions as there would be a nontrivial product of U_s that would act as the identity on A^v.

Finally, we want to count the cardinality of $R(I^n)$. With the help of decomposition and some optimizations, we computed $|R(I^n)|$ with a computer program for $n \leq 5$. We include the results in Table 1 along with the counts of dashings and adinkras. Finding the answer for $n = 6$ seems intractible with an algorithm that is at least linear in the number of solutions. For chromotopologies other than $R(I^n)$, we can still perform similar computations with the help of decomposition. However, doing a case-by-case analysis for different chromotopologies seems uninteresting without unifying principles.

n	dashings	rankings	adinkras
1	2	2	4
2	8	6	48
3	128	38	4864
4	32768	990	32440320
5	2147483648	395094	848457904422912

Tab. 1: Enumeration of dashings, rankings, and adinkras with chromotopology I_c^n.

7 Generalizing Rankings and Discrete Lipschitz Functions

Rankings are easy to generalize directly to bipartite graphs since they do not rely on any other aspects of a chromotopology. Theorems 6.1 and 6.2 both hold for bipartite graphs as stated. In Klein and Zhang we study this generalization, obtaining exact enumerations for $R(G)$ for special families of graphs and constructing a (complicated) generating function whose constant term equals $R(G)$. However, our most interesting result is the following:

Theorem 7.1 (Klein and Zhang) *If the cycle space of G is generated by 4-cycles, then $|R(G)| = (1/3)\chi_A(3)$, where χ_A is the chromatic polynomial of G.*

Many families of graphs satisfy the conditions needed for Theorem 7.1, such as trees, hypercubes (but not their quotients!), and grid graphs. The main strategic advantage, however, is that Theorem 7.1 allows us to borrow techniques from the theory of Tutte/chromatic polynomials. Interestingly, the most promising tools come from statistical mechanics, a branch of physics quite distant from supersymmetry. In particular, the results found by Salas and Sokal in a series of papers (most relevantly Salas and Sokal (2009)) give data that supports our own calculations of $R(G)$ using the transfer-matrix method.

Finally, given a graph G with n vertices, consider the pairs of hyperplanes in $\mathbb{R}^n$ created by $|x_i - x_j| = \pm 1$ for $(i,j) \in E(G)$. Now, if we fix any $x_i = 0$, we get an $(n-1)$-dimensional polytope P_G. The integral points $P_G \cap \mathbb{Z}^n$ are exactly the functions $f\colon V(G) \to \mathbb{Z}$ such that $|f(i) - f(j)| \leq 1$ for $(i,j) \in E(G)$. It seems natural to call these integral functions *discrete Lipschitz functions*; almost identical definitions have occurred in other places, including Jiang and Chen (2011), where they were used to study the No Free Lunch Theorem. There are two seemingly unrelated connections between P_G and $R(G)$:

Theorem 7.2 (Klein and Zhang) *The following hold for a bipartite graph G:*

1. *The vertices of P_G are in bijection with the elements of $R(G)$.*

2. *We have $|P_{G\times I} \cap \mathbb{Z}^{2n-1}| = 2|R(G)|$.*

Besides contributing to this and potential further results about counting rankings, P_G and discrete Lipschitz functions seem very natural objects to study on their own: P_G is the dual polytope to a *root polytope* defined via the graph G, which has been studied in Mészáros (2011). Christian Stump and Vincent Pilaud (Pilaud and Stump) observed that P_G is a special instance of Lam and Postnikov's *alcoved polytopes* found in Lam and Postnikov (2007). Finally, exploring the Ehrhart theory of P_G may be worthwhile.

8 Concluding Remarks

Adinkras are beautiful objects that have given us some very natural mathematical problems where much remain to be done. For sake of brevity, several promising new directions and results have been omitted from this extended abstract. Besides generalizing rankings, we have also started generalizing dashings to arbitrary graphs, finding some similarities with the study of Pfaffian graphs. Recently, the authors of the original literature have used homological techniques to obtain an independent set of results from our own (see Doran et al. (2011)). In a different application of topology, we have obtained a promising notion of Stiefel-Whitney classes for a code and studied the conditions under which they vanish, to emulate obstruction-theoretic interpretations of Stiefel-Whitney classes.

Finally, while we have focused on the mathematics, many potential applications of adinkras to physics are not completely explored. We end with a sketch of the longer discussion in Zhang (2011).

- One may wish to ask which adinkraic representations are irreducible. In the valise case, this is well-understood (see Doran et al. (2008b)) with an elegant answer: irreducible valise adinkraic representations correspond to maximal doubly-even codes. However, there is currently no efficient method for other rankings.

- Even asking what it means for two adinkras to be isomorphic is a subtle question; while it seems to be completely intuitive for the authors of the literature (see Gates et al. (2009) and Douglas et al. (2010)), Zhang (2011) may be the first formal discussion. A natural continuation of this question is how to tell if two adinkras capture isomorphic adinkraic representations, which is not yet completely understood (but known for irreducible representations).

- It would be good to have a theory of adinkras as building blocks of more complex representations and representations of higher dimensions. For example, by direct sums, tensors, and other operations familiar to the Lie algebras setting, it is possible to construct many more representations (see Doran et al. (2008b)), a technique that has been extended to higher dimensions in Hubsch (2011).

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A Hopf-power Markov chain on compositions

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Abstract. In a recent paper, Diaconis, Ram and I constructed Markov chains using the coproduct-then-product map of a combinatorial Hopf algebra. We presented an algorithm for diagonalising a large class of these "Hopf-power chains", including the Gilbert-Shannon-Reeds model of riffle-shuffling of a deck of cards and a rock-breaking model.

A very restrictive condition from that paper is removed in my thesis, and this extended abstract focuses on one application of the improved theory. Here, I use a new technique of lumping Hopf-power chains to show that the Hopf-power chain on the algebra of quasisymmetric functions is the induced chain on descent sets under riffle-shuffling. Moreover, I relate its right and left eigenfunctions to Garsia-Reutenauer idempotents and ribbon characters respectively, from which I recover an analogous result of Diaconis and Fulman (2012) concerning the number of descents under riffle-shuffling.

Résumé. Dans un récent article avec Diaconis et Ram, nous avons construit des chaînes de Markov en utilisant une composition du coproduit et produit d'une algébre de Hopf combinatoire. Nous avons présenté un algorithme pour diagonaliser une large classe de ces "chaînes de Hopf puissance", en particulier nous avons diagonalisé le modèle de Gilbert-Shannon-Reeds de mélange de cartes en "riffle shuffle" (couper en deux, puis intercaler) et un modèle de cassage de pierres.

Dans mon travail de thèse, nous supprimons une condition très restrictive de cet article, et ce papier se concentre sur une application de cette amélioration. Nous utilisons ici une nouvelle technique de projection de chaînes de Hopf puissance pour montrer que la chaîne de Hopf puissance sur l'algèbre des fonctions quasi-symétriques est la chaîne de Markov induite sur les ensembles des descentes dans le "riffle shuffling". De plus, nous faisons le lien entre les fonctions propres à droite et à gauche et respectivement les idempotents de Garsia-Reutenauer et les caractères en rubans, ce qui nous permet de retrouver un résultat analogue à Diaconis et Fulman (2012) concernant le nombre de descentes dans le "riffle shuffling".

Keywords: Quasisymmetric functions, riffle shuffling, descent set, combinatorial Hopf algebras

1 Introduction

The Hopf algebra is a ubiquitous structure in mathematics - having originated in algebraic topology to describe the cohomology of H-spaces, it generalises the group ring in representation theory, and is equivalent to a group scheme in algebraic geometry. Joni and Rota (1979) first introduced Hopf algebras to combinatorics to encode the breaking (coproduct) and assembling (product) of combinatorial objects; since then, many examples of the combinatorial Hopf algebra have been developed, for example in Schmitt (1993), and the theory extended in Aguiar and Mahajan (2010).

It is natural to wonder what happens to a combinatorial object after many iterates of breaking and reassembling. Diaconis et al. (2012) examined this question by building a Markov chain out of the coproduct-then-product operator on the corresponding combinatorial Hopf algebra. Their main examples of these *Hopf-power chains* were inverse shuffling (from the free associative algebra, with states indexed by its usual word basis) and rock-breaking (from the algebra of symmetric functions, with states indexed by the elementary symmetric functions $\{e_\lambda\}$). The advantage of reformulating these familiar Markov chains as Hopf-power chains is to leverage from theorems concerning arbitrary Hopf algebras. In particular, the Eulerian idempotent theory of Patras (1993) and Reutenauer (1986), which holds for any commutative or cocommutative graded Hopf algebra (over a field of characteristic zero), allows the explicit construction of a left and right eigenbasis of the chain in two rather restrictive circumstances: when the states of the chain form a polynomial basis, or when the underlying Hopf algebra is cocommutative and the states form a free basis. These expressions can aid in estimating convergence rates and probabilities of being in certain subsets of the state space - see (Diaconis et al., 2012, Cor. 4.10, Sec. 2.1) respectively for an example and an extensive list of applications.

The thesis of Pang (in preparation) greatly relaxes the condition that the basis of states be polynomial or free, instead requiring simply that no state of degree greater than 1 is primitive. Thus this extension can, for instance, construct and analyse Markov chains on parking functions and binary trees via the Hopf algebras of Hivert et al. (2008), or use the bases with parameters of Lascoux et al. (2011) to deform familiar chains. The example in the present paper has as its states the non-polynomial basis $\{F_I\}$ of fundamental quasisymmetric functions. The new idea of interpreting Hopf algebra morphisms as a lumping of the corresponding Hopf-power chains shows that (Theorem 3.2) this chain on compositions is the induced chain on descent sets under riffle-shuffling. This descent set chain was briefly studied by (Diaconis and Fulman, 2009, Th. 3.2), who gave an upper bound of $\log n$ for the mixing time (n is the number of cards in the deck). Their emphasis was on the induced chain on the number of descents under riffle-shuffling, for which they proved a mixing time of $\frac{1}{2}\log n$.

Extending the ideas of (Diaconis et al., 2012, Sec. 3.5) yields an explicit algorithm for a full right eigenbasis $\{f_I\}$ and left eigenbasis $\{g_I\}$ of this descent set chain, both indexed by compositions of n, where n is the number of cards in the deck. This paper will concentrate on the f_I, g_I when the parts of I are ordered non-increasing, for which the unwieldly general formula simplifies neatly. This subset of "partition eigenfunctions" completely determine the behaviour of the chain if the starting deck has all cards in increasing order, and they have some surprising interpretations:

- (Theorem 4.2) $f_I(J)$ is the coefficient of any permutation with descent set J in the Garsia-Reutenauer idempotent (of the descent algebra) corresponding to I;
- (Theorem 4.8) $g_I(J)$ is the value of the ribbon character (of the symmetric group) corresponding to J on any permutation of cycle type I.

Summing these over partitions of fixed length then recovers the analogous discoveries of Diaconis and Fulman (2012) regarding the Markov chain of the number of descents under riffle-shuffling, see Corollaries 4.7 and 4.12.

We remark that the idea of using quasisymmetric functions to analyse descents under riffle-shuffling is not new: Novelli and Thibon (2012) exploited the dual algebra $\mathbf{Sym}$, of noncommutative symmetric functions, to streamline the results of Diaconis and Fulman (2012); (Hersh and Hsiao, 2009, Sec. 7) specialises their theory of walks on quasisymmetric functions to diagonalise the induced chain of riffle-shuffling on the *idescent set* $\{i|i+1 \text{ occurs earlier than } i\}$. (They phrase their chain as the descent set under left-multiplication of a certain quasisymmetric function whose right-multiplication describes riffle-shuffling; associating the permutation σ with the word $\sigma^{-1}(1)\dots\sigma^{-1}(n)$ instead of $\sigma(1)\dots\sigma(n)$ then exchanges left and right and interprets their chain as the idescent set under shuffling. This results in a genuinely different chain from the one examined in this paper, see the remark in Section 4.1. The remark in (Zhao, 2009, Sec. 2.2) gives further details on the two conventions to notate decks of cards as permutations; to avoid this confusion, this paper will use words instead of permutations.)

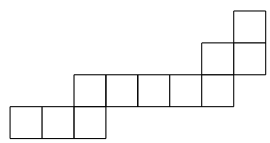
This paper is organised as follows. Section 2 collects together the notation necessary to describe the eigenbasis. Section 3 shows that the Hopf-power chain on quasisymmetric functions is the induced chain on descent sets under riffle-shuffling of distinct cards. Section 4 is devoted to the eigenfunctions.

2 Notation regarding compositions

A *composition* I is a list of positive integers $\left(i_1, i_2, \dots, i_{l(I)}\right)$. Each i_j is a *part* of I. The sum $i_1 + \dots + i_{l(I)}$ is denoted $|I|$, and $l(I)$ is the number of parts in I. So $|(3,5,2,1)| = 11$, $l((3,5,2,1)) = 4$. Forgetting the ordering of the parts of I gives a multiset $\lambda(I) := \left\{i_1, \dots, i_{l(I)}\right\}$. Clearly $\lambda(I) = \lambda(I')$ if and only if I' has the same parts as I, but in a different order. I is a *partition* if its parts are non-increasing, that is, $i_1 \geq i_2 \geq \dots \geq i_{l(I)}$.

The *diagram* of I is a string of $|I|$ dots with a division after the first i_1 dots, another division after the next i_2 dots, *etc.*. The *ribbon shape* of I is a skew-shape (in the sense of tableaux) with i_1 boxes in the bottom row, i_2 boxes in the second-to-bottom row, *etc.*, so that the rightmost square of each row is directly below the leftmost square of the row above. Hence this skew shape contains no 2-by-2 square. The diagram and ribbon shape of $(3,5,2,1)$ are shown below.

$$\cdot\cdot\cdot|\cdot\cdot\cdot\cdot\cdot|\cdot\cdot|\cdot$$



Given compositions $I,\ J$ with $|I| = |J|$, (Gelfand et al., 1995, Sec. 4.8) defines the *decomposition of J relative to I* as the $l(I)$-tuple of compositions $\left(J_1^I, \dots, J_{l(I)}^I\right)$ such that $|J_r^I| = i_r$ and each $l(J_r^I)$ is minimal such that the concatenation $J_1^I \dots J_{l(I)}^I$ refines J. Pictorially, the diagrams of $J_1^I, \dots, J_{l(I)}^I$ are obtained by "splitting" the diagram of J at the points specified by the divisions in the diagram of I. For example, if $I = (4,4,3)$ and $J = (3,5,2,1)$, then $J_1^I = (3,1)$, $J_2^I = (4)$, $J_3^I = (2,1)$.

A composition I is *Lyndon* if the word $i_1 \dots i_{l(I)}$ is lexicographically strictly smaller than its cyclic rearrangements. For example, $(1,1,2,1,2)$ is Lyndon, but $(2,3,2,3)$ and $(3,5,2,1)$ are not. As described by (Lothaire, 1997, Th. 5.1.5, Prop. 5.1.6), the *Lyndon factorisation* $I_{(1)} \dots I_{(k)}$ of I is obtained by taking $I_{(k)}$ to be the lexicographically smallest tail of I, then $I_{(k-1)}$ is the lexicographically smallest tail of I with $I_{(k)}$ removed, and so on. Hence, if $I = (3,5,2,1)$, then $k(I) = 3$ since the Lyndon factors are $I_{(1)} = (3,5)$, $I_{(2)} = (2)$, $I_{(3)} = (1)$. The factors $I_{(r)}$ are important in the general formulae for the full eigenbasis, but this paper will only involve $k(I)$, the number of Lyndon factors in I. If I is a partition, then each part of I is a singleton Lyndon factor, which is why the corresponding eigenfunctions have much simpler expressions. In this case, $k(I) = l(I)$.

3 The Markov chain on descent sets under shuffling

The purpose of this section is to prove that the coproduct-then-product operator on the algebra of quasisymmetric functions encodes the changes in descent set of a deck of distinct cards under riffle-shuffling.

Sections 3.1 and 3.2 review background on the shuffle algebra and the algebra of quasisymmetric functions respectively. Section 3.3 defines the all-important Hopf morphism from the shuffle algebra to $QSym$ to relate the two Hopf-power chains, and explains how to lump other Hopf-power chains by the same reasoning. This allows the ideas of (Diaconis et al., 2012, Sec. 3.5) to give explicit expressions for the eigenbasis, as shown in Section 4.

3.1 The Shuffle algebra and riffle-shuffling

The shuffle algebra $\mathcal{S}$, as defined by Ree (1958), is spanned by words of the form $w = w_1 \dots w_n$, where each $w_i \in \mathbb{N}$. The w_i need not be distinct. $\mathcal{S}$ is multigraded: $\deg(w) = (|\{i : w_i = 1\}|, |\{i : w_i = 2\}|, \dots)$. In other words, the kth component of $\deg(w)$ is the number of times the letter k appears in w. The shuffle algebra also admits a coarser grading: $|w|$ is the number of letters in w. For example, $\deg(12231) = (2, 2, 1)$, $|12231| = 5$.

The product of two words w and w', denoted $m(w \otimes w')$, is the sum of all possible interleavings of their letters, with multiplicity. For example,

$$m(13 \otimes 52) = 1352 + 1532 + 1523 + 5132 + 5123 + 5213;$$

$$m(12 \otimes 231) = 2(12231) + 12321 + 12312 + 21231 + 21321 + 21312 + 2(23112) + 23121.$$

Iterating this gives the a-fold product:

$$m^{[a]} : \mathcal{S}^{\otimes a} \to \mathcal{S}, \quad m^{[1]} := \iota, \quad m^{[a]} := m(m^{[a-1]} \otimes \iota), \tag{1}$$

where ι denotes the identity map. Note that $m = m^{[2]}$. (Reutenauer, 1993, Sec. 1.5) showed that deconcatenation is a compatible coproduct. For example,

$$\Delta(316) = \emptyset \otimes 316 + 3 \otimes 16 + 31 \otimes 6 + 316 \otimes \emptyset.$$

(Here, $\emptyset$ denotes the empty word, which is the unit of $\mathcal{S}$.) The a-fold coproduct is given inductively by:

$$\Delta^{[a]} : \mathcal{S} \to \mathcal{S}^{\otimes a}, \quad \Delta^{[1]} := \iota, \quad \Delta^{[a]} := (\Delta^{[a-1]} \otimes \iota)\Delta, \tag{2}$$

so again $\Delta = \Delta^{[2]}$. As an example,

$$\begin{aligned}\Delta^{[3]}(316) \quad = \quad & \emptyset \otimes \emptyset \otimes 316 + \emptyset \otimes 3 \otimes 16 + 3 \otimes \emptyset \otimes 16 + \emptyset \otimes 31 \otimes 6 + 3 \otimes 1 \otimes 6. \\ & +31 \otimes \emptyset \otimes 6 + \emptyset \otimes 316 \otimes \emptyset + 3 \otimes 16 \otimes \emptyset + 31 \otimes 6 \otimes \emptyset + 316 \otimes \emptyset \otimes \emptyset\end{aligned}$$

Letting the word $w_1 \dots w_n$ represent a deck of cards in the order $w_1, w_2, \dots, w_n$ from top to bottom, the Hopf-square map $\Psi^2 := m\Delta$ represents a Gilbert-Shannon-Reeds shuffle: cut the deck binomially with parameter $\frac{1}{2}$, then drop the cards one by one from either pile, where the chance of dropping from a pile is proportional to the number of cards currently in the pile. Precisely,

$$\Psi^2(w) = m\Delta(w) = \sum_{w'} 2^{|w|} K_2(w, w') w',$$

where $K_2(w, w')$ is the chance of a GSR shuffle applied to w resulting in w'. In other words, the matrix for the operator $2^{-n}\Psi^2$, with respect to the basis of words with n letters, transposes to give the transition matrix of the GSR shuffle. Analogously, the ath *Hopf-power*, $\Psi^a := m^{[a]}\Delta^{[a]}$, describes the a-shuffle of Bayer and Diaconis (1992), where the cards are cut into a piles multinomially (with parameter $\frac{1}{a}$) and then dropped proportional to pile size as before.

The *descent set* of a word $w = w_1 \dots w_n$ is defined to be $D(w) = \{j \in \{1, 2, \dots, |w| - 1\} | w_j > w_{j+1}\}$. It is more convenient in this paper to consider the associated composition of $D(w)$. Hence a word w has *descent composition* $DC(w) = I$ if i_j is the number of letters between the $j-1$th and jth descent, *i.e.* if $w_{i_1+\dots+i_j} > w_{i_1+\dots+i_j+1}$ for all j, and $w_r \leq w_{r+1}$ for all $r \neq i_1 + \dots + i_j$. For example, $D(3521) = \{2, 3\}$ and $DC(3521) = (2, 1, 1)$. Note that no information is lost in passing from $D(w)$ to $DC(w)$, as the divisions in the diagram of $DC(w)$ indicate the positions of descents in w.

3.2 The algebra of quasisymmetric functions

The algebra $QSym$ of quasisymmetric functions was first introduced by Gessel (1984) to study P-partitions. It is a subalgebra of the algebra of polynomials in infinitely many commuting variables $\{x_1, x_2, \dots\}$. Gessel defined two bases of $QSym$, both indexed by compositions. The *monomial quasisymmetric function* M_I associated to a composition $I = \big(i_1, \dots, i_{l(I)}\big)$ is

$$M_I = \sum_{j_1 < \dots < j_{l(I)}} x_{j_1}^{i_1} \dots x_{j_{l(I)}}^{i_{l(I)}},$$

and the *fundamental quasisymmetric function* F_I associated to I is

$$F_I = \sum_{J \geq I} M_J$$

where the sum runs over all partitions J refining I. $QSym$ inherits a grading and a commutative product from the algebra of polynomials, so $\deg(M_I) = \deg(F_I) = |I|$. Malvenuto and Reutenauer (1995) extended this to a Hopf algebra structure by defining the following coproduct:

$$\Delta(M_I) = \sum_{j=0}^{l(I)} M_{(i_1, i_2, \dots, i_j)} \otimes M_{(i_{j+1}, \dots, i_{l(I)})}.$$

Equations (1) and (2) define an a-fold product and a-fold coproduct on $QSym$.

3.3 Lumping riffle-shuffling by descent set

The algebraic relationship between the two Hopf algebras above is:

Theorem 3.1 *There is a morphism of Hopf algebras $\theta : \mathcal{S} \to QSym$ such that, if w is a word with distinct letters, then $\theta(w) = F_{DC(w)}$.*

Proof (sketch): The linear function $\zeta : \mathcal{S} \to \mathbb{R}$ defined by

$$\zeta(w) = \begin{cases} 1 & \text{if } w_1 < w_2 < \dots < w_n \\ 0 & \text{otherwise,} \end{cases}$$

is an algebra homomorphism; now apply the universal construction of (Aguiar et al., 2006, Th. 4.1). □

As θ is a Hopf morphism, it commutes with the ath Hopf-power map Ψ^a. The probability interpretation of this is:

Theorem 3.2 *The descent set process of a deck of n distinct cards under a-shuffling is a Markov chain, whose transition matrix $\bar{K}_{a,n}$ is the transpose of the matrix for the rescaled Hopf-power map $a^{-n}\Psi^a$ on $QSym$, with respect to the basis $\{F_I | |I| = n\}$.*

Proof: Let $K_{a,n}$ be the transition matrix for a-shuffling on a deck of n distinct cards. Then, for any w with n distinct letters,

$$a^{-n}\Psi^a(w) = \sum_{w'} K_{a,n}(w, w')w'.$$

Apply θ of Theorem 3.1 to both sides, remembering that θ and Ψ^a commute:

$$a^{-n}\Psi^a(\theta(w)) = \sum_{w'} K_{a,n}(w, w')\theta(w').$$

As all words involved have distinct letters, Theorem 3.1 yields

$$a^{-n}\Psi^a(F_{DC(w)}) = \sum_{w'} K_{a,n}(w, w')F_{DC(w')} = \sum_{J} \left(\sum_{w':DC(w')=J} K_{a,n}(w, w') \right) F_J. \tag{3}$$

The left hand side of this equation depends only on $DC(w)$, so the same is true of the coefficients $\sum_{w':DC(w')=J} K_{a,n}(w, w')$ on the right. These are the probabilities that, after an a-shuffle, a deck in order w now has descent composition J. Hence the descent set process under shuffling is indeed a Markov chain, and Equation 3 gives the transition probabilities as the (I, J)-entry of the transpose of the matrix for $a^{-n}\Psi^a$. □

Remark. This straightforward argument applies verbatim to lump Hopf-power Markov chains on other graded Hopf algebras. Let $\theta : \mathcal{H} \to \mathcal{H}'$ be a morphism of graded Hopf algebras mapping a basis $\mathcal{B}_\nu$ of the degree ν subspace $\mathcal{H}_\nu$ of $\mathcal{H}$ onto a basis $\mathcal{B}'_{\nu'}$ of some degree ν' subspace $\mathcal{H}'_{\nu'}$ of $\mathcal{H}'$. ($\theta : \mathcal{H}_\nu \to \mathcal{H}'_{\nu'}$ must be surjective, but need not be injective - several elements of $\mathcal{B}_\nu$ may have the same image in $\mathcal{H}'_{\nu'}$, as long as the distinct images are linearly independent.) Then the Hopf-power walk on $\mathcal{B}_\nu$ lumps via θ to the Hopf-power walk on $\mathcal{B}'_{\nu'}$.

4 Explicit formulae for eigenfunctions

Section 4.1 gives the eigenvalues of the descent set chain and their multiplicities. Sections 4.2 and 4.3 detail formulae for the right and left "partition eigenfunctions" respectively, explain how to recover the results of Diaconis and Fulman (2012), and sketch the ideas behind the proofs of the full eigenbases. This strategy also diagonalises Hopf-power chains on a large class of commutative cofree combinatorial Hopf algebras. These eigenfunctions are useful for a variety of probabilistic tasks, as (Diaconis et al., 2012, Sec. 2.1) explains.

4.1 Multiplicity of eigenvalues

As is the case for previously analysed Hopf-power Markov chains, all eigenvalues of this descent set chain are powers of the Hopf-power exponent a. The full eigenbasis algorithm shows that f_I has eigenvalue $a^{k(I)-|I|}$, where $k(I)$ is the number of Lyndon factors in I. A standard generating function argument then rephrases this as:

Theorem 4.1 *The eigenvalues of the ath Hopf-power Markov chain on compositions of n are $1, a^{-1}, a^{-2}, \dots, a^{-n+1}$. The multiplicity of the eigenvalue a^{-n+k} is the coefficient of $x^n y^k$ in $\prod_i \left(1 - yx^i\right)^{-d_i}$, where d_i is the number of Lyndon compositions I with $|I| = i$.*

Remark. The idescent set chain of (Hersh and Hsiao, 2009, Sec. 7) has the same eigenvalues, but there each composition I corresponds to an eigenfunction of eigenvalue $a^{-n+l(I)}$, so the multiplicity of a^{-n+k} in the idescent set chain is the number of compositions of n with length k. This difference in eigenvalue multiplicity suggests the two chains have different convergence rates.

4.2 Right eigenfunctions

All functions f_I in the right eigenbasis are essentially built from the function

$$f(J) := \frac{1}{|J|} \frac{(-1)^{l(J)-1}}{\binom{|J|-1}{l(J)-1}}.$$

Note that $f(J)$ depends only on $|J|$ and $l(J) - 1$, which are respectively the number of dots and the number of divisions in the diagram of J.

Theorem 4.2 below gives an explicit formula for the right eigenfunctions f_I corresponding to a partition I (when I is not a partition, the sum becomes weighted), and relates them to the orthogonal idempotents E_I of the descent algebra. These idempotents refine the more familiar Eulerian idempotent, and were first defined by (Garsia and Reutenauer, 1989, Sec. 3) to classify indecomposable representations of the descent algebra.

Theorem 4.2 *Let I be a partition with $|I| = n$. With f as defined above, the function*

$$f_I(J) := \frac{1}{l(I)!} \sum_{I':\lambda(I')=\lambda(I)} \prod_{r=1}^{l(I')} f\left(J_r^{I'}\right) = \frac{1}{l(I)! i_1 \dots i_{l(I)}} \sum_{I':\lambda(I')=\lambda(I)} \prod_{r=1}^{l(I')} \frac{(-1)^{l\left(J_r^{I'}\right)-1}}{\binom{|J_r^{I'}|-1}{l\left(J_r^{I'}\right)-1}}$$

is a right eigenfunction of eigenvalue $a^{-n+l(I)}$ of the ath Hopf-power Markov chain on compositions. The numbers $f_I(J)$ are the coefficients in the Garsia-Reutenauer idempotent E_I:

$$E_I = \sum_{\sigma \in \mathfrak{S}_n} f_I(DC(\sigma))\sigma.$$

Observe that f itself is a right eigenfunction, that corresponding to the partition with single part. Its eigenvalue is a^{-n+1}, the smallest possible.

Example 4.3 Here's how to apply the algorithm above to calculate $f_{(4,4,3)}((3,5,2,1))$. The I' in the sum are the rearrangements of $(4,4,3)$, which are $(4,4,3)$, $(4,3,4)$ and $(3,4,4)$. The decompositions of $(3,5,2,1)$ with respect to these three I' are:

$$(\cdot\cdot\cdot|\cdot,\ \cdot\cdot\cdot\cdot,\ \cdot\cdot|\cdot) \quad (\cdot\cdot\cdot|\cdot,\ \cdot\cdot\cdot,\ \cdot|\cdot\cdot|\cdot) \quad (\cdot\cdot\cdot,\ \cdot\cdot\cdot\cdot,\ \cdot|\cdot\cdot|\cdot),$$

so

$$f_{(4,4,3)}((3,5,2,1)) = \frac{1}{3!}\left(\frac{-1}{4\binom{3}{1}}\frac{1}{4}\frac{-1}{3\binom{2}{1}} + \frac{-1}{4\binom{3}{1}}\frac{1}{3}\frac{1}{4\binom{3}{2}} + \frac{1}{3}\frac{1}{4}\frac{1}{4\binom{3}{2}}\right) = \frac{7}{5184}.$$

As $f((1)) = 1$, one may omit all r with $|J_r^{I'}| = 1$ from the product in the expression for f_I. This simplifies the calculation of $f_{(i_1,1,1,\ldots,1)}(J)$ to "pulling a window" of length i_1 across the diagram of J and summing the values of f on each position of the window.

Example 4.4 Take $i_1 = 2$, then in the window of length 2, there is either a division or no division. Since $f((2)) = \frac{1}{2}$ and $f((1,1)) = -\frac{1}{2}$, $f_{(2,1,1,\ldots,1)}(J)$ is the sum of $\frac{1}{2}$ for every non-division and $-\frac{1}{2}$ for every division, divided by $(n-1)!$, *i.e.*,

$$f_{(2,1,1,\ldots,1)}(J) = \frac{1}{(n-1)!}\left(\frac{|J|-1}{2} - (l(J)-1)\right).$$

By Theorem 4.1, this is the unique right eigenfunction of eigenvalue $\frac{1}{a}$, the largest eigenvalue after 1. Its lift to the shuffle algebra is the "normalised number of descents" eigenfunction, as discussed in (Diaconis et al., 2012, Ex. 5.8).

Example 4.5 When $i_1 = 3$, calculations of $f(J)$ for J with $|J| = 3$ show that

$$\begin{aligned} f_{(3,1,1,\ldots,1)}(J) \quad = \quad & \frac{1}{3(n-2)!}\Big(\#\text{ (2 consecutive non-divisions) } + \#\text{ (2 consecutive divisions)} \\ & -\frac{1}{2}\#\text{(division followed by non-division) } - \frac{1}{2}\#\text{(non-division followed by division)}\Big). \end{aligned}$$

The associated eigenvalue is $\frac{1}{a^2}$. Since divisions correspond to descents in the shuffle algebra, and non-divisions to ascents, $f_{(3,1,1,\ldots,1)}$ lifts to the shuffle algebra as

$$\begin{aligned} \tilde{f}_{(3,1,1,\ldots,1)}(w) \quad &= \quad \frac{1}{3(n-2)!}\left(\#\text{ straights } - \frac{1}{2}\#\text{ troughs } - \frac{1}{2}\#\text{ peaks}\right) \\ &= \quad \frac{1}{2(n-2)!}\left(\#\text{ straights } - \frac{n-2}{3}\right) \end{aligned}$$

since $\#$ straights $+ \#$ troughs $+ \#$ peaks $= n-2$. (Here, a *straight* is two consecutive ascents or two consecutive descents.) This "normalised number of straights" eigenfunction is $\frac{1}{2(n-2)!}f_-$ in the notation of (Diaconis et al., 2012, Prop.5.10). The full eigenbasis formula shows that the normalised number of peaks and of troughs are also eigenfunctions. Consequently:

Proposition 4.6 *The expected number of straights (resp. peaks, troughs) after l shuffles, starting from a deck with x straights (resp. peaks, troughs), is*

$$(1-4^{-l})\frac{n-2}{3} + 4^{-l}x.$$

The story is similar for larger i_1: $f_{(i_1,1,1,\ldots,1)}$ is the weighted enumeration of "patterns" of length i_1, where pattern J has weight $\frac{f(J)}{(n-i_1+1)!}$. Each of these lifts to an eigenfunction on the shuffle algebra, that is a weighted enumeration of up-down-patterns of length i_1.

Corollary 4.7 *(Diaconis and Fulman, 2012, Cor. 3.2) Let $f_i(j)$ be the coefficient of any permutation with j descents in the ith Eulerian idempoten. Then $\{f_i\}$ is a right eigenbasis for the Markov chain on the number of descents under riffle-shuffling.*

Proof: The key is that the Garsia-Reutenauer idempotents for partitions of a fixed length sum to the corresponding Eulerian idempotent:

$$e_i = \sum_{l(I)=i} E_I = \sum_{\sigma \in \mathfrak{S}_n} \sum_{l(I)=i} f_I(DC(\sigma))\sigma.$$

Since the coefficient of a permutation in the Eulerian idempotent depends only on its number of descents, the function $\sum_{l(I)=i} f_I(J)$ depends only on $l(J)$, and it is a right eigenfunction of eigenvalue a^{-n+i}. By the eigenfunction theory of lumped chains, as in (Levin et al., 2009, Lem. 12.8.i), this descends to a right eigenfunction

$$j \to \sum_{l(I)=i} f_I(J) \text{ for any } J \text{ of length } j$$

on the induced chain on the number of descents, which is the required f_i. □

The full right eigenbasis $\{f_I\}$ comes from applying (Diaconis et al., 2012, Th. 3.16) to $\mathbf{Sym}$, the graded dual of $QSym$. The eigenfunctions are most naturally expressed in terms of $\{\Phi^I\}$, the noncommutative power sum of the second kind; one then uses the explicit change-of-basis matrices of (Gelfand et al., 1995, Sec. 4) to rewrite this in terms of the dual basis to $\{F_I\}$, which is $\{R_I\}$, the noncommutative ribbon symmetric functions.

4.3 Left eigenfunctions

The left eigenfunctions g_I for I a partition are most concisely defined using some representation theory of the symmetric group $\mathfrak{S}_n$, although their calculation is completely combinatorial. Each composition J may be associated to a representation of $\mathfrak{S}_n$ via its ribbon shape J; denote by χ^J the character of this representation.

Theorem 4.8 *Let I be a partition with $|I| = n$. Define $g_I(J) := \chi^J(I)$, the character of $\mathfrak{S}_n$ associated to the ribbon shape J evaluated at a permutation with cycle type I. Then g_I is a left eigenfunction of the ath Hopf-power Markov chain on compositions with eigenvalue $a^{-n+l(I)}$.*

(Ceccherini-Silberstein et al., 2010, Rem. 3.5.18) explains how to calculate $\chi^J(I)$ graphically: find all possible ways of filling the ribbon shape of J with i_1 copies of 1, i_2 copies of 2, *etc.*, such that all copies of each integer are in adjacent cells, and all rows and columns are weakly increasing; then sum over these fillings, weighted by $(-1)^{\Sigma(l_r-1)}$, where l_r is the number of rows containing r. (For general compositions I, the left eigenfunction $g_I(J)$ is a weighted sum over coloured fillings of the ribbon shape of J subject to complex restrictions, and does not have a neat expression in terms of characters.)

Example 4.9 Calculating $g_{(4,4,3)}((3,5,2,1))$ requires filling the ribbon shape of $(3,5,2,1)$ with four copies of 1, four copies of 2 and three copies of 3, subject to the constraints in explained above. Observe that the top square cannot be 1, because then the top four squares must all contain 1, and the fifth square from the top must be equal to or smaller than these. Similarly, the top square cannot be 3, because then the top three squares are all 3s, but the fourth must be equal or larger. Hence 2 must fill the top square, and the only legal way to complete this is

$$\begin{array}{cccccccc} & & & & & & & \boxed{2} \\ & & & & & & \boxed{2} & \boxed{2} \\ & & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{2} & \\ \boxed{3} & \boxed{3} & \boxed{3} & & & & & \end{array}$$

so

$$g_{(4,4,3)}((3,5,2,1)) = (-1)^{(0+2+0)} = 1.$$

Example 4.10 There is only one way to fill any given ribbon shape with n copies of 1, so

$$g_{(n)}(J) = (-1)^{l(J)-1}.$$

Next, take $I = (1,1,\dots,1)$. Then $g_{(1,1,\dots,1)}$ is χ^J evaluated on the identity permutation. A theorem of Foulkes (1980), described in Kerber and Thürlings (1984), translates this to:

Corollary 4.11 *The stationary distribution for the ath Hopf-power Markov chain on compositions is*

$$g_{(1,1,\dots,1)}(J) = \frac{1}{n!} \left|\{w\,|\,|w| = n,\ \deg(w) = (1,1,\dots,1),\ DC(w) = J\}\right|.$$

In other words, the stationary probability of J is the proportion of words with letters are $1,2,\dots,n$ (each appearing exactly once) whose descent composition is J. □

This also follows from the stationary distribution of riffle-shuffling being the uniform distribution.

Corollary 4.12 *(Diaconis and Fulman, 2012, Th. 2.1) Let $g_i(j)$ be the value of the jth Foulkes character of the symmetric group on any permutation with i cycles. Then $\{g_i\}$ is a left eigenbasis for the Markov chain on the number of descents under riffle-shuffling.*

Proof: Each g_I determines a left eigenfunction for the number of descents chain, by summing the values of g_I over all compositions lumping to the same state, see (Barr and Thomas, 1977, Th. 2) for details. So

$$j \to \sum_{l(J)=j} g_I(J) = \sum_{l(J)=j} \chi^J(I)$$

is an eigenfunction of eigenvalue $a^{-n+l(I)}$, and this sum of ribbon characters is by definition the jth Foulkes character. □

The full left eigenbasis is given by essentially applying (Diaconis et al., 2012, Th. 3.15) to products of $\{P_I | I \text{ is Lyndon}\}$, where $\{P_I\}$ is the power sum analogue as defined by (Malvenuto and Reutenauer, 1995, Eq. 2.12), as their Corollary 2.2 states that $\{P_I | I \text{ is Lyndon}\}$ freely generates $QSym$.

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Singularity analysis via the iterated kernel method

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Abstract We provide exact and asymptotic counting formulas for five singular lattice path models in the quarter plane. Furthermore, we prove that these models have a non D-finite generating function.

Résumé Nous présentons des résultats énumératives pour les cinq modèles de marches dans le quart de plan dites “singulière”. Nous prouvons que ces modéles sont non-holonome.

Keywords: Lattice path enumeration, D-finite, generating function, singularities

1 Introduction

The study of lattice path models restricted to the quarter plane has led to some useful innovations in enumeration, including applications of boundary value methods [8, 9, 12], powerful and widely applicable variants of the kernel method [5, 7, 14] and some original computer algebra approaches [3]. In addition to purely enumerative results, there are also results on asymptotic enumeration and the nature of the generating function: either rational, algebraic, transcendental D-finite(i), or other.

A key observation of Bousquet-Mélou and Mishna was that lattice path models with small steps restricted to the quarter plane appeared to be naturally partitioned according to the nature of their generating functions: specifically, they conjectured a test for whether or not the generating function of a given model would satisfy algebraic or linear differential equations. This property is often correlated to other, more combinatorial, qualities of a model. Of the 79 non-isomorphic models, 23 are well studied with D-finite generating functions, 51 are highly suspected to be non D-finite – Bostan *et al.* proved the excursion (walks returning to the origin) generating functions are not D-finite [4], and Kurkova and Raschel proved that the trivariate generating functions marking endpoint [12] are not D-finite. The remaining 5 models are called *singular*, and resist both these strategies. Two of these models had been previously considered [14],

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(i) A univariate series is D-finite if it satisfies a linear differential equation with polynomial coefficients. Additionally, a series is D-finite if and only if its analytic continuation is D-finite.

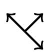
where their univariate generating functions were proven to be non D-finite. We apply this strategy, a relatively direct application of the *iterated kernel method*, a technique modelled on the work of Janse van Rensburg *et al.* in [10], and Bousquet-Mélou and Petkovsec [6] to the final 3 models.

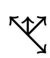
Specifically, the present work proves that the remaining cases are not D-finite (and we note a small correction to the cases already proven). From the method, it is straightforward to determine explicit generating function expressions and asymptotic counting information, in the process identifying many potential singularities for each generating function. In each case, we show that the number of singularities is infinite, and far enough away from the dominating pole that they do not affect the first order asymptotics. The key difficulty here, as was the case in [14], is the justification that these singularities are true poles and are not somehow canceled by some quirk of the expression. This is significant because a D-finite function has a finite number of singularities, and so such a demonstration is a proof of non D-finiteness of the generating function. In the course of our proofs we revisit some older theorems on polynomials that a reader faced with a similar problem may find useful.

In summary, for each of the five singular models we take a unified approach to prove formulas for asymptotic enumeration and determine an explicit expression for the generating function, information which cannot be determined using other known methods. In addition, we prove that the (univariate) counting generating functions are not D-finite. This extended abstract presents the main results, and we we refer the reader to [13] for a completed manuscript.

1.1 The family of singular models

A lattice path model is defined by a set of vectors – the allowable directions in which one can move along the lattice of non-negative integers. We are particularly interested in models which permit only "small" steps, that is, the steps are contained in $\{0, +1, -1\}^2$. We use the notation $NW \equiv (-1, 1), N \equiv (0, 1), NE \equiv (1, 1)$, etc. The family of singular models consists of the following five models, each given with two different representations:

$\mathcal{A} =$  $= \{NW, NE, SE\}$ $\quad \mathcal{B} =$  $= \{NW, N, E, SE\}$ $\quad \mathcal{C} =$  $= \{NW, N, NE, E, SE\}$

$\mathcal{D} =$  $= \{NW, N, SE\}$ $\quad \mathcal{E} =$  $= \{NW, N, NE, SE\}$

Models $\mathcal{A}$ and $\mathcal{D}$ are the two models considered by Mishna and Rechnitzer, and their strategy, known as the iterated kernel method, applies to all of these models. Note that the present work corrects an error found in [14], which does not substantially change the the stated results but which does imply some additional manipulation for the proof.

For each model $\mathcal{S} \in \{\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}\}$ we address the following:

① *What is the number* S_n *of walks of length* n *in the model?*

② *How does* S_n *grow asymptotically when* n *is large?*

③ *Is the generating function* $S(t) = \sum_n S_n t^n$ *D-finite?*

The next section describes how to obtain generating function expressions. This is followed by the asymptotic analysis and non D-finiteness proofs for the symmetric models. We conclude with a summary of the analysis of the asymmetric models.

2 An explicit expression for the generating function

2.1 The functional equation and its kernel

Our central mathematical object is the multivariate generating function $S_{x,y}(t) = \sum_{i,j,n} s_{ij}(n)x^i y^j t^n$, where $s_{ij}(n)$ counts the number of walks of length n ending at the point (i, j) with steps from $\mathcal{S}$. (Throughout, $\mathcal{S}$ is our generic step set.) Our goal is to determine properties of $S(t) \equiv S_{1,1}(t)$, the generating function for the number of walks in the plane.

For each of the five step sets, we associate a polynomial called the *kernel*; for the step set $\mathcal{S}$, define

$$K_S(x, y) = xy - txy \sum_{(i,j)\in\mathcal{S}} x^i y^j.$$

As we restrict ourselves to small steps, the inventory of the steps has the following form

$$\sum_{(i,j)\in\mathcal{S}} x^i y^j = xP_1(y) + P_0(y) + \frac{1}{x}P_{-1}(y) = yQ_1(x) + Q_0(x) + \frac{1}{y}Q_{-1}(x). \tag{1}$$

Thus, $K_S(x, y)$ can be regarded as a quadratic in y (respectively x) whose coefficients contain t, x and the $Q_i(y)$ (resp. t, y, and $P_i(x)$):

$$K_S(x, y) = -txQ_1(x)\, y^2 + (x - xtQ_0(x))\; y - xtQ_{-1}(x). \tag{2}$$

When the model is clear, we omit the subscript S. One common property of the singular models is that they contain the steps NW and SE, and at least one other step – this prevents degeneracy in the quadratic.

Each model admits a functional equation for $S_{x,y}(t)$. We apply the common decomposition that a walk is either the empty walk, or a shorter walk followed by a single step. Taking into account the restrictions on walk location, as well as the fact that substituting $x = 0$ (respectively $y = 0$) into the function $S_{x,y}(t)$ gives the generating function of walks ending on the y-axis (respectively x-axis), we obtain, as many others have before us, the functional equation

$$K(x, y)S_{x,y}(t) = xy + K(x, 0)S_{x,0}(t) + K(0, y)S_{0,y}(t). \tag{3}$$

We are interested in the solutions to the kernel equation of the form:

$$K(x, Y_+(x;t)) = K(x, Y_-(x;t)) = K(X_+(y;t), y) = K(X_-(y;t), y) = 0, \tag{4}$$

and these algebraic functions are easily determined since the kernel is a quadratic:

$$Y_\pm(x;t) = \frac{(1 - tQ_0(x)) \mp \sqrt{(Q_0(x)^2 - 4Q_1(x)Q_{-1}(x))\, t^2 - 2Q_0(x)t + 1}}{2tQ_1(x)} \tag{5}$$

$$X_\pm(y;t) = \frac{(1 - tP_0(y)) \mp \sqrt{(P_0(y)^2 - 4P_1(y)P_{-1}(y))\, t^2 - 2P_0(y)t + 1}}{2tP_1(y)}. \tag{6}$$

There are other function pairs which annihilate the kernel, as we shall see. Remark that the boundary value method begins as we have, with the functional equation (3), but ultimately uses a different parametrization to represent the roots of the kernel, and from there a very different means to get access to the generating function.

The generating function has a natural expression in terms of iterated compositions of the Y and X, hence the name *iterated kernel method.*

2.2 What makes this family special?

Consider the lowest order terms of the roots of the kernel as a power series in t. They are

$$Y_+ = P_{-1}(x)t + O(t^2) \qquad \text{and} \qquad Y_- = \frac{1}{tP_1(x)} - \frac{P_0(x)}{P_1(x)} + O(t),$$

where $P_r(x) = \sum_{(i,r)\in\mathcal{S}} x^i$. Of the 56 (conjectured) non D-finite models only 5 models, precisely the singular family we are studying, have a lowest order term with a positive power in x and t, implying that the infinite sum obtained by the iterated kernel method converges. This prevents the method from being applied to a broader range of models in this context.

2.3 Fast enumeration

Using the series expressions (8) and (9) below, we can generate the first N terms of $S_{1,0}(t)$ and $S_{1,1}(t)$ for each model with $\tilde{O}\left(N^3\right)$ bit-complexity (where the notation $\tilde{O}(\cdot)$ suppresses logarithmic factors), which is an order of magnitude faster than the $\tilde{O}\left(N^4\right)$ bit-complexity of the naive generation algorithm. The key lies in utilizing a linear recurrence for $1/Y_n$ that the iterated kernel method generates for each model (see Table 1). The cost of generating $S_{1,0}(t)$ and $S_{1,1}(t)$ is then dominated by the inversion of the $1/Y_n$, which have summands whose bit-size grows linearly, giving the cubic complexity.

Thus, although the generating function is not D-finite, and hence the coefficients do not satisfy a nice fixed length linear recurrence, we are able to generate the terms in a relatively efficient manner.

3 Symmetric models: $\mathcal{A}, \mathcal{B}, \mathcal{C}$

3.1 An explicit generating function expression

We focus first on the three models $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, as these models are symmetric about the line $x = y$. As such, these models benefit from the relation $S_{x,0} = S_{0,x}$, and Equation (3) can be rewritten as

$$K(x,y)S_{x,y}(t) = xy + K(0,y)S_{0,y}(t) + K(0,x)S_{0,x}(t). \tag{7}$$

Our iterates satisfy $Y_{n+1}(x) = Y_+(Y_n(x))$, $Y_0(x) = x$, and consequently, $K(Y_n, Y_{n+1}) = 0$ for all n by substituting $x = Y_n(x)$ into the kernel relation $K(x, Y_+(x)) = 0$. Thus, when we make this substitution into Equation (7) we find for each n:

$$0 = Y_n(x)Y_{n+1}(x) + K(0, Y_{n+1}(x))S_{0,Y_{n+1}(x)}(t) + K(0, Y_n(x))S_{0,Y_n(x)}(t).$$

We can determine an expression for $K(0,x)S_{0,x}(t)$ by taking an alternating sum of these equations since all of the $K(0, Y_n(x))S_{0,Y_n(x)}(t)$ terms are canceled for $n > 0$ in a telescoping sum:

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty}(-1)^n\Big(Y_n(x)Y_{n+1}(x) + K(0, Y_{n+1}(x))S_{0,Y_{n+1}(x)}(t) + K(0, Y_n)S_{0,Y_n(x)}(t)\Big) \\ &= K(0,x)S_{0,x}(t) + \sum_{n=0}^{\infty}(-1)^n Y_n(x)Y_{n+1}(x). \end{aligned}$$

We rearrange this and evaluate at $x = 1$ to express the counting generating function for walks returning to the axis:

$$S_{0,1}(t) = \frac{1}{t}\sum_{n=0}^{\infty}(-1)^n Y_n(1)Y_{n+1}(1), \tag{8}$$

as $K(0,1) = -t$ for each case considered here. This converges as a power series because in each of these cases $Y_n(x) = O(t^n)$.

Furthermore, substituting $x = 1$ and $y = 1$ into Equation (7) gives the full counting generating function

$$S(t) = \frac{1 - 2tS_{0,1}(t)}{1 - t|\mathcal{S}|} = \frac{1}{1 - t|S|}\left(1 - 2\sum_n (-1)^n Y_n(1)Y_{n+1}(1)\right). \tag{9}$$

We address the robustness of this expression as a complex function in Theorem 2, after we are able to determine an explicit expression for $Y_n(1)$ as a rational function of $Y_1(1)$.

3.2 Asymptotic Enumeration

In each of these cases, the singularity at $\frac{1}{|\mathcal{S}|}$ is dominant.

Theorem 1 *For each model $\mathcal{S}$ in $\{\mathcal{A},\mathcal{B},\mathcal{C}\}$, the sum $R(t) := \sum_n (-1)^n Y_n(1)Y_{n+1}(1)$ is convergent at $t = \frac{1}{|\mathcal{S}|}$. The radius of convergence is bounded below by $t = \frac{1}{p_0+2\sqrt{p_1p_{-1}}}$ where $p_i = P_i(1) = |\{(i,r) : -1 \le r \le 1, (i,r) \in \mathcal{S}\}|$. The dominant singularity for each model $\mathcal{S}$ in $\{\mathcal{A},\mathcal{B},\mathcal{C}\}$ is a simple pole at σ, where $\frac{1}{\sigma} = |\mathcal{S}|$, the number of permitted directions in the model. As a consequence, the number of walks $S_n = [t^n]S(t)$ grows asymptotically like*

$$S_n \sim \kappa_S\left(\frac{1}{\sigma}\right)^n + O\left(\left(p_0 + 2\sqrt{p_1p_{-1}}\right)^n\right),$$

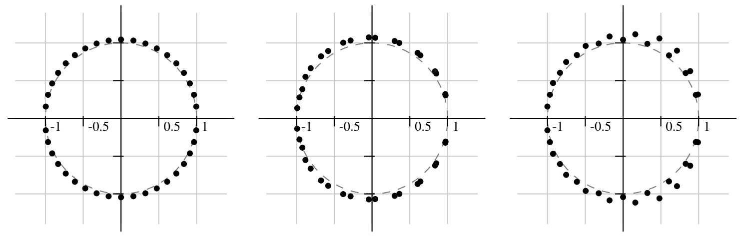
where each κ_S is the constant $R(\sigma)$, which can be calculated to arbitrary precision using Equation (8).

The strategy is to relate the quarter-plane problem to a half-plane problem, which has an explicit general solution given in [1]. We use the expressions that we find for the generating functions to demonstrate the convergence; the results are summarized below.

Model	**Asymptotic estimate for number of walks of length** n	
$\mathcal{A}$	$A_n \sim \kappa_A 3^n + O\left(\left(2\sqrt{2}\right)^n\right)$	$\kappa_A = 0.17317888\ldots$
$\mathcal{B}$	$B_n \sim \kappa_B 4^n + O\left(\left(1+2\sqrt{2}\right)^n\right)$	$\kappa_B = 0.15194581\ldots$
$\mathcal{C}$	$C_n \sim \kappa_C 5^n + O\left(\left(1+2\sqrt{3}\right)^n\right)$	$\kappa_C = 0.38220125\ldots$

3.3 Towards the Non D-Finiteness of $A(t)$, $B(t)$ and $C(t)$

The set of D-finite functions are closed under algebraic substitution. Thus, to prove that the generating functions $A(t)$, $B(t)$ and $C(t)$ are not D-finite, it is equivalent to consider these functions evaluated at $t = q/(1+q^2)$. These turn out to be easier to analyze as the transformation concentrates the singularities around the unit circle. *As such, we shall re-interpret the notation we have introduced this far to be functions of q directly.*



(a) Step set $\mathcal{A}$ (b) Step set $\mathcal{B}$ (c) Step set $\mathcal{C}$

Fig. 1: Plots of the singularities of $Y_{20}(1)|_{t=\frac{q}{1+q^2}}$ for the three symmetric models.

For each model the $Y_n(1)$ terms contribute singularities. A quick glance at an example is very suggestive – see Figure 1 for the singularities of $Y_{20}(1)$ in the q-plane for the three different models. The main difficulty is proving that there is no cancellation; that is, that the singularities in the figure are indeed present in the generating function. To prove this we follow these steps:

Step 1 *Determine an explicit expression for* $Y_n(1)$*;*

Step 2 *Determine a polynomial* $\sigma_n(q)$ *whose set of roots contains the poles of* $Y_n(1)$*;*

Step 3 *Determine a region where there are roots of* $\sigma_n(q)$ *that are truly poles of* $Y_n(1)$*;*

Step 4 *Show that there is no point* ρ *in that region that is a root of both* $\sigma_n(q)$ *and* $\sigma_k(q)$ *for different* n *and* k*;*

Step 5 *Demonstrate that* $S_{1,0}(q/(1+q^2))$ *has an infinite number of singularities and consequently, it is not D-finite. It follows that* $S(t)$ *is not D-finite, by closure under algebraic substitution and the expression in Equation* (9).

3.3.1 Step 1: An explicit expression for Y_n

In this section we find an explicit, non-iterated expression for the functions Y_n. We follow the method of [14] very closely, with the exception that we make the variable substitution earlier in the process. As such, we repeat, that we view all functions as functions of q in this section. From the variable substitution $t = q/(1+q^2)$ in Equation (3), we re-solve the kernel to ensure control over the choice of the branch in the solution. Here are the kernels:

$$
\begin{aligned}
K_A(x,y) &= -q(x^2+1)\,y^2 + x(1+q^2)\,y - qx^2\\
K_B(x,y) &= -q(x+1)\,y^2 + x(-qx+1+q^2)\,y - qx^2\\
K_C(x,y) &= -q(1+x+x^2)\,y^2 + x(-qx+1+q^2)\,y - qx^2.
\end{aligned}
$$

We denote this generically as $K(x,y) = a_2y^2 + a_1y + a_0$, adapting the a_i to each particular model. Each is solved as before to get our initial solutions to $K(x,Y(x)) = 0$. Great care is taken here to ensure that the branch as written remains analytic at 0:

$$Y^A{}_{\pm 1}(x;q) = \frac{x}{2q(1+x^2)} \cdot \left(1 + q^2 \mp \sqrt{1 - 2(2x^2+1)q^2 + q^4}\right)$$
$$Y^B{}_{\pm 1}(x;q) = \frac{x}{2q(1+x)} \cdot \left(1 - qx + q^2 \mp \sqrt{q^4 - 2q^3x + (x^2-4x-2)q^2 - 2qx + 1}\right)$$
$$Y^C{}_{\pm 1}(x;q) = \frac{x}{2q(1+x+x^2)} \cdot \left(1 - qx + q^2 \mp \sqrt{q^4 - 2q^3x - (3x^2+4x+2)q^2 - 2qx + 1}\right).$$

We define the sequence of iterates $\{Y_n(x)\}_{(n)}$ as before: $Y_{n+1}(x) = Y_+(Y_n(x);q), \quad Y_1(x) = Y_+(x;q)$.

For each of these models, examining the coefficients of y in the kernel implies

$$\frac{1}{Y_-(x;q)} + \frac{1}{Y_+(x;q)} = \frac{Y_-(x;q) + Y_+(x;q)}{Y_-(x;q)\cdot Y_+(x;q)} = \frac{-a_1/a_2}{a_0/a_2} = -\frac{a_1}{a_2}. \tag{10}$$

The iterates compose nicely, since for each model $Y_-\left(Y_+(x)\right) = Y_+\left(Y_-(x)\right) = x$.

It turns out to be easier to work with the reciprocal of Y_n, so we define $\overline{Y}_n = \frac{1}{Y_n(1)}$, and view this as a function of q. Equation (10) then converts into a recurrence after the substitution $x = Y_{n-1}(x)$. Specifically, this gives a linear recurrence for the reciprocal function, $\frac{1}{Y_n(x)}$; we are interested in these evaluated at $x = 1$, and the resulting recurrences and their solutions in terms of $\overline{Y}_1$ are summarized below:

$\mathcal{S}$	**Recurrence**	$\overline{Y}^S_n(q)$
$\mathcal{A}$	$\overline{Y}_n = (q+\frac{1}{q})\overline{Y}_{n-1} - \overline{Y}_{n-2}$	$\frac{(q^2-q^{2n})+q(q^{2n}-1)\overline{Y}_1}{q^n(q^2-1)}$
$\mathcal{B},\mathcal{C}$	$\overline{Y}_n = (q+\frac{1}{q})\overline{Y}_{n-1} - \overline{Y}_{n-2} - 1$	$\frac{q(q-1)(q^{2n}-1)\overline{Y}_1^{B,C}+(q-q^n)(2q^{n+1}-q^n+q^2-2q)}{q^n(q+1)(q-1)^2}$

Tab. 1: The recurrences and their solutions for models $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$.

Following the same procedure as above, we obtain a generic expression for $S_{1,0}(t)$, the generating function for the number of walks which return to the axis for model $\mathcal{S}$, which can be applied to all three symmetric walks:

$$S_{1,0}\left(\frac{q}{1+q^2}\right) = (q+1/q)\sum_{n=0}^{\infty}(-1)^n Y_n^S(1) Y_{n+1}^S(1;q). \tag{11}$$

Our careful choice of branches now implies that this is a formal power series. (Remark, this was not the case in [14].) Our expression is robust – an application of the ratio test implies the sum converges everywhere, except possibly on the unit circle and at the poles of the Y_n.

Proposition 2 *For $\mathcal{S} \in \{\mathcal{A},\mathcal{B},\mathcal{C}\}$ the sum $(q+1/q)\sum_{n=0}^{\infty}(-1)^n Y_n^S(1)Y_{n+1}^S(1)$ is convergent for all $q \in \mathbb{C}$ with $|q| \neq 1$, except possibly at the set of points defined by the singularities of the $Y_n^S(1)$ for all n.*

3.3.2 Step 2: The singularities of $Y_n^S(1)$

In order to argue about the singularities, we find a family of polynomials $\sigma_n(q)$ that the roots of $\overline{Y}_n$ satisfy. The polynomials in Table 2 are obtained by manipulating the explicit expressions given above. Unfortunately, extraneous roots are introduced during the algebraic manipulation when an equation is squared to remove the square root present. In fact, the extraneous roots are exactly those which correspond to a negative sign in front of the square root. If one defines $\overline{Y}_{-1} = \overline{Y}_-$ and $\overline{Y}_{-n} = \overline{Y}_{-1} \circ \overline{Y}_{-(n-1)}$ for $n > 1$, then using the argument above one can check that $\overline{Y}_{-n}$ satisfies the same recurrence relation as $\overline{Y}_n$, up to a reversal of the sign in front of the square root. Thus, we see that the set of roots of $\sigma_n(q)$ is simply the union of the sets of roots of $\overline{Y}_n$ and $\overline{Y}_{-n}$.

$\mathcal{S}$	$\sigma_n(q)$
$\mathcal{A}$	$\alpha_n(q) = q^{4n} + q^{2n+2} - 4q^{2n} + q^{2n-2} + 1$
$\mathcal{B}$	$\beta_n(q) = \left(q^{2n-1} + (q^3 - 2q^2 - 2q + 1)q^{n-2} + 1\right)\left(q^{2n+1} + (q^3 - 2q^2 - 2q + 1)q^{n-1} + 1\right)$
$\mathcal{C}$	$\gamma_n(q) = q^2(1 + q^2 - q)(1 + q^{4n}) + q(q^2 - 3q + 1)(q + 1)^2(q^n + q^{3n})$ $+q^{2n}(1 - q^2 - 4q + 14q^3 - 4q^5 - q^4 + q^6)$

Tab. 2: The singularities of Y_n^S in the q-plane satisfy the polynomial $\sigma_n(q)$

Furthermore, we can show that these roots are dense around the unit circle using the results of Beraha, Kahane, and Weiss – specifically, a weakened statement of the Main Theorem of [2].

Corollary 3 *The roots of the families of polynomials $\{\alpha_n(q)\}, \{\beta_n(q)\}$, and $\{\gamma_n(q)\}$ are dense around the unit circle.*

In addition, we can show that $\alpha_n(q)$ and $\gamma_n(q)$ have no roots on the unit circle, except possibly $q = \pm 1$, and if q is a root of $\beta_n(q)$ on the unit circle not equal to 1 then

$$\arg q \in \left[\pi - \arccos\left(\sqrt{2} - \frac{1}{2}\right), \pi\right) \bigcup \left[-\pi, -\pi + \arccos\left(\sqrt{2} - \frac{1}{2}\right)\right) \approx [2.7, 3.1) \cup [-2.7, -3.1).$$

This is complimented by the fact that Rouche's Theorem implies the roots converge to the unit circle as n approaches infinity.

3.3.3 Step 3: Verify that $Y_n(1)$ has some singularities

At this point we have not yet completely established that the $Y_n(1)$ actually have singularities. Theoretically, it is possible that all the roots were added in our manipulations to determine $\sigma_n(q)$ for the different models. As we mentioned above, the roots of $\sigma_n(q)$ are either singularities of $Y_n(1)$ or singularities of $Y_{-n}(1)$. Thus, we prove Lemma 4 which describes at least some region where we are certain to find roots of $\overline{Y}_n$. Experimentally, it seems that the roots are evenly partitioned so that roots outside the unit circle belong to $\overline{Y}_n$ and those inside the unit circle belong to $\overline{Y}_{-n}$, but we do not prove this.

Lemma 4 *If $\arg(q) \in (-\pi/2, -3\pi/8) \cup (3\pi/8, \pi/2)$ then $\overline{Y}_n^S = \overline{Y}_{-n}^S|_{q\mapsto 1/q}$ for all n, for $S \in \{A, B, C\}$. Consequently (using Corollary 3), for an infinite number of n each of Y_n^A, Y_n^B, and Y_n^C admit at least one singularity in the complex q-plane in that region.*

The proof requires only basic manipulations of the formulas. Thus, we can ensure an infinite source of potential singularities for these generating functions.

3.3.4 Step 4: The singularities are distinct

We characterize the roots which are shared between Y_n and Y_k for the three models.

For each of the three models we find the roots of the numerators of our explicit expressions in Table 1 as quadratics in q^n. This determines functions $r_1(q)$ and $r_2(q)$, independent of n, such that $q_c^n = r_1(q_c)$ or $q_c^n = r_2(q_c)$ at any pole q_c of Y_n. Furthermore, suppose q_c is also a pole of Y_k for $k \neq n$ so that $q_c^k = r_1(q_c)$ or $q_c^k = r_2(q_c)$. It is immediate that if $q_c^k = r_1(q_c) = q_c^n$ or $q_c^k = r_2(q_c) = q_c^n$ then q_c must be on the unit circle. Otherwise, we may assume without loss of generality that $q_c^n = r_1(q_c)$ and $q_c^k = r_2(q_c)$. Each model can be considered separately to prove the following result:

Proposition 5 *For models $\mathcal{A}$ and $\mathcal{C}$, if q_c is a pole of Y_n which lies off the unit circle then it is not a pole of Y_k for $k \neq n$. For model $\mathcal{B}$, if q_c is a pole of Y_n off the unit circle then it is not a pole of Y_k for $|k-n| > 1$.*

3.3.5 Step 5: The generating function is not D-finite

Now we tie up all the arguments.

Theorem 6 *The generating functions $A(t)$, $B(t)$, and $C(t)$ of walks in the quarter plane with steps from $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, respectively, are not D-finite.*

Proof: As argued above, it is sufficient to prove the non D-finiteness of $S_{1,0}(t)$ evaluated at $t = q/(1+q^2)$ for each model. In Lemma 4 we have shown that $Y_n^A(1)$, $Y_n^B(1)$, and $Y_n^C(1)$ admit singularities off the unit circle for an infinite number of n. One can show that the singularities are not canceled in the two summands of (11) containing Y_n^A and Y_n^C (respectively the four summands containing Y_{n-1}^B, Y_n^B, and Y_{n+1}^B) so that, as they lie off the unit circle where the remaining summands are analytic by Proposition 5, they give an infinite number of singularities of $A_{1,0}$, $B_{1,0}$, and $C_{1,0}$. □

4 Asymmetric models

The asymmetric models are not substantially different, but when we iterate, we have more functions to keep track of, and eventually cancel. Aside from some irritating bookkeeping, there is no main obstacle to following the strategy of the symmetric models.

4.1 An explicit generating function expression

To obtain the generating function expressions we follow the same path as in the symmetric case: we generate a sequence of equations, each which annihilates the kernel. This opens up the possibility of a telescoping sum expression from which we can find an expression for the generating function of walks returning to the axis in terms of iterates of two functions. An explicit expression for these iterates is obtained by solving some very simple recurrences. We complete these steps for the asymmetric models in this section.

As before, we begin with the main functional equation (3), make the substitution $t = q/(1+q^2)$, and re-arrange to get the kernel equations:

$$\begin{aligned}
\left(xy(1+q^2) - qy^2 - qxy^2 - qx^2\right) D_{x,y}(t) &= xy(1+q^2) - qx^2 D_{x,0}(t) - qy^2 D_{0,y}(t)\\
\left(xy(1+q^2) - qy^2 - qxy^2 - qx^2y^2 - qx^2\right) E_{x,y}(t) &= xy(1+q^2) - qx^2 E_{x,0}(t) - qy^2 E_{0,y}(t),\\
\text{with kernels}\quad K_D(x,y) &= -q(1+x)y^2 + (q^2+1)xy - qx^2\\
K_E(x,y) &= -q(1+x+x^2)y^2 + (1+q^2)xy - qx^2.
\end{aligned}$$

As there is no longer an $x = y$ symmetry, we solve the kernels as functions of both x and y; that is, we find $Y(x)$ satisfying $K(x, Y(x)) = 0$ and also $X(y)$ satisfying $K(X(y), y) = 0$. We have some choice over how we split the solutions over different branches. One such choice of branches is:

$$X^D{}_{\pm}(y;q) = \frac{y}{2q} \cdot \left(1 - qy + q^2 \mp \sqrt{q^4 - 2q^3y + (y^2-2)q^2 - 2qy + 1}\right)$$
$$Y^D{}_{\pm}(x;q) = \frac{x}{2q(1+x)} \cdot \left(1 + q^2 \mp \sqrt{q^4 - 4q^2x - 2q^2 + 1}\right).$$

$$X^E{}_{\pm}(y;q) = \frac{y}{2q(1+y^2)} \cdot \left(1 - qy + q^2 \mp \sqrt{q^4 - 2q^3y - (3y^2+2)q^2 - 2qy + 1}\right)$$
$$Y^E{}_{\pm}(x;q) = \frac{x}{2q(1+x+x^2)} \cdot \left(1 + q^2 \mp \sqrt{q^4 - 2(2x^2+2x+1)q^2 + 1}\right).$$

Next, as we described in the introductory summary, we repeatedly alternate the substitution of the X and Y and create two related sequences of functions:

$$\chi_n(x) = X_+(Y_+(\chi_{n-1}(x);q);q), \quad \chi_0(x) = x \quad \text{and} \quad \Upsilon_n(y) = Y_+(X_+(\Upsilon_{n-1}(x);q);q), \quad \Upsilon_0(y) = y.$$

Simple substitutions yield the kernel relations $K(\chi_n(x), Y_+(\chi_n(x))) = K(X_+(\Upsilon(y)), \Upsilon(y)) = 0$, amongst others. As before, we generate an infinite list of relations by substituting $x = \chi_n(x)$, $y = Y_+(\chi_n(x))$, and then a second infinite list using the substitutions $x = X_+(\Upsilon(y))$, $y = \Upsilon(y)$. Again, we form a telescoping sum, and after some manipulation this results in an expression for the generating functions returning to the axis. For $S \in \{D, E\}$ we have:

$$S_{x,0}\left(\frac{q}{1+q^2}\right) = \frac{q}{1+q^2} \sum_{n\geq 0} \chi_n(x) \cdot \underbrace{(Y_+(\chi_n(x)) - Y_+(\chi_{n-1}(x)))}_{\Delta_L(x;q)} \tag{12}$$

$$S_{0,y}\left(\frac{q}{1+q^2}\right) = \frac{q}{1+q^2} \sum_{n\geq 0} X_+(\Upsilon_n(y)) \cdot \underbrace{(\Upsilon_n(y) - \Upsilon_{n+1}(y))}_{\Delta_R(y;q)}. \tag{13}$$

The two models have identical structure in their generating function, and differ only in their respective functions X_1 and Y_1. Our greatest challenge at this point is keeping track of the various parts:

$$\chi'_0(x) = Y_1(x), \quad \chi'_n(x) = Y_1(\chi_n(x)), \qquad \Upsilon'_n(y) = X_1(\Upsilon_n(y)), \quad \Upsilon'_0(y;q) = X_1(y)$$
$$\Delta_{L,0} = \chi'_0(x), \quad \Delta_{L,n}(x) = \chi'_n(x) - \chi'_{n-1}(x), \qquad \Delta_{R,0} = \Upsilon_0(y), \quad \Delta_{R,n}(y) = \Upsilon_n(y) - \Upsilon_{n+1}(y)$$

For each model, we isolate the left and right hand sides: $L_x^S(q) = qx^2 S_{x,0}(q/(1+q^2))$ and $R_y^S(q) = qy^2 S_{0,y}(q/(1+q^2))$ so that

$$K_S(x,y)S_{x,y}(q/(1+q^2)) = xy(1+q^2) - L_x^S(q) - R_y^S(q).$$

Similar to previous cases, we can use the coefficients of $K_S(x,y)$ and the two identities $Y_{\pm 1}(X_{\mp 1}(y)) = y$ and $X_{\pm 1}(Y_{\mp 1}(x)) = x$ to form the paired up recurrences for the multiplicative inverses of these functions. Here we use the notation that $\overline{F} = \frac{1}{F}$:

$$\overline{\chi}_{\pm n} = (q + 1/q)\overline{\chi'}_{\pm(n-1)} - \overline{\chi}_{\pm(n-1)} - 1, \qquad \overline{\chi'}_{\pm n} = (q+1/q)\overline{\chi'}_{\pm n} - \overline{\chi}_{\pm(n-1)}$$
$$\overline{\Upsilon}_{\pm n} = (q+1/q)\overline{\Upsilon'}^S_{\pm(n-1)} - \overline{\Upsilon}^S_{\pm(n-1)}, \qquad \overline{\Upsilon'}^S_{\pm n} = (q+1/q)\overline{\Upsilon}^S_{\pm n} - \overline{\Upsilon'}^S_{\pm(n-1)} - 1.$$

These recurrences are easily solved, as before, as closed form expressions. Again, we can verify that we have suitable expressions for the analytic continuation of the generating functions for $\mathcal{D}$ and $\mathcal{E}$. These expressions are well suited to exact enumeration, as in the symmetric case, and they also yield asymptotic enumeration formulas.

Theorem 7 (Mishna and Rechnitzer [14]; Proposition 16) *If D_n denotes the number of walks with steps from $\mathcal{D}$ staying in the positive quarter plane, then $D_n \sim \kappa_D \frac{3^n}{\sqrt{n}}(1+o(1))$ where $0 < \kappa_D \leq \sqrt{\frac{3}{\pi}}$.*

Theorem 8 *The function $E(t)$ has a simple singularity at $t = 1/4$, where it has a residue of value $\kappa_E \in \left[\frac{122}{525}, \frac{7}{10}\right]$. The number, E_n, of walks taking steps in $\mathcal{E}$ and staying in the positive quarter plane grows asymptotically as $E_n = \kappa_E \cdot 4^n + O\left((1+2\sqrt{2})^n\right)$.*

Computational evidence given by calculating the series for $E_{1,0}(1/4)$ and $E_{0,1}(1/4)$ to a large number of terms implies that the value of the growth constant is approximately 0.2636, which is consistent with the growth of computationally generated values of E_n for large n.

4.2 The generating functions $D(t)$ and $E(t)$ are not D-finite

The additional sums that arise in our expressions for $D(t)$ and $E(t)$ do not change our fundamental strategy, demonstrating an infinite set of singularities, but here we actually have a simpler argument in hand. In both cases the series $L_1(q)$ has an infinite source of singularities on the imaginary axis, at each of which $R_1(q)$ is analytic.

Lemma 9 *Both of the functions $\overline{\chi}_n^D(q)$ and $\overline{\chi}_n^E(q)$ have a root on the imaginary axis between i and $2i$, when n is even.*

Proof: First, we show that the roots of $\overline{\chi}_n^D(q)$ satisfy the polynomial

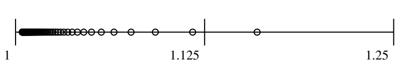
$$\omega_n^D = \left(q^{4n+2} + q^{2n+4} - 4q^{2n+2} + q^{2n} + q^2\right)^2$$

and that the roots of $\overline{\chi}_n^E(q)$ satisfy the polynomial

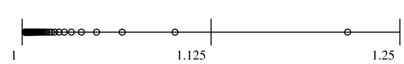
$$\omega_n^E = q^2\left(q^4 - q^2 + 1\right)\left(q^{8n}+1\right) + 2q^2(q^4 - 4q^2 + 1)\left(q^{6n} + q^{2n}\right) + (q^8 - 10q^6 + 24q^4 - 10q^2 + 1)q^{4n}.$$

The rest is straightforward manipulation as in [14]. □

A sampling of these singularities is given in Figure 2.



(a) Step set $\mathcal{D}$



(b) Step set $\mathcal{E}$

Fig. 2: The modulus of the purely imaginary singularities of $D(t)$ and $E(t)$, coming from χ_{2n} for $n = 1, \ldots, 100$.

Theorem 10 *Neither the generating function $D(t)$ nor the generating function $E(t)$ for walks in the quarter plane with steps from $\mathcal{D}$ and $\mathcal{E}$, respectively, are D-finite.*

5 Conclusion

This work addresses a family of lattice path models that have resisted other powerful approaches. There should also be other models, with larger step sizes or in higher dimensions, to which this method may be suitable. We intend to seek them out and to try to apply this method, ideally automating as much as is possible. Finally, we are always in a search to understand the combinatorial nature of D-finite functions. Are there properties inherent to these classes from which we should be able immediately to predict the nature of the generating function? Where does the intuition lie?

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Root-theoretic Young Diagrams, Schubert Calculus and Adjoint Varieties

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Abstract. Root-theoretic Young diagrams are a conceptual framework to discuss existence of a root-system uniform and manifestly nonnegative combinatorial rule for Schubert calculus. Our main results use them to obtain formulas for (co)adjoint varieties of classical Lie type. This case is the simplest after the previously solved (co)minuscule family. Yet our formulas possess both uniform and non-uniform features.

Résumé. Les diagrammes de Young racine-théoriques forment un cadre conceptuel qui permet de discuter l'existence de règles de calcul de Schubert explicitement non-négatives et uniformes sur les systèmes de racines. Notre principal résultat est leur utilisation pour obtenir des formules pour les variétés (co)adjointes de types classiques. C'est le cas le plus simple après celui la famille (co)minuscule, déja résolue. Nos formules possèdent toutefois des propriétés uniformes et non-uniformes.

Keywords: Root-theoretic Young diagrams, Schubert calculus, Adjoint varieties

Overview

This extended abstract concerns the following question:

> Does there exist a *root-system uniform* and *manifestly nonnegative* combinatorial rule for Schubert calculus?

We elaborate on this problem and suggest an approach to it. Let G be a complex reductive Lie group. Fix a choice B of a Borel subgroup and maximal torus T, and let W be its Weyl group: $W \cong N_G(T)/T$. Write $\Phi = \Phi^+ \cup \Phi^-$ to be the partition of roots into positives and negatives, and let Δ be the base of simple roots. Let $\Omega_G = (\Phi^+, \prec)$ denote the canonical poset structure on Φ^+. Suppose $\Delta_P = \{\beta(P)_1, \ldots, \beta(P)_k\} \subseteq \Delta$ identifies the parabolic subgroup P, and set $W_P := W_{\Delta_P}$ as the associated parabolic subgroup of W. Consider the subposet

$$\Lambda_{G/P} = \{\alpha \in \Phi^+ : \beta_i(P) \prec \alpha \text{ for some } i\} \subseteq \Omega_G.$$

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The **Schubert varieties** in G/P are $X_{wW_P} = \overline{B_{-}wP/P}$ where $wW_P \in W/W_P$. Suppose w is the minimal length coset representative of wW_P; w's inversion set $\overline{\lambda}$ sits inside $\Lambda_{G/P}$. Let us write $X_{\overline{\lambda}} := X_{wW_P}$. Call $\overline{\lambda}$ a **root-theoretic Young diagram** (RYD). Let $\mathbb{Y}_{G/P}$ be the set of RYDs for G/P.

The cohomology ring $H^{\star}(G/P, \mathbb{Q})$ has a $\mathbb{Z}$-additive basis of **Schubert classes** $\sigma_{\overline{\lambda}}$. Let $C_{\overline{\lambda},\overline{\mu}}^{\overline{\nu}}(G/P)$ denote the Schubert structure constants for G/P, i.e.,

$$\sigma_{\overline{\lambda}} \cdot \sigma_{\overline{\mu}} = \sum_{\overline{\nu}} C_{\overline{\lambda},\overline{\mu}}^{\overline{\nu}}(G/P)\sigma_{\overline{\nu}}.$$

When G/P is the Grassmannian $Gr_k(\mathbb{C}^n)$, $C_{\overline{\lambda},\overline{\mu}}^{\overline{\nu}} := C_{\overline{\lambda},\overline{\mu}}^{\overline{\nu}}(Gr_k(\mathbb{C}^n))$ is computed by the *Littlewood-Richardson rule*.

Ideally, there is a generalization that computes any $C_{\overline{\lambda},\overline{\mu}}^{\overline{\nu}}(G/P)$ in a cancellation-free fashion, but only in terms of the associated root datum (cancellative formulas are known, see, e.g., [Kn03]). Actually, often the main question is phrased presuming the existence of a rule. However, in that case, what is the qualitative nature of such a putative rule? Is it too much to expect a "counting rule" like the *Littelmann path model*? Should one instead search for a "patchwork" of counting rules and nonnegative recursions through different G/P's for varying G's? How can one classify special cases? Why are some special cases of the problem seemingly harder than others? Finally, if one believes that such a rule does not exist, what are concrete and/or falsifiable reasons for that belief?

Our thesis is that RYDs provide a simple but uniform combinatorial perspective to discuss such questions mathematically, make precise comparisons, and to measure progress towards a rule (uniform or otherwise).

For instance, from this perspective, Grassmannians are special because they sit in the family of G/P's for which the above root-system setup is especially graphical:

(I) $\Lambda_{G/P}$ is a planar poset;

(II) the RYDs are lower order ideals (and in fact classical Young diagrams, thus explaining our nomenclature);

(III) Bruhat order is containment of RYDs.

These properties also hold for all cominuscule G/P's. Together with earlier work of R. Proctor [Pr06], they help demonstrate existence for (co)minuscule Schubert calculus [ThYo09].

Right now, using RYDs is the only known way to solve the existence problem for (co)minuscule G/P. Conversely, it is only for (co)minuscule G/P's that there is a uniform rule. Given this condition, it is therefore sensible to use RYDs to study other families.

We assert the key next case is the family of (co)adjoint G/P's. One reason is that this family extends the (co)minuscule G/P's, see, e.g., [LaMuSe79]. However, in terms of RYDs, it is important that for (co)adjoint varieties, none of the properties (I), (II) or (III) hold in general. Also important is that the failures of these properties are *quantifiably* mild (see Fact 0.1 below).

We obtain positive Schubert calculus rules in the classical (co)adjoint types. These rules have significant (but far from complete) uniformity. Our rules are sufficiently simple to admit nonzeroness criteria extending a simple case of the Horn inequalities, and also to completely classify what numbers occur as structure constants.

Additional complexity of $OG(2,2n)$ comes from the nonplanarity of $\Lambda_{OG(2,2n)}$. To our best knowledge, we give the first complete formula for any G/P with such nonplanarity — and what we find is that it has "patchwork" features for which we have no broad explanation. It first separates out the cases covered by the Pieri rule of [BuKrTa09]. Perhaps surprisingly, it is these "Pieri cases" that would bring unappetizing complications to our rule. Yet, even after removing these cases, the rule exhibits a dependency on the parity of n. This is traceable to the fact that $\Lambda_{\mathbb{Q}^{2n-4}}$ is a subposet of $\Lambda_{OG(2,2n)}$ and that the even-dimensional quadric $\mathbb{Q}^{2n-4}$ has this dependency as well [ThYo09].

It seems plausible to us that the patchwork features of $OG(2,2n)$ are unavoidable if maintaining uniformity with the other (co)adjoint and (co)minuscule varieties. That is, we would infer that our results in this special case challenge the existence of a root-system uniform "counting" rule. Now, there are specific reasons to doubt this interpretation. First, in [ChPe11], RYDs are used to generalize [ThYo09]. Their extension uniformly covers a subset of the Schubert problems in each of the (co)adjoint varieties — but precisely those that are most "cominuscule-like". Second, the "flattening trick" used for the $OG(2,2n)$ problem is patently non-uniform. However, this step is what allows us to make comparisons with the other (co)adjoint formulas. Third, there are alternative and known uniform models such as "chains in Bruhat order", see, e.g., [BerSot98]. However, we reiterate that these alternative approaches are not known to resolve the (co)minuscule case, which from our perspective is the simpler problem.

Definition of (co)adjoint varieties

The following definition is standard. Fix a representation $\rho : G \to GL(V)$ for some finite dimensional complex vector space V. The group G acts on $\mathbb{P}(V)$ through the projection $\pi : V \setminus \{0\} \to \mathbb{P}(V)$. If $\vec{v}$ is a highest weight vector of ρ, then $\pi(G \cdot v) \subseteq \mathbb{P}(V)$ is a homogeneous projective variety, see, e.g., [FuHa04, Section 23.3]. This variety is **adjoint** if ρ is the adjoint representation of G. Adjoint varieties have a root-system theoretic classification, see, e.g., [ChPe11] and the references therein. Then a variety is **coadjoint** if it is adjoint for the dual root system.

Call the highest root of $\Lambda_{G/P}$ the **adjoint root**. If $\overline{\lambda}$ uses it we say $\overline{\lambda}$ is **on** and we write $\overline{\lambda} = \langle \lambda | \bullet \rangle$; otherwise we say $\overline{\lambda}$ is **off** and we write $\overline{\lambda} = \langle \lambda | \circ \rangle$, where λ comprises the roots of $\Lambda_{G/P} \setminus \{\text{adjoint root}\}$ used by $\overline{\lambda}$. We state some facts that are easily checked for the setting of our theorems; cf., [ChPe11, Section 2].

Fact 0.1 *If G/P is adjoint then:*

(i) $|\Lambda_{G/P}|$ *is odd*

(ii) If $\overline{\lambda} = \langle \lambda | \circ \rangle$ *then* $|\overline{\lambda}| < \frac{1}{2}|\Lambda_{G/P}|$

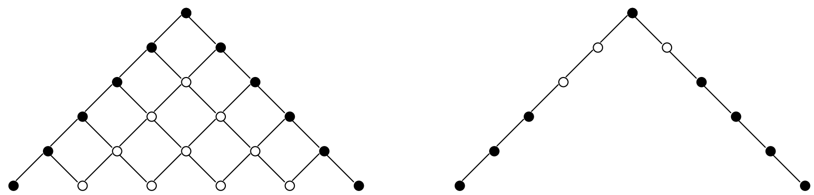
(iii) If $\overline{\lambda} = \langle \lambda | \bullet \rangle$ *then* $|\overline{\lambda}| > \frac{1}{2}|\Lambda_{G/P}|$

(iv) λ *is a lower order ideal in the poset* $\Lambda_{G/P} \setminus \{$*adjoint root*$\}$

For example, point (iv) explains in what sense the failure of (II) above is "mild".

Warmup with the "(line,hyperplane)" flag variety $Fl_{1,n-1;n}$

We begin with a simple case of the adjoint varieties, $G/P = Fl_{1,n-1;n}$. This is the two step partial flag variety $\{\langle 0 \rangle \subset F_1 \subset F_{n-1} \subset \mathbb{C}^n\}$ where F_1 and F_{n-1} have dimensions 1 and $n-1$ respectively. It has dimension $|\Lambda_{G/P}| = 2n - 3$. All two-step flag manifolds have been solved, in a different way, by I. Coskun [Co09]. However, our approach will naturally extend to other (co)adjoint cases.



$\Lambda_{Fl_{1,n-1;n}}$, Ω_{GL_n} and a shape (for $n = 7$)

We denote the shapes $\overline{\lambda}$ by $\langle \lambda_1, \lambda_2 | \bullet / \circ \rangle$, where $0 \leq \lambda_1, \lambda_2 \leq \frac{|\Lambda_{G/P}|-1}{2}$ and $\bullet/\circ$ indicates if $\overline{\lambda}$ is on or off.

We will need some reusable definitions. For any $\nu = (\nu_1, \nu_2) \in \mathbb{Z}^2$ let $\nu^\star = (\nu_1 - 1, \nu_2)$ and $\nu_\star = (\nu_1, \nu_2 - 1)$. Fix $\overline{\lambda}$ and $\overline{\mu}$ and define

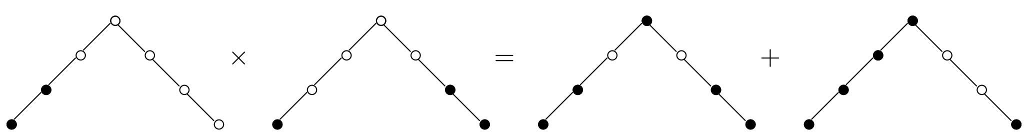
$$\mathbb{A}_{\overline{\lambda},\overline{\mu}}(\nu) = \begin{cases} 0 & \text{if } \overline{\lambda} \text{ and } \overline{\mu} \text{ are on} \\ \sigma_{\langle \nu | \bullet \rangle} & \text{if exactly one of } \overline{\lambda} \text{ or } \overline{\mu} \text{ is on} \\ \sigma_{\langle \nu | \circ \rangle} & \text{if } |\overline{\lambda}| + |\overline{\mu}| < \frac{1}{2}|\Lambda_{G/P}| \\ \sigma_{\langle \nu^\star | \bullet \rangle} + \sigma_{\langle \nu_\star | \bullet \rangle} & \text{otherwise.} \end{cases}$$

Set $\sigma_{\langle \nu | \bullet/\circ \rangle}$, $\sigma_{\langle \nu^\star | \bullet \rangle}$ or $\sigma_{\langle \nu_\star | \bullet \rangle}$ to be zero if ν, $\nu^\star$ or $\nu_\star$ are not in $\left[0, \frac{|\Lambda_{G/P}|-1}{2}\right] \times \left[0, \frac{|\Lambda_{G/P}|-1}{2}\right]$.

The "otherwise" case of the definition of $\mathbb{A}_{\overline{\lambda},\overline{\mu}}(\nu)$ is what we call "adjoint jumping": a nonadjoint root from ν has "jumped" to become the adjoint root. Understanding how this occurs in each type is a key idea needed in the (co)adjoint cases. This reflects the additional complexity coming from the failure of (II).

Proposition 0.2 $\sigma_{\overline{\lambda}} \cdot \sigma_{\overline{\mu}} = \mathbb{A}_{\overline{\lambda},\overline{\mu}}(\lambda + \mu) \in H^\star(Fl_{1,n-1;n}, \mathbb{Q})$.

Example 0.3 *For $n = 5$, the rule gives $\sigma_{\langle 2,0|\circ \rangle} \cdot \sigma_{\langle 1,2|\circ \rangle} = \mathbb{A}_{\langle 2,0|\circ \rangle, \langle 1,2|\circ \rangle}(3,2) = \sigma_{\langle 2,2|\bullet \rangle} + \sigma_{\langle 3,1|\bullet \rangle}$. Pictorially:*



□

Corollary 0.4 $\left\{ C_{\overline{\lambda},\overline{\mu}}^{\overline{\nu}}(Fl_{1,n-1;n}) \right\} = \{0, 1\}$.

In the style of Horn type theorems (see, e.g., [Fu00b]), we give a polytopal characterization of when $C_{\overline{\lambda},\overline{\mu}}^{\overline{\nu}}(Fl_{1,n-1;n}) \neq 0$. Identify

$$\overline{\lambda} = \langle \lambda_1, \lambda_2 | \circ \rangle \text{ with } (\lambda_1, \lambda_2, 0) \in \mathbb{Z}^3 \text{ and } \overline{\lambda} = \langle \lambda_1, \lambda_2 | \bullet \rangle \text{ with } (\lambda_1, \lambda_2, 1) \in \mathbb{Z}^3. \tag{1}$$

Corollary 0.5 *Assume* $\lambda = (\lambda_1, \lambda_2), \mu = (\mu_1, \mu_2), \nu = (\nu_1, \nu_2) \in \mathbb{Z}^2 \cap \left[0, \frac{|\Lambda_{G/P}|-1}{2}\right] \times \left[0, \frac{|\Lambda_{G/P}|-1}{2}\right]$ *and* $\lambda_3, \mu_3, \nu_3 \in \{0,1\}$. *Then* $C^{\overline{\nu}}_{\overline{\lambda},\overline{\mu}}(Fl_{1,n-1;n}) \neq 0$ *if and only if:*

$$\begin{aligned} |\overline{\nu}| &= |\overline{\lambda}| + |\overline{\mu}| \\ \nu_1 &\leq \lambda_1 + \mu_1 \\ \nu_2 &\leq \lambda_2 + \mu_2 \\ \lambda_3 + \mu_3 &\leq \nu_3 \end{aligned}$$

Main theorem for odd orthogonal Grassmannians $OG(2, 2n+1)$ and Lagrangian Grassmannians $LG(2, 2n)$

For the type B_n root system, the adjoint variety $G/P = OG(2, 2n+1)$ is the space of isotropic 2-planes with respect to a non-degenerate symmetric bilinear form on $\mathbb{C}^{2n+1}$. It has dimension $|\Lambda_{G/P}| = 4n - 5$.

$\Lambda_{OG(2,2n+1)}$, $\Omega_{SO_{2n+1}}$ and a shape (for $n = 4$)

The coadjoint partner to $OG(2, 2n+1)$ in the C_n root system is the variety $LG(2, 2n)$ of isotropic 2-planes with respect to a non-degenerate skew-symmetric bilinear form on $\mathbb{C}^{2n}$. Currently, we study the coadjoint variety with RYDs for its adjoint partner. This is analogous to [ThYo09]. We denote the shapes $\overline{\lambda}$ by $\langle\lambda|\bullet/\circ\rangle$, where λ is a partition in $2 \times \left(\frac{|\Lambda_{G/P}|-1}{2}\right)$.

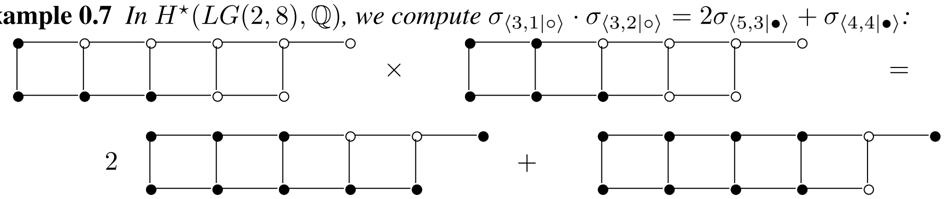
Say $\sigma_{\langle\nu|\bullet/\circ\rangle}$, $\sigma_{\langle\nu^\star|\bullet\rangle}$ or $\sigma_{\langle\nu_\star|\bullet\rangle}$ is zero if ν, $\nu^\star$ or $\nu_\star$ is not a partition in $2 \times \left(\frac{|\Lambda_{G/P}|-1}{2}\right)$. Define $\mathrm{sh}(\overline{\nu})$ to be the number of short roots used by $\overline{\nu}$. The short roots of $\Lambda_{OG(2,2n+1)}$ consist of the middle pair of the nonadjoint roots.

Theorem 0.6

$$\sigma_{\overline{\lambda}} \cdot \sigma_{\overline{\mu}} = \sum_{\nu \subseteq \left(\frac{|\Lambda_{G/P}|+1}{2}, \frac{|\Lambda_{G/P}|-1}{2}\right)} C^{\nu}_{\lambda,\mu} \mathbb{A}_{\overline{\lambda},\overline{\mu}}(\nu) \in H^\star(LG(2,2n), \mathbb{Q}).$$

In $H^\star(OG(2, 2n+1))$*, multiply each coefficient by* $2^{\mathrm{sh}(\overline{\nu})-\mathrm{sh}(\overline{\lambda})-\mathrm{sh}(\overline{\mu})}$.

Example 0.7 *In* $H^\star(LG(2,8), \mathbb{Q})$*, we compute* $\sigma_{\langle 3,1|\circ\rangle} \cdot \sigma_{\langle 3,2|\circ\rangle} = 2\sigma_{\langle 5,3|\bullet\rangle} + \sigma_{\langle 4,4|\bullet\rangle}$*:*



Similarly, in $H^\star(OG(2,9),\mathbb{Q})$, *we compute* $\sigma_{\langle 2,1|\circ\rangle}\cdot\sigma_{\langle 3,2|\circ\rangle}=\sigma_{\langle 5,2|\bullet\rangle}+4\sigma_{\langle 4,3|\bullet\rangle}$. $\square$

Corollary 0.8 $\left\{C^{\overline{\nu}}_{\overline{\lambda},\overline{\mu}}(LG(2,2n))\right\}=\{0,1,2\}$ *and* $\left\{C^{\overline{\nu}}_{\overline{\lambda},\overline{\mu}}(OG(2,2n+1))\right\}=\{0,1,2,4,8\}$.

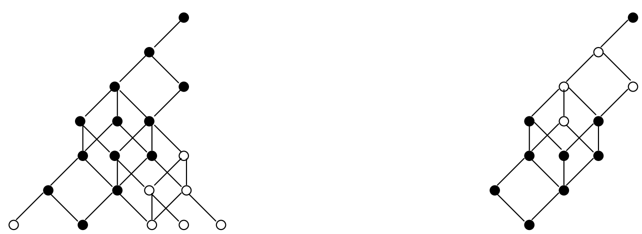
As with the case of $Fl_{1,n-1;n}$, one can describe the nonzero structure constants in terms of a polytope. To do so we make the identification (1).

Corollary 0.9 *Assume* $\lambda=(\lambda_1,\lambda_2),\mu=(\mu_1,\mu_2),\nu=(\nu_1,\nu_2)\subset 2\times\left(\frac{|\Lambda_{G/P}|-1}{2}\right)$ *are partitions and* $\lambda_3,\mu_3,\nu_3\in\{0,1\}$. *Then* $C^{\overline{\nu}}_{\overline{\lambda},\overline{\mu}}(LG(2,2n))\neq 0$ *and* $C^{\overline{\nu}}_{\overline{\lambda},\mu}(OG(2,2n+1))\neq 0$ *if and only if:*

$$\begin{aligned}|\overline{\nu}|&=|\overline{\lambda}|+|\overline{\mu}|\\ \nu_1&\leq\lambda_1+\mu_1\\ \nu_2&\leq\lambda_1+\mu_2\\ \nu_2&\leq\lambda_2+\mu_1\\ \lambda_3+\mu_3&\leq\nu_3\end{aligned}\tag{2}$$

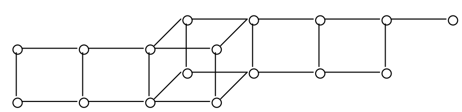
Main theorem for even orthogonal Grassmannians $OG(2,2n)$

The adjoint variety $G/P=OG(2,2n)$ is the space of isotropic 2-planes with respect to a non-degenerate symmetric bilinear form on $\mathbb{C}^{2n}$. It has dimension $|\Lambda_{G/P}|=4n-7$.

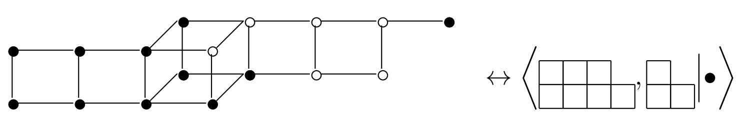


$\Lambda_{OG(2,2n)}$, $\Omega_{SO_{2n}}(\mathbb{C})$ and a shape (for $n=5$)

Here $\Lambda_{G/P}$ is not planar. Consider:

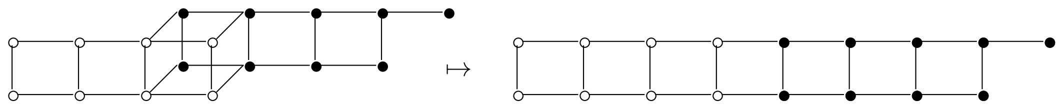
$\Lambda_{OG(2,12)}=$ 

Describe a shape $\overline{\lambda}=\langle\lambda|\bullet/\circ\rangle$ in $\Lambda_{G/P}$ as a triple $\langle\lambda^{(1)},\lambda^{(2)}|\bullet/\circ\rangle$, where $\lambda^{(1)}$ (respectively, $\lambda^{(2)}$) is the Young diagram, in French notation, for the "bottom" (respectively, "top") $2\times\left(\frac{|\Lambda_{G/P}|-1}{4}\right)$ rectangle. For example,



Define $\pi(\lambda)=\lambda^{(1)}+\lambda^{(2)}:=(\lambda_1,\lambda_2)$; the result is a partition inside the $2\times\left(\frac{|\Lambda_{G/P}|-1}{2}\right)$ rectangle.

Consider an auxiliary poset $\Lambda'_{OG(2,2n)}$, a "planarization" of $\Lambda_{OG(2,2n)}$:



$\Lambda_{OG(2,2n)}$ $\qquad\qquad$ $\Lambda'_{OG(2,2n)}$

In the above figure, we have marked the roots of the "top layer" for emphasis.

Shapes of $\Lambda'_{OG(2,2n)}$ are $\overline{\kappa} = \langle \kappa | \bullet/\circ \rangle$ where κ is a partition contained in a $2 \times \left(\frac{|\Lambda_{G/P}|-1}{2}\right)$ rectangle and $\bullet/\circ$ indicates use of the adjoint root in $\Lambda'_{OG(2,2n)}$. Extend π to a map $\Pi : \mathbb{Y}_{OG(2,2n)} \to \mathbb{Y}'_{OG(2,2n)}$ by defining $\Pi(\overline{\lambda}) = \langle \pi(\lambda) | \bullet \rangle$ if $\overline{\lambda}$ is on, and $\Pi(\overline{\lambda}) = \langle \pi(\lambda) | \circ \rangle$ otherwise.

For $\overline{\kappa} \in \mathbb{Y}'_{OG(2,2n)}$, let $\mathrm{fsh}(\overline{\kappa})$ be the number of **fake short roots** used by $\overline{\kappa}$, i.e., the number of roots in the $(n-2)$-th column used by $\overline{\kappa}$. The one exception is that we need $\mathrm{fsh}(\langle n-2, n-2 | \circ \rangle) = 1$. For $\overline{\nu} \in \mathbb{Y}_{OG(2,2n)}$, let $\mathrm{fsh}(\overline{\nu})$ denote $\mathrm{fsh}(\Pi(\overline{\nu}))$.

The map Π is either $1:1$ or $2:1$. In the former case, we identify $\overline{\kappa}$ and $\Pi^{-1}(\overline{\kappa})$. In the latter case, $\Pi^{-1}(\overline{\kappa}) = \{\overline{\kappa}^{\uparrow}, \overline{\kappa}^{\downarrow}\}$ and we call $\overline{\kappa}$ **ambiguous**. Call $\overline{\kappa}^{\uparrow}$ and $\overline{\kappa}^{\downarrow}$ **charged**. If $\overline{\kappa}$ is on (respectively, off), let $\overline{\kappa}^{\downarrow}$ be the shape such that the second part (respectively, first part) of the Young diagram $(\pi^{-1}(\kappa))^{(2)}$ is zero; let $\overline{\kappa}^{\uparrow}$ be the other one. Thus in Example 0.11 below, $\overline{\lambda}$ is up and $\overline{\mu}$ is down. Two charged shapes $\overline{\lambda}$ and $\overline{\mu}$ **match** if their arrows match and are **opposite** otherwise. Let

$$\eta_{\overline{\lambda},\overline{\mu}} = \begin{cases} 2 & \text{if } \overline{\lambda}, \overline{\mu} \text{ are charged and match and } n \text{ is even;} \\ 2 & \text{if } \overline{\lambda}, \overline{\mu} \text{ are charged and opposite and } n \text{ is odd;} \\ 1 & \text{if } \overline{\lambda} \text{ or } \overline{\mu} \text{ are not charged;} \\ 0 & \text{otherwise} \end{cases}$$

Say $\sigma_{\langle \nu | \bullet/\circ \rangle}$, $\sigma_{\langle \nu^{\star} | \bullet \rangle}$ or $\sigma_{\langle \nu_{\star} | \bullet \rangle}$ is zero if ν, $\nu^{\star}$ or $\nu_{\star}$ is not a partition in $2 \times \left(\frac{|\Lambda_{G/P}|-1}{2}\right)$.

Theorem 0.10 *If either $\pi(\lambda)$ or $\pi(\mu)$ equals $(j,0)$ (for some $0 \leq j \leq \frac{|\Lambda_{G/P}|-1}{2}$) then the Schubert expansion of $\sigma_{\overline{\lambda}} \cdot \sigma_{\overline{\mu}} \in H^{\star}(OG(2,2n),\mathbb{Q})$ is obtained by the Pieri rule of [BuKrTa09].*

Otherwise, compute

$$\sigma_{\Pi(\overline{\lambda})} \cdot \sigma_{\Pi(\overline{\mu})} = \sum_{\nu \subseteq \left(\frac{|\Lambda_{G/P}|+1}{2}, \frac{|\Lambda_{G/P}|-1}{2}\right)} C^{\nu}_{\pi(\lambda),\pi(\mu)} \mathbb{A}_{\overline{\lambda},\overline{\mu}}(\nu). \tag{3}$$

(i) Replace any term $\sigma_{\overline{\kappa}}$ that has $\kappa_1 = \frac{|\Lambda_{G/P}|-1}{2}$ by $\eta_{\overline{\lambda},\overline{\mu}} \sigma_{\overline{\kappa}}$

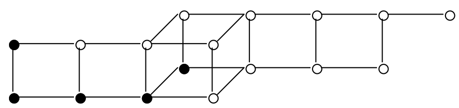
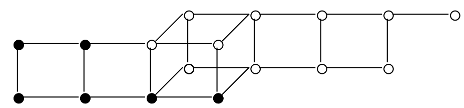
(ii) Next, replace each $\sigma_{\overline{\kappa}}$ by $2^{\mathrm{fsh}(\overline{\kappa}) - \mathrm{fsh}(\overline{\lambda}) - \mathrm{fsh}(\overline{\mu})} \sigma_{\overline{\kappa}}$

(iii) Finally, for any ambiguous $\overline{\kappa}$ replace $\sigma_{\overline{\kappa}}$ by $\frac{1}{2}(\sigma_{\overline{\kappa}^{\uparrow}} + \sigma_{\overline{\kappa}^{\downarrow}})$

The result is a provably integral, and manifestly nonnegative, Schubert basis expansion, which equals $\sigma_{\overline{\lambda}} \cdot \sigma_{\overline{\mu}} \in H^{\star}(OG(2,2n),\mathbb{Q})$.

Integrality is not manifest because of (ii) and (iii); however, it is easy to prove. Rule (i) extends a parity dependency for even-dimensional quadrics, described in [ThYo09]. The point is that the "double tailed diamond" which is $\Lambda_{\mathbb{Q}^{2n-4}}$ sits as a "side" of $\Lambda_{OG(2,2n)}$. Rule (ii) is analogous to our rule for $OG(2, 2n+1)$. Rule (iii) describes how to "disambiguate".

Example 0.11 *We wish to compute* $\sigma_{\overline{\lambda}} \cdot \sigma_{\overline{\mu}} \in H^{\star}(OG(2,12))$ *where:*

$\overline{\lambda} =$  $\overline{\mu} =$ 

Both of these shapes are charged. Here $\pi(\lambda) = (4,1)$ *and* $\pi(\mu) = (4,2)$.

The $\nu \subseteq \left(\frac{|\Lambda_{G/P}|+1}{2}, \frac{|\Lambda_{G/P}|-1}{2}\right) = (9,8)$ *such that* $C^{\nu}_{\pi(\lambda),\pi(\mu)} = 1$ *are* $(8,3), (7,4)$ *and* $(6,5)$. *All other* ν *have* $C^{\nu}_{\pi(\lambda),\pi(\mu)} = 0$. *Thus we compute*

$$
\begin{aligned}
\sigma_{\Pi(\overline{\lambda})} \cdot \sigma_{\Pi(\overline{\mu})} &= \mathbb{A}_{\overline{\lambda},\overline{\mu}}(8,3) + \mathbb{A}_{\overline{\lambda},\overline{\mu}}(7,4) + \mathbb{A}_{\overline{\lambda},\overline{\mu}}(6,5) \\
&= (\langle 7,3|\bullet\rangle + \langle 8,2|\bullet\rangle) + (\langle 6,4|\bullet\rangle + \langle 7,3|\bullet\rangle) + (\langle 5,5|\bullet\rangle + \langle 6,4|\bullet\rangle) \\
&= \langle 8,2|\bullet\rangle + 2\langle 7,3|\bullet\rangle + 2\langle 6,4|\bullet\rangle + \langle 5,5|\bullet\rangle \\
&\mapsto 0\langle 8,2|\bullet\rangle + 2\langle 7,3|\bullet\rangle + 2\langle 6,4|\bullet\rangle + \langle 5,5|\bullet\rangle \qquad \textit{(by (i) and } \eta_{\overline{\lambda},\overline{\mu}} = 0\textit{)} \\
&\mapsto \langle 7,3|\bullet\rangle + 2\langle 6,4|\bullet\rangle + \langle 5,5|\bullet\rangle \qquad \textit{(by (ii) and } \mathrm{fsh}(\overline{\lambda}) = \mathrm{fsh}(\overline{\mu}) = 1\textit{)}
\end{aligned}
$$

Finally, (iii) applies to the ambiguous shape $\langle 6,4|\bullet\rangle$. *Hence we conclude:*

$$\sigma_{\overline{\lambda}} \cdot \sigma_{\overline{\mu}} = \langle 7,3|\bullet\rangle + (\langle 6,4|\bullet\rangle^{\uparrow} + \langle 6,4|\bullet\rangle^{\downarrow}) + \langle 5,5|\bullet\rangle.$$

Each step is nonnegative and integral, in agreement with our theorem. □

Corollary 0.12 $\left\{C^{\overline{\nu}}_{\overline{\lambda},\overline{\mu}}(OG(2,2n))\right\} = \{0,1,2,4,8\}$.

We make the following identifications; cf. (1):

$$\Pi(\overline{\lambda}) = \langle \lambda_1, \lambda_2|\circ\rangle \text{ with } (\lambda_1,\lambda_2,0) \in \mathbb{Z}^3 \text{ and } \Pi(\overline{\lambda}) = \langle \lambda_1,\lambda_2|\bullet\rangle \text{ with } (\lambda_1,\lambda_2,1) \in \mathbb{Z}^3.$$

As with the other cases, we can give a criterion for nonzeroness:

Corollary 0.13 *If either* $\pi(\lambda)$ *or* $\pi(\mu)$ *equals* $(j, 0)$ *(for some* $0 \le j \le \frac{|\Lambda_{G/P}|-1}{2}$*) then nonzeroness of* $C^{\overline{\nu}}_{\overline{\lambda},\overline{\mu}}(OG(2,2n))$ *is determined by the Pieri rule of [BuKrTa09].*

If $\nu_1 = \frac{|\Lambda_{G/P}|-1}{2}$ *then* $C^{\overline{\nu}}_{\overline{\lambda},\overline{\mu}} \neq 0$ *if and only if* $\eta_{\overline{\lambda},\overline{\mu}} \neq 0$ *and the inequalities (2) hold.*

Otherwise, assume $(\lambda_1,\lambda_2), (\mu_1,\mu_2), (\nu_1,\nu_2) \subset 2 \times \left(\frac{|\Lambda_{G/P}|-1}{2}\right)$ *are partitions and* $\lambda_3, \mu_3, \nu_3 \in \{0,1\}$. *Then* $C^{\overline{\nu}}_{\overline{\lambda},\overline{\mu}}(OG(2,2n)) \neq 0$ *if and only if the inequalities (2) hold.*

Concluding remarks

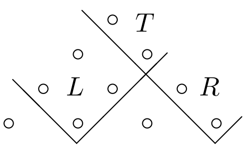
In [SeYo13+] we prove that our rules define an associative ring and agree with known Pieri rules [PrRa96, BuKrTa09]. This implies the claimed results. The remaining cases of classical type are easy.

We will also computationally analyze (co)adjoint Schubert calculus in all the exceptional Lie types. This is achieved by using the cohomology ring presentation of [ChPe11]. In this manner, we can extend the shortroots relationship between cominuscule/adjoint varieties with their minuscule/coadjoint "partners", give information about which numbers appear, and describe which structure constants are nonzero.

In [Se13+], RYDs are used to give a new rule for the Belkale-Kumar coefficients in type A (after [KnPu11]). This rule explains why these Schubert problems are special, from the perspective of RYDs.

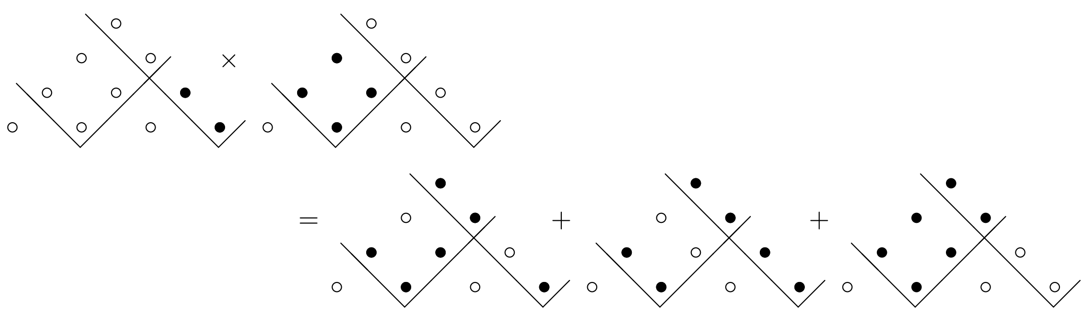
RYDs can also be applied to the study of Kazhdan-Lusztig polynomials; see [WoYo13+].

The "adjoint jumping" that is encoded by "otherwise" case of the operator $\mathbb{A}_{\overline{\lambda},\overline{\mu}}$ appears more generally. A small example: in type A_{n-1} we may take $G/P = Fl_{2,4;5}$. The parabolic subgroup is associated to roots 2 and 4 of the Dynkin diagram of A_4. Thus, $\Lambda_{Fl_{2,4;5}}$ is the overlay of $\Lambda_{Gr_2(\mathbb{C}^5)}$ and $\Lambda_{Gr_4(\mathbb{C}^5)}$. This is depicted below, where one naturally splits $\Lambda_{Fl_{2,4;5}}$ into the three regions "L", "R" and "T".



Schubert classes in $H^\star(Fl_{2,4;5}, \mathbb{Q})$ are indexed by inversion sets $\overline{\lambda}$ consisting of (ordinary) Young diagrams sitting in each region. However, not all such collections of Young diagrams are inversion sets for some permutation. In this example, consider a root β in the top region, and look at the "hook" of roots $\alpha \in \Lambda_{Fl_{2,4;5}}$ that appear diagonally southwest and southeast of β and in region L or R (but not T). A condition just like the one for adjoint varieties appears: β must be used if strictly more than half of these roots α are used by $\overline{\lambda}$. Also β cannot be used if strictly less than half of the roots α are used.

Example 0.14 $\sigma_{12|45|3} \cdot \sigma_{34|12|5} = \sigma_{35|14|2} + \sigma_{34|25|1} + \sigma_{45|12|3} \in H^\star(Fl_{2,4;5}, \mathbb{Q})$. *Pictorially:*



Although $\overline{\lambda}$ and $\overline{\mu}$ use no roots in region T, any $\overline{\nu}$ that appears must use roots in that region. □

Clearly, $\Lambda_{GL_n/P}$ is planar for any GL_n/P. Thus the complexity of this case comes (in part) from finding the analogue of "adjoint jumping". Also, planarity suggests that as a whole, type A Schubert problems are easier than type D problems. If nothing else, this agrees with practical experience.

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Poset vectors and generalized permutohedra (extended abstract)

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Abstract. We show that given a poset P and and a subposet Q, the integer points obtained by restricting linear extensions of P to Q can be explained via integer lattice points of a generalized permutohedron.

Résumé. Nous montrons que, étant donné un poset P et un subposet Q, les points entiers obtenus en restreignant les extensions linéaires de P à Q peuvent être expliqués par les points entiers d'un permutohedron généralisé.

Keywords: Poset, linear extension, associahedron, generalized permutohedra, polytope, integer lattice points

1 Introduction

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition with at most n parts. The ***Young diagram*** of shape λ is the set

$$D_\lambda = \{(i,j) \in \mathbb{N}^2 | 1 \le j \le n, 1 \le i \le \lambda_j\}.$$

A ***Standard Young tableau*** is a bijective map $T : D_\lambda \to \{1, \ldots, |D_\lambda|\}$ which is increasing along rows and down columns, i.e. $T(i,j) < T(i,j+1)$ and $T(i,j) < T(i+1,j)$ Stanley (2000). Standard Young tableaus of $\lambda = (2,2,1)$ is given in Figure 1. In each tableau, the entries at boxes $(1,2)$ and $(2,2)$ are colored with red. The pairs we get from each tableau, are exactly the integer lattice points of a pentagon in the right image of Figure 1. Then one could naturally ask the following question : If we choose some arbitrary boxes inside a Young diagram, and collect the integer vectors we get from the chosen boxes for each standard Young diagram, are they the integer lattice points of some polytope?

Such questions were studied for diagonal boxes of shifted Young diagrams by the first author and the third author in Postnikov (2009) and Croitoru (2008). Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition with at most n parts. The ***shifted Young diagram*** of shape λ is the set

$$SD_\lambda = \{(i,j) \in \mathbb{N}^2 | 1 \le j \le n, j \le i \le n + \lambda_j\}.$$

We think of SD_λ as a collection of boxes with $n + 1 - i - \lambda_i$ boxes in row i, such that the leftmost box of the i-th row is also in the i-th column. A ***shifted standard Young tableau*** is a bijective map $T : SD_\lambda \to$

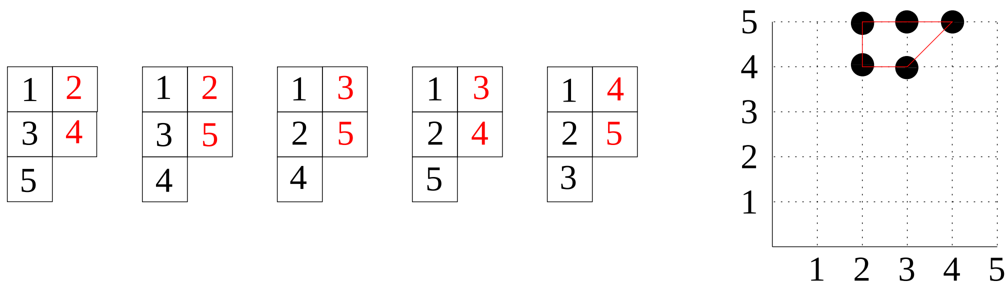

Fig. 1: Example of standard Young tableaus of $\lambda = (2, 2, 1)$, and pairs of entries that occur at $(1, 2)$ and $(2, 2)$ inside the tableaus.

1	2	3	6	7	11
	4	5	8		
		9	10		

Fig. 2: Example of a shifted standard Young tableau, which has diagonal vector $(1, 4, 9)$.

$\{1, \ldots, |SD_\lambda|\}$ which is increasing along rows and down columns, i.e. $T(i, j) < T(i, j+1)$ and $T(i, j) < T(i + 1, j)$. The ***diagonal vector*** of such a tableau T is $Diag(T) = (T(1, 1), T(2, 2), \ldots, T(n, n))$.

Figure 2 is a shifted standard Young tableau for $n = 3, \lambda = (3, 1, 1)$. In Postnikov (2009), the third author showed that when $\lambda = (0, \ldots, 0)$, the diagonal vectors of $SD_\emptyset$ are in bijection with lattice points of $(n-1)$-dimensional ***associahedron*** Ass_{n-1}. Extending this result, the first author, in Croitoru (2008), showed that the diagonal vectors of SD_λ in general, are in bijection with lattice points of a certain deformation of the associahedron.

In this paper, we generalize the previous question for Young diagrams and the previous results for shifted Young diagrams, by looking at an arbitrary poset P in general. More precisely, given an arbitrary poset P, a ***linear extension*** is an order preserving bijection $\sigma : P \to [|P|]$, where $[n]$ is defined to be the set of integers $\{1, \ldots, n\}$. Let Q be a subposet of P and label the elements of Q by $q_1, \ldots, q_{|Q|}$, such that if $q_i < q_j$ in Q, then $i < j$. We call a vector $(\sigma(q_1), \sigma(q_2), \ldots, \sigma(q_{|Q|}))$ obtained in such manner as the (P, Q)***-subposet vector***.

Figure 3 is a poset P with 10 elements. The elements of Q are colored red. We label the elements of Q by q_1, q_2, q_3 by starting from the lowest element going up. Then the (P, Q)-subposet vector we get in this case is $(1, 5, 7)$.

When we are only dealing with the linear extensions of P (when $P = Q$), the connection between linear extensions of posets and generalized permutohedra has been studied in Postnikov et al. (2008) and Morton et al. (2009). In particular, when Q is a chain, we will show that the (P, Q)-subposet vectors are in bijection with a certain deformation of the associahedron (generalized permutohedron). In the general case, we will show that the set of (P, Q)-subposet vectors can be thought as lattice points of a non-convex polytope, obtained by gluing the generalized permutohedra. In section 2, we will go over the basics of

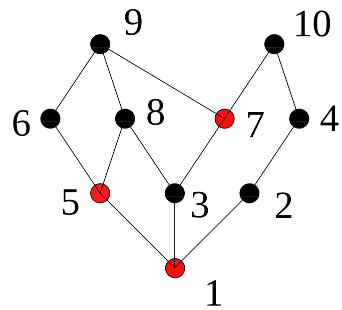

Fig. 3: Example of poset P and a subposet Q, where elements of Q are colored red.

generalized permutohedra, using generalized permutohedra language. In section 3, we will study the case when Q is a chain. In section 4, we will go over the general case when Q is a general subposet of P. In section 5, we give a nice combinatorial method to describe the vertices of the constructed polytope. In this extended abstract, all proofs except the proof of the main theorem will be omitted.

2 Generalized Permutohedron

In this section, we will give an introduction to the ***associahedron*** using ***generalized permutohedra*** language from Postnikov (2009).

Associahedron, also known as the ***Stasheff polytope***, first appeared in the work of Stasheff (1963). Given n letters, think of inserting opening and closing parentheses so that each opening parenthesis is matched to a closing parentheses. Then the associahedron is the convex polytope in which each vertex corresponds to a way of correctly inserting the parentheses, and the edges correspond to single application of the associativity rule. But since we will be working with the integer lattice points of certain realization of an associahedron, we are going to be using a different definition using ***generalized permutohedra***.

The ***permutohedron*** is the polytope obtained by the convex hull of vertices which are formed by permuting the coordinates of the vector $(1, 2, \ldots, n)$. Hence the vertices correspond to each permutation of S_n, and the edges correspond to applying a transposition. The ***generalized permutohedra***, which was introduced in Postnikov (2009), are polytopes that can be obtained by moving vertices of the usual permutohedron so that directions of all edges are preserved. It happens that for a certain class of generalized permutohedra, we can construct them using Minkowski sum of certain simplices.

Let $\Delta_{[n]} = ConvexHull(e_1, \ldots, e_n)$ be the standard coordinate simplex in $\mathbb{R}^n$. For a subset $I \subset [n]$, let $\Delta_I = ConvexHull(e_i | i \in I)$ denote the face of $\Delta_{[n]}$. When $\mathcal{I} = (I_1, \ldots, I_t)$ where I_i's are subsets of $[n]$, we denote $G_{\mathcal{I}}$ to be the Minkowski sum of Δ_{I_i}'s. In other words, we have:

$$G_{\mathcal{I}} := \Delta_{I_1} + \cdots + \Delta_{I_t}.$$

Since the I_i's do not have to be distinct, we could re-write the above sum as

$$G_{\mathcal{I}} := c_1 \Delta_{I_1} + \cdots + c_m \Delta_{I_m},$$

where c_i counts the number of times I_i occurs among $\mathcal{I}$.

For convenience, unless otherwise stated, whenever we use the word generalized permutohedra, we will be referring to the class of polytopes that can be obtained via the construction above. Below are well known cases of generalized permutohedra. For more details, check Section 8 of Postnikov (2009).

Permutohedron : If we set $\mathcal{I}$ to consist of all possible nonempty subsets of $[n]$, then $G_{\mathcal{I}}$ is combinatorially equivalent to the usual permutohedron obtained by permuting the entries of point $(1, \dots, n)$.

Associahedron : If we set $\mathcal{I}$ to consist of all possible intervals of $[n]$ (so that $[i, j] := \{i, i+1, \dots, j\}$ is in $\mathcal{I}$ for all pairs $i < j$), then $G_{\mathcal{I}}$ is combinatorially equivalent to the associahedron.

In this paper, we will mainly be dealing with generalized permutohedra, that can be obtained from the associahedron by deforming the facets.

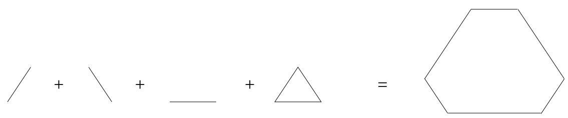

Fig. 4: The permutohedron $G_{(\{1,2\},\{1,3\},\{2,3\},\{1,2,3\})}$.

Figure 4 shows an example of a permutohedron constructed by summing up all subsets of $[3]$. The terms $\Delta_{\{1\}}, \Delta_{\{2\}}, \Delta_{\{3\}}$ are omitted since summing points just corresponds to the translation of the polytope.

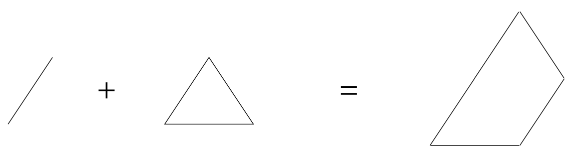

Fig. 5: A deformed associahedron $G_{(\{1,2\},\{1,2,3\})}$.

Figure 12 shows an example of an associahedron constructed by summing $\Delta_{\{1,2\}}, \Delta_{\{2,3\}}$ and $\Delta_{\{1,2,3\}}$. Figure 5 shows an example of a deformed associahedron constructed by summing $\Delta_{\{1,2\}}$ and $\Delta_{\{1,2,3\}}$. One can notice that the polytope in Figure 5 can be obtained from the polytope in Figure 4 or the polytope in Figure 12 by moving around the facets.

Lemma 2.1 (Postnikov (2009) Proposition 6.3) *Let $G_{\mathcal{I}}$ be a generalized permutohedron given by $c_1\Delta_{I_1} + \cdots + c_m\Delta_{I_m}$, where all c_i's are positive integers. Then this polytope has the form $\{(t_1, \dots, t_n) \in \mathbb{R}^n | \sum t_i = z_{[n]}, \sum_{i \in I} t_i \leq z_I, \forall I\}$, where $z_I := \sum_{I_j \cap I \neq \emptyset} c_j$.*

The above lemma allows us to obtain the defining hyperplanes of a generalized permutohedron. For example, if we look at $\Delta_{\{1,2\}} + \Delta_{\{1,2,3\}}$, then this polytope is the collection of points (t_1, t_2, t_3) in $\mathbb{R}^3$ such that:

- $t_1 + t_2 + t_3 = 2$,
- $t_1 + t_2 \leq 2, t_1 + t_3 \leq 2, t_2 + t_3 \leq 2$ and
- $t_1 \leq 2, t_2 \leq 2, t_3 \leq 1$.

3 When Q is a chain

Our goal in this section is to study the (P, Q)-subposet vectors when Q is a chain of P. More precisely, we will show that there is a bijection between (P, Q)-subposet vectors and integer lattice points of a certain generalized permutohedron constructed from the pair (P, Q). Given a (P, Q)-subposet vector $(c_1, \ldots, c_r)$, we are going to look at the vector $(c_1, c_2 - c_1, \ldots, c_r - c_{r-1}, |P| - c_r)$. We define $M_{P,Q}$ to be the collection of such vectors.

Let us denote the elements of Q as $q_1, \ldots, q_r$ such that $q_1 < \cdots < q_r$ in P.

Remark 3.1 *We are going to add a minimal element $\hat{0}$ and a maximal element $\hat{1}$ to P. This does not change the structure of $M_{P,Q}$ or (P, Q)-subposet vectors, since all linear extensions would assign same numbers to $\hat{0}$ and $\hat{1}$. We will denote $\hat{0}$ and $\hat{1}$ as q_0 and q_{r+1} for technical convenience.*

The following is another way to think of vectors of $M_{P,Q}$. An ***order ideal*** of P is a subset I of P such that if $x \in I$ and $y \le x$, then $y \in I$. Now given a linear extension $\sigma : P \to [|P|]$, we define the order ideals J_i to be the collection of elements $p \in P$ such that $\sigma(p) \le \sigma(q_i)$ for $0 \le i \le r+1$. If we define I_i to be $J_i \setminus J_{i-1}$ for $1 \le i \le r+1$, then $(|I_1|, \ldots, |I_{r+1}|)$ is an element of $M_{P,Q}$. Also, any element $(c_1, \ldots, c_{r+1})$ of $M_{P,Q}$ would actually come from some linear extension σ and its corresponding sequence of order ideals $J_0 \subset J_1 \subset \cdots \subset J_{r+1}$.

We define the subset $B_{i,i}$ as:

$$B_{1,1} := \{p \in P | q_0 \le p \le q_1\},$$
$$B_{i,i} := \{p \in P | q_{i-1} < p \le q_i\}, i \ne 1.$$

For $i < j$, we define the set $B_{i,j}$ as:

$$B_{i,j} := \{p \in P | q_{i-1} < p < q_j, q_i \not< p, p \not< q_{j-1}\}.$$

Then we get a decomposition of P into $B_{i,j}$'s for $1 \le i \le j \le r+1$. Let us define the generalized permutohedron $N_{P,Q}$ as:

$$N_{P,Q} := \sum_{1 \le i \le j \le r+1} |B_{i,j}| \Delta_{[i,j]}.$$

Lemma 3.2 *Every integer lattice point of $N_{P,Q}$ is a member of $M_{P,Q}$.*

Proof: Let $p = (p_1, \ldots, p_{r+1})$ be an integer lattice point of $N_{P,Q}$. By proposition 14.12 of Postnikov (2009), p is the sum of $p_{[i,j]}$'s, where each $p_{[i,j]}$ is an integer lattice point of $|B_{i,j}|\Delta_{[i,j]}$. Each $p_{[i,j]}$ can be expressed as $\sum_{k \in [i,j]} b_{i,j,k} e_k$, where $b_{i,j,k}$'s are nonnegative integers such that $\sum_k b_{i,j,k} = |B_{i,j}|$. We then decompose the set $B_{i,j}$ into $B_{i,j,k}$'s such that:

1. for any c and d such that $i \le c < d \le j$, all elements of $B_{i,j,c}$ are smaller than any element of $B_{i,j,d}$ in P and,
2. cardinality of each $B_{i,j,k}$ is given by $b_{i,j,k}$.

Since $p_{[i,j]} = \sum_{k \in [i,j]} b_{i,j,k} e_k$, we have $p = \sum_k \sum_{i,j} b_{i,j,k} e_k$. This tells us that $p_k = \sum_{i,j} b_{i,j,k}$ for all k from 1 to $r+1$. We define the set I_k to be the union of $B_{i,j,k}$'s for all possible i and j's. If $\{\hat{0}\} \subset I_1 \subset I_1 \cup I_2 \subset \cdots \subset I_1 \cup \cdots \cup I_{r+1} = P$ is a chain of order ideals, then we know that $p = (|I_1|, \ldots, |I_r|, |I_{r+1}|)$ is a member of $M_{P,Q}$, due to the argument just after Remark 3.1.

So we need to show that there is some way to decompose P into $B_{i,j,k}$'s such that $I_1, I_1 \cup I_2, \ldots, I_1 \cup \cdots \cup I_{r+1}$ are order ideals of P. In other words, for any pair (x,y) such that $x \in I_k$ and $y \in I_{k'}$ for $k > k'$, we must have $x \not< y$ in P.

For the sake of contradiction, let us assume we do have elements $x \in B_{i,j,k}$ and $y \in B_{i',j',k'}$ such that $k > k'$ but $x < y$ in P. Let us call such pair (x,y) an ***inversion***. Looking at all inversion pairs, construct a collection $\mathcal{C}$ by collecting all (x,y)'s such that $k - k'$ is minimal. And among the pairs of $\mathcal{C}$, find a pair (x,y) such that there does not lie a z such that $(z,x) \in \mathcal{C}$ or $(y,z) \in \mathcal{C}$. Now let us show that we can switch x and y : to put x in $B_{i,j,k'}$ and y in $B_{i',j',k}$ without introducing any new inversions.

We first need to show that $k, k' \in [i,j] \cap [i',j']$. The fact that $x \in B_{i,j,k}, y \in B_{i',j',k'}$ tells us that:

- $q_{i-1} < x \leq q_j$,
- $q_{i'-1} < y \leq q_{j'}$,
- $k \in [i,j]$ and
- $k' \in [i',j']$.

We also get $q_{i-1} \leq q_{i'-1}$ and $q_j \leq q_{j'}$ from $x < y$ and the definition of $B_{i,j}$ and $B_{i',j'}$. Hence we have $i \leq i'$ and $j \leq j'$. Then $k > k'$ allows us to conclude that $k, k' \in [i',j'] \cap [i,j]$.

Next, we are going to show that this switch does not introduce any new inversions. Assume for the sake of contradiction that we get a new inversion (z,x) (The proof for (y,z) case is also similar and will be omitted). Since (z,x) wasn't an inversion before the switch, we have $z < x$ and z has to be in some $I_{k''}$ where $k \geq k'' > k'$. But since $z < x$ implies $z < y$, the minimality of $k - k'$ tells us that $k'' = k$. This implies $(z,y) \in \mathcal{C}$, which contradicts the condition for our choice of (x,y).

By repeating the switching process, we can get $I_1, \ldots, I_{r+1}$ such that $I_1, I_1 \cup I_2, \ldots, I_1 \cup \cdots \cup I_{r+1}$ are all order ideals. This switching process does not change the cardinality of any I_i for $1 \leq i \leq r+1$, so $|I_i| = p_i$ for all $1 \leq i \leq r+1$. Hence we get the desired result that $p \in M_{P,Q}$.

□

Theorem 3.3 *The collection $M_{P,Q}$ is exactly the set of integer lattice points of the generalized permutohedron $N_{P,Q}$.*

Since each (P,Q)-subposet vector $(c_1, \ldots, c_r)$ corresponds to a point $(c_1, c_2 - c_1, \ldots, c_r - c_{r-1}, |P| - c_r)$ of $M_{P,Q}$, the above theorem allows us to conclude that:

Corollary 3.4 *When P is a poset and Q is a chain in P, the collection of (P,Q)-subposet vectors are in bijection with integer lattice points of the generalized permutohedron $N_{P,Q}$.*

Actually, we can say a bit more about (P,Q)-subposet vectors. Let us define Δ'_I to be the simplex obtained by sending each point $(x_1, \cdots, x_{r+1})$ of a simplex Δ_I in $\mathbb{R}^{r+1}$ to $(x_1, x_1 + x_2, \ldots, x_1 + \cdots + x_r)$ in $\mathbb{R}^r$. In other words,

- if $r+1 \notin I$, then Δ'_I is the convex hull of $e_1 + \cdots + e_i$'s for $i \in I$ and,
- if $r+1 \in I$, then Δ'_I is the convex hull of the origin and $e_1 + \cdots + e_i$'s for $i \in I \setminus \{r+1\}$.

Then we can describe the set of (P,Q)-subposet vectors more precisely:

Corollary 3.5 *The (P, Q)-subposet vectors are exactly the integer lattice points of the polytope*

$$N'_{P,Q} := \sum_{1 \le i \le j \le r+1} |B_{i,j}| \Delta'_{[i,j]}$$

.

Let us end with an example. Consider a poset P given by Figure 6. The chain Q is chosen as the elements labeled a and b. We label $\hat{0} = q_0, a = q_1, b = q_2, \hat{1} = q_3$. If we restrict all possible linear extensions of P to q_1 and q_2, we get integer vectors $(2,4), (2,5), (3,4), (3,5), (4,5)$. These points are exactly the (P, Q)-subposet vectors. We have $\hat{0}, a \in B_{1,1}, b \in B_{2,2}, \hat{1} \in B_{3,3}, y \in B_{1,2}, z \in B_{1,3}$. So $N'_{P,Q} = 2\Delta_{[1]} + 1\Delta_{[2]} + \Delta_{[3]} + \Delta_{[1,2]} + \Delta_{[1,3]}$ and this gives us a pentagon, where all integer lattice points are exactly the elements of $M_{P,Q}$.

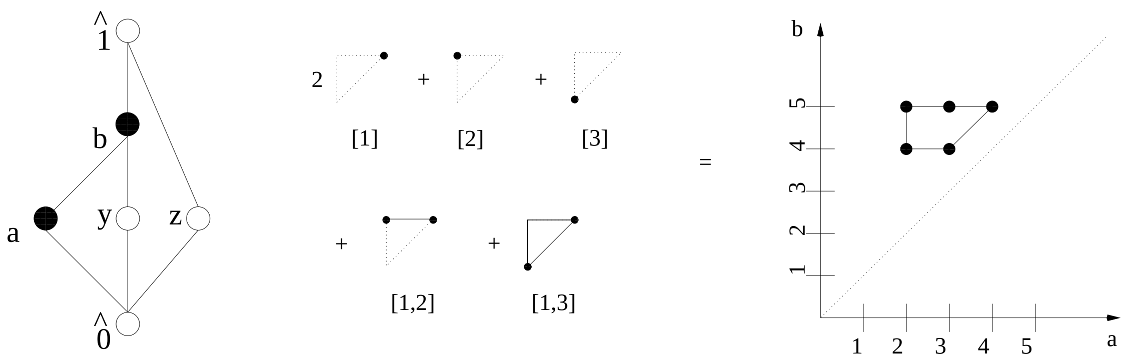

Fig. 6: A poset P and chain Q given by the black-colored elements. The summands of $N'_{P,Q}$ and the polytope $N'_{P,Q}$ with its integer lattice points.

4 For Q in general

In this section, we are going to study the (P, Q)-subposet vectors when Q is not necessarily a chain of P. The (P, Q)-subposet vectors are integer lattice points in a union of polytopes combinatorially equivalent to generalized permutohedron. Then we are going to show that there is a nonconvex, contractible polytope that can be obtained by gluing those polytopes nicely.

We will start with an example in Figure 7. One can notice that the points are grouped into two parts, depending on which of a or b is bigger. Let us add the relation $a > b$ to P and Q respectively to get P_1 and Q_1. Similarly, let us add the relation $a < b$ to P and Q respectively to get P_2 and Q_2. Then as one can see from Figure 8, one group of points of (P, Q)-subposet vectors come from (P_1, Q_1)-subposet vectors and the other comes from (P_2, Q_2)-subposet vectors.

If we look at the line $x_a = x_b$ in Figure 8, the nearby faces of N'_{P_1,Q_1} and N'_{P_2,Q_2} look identical. More precisely, the intersection of N'_{P_1,Q_1} with $x_a - x_b = 1$ and the intersection of N'_{P_2,Q_2} with $x_a - x_b = -1$ looks identical. And that face looks exactly like N'_{P_3,Q_3} where P_3 and Q_3 are obtained from P and Q by identifying a and b, as in Figure 9.

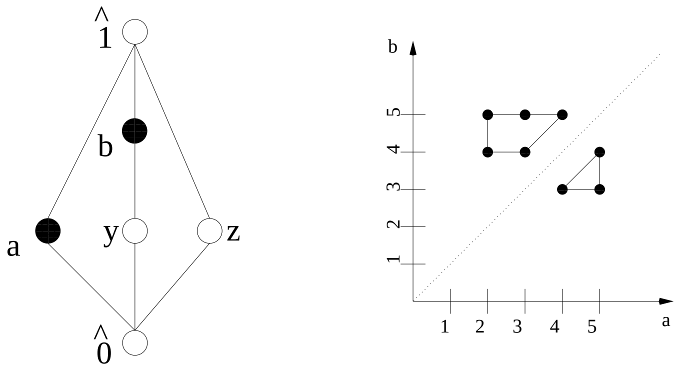

Fig. 7: A poset P and subposet Q given by the black-colored elements. (P, Q)-subposet vectors are drawn on the right.

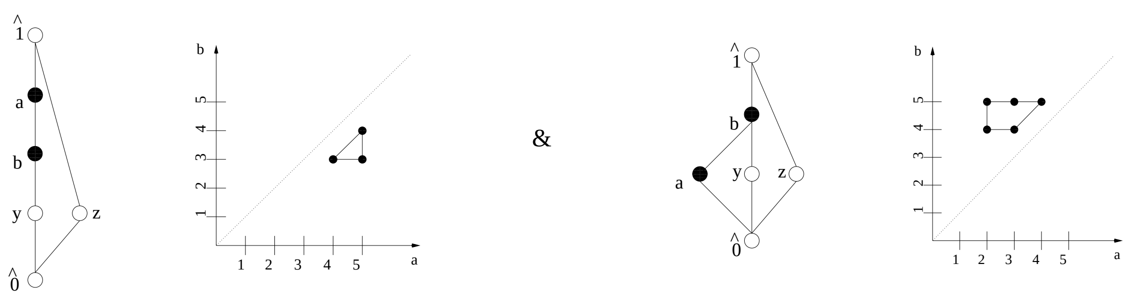

Fig. 8: P_1 and Q_1 are obtained from P and Q by adding $a > b$. P_2 and Q_2 are obtained from P and Q by adding $a < b$.

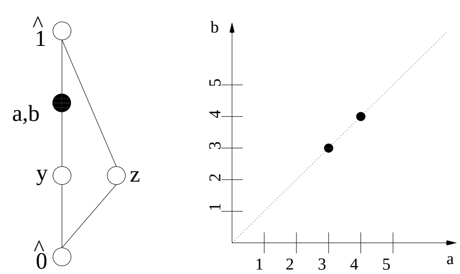

Fig. 9: P_3 and Q_3 are obtained from P and Q by identifying a and b.

This suggests that we can glue together N'_{P_1,Q_1} and N'_{P_2,Q_2} along N'_{P_3,Q_3}. We translate N'_{P_1,Q_1} by negating 1 from x_a and N'_{P_2,Q_2} by negating 1 from x_b. Then we get a polyhedra as in Figure 10, which we will call the ***posetohedron*** of the pair (P, Q).

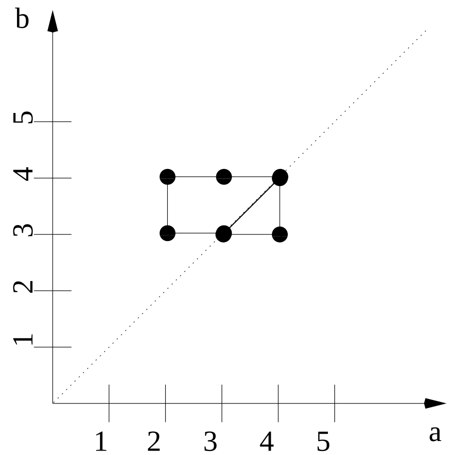

Fig. 10: N'_{P_1,Q_1} and N'_{P_2,Q_2} glued together.

Now we will describe the above procedure formally. Recall that we denote the elements of Q by $q_1, \dots, q_r$, by choosing some linear extension on Q. We are going to associate a hyperplane arrangement given by hyperplanes $x_i - x_j = 0$ for all pairs $1 \le i \le j \le r$, and denote this by $\mathcal{A}_Q$. Each chamber in $\mathcal{A}_Q$ corresponds to an ordering $x_{w(1)} < \cdots < x_{w(r)}$ where $w \in S_r$. So from now on, we will identify the chambers with their corresponding permutations. We will say that w is ***valid*** if $q_{w(1)} < \cdots < q_{w(r)}$ is a valid total ordering of elements of Q. For (P, Q)-subposet vectors coming from linear extensions compatible with the ordering $q_{w(1)} < \cdots < q_{w(r)}$, we denote them (P, Q, w)-subposet vectors.

Then the set of (P, Q)-subposet vectors is just the disjoint union of (P, Q, w)-subposet vectors for all $w \in S_r$, since (P, Q, w)-subposet vectors lie in the interior of chamber w. If we add the relation $q_{w(1)} < \cdots < q_{w(r)}$ to P and Q to get P_w and Q_w respectively, (P, Q, w)-subposet vectors are exactly the integer lattice points of N'_{P_w,Q_w}. We will call such polytope as a ***block***.

We want to show that if we translate each block N'_{P_w,Q_w} by $-\sum_i (i-1)e_{w(i)}$, we get a polytopal complex. In other words, we want to show that under this translation, the blocks glue nicely, especially that the intersection of any collection of blocks is a common face of all such blocks.

We start with an obvious property.

Lemma 4.1 *Let w and v be two different permutations. The translated blocks $N'_{P_w,Q_w} - \sum_i (i-1)e_{w(i)}$ and $N'_{P_v,Q_v} - \sum_i (i-1)e_{v(i)}$ have disjoint interiors.*

We now check that a nonempty intersection of any collection of blocks is a common face of the blocks. We define a ***face*** of $\mathcal{A}_Q$ to be a face of one of the polyhedron, obtained by taking the closure of a chamber. The faces of $\mathcal{A}_Q$ are in bijection with the faces of a permutohedron under duality. The intersection of some given set of translated blocks happen inside a face of $\mathcal{A}_Q$. Fix a face F of $\mathcal{A}_Q$ and we will use $\mathcal{C}_F$

to denote the set of chambers whose closure contains F. A face F which has dimension d corresponds to an ordered partition of $[r]$ into d parts according to Ziegler (1995). To be more precise, F of dimension d corresponds to some ordered partition $\Pi = (\Pi_1, \ldots, \Pi_d)$ and this translates to a partial ordering where $a < b$ if $a \in \Pi_i, b \in \Pi_j$ for $i < j$. The chambers of $\mathcal{C}_F$ are chambers corresponding to the total order compatible with this partial order.

By reordering the coordinates, we may assume that F corresponds to the ordered partition $([1..i_1], [i_1+1..i_1+i_2], \ldots, [i_1+\cdots+i_{d-1}+1..i_1+\cdots+i_d])$. Then each chamber of $\mathcal{C}_F$ correspond to a total ordering $w_1(1) < \cdots < w_1(i_1) < w_2(i_1+1) < \cdots < w_2(i_1+i_2) < \cdots < w_d(i_1+\cdots+i_d)$ where $w_k \in S_{i_k}$ for each $1 \leq k \leq d$.

First by showing that the blocks glue nicely when F is a facet of $\mathcal{A}_Q$, we can extend that to show that all the blocks glue nicely. Hence we get the following result:

Proposition 4.2 *The collection of translated blocks $N'_{P_w,Q_w} - \sum_i (i-1)e_{w(i)}$'s form a polytopal complex.*

We will call the support of the polytopal complex the (P,Q)-posetohedron. When Q is not a chain, then (P,Q)-posetohedron is not convex in general, as one can see from Figure 11.

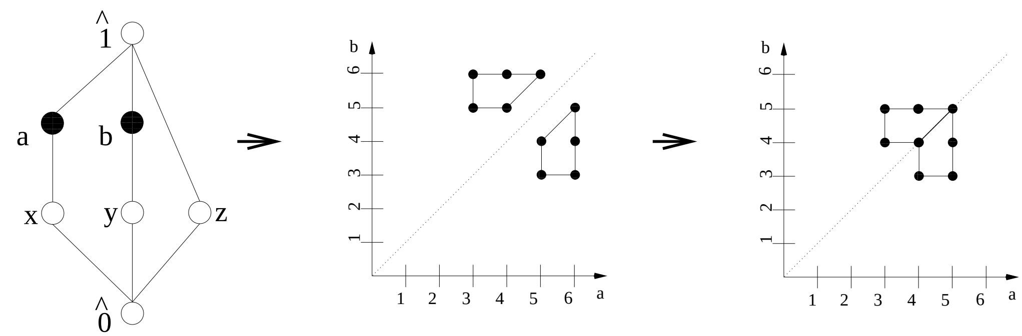

Fig. 11: Example of a posetohedron that is not convex.

Proposition 4.3 *The (P,Q)-posetohedron is contractible.*

Hence we get a non-convex polytope from (P,Q)-subposet vectors, and this polytope turns out to be contractible.

Problem 4.4 *Is there some interesting topological property of a (P,Q)-posetohedron that depends on the combinatorics of P and Q?*

5 Describing the vertices of a posetohedron

In this section we use the machinery from Postnikov (2009) to give a description for the vertices of $N_{P,Q}$. Since the general case is obtained by gluing the posetohedra when Q is a chain, we will restrict ourselves to when $Q = \{\hat{0} = q_0 < q_1 < \ldots < q_{r+1} = \hat{1}\}$ is a chain.

Recall that a generalized permutohedron can be expressed by $\sum_{1 \leq i,j \leq r+1} c_{i,j} \Delta_{[i,j]}$, where $c_{i,j}$ are nonnegative integers. In case when $c_{i,j} > 0$ for all i and j, the vertices of of the polytope are in bijection with plane binary trees on $[r+1]$ with the ***binary search labeling*** Knuth (1998). Binary search labeling is the unique labeling of the tree nodes such that the label of any node is greater than that of any left descendant, and less than that of any right descendant. Let T be such a binary tree, and identify any of its nodes with its labeling. Extending Corollary 8.2 of Postnikov (2009), we get:

Lemma 5.1 *The vertex $v_T = (t_1, ..., t_{r+1})$ of a generalized permutohedron $\sum_{1\le i,j\le r+1} c_{i,j}\Delta_{[i,j]}$ is given by*

$$t_k = \sum_{l_k \le i \le k \le j \le r_k} c_{i,j},$$

where l_k, r_k are such that the interval $[l_k, r_k]$ is exactly the set of descendants of k in T.

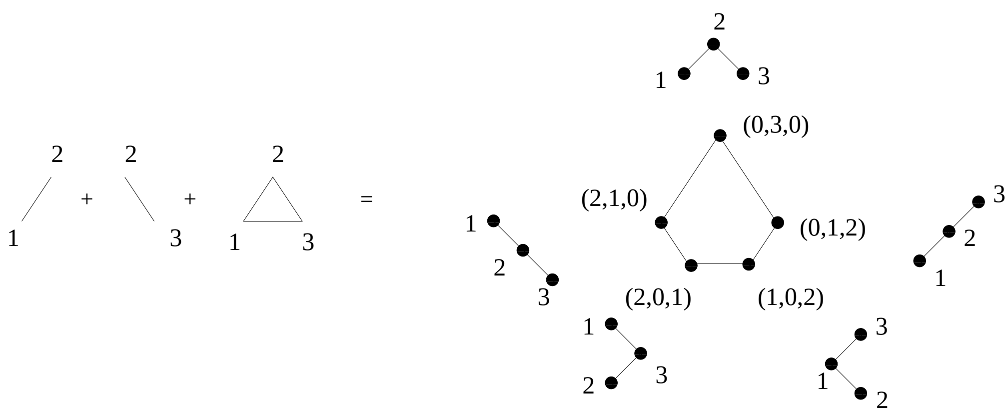

Fig. 12: Using binary search labeling to describe the vertices of Ass_3.

There is a well-known bijection between plane binary trees on $[r+1]$ and subdivisions of the shifted triangular $(r+1)$-by-$(r+1)$ shape D_{r+1} into rectangles, each touching a diagonal box. The nice feature about this bijection is that if we denote by R_k the rectangle containing the kth diagonal box, then

$$l_k \le i \le k \le j \le r_k \iff (i,j) \in R_k.$$

Figure 13 shows an example of this bijection.

Now we have a a convenient way to visualize the result of 5.1 as follows:

Write the numbers $|B_{i,j}|$ in the boxes of the triangular shape D_{r+1}:

Corollary 5.2 *Consider a subdivision Ξ of D_{r+1} into rectangles $R_1, ..., R_{r+1}$ with $(i,i) \in R_i$. Then a vertex $v_\Xi = (t_1, ..., t_{r+1})$ of $N_{P,Q}$ is given by*

$$t_k = \sum_{(i,j)\in R_k} |B_{i,j}|$$

The map $\Xi \mapsto v_\Xi$ is always surjective, and it is a bijection if and only if $|B_{i,j}| > 0$ for all $i < j$.

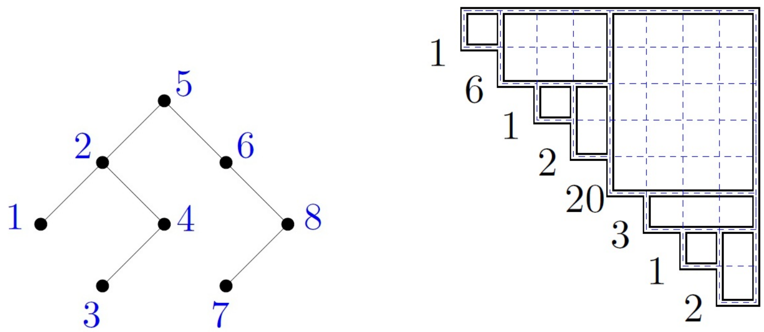

Fig. 13: A binary search labeling and a corresponding subdivision.

This corollary also suggests a nice way of constructing linear extensions of P, whose (P, Q)-subposet vector is the vertex v_Ξ of the posetohedron: Fill rectangle R_k with the numbers $t_1 + ... + t_{k-1} + 1, ..., t_1 + ... + t_k$ (i.e. construct an order preserving bijection $\sigma_k : \cup_{(i,j)\in R_k} B_{i,j} \to [t_1 + ... + t_{k-1} + 1, t_1 + ... + t_k]$ for each k, and then combine the σ_k's to produce a linear extension of P).

Note that for each of the $C_{r+1} = \frac{1}{r+2}\binom{2(r+1)}{r+1}$ subdivisions Ξ of D_{r+1} produces a vertex of $N_{P,Q}$, but some of these will coincide if some of the $|B_{i,j}|$ are 0.

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On the Topology of the Cambrian Semilattices (Extended Abstract)

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Abstract. For an arbitrary Coxeter group W, David Speyer and Nathan Reading defined Cambrian semilattices C_γ as certain sub-semilattices of the weak order on W. In this article, we define an edge-labeling using the realization of Cambrian semilattices in terms of γ-sortable elements, and show that this is an EL-labeling for every closed interval of C_γ. In addition, we use our labeling to show that every finite open interval in a Cambrian semilattice is either contractible or spherical, and we characterize the spherical intervals, generalizing a result by Nathan Reading.

Résumé. Pour tout groupe de Coxeter W, David Speyer et Nathan Reading ont défini les demi-treillis Cambriens comme certains sous-demi-treillis de l'ordre faible sur W. Dans cet article, nous définissons un étiquetage des arêtes basé sur la réalisation des demi-treillis Cambriens en termes d'éléments γ-triables, et prouvons que c'est un étiquetage EL pour tout intervalle fermé de C_γ. Nous utilisons de plus cet étiquetage pour montrer que tout intervalle ouvert fini dans un demi-treillis Cambrien est soit contractible soit sphérique, et nous caractérisons les intervalles sphériques, généralisant ainsi un résultat de Nathan Reading.

Keywords: Coxeter Groups, Weak Order, Cambrian Semilattices, EL-Shellability

1 Introduction

In [6, Theorem 9.6] Anders Björner and Michelle Wachs showed that the Tamari lattice T_n, introduced in [17], can be regarded as the subposet of the weak-order lattice on the symmetric group $\mathfrak{S}_n$, consisting of 312-avoiding permutations. More precisely, there exists a lattice homomorphism $\sigma : \mathfrak{S}_n \to T_n$ such that T_n is isomorphic to the subposet of the weak-order lattice on $\mathfrak{S}_n$ consisting of the bottom elements in the fibers of σ. In [13], the map σ was realized as a map from $\mathfrak{S}_n$ to the triangulations of an $(n+2)$-gon, where the partial order on the latter is given by diagonal flips. It was shown that the fibers of σ induce a congruence relation on the weak-order lattice on $\mathfrak{S}_n$, and that the Tamari lattice is isomorphic to the lattice quotient induced by this congruence. Moreover, it was observed that different embeddings of the $(n+2)$-gon in the plane yield different lattice quotients of the weak-order lattice on $\mathfrak{S}_n$. The realization of $\mathfrak{S}_n$ as the Coxeter group A_{n-1} was then used to connect the embedding of the $(n+2)$-gon in the plane with a Coxeter element of A_{n-1}. This connection eventually led to the definition of Cambrian

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lattices, which can analogously be defined for an arbitrary finite Coxeter group W as lattice quotients of the weak-order lattice on W with respect to certain lattice congruences induced by orientations of the Coxeter diagram of W, see [14].

In [15], Nathan Reading and David Speyer generalized the idea of Cambrian lattices to infinite Coxeter groups. Since there exists no longest element in an infinite Coxeter group, the weak order constitutes only a (meet)-semilattice. Using the realization of the Cambrian lattices in terms of Coxeter-sortable elements, which was first described in [14] and later extended in [15], the analogous construction as in the finite case yields a quotient semilattice of the weak-order semilattice, the so-called *Cambrian semilattice*.

This article is dedicated to the investigation of the topological properties of the order complex of the proper part of closed intervals in a Cambrian semilattice. One (order-theoretic) tool to investigate these properties is EL-shellability, which was introduced in [1], and further developed in [4, 5, 6]. The fact that a poset is EL-shellable implies a number of properties of the associated order complex: this order complex is Cohen-Macaulay, it is homotopy equivalent to a wedge of spheres and the dimensions of its homology groups can be computed from the labeling. The main results of the present article are the following.

Theorem 1.1 *Every closed interval in C_γ is EL-shellable for every (possibly infinite) Coxeter group W and every Coxeter element $\gamma \in W$.*

We prove this result uniformly using the realization of C_γ in terms of Coxeter-sortable elements, and thus our proof does not require W to be finite or even crystallographic. For finite crystallographic Coxeter groups, Theorem 1.1 is implied by [9, Theorem 4.17]. Colin Ingalls and Hugh Thomas considered in [9] the category of finite dimensional representations of an orientation of the Coxeter diagram of a finite crystallographic Coxeter group W. However, their approach cannot be applied to non-crystallographic or to infinite Coxeter groups. Recently, Vincent Pilaud and Christian Stump gave a proof of Theorem 1.1 for finite Coxeter groups, by investigating increasing flip posets of certain subword complexes, see [11].

Finally, using the fact that every closed interval of C_γ is EL-shellable, we are able to determine the homotopy type of the proper parts of these intervals by counting the number of falling chains with respect to our labeling. It turns out that every open interval is either contractible or spherical, *i.e.* homotopy equivalent to a sphere. We can further characterize which intervals of C_γ are contractible and which are spherical, as our second main result shows. Recall that a closed interval $[x, y]$ in a lattice is called *nuclear* if y is the join of atoms of $[x, y]$.

Theorem 1.2 *Let W be a (possibly infinite) Coxeter group and let $\gamma \in W$ be a Coxeter element. Every finite open interval in the Cambrian semilattice C_γ is either contractible or spherical. Furthermore, a finite open interval $(x, y)_\gamma$ is spherical if and only if the corresponding closed interval $[x, y]_\gamma$ is nuclear.*

For finite Coxeter groups, Theorem 1.2 is implied by concatenating [12, Theorem 1.1] and [12, Propositions 5.6 and 5.7]. Nathan Reading's approach in the cited article was to investigate fan posets of central hyperplane arrangements. He then showed that for a finite Coxeter group W the Cambrian lattices can be viewed as fan posets of a fan induced by certain regions of the Coxeter arrangement of W which are determined by orientations of the Coxeter diagram of W. The tools Nathan Reading developed in [12] apply to a much larger class of fan posets, but cannot be applied directly to infinite Coxeter groups.

The proofs of Theorems 1.1 and 1.2 are obtained completely within the framework of Coxeter-sortable elements and thus have the advantage that they are uniform and direct.

This article is organized as follows. In Section 2, we recall the necessary order-theoretic concepts, as well as the definition of EL-shellability. Furthermore, we recall the definition of Coxeter groups, and the construction of the Cambrian semilattices. In Section 3, we define a labeling of the Hasse diagram of a Cambrian semilattice and give a case-free proof that this labeling is indeed an EL-labeling for every closed interval of this semilattice, thus proving Theorem 1.1. In Section 4, we prove Theorem 1.2, by counting the falling maximal chains with respect to our labeling and by applying [5, Theorem 5.9] which relates the number of falling maximal chains in a poset to the homotopy type of the corresponding order complex. The characterization of the spherical intervals of C_γ follows from Theorem 4.3.

The present article is an extended abstract of [10], and we have thus omitted most of the proofs and some illustrating examples. They can be found at the corresponding places in the original article.

2 Preliminaries

In this section, we recall the necessary definitions, which are used throughout the article. For further background on posets, we refer to [7] or to [16] where in addition some background on lattices and lattice congruences is provided. An introduction to poset topology can be found in either [2] or [18]. For more background on Coxeter groups, we refer to [3] and [8].

2.1 Posets and EL-Shellability

Let $(P, \leq_P)$ be a finite partially ordered set (*poset* for short). We say that P is *bounded* if it has a unique minimal and a unique maximal element, which we usually denote by $\hat{0}$ and $\hat{1}$, respectively. For $x, y \in P$, we say that y *covers* x (and write $x \lessdot_P y$) if $x \leq_P y$ and there is no $z \in P$ such that $x <_P z <_P y$. We denote the set of all covering relations of P by $\mathcal{E}(P)$. For $x, y \in P$ with $x \leq_P y$, we define the closed interval $[x, y]$ to be the set $\{z \in P \mid x \leq_P z \leq_P y\}$. Similarly, we define the open interval $(x, y) = \{z \in P \mid x <_P z <_P y\}$. A chain $c : x = p_0 \leq_P p_1 \leq_P \cdots \leq_P p_s = y$ is called *maximal* if $(p_i, p_{i+1}) \in \mathcal{E}(P)$ for every $0 \leq i \leq s-1$. Let $(P, \leq_P)$ be a bounded poset and let $c : \hat{0} = p_0 \lessdot_P p_1 \lessdot_P \cdots \lessdot_P p_s = \hat{1}$ be a maximal chain of P. Given another poset $(\Lambda, \leq_\Lambda)$, a map $\lambda : \mathcal{E}(P) \to \Lambda$ is called *edge-labeling of* P. We denote the sequence $\big(\lambda(p_0, p_1), \lambda(p_1, p_2), \ldots, \lambda(p_{s-1}, p_s)\big)$ of edge-labels of c by $\lambda(c)$. The chain c is called *rising* (respectively *falling*) if $\lambda(c)$ is a strictly increasing (respectively weakly decreasing) sequence. For two words $(p_1, p_2, \ldots, p_s)$ and $(q_1, q_2, \ldots, q_t)$ in the alphabet Λ, we write $(p_1, p_2, \ldots, p_s) \leq_{\Lambda^*} (q_1, q_2, \ldots, q_t)$ if and only if either

$$\begin{aligned} p_i &= q_i, && \text{for } 1 \leq i \leq s \text{ and } s \leq t, \quad \text{or} \\ p_i &<_\Lambda q_i, && \text{for the least } i \text{ such that } p_i \neq q_i. \end{aligned}$$

A maximal chain c of P is called *lexicographically first* among all maximal chains of P if for every other maximal chain c' of P we have $\lambda(c) \leq_{\Lambda^*} \lambda(c')$. An edge-labeling of P is called *EL-labeling* if for every closed interval $[x, y]$ in P there exists a unique rising maximal chain which is lexicographically first among all maximal chains in $[x, y]$. A bounded poset that admits an EL-labeling is called *EL-shellable*. Finally, we recall that the Möbius function μ of P is the map $\mu : P \times P \to \mathbb{Z}$ defined recursively by

$$\mu(x, y) = \begin{cases} 1, & x = y \\ -\sum_{x \leq_P z <_P y} \mu(x, z), & x <_P y \\ 0, & \text{otherwise.} \end{cases}$$

A remarkable property of EL-shellable posets is that we can compute the value of the Möbius function for every closed interval of P from the labeling, as is stated in the following proposition[(i)].

Proposition 2.1 ([5, Proposition 5.7]) *Let $(P, \leq_P)$ be an EL-shellable poset, and let $x, y \in P$ with $x \leq_P y$. Then,*

$$\mu(x,y) = \textit{number of even length falling maximal chains in } [x,y] \\ - \textit{ number of odd length falling maximal chains in } [x,y].$$

2.2 Coxeter Groups and Weak Order

Let W be a (possibly infinite) group, which is generated by the finite set $S = \{s_1, s_2, \ldots, s_n\}$. Let $m = (m_{i,j})_{1 \leq i,j \leq n}$ be a symmetric $(n \times n)$-matrix, where the entries are either positive integers or the formal symbol ∞, and which satisfies $m_{i,i} = 1$ for all $1 \leq i \leq n$, and $m_{i,j} \geq 2$ otherwise. (We use the convention that ∞ is formally larger than any natural number.) We call W a *Coxeter group* if it has the presentation

$$W = \langle S \mid (s_i s_j)^{m_{i,j}} = \varepsilon, \text{ for } 1 \leq i, j \leq n \rangle,$$

where $\varepsilon \in W$ denotes the identity. We interpret the case $m_{i,j} = \infty$ as stating that there is no relation between the generators s_i and s_j, and we call the matrix m the *Coxeter matrix of W*. The *Coxeter diagram of W* is the graph $G = (V, E)$, with $V = S$ and $E = \big\{\{s_i, s_j\} \mid m_{i,j} \geq 3\big\}$. In addition, an edge $\{s_i, s_j\}$ of G is labeled by the value $m_{i,j}$ if and only if $m_{i,j} \geq 4$.

Since S is a generating set of W, we can write every element $w \in W$ as a product of the elements in S, and we call such a word a *reduced word for w* if it has minimal length. More precisely, define the *word length* on W (with respect to S) as

$$\ell_S : W \to \mathbb{N}, \quad w \mapsto \min\{k \mid w = s_{i_1} s_{i_2} \cdots s_{i_k} \text{ and } s_{i_j} \in S \text{ for all } 1 \leq j \leq k\}.$$

If $\ell_S(w) = k$, then every product of k generators which yields w is a reduced word for w. Define the *(right) weak order of W* by

$$u \leq_S v \quad \text{if and only if} \quad \ell_S(v) = \ell_S(u) + \ell_S(u^{-1}v).$$

The poset $(W, \leq_S)$ is a graded meet-semilattice, the so-called *weak-order semilattice of W*, and ℓ_S is its rank function. Further, $(W, \leq_S)$ is *finitary* meaning that every closed interval of $(W, \leq_S)$ is finite. In the case where the group W is finite, there exists a unique longest word w_o of W, and $(W, \leq_S)$ is a lattice.

2.3 Coxeter-Sortable Words

Let $\gamma = s_1 s_2 \cdots s_n \in W$ be a Coxeter element, and define the half-infinite word

$$\gamma^\infty = s_1 s_2 \cdots s_n | s_1 s_2 \cdots s_n | \cdots .$$

The vertical bars in the representation of γ^∞ are "dividers", which have no influence on the structure of the word, but shall serve for a better readability. Clearly, every reduced word for $w \in W$ can be considered as a subword of γ^∞. Among all reduced words for w, there is a unique reduced word, which is lexicographically first as a subword of γ^∞. This reduced word is called the *γ-sorting word of w*.

(i) Actually, Proposition 5.7 in [5] is stated for posets admitting a so-called *CR-labeling*. EL-shellable posets are a particular instance of this class of posets, and for the scope of this article it is sufficient to restrict our attention to these.

Example 2.2 *Consider the Coxeter group $W = \mathfrak{S}_5$, generated by $S = \{s_1, s_2, s_3, s_4\}$, where s_i corresponds to the transposition $(i, i+1)$ for all $i \in \{1,2,3,4\}$ and let $\gamma = s_1s_2s_3s_4$. Clearly, s_1 and s_4 commute. Hence, $w_1 = s_1s_2|s_1s_4$ and $w_2 = s_1s_2s_4|s_1$ are reduced words for the same element $w \in W$. Considering w_1 and w_2 as subwords of γ^∞, we find that w_2 is a lexicographically smaller subword of γ^∞ than w_1 is. There are six other reduced words for w, namely*

$$\begin{aligned} w_3 &= s_1s_4|s_2|s_1, & w_4 &= s_4|s_1s_2|s_1, & w_5 &= s_4|s_2|s_1s_2,\\ w_6 &= s_2s_4|s_1s_2, & w_7 &= s_2|s_1s_4|s_2, & w_8 &= s_2|s_1s_2s_4. \end{aligned}$$

It is easy to see that among these w_2 is the lexicographically first subword of γ^∞, and hence w_2 is the γ-sorting word of w.

In the following, we consider only γ-sorting words, and write

$$w = s_1^{\delta_{1,1}} s_2^{\delta_{1,2}} \cdots s_n^{\delta_{1,n}} \mid s_1^{\delta_{2,1}} s_2^{\delta_{2,2}} \cdots s_n^{\delta_{2,n}} \mid \cdots \mid s_1^{\delta_{l,1}} s_2^{\delta_{l,2}} \cdots s_n^{\delta_{l,n}}, \tag{1}$$

where $\delta_{i,j} \in \{0,1\}$ for $1 \leq i \leq l$ and $1 \leq j \leq n$. For each $i \in \{1,2,\ldots,l\}$, we say that

$$b_i = \{s_j \mid \delta_{i,j} = 1\} \subseteq S$$

is the *i-th block of w*. We consider the blocks of w sometimes as sets and sometimes as subwords of γ, depending on how much structure we need. We say that w is *γ-sortable* if and only if $b_1 \supseteq b_2 \supseteq \cdots \supseteq b_l$. In the previous example, we have seen that $w_2 = s_1s_2s_4|s_1$ is a γ-sorting word in W with $b_1 = \{s_1, s_2, s_4\}$ and $b_2 = \{s_1\}$. Since $b_2 \subseteq b_1$, we see that w_2 is indeed γ-sortable.

By definition, the set of γ-sortable words of W does not depend on the choice of the reduced word for γ. Furthermore, the γ-sortable words of W are characterized by a recursive property which we will describe next. A generator $s \in S$ is called *initial in* γ if it is the first letter in some reduced word for γ. For some subset $J \subseteq S$, we denote by W_J the parabolic subgroup of W generated by the set J, and for $s \in S$ we abbreviate $\langle s \rangle = S \setminus \{s\}$. For $w \in W$, and $J \subseteq S$, we denote by w_J the restriction of w to the parabolic subgroup W_J.

Proposition 2.3 ([15, Proposition 2.29]) *Let W be a Coxeter group, γ a Coxeter element and let s be initial in γ. Then an element $w \in W$ is γ-sortable if and only if*

(i) $s \leq_S w$ and sw is $s\gamma s$-sortable, or

(ii) $s \not\leq_S w$ and w is an $s\gamma$-sortable word of $W_{\langle s \rangle}$.

2.4 Cambrian Semilattices

In [15, Section 7] the *Cambrian semilattice* C_γ was defined as the sub-semilattice of the weak order on W consisting of all γ-sortable elements. That C_γ is well-defined follows from [15, Theorem 7.1]. It turns out that C_γ is not only a sub-semilattice of the weak order, but also a quotient semilattice. The key role in the proof of this property is played by the projection $\pi_\downarrow^\gamma$ which maps every word $w \in W$ to the unique largest γ-sortable element below w. More precisely if s is initial in γ, then define

$$\pi_\downarrow^\gamma(w) = \begin{cases} s\pi_\downarrow^{s\gamma s}(sw), & \text{if } s \leq_S w\\ \pi_\downarrow^{s\gamma}(w_{\langle s \rangle}), & \text{if } s \not\leq_S w, \end{cases} \tag{2}$$

and set $\pi_{\downarrow}^{\gamma}(\varepsilon)=\varepsilon$, see [15, Section 6].

Theorem 7.3 in [15] implies that $\pi_{\downarrow}^{\gamma}$ is a semilattice homomorphism from the weak-order semilattice on W to C_γ, and C_γ can be considered as the quotient semilattice of the weak order modulo the semilattice congruence θ_γ induced by the fibers of $\pi_{\downarrow}^{\gamma}$. This semilattice congruence is called *Cambrian congruence*. Since the lack of a maximal element is the only obstruction for the weak order to be a lattice, it follows immediately that the restriction of $\pi_{\downarrow}^{\gamma}$ (and hence θ_γ) to closed intervals of the weak order yields a lattice homomorphism (and hence a lattice congruence).

In the remainder of this article, we switch frequently between the weak-order semilattice on W and the Cambrian semilattice C_γ. In order to point out properly which semilattice we consider, we denote the order relation of the weak-order semilattice by $\leq_S$, and the order relation of C_γ by $\leq_\gamma$. Analogously, we denote a closed (respectively open) interval in the weak-order semilattice by $[u,v]_S$ (respectively $(u,v)_S$), and a closed (respectively open) interval in C_γ by $[u,v]_\gamma$ (respectively $(u,v)_\gamma$).

3 EL-Shellability of the Closed Intervals in C_γ

In this section, we define an edge-labeling of C_γ, discuss some of its properties and eventually prove Theorem 1.1.

3.1 The Labeling

Define for every $w\in W$ the set of positions of the γ-sorting word of w as

$$\alpha_\gamma(w)=\big\{(i-1)\cdot n+j\mid \delta_{i,j}=1\big\}\subseteq\mathbb{N},$$

where the $\delta_{i,j}$'s are the exponents from (1). We remark that the set of positions of w depends not only on the choice of the Coxeter element γ, but also on the choice of the reduced word of γ.

Example 3.1 *Let $W=\mathfrak{S}_4$, $\gamma=s_1s_2s_3$ and consider $u=s_1s_2s_3|s_2$, and $v=s_2s_3|s_2|s_1$. Then, $\alpha_\gamma(u)=\{1,2,3,5\}$, and $\alpha_\gamma(v)=\{2,3,5,7\}$, where $u\in C_\gamma$, while $v\notin C_\gamma$.*

It is not hard to see that an element $w\in W$ lies in C_γ if and only if the following holds: if $i\in\alpha_\gamma(w)$ and $i>n$, then $i-n\in\alpha_\gamma(w)$. In the previous example, we see that $\alpha_\gamma(u)$ contains both 5 and 2, while $\alpha_\gamma(v)$ does not contain $7-3=4$.

Lemma 3.2 *Let $u,v\in W$ with $u\leq_S v$. Then $\alpha_\gamma(u)$ is a subset of $\alpha_\gamma(v)$.*

Denote by $\mathcal{E}(C_\gamma)$ the set of covering relations of C_γ, and define an edge-labeling of C_γ by

$$\lambda_\gamma:\mathcal{E}(C_\gamma)\to\mathbb{N},\quad (u,v)\mapsto\min\{i\mid i\in\alpha_\gamma(v)\smallsetminus\alpha_\gamma(u)\}. \tag{3}$$

Figure 1 shows the Hasse diagram of a Cambrian lattice C_γ of the Coxeter group A_3, together with the labels defined by the map λ_γ.

3.2 Properties of the Labeling

We notice that the definition of λ_γ depends on a specific reduced word for γ. The following lemma shows that the structural properties of λ_γ are independent of the choice of reduced word for γ.

Lemma 3.3 *Let $\gamma\in W$ be a Coxeter element, and let $u,v\in C_\gamma$ with $u\leq_\gamma v$. The number of maximal falling and rising chains in $[u,v]_\gamma$ does not depend on the choice of a reduced word for γ.*

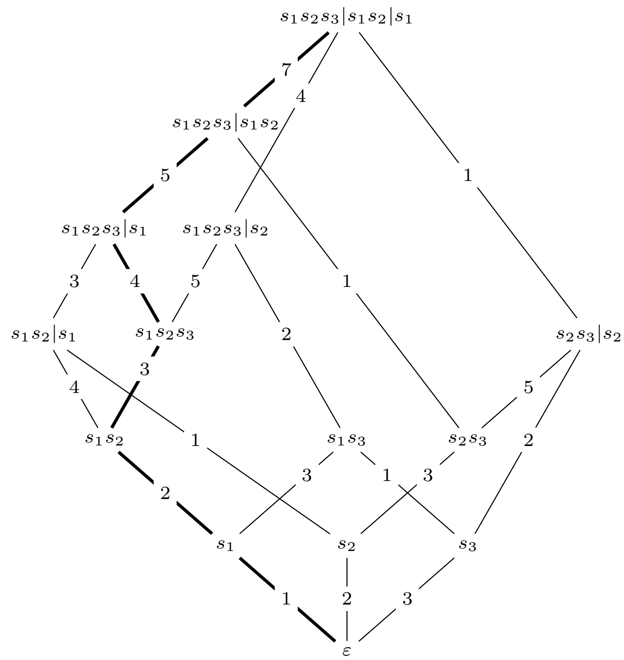

Fig. 1: An A_3-Cambrian lattice with the labeling as defined in (3).

Whenever we use an initial letter s of γ in the remainder of this article, we consider λ_γ with respect to a fixed reduced word for γ which has s as its first letter. The previous lemma implies that this can be done without loss of generality.

Lemma 3.4 *Let C_γ be a Cambrian semilattice, and let $u, v \in C_\gamma$ such that $u \leq_\gamma v$. Let $i_0 = \min\{i \mid i \in \alpha_\gamma(v) \setminus \alpha_\gamma(u)\}$. Then the following hold.*

(i) The label i_0 appears in every maximal chain of the interval $[u, v]_\gamma$.

(ii) The labels of a maximal chain in $[u, v]_\gamma$ are distinct.

The γ-sortable words of W are defined recursively as described in Proposition 2.3. Thus we need to investigate how our labeling behaves with respect to this recursion.

Lemma 3.5 *Let W be a Coxeter group and let $\gamma \in W$ be a Coxeter element. For $u, v \in C_\gamma$ with $u \lessdot_\gamma v$ and for $s \in S$ initial in γ, we have*

$$\lambda_\gamma(u,v) = \begin{cases} 1, & \textit{if } s \not\leq_S u \textit{ and } s \leq_S v, \\ \lambda_{s\gamma s}(su, sv) + 1, & \textit{if } s \leq_S u, \\ \lambda_{s\gamma}(u_{\langle s \rangle}, v_{\langle s \rangle}) + k, & \textit{if } s \not\leq_S v \textit{ and the first position where } u \textit{ and } v \\ & \textit{differ is in their } k\textit{-th block.} \end{cases}$$

3.3 Proof of Theorem 1.1

We will prove Theorem 1.1 by showing that the map λ_γ defined in (3) is an EL-labeling for every closed interval in C_γ. In particular we show the following.

Theorem 3.6 *Let $u, v \in C_\gamma$ with $u \leq_\gamma v$. Then the map λ_γ defined in* (3) *is an EL-labeling for $[u, v]_\gamma$.*

We notice in view of Lemma 3.3 that the statement of Theorem 3.6 does not depend on a reduced word for γ, even though our labeling does. For the proof of Theorem 3.6, we need one more technical lemma. This lemma uses many of the deep results on Cambrian semilattices developed in [15], and requires the following alternative characterization of the (right) weak order on W. Let $T = \{wsw^{-1} \mid w \in W, s \in S\}$, and define for $w \in W$, the *(left) inversion set of* w as $\mathrm{inv}(w) = \{t \in T \mid \ell_S(tw) \leq \ell_S(w)\}$. It is the statement of [3, Proposition 3.1.3] that $u \leq_S v$ if and only if $\mathrm{inv}(u) \subseteq \mathrm{inv}(v)$. Moreover, for $w \in W$, we say that $t \in \mathrm{inv}(w)$ is called a *cover reflection of* w if there exists some $s \in S$ with $tw = ws$. We denote by $\mathrm{cov}(w)$ the set of all cover reflections of w.

Lemma 3.7 *Let $u, v \in C_\gamma$ with $u \leq_\gamma v$ and let s be initial in γ. If $s \not\leq_\gamma u$ and $s \leq_\gamma v$, then the join $s \vee_\gamma u$ covers u in C_γ.*

Proof: First of all, since $s \leq_\gamma v$ and $u \leq_\gamma v$, we conclude from [15, Theorem 7.1] that $s \vee_\gamma u$ exists, and set $z = s \vee_\gamma u$. By assumption, we have $u = \pi_\downarrow^{s\gamma}(u_{\langle s\rangle}) \in W_{\langle s\rangle}$, and Proposition 2.3 implies $u = u_{\langle s\rangle}$. We deduce then from [15, Lemma 2.23] that $\mathrm{cov}(z) = \{s\} \cup \mathrm{cov}(u)$. Therefore s is a cover reflection of z, and it follows from [15, Proposition 5.4 (i)] that $z = s \vee_\gamma z_{\langle s\rangle}$, and [15, Proposition 5.4 (ii)] implies that $\mathrm{cov}(z) = \{s\} \cup \mathrm{cov}(z_{\langle s\rangle})$. Hence, $\mathrm{cov}(u) = \mathrm{cov}(z_{\langle s\rangle})$, and [15, Theorem 8.9 (iv)] implies $u = z_{\langle s\rangle}$. (The required fact that $z_{\langle s\rangle}$ is γ-sortable follows from [15, Propositions 3.13 and 6.10].)

On the other hand, it follows from the definition of a cover reflection that there exists an element $z' = sz \in W$ with $z' \lessdot_S z$, thus $z'_{\langle s\rangle} \leq_S z_{\langle s\rangle}$, see [15, Section 2.5]. Furthermore we have that $\mathrm{inv}(z') = \mathrm{inv}(z) \setminus \{s\}$, and since $\mathrm{inv}(s) = \{s\}$, Proposition 3.1.3 in [3] implies $s \not\leq_S z'$. Hence, by definition of $\pi_\downarrow^\gamma$, see (2), we have $\pi_\downarrow^\gamma(z') = \pi_\downarrow^{s\gamma}(z'_{\langle s\rangle}) \in W_{\langle s\rangle}$, and $\pi_\downarrow^\gamma(z') \lessdot_\gamma z$. Since $\pi_\downarrow^{s\gamma}$ is order-preserving, see [15, Theorem 6.1], we conclude from $z'_{\langle s\rangle} \leq_S z_{\langle s\rangle}$ that $\pi_\downarrow^{s\gamma}(z'_{\langle s\rangle}) \leq_S \pi_\downarrow^{s\gamma}(z_{\langle s\rangle})$. Hence,

$$\pi_\downarrow^\gamma(z') = \pi_\downarrow^{s\gamma}(z'_{\langle s\rangle}) \leq_S \pi_\downarrow^{s\gamma}(z_{\langle s\rangle}) = \pi_\downarrow^{s\gamma}(u) = \pi_\downarrow^{s\gamma}(u_{\langle s\rangle}) = \pi_\downarrow^\gamma(u) = u.$$

Since $\pi_\downarrow^\gamma(z') \lessdot_\gamma z$ and $u <_\gamma z$, the previous implies $u = \pi_\downarrow^\gamma(z')$ and thus $u \lessdot_\gamma z$. □

Proof of Theorem 3.6: Let $[u, v]_\gamma$ be a closed interval of C_γ. Since the weak order on W is finitary, it follows that $[u, v]_\gamma$ is a finite lattice. We show that there exists a unique maximal rising chain in $[u, v]_\gamma$ which is the lexicographically first among all maximal chains in this interval.

We proceed by induction on length and rank, using the recursive structure of γ-sortable words, see Proposition 2.3. We assume that $\ell_S(v) \geq 3$, and that W is a Coxeter group of rank ≥ 2, since the result is trivial otherwise. Say that W is of rank n, and say that $\ell_S(v) = k$. Suppose that the induction hypothesis is true for all parabolic subgroubs of W of rank $< n$ and suppose that for every closed interval $[u', v']_\gamma$ of C_γ with $\ell_S(v') < k$, there exists a unique rising maximal chain from u' to v' which is lexicographically first among all maximal chains in $[u', v']_\gamma$. We show that there is a unique rising maximal chain in the interval $[u, v]_\gamma$ wich is lexicographically first among all maximal chains in $[u, v]_\gamma$. For s initial in γ, we distinguish two cases: (1) $s \leq_\gamma v$ and (2) $s \not\leq_\gamma v$.

(1a) Suppose first that $s \leq_\gamma u$ as well. Then, s is the first letter in the γ-sorting word of every element in $[u,v]_\gamma$. It follows from [15, Proposition 2.18] and Proposition 2.3 that the interval $[u,v]_\gamma$ is isomorphic to the interval $[su,sv]_{s\gamma s}$. Moreover, Lemma 3.5 implies that for a covering relation $x \lessdot_\gamma y$ in $[u,v]_\gamma$ we have $\lambda_\gamma(x,y) = \lambda_{s\gamma s}(sx,sy)+1$. Say that $c' : su = sx_0 \lessdot_{s\gamma s} sx_1 \lessdot_{s\gamma s} \cdots \lessdot_{s\gamma s} sx_t = sv$ is the unique rising maximal chain in $[su,sv]_{s\gamma s}$. (This chain exists by induction, since $\ell_S(sv) < \ell_S(v)$.) Then, the chain $c : u = x_0 \lessdot_\gamma x_1 \lessdot_\gamma \cdots \lessdot_\gamma x_t = v$ is a maximal chain in $[u,v]_\gamma$ and clearly rising. With Lemma 3.5, we find that c is the unique rising chain and every other maximal chain in $[u,v]_\gamma$ is lexicographically larger than c.

(1b) Suppose now that $s \not\leq_\gamma u$. Since $s \leq_\gamma v$ and $u \leq_\gamma v$ the join $u_1 = s \vee_\gamma u$ exists and lies in $[u,v]_\gamma$. Lemma 3.7 implies that $u \lessdot_\gamma u_1$. Consider the interval $[u_1,v]_\gamma$. Then $s \leq_\gamma u_1$ and analogously to (1a) we can find a unique maximal rising chain $c' : u_1 \lessdot_\gamma u_2 \lessdot_\gamma \cdots \lessdot_\gamma u_t = v$ in $[u_1,v]_\gamma$ which is lexicographically first. Moreover, $\min\{i \mid i \in \alpha_\gamma(v) \smallsetminus \alpha_\gamma(u_1)\} > 1$, since $s \leq_\gamma u_1 \leq_\gamma v$. By definition of our labeling, the label 1 cannot appear as a label in any chain in the interval $[u_1,v]_\gamma$. On the other hand, it follows from Lemma 3.5 that $\lambda_\gamma(u,u_1) = 1$. Thus, the chain $c : u \lessdot_\gamma u_1 \lessdot_\gamma u_2 \lessdot_\gamma \cdots \lessdot_\gamma u_t = v$ is maximal and rising in $[u,v]_\gamma$. Suppose that there is another element u' that covers u in $[u,v]_\gamma$ such that $\lambda_\gamma(u,u') = 1$. Then, by definition of λ_γ, it follows that s appears in the γ-sorting word of u'. In particular, since s is initial in γ, we deduce that $s \leq_\gamma u'$. Therefore u' is above both s and u in C_γ. By the uniqueness of joins and the definition of u_1 it follows that $u_1 = u'$. Thus c is the lexicographically smallest maximal chain in $[u,v]_\gamma$. Finally, Lemma 3.4 implies that c is the unique maximal rising chain.

(2) Since $s \not\leq_\gamma v$, it follows that no element of $[u,v]_\gamma$ contains the letter s in its γ-sorting word. We consider the parabolic Coxeter group $W_{\langle s\rangle}$ (generated by $S \setminus \{s\}$) and the Coxeter element $s\gamma$. It follows from Proposition 2.3 that the interval $[u,v]_\gamma$ is isomorphic to the interval $[u_{\langle s\rangle}, v_{\langle s\rangle}]_{s\gamma}$ in $W_{\langle s\rangle}$. Since the rank of $W_{\langle s\rangle}$ is $n-1 < n$, by induction there exists a unique maximal rising chain $c' : u_{\langle s\rangle} = (x_0)_{\langle s\rangle} \lessdot_{s\gamma} (x_1)_{\langle s\rangle} \lessdot_{s\gamma} \cdots \lessdot_{s\gamma} (x_t)_{\langle s\rangle} = v_{\langle s\rangle}$ which is lexicographically first among all maximal chains in $[u_{\langle s\rangle}, v_{\langle s\rangle}]_{s\gamma}$. The result then follows with Lemma 3.5. □

Proof of Theorem 1.1: This follows by definition from Theorem 3.6. □

4 Applications

In [12], Nathan Reading investigated, among others, the topological properties of open intervals in so-called *fan posets*. A fan poset is a certain partial order defined on the maximal cones of a complete fan of regions of a real hyperplane arrangement. For a finite Coxeter group W and a Cambrian congruence θ, the *Cambrian fan* $\mathcal{F}_\theta$ is the complete fan induced by certain cones in the Coxeter arrangement $\mathcal{A}_W$ of W. More precisely, each such cone is a union of regions of $\mathcal{A}_W$ which correspond to elements of W lying in the same congruence class of θ. It is the assertion of [12, Theorem 1.1], that a Cambrian lattice of W is the fan poset associated to the corresponding Cambrian fan. The following theorem is a concatenation of [12, Theorem 1.1] and [12, Propositions 5.6 and 5.7]. In fact, Propositions 5.6 and 5.7 in [12] imply this result for a much larger class of fan posets.

Theorem 4.1 *Let W be a finite Coxeter group and let $\gamma \in W$ be a Coxeter element. Every open interval in the Cambrian lattice C_γ is either contractible or spherical.*

It is well-known that the reduced Euler characteristics of the order complex of an open interval (x,y) in a poset determines $\mu(x,y)$, see for instance [16, Proposition 3.8.6]. Hence, it follows immediately from

Theorem 4.1 that for γ-sortable elements x and y in a finite Coxeter group W satisfying $x \leq_\gamma y$, we have $|\mu(x,y)| \leq 1$, as was already remarked in [13, pp. 4-5]. In light of Proposition 2.1 and Theorem 3.6, we can extend this statement to compute the Möbius function of closed intervals in the Cambrian semilattice C_γ, by counting the falling maximal chains with respect to the labeling defined in (3).

Theorem 4.2 *Let W be a (possibly infinite) Coxeter group and $\gamma \in W$ a Coxeter element. For $u, v \in C_\gamma$ with $u \leq_\gamma v$, we have $|\mu(u,v)| \leq 1$.*

Proof: In view of Proposition 2.1 it is enough to show that the interval $[u,v]_\gamma$ has at most one maximal falling chain. We use similar arguments as in the proof of Theorem 3.6 and proceed by induction on length and rank. Again, for s initial in γ, we distinguish the following two cases: $s \leq_\gamma v$ and $s \not\leq_\gamma v$. Here we discuss only the special case where $s \leq_\gamma v$ and $s \not\leq_\gamma u$. (The others follow by applying the same methods as in the proof of Theorem 3.6.) It follows from Lemma 3.4 that a maximal chain $u = c_0 \lessdot_\gamma c_1 \lessdot_\gamma \cdots \lessdot_\gamma c_{t-1} \lessdot c_t = v$ of $[u,v]_\gamma$ can be falling only if $\lambda_\gamma(c_{t-1}, v) = 1$. Hence, if there is no element $v_1 \in (u,v)_\gamma$, with $v_1 \lessdot v$ satisfying $\lambda_\gamma(v_1, v) = 1$, then the interval $[u,v]_\gamma$ has no maximal falling chain, which means that $\mu(u,v) = 0$. Otherwise, consider the interval $[u, v_1]_\gamma$. By the choice of v_1, it follows that every maximal falling chain in $[u,v_1]_\gamma$ can be extended to a maximal falling chain in the interval $[u,v]_\gamma$. Conversely, every maximal falling chain in $[u,v]_\gamma$ can be restricted to a maximal falling chain in $[u,v_1]_\gamma$. Therefore, since $\ell_S(v_1) < \ell_S(v)$, we deduce from the induction hypothesis that the interval $[u,v_1]_\gamma$ has at most one maximal falling chain. Thus $|\mu(u,v)| \leq 1$. □

In addition Propositions 5.6 and 5.7 in [12] characterize the open intervals in a (finite) Cambrian lattice which are contractible, and those which are spherical in the following way: an interval $[u,v]_\gamma$ in C_γ is called *nuclear* if the join of the upper covers of u is precisely v. Nathan Reading showed that the nuclear intervals are precisely the spherical intervals. With the help of our labeling, we can generalize this characterization to infinite Coxeter groups.

Theorem 4.3 *Let $u, v \in C_\gamma$ with $u \leq_\gamma v$ and let k denote the number of atoms of the interval $[u,v]_\gamma$. Then, $\mu(u,v) = (-1)^k$ if and only if $[u,v]_\gamma$ is nuclear.*

For the proof of Theorem 4.3, we need the following lemma.

Lemma 4.4 *Let $u, v \in C_\gamma$ with $u \leq_\gamma v$, and let s be initial in γ. Suppose further that $s \not\leq_\gamma u$, while $s \leq_\gamma v$. Then the following are equivalent:*

1. *The interval $[u,v]_\gamma$ is nuclear.*
2. *There exists an element $v' \in [u,v]_\gamma$ satisfying $s \not\leq_\gamma v' \lessdot_\gamma v$, and the interval $[u,v']_\gamma$ is nuclear.*

Proof of Theorem 4.3: In view of Proposition 2.1, we need to show that $[u,v]_\gamma$ has a falling chain if and only if $[u,v]_\gamma$ is nuclear. We use similar arguments as in the proof of Theorem 3.6 and proceed by induction on length and rank. For the inductive step we distinguish two cases: (1) $s \not\leq_\gamma v$ and (2) $s \leq_\gamma v$, where s initial in γ. Here we discuss the special case where $s \not\leq_\gamma u$, while $s \leq_\gamma v$. If $[u,v]_\gamma$ is nuclear, the result follows by combining Lemmas 3.5,4.4, Theorem 4.2 and by applying induction on the rank of W. Conversely, suppose that there exists a maximal falling chain $c : u = x_0 \lessdot_\gamma x_1 \lessdot_\gamma \cdots \lessdot_\gamma x_t = v$ in $[u,v]_\gamma$, and let $A = \{w \in C_\gamma \mid u \lessdot_\gamma w \text{ and } w \leq_\gamma v\}$ denote the set of atoms of $[u,v]_\gamma$. It follows then that the chain $c' : u = x_0 \lessdot_\gamma x_1 \lessdot_\gamma \cdots \lessdot_\gamma x_{t-1}$ is falling, thus by induction we can conclude that the interval $[u, x_{t-1}]_\gamma$ is nuclear. We deduce from Lemma 3.4 that $s \not\leq_\gamma x_{t-1}$, and since $x_{t-1} \lessdot_\gamma v$, it follows from Lemma 4.4 that $[u, v]_\gamma$ is nuclear. This completes the proof of the theorem. $\square$

Proof of Theorem 1.2: It follows directly from Theorem 1.1, [5, Theorem 5.9] and Theorem 4.2. The characterization of the spherical intervals is an immediate consequence of Theorem 4.2. $\square$

We conclude this article with a short example of an infinite Coxeter group.

Example 4.5 *Consider the affine Coxeter group $\tilde{A}_2$, which is generated by the set $\{s_1, s_2, s_3\}$ satisfying $(s_1s_2)^3 = (s_1s_3)^3 = (s_2s_3)^3 = \varepsilon$, as well as $s_1^2 = s_2^2 = s_3^2 = \varepsilon$. Consider the Coxeter element $\gamma = s_1s_2s_3$. Figure 2 shows the sub-semilattice of the Cambrian semilattice C_γ consisting of all γ-sortable elements of $\tilde{A}_2$ of length ≤ 7. We encourage the reader to verify Theorem 3.6 and Theorem 4.2.*

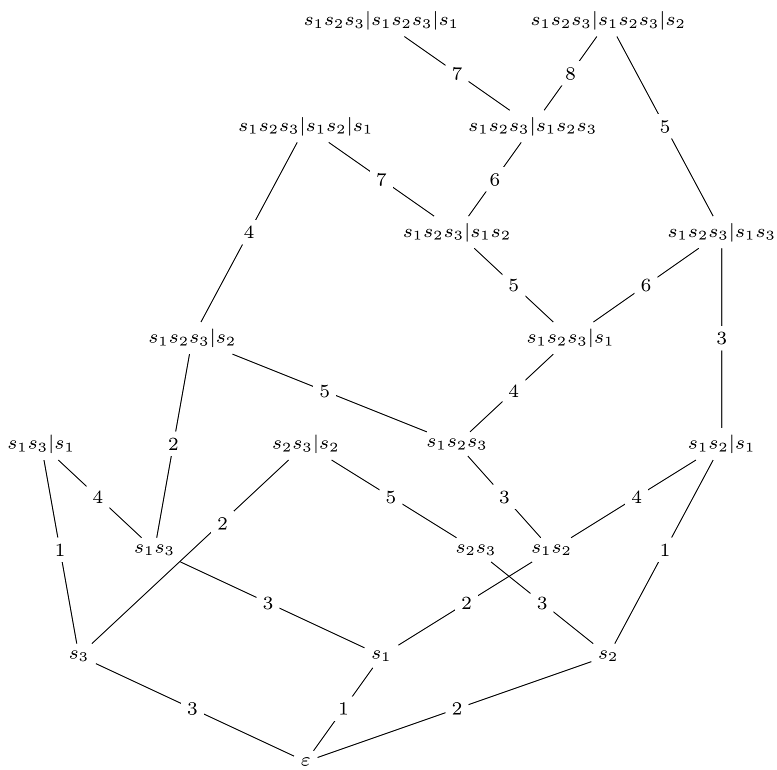

Fig. 2: The first seven ranks of an $\tilde{A}_2$-Cambrian semilattice, with the labeling as defined in (3).

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The Eulerian polynomials of type D have only real roots

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Abstract. We give an intrinsic proof of a conjecture of Brenti that all the roots of the Eulerian polynomial of type D are real and a proof of a conjecture of Dilks, Petersen, and Stembridge that all the roots of the affine Eulerian polynomial of type B are real, as well.

Résumé. Nous prouvons, de façon intrinsèque, une conjecture de Brenti affirmant que toutes les racines du polynôme eulérien de type D sont réelles. Nous prouvons également une conjecture de Dilks, Petersen, et Stembridge que toutes les racines du polynôme eulérien affine de type B sont réelles.

Keywords: Eulerian polynomials, Coxeter group of type D, inversion sequences, polynomials with only real roots.

1 Overview

Let $\mathfrak{S}_n$ denote the group of all permutations of the set $[n] = \{1, 2, \ldots, n\}$. The *descent set* of a permutation π in $\mathfrak{S}_n$ (given in its one-line notation $\pi = \pi_1 \cdots \pi_n$) is defined as

$$\operatorname{Des}\pi = \{i \in [n-1] : \pi_i > \pi_{i+1}\}.$$

The *Eulerian polynomial*, $S_n(x)$, is the generating function for the statistic $\operatorname{des}\pi = |\operatorname{Des}\pi|$ on $\mathfrak{S}_n$:

$$S_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\operatorname{des}\pi}. \tag{1}$$

In addition to its many notable properties, $S_n(x)$ has only real roots, which implies that its coefficient sequence, the Eulerian numbers, is unimodal and log-concave.

The notion of descents, and therefore Eulerian polynomials, extends to all finite Coxeter groups. Brenti conjectured (Conjecture 5.1 in Brenti (1994)) that the Eulerian polynomials for all finite Coxeter groups have only real roots. He proved this to be the case for the exceptional groups and for type B. The main goal of this paper is to prove the last missing part of Brenti's conjecture, for type D groups:

Conjecture 1.1 (Conjecture 5.2 in Brenti (1994)) *The type D Eulerian polynomials have only real roots.*

To be precise, we view the Coxeter group of type B (resp. D) of rank n, denoted by $\mathfrak{B}_n$ (resp. $\mathfrak{D}_n$), as the set of signed (resp. even-signed) permutations of the set $[n]$. The type B and D descents have the following simple combinatorial interpretation (see Brenti (1994); Björner and Brenti (2005)). For a signed (resp. even-signed) permutation σ given in its "window notation" $(\sigma_1, \ldots, \sigma_n)$, let

$$\mathrm{Des}_B\,\sigma = \{i \in [n-1] : \sigma_i > \sigma_{i+1}\} \cup \{0 : \text{if } \sigma_1 < 0\}, \tag{2}$$

$$\mathrm{Des}_D\,\sigma = \{i \in [n-1] : \sigma_i > \sigma_{i+1}\} \cup \{0 : \text{if } \sigma_1 + \sigma_2 < 0\}. \tag{3}$$

The type B and type D Eulerian polynomials are, respectively,

$$B_n(x) = \sum_{\sigma \in \mathfrak{B}_n} x^{\mathrm{des}_B\,\sigma} \quad \text{and} \quad D_n(x) = \sum_{\sigma \in \mathfrak{D}_n} x^{\mathrm{des}_D\,\sigma},$$

where $\mathrm{des}_B\,\sigma = |\mathrm{Des}_B\,\sigma|$ and $\mathrm{des}_D\,\sigma = |\mathrm{Des}_D\,\sigma|$.

Our approach is novel and general. It applies not only to type D, but also—as we will see—to the type A and type B Eulerian polynomials. Moreover, we will apply our method to the *affine* Eulerian polynomials proposed by Dilks, Petersen, and Stembridge (2009). In doing so, in Section 8, we will resolve another conjecture:

Conjecture 1.2 (Dilks et al. (2009)) *The affine Eulerian polynomials of type B have only real roots.*

The type A and C affine Eulerian polynomials are multiples of the classical Eulerian polynomial and hence, have only real roots. However, the affine type D case remains open. See discussion in Section 8.

Our method makes use of the $\boldsymbol{s}$-inversion sequences and their *ascent* statistic, defined in the Section 3. These were inspired by lecture hall partitions Bousquet-Mélou and Eriksson (1997) and introduced in Savage and Schuster (2012). The method works as follows.

- First, encode each element, w, of the Coxeter group as an $\boldsymbol{s}$-inversion sequence, e, in such a way that the descent set of w is the same as the ascent set of e (Sections 2,3 and 4).

- Secondly, observe that (a refinement of) the generating polynomial for the number of ascents over inversion sequences satisfies a recurrence of a certain form (Section 5).

- Finally, show that the polynomials defined by such recurrences are "compatible" (a notion closely related to interlacing) to deduce that the Eulerian polynomials have all roots real (Sections 6, 7).

2 Type D Eulerian polynomials

Let us start with a simple observation. Note that the type D descent statistic, Des_D, defined in (3) can be extended to all signed permutations. Furthermore, Des_D is equidistributed over even-signed and odd-signed permutations. In other words, we have the following equality.

Proposition 2.1 *For $n \geq 2$,*

$$\sum_{\sigma \in \mathfrak{B}_n} x^{\mathrm{des}_D\,\sigma} = 2 \sum_{\sigma \in \mathfrak{D}_n} x^{\mathrm{des}_D\,\sigma}.$$

Proof: The involution on $\mathfrak{B}_n$ that swaps the values 1 and -1 in (the window notation of) $\sigma \in \mathfrak{B}_n$ is a bijection between $\mathfrak{D}_n$ and $\mathfrak{B}_n \setminus \mathfrak{D}_n$ that preserves the type D descent statistic.

□

Therefore, in order to avoid dealing with the parity of the signs and to allow for simpler recurrences, we will be working instead with the polynomial

$$T_n(x) = \sum_{\sigma \in \mathfrak{B}_n} x^{\mathrm{des}_D \sigma}. \tag{4}$$

Clearly, $T_n(x)$ has only real roots if and only if $D_n(x)$ does. In what follows, we will restrict our attention to permutations and signed permutations, with the goal of showing $T_n(x)$ has all real roots.

3 $\boldsymbol{s}$-Inversion sequences and $\boldsymbol{s}$-Eulerian polynomials

For a sequence $\boldsymbol{s} = s_1, s_2, \ldots$ of positive integers, the set $I_n^{(\boldsymbol{s})}$ of $\boldsymbol{s}$-inversion sequences of length n is defined by

$$I_n^{(\boldsymbol{s})} = \{e \in \mathbb{Z}^n : 0 \leq e_i < s_i\}.$$

The *ascents* of an inversion sequence $e \in I_n^{(\boldsymbol{s})}$ are the elements of the set

$$\operatorname{Asc} e = \left\{ i \in [n-1] : \frac{e_i}{s_i} < \frac{e_{i+1}}{s_{i+1}} \right\} \cup \{0 : \text{if } e_1 > 0\}.$$

The $\boldsymbol{s}$-Eulerian polynomial, $E_n^{(\boldsymbol{s})}(x)$, is the generating polynomial for the ascent statistic $\operatorname{asc} e = |\operatorname{Asc} e|$ on $I_n^{(\boldsymbol{s})}$:

$$E_n^{(\boldsymbol{s})}(x) = \sum_{e \in I_n^{(\boldsymbol{s})}} x^{\operatorname{asc} e}.$$

Our main result is the following theorem which we will prove in Section 6.

Theorem 3.1 *For any $n \geq 1$ and any sequence $\boldsymbol{s}$ of positive integers, $E_n^{(\boldsymbol{s})}(x)$ has only real roots.*

Consequently, to show the real-rootedness of a family of polynomials, it suffices to show that it is equal to $E_n^{(\boldsymbol{s})}(x)$ for some sequence $\boldsymbol{s}$ of positive integers. For example, the type A and type B Eulerian polynomials have the following form (we defer the proof to Section 4).

Proposition 3.2 (Savage and Schuster (2012)) *For $n \geq 1$,*

$$S_n(x) = E_n^{(1,2,\ldots,n)}(x), \tag{5}$$
$$B_n(x) = E_n^{(2,4,\ldots,2n)}(x), \tag{6}$$

from which it follows by Theorem 3.1 that $S_n(x)$ and $B_n(x)$ have only real roots.

In this paper, we will show how to adapt this idea to type D and affine type B Eulerian polynomials.

4 Inversion sequence representation of (signed) permutations

To simplify notation for inversion sequences, let

$$I_n = I_n^{(1,2,\ldots,n)} \text{ and } I_n^B = I_n^{(2,4,\ldots,2n)}.$$

In this section, we will prove bijections that imply not only the inversion sequence representations for $\mathfrak{S}_n$ (5) and $\mathfrak{B}_n$ (6), but also the following, novel, inversion sequence representation for $\mathfrak{D}_n$.

Proposition 4.1 *For* $n \geq 1$,

$$T_n(x) = \sum_{e \in I_n^B} x^{\operatorname{asc}_D e},$$

where $\operatorname{asc}_D e = |\operatorname{Asc}_D e|$ *is the number of type* D *ascents of* $e = (e_1, \ldots, e_n) \in I_n^B$ *given by*

$$\operatorname{Asc}_D e = \left\{ i \in [n-1] : \frac{e_i}{i} < \frac{e_{i+1}}{i+1} \right\} \cup \left\{ 0 : \text{if } e_1 + \frac{e_2}{2} \geq \frac{3}{2} \right\}. \tag{7}$$

4.1 An inversion sequence for permutations

We will make use of the following bijection between permutations and inversion sequences (see, for example, Lemma 1 in (Savage and Schuster, 2012)). We note that several variants of this map are known under different names: inversion table, Lehmer code, etc.

Lemma 4.2 *The mapping* $\phi : \mathfrak{S}_n \to I_n$ *defined by* $\phi(\pi_1 \pi_2 \cdots \pi_n) = (t_1, t_2, \ldots, t_n)$, *where*

$$t_i \ = \ |\{j \in [i-1] : \pi_j > \pi_i\}|$$

is a bijection satisfying

$\pi_i > \pi_{i+1}$ *if and only if* $t_i < t_{i+1}$, *for* $i \in [n-1]$.

Proof: First, $\pi_i > \pi_{i+1}$ if and only if the set $\{j \in [i-1] \mid \pi_j > \pi_i\}$ is a proper subset of the set $\{j \in [i] \mid \pi_j > \pi_{i+1}\}$, which happens if and only if $t_i < t_{i+1}$.

Clearly, $\phi(\mathfrak{S}_n) \subseteq I_n$. In particular, $t_1 = 0$ and $t_n = n - \pi_n$. It is clear that ϕ is a bijection for $n = 1$. Let $(t_1, \ldots t_n) \in I_n$ for some $n > 1$ and assume that ϕ is a bijection for smaller dimensions. Let $\pi_1 \cdots \pi_{n-1} = \phi^{-1}(t_1, \ldots t_{n-1})$. Then $\phi^{-1}(t_1, \ldots, t_n) = \pi'_1 \cdots \pi'_n$, where $\pi'_n = n - t_n$ and $\pi'_i = \pi_i + \chi(\pi_i \geq \pi_n)$ for $i \in [n-1]$, where $\chi(P) = 1$ if P is true and 0, otherwise. □

We will also use the following basic but useful observation.

Proposition 4.3 *Let* a, b, p *be nonnegative integers such that* $0 \leq a/p < 1$ *and* $0 \leq b/(p+1) < 1$. *Then*

$$\frac{a}{p} < \frac{b}{p+1} \iff a < b.$$

Proof: If $a < b$, then $a + 1 \leq b$. Thus, since $a < p$, $(p+1)a = pa + a < pa + p = p(a+1) \leq pb$. So, $(p+1)a < pb$. Conversely, if $a \geq b$, then $(p+1)a \geq (p+1)b > pb$, so $(p+1)a > pb$. □

Remark 4.4 *The proposition does not hold without the hypothesis. For example,* $4 < 5$, *but* $4/3 > 5/4$.

Proof of (5) **in Proposition 3.2:** Follows from Lemma 4.2 and Proposition 4.3. □

4.2 The bijection for signed permutations and its properties

Clearly, the set of signed permutations, $\mathfrak{B}_n$, has the same cardinality as the set of "type B" inversion sequences, I_n^B. Next, we define a bijection between these sets that maps type B descents in signed permutations to ascents in the inversion sequences. We will prove several other properties of Θ as well. Some will be used to establish the real-rootedness of $T_n(x)$—and hence $D_n(x)$—and others will be needed in Section 8 for the affine Eulerian polynomials. Throughout this paper we will assume the natural ordering of integers,

$$-n < \cdots < -1 < 0 < 1 < \cdots < n.$$

Theorem 4.5 *For $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathfrak{B}_n$, let $(t_1, \ldots t_n) = \phi(|\sigma_1| \cdots |\sigma_n|)$ where ϕ is the map defined in Lemma 4.2 and $|\sigma_1| \cdots |\sigma_n|$ denotes the underlying permutation in $\mathfrak{S}_n$. Define the map $\Theta : \mathfrak{B}_n \to I_n^B$ as follows. Let*

$$\Theta(\sigma) = (e_1, \ldots, e_n),$$

where, for each $i \in [n]$, $e_i = \begin{cases} t_i & \text{if } \sigma_i > 0, \\ 2i - 1 - t_i & \text{if } \sigma_i < 0. \end{cases}$

Then Θ is a bijection satisfying the following properties.

1. *$\sigma_1 < 0$ if and only if $e_1 > 0$.*
2. *$\sigma_n > 0$ if and only if $e_n < n$.*
3. *$\sigma_1 + \sigma_2 < 0$ if and only if $e_1 + \frac{e_2}{2} \geq \frac{3}{2}$.*
4. *$\sigma_i > \sigma_{i+1}$ if and only if $\frac{e_i}{i} < \frac{e_{i+1}}{i+1}$, for $i \in [n-1]$.*
5. *$\sigma_{n-1} + \sigma_n > 0$ if and only if $\frac{e_{n-1}}{n-1} + \frac{e_n}{n} < \frac{2n-1}{n}$.*

Proof: Θ is a bijection since ϕ is. Note that $\sigma_i < 0$ if and only if $e_i \geq i$ which proves 1. and 2.

3. It is not too hard to see that it is sufficient to verify this claim for all $\sigma \in \mathfrak{B}_2$. See Table 1.

$\sigma \in \mathfrak{B}_2$	$e \in I_2^B$	$\mathrm{Asc}_D\, e$	$\mathrm{asc}_D\, e$
(1,2)	(0,0)	$\{\ \}$	0
(-1,2)	(1,0)	$\{\ \}$	0
(2,1)	(0,1)	$\{1\}$	1
(-2,1)	(1,1)	$\{0\}$	1
(2,-1)	(0,2)	$\{1\}$	1
(-2,-1)	(1,2)	$\{0\}$	1
(1,-2)	(0,3)	$\{0,1\}$	2
(-1,-2)	(1,3)	$\{0,1\}$	2

Tab. 1: An example of the bijection for $n = 2$.

4. To prove this claim, we consider four cases, based the signs of σ_i and σ_{i+1}.

(a) If $\sigma_i > 0$ and $\sigma_{i+1} > 0$. Then $e_i = t_i < i$ and $e_{i+1} = t_{i+1} < i+1$. By Lemma 4.2, $\sigma_i > \sigma_{i+1}$ if and only if $t_i < t_{i+1}$, i.e, if and only if $e_i < e_{i+1}$. By Proposition 4.3, this is equivalent to $e_i/i < e_{i+1}/(i+1)$.

(b) If $\sigma_i < 0$ and $\sigma_{i+1} < 0$. Then $e_i = 2i-1-t_i$ and $e_{i+1} = 2(i+1)-1-t_{i+1}$. Now $\sigma_i > \sigma_{i+1}$ if and only if $|\sigma_i| < |\sigma_{i+1}|$, which, applying Lemma 4.2, is equivalent to $t_i \geq t_{i+1}$.

If $t_i \geq t_{i+1}$,

$$\frac{e_i}{i} = 2 - \frac{t_i+1}{i} \leq 2 - \frac{t_{i+1}+1}{i} < 2 - \frac{t_{i+1}+1}{i+1} = \frac{e_{i+1}}{i+1}.$$

On the other hand, if $t_i < t_{i+1}$, then $t_i + 1 \leq t_{i+1}$ and by Proposition 4.3, $t_{i+1}/i < (t_{i+1}+1)/(i+1)$, so

$$\frac{e_i}{i} = 2 - \frac{t_i+1}{i} \geq 2 - \frac{t_{i+1}}{i} > 2 - \frac{t_{i+1}+1}{i+1} = \frac{e_{i+1}}{i+1}.$$

(c) If $\sigma_i < 0 < \sigma_{i+1}$. In this case, $e_i = 2i-1-t_i$ and $e_{i+1} = t_{i+1} \leq i$. Since $t_i \leq i-1$, $e_i \geq 2i-1-(i-1) = i$. Thus we have

$$\frac{e_i}{i} \geq 1 > \frac{i}{i+1} \geq \frac{e_{i+1}}{i+1}.$$

(d) If $\sigma_i > 0 > \sigma_{i+1}$. In this case, $e_i = t_i < i$ and $e_{i+1} = 2(i+1)-1-t_{i+1}$. Since $t_{i+1} \leq i$, $e_{i+1} \geq 2(i+1)-1-(i) = i+1$. Thus we have

$$\frac{e_i}{i} < 1 \leq \frac{e_{i+1}}{i+1}.$$

5. The proof of this claim is a tedious case analysis which we defer to the full paper. We make use of the fifth claim only at the end of the paper where we propose an inversion sequence characterization of the type D affine Eulerian polynomials.

□

Proof of (6) **in Proposition 3.2:** Follows from Theorem 4.5 (parts 1 and 4). □

Remark 4.6 *The bijection Θ is different from the one given in (Pensyl and Savage, 2013).*

Proof of Proposition 4.1: Follows from Theorem 4.5 (parts 3 and 4). □

For example, for $n = 2$, from Table 1, $T_2(x) = 2 + 4x + 2x^2 = 2D_2(x)$, as expected.

5 A recurrence for refined Eulerian polynomials

The inversion sequence representation of $E_n^{(\boldsymbol{s})}(x)$ and $T_n(x)$ allows us to refine these polynomials as follows. Let $\chi(P) = 1$ if P is true and let $\chi(P) = 0$, otherwise. Define

$$E_{n,i}^{(\boldsymbol{s})}(x) = \sum_{e \in I_n^{(\boldsymbol{s})}} \chi(e_n = i) \cdot x^{\operatorname{asc} e}, \tag{8}$$

$$T_{n,i}(x) = \sum_{e \in I_n^B} \chi(e_n = i) \cdot x^{\operatorname{asc}_D e}. \tag{9}$$

Clearly, $E_n^{(\boldsymbol{s})}(x) = \sum_{i=0}^{s_i} E_{n,i}^{(\boldsymbol{s})}(x)$ and $T_n(x) = \sum_{i=0}^{2n-1} T_{n,i}(x)$. We have the following recurrences.

Proposition 5.1 *Let $\boldsymbol{s} = \{s_i\}_{i=1}^{\infty}$. For $n \geq 1$ and $0 \leq i < s_{n+1}$,*

$$E_{n+1,i}^{(\boldsymbol{s})}(x) = \sum_{\ell=0}^{\lceil ni/(n+1)\rceil - 1} x E_{n,\ell}^{(\boldsymbol{s})}(x) + \sum_{\ell=\lceil ni/(n+1)\rceil}^{s_n - 1} E_{n,\ell}^{(\boldsymbol{s})}(x),$$

with initial conditions $E_{1,0}^{(\boldsymbol{s})}(x) = 1$ and $E_{1,i}^{(\boldsymbol{s})}(x) = x$ for $0 < i < s_1$.

Proof: Omitted. □

Proposition 5.2 *For $n \geq 2$ and $0 \leq i < 2(n+1)$,*

$$T_{n+1,i}(x) = \sum_{\ell=0}^{\lceil ni/(n+1)\rceil - 1} x T_{n,\ell}(x) + \sum_{\ell=\lceil ni/(n+1)\rceil}^{2n-1} T_{n,\ell}(x),$$

with initial conditions $T_{2,0}(x) = 2$, $T_{2,1}(x) = T_{2,2}(x) = 2x$, and $T_{2,3}(x) = 2x^2$.

Proof: The initial conditions can be checked from the Table 1. For $n \geq 2$ and $e = (e_1, \ldots, e_{n+1}) \in I_{n+1}^B$ with $e_{n+1} = i$, let $\ell = e_n$. Then by the definition of the type D ascent set, $n+1 \in \operatorname{Asc}_D e$ if and only if $\ell/n < i/(n+1)$ or, equivalently, whenever $0 \leq \ell \leq \lceil ni/(n+1)\rceil - 1$. So,

$$x^{\operatorname{asc}_D e} = \begin{cases} x^{1+\operatorname{asc}_D(e_1,\ldots,e_n)} & \text{if } 0 \leq \ell \leq \lceil ni/(n+1)\rceil - 1 \\ x^{\operatorname{asc}_D(e_1,\ldots,e_n)} & \text{if } \lceil ni/(n+1)\rceil \leq \ell < 2n. \end{cases}$$

□

In Section 7, we will show that $T_{n,i}(x)$ has real roots for all $0 \leq i < 2n$ and so also does $T_n(x)$.

6 Preserving real-rootedness via compatible polynomials

A classical way to show that a recurrence given by a linear combination of two polynomials preserves real-rootedness is to show that the roots of the two polynomials interlace. We say that two real-rooted polynomials $f(x) = \prod_{i=1}^{\deg f}(x - x_i)$ and $g(x) = \prod_{j=1}^{\deg g}(x - \xi_j)$ interlace if their roots alternate, formally,

$$\cdots \leq x_2 \leq \xi_2 \leq x_1 \leq \xi_1.$$

Note that this requires the degrees to satisfy the following inequalities: $\deg f \le \deg g \le \deg f + 1$. In particular, the order of polynomials is important.

Interlacing implies real-rootedness by the following theorem.

Theorem 6.1 (Satz 5.2 in Obreschkoff (1963)) *Let $f, g \in \mathbb{R}[x]$. Then f and g interlace if and only if their arbitrary linear combination, $c_1 f(x) + c_2 g(x)$ for all $c_1, c_2 \in \mathbb{R}$ has only real roots.*

Unfortunately, the interlacing property cannot be extended to linear combinations of more than two polynomials (as pointed out in Chudnovsky and Seymour (2007)). So, instead, we will be working with a weaker property, called compatibility, that can be defined for an arbitrary number of polynomials. In fact, these properties are closely related as we will see in Lemmas 6.2 and 6.4.

Following Chudnovsky and Seymour (2007), we call a set of polynomials $f_1, \ldots, f_m \in \mathbb{R}[x]$ *compatible* if their arbitrary conic combination, i.e., $\sum_i c_i f_i(x)$ for $c_i \ge 0$, has real roots only. We say that $f_1, \ldots, f_m$ are *pairwise compatible* if f_i and f_j are compatible for all $1 \le i < j \le m$. Several useful properties of compatible polynomials were summarized in the following lemma.

Lemma 6.2 (3.6 in Chudnovsky and Seymour (2007)) *Let $f_1(x), \ldots, f_k(x)$ be polynomials with positive leading coefficients and all roots real. The following four statements are equivalent:*

- *$f_1, \ldots, f_k$ are pairwise compatible,*
- *for all s, t such that $1 \le s < t \le k$, the polynomials f_s, f_t have a common interlacer,*
- *$f_1, \ldots, f_k$ have a common interlacer,*
- *$f_1, \ldots, f_k$ are compatible,*

where f and g have a common interlacer if there is a polynomial h such that h and f interlace and also h and g interlace.

Next we give a transformation that maps a set of compatible polynomials to another set of compatible polynomials under the following conditions.

Theorem 6.3 *Given a set of polynomials $f_1, \ldots, f_m \in \mathbb{R}[x]$ with positive leading coefficients that satisfy for all $1 \le i < j \le m$ that*

(a) $f_i(x)$ and $f_j(x)$ are compatible, and

(b) $xf_i(x)$ and $f_j(x)$ are compatible

define another set of polynomials $g_1, \ldots, g_{m'} \in \mathbb{R}[x]$ by the equations

$$g_k(x) = \sum_{\ell=0}^{t_k-1} x f_\ell(x) + \sum_{\ell=t_k}^{m} f_\ell(x), \quad \text{for } 1 \le k \le m'$$

where $0 \le t_0 \le t_1 \le \ldots \le t_{m'} \le m$. Then, for all $1 \le i < j \le m'$ we have that

(a') $g_i(x)$ and $g_j(x)$ are compatible, and

(b') $xg_i(x)$ and $g_j(x)$ are compatible.

Proof: We first show (a'), i.e., that the polynomial $c_i g_i(x) + c_j g_j(x)$ has only real roots for all $c_i, c_j \geq 0$. By the definition of $g_i(x)$, $g_j(x)$ and the assumption that $t_i \leq t_j$ it is clear that

$$c_i g_i(x) + c_j g_j(x) = \sum_{\alpha=0}^{t_i-1} (c_i x + c_j x) f_\alpha(x) + \sum_{\beta=t_i}^{t_j-1} (c_i + c_j x) f_\beta(x) + \sum_{\gamma=t_j}^{m} (c_i + c_j) f_\gamma(x),$$

that is, $c_i g_i(x) + c_j g_j(x)$ can be written as a conic combination of the following polynomials, which we group into three (possibly empty) sets:

$$\{x f_\alpha(x) : 0 \leq \alpha < t_i\} \,\cup\, \{(c_i + c_j x) f_\beta(x) : t_i \leq \beta < t_j\} \,\cup\, \{f_\gamma(x) : t_j \leq \gamma \leq m\}\ .$$

Therefore, it suffices to show that these m polynomials are compatible. In fact, by Lemma 6.2, it is equivalent to show that they are pairwise compatible. This is what we do next.

First, two polynomials from the same sets are compatible by (a). Secondly, a polynomial from the first set is compatible with another from the third set by (b), since $\alpha < \gamma$. To show compatibility between a polynomial from the first set and one from the second, we need that $ax f_\alpha(x) + b(c_i + c_j x) f_\beta(x)$ has only real roots for all $a, b, c_i, c_j \geq 0$ and $\alpha < \beta$. Note that this expression is a conic combination of $x f_\alpha(x)$, $x f_\beta(x)$, and $f_\beta(x)$. Since $\alpha < \beta$, these three polynomials are again pairwise compatible by (a) and (b) (and the basic fact the $f(x)$ and $x f(x)$ are compatible), and hence compatible, by Lemma 6.2. Finally, the compatibility of a polynomial in the second set and one in the third set follows by a similar argument, exploiting the fact that, $x f_\beta(x)$, $f_\beta(x)$, and $f_\gamma(x)$ are pairwise compatible for $\beta < \gamma$.

Now we are left to show (b'), that $x g_i(x)$ and $g_j(x)$ are compatible for all $i < j$. Similarly as before, $c_i x g_i(x) + c_j g_j(x)$ is real-rooted for all $c_i, c_j \geq 0$ if

$$\{x(c_i x + c_j) f_\alpha(x) : 0 \leq \alpha < t_i\} \cup \{(c_i + c_j) x f_\beta(x) : t_i \leq \beta < t_j\} \cup \{(c_i x + c_j) f_\gamma(x) : t_j \leq \gamma \leq m\}$$

is a set of compatible polynomials. This follows from analogous reasoning to the above. Two polynomials from the same subsets are compatible by (a). Considering one from the first and one from the third: $x f_\alpha(x)$ and $f_\gamma(x)$ are compatible by (b). Similarly, $x^2 f_\alpha(x)$, $x f_\alpha(x)$, and $x f_\beta(x)$ are pairwise compatible which settles the case when we have a polynomial from the first and one from the second subset. Finally, $x f_\beta(x)$, $x f_\gamma(x)$, and $f_\gamma(x)$ are compatible, settling the case of one polynomial from the second subset and one from the third. □

Proof of Theorem 3.1: We use induction on n. When $n = 1$, for $0 \leq i \leq j < s_1$, $(E_{1,i}^{(\boldsymbol{s})}(x), E_{1,j}^{(\boldsymbol{s})}(x)) \in \{(1,1),\ (1,x),\ (x,x)\}$ and thus $(xE_{1,i}^{(\boldsymbol{s})}(x), E_{1,j}^{(\boldsymbol{s})}(x)) \in \{(x,1),\ (x,x),\ (x^2,x)\}$. Clearly, each of the pairs of polynomials $(1,1)$, $(1,x)$, (x,x), (x^2,x), is compatible. From Proposition 5.1 we see that the polynomials $E_{n,i}^{(\boldsymbol{s})}(x)$ satisfy a recurrence of the form required in Theorem 6.3, hence, by induction, they are real-rooted for all $0 \leq i < 2n$. In particular, $E_n^{(\boldsymbol{s})}(x)$ has only real roots for $n \geq 1$ (and arbitrary sequence $\boldsymbol{s}$).

□

6.1 Connection to interlacing

The condition of Theorem 6.3 can be simplified since our polynomials have positive coefficients.

Lemma 6.4 *Let $f, g \in \mathbb{R}[x]$ be polynomials with positive coefficients. Then the following are equivalent:*

- *$f(x)$ and $g(x)$ are compatible, and $xf(x)$ and $g(x)$ are also compatible.*
- *$f(x)$ and $g(x)$ interlace.*

Proof: Let $n_f(x_0)$ denote the number of roots of the polynomial f in the interval $[x_0, \infty)$. There is an equivalent formulation for both compatibility and interlacing in terms of this notion. First, f and g are compatible if and only if $|n_f(x_0) - n_g(x_0)| \leq 1$ for all $x_0 \in \mathbb{R}$ (see 3.5 in Chudnovsky and Seymour (2007)). Secondly, by definition, f and g interlace, if and only if $0 \leq n_g(x_0) - n_f(x_0) \leq 1$ for all $x_0 \in \mathbb{R}$. We also have that $n_{xf}(x_0) = n_f(x_0) + \chi(x_0 \leq 0)$. Therefore, the following equivalence settles the lemma (since all roots of f and g are nonpositive, we can assume that $x_0 \leq 0$):

$$|n_f(x_0) - n_g(x_0)| \leq 1 \text{ and } |n_f(x_0) + 1 - n_g(x_0)| \leq 1 \iff 0 \leq n_g(x_0) - n_f(x_0) \leq 1 .$$

□

Remark 6.5 *Lemma 6.4 appeared (without a proof) as Lemma 3.4 in Wagner (2000).*

7 The Eulerian polynomials of type D have only real roots

Now we are in position to prove Conjecture 1.1.

Theorem 7.1 *For $n \geq 2$, the polynomial $T_n(x)$ has only real roots. In fact, for $0 \leq i < 2n$, $T_{n,i}(x)$ has only real roots.*

Proof: Clearly, $T_2(x) = 2(x+1)^2$ has only real roots, but $T_{2,0}(x) = 2, T_{2,1}(x) = T_{2,2}(x) = 2x, T_{2,3}(x) = 2x^2$ fail to be compatible. Using the recurrence given in Proposition 5.2 we can compute $T_{n,i}$ for $n = 3$. It is easy to check that $T_{3,0} = 2(x+1)^2, T_{3,1}(x) = 2x(x+3), T_{3,2}(x) = T_{3,3}(x) = 4x(x+1), T_{3,4}(x) = 2x(3x+1), T_{3,5}(x) = 2x(x+1)$ are compatible polynomials—hence $T_3(x)$ has only real roots—but $xT_{3,0}(x)$ and $T_{3,1}(x)$ do not interlace and thus they don't satisfy the assumption needed for Theorem 6.3. However, iterating one more time, we obtain the following polynomials.

$$\begin{array}{lll}
T_{4,0}(x) = & 2(x+1)(x^2+10x+1) & \{-9.899, -1, -0.101\} \\
T_{4,1}(x) = & 4x(x+1)(x+5) & \{-5, -1, 0\} \\
T_{4,2}(x) = & 2x(3x^2+14x+7) & \{-4.097, -0.569, 0\} \\
T_{4,3}(x) = & 2x(5x^2+14x+5) & \{-2.380, -0.420, 0\} \\
T_{4,4}(x) = & 2x(5x^2+14x+5) & \{-2.380, -0.420, 0\} \\
T_{4,5}(x) = & 2x(7x^2+14x+3) & \{-1.756, -0.244, 0\} \\
T_{4,6}(x) = & 4x(x+1)(5x+1) & \{-1, -0.2, 0\} \\
T_{4,7}(x) = & 2x(x+1)(x^2+10x+1) & \{-9.899, -1, -0.101, 0\}.
\end{array}$$

One can check the roots explicitly (approximate values are given above for the reader's convenience) to see that $T_{4,i}(x)$ and $T_{4,i+1}(x)$ interlace for all $0 \leq i \leq 6$. By Lemma 6.4, this means that the polynomials $T_{4,0}(x), \ldots, T_{4,7}(x)$ are compatible and also that $xT_{4,i}(x)$ and $T_{4,j}(x)$ are compatible for $0 \leq i < j \leq 7$. Therefore, by induction on n, and successive applications of Theorem 6.3 we get that for all $n \geq 4$, $\{T_{n,i}(x)\}_{0 \leq i \leq 2n-1}$ is a set of pairwise interlacing polynomials. In particular, this implies that they are compatible, hence $T_n(x)$ has only real roots for all $n \geq 4$ as well. □

8 Further implications: real-rooted affine Eulerian polynomials

Dilks, Petersen, and Stembridge (2009) recently defined Eulerian-like polynomials associated to irreducible affine Weyl groups and proposed two companion conjectures to Brenti's conjecture. In this section, we prove one of them. The affine Eulerian polynomial of type B is defined in (Dilks et al., 2009, Section 5.3) as the generating function of the "affine descents" over the corresponding finite Weyl group, $\mathfrak{B}_n$,

$$\widetilde{B}_n(x) = \sum_{\sigma \in \mathfrak{B}_n} x^{\widetilde{\operatorname{des}}_B \sigma},$$

where for a signed permutation $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathfrak{B}_n$ the affine descent statistic is computed as

$$\widetilde{\operatorname{des}}_B \sigma = \chi(\sigma_1 < 0) + |\{i \in [n-1] : \sigma_i > \sigma_{i+1}\}| + \chi(\sigma_{n-1} + \sigma_n > 0).$$

We now prove Conjecture 1.2. Notice the affine Eulerian polynomial of type B is intimately related to the type D Eulerian polynomial in the following way.

Theorem 8.1 *For $n \geq 2$,*

$$\widetilde{B}_n(x) = T_{n+1,n+1}(x)\,,$$

where $T_{n,i}(x)$ is the refined Eulerian polynomial of type D defined in (9).

Proof: It is easy to see, under the involution $(\sigma_1, \ldots, \sigma_n) \mapsto (-\sigma_n, \ldots, -\sigma_1)$, that $\widetilde{\operatorname{des}}_B$ has the same distribution over $\mathfrak{B}_n$ as the statistic

$$\widetilde{\operatorname{stat}}_B \sigma = \chi(\sigma_n > 0) + |\{i \in [n-1] : \sigma_i > \sigma_{i+1}\}| + \chi(\sigma_2 + \sigma_1 < 0).$$

From Theorem 4.5 part 3 it follows that $\sigma_2 + \sigma_1 < 0$ is equivalent to $e_1 + e_2/2 > 3/2$ and from part 2 we have that $\sigma_n > 0$ if and only if $e_n < n$. Note that $e_n < n$ is equivalent to $e_n/n < 1 = (n+1)/(n+1)$. So, $\widetilde{B}_n(x) = T_{n+1,n+1}(x)$. □

Corollary 8.2 *For $n \geq 2$, $\widetilde{B}_n(x)$ has only real roots.*

Proof: Follows from the fact that $T_{n,i}(x)$ have only real roots (see Theorem 7.1). □

We should mention that there is an analogous conjecture for type D which remains unsolved.

Conjecture 8.3 (Dilks et al. (2009)) *Let*

$$\widetilde{\operatorname{des}}_D \sigma = \chi(\sigma_1 + \sigma_2 < 0) + |\{i \in [n-1] : \sigma_i > \sigma_{i+1}\}| + \chi(\sigma_{n-1} + \sigma_n > 0)\,.$$

Then the affine Eulerian polynomial of type D

$$\sum_{\sigma \in \mathfrak{D}_n} x^{\widetilde{\operatorname{des}}_D}$$

has only real roots.

By Theorem 4.5 (parts 2, 4 and 5) we can at least express the type D affine Eulerian polynomial in terms of ascent statistics on inversion sequences.

Corollary 8.4 *The type D affine Eulerian polynomial satisfies*

$$2\sum_{\sigma\in\mathfrak{D}_n} x^{\widetilde{\operatorname{des}}_D \sigma} = \sum_{e\in I_n^B} x^{\widetilde{\operatorname{asc}}_D e},$$

where the type D affine ascent statistic is given by

$$\widetilde{\operatorname{asc}}_D e = \chi\left(e_1 + \frac{e_2}{2} \geq \frac{3}{2}\right) + \left|\left\{i \in [n-1] : \frac{e_i}{i} < \frac{e_{i+1}}{i+1}\right\}\right| + \chi\left(\frac{e_{n-1}}{n-1} + \frac{e_n}{n} < \frac{2n-1}{n}\right).$$

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On Kerov polynomials for Jack characters[†]

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Abstract. We consider a deformation of Kerov character polynomials, linked to Jack symmetric functions. It has been introduced recently by M. Lassalle, who formulated several conjectures on these objects, suggesting some underlying combinatorics. We give a partial result in this direction, showing that some quantities are polynomials in the Jack parameter α with prescribed degree.

Our result has several interesting consequences in various directions. Firstly, we give a new proof of the fact that the coefficients of Jack polynomials expanded in the monomial or power-sum basis depend polynomially in α. Secondly, we describe asymptotically the shape of random Young diagrams under some deformation of Plancherel measure.

Résumé. On considère une déformation des polynômes de Kerov pour les caractères du groupe symétrique. Cette déformation est liée aux polynômes de Jack. Elle a été récemment définie par M. Lassalle, qui a proposé plusieurs conjectures sur ces objets, suggérant ainsi l'existence d'une combinatoire sous-jacente. Nous donnons un résultat partiel dans cette direction, en montrant que certaines quantités sont des polynômes (dont on contrôle les degrés) en fonction du paramètre de Jack α.

Notre résultat a des conséquences intéressantes dans des directions diverses. Premièrement, nous donnons une nouvelle preuve de la polynomialité (toujours en fonction de α) des coefficients du développement des polynômes de Jack dans la base monomiale. Deuxièmement, nous décrivons asymptotiquement la forme de grands diagrammes de Young distribués selon une déformation de la mesure de Plancherel.

Keywords: Jack polynomials; Kerov's polynomials; free cumulants; Young diagrams

1 Introduction

1.1 Polynomiality of Jack polynomials

In a seminal paper [9], H. Jack introduced a family of symmetric functions $J_\lambda^{(\alpha)}$ depending on an additional parameter α called *Jack polynomials*. Up to multiplicative constants, for $\alpha = 1$, Jack polynomials coincide with Schur polynomials. Over the time, it has been shown that several results concerning Schur polynomials can be generalized in a rather natural way to Jack polynomials (Section (VI,10) of I.G. Macdonald's book [20] gives a few results of this kind).

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One of the most surprising features of Jack polynomials is that they have several equivalent classical definitions, but none of them makes obvious the fact that the coefficients of their expansion on the monomial basis are polynomials in α (by construction, they are only rational functions). This property has been established by Lapointe and Vinet [16]. One of the result of this paper is a new proof of Lapointe-Vinet theorem.

Theorem 1.1 (Lapointe and Vinet [16]) *The coefficients of the expansion of Jack polynomials in the monomial basis are polynomials in α.*

This theorem is proved in Section 3.2. We believe that this new proof is interesting in itself, because it relies on a very different approach to Jack polynomials.

To be comprehensive on the subject, let us mention that the coefficients of these polynomials are in fact non-negative integers. This result had been conjectured by R. Stanley and I. Macdonald; see *e.g.* [20, VI, equation (10.26?)]. It was proved by Knop and Sahi [15], shortly after Lapointe-Vinet's paper. We are unfortunately unable to prove this stronger result with our methods.

1.2 Dual approach

We will later define *Jack character* to be equal (up to some simple normalization constant) to the coefficient $[p_\mu]J_\lambda$ in the expansion of the Jack polynomial J_λ in the basis of power-sum symmetric functions. The idea of the *dual approach* is to consider Jack characters as a function of λ and not as a function of μ as usual. In more concrete words, we would like to express the Jack character as a sum of some quantities depending on λ over some combinatorial set depending on μ (in Knop-Sahi's result, it is roughly the opposite).

Inspired by the case $\alpha = 1$ (which corresponds to the usual characters of the symmetric groups), Lassalle [18] suggested to express Jack characters in terms of, so called, free cumulants of the transition measure of the Young diagram λ. This expression, called *Kerov polynomials for Jack characters*, involves rational functions in α, which are conjecturally polynomials with non-negative integer coefficients in α and $\beta = 1 - \alpha$ (see [18, Conjecture 1.2]); we refer to this as Lassalle's conjecture. This suggests the existence of a combinatorial interpretation. A result of this type holds true in the case $\alpha = 1$, see [5].

In this paper, we prove a part of Lassalle's conjecture, that is the polynomiality in α (but neither the non-negativity, nor the integrity) of the coefficients.

Theorem 1.2 *The coefficients of Kerov polynomials for Jack characters are polynomials in α with rational coefficients.*

This theorem with a precise bound on the degree of these polynomials is stated in Section 3.1. In this extended abstract, we only give the guidelines of the proof.

1.3 Applications

Our bounds for degrees of coefficients of Kerov polynomials for Jack characters imply in particular that some coefficients (corresponding to the leading term for some gradation) are independent on α. In Section 4, we use this simple remark to describe asymptotically the shape of random Young diagrams whose distribution is a deformation of Plancherel measure.

Another consequence of our results is a uniform proof of the polynomiality of structure constants of several meaningful algebras. This allows us to solve some conjectures of Matsumoto [21] and to give a partial answer to the Matching-Jack conjecture of I. Goulden and D. Jackson [7]. Due to the lack of space, we will not present these results in this extended abstract. They can be found in [4, Section 4].

Outline of the paper. Section 2 gives all necessary definitions and background; in particular we recall the notions of free cumulants and Kerov polynomials. In Section 3 we sketch the proof of Theorem 1.2 and Theorem 1.1. Finally, we state and prove our results on large Young diagrams in Section 4.

2 Jack characters and Kerov polynomials

2.1 Partitions and symmetric functions

We begin with a few classical definitions and notations.

A *partition* λ of n (denote it by $\lambda \vdash n$) is a non-increasing list $(\lambda_1, \dots, \lambda_\ell)$ of positive integers of sum equal to n. Then n is called the *size* of λ and denoted by $|\lambda|$ and the number ℓ is the *length* of the partition (denoted by $\ell(\lambda)$).

We also consider the graded ring of symmetric functions Sym. Recall that its homogeneous component Sym_n of degree n admits several classical bases: the monomials $(m_\lambda)_{\lambda\vdash n}$, the power-sums $(p_\lambda)_{\lambda\vdash n}$, each indexed by partitions of n. All the definitions can be found in [20, Chapter I].

Jack polynomials are symmetric functions indexed by partitions and depending on a parameter (α). There exist several normalizations for Jack polynomials in the literature. We shall work with the one denoted by J in the book of Macdonald [20, VI, (10.22)] and use the same notation as he does. For a fixed value of the parameter α, the family $(J_\lambda^{(\alpha)})_{\lambda\vdash n}$ forms a basis of Sym_n.

2.2 Jack characters

As power-sum symmetric functions $(p_\rho)_{\rho\vdash n}$ form a basis of Sym_n, we can expand the Jack polynomial $J_\lambda^{(\alpha)}$ in that base. For $\lambda \vdash n$, there exist (unique) coefficients $\theta_\rho^{(\alpha)}(\lambda)$ such that

$$J_\lambda^{(\alpha)} = \sum_{\substack{\rho: \\ |\rho|=|\lambda|}} \theta_\rho^{(\alpha)}(\lambda)\, p_\rho. \tag{1}$$

Then we can define *Jack characters* by the formula:

$$Ch_\mu^{(\alpha)}(\lambda) = \alpha^{-\frac{|\mu|-\ell(\mu)}{2}} \binom{|\lambda| - |\mu| + m_1(\mu)}{m_1(\mu)} z_\mu\, \theta^{(\alpha)}_{\mu,1^{|\lambda|-|\mu|}}(\lambda),$$

where $m_i(\mu)$ denotes the multiplicity of i in the partition μ and $z_\mu = \mu_1\mu_2 \cdots m_1(\mu)!\, m_2(\mu)! \cdots$.

In the case $\alpha = 1$, Jack polynomials correspond, up to some normalization constants, to Schur symmetric functions. The coefficients of the latter in the basis of the power-sum symmetric functions are known to be equal to the irreducible characters of the symmetric groups; see [20, Section I,7] (this explains the name *characters* in the general case, even if, except for $\alpha = 1/2, 1, 2$, these quantities have no known representation-theoretical interpretation). It means that Jack characters with parameter $\alpha = 1$ correspond, up to some numerical factors, to character values of the symmetric groups.

This normalization corresponds in fact to the one used by Kerov and Olshanski in [13]. These *normalized characters* – following the denomination of Kerov and Olshanski – have plenty of interesting properties; for example when considered as functions on the set of Young diagrams $\lambda \mapsto Ch_\mu^{(1)}(\lambda)$, they form a linear basis (when μ runs over the set of all partitions) of the algebra $\Lambda^\star$ of shifted symmetric functions , which is very rich in structure.

Jack characters have been first considered by M. Lassalle in [17]. Note that the normalization used here is different that the one of these papers. The reason of this new choice of normalization will be clear later.

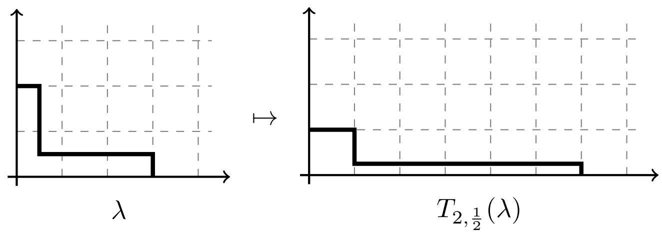

Fig. 1: Example of stretched Young diagram.

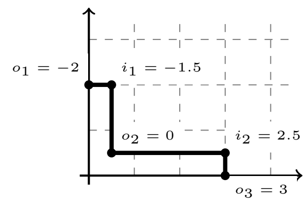

Fig. 2: A generalized Young diagram L with the corresponding set $\mathbb{O}_L$ and $\mathbb{I}_L$.

2.3 Generalized Young diagrams and Kerov interlacing coordinates

In this section, we will see different ways of representing Young diagrams and even more general objects related to them. Let us consider a zigzag line L going from a point $(0, y)$ on the y-axis to a point $(x, 0)$ on the x-axis. We assume that every piece is either an horizontal segment from left to right or a vertical segment from top to bottom. A Young diagram can be seen as such a zigzag line: just consider its border. Therefore, we call these zigzag lines *generalized Young diagrams*.

We will be in particular interested in the following generalized Young diagrams. Let λ be a (generalized) Young diagram and s and t two positive real numbers. We denote by $T_{s,t}(\lambda)$ the broken line obtained by stretching λ horizontally by a factor s and vertically by a factor t (see Figure 1; we use french convention to draw Young diagrams). These *anisotropic* Young diagrams have been introduced by S. Kerov in [11]. In the special case $s = t$, we denote by $D_s(\lambda) = T_{s,s}(\lambda)$ the *dilated* Young diagram.

The content of a point of a plane is the difference of its x-coordinate and its y-coordinate. We denote by $\mathbb{O}_L$ the sets of contents of the *outer corners* of L, that is corners which are points of L connecting a horizontal line on the left with vertical line on the bottom. Similarly, the set $\mathbb{I}_L$ is defined as the contents of the *inner corners*, that is corners which are points of L connecting a horizontal line on the right with vertical line above. An example is given on Figure 2. The denomination *inner/outer* may seem strange, but it refers to the fact that the box in the corner is inside or outside the diagram.

A generalized Young diagram can also be seen as a function on the real line. Indeed, if one rotates the zigzag line counterclockwise by $45°$ and scale it by a factor $\sqrt{2}$ (so that the new x-coordinate corresponds to contents), then it can be seen as the graph of a piecewise affine continuous function with slope ± 1. We denote this function by $\omega(\lambda)$. Therefore, we shall call *continuous Young diagram* a Lipshitz continuous function ω with Lipshitz constant 1 such that $\omega(x) = |x|$ for $|x|$ big enough . This notion will be used in Section 4 to describe the limit shape of Young diagrams.

2.4 Polynomial functions on the set of Young diagrams

If k is a positive integer, one can consider the power sum symmetric function p_k, evaluated on the difference of alphabets $\mathbb{O}_L - \mathbb{I}_L$. By definition, it is a function on generalized Young diagrams given by:

$$L \mapsto p_k(\mathbb{O}_L - \mathbb{I}_L) := \sum_{o \in \mathbb{O}_L} o^k - \sum_{i \in \mathbb{I}_L} i^k.$$

As any symmetric function can be written (uniquely) in terms of p_k, we can define $f(\mathbb{O}_L - \mathbb{I}_L)$ for any symmetric function f as follows. Expand f on the power-sum basis $f = \sum_\rho a_\rho p_{\rho_1} \cdots p_{\rho_\ell}$ for some family of scalars (a_ρ) indexed by partitions. Then, by definition

$$f(\mathbb{O}_L - \mathbb{I}_L) = \sum_{\rho \text{ partition}} a_\rho p_{\rho_1}(\mathbb{O}_L - \mathbb{I}_L) \cdots p_{\rho_\ell}(\mathbb{O}_L - \mathbb{I}_L).$$

This convenient notation is classical in lambda-ring calculus.

Consider the set of functions $\{\lambda \mapsto f(\mathbb{O}_L - \mathbb{I}_L)\}$, where f describes the set of symmetric functions. This is a subalgebra of the algebra of functions on the set of all Young diagrams. Following S. Kerov and G. Olshanski, we shall call it the algebra of *polynomial functions* and denote it by $\Lambda^\star$.

V. Ivanov and G. Olshanski [8, Corollary 2.8] have shown that the normalized characters $(\lambda \mapsto Ch_\mu^{(1)}(\lambda))_\mu$ form a linear basis of this algebra and $(\lambda \mapsto p_k(\mathbb{O}_\lambda - \mathbb{I}_\lambda))_{k\geq 2}$ forms an algebraic basis of $\Lambda^\star$ (for all diagrams λ, one has $p_1(\mathbb{O}_\lambda - \mathbb{I}_\lambda) = 0$). This algebra admits several other characterization: for instance it corresponds to the algebra of shifted symmetric functions (see [8, Section 1 and 2]).

All this has a natural extension for a general parameter α.

We say that F is an *α-polynomial function* on the set of (generalized) Young diagrams if

$$\lambda \mapsto F(T_{\sqrt{\alpha}^{-1},\sqrt{\alpha}}(\lambda))$$

is a polynomial function. The ring of α polynomial functions is denoted by $\Lambda_\star^{(\alpha)}$. Then $(\lambda \mapsto Ch_\mu^{(\alpha)}(\lambda))_\mu$ forms a linear basis of $\Lambda_\star^{(\alpha)}$. This is a consequence of a result of M. Lassalle [17, Proposition 2].

Remark 1 *Lassalle's result is in fact formulated in terms of shifted symmetric functions, but as mentioned above, it is proved in [8, Section 1 and 2] that they correspond to polynomial functions.*

Fact 2 *With the definitions above, it should be clear that polynomial functions are defined on* generalized *Young diagrams. They can in fact also be canonically extended to* continuous *Young diagrams; see [1, Section 1.2]. This will be useful in Section 4.*

2.5 Transition measure and free cumulants

S. Kerov [10] introduced the notion of *transition measure* of a Young diagram. This probability measure μ_λ associated to λ is defined by its Cauchy transform

$$G_{\mu_\lambda}(z) = \int_{\mathbb{R}} \frac{d\mu_\lambda(x)}{z-x} = \frac{\prod_{i\in\mathbb{I}_\lambda} z-i}{\prod_{o\in\mathbb{O}_\lambda} z-o}.$$

Its moments are $h_k(\mathbb{O}_\lambda - \mathbb{I}_\lambda)$, where h_k is the complete symmetric function of degree k, hence they are polynomial functions on the set of Young diagrams; we will denote them by $M_k^{(1)}$.

In Voiculescu's free probability it is very convenient to associate to a probability measure μ a sequence of numbers $(R_k(\mu))_{k\geq 1}$ called *free cumulants* [26]. The free cumulants of the transition measure of Young diagrams appeared first in the work of P. Biane [1] and play an important role in the asymptotic representation theory. As explained by M. Lassalle (look at the case $\alpha = 1$ of [18, Section 5]), they can be expressed as

$$R_k^{(1)}(\lambda) := R_k(\mu_\lambda) = e_k^\star(\mathbb{O}_\lambda - \mathbb{I}_\lambda)$$

for some homogeneous symmetric function $e_k^\star$ of degree k. Note also that $(e_k^\star)_k$ as well as complete symmetric functions $(h_k)_k$ are algebraic basis of symmetric functions and, hence $(R_k^{(1)})_{k\geq 2}$ as well as

$(M_k^{(1)})_{k\geq 2}$ are algebraic basis of ring of polynomial functions on the set of Young diagrams ($R_1^{(1)} = M_1^{(1)}$ is the null function).

Fact 3 *It is easy to see that, as h_k and $e_k^\star$ are homogeneous symmetric functions, the corresponding polynomial functions $M_k^{(1)}$ and $R_k^{(1)}$ are compatible with dilations. Namely*

$$M_k^{(1)}\big(D_s(\lambda)\big) = s^k M_k^{(1)}\big(\lambda\big); \quad R_k^{(1)}\big(D_s(\lambda)\big) = s^k R_k^{(1)}\big(\lambda\big).$$

Using the relevant definitions, the α-anisotropic moments and free cumulants defined by

$$M_k^{(\alpha)}(\lambda) = M_k^{(1)}\left(T_{\sqrt{\alpha},\sqrt{\alpha}^{-1}}(\lambda)\right),$$
$$R_k^{(\alpha)}(\lambda) = R_k^{(1)}\left(T_{\sqrt{\alpha},\sqrt{\alpha}^{-1}}(\lambda)\right)$$

are α-polynomial and the families $(M_k^{(\alpha)})_{k\geq 2}$ and $(R_k^{(\alpha)})_{k\geq 2}$ are two algebraic basis of the algebra $\Lambda^\star_{(\alpha)}$.

2.6 Kerov polynomials

Recall that Jack characters $Ch_\mu^{(\alpha)}$ are α-polynomial functions hence can be expressed in terms of the two algebraic bases above.

Definition-Proposition 2.1 *Let μ be a partition and $\alpha > 0$ be a fixed real number. There exist unique polynomials $L_\mu^{(\alpha)}$ and $K_\mu^{(\alpha)}$ such that, for every λ,*

$$Ch_\mu^{(\alpha)}(\lambda) = L_\mu^{(\alpha)}\left(M_2^{(\alpha)}(\lambda), M_3^{(\alpha)}(\lambda), \cdots\right),$$
$$Ch_\mu^{(\alpha)}(\lambda) = K_\mu^{(\alpha)}\left(R_2^{(\alpha)}(\lambda), R_3^{(\alpha)}(\lambda), \cdots\right).$$

The polynomials $K_\mu^{(\alpha)}$ have been introduced by S. Kerov in the case $\alpha = 1$ [12] and by M. Lassalle in the general case [18]. Once again, we emphasize that our normalizations are different from his.

From now on, when it does not create any confusion, we suppress the superscript (α).

We present a few examples of polynomials K_μ. This data has been computed using the one given in [18, page 2230]

$$\begin{aligned}
K_{(1)} &= R_2,\\
K_{(2)} &= R_3 + \gamma R_2,\\
K_{(3)} &= R_4 + 3\gamma R_3 + (1+2\gamma^2)R_2,\\
K_{(4)} &= R_5 + \gamma(6R_4 + R_2^2) + (5+11\gamma^2)R_3 + (7\gamma + 6\gamma^3)R_2,\\
K_{(2,2)} &= R_3^2 + 2\gamma R_3 R_2 - 4R_4 + (\gamma^2 - 2)R_2^2 - 10\gamma R_3 - (6\gamma^2+2)R_2.
\end{aligned}$$

where we set $\gamma = \frac{1-\alpha}{\sqrt{\alpha}}$. A few striking facts appear on these examples:

- All coefficients are polynomials in the auxiliary parameter γ: the sketch of the proof of this fact will be presented in the next section with explicit bounds on the degrees.

- For one part partition, polynomials $K_{(r)}$ have non-negative coefficients. We are unfortunately unable to prove this statement, which is a more precise version of [18, Conjecture 1.2]. A similar conjecture holds for several part partitions, see [18, Conjecture 1.2].

Remark 4 *This facts explain our changes of normalization. In Lassalle's work, the non-negativity of the coefficients is hidden: he has to use two variables α and $\beta = 1 - \alpha$ and choose a "natural" way to write all quantities in terms of α and β. Using our normalizations and the parameter γ, the non-negativity of the coefficients appears directly.*

3 Polynomiality

3.1 Main result

Theorem 3.1 *The coefficient of M_ρ in Jack character polynomial L_μ is a polynomial in γ of degree smaller or equal to*

$$\min\big(|\mu| + \ell(\mu) - |\rho|, |\mu| - \ell(\mu) - (|\rho| - 2\ell(\rho))\big).$$

Moreover, it has the same parity as the integer $|\mu| + \ell(\mu) - |\rho|$.

The same is true for the coefficient of R_ρ in K_μ.

We do not prove this theorem in this extended abstract. The proof is of course available in the long version of the paper [4, Section 3]. Here, we are going to present a guidelines of this proof.

The difficulty is that the proof of the existence of the polynomials L_μ and K_μ (Proposition 2.1) is not constructive. However, M. Lassalle gives an algorithm to compute the polynomial K_μ [18, Section 9], but his algorithm involves an induction on the size of the partition $|\mu|$. The coefficients of K_μ are the solutions of an overdetermined linear system involving the coefficients of $K_{\mu'}$, for some partitions μ' with $|\mu'| < |\mu|$. His algorithm can be easily adapted to L_μ [4, Section 3.2].

Our proof relies on this work and on the two following important facts:

- the linear system computing the coefficients of L_μ contains a triangular subsystem (this is not true with K_μ);
- with our normalization of Jack characters and anisotropic moments, the diagonal coefficients of this linear subsystem are independent of γ (and hence invertible in $\mathbb{Q}[\gamma]$).

The polynomiality in γ follows from these two facts. To obtain the bound on the degree, one has to look carefully at the degrees of the coefficients of the linear system.

Recall that our normalization is different from the one used by M. Lassalle. After a simple rewriting game [4, Section 3.6], we can see that Theorem 3.1 implies that the coefficients of L_μ and K_μ with Lassalle's normalizations are polynomials in α (that is the statement of Theorem 1.2).

3.2 Lapointe-Vinet theorem

In this section, we prove that $\theta_\mu(\lambda)$ is a polynomial in α. This result was already known (see Introduction), but in our opinion it illustrates that Lassalle's approach to Jack polynomials is relevant.

To deduce this from the results above, one has to see how $M_k(\lambda)$ depends on α.

Lemma 3.2 *Let $k \geq 2$ be an integer and λ be a partition. Then $\sqrt{\alpha}^{k-2} M_k(\lambda)$ is a polynomial in α with integer coefficients.*

Proof: We use induction over $|\lambda|$. Let $o = (x, y)$ be an outer corner of λ, we denote by $\lambda^{(o)}$ the diagram obtained from λ by adding a box at place o. Comparing the corner of $\lambda^{(o)}$ and λ, we get that:

$$\mathbb{O}_{\lambda^{(o)}} - \mathbb{I}_{\lambda^{(o)}} = \mathbb{O}_{\lambda} - \mathbb{I}_{\lambda} + \{x - (y+1)\} + \{x + 1 - y\} - \{x - y\}$$

(for readers not used to λ-ring, this equality can be understood as equality between formal sums of elements in the set). After dilatation, we get

$$\mathbb{O}_{T_{\sqrt{\alpha},\sqrt{\alpha}^{-1}}(\lambda^{(o)})} - \mathbb{I}_{T_{\sqrt{\alpha},\sqrt{\alpha}^{-1}}(\lambda^{(o)})} = \mathbb{O}_{T_{\sqrt{\alpha},\sqrt{\alpha}^{-1}}(\lambda)} - \mathbb{I}_{T_{\sqrt{\alpha},\sqrt{\alpha}^{-1}}(\lambda)} + \{z_o - \frac{1}{\sqrt{\alpha}}\} + \{z_o + \alpha\} - \{z_o\},$$

and $z_o = \sqrt{\alpha}x - y/\sqrt{\alpha}$ is the content of the considered corner in $T_{\sqrt{\alpha},\sqrt{\alpha}^{-1}}(\lambda)$.

By a standard λ-ring computations (see [18, Proposition 8.3]), this yields

$$M_k(\lambda^{(o)}) - M_k(\lambda) = \sum_{\substack{r\geq 1, s,t\geq 0,\\ 2r+s+t\leq k}} z_o^{k-2r-s-t} \binom{k-t-1}{2r+s-1} \binom{r+s-1}{s} (-\gamma)^s M_t(\lambda),$$

which can be rewritten as

$$\sqrt{\alpha}^{k-2} M_k(\lambda^{(o)}) - \sqrt{\alpha}^{k-2} M_k(\lambda) = \sum_{\substack{r\geq 1, s,t\geq 0,\\ 2r+s+t\leq k}} \alpha^r (\sqrt{\alpha} z_o)^{k-2r-s-t} \binom{k-t-1}{2r+s-1} \binom{r+s-1}{s} (\alpha - 1)^s \sqrt{\alpha}^{t-2} M_t(\lambda).$$

But $\sqrt{\alpha} z_o = \alpha x - y$ is a polynomial in α with integer coefficients. Thus the induction is immediate. □

Now we write, for $\mu, \lambda \vdash n$,

$$z_\mu \theta_\mu(\lambda) = \alpha^{\frac{|\mu|-\ell(\mu)}{2}} Ch_\mu(\lambda) = \sum_\rho \alpha^{\frac{|\mu|-\ell(\mu)-(|\rho|-2\ell(\rho))}{2}} a^\mu_\rho \left(\prod_{i\leq \ell(\rho)} \sqrt{\alpha}^{\rho_i - 2} M_{\rho_i}(\lambda) \right).$$

The quantities $\alpha^{\frac{|\mu|-\ell(\mu)-(|\rho|-2\ell(\rho))}{2}} a^\mu_\rho$ and $\sqrt{\alpha}^{\rho_i-2} M_{\rho_i}(\lambda)$ are polynomials in α (by Theorem 3.1 and Lemma 3.2), hence $\theta_\mu(\lambda)$ is a polynomial in α, which proves Theorem 1.1.

3.3 Gradation

Looking at Theorem 3.1 it makes natural to consider some gradations on $\Lambda_\star^{(\alpha)}$. This structure will also be useful in the next section.

The ring $\Lambda_\star^{(\alpha)}$ of α-polynomial functions can be endowed with a gradation by deciding that M_k is a homogeneous function of degree k: as $(M_k)_{k\geq 2}$ is an algebraic basis of $\Lambda_\star^{(\alpha)}$, any choice of degree for M_k (for all $k \geq 2$) defines uniquely a gradation on $\Lambda_\star^{(\alpha)}$. Then R_k is also a homogeneous function of degree k, thanks to the moment-free cumulant relations, see *e.g.* [1, Section 2.4].

Theorem 3.1 shows that Ch_μ has at most degree $|\mu| + \ell(\mu)$ (this has also been proved by M. Lassalle [18, Proposition 9.2 (ii)]). Note that Ch_μ is not homogeneous in general. Moreover, its component of degree $|\mu| + \ell(\mu)$ does not depend on α. As this dominant term is known in the case $\alpha = 1$ (see for example [24, Theorem 4.9]), one obtains the following result (which extends [18, Theorem 10.2]):

$$Ch_\mu = \prod_{i=1}^{\ell(\mu)} R_{\mu_i+1} + \text{smaller degree terms.}$$

In particular, Ch_μ has exactly degree $|\mu| + \ell(\mu)$.

Consider the subspace $V_{\leq d} \subset \Lambda_\star^{(\alpha)}$ of elements of degree less or equal to d. Its dimension is the number of partitions ρ of size less or equal to d with no parts equal to 1. By removing 1 from every part of ρ, we see that this is also the number of partitions μ such that $|\mu| + \ell(\mu) \leq d$. But the latter index the functions Ch_μ lying in $V_{\leq d}$. Hence,

$$V_{\leq d} = \mathrm{Vect}\left(\{Ch_\mu, |\mu| + \ell(\mu) \leq d\}\right)$$

and the degree of an element in $\Lambda_\star^{(\alpha)}$ can be determined as follows:

$$\deg\left(\sum_\mu a_\mu Ch_\mu\right) = \max_{\mu : a_\mu \neq 0} |\mu| + \ell(\mu). \tag{2}$$

Remark 5 *The algebra $\Lambda_\star^{(\alpha)}$ admits other relevant gradations, see [4, Sections 3.5 and 3.8].*

4 Application: asymptotics of large Young diagrams

We consider the following deformation of the Plancherel measure $\mathbb{P}_n^{(\alpha)}(\lambda) = \frac{\alpha^n n!}{j_\lambda^{(\alpha)}}$, where $j_\lambda^{(\alpha)}$ is the following deformation of the square of the hook products:

$$j_\lambda^{(\alpha)} = \prod_{\square \in \lambda} (\alpha a(\square) + \ell(\square) + 1)(\alpha a(\square) + \ell(\square) + \alpha). \tag{3}$$

Here, $a(\square)$ and $\ell(\square)$ are respectively the arm and leg length of the box as defined in [20, Chapter I]. The probability measure $\mathbb{P}_n^{(\alpha)}$ on Young diagrams of size n has appeared recently in several research papers [3, 6, 23, 21] and is presented as an important area of research in Okounkov's survey on random partitions [22, § 3.3]. When $\alpha = 1$, it specializes to the well-known Plancherel measure for the symmetric groups.

The following property, which corresponds to the case $\pi = (1^n)$ in [21, Equation (8.4)], characterizes the Jack-Plancherel measure:

$$\mathbb{E}_{\mathbb{P}_n^{(\alpha)}}(\theta_\mu^{(\alpha)}(\lambda)) = \delta_{\mu,1^n},$$

where λ is a random variable distributed according to $\mathbb{P}_n^{(\alpha)}$.

Using the definition of Ch_μ we have:

$$\mathbb{E}_{\mathbb{P}_n^{(\alpha)}}(Ch_\mu) = \begin{cases} n(n-1)\cdots(n-k+1) & \text{if } \mu = 1^k \text{ for some } k \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

As Ch_μ is a linear basis of $\Lambda_\star^{(\alpha)}$, it implies the following lemma (which is an analogue of [23, Theorem 5.5] with another gradation).

Lemma 4.1 *Let F be an α-polynomial function. Then $\mathbb{E}_{\mathbb{P}_n^{(\alpha)}}(F)$ is a polynomial in n of degree at most $\deg(F)/2$.*

Proof: It is enough to verify this lemma on the basis Ch_μ because of equation (2). But in this case $\mathbb{E}_{\mathbb{P}_n^{(\alpha)}}(F)$ is explicit (see formula above) and the lemma is obvious (recall that $\deg(Ch_\mu) = |\mu| + \ell(\mu)$; see Section 3.3). □

Let $(\lambda^n)_{n\geq 1}$ be a sequence of random partitions, where λ^n has distribution $\mathbb{P}_n^{(\alpha)}$. In the case $\alpha = 1$, it has been proved in 1977 separately by Logan and Shepp [19] and Kerov and Vershik [14] that, in probability,

$$\lim_{n\to\infty} \left\| \omega(D_{1/\sqrt{n}}(\lambda^n)) - \Omega \right\| = 0, \tag{4}$$

where Ω is the limit shape given explicitly as follows:

$$\Omega(x) = \begin{cases} |x| & \text{if } |x| \geq 2; \\ \frac{2}{\pi}\left(x \cdot \arcsin\frac{x}{2} + \sqrt{4-x^2}\right) & \text{otherwise.} \end{cases}$$

Recall from Section 2.3 that $D_s(\lambda)$ is the Young diagram λ *dilated* by a factor s and $\omega(\lambda)$ is by definition the function whose graphical representation is the border of λ, rotated by $45°$ (see Section 2.3) and stretched by $\sqrt{2}$.

In the general α case, we have the following weak convergence result:

Proposition 4.2 *For any 1-polynomial function $F \in \Lambda_\star^{(1)}$, when n tends to infinity, one has*

$$F\big(T_{\sqrt{\alpha/n},1/\sqrt{n\alpha}}(\lambda^n)\big) \xrightarrow{\mathbb{P}_n^{(\alpha)}} F(\Omega),$$

where $\xrightarrow{\mathbb{P}_n^{(\alpha)}}$ means convergence in probability.

Proof: As $(R_k^{(1)})_{k\geq 2}$ is an algebraic basis of $\Lambda_\star^{(1)}$, it is enough to prove the proposition for any $R_k^{(1)}$.

Let μ be partition. As mentioned at the beginning of the section, one has:

$$\prod_{i\leq \ell(\mu)} R^{(\alpha)}_{\mu_i+1} = Ch_\mu + \text{ terms of degree at most } |\mu| + \ell(\mu) - 1. \tag{5}$$

Together with Lemma 4.1 and the formula for $\mathbb{E}_{\mathbb{P}_n^{(\alpha)}}(Ch_\mu)$, this implies:

$$\mathbb{E}_{\mathbb{P}_n^{(\alpha)}}\left(\prod_{i\leq \ell(\mu)} R^{(\alpha)}_{\mu_i+1}\right) = \begin{cases} n(n-1)\cdots(n-k+1) + O(n^{k-1}) & \text{if } \mu = 1^k \text{ for some } k; \\ o(n^{\frac{|\mu|+\ell(\mu)}{2}}) & \text{otherwise.} \end{cases}$$

In particular, one has that

$$\mathbb{E}_{\mathbb{P}_n^{(\alpha)}}\left(R_k^{(\alpha)}(D_{1/\sqrt{n}}(\lambda^n))\right) = \frac{1}{n^{k/2}} \mathbb{E}_{\mathbb{P}_n^{(\alpha)}}(R_k^{(\alpha)}) = \delta_{k,2} + O\left(\frac{1}{\sqrt{n}}\right);$$

$$\mathrm{Var}_{\mathbb{P}_n^{(\alpha)}}\left(R_k^{(\alpha)}(D_{1/\sqrt{n}}(\lambda^n))\right) = \frac{1}{n^k}\left(\mathbb{E}_{\mathbb{P}_n^{(\alpha)}}\left((R_k^{(\alpha)})^2\right) - \mathbb{E}_{\mathbb{P}_n^{(\alpha)}}(R_k^{(\alpha)})^2\right) = O\left(\frac{1}{n}\right).$$

Thus, for each k, $R_k^{(\alpha)}(D_{1/\sqrt{n}}(\lambda^n))$ converges in probability towards $\delta_{k,2}$. But, by definition

$$R_k^{(\alpha)}(D_{1/\sqrt{n}}(\lambda^n)) = R_k^{(1)}\big(T_{\sqrt{\alpha/n},1/\sqrt{n\alpha}}(\lambda^n)\big)$$

and $(\delta_{k,2})_{k\geq 2}$ is the sequence of free cumulants of the continuous diagram Ω (see [2, Section 3.1]), *i.e.* $\delta_{k,2} = R_k^{(1)}(\Omega)$. □

Roughly speaking, Proposition 4.2 means that, the stretched Young diagram $T_{\sqrt{\alpha/n},1/\sqrt{n\alpha}}(\lambda^n)$ converges weakly towards Ω (in probability). So this result already means that the considered diagrams admit some limit shape.

However, it would be desirable to obtain a result with uniform convergence, which is a more natural notion of convergence. This can be done thanks to the following lemma.

Lemma 4.3 *There exists a constant C such that*

$$\lim_{n\to\infty} \mathbb{P}\left[\max\left(\frac{c(\lambda^n)}{\sqrt{n}};\frac{r(\lambda^n)}{\sqrt{n}}\right) \leq C\right] = 1,$$

where, for each n, the diagram λ^n is chosen randomly with distribution $\mathbb{P}_n^{(\alpha)}$ and $r(\lambda^n)$ and $c(\lambda^n)$ are respectively its numbers of rows and columns.

The proof of this lemma is quite technical and relies on the explicit formula (3) for $\mathbb{P}_n^{(\alpha)}$. It can be found in [4, Section 6.4]. We can now state the uniform convergence result.

Theorem 4.4 *For each n, let λ^n be a random Young diagram of size n distributed with α-Plancherel measure. Then, in probability,*

$$\lim_{n\to\infty} \left\|\omega\big(T_{\sqrt{\alpha/n},1/\sqrt{n\alpha}}(\lambda^n)\big) - \Omega\right\| = 0.$$

Proof: It follows from Proposition 4.2 and Lemma 4.3 by the same argument as the one given in [8, Theorem 5.5]. □

The idea of using polynomial functions to study the asymptotic shape of Young diagrams has been developped by S. Kerov (see [8]). In the case $\alpha = 1$, he gave more precise result that what we have here: he proved that for any polynomial function F, the quantity $F(\lambda^n)$ has Gaussian fluctuations. A better understanding of polynomials K_μ could lead to a proof of a similar phenomena in the general α case, using the ideas introduced in [25]. Let us mention the existence of a partial result (corresponding to $F = Ch^{(2)}$) obtained by J. Fulman [6, Theorem 1.2] by another method.

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Structure coefficients of the Hecke algebra of $(\mathcal{S}_{2n}, \mathcal{B}_n)$

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Abstract. The Hecke algebra of the pair $(\mathcal{S}_{2n}, \mathcal{B}_n)$, where $\mathcal{B}_n$ is the hyperoctahedral subgroup of $\mathcal{S}_{2n}$, was introduced by James in 1961. It is a natural analogue of the center of the symmetric group algebra. In this paper, we give a polynomiality property of its structure coefficients. Our main tool is a combinatorial universal algebra which projects on the Hecke algebra of $(\mathcal{S}_{2n}, \mathcal{B}_n)$ for every n. To build it, we introduce new objects called partial bijections.

Résumé. L'algèbre de Hecke de la paire $(\mathcal{S}_{2n}, \mathcal{B}_n)$, où $\mathcal{B}_n$ est le sous-groupe hyperoctaédral de $\mathcal{S}_{2n}$, a été introduite par James en 1961. C'est un analogue naturel du centre de l'algèbre du groupe symétrique. Dans ce papier, on donne une propriété de polynomialité de ses coefficients de structure. On utilise une algèbre universelle construite d'une facon combinatoire et qui se projette sur toutes les algèbres de Hecke de $(\mathcal{S}_{2n}, \mathcal{B}_n)$. Pour la construire, on introduit de nouveaux objets appelés bijections partielles.

Keywords: Hecke algebra of $(\mathcal{S}_{2n}, \mathcal{B}_n)$, partial bijections, structure coefficients

This paper is an extended abstract of [Tou12], which contains all detailed proofs and will be submitted elsewhere.

1 Introduction

The center of the symmetric group algebra of n, denoted $Z(\mathbb{C}[\mathcal{S}_n])$, is a classical object in algebraic combinatorics. It is linearly generated by elements $\mathcal{Z}_\lambda$, indexed by partitions of n, which are the sums of permutations of n with cycle-type λ. The structure coefficients $c^\rho_{\lambda\delta}$ describe the product in this algebra:

$$\mathcal{Z}_\lambda \mathcal{Z}_\delta = \sum_{\rho \text{ partition of } n} c^\rho_{\lambda\delta} \mathcal{Z}_\rho.$$

In other words, $c^\rho_{\lambda\delta}$ counts the number of pairs of permutations (x, y) with cycle-type λ and δ such that $x \cdot y = z$ for a fixed permutation z with cycle-type ρ. It is known, see [Cor75], that these coefficients also count numbers of graphs drawn on oriented surfaces with some additional conditions. One of the tools used to calculate these coefficients is the representation theory of the symmetric group, see [JV90, Lemma 3.3]. In [GS98, Theorem 2.1], Goupil and Schaeffer gave a cumbersome formula for $c^\rho_{\lambda\delta}$ if one of the partitions λ, δ and ρ is equal to (n). There are no formulas for $c^\rho_{\lambda\delta}$ in general.

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In 1958, Farahat and Higman proved the polynomiality of the coefficients $c^{\rho}_{\lambda\delta}$ in n when λ, δ and ρ are fixed partitions, completed with parts equal to 1 to get partitions of n, see [FH59, Theorem 2.2]. This result is also proved by Ivanov and Kerov in [IK99] through the introduction of partial permutations. This proof provides a combinatorial description of these coefficients.

Here, we consider the Hecke algebra of the pair $(\mathcal{S}_{2n}, \mathcal{B}_n)$, where $\mathcal{B}_n$ is the hyperoctahedral group (see definition in section 2.2). It was introduced by James in [Jam61] and it also has a basis indexed by partitions of n. This algebra is a natural analogue of $Z(\mathbb{C}[\mathcal{S}_n])$ for several reasons. Goulden and Jackson proved in [GJ96] that its structure coefficients count graphs drawn on non-oriented surfaces. To get formulas for these coefficients, zonal spherical functions are used instead of irreducible characters of the symmetric group, see [Mac95, Section VII, 2].

In this paper we give a polynomiality property of the structure coefficients of the Hecke algebra of $(\mathcal{S}_{2n}, \mathcal{B}_n)$. We prove that these coefficients can be written as the product of the number $2^n n!$ with a polynomial in n. In some specific basis, this polynomial has non-negative coefficients. Our proof is inspired by the construction of Ivanov and Kerov in [IK99]. However, we had to face some difficulties that do not appear in their work. In the proof, we introduce new combinatorial objects called partial bijections of n. These objects allow us to build in a combinatorial way a universal algebra which projects on Hecke algebra of $(\mathcal{S}_{2n}, \mathcal{B}_n)$ for every n. It also gives us a combinatorial description of the coefficients of the relevant polynomials.

A weaker version of our polynomiality result (without non-negativity of the coefficients) for the structure coefficients of Hecke algebra of $(\mathcal{S}_{2n}, \mathcal{B}_n)$ has been established by indirect approach using Jack polynomials in [DF12, Proposition 4.4]. There is no combinatorial description in that proof. In [AC10], Aker and Can considered the same question, but their article contains a mistake (the coefficient $2^n n!$ does not appear in their result).

The paper is organized as follows. In section 2, we put on all necessary definitions to describe the Hecke algebra of $(\mathcal{S}_{2n}, \mathcal{B}_n)$. Then, we give our main result about its structure coefficients. We start section 3 by introducing partial bijections of n then we construct our universal algebra. We use this algebra in section 4 to prove Theorem 2.1.

2 Definitions and statement of the main result

2.1 Partitions

Since partitions index bases of the algebras studied in this paper, we recall the main definitions. A *partition* λ is a sequence of integers $(\lambda_1, \lambda_2, \dots)$ where $\lambda_1 \geq \lambda_2 \geq \dots \geq 1$. The λ_i are called the *parts* of λ and the *size* of λ, denoted by $|\lambda|$, is the sum of all its parts. If $|\lambda| = n$, we say that λ is a partition of n. We will also use the exponential notation $\lambda = (1^{m_1(\lambda)}, 2^{m_2(\lambda)}, 3^{m_3(\lambda)}, \dots)$, where $m_i(\lambda)$ is the number of parts equal to i in the partition λ. If λ and δ are two partitions we define the *union* $\lambda \cup \delta$ as the following partition:

$$\lambda \cup \delta = (1^{m_1(\lambda)+m_1(\delta)}, 2^{m_2(\lambda)+m_2(\delta)}, 3^{m_3(\lambda)+m_3(\delta)}, \dots).$$

A partition is called *proper* if it does not have any part equal to 1. The proper partition associated to a partition λ is the partition $\bar{\lambda} := \lambda \setminus (1^{m_1(\lambda)}) = (2^{m_2(\lambda)}, 3^{m_3(\lambda)}, \dots)$.

2.2 Permutations and Coset type

We will denote by $[n]$ the set $\{1, \ldots, n\}$. A *permutation* of n is a bijection between the set $[n]$ and itself. For a permutation ω, we use the line notation $\omega_1\, \omega_2\, \cdots\, \omega_n$, where $\omega_i = \omega(i)$. The set $\mathcal{S}_n$ of all permutations of n is a group for the composition called the *symmetric group* of size n.

To each permutation ω of $2n$ we associate a graph $\Gamma(\omega)$ with $2n$ vertices located on a circle. Each vertex is labelled by two labels (exterior and interior). The exterior labels run through natural numbers from 1 to $2n$ around the circle. The interior label of the vertex with exterior label i is $\omega(i)$. We link the vertices with exterior (resp. interior) labels $2i-1$ and $2i$ by an exterior (resp. interior) edge. Since exterior and interior edges alternate, the graph $\Gamma(\omega)$ has only cycles of even lengths $2\lambda_1 \geq 2\lambda_2 \geq 2\lambda_3 \geq \cdots$. The *coset-type* of ω denoted by $ct(\omega)$ is the partition $(\lambda_1, \lambda_2, \lambda_3, \ldots)$ of n.

Example 2.1. *The graph* $\Gamma(\omega)$ *associated to the permutation* $\omega = 2\,4\,9\,3\,1\,10\,5\,8\,6\,7 \in \mathcal{S}_{2n}$ *is drawn on Figure 1. It has two cycles of length 6 and 4, so* $ct(\omega) = (3, 2)$.

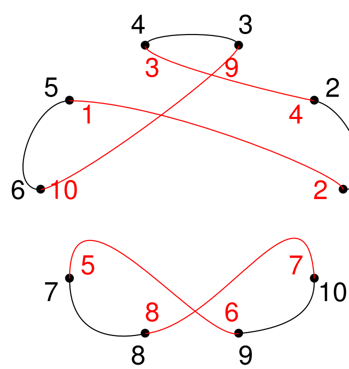

Fig. 1: The graph $\Gamma(\omega)$ from Example 2.1.

For every $k \geq 1$, we set $\rho(k) := \{2k-1, 2k\}$. The *hyperoctahedral group* $\mathcal{B}_n$ is the subgroup of $\mathcal{S}_{2n}$ of permutations ω such that, for every $1 \leq k \leq n$, there exists $1 \leq k' \leq n$ with $\omega(\rho(k)) = \rho(k')$. In other words $\mathcal{B}_n = \{\omega \in \mathcal{S}_{2n} \mid ct(\omega) = (1^n)\}$. For example, $4\,3\,1\,2\,6\,5 \in \mathcal{B}_3$.

A $\mathcal{B}_n$*-double coset* of $\mathcal{S}_{2n}$ is the set $\mathcal{B}_n x \mathcal{B}_n = \{bxb'\ ;\ b, b' \in \mathcal{B}_n\}$ for some $x \in \mathcal{S}_{2n}$. It is known, see [Mac95, page 401], that two permutations of $\mathcal{S}_{2n}$ are in the same $\mathcal{B}_n$-double coset if and only if they have the same coset-type. Thus, if $x \in \mathcal{S}_{2n}$ has coset-type λ, we have:

$$\mathcal{B}_n x \mathcal{B}_n = \{y \in \mathcal{S}_{2n} \text{ such that } ct(y) = \lambda\}.$$

2.3 The Hecke algebra of $(\mathcal{S}_{2n}, \mathcal{B}_n)$

The *symmetric group algebra of* n denoted by $\mathbb{C}[\mathcal{S}_n]$ is the algebra over $\mathbb{C}$ linearly generated by all permutations of n. The group $\mathcal{B}_n \times \mathcal{B}_n$ acts on $\mathbb{C}([\mathcal{S}_{2n}])$ by the following action: $(b, b') \cdot x = bxb'$, called the $\mathcal{B}_n$-double action. The *Hecke algebra* of $(\mathcal{S}_{2n}, \mathcal{B}_n)$ denoted by $\mathbb{C}[\mathcal{B}_n \setminus \mathcal{S}_{2n}/\mathcal{B}_n]$ is the sub-algebra of $\mathbb{C}[\mathcal{S}_{2n}]$ of elements invariant under the $\mathcal{B}_n$-double action. Recall that $\mathcal{B}_n$-double cosets are indexed by partitions of n. Therefore, the set

$$\{K_\lambda(n) : \lambda \text{ proper partition}, |\lambda| \leq n\}$$

forms a basis for $\mathbb{C}[\mathcal{B}_n \setminus \mathcal{S}_{2n}/\mathcal{B}_n]$, where $K_\lambda(n)$ is the sum of all permutations of $\mathcal{S}_{2n}$ with coset-type $\lambda \cup 1^{n-|\lambda|}$. So, for any two proper partitions λ and δ with size at most n, there exist complex numbers $\alpha_{\lambda\delta}^{\rho}(n)$ such that:

$$K_\lambda(n) \cdot K_\delta(n) = \sum_{\substack{\rho \text{ proper partition} \\ |\rho| \leq n}} \alpha_{\lambda\delta}^{\rho}(n) K_\rho(n). \tag{1}$$

2.4 Main result

We give in this paper a polynomiality property of the structure coefficients of the Hecke algebra of $(\mathcal{S}_{2n}, \mathcal{B}_n)$ that appear in (1). More precisely, we prove the following theorem. We will use the standard notation $(n)_k := \frac{n!}{(n-k)!} = n(n-1)\cdots(n-k+1)$.

Theorem 2.1. *Let λ, δ and ρ be three proper partitions, we have:*

$$\alpha_{\lambda\delta}^{\rho}(n) = \begin{cases} 2^n n! f_{\lambda\delta}^{\rho}(n) & \text{if} \quad n \geq |\rho|, \\ 0 & \text{if} \quad n < |\rho|, \end{cases}$$

where $f_{\lambda\delta}^{\rho}(n) = \sum_{j=0}^{|\lambda|+|\delta|-|\rho|} a_j (n-|\rho|)_j$ is a polynomial in n with $a_j \in \mathbb{Q}^+$.

Example 2.2. *Let us compute the structure coefficient $\alpha_{(2)(2)}^{\emptyset}(n)$. We have:*

$$K_{(2)}(n) = \sum_{\substack{\omega \in \mathcal{S}_{2n} \\ ct(\omega) = (2) \cup (1^{n-2})}} \omega.$$

To find the coefficient of $K_\emptyset(n)$ in $K_{(2)}(n) \cdot K_{(2)}(n)$, we fix a permutation with coset-type (1^n), for example Id_{2n}, and we look in how many ways we can obtain Id_{2n} as a product of two elements $\sigma \cdot \beta$ where $ct(\sigma) = ct(\beta) = (2, 1^{n-2})$. Thus we are looking for the number of permutations $\sigma \in \mathcal{S}_{2n}$ such that $ct(\sigma) = ct(\sigma^{-1}) = (2, 1^{n-2})$. But, for any $\sigma \in \mathcal{S}_{2n}$ with $ct(\sigma) = (2, 1^{n-2})$, its inverse has the same coset-type. Therefore $\alpha_{(2)(2)}^{\emptyset}(n)$ is the number of permutations of coset-type $(2, 1^{n-2})$, which is by [Mac95, page 402]

$$\frac{(2^n n!)^2}{2^{n-1}(2 \cdot (n-2)!)} = n(n-1)2^n n!.$$

2.5 Major steps of the proof

The idea of the proof is to build a universal algebra $\mathcal{A}_\infty$ over $\mathbb{C}$ satisfying the following properties:

1. For every $n \in \mathbb{N}^*$, there exists a morphism of algebras $\theta_n : \mathcal{A}_\infty \longrightarrow \mathbb{C}[\mathcal{B}_n \setminus \mathcal{S}_{2n}/\mathcal{B}_n]$.
2. Every element x in $\mathcal{A}_\infty$ is written in a unique way as an infinite linear combination of elements T_λ, indexed by partitions. For any two partitions λ and δ, there exist non-negative rational numbers $b_{\lambda\delta}^{\rho}$ such that:

$$T_\lambda * T_\delta = \sum_{\rho \text{ partition}} b_{\lambda\delta}^{\rho} T_\rho. \tag{2}$$

3. The morphism θ_n sends T_λ to a multiple of $K_{\bar{\lambda}}(n)$.

To build $\mathcal{A}_\infty$, we introduce new combinatorial objects called *partial bijections.* For every $n \in \mathbb{N}^*$, we construct an algebra $\mathcal{A}_n$ using the set of partial bijections of size n. The algebra $\mathcal{A}_\infty$ is defined as the projective limit of this sequence $(\mathcal{A}_n)$.

The projection $p_n : \mathcal{A}_\infty \to \mathcal{A}_n$ involves coefficients which are polynomials in n. By defining the extension of a partial bijection of n to the set $[2n]$, we construct a morphism from $\mathcal{A}_n$ to $\mathbb{C}[\mathcal{B}_n \setminus \mathcal{S}_{2n}/\mathcal{B}_n]$. Its coefficients involve the number $2^n n!$. It turns out that the morphism θ_n is the composition of those two morphisms:

$$\begin{tikzcd}
\mathcal{A}_\infty \arrow[dr, "p_n"] \arrow[dd, "\theta_n"'] & \\
& \mathcal{A}_n \arrow[dl] \\
\mathbb{C}[\mathcal{B}_n \setminus \mathcal{S}_{2n}/\mathcal{B}_n] &
\end{tikzcd}$$

The final step consists in applying the chain of morphisms in the diagram above to equation (2).

Remark. This method is based on Ivanov and Kerov's one to get the polynomiality of the structure coefficients of the center of the symmetric group algebra (see [IK99] for more details). Nevertheless, our construction is more complicated, mainly because a partial bijection does not have a unique trivial extension to a given set, see Definition 3.2.

3 The partial bijection algebra

In this section we define the set of partial bijections of n. With this set, we build the algebras and morphisms that appear in the diagram above.

3.1 Definition

We start by defining partial bijections of n and the partial bijection algebra. Then, we introduce the notion of trivial extension of a partial bijection of n and we use it to build a morphism between the partial bijection algebra of n and the symmetric group algebra of $2n$.

For $n \in \mathbb{N}^*$, $\mathbf{P}_n$ denotes the following set:

$$\mathbf{P}_n := \{\rho(k_1) \cup \cdots \cup \rho(k_i) \mid 1 \leq i \leq n,\ 1 \leq k_1 < \cdots < k_i \leq n\}.$$

Definition 3.1. A *partial bijection* of n is a triple $(\sigma, d, d^{'})$ where $d, d^{'} \in \mathbf{P}_n$ and $\sigma : d \longrightarrow d^{'}$ is a bijection. We denote by Q_n the set of all partial bijections of n.

It should be clear that

$$|Q_n| = s_n = \sum_{k=0}^{n} \binom{n}{k}^2 (2k)!.$$

A permutation σ of $2n$ can be written as $(\sigma, [2n], [2n])$, so the set $\mathcal{S}_{2n}$ can be considered as a subset of Q_n.

For any partial bijection α, we will use the convention that σ (resp. d, $d^{'}$) is the first (resp. second, third) element of the triple defining α. The same convention holds for $\tilde{\alpha}$, α_i, $\hat{\alpha}$ ...

Observation. In the same way as in section 2.2, we can associate to each partial bijection α of n a graph $\Gamma(\alpha)$ with $|d|$ vertices placed on a circle. The exterior (resp. interior) labels are the elements of the set d (resp. $d^{'}$). Since the sets d and $d^{'}$ are in $\mathbf{P}_n$, we can link $2i$ with $2i-1$ as in the case $d = d^{'} = [2n]$. So, the definition of coset-type extends naturally to partial bijection. We denote by $ct(\alpha)$ or $ct(\sigma)$ the coset-type of a partial bijection α.

Definition 3.2. Let $(\sigma, d, d^{'})$ and $(\tilde{\sigma}, \tilde{d}, \tilde{d}^{'})$ be two partial bijections of n. We say that $(\tilde{\sigma}, \tilde{d}, \tilde{d}^{'})$ is a *trivial extension* of $(\sigma, d, d^{'})$ if:

$$d \subseteq \tilde{d},\ \tilde{\sigma}_{|_d} = \sigma \text{ and } ct(\tilde{\sigma}) = ct(\sigma) \cup \left(1^{\frac{|\tilde{d}\setminus d|}{2}}\right).$$

We denote by $P_\alpha(n)$ the set of all trivial extensions of α in Q_n.

Lemma 3.1. *Let α be a partial bijection of n and X an element of $\mathbf{P}_n$ such that $d \subseteq X$. The number of trivial extension $\tilde{\alpha}$ such that $\tilde{d} = X$ is*

$$(2n-|d|)\cdot(2n-|d|-2)\cdots(2n-|d|-|X\setminus d|+2) = 2^{\frac{|X\setminus d|}{2}}\left(n-\frac{|d|}{2}\right)_{\frac{|X\setminus d|}{2}}.$$

We have the same formula for the number of trivial extension $\tilde{\alpha}$ such that $\tilde{d}' = X$.

Proof. We proceed by induction on the size of $X \setminus d$. If $|X\setminus d| = 2$, suppose that $X \setminus d = \rho(k)$ for some $k \in [n]$. There are $2n-|d|$ possible values for $\tilde{\sigma}(2k-1)$. If $\tilde{\sigma}(2k-1) = 2k'-1$ (resp. $2k'$), we have $\tilde{\sigma}(2k) = 2k'$ (resp. $2k'-1$). So, the number of trivial extensions $\tilde{\alpha}$ such that $\tilde{d} = X$ is $2n-|d|$.

We suppose that we have the result for $|X\setminus d| \leq 2(r-1)$. Let X be a set such that $|X\setminus d| = 2r$. We fix an element $2i-1$ of $X\setminus d$. Trivial extensions $\tilde{\alpha}$ such that $\tilde{d} = X$ are obtained as follows:

- first, take all trivial extensions α_1 such that $d_1 = X \setminus \rho(i)$. Since $\left|\big(X\setminus\rho(i)\big)\setminus d\right| = 2(r-1)$, the number of these trivial extensions is by induction $2^{r-1}(n-\frac{|d|}{2})_{r-1}$.
- second, for every trivial extension α_1, take all trivial extensions $\widetilde{\alpha_1}$ such that $\widetilde{d_1} = X$. For a fixed α_1, the number of trivial extensions $\widetilde{\alpha_1}$ such that $\widetilde{d_1} = X$ is, by the base case of induction, $2n - |X\setminus\rho(i)| = 2n-|d|-2r+2$.

Every trivial extension $\tilde{\alpha}$ is obtained exactly once: as a trivial extension of α_1, where α_1 is $\tilde{\alpha}_{|_{X\setminus\rho(i)}}$. Thus, the number of trivial extension $\tilde{\alpha}$ such that $\tilde{d} = X$ is the product:

$$2^{r-1}\left(n-\frac{|d|}{2}\right)_{r-1}\cdot(2n-|d|-2r+2) = 2^r\left(n-\frac{|d|}{2}\right)_r.$$

This ends our induction and proves the first part of lemma. The proof of the second part (number of trivial extension $\tilde{\alpha}$ such that $\tilde{d}' = X$) is similar. $\square$

Consider $\mathcal{D}_n = \mathbb{C}[Q_n]$ the vector space with basis Q_n. We want to endow it with an algebra structure. Let α_1 and α_2 be two partial bijections. If $d_1 = d_2'$, we can compose α_1 and α_2 and we define $\alpha_1 * \alpha_2 = \alpha_1 \circ \alpha_2 = (\sigma_1\circ\sigma_2, d_2, d_1')$. Otherwise, we need to extend α_1 and α_2 to partial bijections $\widetilde{\alpha_1}$ and $\widetilde{\alpha_2}$ such that $\widetilde{d_1} = \widetilde{d_2'}$. Since there exist several trivial extensions of α_1 and α_2, a natural choice is to take the average of the composition of all possible trivial extensions. Let $E_{\alpha_1}^{\alpha_2}(n)$ be the following set:

$$E_{\alpha_1}^{\alpha_2}(n) \quad := \quad \{(\widetilde{\alpha_1}, \widetilde{\alpha_2}) \in P_{\alpha_1}(n)\times P_{\alpha_2}(n) \text{ such that } \widetilde{d_1} = \widetilde{d_2'} = d_1\cup d_2'\}.$$

Elements of $E^{\alpha_2}_{\alpha_1}(n)$ are schematically represented on Figure 2. We define the product of α_1 and α_2 as follows:

$$\alpha_1 * \alpha_2 := \frac{1}{|E^{\alpha_2}_{\alpha_1}(n)|} \sum_{(\widetilde{\alpha_1},\widetilde{\alpha_2})\in E^{\alpha_2}_{\alpha_1}(n)} \widetilde{\alpha_1} \circ \widetilde{\alpha_2}. \tag{3}$$

By Lemma 3.1, we can see that $|E^{\alpha_2}_{\alpha_1}(n)| = 2^{\frac{|d'_2\setminus d_1|}{2}+\frac{|d_1\setminus d'_2|}{2}}.(n-\frac{|d'_1|}{2})_{(\frac{|d'_2\setminus d_1|}{2})}.(n-\frac{|d_2|}{2})_{(\frac{|d_1\setminus d'_2|}{2})}$.

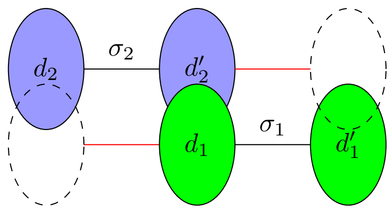

Fig. 2: Schematic representation of elements of $E^{\alpha_2}_{\alpha_1}(n)$.

Proposition 3.2. *The product $*$ is associative and $\mathcal{D}_n$ is a (non-unital) algebra.*

We shall not present the proof here since it is technical, and uses the same type of arguments as the proof of Proposition 3.3.

Proposition 3.3. *The following function defines a morphism of algebras:*

$$\begin{array}{cccc} \psi_n & : & \mathbb{C}[Q_n] & \rightarrow & \mathbb{C}[\mathcal{S}_{2n}] \\ & & \alpha & \mapsto & \frac{1}{2^{n-\frac{|d|}{2}}(n-\frac{|d|}{2})!} \sum_{\hat{\alpha}\in\mathcal{S}_{2n}\cap P_\alpha(n)} \hat{\sigma} \end{array}.$$

Proof. Let α_1 and α_2 be two basis elements of $\mathbb{C}[Q_n]$. We refer to Figure 2 and denote $2b = |d_2| = |d'_2|, 2c = |d_1| = |d'_1|$ and $2e = |d'_2 \cap d_1|$. We first prove that:

$$\begin{aligned} &\sum_{\widehat{\alpha_1}\in\mathcal{S}_{2n}\cap P_{\alpha_1}(n)} \sum_{\widehat{\alpha_2}\in\mathcal{S}_{2n}\cap P_{\alpha_2}(n)} \widehat{\sigma_1}\circ\widehat{\sigma_2} \\ &\quad = 2^{n-(b+c-e)}(n-(b+c-e))! \sum_{(\widetilde{\alpha_1},\widetilde{\alpha_2})\in E^{\alpha_1}_{\alpha_2}(n)} \sum_{\widehat{\widetilde{\alpha_1}\circ\widetilde{\alpha_2}}\in\mathcal{S}_{2n}\cap P_{\widetilde{\alpha_1}\circ\widetilde{\alpha_2}}(n)} \widehat{\widetilde{\sigma_1}\circ\widetilde{\sigma_2}}. \end{aligned} \tag{4}$$

We fix $(\widetilde{\alpha_1},\widetilde{\alpha_2}) \in E^{\alpha_2}_{\alpha_1}(n)$ and $\omega \in \mathcal{S}_{2n} \cap P_{\widetilde{\alpha_1}\circ\widetilde{\alpha_2}}(n)$, *i.e.*:

$$\omega_{|_{\widetilde{d_2}}} = \widetilde{\sigma_1}\circ\widetilde{\sigma_2} \text{ and } ct(\omega) = ct(\tilde{\sigma}_1\circ\tilde{\sigma}_2) \cup (1^{(n-(b+c-e))}).$$

We look for the number of permutations $\widehat{\sigma_1}$ and $\widehat{\sigma_2}$ in $\mathcal{S}_{2n}\cap P_{\alpha_1}(n)$ and $\mathcal{S}_{2n}\cap P_{\alpha_2}(n)$ such that $\widehat{\sigma_1}\circ\widehat{\sigma_2} = \omega$. In this equation, $\widehat{\sigma_2}$ determines $\widehat{\sigma_1}$. But the condition $\omega_{|_{\widetilde{d_2}}} = \widetilde{\sigma_1}\circ\widetilde{\sigma_2}$ gives the values of $\widehat{\sigma_2}$ on $\widetilde{d_2}$ ($\widehat{\sigma_2}(x) = \sigma_2(x)$ if $x \in d_2$ and $\widehat{\sigma_2}(x) = \sigma_1^{-1}(\omega(x))$ if $x \in \tilde{d_2}\setminus d_2$). Thus, the number of ways to choose $\widehat{\sigma_2}$ is the number of ways to extend trivially $\widetilde{\sigma_2}$ to be a permutation of $2n$, which is $2^{n-(b+c-e)}(n-(b+c-e))!$

by Lemma 3.1. This proves equation (4).
Now we have:

$$\begin{aligned}\psi_n(\alpha_1)\psi_n(\alpha_2) &= \frac{1}{2^{2n-b-c}(n-c)!(n-b)!}\sum_{\widehat{\alpha_1}\in\mathcal{S}_{2n}\cap P_{\alpha_1}(n)}\ \sum_{\widehat{\alpha_2}\in\mathcal{S}_{2n}\cap P_{\alpha_2}(n)}\widehat{\sigma_1}\circ\widehat{\sigma_2}\\ &= \frac{(n-b-c+e)!}{2^{n-e}(n-c)!(n-b)!}\sum_{(\widetilde{\alpha_1},\widetilde{\alpha_2})\in E^{\alpha_1}_{\alpha_2}(n)}\ \sum_{\widehat{\widetilde{\alpha_1}\circ\widetilde{\alpha_2}}\in\mathcal{S}_{2n}\cap P_{\widetilde{\alpha_1}\circ\widetilde{\alpha_2}}(n)}\widehat{\widetilde{\sigma_1}\circ\widetilde{\sigma_2}}.\end{aligned}\tag{5}$$

On the other hand:

$$\psi_n(\alpha_1*\alpha_2)=\frac{1}{2^{b+c-2e}(n-c)_{(b-e)}(n-b)_{(c-e)}}\sum_{(\widetilde{\alpha_1},\widetilde{\alpha_2})\in E^{\alpha_1}_{\alpha_2}(n)}\psi_n\big((\widetilde{\sigma_1}\circ\widetilde{\sigma_2},\widetilde{d_2},\widetilde{d'_1})\big).$$

But

$$\psi_n\big((\widetilde{\sigma_1}\circ\widetilde{\sigma_2},\widetilde{d_2},\widetilde{d'_1})\big)=\frac{1}{2^{n-(b+c-e)}(n-(b+c-e))!}\sum_{\widehat{\widetilde{\alpha_1}\circ\widetilde{\alpha_2}}\in\mathcal{S}_{2n}\cap P_{\widetilde{\alpha_1}\circ\widetilde{\alpha_2}}(n)}\widehat{\widetilde{\sigma_1}\circ\widetilde{\sigma_2}}.$$

Thus,

$$\psi_n(\alpha_1*\alpha_2)=\frac{(n-b-c+e))!}{2^{n-e}(n-c)!(n-b)!}\sum_{(\widetilde{\alpha_1},\widetilde{\alpha_2})\in E^{\alpha_1}_{\alpha_2}(n)}\ \sum_{\widehat{\widetilde{\alpha_1}\circ\widetilde{\alpha_2}}\in\mathcal{S}_{2n}\cap P_{\widetilde{\alpha_1}\circ\widetilde{\alpha_2}}(n)}\widehat{\widetilde{\sigma_1}\circ\widetilde{\sigma_2}}.\tag{6}$$

Comparing equations (5) and (6), we see that for any two partial bijections α_1 and α_2 of n, we have $\psi_n(\alpha_1*\alpha_2)=\psi_n(\alpha_1)\psi_n(\alpha_2)$. In other words, ψ_n is a morphism of algebras. $\square$

3.2 Action of $\mathcal{B}_n\times\mathcal{B}_n$ on $\mathcal{D}_n$

In this section, we construct the algebra $\mathcal{A}_n$ as the algebra of invariant elements by an action of $\mathcal{B}_n\times\mathcal{B}_n$ on $\mathcal{D}_n$.

Definition 3.3. The group $\mathcal{B}_n\times\mathcal{B}_n$ acts on Q_n by:

$$(a,b)\bullet(\sigma,d,d')=(a\sigma b^{-1},b(d),a(d')),$$

for any $(a,b)\in\mathcal{B}_n\times\mathcal{B}_n$ and $(\sigma,d,d')\in Q_n$.

We can extend this action by linearity to get an action of $\mathcal{B}_n\times\mathcal{B}_n$ on $\mathcal{D}_n$. This action is compatible with the product of Q_n. Namely, we can prove that, for any $(a,b)\in\mathcal{B}_n\times\mathcal{B}_n$ and for any partial bijections α_1,α_2 of n, we have:

$$(a,b)\bullet(\alpha_1*\alpha_2)=((a,id)\bullet\alpha_1)*((id,b)\bullet\alpha_2).\tag{7}$$

We consider the set $\mathcal{A}_n$ of invariant elements by the action of $\mathcal{B}_n\times\mathcal{B}_n$ on $\mathcal{D}_n$:

$$\mathcal{A}_n=\mathcal{D}_n^{\mathcal{B}_n\times\mathcal{B}_n}=\{x\in\mathcal{D}_n\mid(a,b)\bullet x=x\ \text{ for any }(a,b)\in\mathcal{B}_n\times\mathcal{B}_n\}.$$

Due to equation (7), we can see that $\mathcal{A}_n$ is an algebra. The following result is easy to check.

Proposition 3.4. *The elements* $(S_{\lambda,n})_{|\lambda|=r\leq n}$, *where* $S_{\lambda,n} = \sum_{\alpha\in Q_n, ct(\alpha)=\lambda} \alpha$, *form a basis for the algebra* $\mathcal{A}_n$.

Corollary 3.5. *If* λ *and* δ *are two partitions such that* $|\lambda|, |\delta| \leq n$, *there exist unique constants* $c^{\rho}_{\lambda\delta}(n) \in \mathbb{C}$ *such that:*

$$S_{\lambda,n} * S_{\delta,n} = \sum_{\substack{\rho \text{ partition} \\ \max(|\lambda|,|\delta|)\leq|\rho|\leq\min(|\lambda|+|\delta|,n)}} c^{\rho}_{\lambda\delta}(n) S_{\rho,n}.$$

Proof. We only have to prove the inequalities on the size of ρ. Let α_1 and α_2 be two partial bijections of n with coset-type λ and δ. By definition (see Figure 2), every partial bijection of n that appears in the sum of the product $\alpha_1 * \alpha_2$ has some coset-type ρ with $|\rho| = \frac{|d_1\cup d_2'|}{2}$. But

$$\max\big(\frac{|d_1|}{2}, \frac{|d_2'|}{2}\big) = \max(|\lambda|, |\delta|) \leq |\rho| = \frac{|d_1 \cup d_2'|}{2} \leq \frac{|d_1| + |d_2'|}{2} = |\lambda| + |\delta|. \qquad \square$$

Lemma 3.6. *Let* λ *be a partition such that* $|\lambda| = r \leq n$. *We have:*

$$\psi_n(S_{\lambda,n}) = \frac{1}{2^{n-|\lambda|}(n-|\lambda|)!}\binom{n-|\bar{\lambda}|}{m_1(\lambda)} K_{\bar{\lambda}}(n).$$

Proof. For a partial bijection $\alpha \in Q_n$ such that $ct(\alpha) = \lambda$, we have:

$$\psi_n(\alpha) = \frac{1}{2^{n-|\lambda|}(n-|\lambda|)!}\sum_{\hat{\alpha}\in\mathcal{S}_{2n}\cap P_\alpha(n)} \hat{\sigma}.$$

To conclude the proof, note that for a fixed permutation $\omega \in K_{\bar{\lambda}}(n)$, the number of partial bijections $\alpha \in A_{\lambda,n}$ such that ω is a trivial extension of α is $\binom{n-|\bar{\lambda}|}{m_1(\lambda)}$. $\square$

This lemma implies that $\psi_n(\mathcal{A}_n) \subseteq \mathbb{C}[\mathcal{B}_n/\mathcal{S}_{2n}\backslash\mathcal{B}_n]$. The morphism $\mathcal{A}_n \to \mathbb{C}[\mathcal{B}_n/\mathcal{S}_{2n}\backslash\mathcal{B}_n]$ mentioned in Section 2.5 is the morphism $\psi_{n_{|\mathcal{A}_n}}$.

3.3 Projective limits

In this paragraph, we first show that the sequence $(\mathcal{A}_n)$ admits a projective limit $\mathcal{A}_\infty$ by giving a morphism from $\mathcal{A}_{n+1}$ to $\mathcal{A}_n$. Then, we prove in Proposition 3.8 that every element of $\mathcal{A}_\infty$ is written in a unique way as infinite linear combination of elements indexed by partitions.

Lemma 3.7. *The function* φ_n *defined as follows:*

$$\begin{array}{rcll} \varphi_n : & \mathcal{A}_{n+1} & \to & \mathcal{A}_n \\ & S_{\lambda,n+1} & \mapsto & \begin{cases} \frac{n+1}{(n+1-|\lambda|)} S_{\lambda,n} & \text{if} \quad |\lambda| = r < n+1, \\ 0 & \text{if} \quad |\lambda| = n+1, \end{cases} \end{array}$$

is a morphism of algebras.

Proof. Omitted for brevity. $\square$

Let $\mathcal{A}_\infty$ be the projective limit of $(\mathcal{A}_n, \varphi_n)$:

$$\mathcal{A}_\infty = \{(a_n)_{n\geq 1} \mid \text{for every } n \geq 1, a_n \in \mathcal{A}_n \text{ and } \varphi_n(a_{n+1}) = a_n\}.$$

For every partition λ, we define the sequence T_λ as follows:

$$T_\lambda = (T_\lambda)_{n\geq 1} = \begin{cases} 0 & \text{if} \quad n < |\lambda|, \\ \frac{1}{\binom{n}{|\lambda|}} S_{\lambda,n} & \text{if} \quad n \geq |\lambda|. \end{cases}$$

We can prove easily the following proposition.

Proposition 3.8. *Every element $a \in \mathcal{A}_\infty$ is written in a unique way as infinite linear combination of elements T_λ.*

This proposition shows that the algebra $\mathcal{A}_\infty$ satisfies the second property required in section 2.5. In particular, $T_\lambda * T_\delta$ writes as linear combination of elements T_ρ. We can be more precise.

Corollary 3.9. *Let λ and δ be two partitions. There exist unique constants $b_{\lambda\delta}^{\rho} \in \mathbb{Q}^+$ such that:*

$$T_\lambda * T_\delta = \sum_{\substack{\rho \text{ partition} \\ \max(|\lambda|,|\delta|) \leq |\rho| \leq |\lambda|+|\delta|}} b_{\lambda\delta}^{\rho} T_\rho.$$

Proof. The conditions on the size of partitions ρ in the sum index are obtained by Corollary 3.5. We may check that $b_{\lambda\delta}^{\rho} = \frac{c_{\lambda\delta}^{\rho}(|\rho|)}{\binom{|\rho|}{|\lambda|}\binom{|\rho|}{|\delta|}}$, which explains that $b_{\lambda\delta}^{\rho} \in \mathbb{Q}^+$. $\square$

4 Proof of Theorem 2.1

In the previous section, we built all algebras and morphisms that we need in order to prove Theorem 2.1.

Let λ and δ be two proper partitions, by Corollary 3.9, we have:

$$T_\lambda * T_\delta = \sum_{\substack{\rho \text{ partition} \\ \max(|\lambda|,|\delta|) \leq |\rho| \leq |\lambda|+|\delta|}} b_{\lambda\delta}^{\rho} T_\rho.$$

Recall that this is an equality of sequences. Taking the n-th term, we have:

$$\frac{1}{\binom{n}{|\lambda|}} S_{\lambda,n} * \frac{1}{\binom{n}{|\delta|}} S_{\delta,n} \quad = \sum_{\substack{\rho \text{ partition} \\ \max(|\lambda|,|\delta|) \leq |\rho| \leq \min(|\lambda|+|\delta|, n)}} b_{\lambda\delta}^{\rho} \frac{1}{\binom{n}{|\rho|}} S_{\rho,n}.$$

By applying ψ_n we obtain (see Lemma 3.6):

$$\frac{1}{2^{n-|\lambda|}(n-|\lambda|)!} K_\lambda(n) \cdot \frac{1}{2^{n-|\delta|}(n-|\delta|)!} K_\delta(n) =$$

$$\sum_{\substack{\rho \text{ partition} \\ \max(|\lambda|,|\delta|) \leq |\rho| \leq \min(|\lambda|+|\delta|, n)}} b_{\lambda\delta}^{\rho} \frac{\binom{n}{|\lambda|}\binom{n}{|\delta|}}{\binom{n}{|\rho|} 2^{n-|\rho|}(n-|\rho|)!} \binom{n-|\bar{\rho}|}{m_1(\rho)} K_{\bar{\rho}}(n).$$

After simplification, we get:

$$K_\lambda(n) \cdot K_\delta(n) = \sum_{\substack{\rho \text{ partition} \\ \max(|\lambda|,|\delta|) \leq |\rho| \leq \min(|\lambda|+|\delta|,n)}} b_{\lambda\delta}^{\rho} \frac{(|\rho|)_{|\bar{\rho}|}}{|\lambda|!|\delta|!} 2^{n+|\rho|-|\lambda|-|\delta|} n!(n - |\bar{\rho}|)_{m_1(\rho)} K_{\bar{\rho}}(n).$$

Fact. Any partition ρ such that $|\rho| \leq \min(|\lambda| + |\delta|, n)$ can be written in a unique way as $\rho = \tau \cup (1^j)$, where τ is a proper partition and $j \leq \min(|\lambda| + |\delta|, n) - |\tau|$.

Using this fact, the product can be written as follows:

$$K_\lambda(n) \cdot K_\delta(n) = \sum_{\substack{\tau \text{ proper partition} \\ |\tau| \leq \min(|\lambda|+|\delta|,n)}} \alpha_{\lambda\delta}^{\tau}(n) K_\tau(n),$$

where

$$\begin{aligned} \alpha_{\lambda\delta}^{\tau}(n) &= \frac{1}{|\lambda|!|\delta|!} \sum_{j=0}^{\min(|\lambda|+|\delta|,n)-|\tau|} b_{\lambda\delta}^{\tau\cup(1^j)} n!(n-|\tau|)_j (|\tau|+j)_{|\tau|} 2^{n+|\tau|+j-|\lambda|-|\delta|} \\ &= \frac{2^n n!}{|\lambda|!|\delta|!} \sum_{j=0}^{|\lambda|+|\delta|-|\tau|} b_{\lambda\delta}^{\tau\cup(1^j)} (n-|\tau|)_j (|\tau|+j)_{|\tau|} 2^{|\tau|+j-|\lambda|-|\delta|}. \end{aligned}$$

The change of sum index in the last equality comes from the fact that if $n < |\lambda| + |\delta|$, we have:

$$(n - |\tau|)_j = 0 \text{ for any } j \text{ with } n - |\tau| < j \leq |\lambda| + |\delta| - |\tau|.$$

This ends the proof of Theorem 2.1.

Corollary 4.1. *If λ, δ and ρ are three proper partitions such that $|\rho| = |\lambda| + |\delta|$, then:*

$$\alpha_{\lambda\delta}^{\rho}(n) = b_{\lambda\delta}^{\rho} \frac{|\rho|!}{|\lambda|!|\delta|!} 2^n n! = c_{\lambda\delta}^{\rho}(|\rho|) \frac{|\lambda|!|\delta|!}{(|\lambda|+|\delta|)!} 2^n n!.$$

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Word symmetric functions and the Redfield-Pólya theorem[†]

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Abstract. We give noncommutative versions of the Redfield-Pólya theorem in WSym, the algebra of word symmetric functions, and in other related combinatorial Hopf algebras.

Résumé. Nous donnons des versions non-commutatives du théorème d'énumération de Redfield-Pólya dans WSym, l'algèbre des fonctions symétriques sur les mots, ainsi que dans d'autres algèbres de Hopf combinatoires.

Keywords: Symmetric functions, Redfield-Pólya theorem, Combinatorial Hopf algebras, Operads

1 Introduction

The Redfield-Pólya enumeration theorem (R-P theorem) is one of the most exciting results in combinatorics of the twentieth century. It was first published by John Howard Redfield (18) in 1927 and independently rediscovered by George Pólya ten years later (17). Their motivation was to generalize Burnside's lemma on the number of orbits of a group action on a set (see *e.g* (3)). Note that Burnside attributed this result to Frobenius (8) and it seems that the formula was prior known to Cauchy. Although Redfield found the theorem before Pólya, it is often attributed only to Pólya. This is certainly due to the fact that Pólya popularized the result by providing numerous applications to counting problems and in particular to the enumeration of chemical compounds. The theorem is a result of group theory but there are important implications in many disciplines (chemistry, theoretical physics, mathematics — in particular combinatorics and enumeration *etc.*) and its extensions lead to Andrés Joyal's combinatorial species theory (1).
Consider two sets X and Y (X finite) and let G be a finite group acting on X. For a map $f : X \to Y$, define the vector $v_f = (\#\{x : f(x) = y\})_{y\in Y} \in \mathbb{N}^Y$. The R-P theorem deals with the enumeration of the maps f having a given $v_f = v$ (v fixed) up to the action of the group G. The reader can refer to (11) for proof, details, examples and generalizations of the R-P theorem.

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Algebraically, the theorem can be pleasantly stated in terms of symmetric functions (see *e.g.* (15, 12)); the cycle index polynomial is defined in terms of power sum symmetric functions and, whence writing in the monomial basis, the coefficients count the number of orbits of a given type. From the seminal paper (9), many combinatorial Hopf algebras have been discovered and investigated. The goal is to mimic the combinatorics and representation theory related to symmetric functions in other contexts. This paper asks the question of the existence of combinatorial Hopf algebras in which the R-P theorem can naturally arise. In this sense, the article is the continuation of (6, 7) in which the authors investigated some Hopf algebras in the aim to study the enumeration of bipartite graphs up to the permutations of the vertices.
Each function $f : X \to Y$ can be encoded by a word of size $\#X$ on an alphabet $A_Y = \{a_y : y \in Y\}$. Hence, intuitively, we guess that the R-P theorem arises in a natural way in the algebra of words $\mathbb{C}\langle A\rangle$. The Hopf algebra of word symmetric functions WSym has been studied in (19, 2, 10). In S Section 2, we recall the basic definitions and properties related to this algebra and propose a definition for the specialization of an alphabet using the concept of operad (13, 14). In Section 3, we construct and study other related combinatorial Hopf algebras. In Section 4, we investigate the analogues of the cycle index polynomials in these algebras and give two noncommutative versions of the R-P theorem. In particular, we give a word version and a noncommutative version. Finally, in Subsection 4.5, we propose a way to raise Harary-Palmer type enumerations (the functions are now enumerated up to an action of G on X and an action of another group H on Y) in WSym. For this last equality, we need the notion of specialization defined in Section 2.

2 Word symmetric functions

2.1 Basic definitions and properties

Consider the family $\Phi := \{\Phi^\pi\}_\pi$ whose elements are indexed by set partitions of $\{1, \dots, n\}$ (we will denote $\pi \Vdash n$). The algebra WSym (19) is generated by Φ for the shifted concatenation product: $\Phi^\pi\Phi^{\pi'} = \Phi^{\pi\pi'[n]}$ where π and π' are set partitions of $\{1, \dots, n\}$ and $\{1, \dots, m\}$, respectively, and $\pi'[n]$ means that we add n to each integer occurring in π'. Other bases are known, for example, the word monomial functions defined by

$$\Phi^\pi = \sum_{\pi \le \pi'} M_{\pi'}$$

where $\pi \le \pi'$ indicates that π is finer than π', *i.e.*, that each block of π' is a union of blocks of π.

WSym is a Hopf algebra when endowed with the shifted concatenation product and the following coproduct, where $\mathrm{std}(\pi)$ means that for all i, we replace the ith smallest integer in π by i:

$$\Delta M_\pi = \sum_{\substack{\pi' \cup \pi'' = \pi \\ \pi' \cap \pi'' = \varnothing}} M_{\mathrm{std}(\pi')} \otimes M_{\mathrm{std}(\pi'')}.$$

Note that the notion of standardization makes sense for more general objects. If S is a total ordered set, the standardized $\mathrm{std}(\ell)$, for any list ℓ of n elements of S, is classicaly the permutation $\sigma = [\sigma_1, \dots, \sigma_n]$ verfying $\sigma[i] > \sigma[j]$ if $\ell[i] > \ell[j]$ or if $\ell[i] = \ell[j]$ and $i > j$. Now if the description of an object o contains a list ℓ , the standardized $\mathrm{std}(o)$ is obtained by replacing ℓ by $\mathrm{std}(\ell)$ in o.

Let A be an infinite alphabet. The algebra WSym is isomorphic to $\mathrm{WSym}(A)$, the subalgebra of $\mathbb{C}\langle A\rangle$ defined by Rosas and Sagan (19) and constituted by the polynomials which are invariant by permutation of the letters of the alphabet. The explicit isomorphism sends each Φ^π to the polynomial $\Phi^\pi(A) := \sum_w w$ where the sum is over the words $w = w_1 \cdots w_n$ $(w_1, \ldots, w_n \in A)$ such that if i and j are in the same block of π then $w_i = w_j$. Under this isomorphism, each M_π is sent to $M_\pi(A) = \sum_w w$ where the sum is over the words $w = w_1 \cdots w_n$ $(w_1, \ldots, w_n \in A)$ such that $w_i = w_j$ if and only if i and j are in the same block of π. In the sequel, when there is no ambiguity, we will identify the algebras WSym and $\mathrm{WSym}(A)$. With the realization explained above, the coproduct of WSym consists of identifying the algebra $\mathrm{WSym} \otimes \mathrm{WSym}$ with $\mathrm{WSym}(A + B)$, where A and B are two noncommutative alphabets such that A commutes with B, by setting $f(A)g(B) \sim f \otimes g$ (see. It is a cocommutative coproduct for which the polynomials $\Phi^{\{1,\ldots,n\}}$ are primitive. Endowed with this coproduct, WSym has a Hopf structure which has been studied by Hivert *et al.* (10) and Bergeron *et al.* (2).

2.2 What are virtual alphabets in WSym?

We consider the set $\mathfrak{C}$ of set compositions together with additional elements $\{\mathfrak{o}_m : m > 1\}$ (we will also set $\mathfrak{o}_0 = []$) and a unity $\mathbf{1}$. This set is a naturally bigraded set: if $\mathfrak{C}_n^m$ denotes the set of compositions of $\{1, \ldots, n\}$ into m subsets, we have $\mathfrak{C} = \{\mathbf{1}\} \cup \bigcup_{n,m \in \mathbb{N}} \mathfrak{C}_n^m \cup \{\mathfrak{o}_m : m > 1\}$. We will also use the notations $\mathfrak{C}_n$ (resp. $\mathfrak{C}^m$) to denote the set of compositions of $\{1, \ldots, n\}$ (resp. the set compositions into m subsets together with $\mathfrak{o}_m$) with the special case: $\mathbf{1} \in \mathfrak{C}^1$. The formal space $\mathbb{C}[\mathfrak{C}^m]$ is naturally endowed with a structure of right $\mathbb{C}[\mathfrak{S}_m]$-module; the permutations acting by permuting the blocks of each composition and letting $\mathfrak{o}_m$ invariant. For simplicity, we will denote also by $\mathfrak{C}$ the collection ($\mathbb{S}$-module, see *e.g.*(13))$[\mathbb{C}[\mathfrak{C}^0], \mathbb{C}[\mathfrak{C}^1], \ldots, \mathbb{C}[\mathfrak{C}^m]]$.

For each $1 \le i \le k$, we define partial compositions $\circ_i : \mathfrak{C}^k \times \mathfrak{C}^{k'} \to \mathfrak{C}^{k+k'-1}$ by:

1. If $\Pi = [\pi_1, \ldots, \pi_k]$ and $\Pi' = [\pi'_1, \ldots, \pi'_{k'}]$ then
$$\Pi \circ_i \Pi' = \begin{cases} [\pi_1, \ldots, \pi_{i-1}, \pi'_1[\pi_i], \ldots, \pi'_{k'}[\pi_i], \pi_{i+1}, \ldots, \pi_k] & \text{if } \Pi' \in \mathfrak{C}_{\#\pi_i} \\ \mathfrak{o}_{k+k'-1} & \text{otherwise,} \end{cases}$$
where $\pi'_j[\pi_i] = \{i_{j_1}, \ldots, i_{j_p}\}$ if $\pi'_j = \{j_1, \ldots, j_p\}$ and $\pi_i = \{i_1, \ldots, i_k\}$ with $i_1 < \cdots < i_k$;
2. $\Pi \circ_i \mathfrak{o}_{k'} = \mathfrak{o}_k \circ_i \Pi' = \mathfrak{o}_{k+k'-1}$ for each $\Pi \in \mathfrak{C}^k$ and $\Pi' \in \mathfrak{C}^{k'}$;
3. $\mathbf{1} \circ_1 \Pi' = \Pi'$ and $\Pi \circ_i \mathbf{1} = \Pi$ for each $\Pi \in \mathfrak{C}^k$ and $\Pi' \in \mathfrak{C}^{k'}$.

Proposition 2.1 *The $\mathbb{S}$-module $\mathfrak{C}$ (*i.e.* each graded component $\mathfrak{C}_n$ is a $\mathfrak{S}_n$-module (13)) endowed with the partial compositions $\circ_i$ is an operad in the sense of Martin Markl (14), which means the compositions satisfy:*

1. *(Associativity) For each $1 \le j \le k$, $\Pi \in \mathfrak{C}^k$, $\Pi' \in \mathfrak{C}^{k'}$ and $\Pi'' \in \mathfrak{C}^{k''}$:*
$$(\Pi \circ_j \Pi') \circ_i \Pi'' = \begin{cases} (\Pi \circ_i \Pi'') \circ_{j+k''-1} \Pi', & \textit{for } 1 \le i < j, \\ \Pi \circ_J (\Pi' \circ_{i-j+1} \Pi''), & \textit{for } j \le i < k' + j, \\ (\Pi \circ_{i-k'+1} \Pi'') \circ_j \Pi', & \textit{for } j + k' \le i < k + k' - 1, \end{cases}$$
2. *(Equivariance) For each $1 \le i \le m$, $\tau \in \mathfrak{S}_m$ and $\sigma \in S_n$, let $\tau \circ_i \sigma \in \mathfrak{S}_{m+n-1}$ be given by inserting the permutation σ at the ith place in τ. If $\Pi \in \mathfrak{C}^k$ and $\Pi' \in \mathfrak{C}^{k'}$ then $(\Pi\tau) \circ_i (\Pi'\sigma) = (\Pi \circ_{\tau(i)} \Pi')(\tau \circ_i \sigma)$.*

3. *(Unitality)* $\mathbf{1} \circ_1 \Pi' = \Pi'$ *and* $\Pi \circ_i \mathbf{1} = \Pi$ *for each* $\Pi \in \mathfrak{C}^k$ *and* $\Pi' \in \mathfrak{C}^{k'}$.

Let $V = \bigoplus_n V_n$ be a graded space over $\mathbb{C}$. We will say that V is a (symmetric) $\mathfrak{C}$-module, if there is an action of $\mathfrak{C}$ on V which satisfies:

1. Each $\Pi \in \mathfrak{C}^m$ acts as a linear application $V^m \to V$.
2. (Compatibility with the graduation) $[\pi_1, \ldots, \pi_m](V_{j_1}, \ldots, V_{j_m}) = \mathbf{0}$ if $\#\pi_i \neq j_i$ for some $1 \leq i \leq m$ and $\mathfrak{o}_m(V_{j_1}, \ldots, V_{j_m}) = \mathbf{0}$. Otherwise, $[\pi_1, \ldots, \pi_m] \in \mathfrak{C}_n$ sends $V_{\#\pi_1} \times \cdots \times V_{\#\pi_m}$ to V_n. Note also the special case $\mathbf{1}(v) = v$ for each $v \in V$.
3. (Compatibility with the compositions)
$$(\Pi \circ_i \Pi')(v_1, \ldots, v_{k+k'-1}) = \Pi(v_1, \ldots, v_{i-1}, \Pi'(v_i, \ldots, v_{i+k'-1}), v_{i+k'}, \ldots, v_{k+k'-1}).$$
4. (Symmetry) $\Pi(v_1, \ldots, v_k) = (\Pi\sigma)(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$.

If V is generated (as a $\mathfrak{C}$-module) by $\{v_n : n \geq 1\}$ with $v_n \in V_n$, then setting $v^{\{\pi_1, \ldots, \pi_m\}} = [\pi_1, \ldots, \pi_m](v_{\#\pi_1}, \ldots, v_{\#\pi_m})$, we have $V_n = \mathrm{span}\{v^\pi : \pi \Vdash n\}$. Note that the existence of v^π follows from the point 4 of the definition of a $\mathfrak{C}$-module.

Example 2.2 If A is a noncommutative alphabet, the algebra $\mathbb{C}\langle A\rangle$ can be endowed with a structure of $\mathfrak{C}$-module by setting

$$[\pi_1, \ldots, \pi_m](w_1, \ldots, w_m) = \begin{cases} \sqcup\!\sqcup_{[\pi_1, \ldots, \pi_m]}(w_1, \ldots, w_m) & \text{if } w_i \in A^{\#\pi_i} \text{ for each } 1 \leq i \leq m \\ 0 & \text{otherwise} \end{cases}$$

where $w_1, \ldots, w_m \in A^*$ and $\sqcup\!\sqcup_{[\pi_1, \ldots, \pi_m]}(w_1, \ldots, w_m) = a_1 \ldots a_n$ is the only word of $A^{\#\pi_1 + \cdots + \#\pi_m}$ such that for each $1 \leq i \leq m$, if $\pi_i = \{j_1, \ldots, j_\ell\}$, $a_{j_1} \ldots a_{j_\ell} = w_j$.
Note that $\mathbb{C}[A]$ has also a structure of $\mathfrak{C}$-module defined by $[\pi_1, \ldots, \pi_m](x_1, \ldots, x_m) = x_1 \ldots x_m$ if x_i is a monomial of degree $\#\pi_i$.

Proposition 2.3 WSym *is a* $\mathfrak{C}$*-module.*

Proof: We define the action of $\mathfrak{C}$ on the power sums by

$$[\pi_1, \ldots, \pi_m](\Phi^{n_1}, \ldots, \Phi^{n_m}) = \begin{cases} \Phi^{\{\pi_1, \ldots, \pi_m\}} & \text{if } n_i = \#\pi_i \text{ for each } 1 \leq i \leq m \\ 0 & \text{otherwise.} \end{cases}$$

and extend it linearly to the spaces WSym_n. Since this action is compatible with the realization:

$$[\pi_1, \ldots, \pi_m](\Phi^{n_1}, \ldots, \Phi^{n_m})(A) = \sqcup\!\sqcup_{[\pi_1, \ldots, \pi_m]}(\Phi^{n_1}(A), \ldots, \Phi^{n_m}(A))$$

(the definition of $\sqcup\!\sqcup_\pi$ is given in Example 2.2) and WSym is obviously stable by the action of $\mathfrak{C}$, $\mathrm{WSym}(A)$ is a sub-$\mathfrak{C}$-module of $\mathbb{C}\langle A\rangle$. Hence, WSym is a $\mathfrak{C}$-module. ■

A morphism of $\mathfrak{C}$-module is a linear map φ from a $\mathfrak{C}$-module V_1 to another $\mathfrak{C}$-module V_2 satisfying $\varphi\Pi(v_1, \ldots, v_m) = \Pi(\varphi(v_1), \ldots, \varphi(v_m))$ for each $\Pi \in \mathfrak{C}^m$. In this context, a virtual alphabet (or a specialization) is defined by a morphism of $\mathfrak{C}$-module φ from WSym to a $\mathfrak{C}$-module V. The image of a word symmetric function f will be denoted by $f[\varphi]$. The $\mathfrak{C}$-module WSym is free in the following sense:

Proposition 2.4 *Let V be generated by $v_1 \in V_1, v_2 \in V_2, \dots, v_n \in V_n, \dots$ as a $\mathfrak{C}$-module (or equivalently, $V_n = \mathrm{span}\{v^\pi : \pi \Vdash n\}$). There exists a morphism of $\mathfrak{C}$-module $\varphi : \mathrm{WSym} \to V$, which sends Φ^n to v_n.*

Proof: This follows from the fact that $\{\Phi^\pi : \pi \Vdash n\}$ is a basis of WSym_n. Let $\varphi : \mathrm{WSym}_n \to V_n$ be the linear map such that $\varphi(\Phi^\pi) = V^\pi$; this is obviously a morphism of $\mathfrak{C}$-module. ■

For convenience, we will write $\mathrm{WSym}[\varphi] = \varphi\mathrm{WSym}$.

Example 2.5

1. Let A be an infinite alphabet. The restriction to $\mathrm{WSym}(A)$ of the morphism of algebra $\varphi : \mathbb{C}\langle A\rangle \to \mathbb{C}[A]$ defined by $\varphi(a) = a$ is a morphism of $\mathfrak{C}$-module sending $\mathrm{WSym}(A)$ to $Sym(A)$ (the algebra of symmetric functions on the alphabet A). This morphism can be defined without the help of alphabets, considering that Sym is generated by the power sums $p_1, \dots, p_n, \dots$ with the action $[\pi_1, \dots, \pi_m](p_{\#\pi_1}, \dots, p_{\#\pi_m}) = p_{\#\pi_1} \cdots p_{\#\pi_m}$.
2. Let A be any alphabet (finite or not). If V is a sub-$\mathfrak{C}$-module of $\mathbb{C}\langle A\rangle$ generated by the homogeneous polynomials $P_n \in \mathbb{C}[A^n]$ as a $\mathfrak{C}$-module, the linear map sending, for each $\pi = \{\pi_1, \dots, \pi_m\}$, Φ^π to $\sqcup\!\sqcup_\Pi[P_{\#\pi_1}, \dots, P_{\#\pi_m}]$, where $\Pi = [\pi_1, \dots, \pi_m]$, is a morphism of $\mathfrak{C}$-module.

3 The Hopf algebra of set partitions into lists

3.1 Set partitions into lists

A *set partition into lists* is an object which can be constructed from a set partition by ordering each block. For example, $\{[1, 2, 3], [4, 5]\}$ and $\{[3, 1, 2], [5, 4]\}$ are two distinct set partitions into lists of the set $\{1, 2, 3, 4, 5\}$. The number of set partitions into lists of an n-element set (or set partitions into lists of size n) is given by Sloane's sequence A000262 (20). If Π is a set partition into lists of $\{1, \dots, n\}$, we will write $\Pi \Vvdash n$. We will denote by $\mathrm{cycle\_support}(\sigma)$ the *cycle support* of a permutation σ, *i.e.*, the set partition associated to its cycle decomposition. For instance, $\mathrm{cycle\_support}(325614) = \{\{135\}, \{2\}, \{4, 6\}\}$. A set partition into lists can be encoded by a set partition and a permutation in view of the following easy result:

Proposition 3.1 *For all n, the set partitions into lists of size n are in bijection with the pairs (σ, π) where σ is a permutation of size n and π is a set partition which is less fine than or equal to the cycle support of σ.*

Indeed, from a set partition π and a permutation σ, we obtain a set partition into lists Π by ordering the elements of each block of π so that they appear in the same order as in σ.

Example 3.2 Starting the set partition $\pi = \{\{1, 4, 5\}, \{6\}, \{3, 7\}, \{2\}\}$ and the permutation $\sigma = 4271563$, we obtain the set partition into lists $\Pi = \{[4, 1, 5], [7, 3], [6], [2]\}$.

3.2 Construction

Let $\Pi \Vvdash n$ and $\Pi' \Vvdash n'$ be two set partitions into lists. Then, we set $\Pi \uplus \Pi' = \Pi \cup \{[l_1 + n, \dots, l_k + n] : [l_1, \dots, l_k] \in \Pi'\} \Vvdash n + n'$. Let $\Pi' \subset \Pi \Vvdash n$, since the integers appearing in Π' are all distinct, the standardized $\mathrm{std}(\Pi')$ of Π' is well defined as the unique set partition into lists obtained by replacing the ith smallest integer in Π by i. For example,$\mathrm{std}(\{[5, 2], [3, 10], [6, 8]\}) = \{[3, 1], [2, 6], [4, 5]\}$.

Definition 3.3 *The Hopf algebra* BWSym *is formally defined by its basis* (Φ^{Π}) *where the* Π *are set partitions into lists, its product* $\Phi^{\Pi}\Phi^{\Pi'} = \Phi^{\Pi\uplus\Pi'}$ *and its coproduct* $\Delta(\Phi^{\Pi}) = \sum \Phi^{\mathrm{std}(\Pi')} \otimes \Phi^{\mathrm{std}(\Pi'')}$, *where the sum is over the* (Π', Π'') *such that* $\Pi' \cup \Pi'' = \Pi$ *and* $\Pi' \cap \Pi'' = \emptyset$.

Following Section 3.1 and for convenience, we will use alternatively Φ^{Π} and $\Phi^{\binom{\sigma}{\pi}}$ to denote the same object.
We define $M_{\Pi} = M_{\binom{\sigma}{\pi}}$ by setting $\Phi^{\binom{\sigma}{\pi}} = \sum\limits_{\pi\leq\pi'} M_{\binom{\sigma}{\pi}'}$. The formula being diagonal, it defines M_{Π} for any Π and proves that the family $(M_{\Pi})_{\Pi}$ is a basis of BWSym. Consider for instance $\{[3,1],[2]\} \sim \begin{pmatrix} 321 \\ \{\{1,3\},\{2\}\} \end{pmatrix}$, we have $\Phi^{\{[3,1],[2]\}} = M_{\{[3,1],[2]\}} + M_{\{[3,2,1]\}}$.

For any set partition into lists Π, let $s(\Pi)$ be the corresponding classical set partition. Then, the linear application ϕ defined by $\phi(\Phi^{\Pi}) = \Phi^{s(\Pi)}$ is obviously a morphism of Hopf algebras. As an associative algebra, BWSym has also algebraical links with the algebra FQSym (4). Recall that this algebra is defined by its basis (E^{σ}) whose product is $E^{\sigma}E^{\tau} = E^{\sigma/\tau}$, where σ/τ is the word obtained by concatening σ and the word obtained from τ by adding the size of σ to all the letters (for example $321/132 = 321465$).
The subspace V of $\mathrm{WSym}\otimes\mathrm{FQSym}$ linearly spanned by the $\Phi^{\pi}\otimes E^{\sigma}$ such that the cycle supports of σ is finer than π is a subalgebra, and the linear application sending $\Phi^{\pi,\sigma}$ to $\Phi_{\pi}\otimes E^{\sigma}$ is an isomorphism of algebras. Moreover, when the set of cycle supports of σ is finer than π, $M_{\pi}\otimes E^{\sigma}$ also belongs to V and is the image of $M_{\binom{\sigma}{\pi}}$.
The linear application from BWSym to FQSym which sends $M_{\{[\sigma]\}}$ to E^{σ} and M_{Π} to 0 if $\mathrm{card}(\Pi) > 1$, is also a morphism of algebras.

3.3 Realization

Let $A^{(j)} = \{a_i^{(j)} \mid i > 0\}$ be an infinite set of bi-indexed noncommutative variables, with the order relation defined by $a_i^{(j)} < a_{i'}^{(j)}$ if $i < i'$. Let $A = \bigcup_j A^{(j)}$. Consider the set partition into lists $\Pi = \{L^1, L^2, \ldots\} = \{[l_1^1, l_2^1, \ldots, l_{n_1}^1], [l_1^2, l_2^2, \ldots, l_{n_2}^2], \ldots\} \Vdash n$. Then, one obtains a *polynomial realization* $\mathrm{BWSym}(A)$ by identifying Φ^{Π} with $\Phi^{\Pi}(A)$, the sum of all the monomials $a_1 \ldots a_n$ (where the a_i are in A) such that $k = k'$ implies $a_{l_t^k}, a_{l_s^{k'}} \in A^{(j)}$ for some j, and for each k, $\mathrm{std}(a_{i_1} \ldots a_{i_{n_k}}) = \mathrm{std}(l_1^k \ldots l_{n_k}^k)$ with $\{l_1^k, \ldots, l_{n_k}^k\} = \{i_1 < \cdots < i_{n_k}\}$ (The "B" of BWSym is for "bi-indexed letters"). The coproduct Δ can be interpreted by identifying $\Delta(\Phi^{\Pi})$ with $\Phi^{\Pi}(A+B)$ as in the case of WSym. Here, if $a \in A$ and $b \in B$ then a and b are not comparable.

Proposition 3.4 *The Hopf algebras* BWSym *and* $\mathrm{BWSym}(A)$ *are isomorphic.*

Now, let $M'_{\Pi}(A)$ be the sum of all the monomials $a_1 \ldots a_n$, $a_i \in A$, such that $a_{l_t^k}$ and $a_{l_s^{k'}}$ belong in the same $A^{(j)}$ if and only if $k = k'$, and for each k, $\mathrm{std}(a_{i_1} \ldots a_{i_{n_k}}) = \mathrm{std}(l_1^k \ldots l_{n_k}^k)$ with $\{l_1^k, \ldots, l_{n_k}^k\} = \{i_1 < \cdots < i_{n_k}\}$. For example, the monomial $a_1^{(1)}a_1^{(1)}a_2^{(1)}$ appears in the expansion of $\Phi_{\{[1,3],[2]\}}$, but not in the one of $M'_{\{[1,3],[2]\}}$. The M'_{Π} form a new basis (M'_{Π}) of BWSym. Note that this basis is not the same as (M_{Π}). For example, one has

$$\Phi^{\{[1],[2]\}} = M_{\{[1],[2]\}} + M_{\{[1,2]\}} = M'_{\{[1],[2]\}} + M'_{\{[1,2]\}} + M'_{\{[2,1]\}}.$$

Consider the basis F_{σ} of FQSym defined in (4). The linear application, from BWSym to FQSym, which sends $M'_{\{[\sigma]\}}$ to $F_{(\sigma^{-1})}$, and M'_{Π} to 0 if $\mathrm{card}(\Pi) > 1$, is a morphism of algebras.

3.4 Related Hopf algebras

By analogy with the construction of BWSym, we define a "B" version for each of the algebras Sym, QSym and WQSym. In this section, we sketch briefly how to construct them; the complete study is deferred to a forthcoming paper.

As usual when $L = [\ell_1, \dots, \ell_k]$ and $M = [m_1, \dots, m_2]$ are two lists, the shuffle product is defined recursively by $[\,] \sqcup\!\sqcup L = L \sqcup\!\sqcup [\,] = \{L\}$ and $L \sqcup\!\sqcup M = [\ell_1].([\ell_2, \dots, \ell_k] \sqcup\!\sqcup M) \cup [m_1].(L \sqcup\!\sqcup [m_2, \dots, m_k])$. The algebra of biword quasi-symmetric functions (BWQSym) has its bases indexed by set compositions into lists. The algebra is defined as the vector space spanned by the formal symbols Φ_Π, where Π is a composition into lists of the set $\{1, \dots, n\}$ for a given n, together with the product $\Phi_\Pi \Phi_{\Pi'} = \sum_{\Pi'' \in \Pi \sqcup\!\sqcup \Pi'[n]} \Phi_{\Pi''}$, where $\Pi'[n]$ means that we add n to each of the integers in the lists of Π' and Π is a composition into lists of $\{1, \dots, n\}$. Endowed with the coproduct defined by$\Delta(\Phi_\Pi) = \sum_{\Pi'.\Pi''=\Pi} \Phi_{\mathrm{std}(\Pi')} \otimes \Phi_{\mathrm{std}(\Pi'')}$, BWQSym has a structure of Hopf algebra. Note that $\mathrm{BWQSym} = \bigoplus_n \mathrm{BWQSym}_n$ is naturally graded; the dimension of the graded component BWQSym_n is $2^{n-1}n!$ (see sequence A002866 in (20)).

The algebra $\mathrm{BSym} = \bigoplus_n \mathrm{BSym}_n$ is a graded cocommutative Hopf algebra whose bases are indexed by multisets of permutations. Formally, we set $\mathrm{BSym}_n = \mathrm{span}\{\phi^{\{\sigma_1, \dots, \sigma_k\}} : \sigma_i \in \mathfrak{S}_{n_i}, n_1 + \dots + n_k = n\}$, $\phi^{S_1}.\phi^{S_2} = \phi^{S_1 \cup S_2}$ and for any permutation σ, $\phi^{\{\sigma\}}$ is primitive. The dimensions of the graded components are given by the sequence A077365 of (20).

Finally, $\mathrm{BQSym} = \bigoplus_n \mathrm{BQSym}_n$ is generated by $\phi_{[\sigma_1, \dots, \sigma_k]}$, its product is $\phi_L \phi_{L'} = \sum_{L'' \in L \sqcup\!\sqcup L'} \phi_{L''}$ and its coproduct is $\Delta(\phi_L) = \sum_{L=L'.L''} \phi_{L'} \otimes \phi_{L''}$. The dimension of the graded component BQSym_n is given by Sloane's sequence A051296 (20).

4 On the R-P theorem

4.1 R-P theorem and symmetric functions

Consider two sets X and Y such that X is finite ($\#X = n$), together with a group $G \subset \mathfrak{S}_n$ acting on X by permuting its elements. We consider the set Y^X of the maps $X \to Y$. The type of a map f is the vector of the multiplicities of its images; more precisely, $\mathrm{type}(f) \in \mathbb{N}^Y$ with $\mathrm{type}(f)_y = \#\{x \in X : f(x) = y\}$. For instance, consider $X = \{a, b, c, d, e\}$, $Y = \{0, 1, 2\}$, $f(a) = f(c) = f(d) = 1$, $f(b) = 2$, $f(c) = 0$: we have $\mathrm{type}(f) = [1^0, 3^1, 1^2]$. The action of G on X induces an action of G on Y^X. Obviously, the type of a function is invariant for the action of G. Then all the elements of an orbit of G in Y^X have the same type, so that the type of an orbit will be the type of its elements. The question is: how to count the number $\mathfrak{n}_I$ of orbits for the given type I? Note that, if λ_I denotes the (integer) partition obtained by removing all the zeros in I and reordering its elements in the decreasing order, $\lambda_I = \lambda_{I'}$ implies $\mathfrak{n}_I = \mathfrak{n}_{I'}$; it suffices to understand how to compute $\mathfrak{n}_\lambda$ when λ is a partition. The Redfield-Pólya theorem deals with this problem and its main tool is the cycle index:

$$Z_G := \frac{1}{\#G} \sum_{\sigma \in G} p^{\mathrm{cycle\_type}(\sigma)},$$

where $\mathrm{cycle\_type}(\sigma)$ is the (integer) partition associated to the cycle of σ (for instance $\sigma = 325614 = (135)(46)$, $\mathrm{cycle\_type}(\sigma) = [3, 2, 1]$). When $\lambda = [\lambda_1, \dots, \lambda_k]$ is a partition, p^λ denotes the (commutative) symmetric function $p^\lambda = p_{\lambda_1} \dots p_{\lambda_k}$ and p_n is the classical power sum symmetric function.

The Redfield-Pólya theorem states:

Theorem 4.1 *The expansion of Z_G on the basis (m_λ) of monomial symmetric functions is given by*

$$Z_G = \sum_\lambda \mathfrak{n}_\lambda m_\lambda.$$

Example 4.2 Suppose that we want to enumerate the non-isomorphic non-oriented graphs on three vertices. The symmetric group $\mathfrak{S}_3$ acting on the vertices induces an action of the group

$$G := \{123456, 165432, 345612, 321654, 561234, 543216\} \subset \mathfrak{S}_6$$

on the edges. The construction is not unique. We obtain the group G by labelling the 6 edges from 1 to 6. Hence, to each permutation of the vertices corresponds a permutation of the edges. Here, the 1 labels the loop from the vertex 1 to itself, 2 labels the edge which links the vertices 1 and 2, 3 is the loop from the vertex 2 to itself, 4 labels the edge from the vertex 2 to the vertex 3, 5 is the loop from the vertex 3 to itself, finally, 6 links the vertices 1 and 3. The cycle index of G is

$$Z_G \quad = \quad \tfrac{1}{6}(p_1^6 + 3p_2^2p_1^2 + 2p_3^2) = m_6 + 2m_{51} + 4m_{42} + 6m_{411} + 6m_{33} + \ldots$$

The coefficient 4 of m_{42} means that there exists 4 non-isomorphic graphs with 4 edges coloured in blue and 2 edges coloured in red.

4.2 Word R-P theorem

If σ is a permutation, we define $\Phi^\sigma := \Phi^{\mathrm{cycle\_support}(\sigma)}$. Now for our purpose, a map $f \in Y^X$ will be encoded by a word w : we consider an alphabet $A = \{a_y : y \in Y\}$, the elements of X are relabelled by $1, 2, \ldots, \#X = n$ and w is defined as the word $b_1 \ldots b_n \in A^n$ such that $b_i = a_{f(i)}$. With these notations, the action of G on Y^X is encoded by the action of the permutations of G on the positions of the letters in the words of A^n.

It follows that for any permutation $\sigma \in G$, one has

$$\Phi^\sigma = \sum_{w\sigma = w} w. \tag{1}$$

The cycle support polynomial is defined by $\mathrm{Z}_G := \sum_{\sigma\in G} \Phi^\sigma$. From (1) we deduce

$$\mathrm{Z}_G = \sum_w \#\mathrm{Stab}_G(w)w$$

where $\mathrm{Stab}_G(w) = \{\sigma \in G : w\sigma = w\}$ is the subgroup of G which stabilizes w. In terms of monomial functions, this yields :

Theorem 4.3

$$\mathrm{Z}_G = \sum_\pi \#\mathrm{Stab}_G(w_\pi) M_\pi$$

where w_π is any word $a_1 \ldots a_n$ such that $a_i = a_j$ if and only if $i, j \in \pi_k$ for some $1 \le k \le n$.

Example 4.4 Consider the same example as in Example 4.2. Each graph is now encoded by a word $a_1a_2a_3a_4a_5a_6$: the letter a_1 corresponds to the colour of the vertex 1, the letter 2 to the colour of the vertex 2 and so on.
The cycle support polynomial is

$$\begin{aligned}\mathrm{Z}_G \quad := \quad & \Phi^{\{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\}\}} + \Phi^{\{\{2,6\},\{3,5\},\{1\},\{4\}\}} \\ & + \Phi^{\{\{1,3\},\{4,6\},\{2\},\{5\}\}} + \Phi^{\{\{1,5\},\{2,4\},\{3\},\{6\}\}} + 2\Phi^{\{\{1,3,5\},\{2,4,6\}\}}.\end{aligned}$$

The coefficient of $M_{\{\{2,6\},\{3,5\},\{1\},\{4\}\}}$ in Z_G is 2 because it appears only in $\Phi^{\{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\}\}}$ and $\Phi^{\{\{2,6\},\{3,5\},\{1\},\{4\}\}}$. The monomials of $M_{\{\{2,6\},\{3,5\},\{1\},\{4\}\}}$ are of the form *abcdbcb*, where a, b, c and d are four distinct letters. The stabilizer of *abcdcb* in G is the two-element subgroup $\{123456, 165432\}$.

Note that the cycle support polynomial has already appeared in the literature on the work of Sagan and Gebhard (5) on a slightly different setting which is a special case of our purpose.

4.3 From word R-P theorem to R-P theorem

The aim of this section is to link the numbers $\mathfrak{n}_I$ of Section 4.1 and the numbers $\#\mathrm{Stab}_G(w)$ appearing in Section 4.2.
If w is a word we will denote by $\mathrm{orb}_G(w)$ its orbit under the action of G. The Orbit-stabilizer theorem (see *e.g.*(3)) together with Lagrange's theorem gives:

$$\#G = \#\mathrm{orb}_G(w)\#\mathrm{Stab}_G(w) \tag{2}$$

Denote by $\Lambda(\pi)$ the unique integer partition defined by $(\#\pi_1, \dots, \#\pi_k)$ if $\pi = \{\#\pi_1, \dots, \pi_k\}$ with $\#\pi_1 \geq \#\pi_2 \geq \cdots \geq \#\pi_k$. If $\lambda = (m^{k_m}, \dots, 2^{k_2}, 1^{k_1})$ we set $\lambda^! = k_m! \dots k_2! k_1!$. The shape of a word w is the unique set partition $\pi(w)$ such that w is a monomial of $M_{\pi(w)}$. Note that all the orbits of words with a fixed shape π have the same cardinality. Let $\pi = \{\pi_1, \dots, \pi_k\}$ and $A_k = \{a_1, \dots, a_k\}$ be an alphabet of size k. we will denote by S_{π,A_k} the set of words w with size k such that the number of letter a_i in w is $\#\pi_i$. All the words of S_{π,A_k} have the same coefficients in Z_G and this set splits into $\frac{\Lambda(\pi)^!}{\#\mathrm{orb}(w_\pi)}$ orbits of size $\#\mathrm{orb}(w_\pi)$

$$\mathfrak{n}_\lambda = \sum_{\Lambda(\pi)=\lambda} \frac{\lambda^!}{\#\mathrm{orb}_G(w_\pi)} = \sum_{\Lambda(\pi)=\lambda} \frac{\lambda^! \#\mathrm{Stab}_G(w_\pi)}{\#G}. \tag{3}$$

If we consider the morphism of algebra $\theta : \mathrm{WSym} \to Sym$ which sends Φ_n to p_n, we have $\theta(M_\pi) = \Lambda(\pi)^! m_\lambda$. Hence, we have

$$\frac{1}{\#G}\theta(\mathrm{Z}_G) = \sum_\lambda \left(\sum_{\Lambda(\pi)=\lambda} \frac{\lambda^!}{\#\mathrm{orb}_G(w_\pi)} \right) m_\lambda = \sum_\lambda \mathfrak{n}_\lambda m_\lambda$$

as expected by the Redfield-Pólya theorem (Theorem 4.1).

4.4 R-P theorem without multiplicities

Examining with more details Example 4.4, the coefficient 2 of $M_{\{2,6\},\{3,5\},\{1\},\{4\}}$ in Z_G follows from the group $\{123456, 165432\}$ of order two which stabilizes *abcdcb*. In terms of set partitions into lists, this can be interpreted by $M_{\{[2,6],[3,5],[1],[4]\}} + M_{\{[6,2],[5,3],[1],[4]\}} \to 2M_{\{\{2,6\},\{3,5\},\{1\},\{4\}\}}$. We deduce the following version (without multiplicities) of Theorem 4.3 in BWSym.

Theorem 4.5 *Let G be a permutation group. We have*

$$\mathbb{Z}_G := \sum_{\sigma\in G} \Phi^{\binom{\sigma}{\text{cycle\_support}(\sigma)}} = \sum_{\pi} \sum_{\sigma\in Stab_\pi(G)} M_{\binom{\sigma}{\pi}}.$$

Consider again Example 4.4.

$$\begin{aligned}\mathbb{Z}_G = \quad & \Phi^{\binom{123456}{\{\{1\}\{2\}\{3\}\{4\}\{5\}\{6\}\}}} + \Phi^{\binom{165432}{\{\{1\}\{26\}\{35\}\{4\}\}}} + \Phi^{\binom{345612}{\{\{135\}\{246\}\}}} \\ + \quad & \Phi^{\binom{321654}{\{\{13\}\{2\}\{46\}\{5\}\}}} + \Phi^{\binom{561234}{\{\{135\}\{246\}\}}} + \Phi^{\binom{543216}{\{\{15\}\{234\}\{6\}\}}}.\end{aligned}$$

When expanded in the monomial M basis, there are exactly 2 terms of the form $M_{\binom{\sigma}{\{\{2,6\},\{3,5\},\{1\},\{4\}\}}}$ (for $\sigma = 123456$ and $\sigma = 165432$). Note that we can use another realization which is compatible with the space but not with the Hopf algebra structure. It consists to set $\widetilde{\Phi}^{\binom{\sigma}{\pi}} := \sum_w \binom{\sigma}{w}$, where the sum is over the words $w = w_1 \dots w_n$ ($w_i \in A$) such that if i and j are in the same block of π then $w_i = w_j$. If we consider the linear application $\widetilde{\psi}$ sending $\Phi^{\binom{\sigma}{\pi}}$ to $\widetilde{\Phi}^{\binom{\sigma}{\pi}}$, $\widetilde{\psi}$ sends $M_{\binom{\sigma}{\pi}}$ to $\widetilde{M}_{\binom{\sigma}{\pi}} := \sum_w \binom{\sigma}{w}$, where the sum is over the words $w = w_1 \dots w_n$ ($w_i \in A$) such that i and j are in the same block of π if and only if $w_i = w_j$. Let w be a word, the set of permutations σ such that $\binom{\sigma}{w}$ appears in the expansion of $\widetilde{\psi}(\mathbb{Z}_G)$ is the stabilizer of w in G. The linear application sending each biword $\binom{\sigma}{w}$ to w sends $\widetilde{\Phi}^{\binom{\sigma}{\pi}}$ to Φ^π and $\sum_{\sigma\in\mathrm{Stab}_G(w)} \binom{\sigma}{w}$ to $\#\mathrm{Stab}_G(w)w$. Note that $\#\mathrm{Stab}_G(w)$ is also the coefficient of the corresponding monomial $M_{\pi(w)}$ in the cycle support polynomial Z_G. For instance, we recover the coefficient 2 in Example 4.4 from the biwords $\binom{123456}{abcdcb}$ and $\binom{165432}{abcdcb}$ in $\widetilde{\psi}(\mathbb{Z}_G)$.

4.5 WSym *and Harary-Palmer type enumerations*

Let $A := \{a_1, \dots, a_m\}$ be a set of formal letters and $I = [i_1, \dots, i_k]$ a sequence of elements of $\{1, \dots, m\}$. We define the virtual alphabet A_I by

$$\Phi^n(A_I) := (a_{i_1} \dots a_{i_k})^{\frac{n}{k}} + (a_{i_2} \dots a_{i_k} a_{i_1})^{\frac{n}{k}} + \dots + (a_{i_k} a_1 \dots a_{i_{k-1}})^{\frac{n}{k}},$$

if k divides n and 0 otherwise. If $\sigma \in \mathfrak{S}_m$ we define the alphabet A_σ as the formal sum of the alphabets A_c associated to its cycles:

$$\Phi^{\{1\dots n\}}[A_\sigma] := \sum_{c \text{ cycle in } \sigma} \Phi^n[A_c].$$

From Example 2.5.2, the set $\{\Phi^{\{1,\dots,n\}}[A_\sigma] : n \in \mathbb{N}\}$ generates the sub-$\mathfrak{C}$-module $\mathrm{WSym}[A_\sigma]$ of $\mathbb{C}\langle A\rangle$ (the composition Π acting by $\sqcup\!\sqcup_\Pi$).
Let $H \subset \mathfrak{S}_m$ and $G \subset \mathfrak{S}_n$ be two permutation groups. We define $\mathrm{Z}(H; G) := \sum_{\tau\in H} \Phi^G[A_\tau]$.

Proposition 4.6 *We have:*

$$\mathrm{Z}(H; G) = \sum_{w\in A^n} \#Stab_{H,G}(w)w$$

where $Stab_{H,G}(w)$ denotes the stabilizer of w under the action of $H\times G$ (H acting on the left on the names of the variables a_i and G acting on the right on the positions of the letters in the word); equivalently, $Stab_{H,G}(a_{i_1} \dots a_{i_k}) = \{(\tau, \sigma) \in H \times G : a_{\tau(i_{\sigma(j)})}$ for each $1 \le j \le n\}$.

Hence, from Burnside's classes formula, sending each variable to 1 in $\mathrm{Z}(H;G)$, we obtain the number of orbits of $H \times G$.

Example 4.7 Consider the set of the non-oriented graphs without loop whose edges are labelled by three colours. Suppose that we consider the action of the group $H = \{123, 231, 312\} \subset \mathfrak{S}_n$ on the colours. We want to count the number of graphs up to permutation of the vertices ($G = \mathfrak{S}_3$) and the action of H on the edges. There are three edges, and each graph will be encoded by a word $a_{i_1}a_{i_2}a_{i_3}$ where i_j denotes the colour of the edge j. We first compute the specialization $\Phi^{\{1\dots n\}}[A_\sigma]$ for $1 \le n \le 3$ and $\sigma \in H$. We find $\Phi^{\{1\}}[A_{123}] = a_1+a_2+a_3$, $\Phi^{\{1\}}[A_{231}] = \Phi^{\{1\}}[A_{312}] = 0$, $\Phi^{\{1,2\}}[A_{123}] = a_1^2+a_2^2+a_3^2$, $\Phi^{\{1,2\}}[A_{231}] = \Phi^{\{1,2\}}[A_{312}] = 0$, $\Phi^{\{1,2,3\}}[A_{123}] = a_1^3 + a_2^3 + a_3^3$, $\Phi^{\{1,2,3\}}[A_{213}] = a_2a_3a_1 + a_3a_1a_2 + a_1a_2a_3$, and $\Phi^{\{1,2,3\}}[A_{312}] = a_1a_3a_2+a_3a_2a_1+a_2a_1a_3$. Now, we deduce the values of the other $\Phi^\pi[A_\sigma]$ with $\pi \Vdash 3$ and $\sigma \in H$ by the action of $\sqcup\!\sqcup_\Pi$. For instance:

$$\begin{array}{rcl}\Phi^{\{\{1,2\},\{3\}\}}[A_{123}] &=& \sqcup\!\sqcup_{[\{1,2\},\{3\}]}\left(\Phi^{\{1,2\}}[A_{123}], \Phi^{\{1\}}[A_1]\right)\\ &=& a_1^3 + a_1^2a_2 + a_1^2a_3 + a_2^2a_1 + a_2^3 + a_2^2a_3 + a_3^2a_1 + a_3^2a_2 + a_3^3.\end{array}$$

We find also

$$\begin{array}{r}\Phi^{\{\{1,3\},\{2\}\}}[A_{123}] = a_1^3 + a_1a_2a_1 + a_1a_3a_1 + a_2a_1a_2 + a_2^3 + a_2a_3a_2 + a_3a_1a_3 + a_3a_2a_3 + a_3^3,\\ \Phi^{\{\{1\},\{2,3\}\}}[A_{123}] = a_1^3 + a_2a_1^2 + a_3a_1^2 + a_1a_2^2 + a_2^3 + a_3a_2^2 + a_1a_3^2 + a_2a_3^2 + a_3^3,\\ \Phi^{\{\{1\},\{2\},\{3\}\}}[A_{123}] = (a_1+a_2+a_3)^3.\end{array}$$

The other $\Phi^\pi[A_\sigma]$ are zero. Hence,

$$\begin{array}{rcl}\mathrm{Z}[H;\mathfrak{S}_3] &=& \Phi^{123}[A_{123}] + \Phi^{132}[A_{123}] + \Phi^{213}[A_{123}] + \Phi^{321}[A_{123}] +\\ && \Phi^{231}[A_{231}] + \Phi^{231}[A_{312}] + \Phi^{312}[A_{231}] + \Phi^{312}[A_{312}]\\ &=& 6(a_1^3+a_2^3+a_3^3) + 2\sum_{i\neq j} a_i^2a_j + 2\sum_{i\neq j} a_ja_ia_j + 2\sum_{i\neq j} a_ja_i^2\\ && +3(a_1a_2a_3 + a_2a_3a_1 + a_3a_1a_2) + 3(a_1a_3a_2 + a_3a_2a_1 + a_2a_1a_3).\end{array}$$

The coefficient 3 of $a_1a_2a_3$ means that the word is invariant under the action of three pairs of permutations (here $(123,123)$, $(231,312)$, $(312,231)$). Setting $a_1 = a_2 = a_3 = 1$, we obtain $\mathrm{Z}[H,\mathfrak{S}_3] = 18 \times 4$: 18 is the order of the group $H \times \mathfrak{S}_3$ and 4 is the number of orbits: $\{a_1^3, a_2^3, a_3^3\}$, $\{a_1^2a_2, a_1a_2a_1, a_2a_1^2, a_2^2a_3, a_2a_3a_2, a_3a_2^2, a_3^2a_1, a_3a_1a_3, a_1a_3^2\}$, $\{a_2^2a_1, a_2a_1a_2, a_1a_2^2, a_1^2a_3, a_1a_3a_1, a_3a_1^2, a_3^2a_2, a_3a_2a_3, a_1a_3^2\}$, and $\{a_1a_2a_3, a_1a_3a_2, a_2a_1a_3, a_2a_3a_1, a_3a_1a_2, a_3a_2a_1\}$.

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Extending the parking space

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Abstract. The action of the symmetric group S_n on the set Park_n of parking functions of size n has received a great deal of attention in algebraic combinatorics. We prove that the action of S_n on Park_n extends to an action of S_{n+1}. More precisely, we construct a graded S_{n+1}-module V_n such that the restriction of V_n to S_n is isomorphic to Park_n. We describe the S_n-Frobenius characters of the module V_n in all degrees and describe the S_{n+1}-Frobenius characters of V_n in extreme degrees. We give a bivariate generalization $V_n^{(\ell,m)}$ of our module V_n whose representation theory is governed by a bivariate generalization of Dyck paths. A Fuss generalization of our results is a special case of this bivariate generalization.

Résumé. L'action du groupe symétrique S_n sur l'ensemble Park_n des fonctions de stationnement de longueur n a reçu beaucoup d'attention dans la combinatoire algébrique. Nous démontrons que l'action de S_n sur Park_n s'étend à une action de S_{n+1}. Plus précisément, nous construisons un gradué S_{n+1}-module V_n telles que la restriction de S_n est isomorphe à Park_n. Nous décrivons la S_n-Frobenius caractères des modules V_n à tous les degrés et décrivent le S_{n+1}-Frobenius caractères de V_n en degrés extrêmes. Nous donnons une généralisation bivariée $V_n^{(\ell,m)}$ de notre module V_n dont la représentation théorie est régi par une généralisation bivariée des chemins de Dyck. Une généralisation Fuss de nos résultats est un cas particulier de cette généralisation bivariée.

Keywords: parking functions, symmetric group, Dyck paths, representation, matriod

1 Introduction

This paper is about extending the visible permutation action of S_n on the space Park_n spanned by parking functions of size n to a hidden action of the larger symmetric group S_{n+1}. The S_{n+1}-module we construct will be a subspace of the coordinate ring of the reflection representation of type A_n and will inherit the polynomial grading of this coordinate ring. Using statistics on Dyck paths, we will give an explicit combinatorial formula for the graded S_n-Frobenius character of our module and will describe the extended S_{n+1}-Frobenius character in extreme degrees.

As far as the authors know, this is the first example and proof of an extension of Park_n to S_{n+1} an S_{n+1}-module.

We remark that our result is the 'best possible' in two senses. First, it is not always possible to extend Park_n to an S_{n+2}-module; for example, the action of S_4 on Park_4 does not extend to an action of S_6.

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Also, from a combinatorial point of view, one may be interested in extending the action of S_n on Park_n to a *permutation* action of the larger symmetric group S_{n+1}. Our extended module is graded, but is not a permutation module. However, it is impossible to extend the action of S_4 on Park_4 to a permutation action of S_5.

2 Background and Main Results

A length n sequence $(a_1, \ldots, a_n)$ of positive integers is called a *parking function of size* n if its nondecreasing rearrangement $(b_1 \leq \cdots \leq b_n)$ satisfies $b_i \leq i$ for all i. Parking functions were introduced by Konheim and Weiss [KW66] in the context of computer science, but have seen much application in algebraic combinatorics with connections to Catalan combinatorics, Shi hyperplane arrangements, diagonal coinvariant rings, and rational Cherednik algebras. The set of parking functions of size n is famously counted by $(n+1)^{n-1}$. The $\mathbb{C}$-vector space Park_n spanned by the set of parking functions of size n carries a natural permutation action of the symmetric group S_n on n letters:

$$w.(a_1, \ldots, a_n) = (a_{w(1)}, \ldots, a_{w(n)}) \tag{1}$$

for $w \in S_n$ and $(a_1, \ldots, a_n) \in \mathrm{Park}_n$.

A *partition* λ of a positive integer n is a weakly decreasing sequence $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k)$ of nonnegative integers which sum to n. We write $\lambda \vdash n$ to mean that λ is a partition of n and define $|\lambda| := n$. We call k the *length* of the partition λ. The *Ferrers diagram* of λ consists of λ_i left justified boxes in the i^{th} row from the top ('English notation'). If λ is a partition, we define a new partition $\mathrm{mult}(\lambda)$ whose parts are obtained by listing the (positive) part multiplicities in λ in weakly decreasing order. For example, we have that $\mathrm{mult}(4, 4, 3, 3, 3, 1, 0, 0) = (3, 2, 2, 1)$.

We will make use of two partial orders on partitions in this paper. The first partial order is *Young's lattice* with relations given by $\lambda \subseteq \mu$ if $\lambda_i \leq \mu_i$ for all $i \geq 1$ (where we append an infinite string of zeros to the ends of λ and μ so that these inequalities make sense). Equivalently, we have that $\lambda \subseteq \mu$ if and only if the Ferrers diagram of λ fits inside the Ferrers diagram of μ. *Dominance order* on partitions is defined by $\lambda \preceq \mu$ if for all $i \geq 1$ we have the inequality of partial sums $\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i$ (where we again append an infinite string of zeros to the ends of λ and μ). Observe that either of the relations $\lambda \subseteq \mu$ or $\lambda \preceq \mu$ imply that $|\lambda| \leq |\mu|$.

For a partition $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$, we let S_λ denote the *Young subgroup* $S_{\lambda_1} \times \cdots \times S_{\lambda_k}$ of S_n. We denote by M^λ the coset representation of S_n given by $M^\lambda := \mathrm{Ind}_{S_\lambda}^{S_n}(\mathbf{1}_{S_\lambda}) \cong_{S_n} \mathbb{C}S_n/S_\lambda$ and we denote by S^λ the irreducible representation of S_n labeled by the partition λ.

Let R_n denote the $\mathbb{C}$-vector space of class functions $S_n \to \mathbb{C}$. Identifying modules with their characters, the set $\{S^\lambda : \lambda \vdash n\}$ forms a basis of R_n. The graded vector space $R := \bigoplus_{n \geq 0} R_n$ attains the structure of a $\mathbb{C}$-algebra via the induction product $S^\lambda \circ S^\mu := \mathrm{Ind}_{S_n \times S_m}^{S_{n+m}}(S^\lambda \otimes_{\mathbb{C}} S^\mu)$, where $\lambda \vdash n$ and $\mu \vdash m$.

We denote by Λ the ring of symmetric functions (in an infinite set of variables $X_1, X_2, \ldots$, with coefficients in $\mathbb{C}$). The $\mathbb{C}$-algebra Λ is graded and we denote by Λ_n the homogeneous piece of degree n. Given a partition λ, we denote the corresponding Schur function by s_λ and the corresponding complete homogeneous symmetric function by h_λ.

The *Frobenius character* is the graded $\mathbb{C}$-algebra isomorphism $\mathrm{Frob} : R \to \Lambda$ induced by setting $\mathrm{Frob}(S^\lambda) = s_\lambda$. It is well known that we have $\mathrm{Frob}(M^\lambda) = h_\lambda$. Generalizing slightly, if $V = \bigoplus_{k \geq 0} V(k)$ is a graded S_n-module, define $\mathrm{grFrob}(V; q) \in \Lambda \otimes_{\mathbb{C}} \mathbb{C}[[q]]$ to be the formal power series in q with coefficients in Λ given by $\mathrm{grFrob}(V; q) := \sum_{k \geq 0} \mathrm{Frob}(V(k)) q^k$.

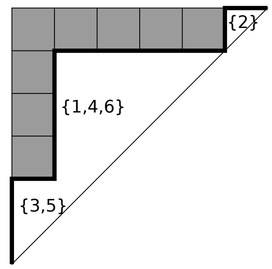

Fig. 1: A Dyck path of size 6.

A *Dyck path of size* n is a lattice path D in $\mathbb{Z}^2$ consisting of vertical steps $(0,1)$ and horizontal steps $(1,0)$ which starts at $(0,0)$, ends at (n,n), and stays weakly above the line $y=x$. A maximal continguous sequence of vertical steps in D is called a *vertical run* of D.

We will associate two partitions to a Dyck path D of size n. The *vertical run partition* $\lambda(D) \vdash n$ is obtained by listing the (positive) lengths of the vertical runs of D in weakly decreasing order. For example, if D is the Dyck path in Figure 1, then $\lambda(D) = (3,2,1)$. The *area partition* $\mu(D)$ is the partition of length n whose Ferrers diagram is the set of boxes to the upper left of D in the $n \times n$ square with lower left coordinate at the origin. For example, if D is the Dyck path of size 6 in Figure 1, then $\mu(D) = (5,1,1,1,0,0)$. The boxes in the Ferrers diagram of $\mu(D)$ are shaded. We define the *area* statistic[(i)] on Dyck paths by $\mathrm{area}(D) = |\mu(D)|$. For the Dyck path in our running exampe, $\mathrm{area}(D) = 8$. By construction, we have that $\mathrm{mult}(\mu(D)) = \lambda(D)$ for any Dyck path D of size n.

Dyck paths of size n can be used to obtain a decomposition of Park_n as a direct sum of coset modules M^λ. In particular, let D be a Dyck path of size n. A *labeling* of D assigns each vertical run of D to a subset of $[n] := \{1,2,\dots,n\}$ of size equal to the length of that vertical run such that every letter in $[n]$ appears exactly once as a label of a vertical run. Figure 1 shows an example of a labeled Dyck path of size 6, where the subsets labeling the vertical runs are placed just to the right of the runs.

The set of labeled Dyck paths of size n carries an action of S_n given by label permutation. There is an S_n-equivariant bijection from the set of labeled Dyck paths D of size n to parking functions $(a_1,\dots,a_n)$ of size n given by letting a_i be one greater than the x-coordinate of the vertical run of D labeled by i. For example, the labeled Dyck path in Figure 1 corresponds to the parking function $(2,6,1,2,1,2) \in \mathrm{Park}_6$. Since any fixed labeled Dyck path of size D generates a cyclic S_n-module isomorphic to $M^{\lambda(D)}$, it is immediate that the parking space Park_n decomposes into coset representations as

$$\mathrm{Park}_n \cong_{S_n} \bigoplus_D M^{\lambda(D)}, \tag{2}$$

where the direct sum is over all Dyck paths D of size n. Equivalently, we have that the Frobenius character of Park_n is given by $\mathrm{Frob}(\mathrm{Park}_n) = \sum_D h_{\lambda(D)}$. For example, the 5 Dyck paths of size 3 shown in Figure 3 lead to the Frobenius character

$$\mathrm{Frob}(\mathrm{Park}_3) = h_{(3)} + 3h_{(2,1)} + h_{(1,1,1)}. \tag{3}$$

(i) Many authors instead define the area of a Dyck path D to be the number of complete lattice squares between D and the line $y=x$, so that our statistic would be the ‘coarea’.

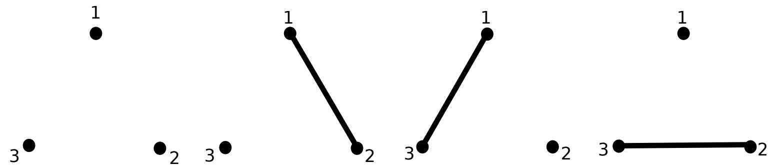

Fig. 2: The four slim subgraphs of K_3.

The vector space underlying the S_{n+1}-module which will extend Park_n is a subspace of the polynomial ring $\mathbb{C}[x_1, \dots, x_{n+1}]$ in $n+1$ variables and first studied in the work of Postnikov and Shapiro [PS04]. Let K_{n+1} denote the complete graph on the vertex set $[n+1]$. Given an edge $e = (i < j)$ in K_{n+1}, we associate the polynomial weight $p(e) := x_i - x_j \in \mathbb{C}[x_1, \dots, x_{n+1}]$. A subgraph $G \subseteq K_{n+1}$ (identified with its edge set) gives rise to the polynomial weight $p(G) := \prod_{e \in G} p(e)$. Following Postnikov and Shapiro, we call a subgraph $G \subseteq K_{n+1}$ *slim* if the complement edge set $K_{n+1} - G$ is a connected graph on the vertex set $[n+1]$.

Definition 1 *Denote by V_n the $\mathbb{C}$-linear subspace of $\mathbb{C}[x_1, \dots, x_{n+1}]$ given by*

$$V_n := \mathrm{span}\{p(G) \,:\, G \textit{ is a slim subgraph of } K_{n+1}\}. \tag{4}$$

Let $V_n(k)$ denote the homogeneous piece of V_n of polynomial degree k; the space $V_n(k)$ is spanned by those polynomials $p(G)$ corresponding to slim subgraphs G of K_{n+1} with k edges.

In the case $n = 2$, Figure 2 shows that four slim subgraphs of the complete graph K_3. From left to right, the corresponding polynomials are $1, x_1 - x_2, x_1 - x_3$, and $x_2 - x_3$. It follows that $V_2(0) = \mathrm{span}\{1\}$ and $V_2(1) = \mathrm{span}\{x_1 - x_2, x_1 - x_3, x_2 - x_3\}$. Observe that the graded Frobenius character of V_2 is $\mathrm{grFrob}(V_2; q) = s_{(3)} q^0 + s_{(2,1)} q^1$. By the branching rule for symmetric groups (see [Sag01]), we have that $\mathrm{grFrob}(\mathrm{Res}^{S_3}_{S_2}(V_2); q) = s_{(2)} q^0 + (s_{(2)} + s_{(1,1)}) q^1$. Setting $q = 1$ yields $\mathrm{Frob}(\mathrm{Res}^{S_3}_{S_2}(V_2)) = 2s_{(2)} + s_{(1,1)}$, which agrees with the Frobenius character of Park_2.

While the set of polynomials $\{p(G) \,:\, G \text{ is a slim subgraph of } K_{n+1}\}$ is linearly dependent in general, a basis for V_n can be constructed using standard matroid theoretic results. Fix a total order on the edge set of K_{n+1}. Given a spanning tree T of K_{n+1}, the *external activity* $\mathrm{ex}(T)$ of T is the set of edges $e \in K_{n+1}$ such that e is the minimal edge of the unique cycle in $T \cup \{e\}$. A basis of V_n is given by $\{p(K_{n+1} - (T \cup \mathrm{ex}(T))) \,:\, T \text{ is a spanning tree of } K_{n+1}\}$. It follows immediately from Cayley's theorem that $\dim V_n = (n+1)^{n-1}$.

Since the slimness of a subgraph is preserved under the action of S_{n+1} on the vertex set $[n+1]$ and $p(G)$ is homogeneous of degree equal to the number of edges in G, it follows that $V_n = \bigoplus_{k \geq 0} V_n(k)$ is a graded S_{n+1}-submodule of the polynomial ring $\mathbb{C}[x_1, \dots, x_{n+1}]$. In fact, the space V_n sits inside the copy of the coordinate ring of the reflection representation of type A_n sitting inside $\mathbb{C}[x_1, \dots, x_{n+1}]$ generated by $x_i - x_{i+1}$ for $1 \leq i \leq n$.

The following result was conjectured by the first author. We postpone its proof, along with the proofs of the other results in this section, to Section 3.

Theorem 2 *Embed S_n into S_{n+1} by letting S_n act on the first n letters. We have that*

$$\mathrm{Res}^{S_{n+1}}_{S_n}(V_n(k)) \cong_{S_n} \bigoplus_D M^{\lambda(D)}, \tag{5}$$

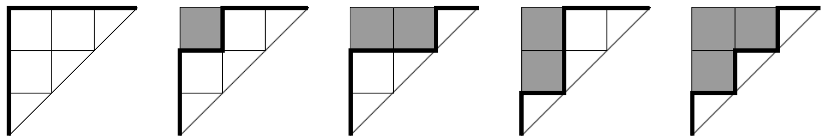

Fig. 3: The 5 Dyck paths of size 3. From left to right, their contributions to the graded Frobenius character $\mathrm{grFrob}(\mathrm{Res}^{S_4}_{S_3}(V_3); q)$ are $h_{(3)}q^0$, $h_{(2,1)}q^1$, $h_{(2,1)}q^2$, $h_{(2,1)}q^2$, and $h_{(1,1,1)}q^3$.

where the direct sum is over all Dyck paths of size n and area k. In particular, by Equation 2 we have that

$$\mathrm{Res}^{S_{n+1}}_{S_n}(V_n) \cong_{S_n} \mathrm{Park}_n. \tag{6}$$

Equivalently, we have that $\mathrm{grFrob}(\mathrm{Res}^{S_{n+1}}_{S_n}(V_n); q) = \sum_D q^{\mathrm{area}(D)} h_{\lambda(D)}$, where the sum is over all Dyck paths D of size n. For example, computing the area and run partitions of the 5 Dyck paths of size 3 shown in Figure 3 shows that

$$\mathrm{grFrob}(\mathrm{Res}^{S_4}_{S_3}(V_3); q) = h_{(3)}q^0 + h_{(2,1)}q^1 + 2h_{(2,1)}q^2 + h_{(1,1,1)}q^3. \tag{7}$$

Postnikov and Shapiro showed that the dimension of the vector space V_n is equal to $(n+1)^{n-1}$, however the S_n-module structure of V_n has remained unstudied. Indeed, Theorem 2 is the first description of the S_n-module structure of V_n.

It is natural to ask for an explicit description of the S_{n+1}-structure of V_n or of its graded pieces $V_n(k)$. This problem is open in general, but we can describe the extended structure of $V_n(k)$ in the extreme degrees $k = 0, 1, \ldots, n$ as well as $k = \binom{n}{2}$. Let C_{n+1} be the cyclic subgroup of S_{n+1} generated by the long cycle $c := (1, 2, \ldots, n+1)$ and let ζ be the linear representation of C_{n+1} which sends c to $e^{\frac{2\pi i}{n+1}}$. Mackey's Theorem can be used to prove that the *Lie representation* $\mathrm{Lie}_n := \mathrm{Ind}^{S_{n+1}}_{C_{n+1}}(\zeta)$ of S_{n+1} satisfies $\mathrm{Res}^{S_{n+1}}_{S_n}(\mathrm{Lie}_n) \cong_{S_n} \mathbb{C}[S_n]$. Stanley proved that the Lie representation arises as the action of S_{n+1} on the top poset cohomology of the lattice of set partitions of $[n+1]$, tensored with the sign representation [Sta82].

Proposition 3 *The module $V_n(0)$ carries the trivial representation of S_{n+1}, the module $V_n(1)$ carries the reflection representation of S_{n+1}, and in general $V_n(k) = \mathrm{Sym}^k(V_n(1))$ for $k < n$. The module $V_n(\binom{n}{2}) = V_n(\mathrm{top})$ carries the Lie representation of S_{n+1} tensor the sign representation.*

The first part of this result is optimal in the sense that if $k \geq n$ then $V_n(k)$ is a proper subspace of $\mathrm{Sym}^k(V_n(1))$.

We will prove a bivariate generalization of Theorem 2 which includes a 'Fuss generalization' as a special case. Given $\ell, m, n > 0$, define a *(ℓ, m)-Dyck path of size n* to be a lattice path D in $\mathbb{Z}^2$ consisting of vertical steps $(0, 1)$ and horizontal steps $(1, 0)$ which starts at $(-\ell + 1, 0)$, ends at (mn, n), and stays weakly above the line $y = \frac{x}{m}$. Taking $\ell = m = 1$, we recover the classical notion of a Dyck path of size n. Taking $\ell = 1$ and m general, the $(1, m)$-Dyck paths are the natural Fuss extension of Dyck paths. As before, we define the *vertical run partition* $\lambda(D) \vdash n$ of an (ℓ, m)-Dyck path D of size n to be the partition obtained by listing the lengths of the vertical runs of D in weakly decreasing order. We also define the *area partition* $\mu(D)$ to be the length n partition whose Ferrers diagram fits between D and a

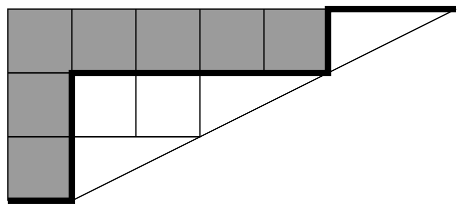

Fig. 4: A $(2,2)$-Dyck path of size 3.

$(\ell - 1 + mn) \times n)$ rectangle with lower left hand coordinate $(-\ell + 1, 0)$. The *area* of D is defined by $\mathrm{area}(D) := |\mu(D)|$. We have that $\mathrm{mult}(\mu(D)) = \lambda(D)$.

Figure 4 shows an example of a $(2,2)$-Dyck path of size 3. The path D starts at $(-1,0)$, ends at $(6,3)$, and stays above the line $y = \frac{x}{2}$. We have that $\lambda(D) = (2,1) \vdash 3$, $\mu(D) = (5,1,1)$, and $\mathrm{area}(D) = 7$.

Let $K_{n+1}^{(\ell,m)}$ be the multigraph on the vertex set $[n+1]$ with m edges between i and j for all $1 \leq i < j \leq n$ and ℓ edges between i and $n+1$ for all $1 \leq i \leq n$. We call a sub-multigraph G of $K_{n+1}^{(\ell,m)}$ *slim* if the multi-edge set difference $K_{n+1}^{(\ell,m)} - G$ is a connected multigraph on $[n+1]$. We extend the polynomial weight $p(G) \in \mathbb{C}[x_1, \ldots, x_{n+1}]$ to multigraphs G in the obvious way.

Definition 4 *Let $V_n^{(\ell,m)}$ be the $\mathbb{C}$-linear subspace of $\mathbb{C}[x_1, \ldots, x_{n+1}]$ given by the span*

$$V_n^{(\ell,m)} := \mathrm{span}\{p(G) \,:\, G \text{ is a slim sub-multigraph of } K_{n+1}^{(\ell,m)}\}. \tag{8}$$

As in the case $m = \ell = 1$, the space $V_n^{(\ell,m)}$ is stable under the action of S_n and is a graded S_n-representation with respect to the standard polynomial degree. When $\ell = m$, $V_n^{(\ell,m)}$ also caries an action of S_{n+1}. Postnikov and Shapiro showed that the dimension of $V_n^{(\ell,m)}$ is $(mn + \ell)^{n-1}$ [PS04]. Let $V_n^{(\ell,m)}(k)$ be the degree k piece of $V_n^{(\ell,m)}$.

Theorem 5 *We have that*

$$(V_n^{(\ell,m)}(k)) \cong_{S_n} \bigoplus_D M^{\lambda(D)}, \tag{9}$$

where the direct sum is over all (ℓ,m)-Dyck paths of size n and area k. For $k < n$, $V_n^{(\ell,m)}(k) = \mathrm{Sym}^k(V_n^{(\ell,m)}(1))$.

When $\ell = m$, the top degree piece of $V_n^{(\ell,m)}$ is isomorphic, as an S_{n+1}-module, to $\mathrm{Lie}_n \otimes \mathrm{sign}^{\otimes \ell}$.

While the degree 0 and 1 pieces of $V_n^{(\ell,m)}$ have S_{n+1}-structure given by the trivial representation and the reflection representation, respectively, the authors do not know of a nice expression for the extended Frobenius character in other degrees.

3 Proofs

While Theorem 5 implies Theorem 2, the proof of Theorem 5 is a straightforward extension of the proof of Theorem 2 and it will be instructive to prove Theorem 2 first.

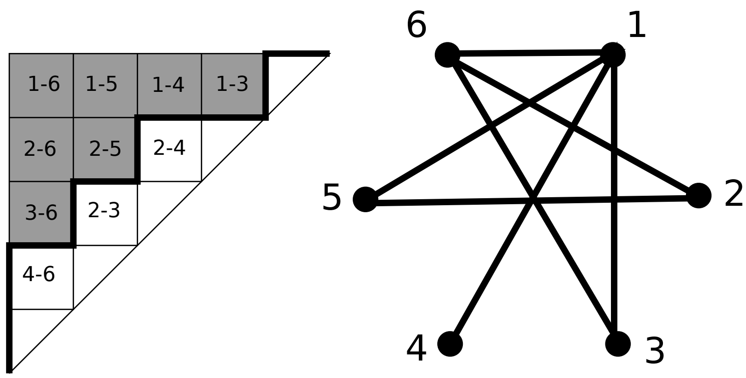

Fig. 5: A Dyck path D of size 5 and the associated subgraph $G(D)$ of K_6.

The first step in the proof of Theorem 2 is to relate the modules on both sides of the claimed isomorphism by associating a subgraph $G(D)$ of K_{n+1} and a polynomial $p(D) \in \mathbb{C}[x_1, \ldots, x_{n+1}]$ to any Dyck path D of size n. We start by labeling the 1×1 box b which is completely above the line $y = x$ with the edge $e(b) = (n - j, n - i)$ in K_{n+1}, where (i, j) is the upper left coordinate of b. See Figure 5 for an example of this labeling in the case $n = 5$. We let $G(D)$ be the subgraph of K_{n+1} consisting of those edges $e(b)$ for which the box b is to the upper left of the path D. In Figure 5, the shaded boxes above the path D each contribute an edge to the subgraph $G(D)$ and we have that $G(D) = \{1-6, 1-5, 1-4, 1-3, 2-6, 2-5, 3-6\}$.

Lemma 6 *The subgraph $G(D)$ is slim for any Dyck path D.*

Proof: The subgraph $G(D)$ contains none of the edges in the path $1 - 2 - \cdots - (n+1)$. □

By Lemma 6, the polynomial $p(D) := p(G(D))$ is contained in V_n. For example, if $n = 5$ and D is the Dyck path shown in Figure 5, we have that

$$p(D) = (x_1 - x_6)(x_1 - x_5)(x_1 - x_4)(x_1 - x_3)(x_2 - x_6)(x_2 - x_5)(x_3 - x_6) \in V_5. \tag{10}$$

By construction, for any Dyck path D the polynomial $p(D)$ is homogeneous with degree equal to $\mathrm{area}(D)$.

In order to prove the direct sum decomposition in Theorem 2, we will show that the polynomials $p(D)$ project nicely onto a certain subspace of $\mathbb{C}[x_1, \ldots, x_{n+1}]$. Since Theorem 2 only concerns the restriction of V_n to S_n, it is natural to consider a subspace of $\mathbb{C}[x_1, \ldots, x_{n+1}]$ which is closed under the action of S_n but not of S_{n+1}.

Let $st_n := (n-1, n-2, \ldots, 1)$ be the *staircase partition* of length $n - 1$. We call a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ *sub-staircase* if $\lambda \subseteq st_n$ (observe that this definition has tacit dependence on n). For any Dyck path D of size n, the partition $\mu(D)$ is sub-staircase.

For a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, we use the shorthand $x^\lambda := x_1^{\lambda_1} \cdots x_n^{\lambda_n} \in \mathbb{C}[x_1, \ldots, x_n]$. We call a monomial $x_1^{d_1} \cdots x_{n+1}^{d_{n+1}}$ in the variables $x_1, \ldots, x_{n+1}$ *sub-staircase* if there exists a permutation $w \in S_n$ and a sub-staircase partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n) \vdash n$

$$x_1^{d_1} \cdots x_{n+1}^{d_{n+1}} = w.x^\lambda. \tag{11}$$

In particular, the variable x_{n+1} does not appear in any sub-staircase monomial. If the monomial $x_1^{d_1} \cdots x_{n+1}^{d_{n+1}}$ is sub-staircase, the partition λ is uniquely determined from the monomial; call this the *exponent partition* of the monomial. Let $W_n \subset \mathbb{C}[x_1, \ldots, x_{n+1}]$ be the $\mathbb{C}$-linear span of all sub-staircase monomials. The subspace W_n is closed under the action of S_n, but not under the action of S_{n+1}.

In the case $n = 3$, the S_3-orbits of the 16 staircase monomials in $\mathbb{C}[x_1, \ldots, x_4]$ are shown in the following table, where the left column shows a representative from each orbit.

1	
x_1	x_2, x_3
x_1^2	x_2^2, x_3^2
x_1x_2	x_1x_3, x_2x_3
$x_1^2x_2$	$x_1^2x_3, x_2^2x_1, x_2^2x_3, x_3^2x_1, x_3^2x_2$

The S_3-orbits are parametrized by sub-staircase partitions $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ and each orbit contains a unique representative of the form x^λ. The staircase monomials form a linear basis of W_3 and the cyclic S_3-submodule of W_3 generated by x^λ is isomorphic to $M^{\mathrm{mult}(\lambda)}$. The natural bijection between exponent vectors and parking functions affords an isomorphism $W_3 \cong_{S_3} \mathrm{Park}_3$. These observations generalize in a straightforward way to the following lemma, whose proof is left to the reader.

Lemma 7 *The set of sub-staircase monomials forms a linear basis for W_n and is closed under the action of S_n. The S_n-orbits are parametrized by sub-staircase partitions λ, and the orbit labeled by λ has a unique monomial of the form x^λ. The cyclic S_n-submodule of W_n generated by x^λ is isomorphic to $M^{\mathrm{mult}(\lambda)}$ and we have that $W_n \cong_{S_n} \mathrm{Park}_n$.*

With Lemma 7 in mind, we will construct a graded S_n-module isomorphism $V_n \xrightarrow{\sim} W_n$. We define a graded S_n-module homomorphism $\phi : V_n \to W_n$ by the following composition:

$$\phi : V_n \hookrightarrow \mathbb{C}[x_1, \ldots, x_{n+1}] \twoheadrightarrow \mathbb{C}[x_1, \ldots, x_n] \twoheadrightarrow W_n, \tag{12}$$

where the first map is inclusion, the second is the specialization $x_{n+1} = 0$, and the third linear map fixes the space W_n pointwise and sends monomials which are not sub-staircase to zero.

We want to show that ϕ is an isomorphism. Postnikov and Shapiro showed that $\dim(W_n) = \dim(V_n) = (n+1)^{n-1}$ [PS04], so it is enough to show that ϕ is surjective. We will do this by analyzing the polynomials $\phi(p(D))$, where D is a Dyck path of size n.

The set of sub-staircase partitions forms an order ideal in dominance order. The next lemma states that the transition matrix between the set $\{\phi(p(D)) : D \text{ a Dyck path of size } n\}$ expands in the monomial basis of W_n given by $\{x^\lambda : \lambda \text{ sub-staircase}\}$ in a unitriangular way with respect to any linear extension of dominance order (where we associate $\phi(p(D))$ with the partition $\mu(D)$).

Lemma 8 *Let D be a Dyck path of size n. There exist integers $c_{\lambda,w} \in \mathbb{Z}$ such that*

$$\phi(p(D)) = x^{\mu(D)} + \sum_{\substack{\lambda \prec \mu(D) \\ |\lambda| = |\mu(D)| \\ w \in S_n}} c_{\lambda,w} w.x^\lambda. \tag{13}$$

Proof: By definition, we have that

$$p(D) = \prod_{e=(i<j) \in G(D)} (x_i - x_j), \tag{14}$$

so (up to sign) a typical monomial in the expansion of $p(D)$ is obtained by choosing an endpoint of every edge in $G(D)$ and multiplying the corresponding variables. The map ϕ kills any monomial which contains the variable x_{n+1}, so up to sign a typical monomial in $\phi(p(D))$ is obtained by choosing an endpoint of each edge in $G(D)$ and multiplying the corresponding variables such that whenever $G(D)$ has an edge of the form $(i < n+1)$, we choose the smaller endpoint i. The result follows from the construction of $G(D)$ and the definition of dominance order. $\square$

As an example of Lemma 8, consider the case $n = 5$ and let the Dyck path D be shown in Figure 5 with $\mu(D) = (4, 2, 1)$. To calculate $\phi(p(D))$, we set $x_6 = 0$ in the product formula for $p(D)$ given in Equation 10 and expand. The resulting polynomial is

$$\phi(p(D)) = x_1(x_1 - x_5)(x_1 - x_4)(x_1 - x_3)x_2(x_2 - x_5)x_3 \tag{15}$$

$$= x_1^4 x_2^2 x_3 + \text{terms involving sub-staircase monomials with exponent partition} \prec (4, 2, 1). \tag{16}$$

We are ready to complete the proof of Theorem 2.

Proof of Theorem 2: By Lemma 7, the set of sub-staircase monomials forms a linear basis of W_n, so Lemma 8 implies that the S_n-module homomorphism $\phi : V_n \to W_n$ is surjective. Since $\dim(V_n) = \dim(W_n)$, this implies that ϕ is also injective and gives an isomorphism $\mathrm{Res}^{S_{n+1}}_{S_n}(V_n) \cong_{S_n} \mathrm{Park}_n$. To prove the graded isomorphism in Theorem 2, it is enough to observe that $\mathrm{mult}(\mu(D)) = \lambda(D)$ for any Dyck path D and apply Lemmas 7 and 8 together with the fact that ϕ is graded. $\square$

It may be tempting to guess that $p(D)$ generates a cyclic S_n-submodule of V_n isomorphic to $M^{\lambda(D)}$, but this is false in general. The reason for this is that while the 'leading term' in the expansion of $\phi(p(D))$ in Lemma 8 generates the submodule $M^{\lambda(D)}$ under the action of S_n, the other terms in this expansion can cause $\phi(p(D))$ to generate a different cyclic submodule.

We are ready to prove the claimed S_{n+1}-structure of the extreme degrees of the graded module $V_n(k)$.

Proof of Proposition 3: It is clear from the definitions that $V_n(0)$ carries the trivial representation of S_{n+1}. The space $V_n(1)$ has basis given by the polynomials $x_1 - x_2, x_2 - x_3, \ldots, x_n - x_{n+1}$ and hence carries the reflection representation of S_{n+1} (i.e., the irreducible S_{n+1}-module corresponding to the partition $(n, 1)$). Since $V_n \subseteq \mathrm{Sym}(V_n(1))$ we are claiming that in degree $k < n$ this is an equality. The Hilbert series of V_n is the Tutte polynomial evaluation $q^{\binom{n+1}{2} - n} T_{K_{n+1}}(1, 1/q)$ and so we must prove that the first $n - 1$ terms of this sum are the binomial coefficents $\binom{n+k-1}{k}$. There is nothing special about K_{n+1} in this claim and we will prove a more general statement in Lemma 9.

To prove that $V_n(\mathrm{top})$ is isomorphic to $\mathrm{Lie}_{n+1} \otimes \mathrm{sign}$ we reason as follows. The space $V_n(\mathrm{top})$ is spanned by those $p(G)$ where the complementary subgraph $K_{n+1} \setminus G$ is connected and has n edges.

Let $\mathcal{A}_n$ denote the braid arrangementin $\mathbb{C}^{n+1}$, which is the union of those hyperplanes with at least two coordinates equal. Let $H^*(\mathbb{C}^{n+1} \setminus \mathcal{A}_n; \mathbb{C})$ denote the (complexified) de Rham cohomology ring of its complement. Consider, now, the linear map $c : V_n(\text{top}) \to H^n(\mathbb{C}^{n+1} \setminus \mathcal{A}_n)$ that sends

$$p(G) \mapsto p(G) \cdot d(x_1 - x_2) \wedge d(x_2 - x_3) \wedge \cdots \wedge d(x_n - x_{n+1}) / \prod_{1 \le i < j \le n} (x_i - x_j).$$

This is an isomorphism of vector spaces, since it is division by the Vandermond product, followed by multiplication by the n-form. To see that c is equivariant notice that $\bigwedge^n V_n(1)$ carries the sign representation of S_{n+1}, because it is 1 dimensional and non-trivial. Likewise does the one dimensional representation spanned by the Vadnermond product. It follows that the signs introduced by multiplication by the n-form and division by the Vandermond cancel, and c is equivariant.

Finally, the top degree cohomology of the complement $\mathbb{C}^{n+1} \setminus \mathcal{A}_n$ is known to be S_{n+1}-isomorphic to the top degreee *Whitney homology* of its lattice of flats. The lattice of flats of $\mathcal{A}_n$ is the partition lattice Π_{n+1} and by a result of Stanley [Sta82] (beautifuly recounted by Wachs in [Wac07]), the top degree Whitney homology of the partition lattice Π_{n+1} is $\mathrm{Lie}_{n+1} \otimes \mathrm{sign}$. □

Lemma 9 *Let G be a connected graph on v vertices with e edges. Denote the Tutte polynomial of G by $T_G(x, y)$. Then, the polynomial $q^{e-v+1} T_G(1, 1/q)$ takes the form,*

$$1 + (v-1)q + \binom{v}{2} q^2 + \binom{v+1}{3} q^3 + \cdots + \binom{(v-1) + (v-2) - 1}{v-2} q^{v-2} + O(q^{v-1}).$$

Proof: We write $T_G(x, y)$ in terms of the two variable *coboundary polynomial*, $\overline{\chi}_G(\lambda, \nu)$. This is the sum

$$\overline{\chi}_G(\lambda, \nu) = \frac{1}{\lambda} \sum_{i=0}^{e} c_i(G; \lambda) \nu^i$$

where $c_i(G; \lambda)$ is the number of ways to color the vertices of G with λ colors and exactly i monochromatic edges. It is a fact that this is a polynomial in λ and ν. Now by [Whi92, Theorem 6.3.26],

$$q^{e-v+1} T_G(1, 1/q) = \frac{q^e}{(1-q)^{v-1}} \overline{\chi}_G(0, 1/q).$$

Thus, to prove the first part of the lemma we will show that $c_i(G; \lambda) = 0$ for $e - v + 1 < i < e$, and that $c_e(G; \lambda) = \lambda$. Suppose that we have colored the vertices of G and we have more than $e - v + 1$ monochromatic edges. Then the collection of monochromatic edges forms a connected subgraph of G. It follows that all vertices of G are colored the same and hence all edges of G are monochromatic. This means that $c_i(G; \lambda) = 0$ unless $i = e$. That $c_e(G; \lambda) = \lambda$ is clear. □

The proof of Theorem 5 is a straight-forward extension of the proof of Theorem 2 and is only sketched.

Proof of Theorem 5, sketch: Given any (ℓ, m)-Dyck path D of size n we associate a sub-multigraph $G(D)$ of $K_{n+1}^{(\ell,m)}$ by letting every box which contributes to $\mathrm{area}(D)$ correspond to a single edge in the multigraph $G(D)$; the labeling which accomplishes this is shown in Figure 6 in the case $(\ell, m) = (3, 2)$ and $n = 4$. For general ℓ, m, and n, we label the boxes in the i^{th} row from the top from left to right with $(\ell + m - 2)$ copies of the edge $i - (n+1)$, m copies of the edge $i - n$, m copies of the edge $i - (n-1), \ldots,$ m copies of the edge $i - (i+2)$, and $(m-1)$ copies of the edge $i - (i+1)$.

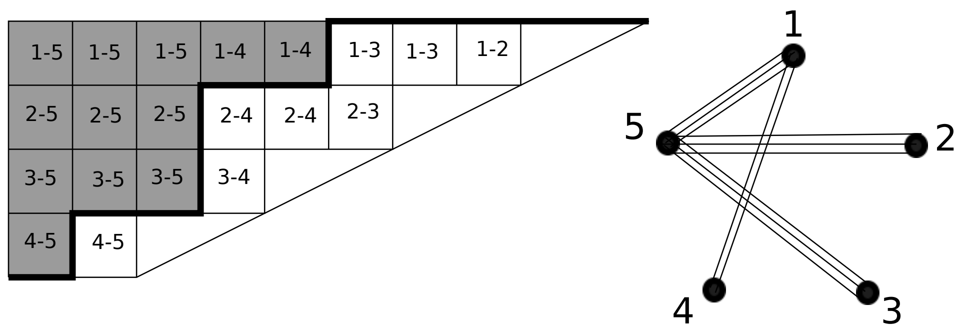

Fig. 6: A $(3,2)$-Dyck path D of size 4 and the associated sub-multigraph $G(D)$ of $K_4^{(3,2)}$.

For any (ℓ,m)-Dyck path D of size n, the multigraph complement of $G(D)$ within $K_n^{(\ell,m)}$ contains each of the edges in the path $1-2-\cdots-n-(n+1)$ with multiplicity at least one. Therefore, the sub-multigraph $G(D)$ is slim and the polynomial $p(D) := p(G(D))$ is contained in $V_n^{(\ell,m)}$.

We say that a partition λ with n parts is *sub-(ℓ,m)-staircase* if in Young's lattice we have the relation $\lambda \subseteq (\ell-1+m(n-1), \ell-1+m(n-2), \ldots, \ell-1)$. A monomial $x_1^{d_1}\cdots x_{n+1}^{d_{n+1}}$ is *sub-(ℓ,m)-staircase* if there exists $w \in S_n$ and a sub-(ℓ,m)-staircase partition λ such that

$$x_1^{d_1}\cdots x_{n+1}^{d_{n+1}} = x_{w(1)}^{\lambda_1}\cdots x_{w(n)}^{\lambda_n}. \tag{17}$$

Let $W_n^{(\ell,m)}$ be the subspace of $\mathbb{C}[x_1,\ldots,x_{n+1}]$ spanned by the set of all sub-(ℓ,m)-staircase monomials. We have that $W_n^{(\ell,m)}$ is closed under the action of S_n and the degree k homogeneous piece of $W_n^{(\ell,m)}$ is isomorphic as an S_n-module to the direct sum on the right hand side of the isomorphism asserted in Theorem 5.

The isomorphism in Theorem 5 is proven by showing that the graded S_n-module homomorphism $\phi^{(\ell,m)} : V_n^{(\ell,m)} \to W_n^{(\ell,m)}$ given by the composite

$$\phi^{(\ell,m)} : V_n^{(\ell,m)} \hookrightarrow \mathbb{C}[x_1,\ldots,x_{n+1}] \twoheadrightarrow \mathbb{C}[x_1,\ldots,x_n] \twoheadrightarrow W_n^{(\ell,m)} \tag{18}$$

is an isomorphism, where the first map is inclusion, the second is the evaluation $x_{n+1}=0$, and the third fixes $W_n^{(\ell,m)}$ pointwise and sends every monomial which is not sub-(ℓ,m)-staircase to zero.

Postnikov and Shapiro proved that the vector space $V_n^{(\ell,m)}$ has dimension $(\ell+mn)^{n-1}$ [PS04]. A standard counting argument shows that there are $(\ell+mn)^{n-1}$ sub-(ℓ,m)-staircase monomials, so we have that $\dim(V_n^{(\ell,m)}) = \dim(W_n^{(\ell,m)})$. Therefore, to show that $\phi^{(\ell,m)}$ is a graded isomorphism of S_n-modules, it is enough to show that $\phi^{(\ell,m)}$ is surjective.

To show that $\phi^{(\ell,m)}$ is surjective, we prove a generalization of Lemma 8 which states that for any (ℓ,m)-Dyck path D of size n, the monomial expansion of $\phi^{(\ell,m)}(p(D))$ has the form

$$\phi^{(\ell,m)}(p(D)) = x^{\mu(D)} + \text{terms involving monomials whose exponent partitions are} \prec \mu(D), \tag{19}$$

where we extend the definition of $\mu(D)$ to (ℓ,m)-Dyck paths of size n in the obvious way. This triangularity result implies that $\phi^{(\ell,m)}$ is surjective, and dimension counting implies that $\phi^{(\ell,m)}$ is a graded S_n-module isomorphism. Theorem 5 follows.

The claim about $V_n^{(\ell,m)}$ in low degree follows since $V_n \subset V_n^{(\ell,m)}$. The claim about $V_n^{(\ell,\ell)}(\text{top})$ follows since this space is isomorphic to $V_n(\text{top})$, the isomorphism being division by $\prod_{i<j}(x_i - x_j)^{\ell-1}$. $\square$

4 Concluding Remarks

In this paper we constructed a graded S_{n+1}-module V_n which satisfies $\mathrm{Res}^{S_{n+1}}_{S_n}(V_n) \cong_{S_n} \mathrm{Park}_n$. As we mentioned in Section 1, there does not exist an S_6-module M such that $\mathrm{Res}^{S_6}_{S_4}(M) \cong_{S_4} \mathrm{Park}_4$, so we cannot hope for an extension of Park_n to a symmetric group of higher rank than $n+1$ in general.

On the other hand, we identified the top degree $V_n(\text{top})$ of V_n with the Lie representation Lie_n of S_{n+1} tensor the sign representation. Whitehouse [Whi97] proved that the representation Lie_n extends to S_{n+2}. This suggests the following problem.

Problem 10 *For which values of n and k does $V_n(k)$ extend to a representation of S_{n+2}?*

By Whitehouse's result, for any $n > 0$, the k-value $k = \binom{n-1}{2}$ leads to an extension as in Problem 10. Also, since $V_n(0)$ is the trivial representation of S_{n+1}, one can take $k = 0$ and n arbitrary. On the other hand, if $k = 1$ we have that $V_n(1)$ is the reflection representation of S_{n+1}. For $n > 3$, this representation is not the restriction of any S_{n+2}-module.

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Kazhdan–Lusztig polynomials of boolean elements

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Abstract. We give closed combinatorial product formulas for Kazhdan–Lusztig poynomials and their parabolic analogue of type q in the case of boolean elements, introduced in [M. Marietti, Boolean elements in Kazhdan–Lusztig theory, J. Algebra 295 (2006)], in Coxeter groups whose Coxeter graph is a tree. Such formulas involve Catalan numbers and use a combinatorial interpretation of the Coxeter graph of the group. In the case of classical Weyl groups, this combinatorial interpretation can be restated in terms of statistics of (signed) permutations. As an application of the formulas, we compute the intersection homology Poincaré polynomials of the Schubert varieties of boolean elements.

Résumé. Nous donnons des formules combinatories pour les polynômes de Kazhdan-Lusztig et leurs analogues parabolique de type q pour les éléments booléens, introduite dans [M. Marietti, Boolean elements in Kazhdan–Lusztig theory, J. Algebra 295 (2006)], dans les groupes de Coxeter dont le graphe de Coxeter est un arbre. Ces formules utilisent les nombres de Catalan et une interprétation combinatoire des graphes du groupe de Coxeter. Dans le cas des groupes de Weyl classiques, cette interprétation combinatoire peut être reformulée en termes de statistiques de permutations avec signe. Avec ces formules, on peut calculer le polynôme de l'intersection homologie de Poincaré pour la variété de Schubert de booléen éléments.

Keywords: Coxeter groups, Kazhdan–Lusztig polynomials, Boolean elements, Poincaré polynomials

1 Introduction

In their fundamental paper Kazhdan and Lusztig (1979) defined, for every Coxeter group W, a family of polynomials, indexed by pairs of elements of W, which have become known as the Kazhdan–Lusztig polynomials of W (see, e. g., (Humphreys, 1990, Chapter 7) or (Björner and Brenti, 2005, Chapter 5)). These polynomials play an important role in several areas of mathematics, including the algebraic geometry and topology of Schubert varieties and representation theory (see, e. g., (Björner and Brenti, 2005, Chapter 5), and the references cited there). In particular, their coefficients gives the dimensions of the intersection cohomology modules for Schubert varieties (see, e. g., Kazhdan and Lusztig (1980)).

In order to find a method for the computation of the dimensions of the intersection cohomology modules corresponding to Schubert varieties in G/P, where P is a parabolic subgroup of the Kac-Moody group G,

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(Deodhar (1987)) introduced two parabolic analogues of these polynomials which correspond to the roots $x = q$ and $x = -1$ of the equation $x^2 = q + (q-1)x$. These parabolic Kazhdan–Lusztig polynomials reduce to the ordinary ones for the trivial parabolic subgroup and are also related to them in other ways (see, e. g., Proposition 2.2 below). Besides these connections the parabolic polynomials also play a direct role in several areas including the theories of generalized Verma modules (Casian and Collingwood (1987)), tilting modules (Soergel (1997a), Soergel (1997b)) and Macdonald polynomials(Haglund et al. (2005a), Haglund et al. (2005b)).

The purpose of this work is to give explicit combinatorial product formulas for all (parabolic and ordinary) Kazhdan-Lusztig polynomials indexed by pairs of boolean elements (see Section 2 for the definition) in all Coxeter groups whose Coxeter graph is a tree. Our results show that all such polynomials have nonnegative coefficients, conjectured by Kazhdan and Lusztig (1979), and give a combinatorial interpretation of them in terms of Catalan numbers and the Coxeter graph of the group. In the case of classical Weyl groups, this combinatorial interpretation can be restated in terms of excedances and other statistics of (signed) permutations. Our results also confirm a conjecure of Brenti on the parabolic Kazhdan-Lusztig polynomials of type q (see Corollary 3.3 below).

2 Definitions, notation and preliminaries

We let $\mathbb{P} := \{1, 2, 3, \dots\}$, $\mathbb{N} := \mathbb{P} \cup \{0\}$, $\mathbb{Z} := \mathbb{N} \cup \{-1, -2, \dots\}$. For all $m, n \in \mathbb{Z}$, $m \leq n$ we set $[m, n] := \{m, m+1, \dots, n\}$ and $[n] := [1, n]$. Given a set A we denote by $\#A$ its cardinality.

We follow (Stanley, 1997, Chapter 3) for poset notation and terminology. In particular, given a poset $(P, \leq)$ and $u, v \in P$ we let $[u, v] := \{w \in P | u \leq w \leq v\}$ and call this an *interval* of P. We say that v *covers* u, denoted $u \lhd v$ (or, equivalently, that u is *covered* by v) if $\#[u, v] = 2$.

We follow Humphreys (1990) for general Coxeter groups notation and terminology. Given a Coxeter system (W, S) and $u \in W$ we denote by $l(u)$ the length of u in W, with respect to S, i. e. the minimal length of words $s_{i_1} \cdots s_{i_k} = u$ whose alphabet is S (such minimal words are called reduced). Given $u, v \in W$ we denote by $l(u, v) = l(v) - l(u)$. We let $D_R(u) := \{s \in S | l(us) < l(u)\}$ the set of the right descents of u, $D_L(u) := \{s \in S | l(su) < l(u)\}$ the set of the left descents of u and we denote by ϵ the identity of W. Given $J \subseteq S$ we let W_J the parabolic subgroup generated by J and

$$W^J := \{u \in W | l(su) > l(u) \text{ for all } s \in J\} \tag{1}$$

Note that $W^{\emptyset} = W$ (the above definition is a little bit different from the classical one given in (Björner and Brenti, 2005, Definition 2.4.2)). If W_J is finite, then we denote by $w_0(J)$ its longest element. We will always assume that W^J is partially ordered by *Bruhat order*. Recall (see e.g. (Humphreys, 1990, Chapter 5.9 and 5.10)) that this means that $x \leq y$ if and only if for one reduced word of y (equivalently for all) there exists a subword that is a reduced word of x. Given $u, v \in W^J$, $u \leq v$ we let

$$[u, v]^J := \{w \in W^J | u \leq w \leq v\},$$

and $[u, v] := [u, v]^{\emptyset}$.

For $J \subseteq S$, $x \in \{-1, q\}$ and $u, v \in W^J$ we denote by $P^{J,x}_{u,v}(q)$ the parabolic Kazhdan–Lusztig polynomials in W^J of type x (we refer the reader to Deodhar (1987) for the definitions of these polynomials, see also Proposition 2.2 below). We denote by $P_{u,v}(q)$ the ordinary Kazhdan–Lusztig polynomials.

For $u, v \in W^J$ let $\mu_{J,q}(u,v)$ be the coefficient of $q^{\frac{1}{2}(l(u,v)-1)}$ in $P^{J,q}_{u,v}(q)$ (so $\mu_{J,q}(u,v) = 0$ when $l(v) - l(u)$ is even). It is well known that if $u, v \in W^J$ then $\mu_{J,q}(u,v) = \mu(u,v)$, the coefficient of $q^{\frac{1}{2}(l(u,v)-1)}$ in $P_{u,v}(q)$ (see Corollary 2.1 below). The following result is due to Deodhar, and we refer the reader to Deodhar (1987) for its proof.

Proposition 2.1 *Let (W,S) be a Coxeter system, $J \subseteq S$, and $u, v \in W^J, u \leq v$. Then for each $s \in D_R(v)$ we have that*

$$P^{J,q}_{u,v}(q) = \widetilde{P}_{u,v} - \widetilde{M}_{u,v} \tag{2}$$

where

$$\widetilde{P}_{u,v} = \begin{cases} P^{J,q}_{us,vs} + qP^{J,q}_{u,vs} & \text{if } us < u; \\ qP^{J,q}_{us,vs} + P^{J,q}_{u,vs} & \text{if } u < us \in W^J; \\ 0 & \text{if } u < us \notin W^J. \end{cases}$$

and

$$\widetilde{M}_{u,v} = \sum_{u \leq w < vs | ws < w} \mu(w, vs) q^{\frac{l(w,v)}{2}} P^{J,q}_{u,w}(q).$$

The parabolic Kazhdan–Lusztig polynomials are related to their ordinary counterparts in several ways, including the following one, which may be taken as their definition in most cases.

Proposition 2.2 *Let (W,S) be a Coxeter system, $J \subseteq S$ and $u, v \in W^J$. Then we have that*

$$P^{J,q}_{u,v}(q) = \sum_{w \in W_J} (-1)^{l(w)} P_{wu,v}(q).$$

Moreover, if W_J is finite, then

$$P^{J,-1}_{u,v}(q) = P_{w_0(J)u, w_0(J)v}(q).$$

A proof of this result can be found in Deodhar (1987) (see Proposition 3.4, and Remark 3.8). Since for all $w \in W_J$ and $u \in W^J$ we have $l(wu) = l(w) + l(u)$ by (Björner and Brenti, 2005, Proposition 2.4.4), then the degree of $P_{wu,v}(q)$ in Proposition 2.2 is less than $\frac{1}{2}(l(u,v) - 1)$ except when $w = \epsilon$. Therefore we have

Corollary 2.1 *For any $J \subseteq S$ and $u, v \in W^J$ we have*

$$\mu_{J,q}(u,v) = \mu(u,v).$$

Proposition 2.3 *Let (W,S) a Coxeter system and $J \subseteq S$. Let $u, v \in W^J$ and $s \in D_R(v)$.*

a) If $us \notin W^J$ then $P^{J,q}_{u,v}(q) = 0$;

b) if $us \in W^J$ then $P^{J,q}_{us,v}(q) = P^{J,q}_{u,v}(q)$;

c) if $\mu(u,v) \neq 0$ then $D_R(v) \subseteq D_R(u)$ and $D_L(v) \subseteq D_L(u)$.

In the rest of the paper we will consider parabolic Kazhdan–Lusztig polynomials of type q. Therefore we will write $P^J_{u,v}$ instead of $P^{J,q}_{u,v}$.

Let (W,S) be any Coxeter system and t be a reflection in W. Following Marietti (Marietti (2002), Marietti (2006) and Marietti (2010)), we say that t is a *boolean reflection* if it admits a *boolean expression*, which is, by definition, a reduced expression of the form $s_1 \cdots s_{n-1} s_n s_{n-1} \cdots s_1$ with $s_k \in S$, for all $k \in \{1, \dots n\}$ and $s_i \neq s_j$ if $i \neq j$. We say that $u \in W$ is a *boolean element* if u is smaller than a boolean reflection in the Bruhat order. Let $\overline{v}$ be a reduced word of a boolean element and $s \in S$, we denote by $\overline{v}(s)$ the number of occurrences of s in $\overline{v}$.

Given a Coxeter system (W, S), the Coxeter graph of W is a graph whose vertex set is S and two vertices s, s' are joined by an edge if $ss' \neq s's$. We label this edge with $m(s, s')$, the smallest positive integer such that $(ss')^{m(s,s')} = \epsilon$ ($m(s, s') = \infty$ if there is no such integer). We say that W is a *tree-Coxeter group* if its Coxeter graph is a tree.

fai o nuova sezione o breve intro For any generator $s_i \in S$ we denote by $S^i = S \setminus \{s_i\}$ and by $\mathrm{com}(s_i)$ the subset of S which contains all elements commuting with s_i different from s_i.

Lemma 2.1 *Let $u, v \in W^J$ such that $s_i u, s_i v \in W^J_{S^i}$ (i. e. there exist reduced words for u, v starting with s_i and with no other occurrences of s_i). Then*

$$P^J_{u,v} = P^{J \cap \mathrm{com}(s_i)}_{s_i u, s_i v}.$$

Lemma 2.2 *Let $u, v \in W^J$ be such that $u, s_i v \in W_{S^i}$ (i. e. there are no occurrences of s_i in any reduced expression of u and $s_i v$). Then*

$$P^J_{u,v} = \begin{cases} P^J_{u,s_i v} & \text{if } s_i v \in W^J \\ 0 & \text{otherwise} \end{cases}$$

We now introduce a family of numbers which are used in the next section. The *Catalan triangle* is a triangle of numbers formed in the same manner as Pascal's triangle, except that no number may appear on the left of the first element (see (OEI, sequence A008313)).

$$\begin{array}{cccccccccc}
1 & & & & & & & & & \\
 & 1 & & & & & & & & \\
1 & & 1 & & & & & & & \\
 & 2 & & 1 & & & & & & \\
2 & & 3 & & 1 & & & & & \\
 & 5 & & 4 & & 1 & & & & \\
5 & & 9 & & 5 & & 1 & & & \\
 & 14 & & 14 & & 6 & & 1 & & \\
14 & & 28 & & 20 & & 7 & & 1 & \\
 & 42 & & 48 & & 27 & & 8 & & 1
\end{array}$$

Let $h \geq 1$. We set

$$f_h(q) = \sum_{i=0}^{[\frac{h}{2}]} C(h, i) q^{[\frac{h}{2}]-i}$$

where $[h]$ denotes the integer part of h and $C(h, i)$ is the i-th number in the h-th row (here we start the enumeration from 0). For example $f_4(q) = 2q^2 + 3q + 1$; $f_7(q) = 14q^3 + 14q^2 + 6q + 1$. Note that in the first column we find the classical Catalan numbers (see (OEI, sequence A008313) for details).

3 Parabolic Kazhdan–Lusztig polynomials

Let (W, S) be a tree-Coxeter group. Let $t = s_{i_1} \cdots s_{i_{n-1}} s_{i_n} s_{i_{n-1}} \cdots s_{i_1}$ be a boolean reflection. Consider the Coxeter graph G and represent it as a rooted tree with root the vertex corresponding to the generator s_{i_n}. In this paper all the roots will be depicted on the right of their graphs. In Figure 1 we give the Coxeter graph of the affine Weyl group $\widetilde{D}_{11}$.

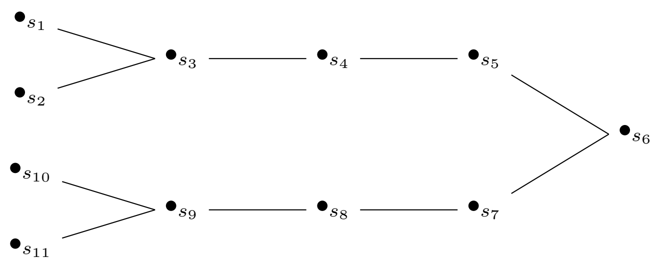


Fig. 1: The Coxeter graph of $\widetilde{D}_{11}$ with root s_6, corresponding to the reflection $t = s_1 s_2 \cdots s_5 s_{10} s_{11} s_9 s_8 s_7 s_6 s_7 s_8 s_9 s_{11} s_{10} s_5 \ldots s_2 s_1$.

According to such rooted graph we say that s_j is on the right (respectively on the left) of s_i if and only if there exists an edge joining them and the only path from s_i to s_n contains s_j.

Let w be a word in the alphabet S and $s \in S$. We denote by $w(s)$ the number of occurrences of s in w. Let $u, v \in W$ be such that $u, v \leq t$. Let $\overline{u}, \overline{v}$ be the unique reduced expressions of u, v satisfying the following properties

- $\overline{v}$ is a subword of $s_1 \cdots, s_{n-1} s_n s_{n-1} \cdots s_1$ and if i is such that $\overline{v}(s_i) = 1$ and $\overline{v}(s_j) = 0$, where s_j is the only element on the right of s_i, then we choose the subword with s_i in the leftmost admissible position;

- $\overline{u}$ is a subword of $\overline{v}$ and if i is such that $\overline{u}(s_i) = 1$ and $\overline{u}(s_j) = 0$, we apply the same above rule.

Here we give an example. Let $t = s_1 s_2 \cdots s_5 s_{10} s_{11} s_9 s_8 s_7 s_6 s_7 s_8 s_9 s_{11} s_{10} s_5 \ldots s_2 s_1$ in $\widetilde{D}_{11}$, see Figure 1. Let $v = s_4 s_5 s_{10} s_{11} s_6 s_7 s_8 s_9 s_5 s_4 s_2 s_1$ and $u = s_8 s_6 s_1$ then $\overline{v} = s_1 s_2 s_4 s_5 s_{10} s_{11} s_6 s_7 s_8 s_9 s_5 s_4$ and $\overline{u} = s_1 s_6 s_8$.

Now we give a graphical representation of the pair $(\overline{v}, \overline{u})$. We start from the rooted tree of the Coxeter graph and we substitute for each vertex a table with one column and two rows. In the first row we write $\overline{v}(s_j)$ (s_j is the element associated to the vertex); in the second row we write $\overline{u}(s_j)$. In the case $\overline{v}(s_j) = 1$, it is possible that s_j is on the left or on the right of s_n (the root) as subword of t. We distinguish the two cases by writing 1_l if s_j is on the left of s_n, and 1_r otherwise. By convention we write 1_l in the root s_n if $\overline{v}(s_n) \neq 0$. We apply the same rule to the second row. Moreover, in the first row, we use capital letter R instead of r if the second row of the column to the right does not contain 0.

We mark the column corresponding to s_j with $\circ$ if $j \in J$ and with $\times$ if $j \notin J$. Finally, if a vertex s_j has only one vertex on the left then we write the two corresponding columns in same table. In Figure 2 we give the graphical representation of the pair $(\overline{v}, \overline{u})$ in $\widetilde{D}_{11}$, with $J = \{s_5, s_7\}$.

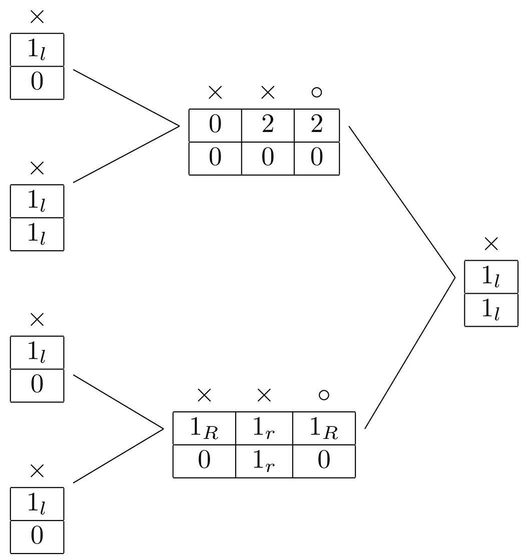
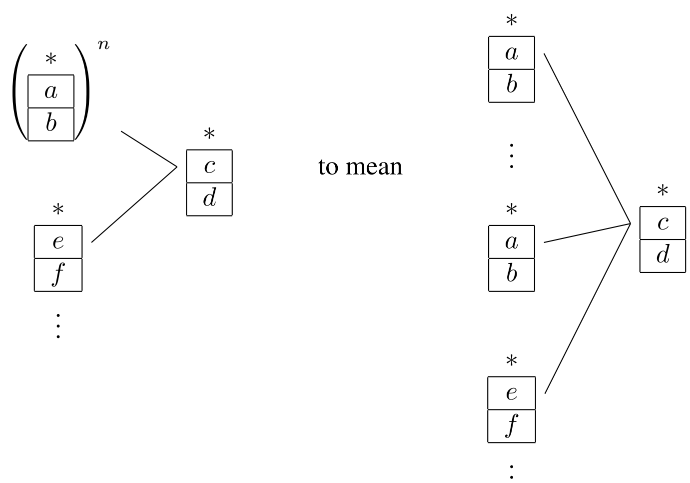


Fig. 2: Diagram of $(\overline{v} = s_1 s_2 s_4 s_5 s_{11} s_{10} s_6 s_7 s_8 s_9 s_5 s_4, \overline{u} s_1 s_6 s_8)$ in $\widetilde{D}_{11}$.

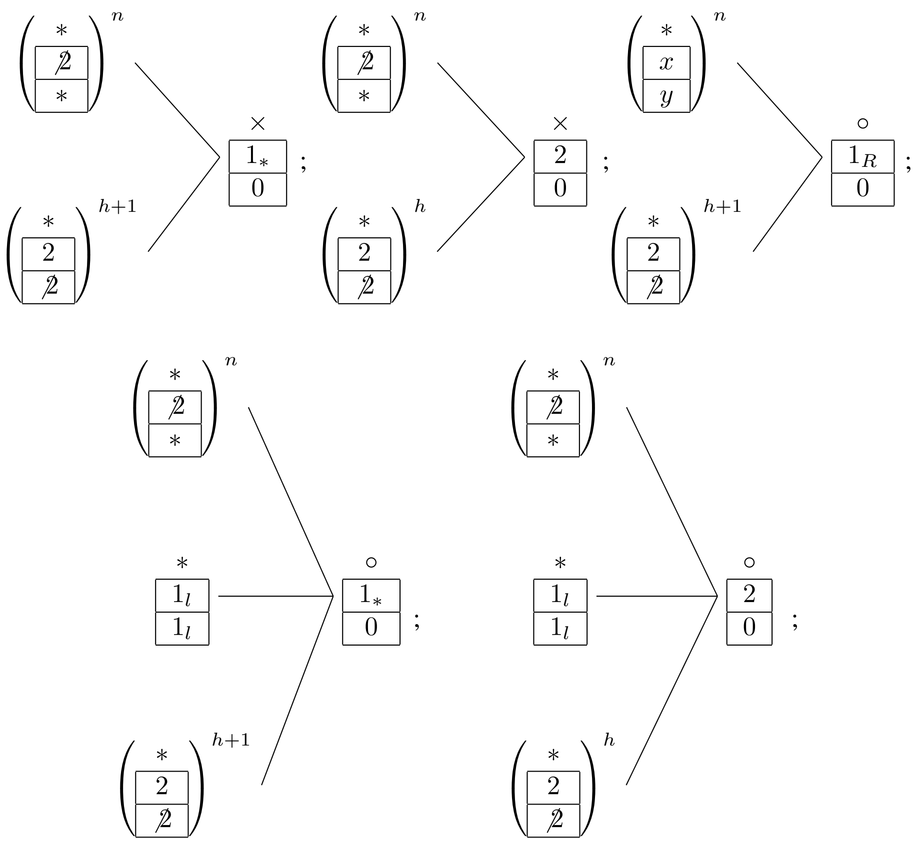
In the sequel a symbol $*$ denotes the possibility to have arbitrary entries in the cell. A symbol such as $\not{1}_l$, $\not{0}$, etc. means that the value in the cell is not 1_l, 0, etc. Moreover we will be interested in subdiagrams of such representations, i. e. diagrams obtained by deleting one or more columns. Since the order of the tables from top to bottom is not important (while the order from left to right is fundamental), we use the following notation



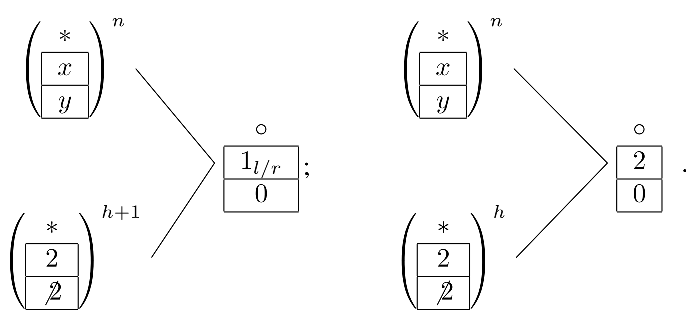
(3)

where the column with entries a, b is repeated n times. Now we give all the definitions necessary to Theorem 3.1.

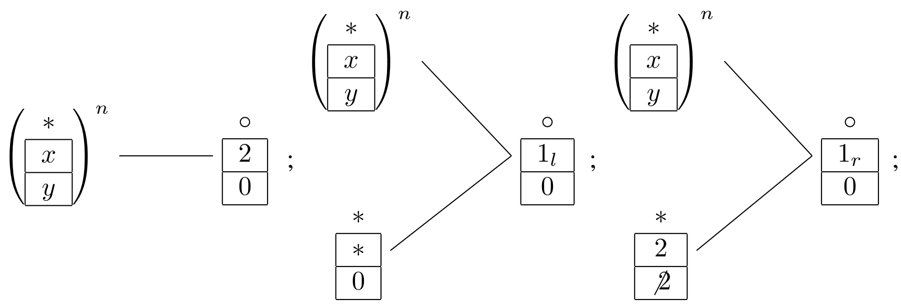
Given a pair $(\overline{v}, \overline{u})$ in W, we let $a_h(\overline{u}, \overline{v})$ be the number of subdiagrams in the diagram of $(\overline{u}, \overline{v})$ of one of the following type:

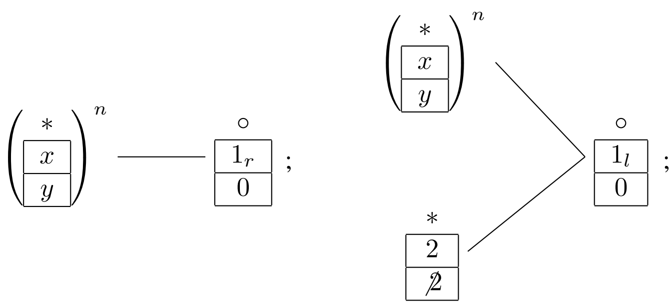


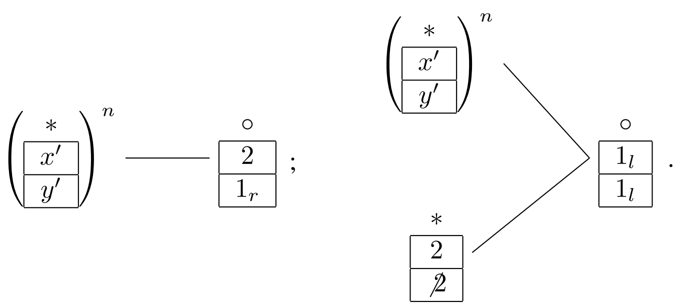
We define $b_h(\overline{u}, \overline{v})$ be the number of subdiagrams in the diagram of $(\overline{u}, \overline{v})$ of one of the following type:



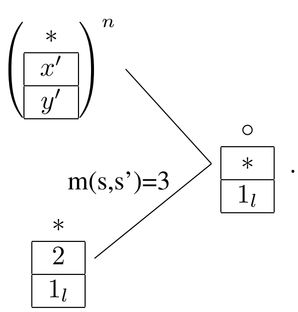
We set $c(\overline{u},\overline{v})$ be the number of subdiagrams in the diagram of $(\overline{u},\overline{v})$ of one of the following type:







Finally, we set $c'(\overline{u},\overline{v})$ be the number of subdiagrams of the diagram of $(\overline{u},\overline{v})$ of the following type:



In all previous diagrams n is an arbitrary non–negative integer and $(x, y) \in P_1$, $(x', y') \in P_1 \cup P_2$ with $P_1 = \{(1_l, 0), (1_r, 0), (1_r, 1_r), (2, 1_r)\}$, $P_2 = \{(1_R, 0), (1_R, 1_r), (2, 0)\}$. In each diagram (x, y), (x', y'), $(\not 2, *)$ or $(2, \not 2)$ are not necessarily the same pair for all $n \geq 0$ (or $h \geq 0$) columns. We can now state the main result of this work.

Theorem 3.1 *Let $J \subseteq S$, $u, v \in W^J$ and set $\overline{c}(\overline{u}, \overline{v}) = c(\overline{u}, \overline{v}) + c'(\overline{u}, \overline{v})$. We have*

$$P^J_{u,v}(q) = \begin{cases} \prod_{h\geq 1} f_{h+1}^{a_h}(f_{h+1} - 1)^{b_h} & \textit{if } \overline{c}(\overline{u}, \overline{v}) = 0 \\ 0 & \textit{otherwise} \end{cases}$$

Corollary 3.1 *Let $J \subseteq S$, $u, v \in W^J$ with $l(v) - l(u) \geq 3$ odd. Then $\mu(u, v) \neq 0$ if and only if the entries in each column of the diagram of $(\overline{u}, \overline{v})$ are equal except for exactly one subdiagram which is*

$$\left(\begin{array}{c} * \\ \hline \multicolumn{1}{|c|}{2} \\ \hline \multicolumn{1}{|c|}{1_l} \\ \hline \end{array}\right)^{h+1} \text{———} \begin{array}{c} * \\ \hline \multicolumn{1}{|c|}{\not 0} \\ \hline \multicolumn{1}{|c|}{0} \\ \hline \end{array} \quad \textit{or} \quad \left(\begin{array}{c} * \\ \hline \multicolumn{1}{|c|}{2} \\ \hline \multicolumn{1}{|c|}{1_l} \\ \hline \end{array}\right)^{h} \text{———} \begin{array}{ccc} * & & * \\ \hline \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{\dots} & \multicolumn{1}{c|}{2} \\ \hline \multicolumn{1}{|c|}{0} & \multicolumn{1}{c|}{\dots} & \multicolumn{1}{c|}{0} \\ \hline \end{array}$$

In this case $\mu(u, v) = C([\frac{h+1}{2}])$, the $[\frac{h+1}{2}]$-th Catalan number.

In the case of the classical Kazhdan–Lusztig polynomials, Theorem 3.1 becomes much simpler.

Corollary 3.2 *Let W be a tree-Coxeter group and $u, v \in W$ be boolean elements. Then $P_{u,v}(q) = \prod_{h\geq 1} f_{h+1}^{a_h}$, where a_h is defined before Theorem 3.1.*

For example, the Kazhdan–Lusztig polynomial of the pair (u, v) depicted in Figure 2 is $P^J_{u,v} = f_2(q) - 1 = q$, since $a_h = 0$ for all $h \geq 0$, $b_1 = 1$ and $b_h = 0$ for all $h \neq 1$.

Remark 3.1 *Theorem 3.1 implies result in (Marietti, 2010, Theorem 5.2).*

We give the following easy consequence of Theorem 3.1 which proves, in the case of boolean elements, a conjecture of Brenti (private communication).

Corollary 3.3 *Let $I \subseteq J$ and $u, v \in W^J$. Then*

$$P^J_{u,v}(q) \leq P^I_{u,v}(q)$$

in the coefficientwise comparison.

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4 Poincaré polynomials

Given $v \in W$, let $F_v(q) = \sum_{u\leq v} q^{l(v)} P_{u,v}$. It is well known that, if W is any finite Coxeter or affine Weyl group, $F_v(q)$ is the intersection homology Poincaré polynomial of the Schubert variety indexed by v (see Kazhdan and Lusztig (1980)). In this section we compute the Poincaré polynomial for any boolean element in a Coxeter group whose Coxeter graph is a tree with at most one vertex having more than two adjacent vertices (such groups include all classical finite Coxeter and affine Weyl groups except $\widetilde{A}_n$ and $\widetilde{D}_n$).

Let $v \in W$ be a boolean element and consider the diagram of $(\epsilon_W, \overline{v})$. For convenience we will not depict the second row of each column which is always 0 and we omit all symbols $\times$. We will call it the diagram of v.

Let v be a boolean element and let s be the element of S associated to one of the leftmost vertices in the diagram of v. We set $F^{\setminus}_{v,s} = \sum q^{l(v)} P_{u,v}$ where the sum runs over all elements $u \leq v$ such that $\overline{u}(s) \neq 0$ and $F^{0}_{v,s} = \sum q^{l(v)} P_{u,v}$ where the sum runs over all elements $u \leq v$ such that $\overline{u}(s) = 0$.

Now consider a diagram d. Delete all entries equal to 0 and delete all edges whose left vertex is not a cell containing 2. Let $d_1, \ldots, d_k$ be the remaining connected components. We refer to them as the *essential components* of d.

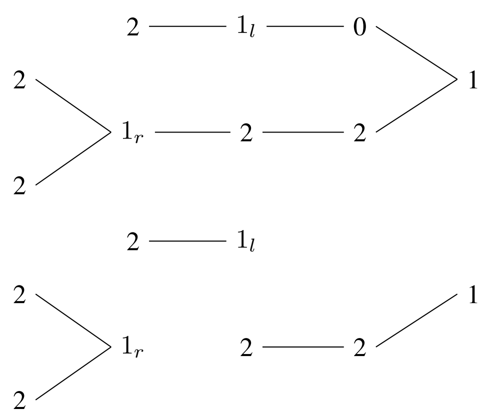

Fig. 3: A diagram and its essential components.

Lemma 4.1 *Let $v \in W$ be a boolean element and let d be the diagram of $\overline{v}$. Let $d_1, \ldots, d_k$ be the essential components of the diagram d and $v_1, \ldots, v_k$ be the boolean reflections corresponding to $d_1, \ldots, d_k$. Then*

$$F_v(q) = \prod_{i=1}^{k} F_{v_i}(q).$$

Proposition 4.1 *Let W be a Coxeter group such that its Coxeter graph is a tree and all vertices except at most one have degree less than 3. Denote with w such exceptional vertex. Let $v \in W$ be a boolean element. Then*

$$F_v(q) = (1 + q + q^2)^{k-1}\big(q(1+q)^{h+1} + f_h(q)\big)(1+q)^{l(v)-2k-h-2},$$

where k is the number of essential components of the diagram d of v with at least two vertices and h is the number of entries equal to 2 in the adjacent cells of w (also consider the cell on the right).

The formula is also true when there is no vertex of degree greater than 2: in this case let w be any vertex of degree 2.

Acknowledgements

At the end of the manuscript, right before the bibliography you might want to place an acknowledgement. This can be easily done by using the command `\acknowledgements` as you can see here.

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Asymptotic properties of some minor-closed classes of graphs

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Abstract. Let $\mathcal{A}$ be a minor-closed class of labelled graphs, and let $\mathcal{G}_n$ be a random graph sampled uniformly from the set of n-vertex graphs of $\mathcal{A}$. When n is large, what is the probability that $\mathcal{G}_n$ is connected? How many components does it have? How large is its biggest component? Thanks to the work of McDiarmid and his collaborators, these questions are now solved when all excluded minors are 2-connected.

Using exact enumeration, we study a collection of classes $\mathcal{A}$ excluding non-2-connected minors, and show that their asymptotic behaviour is sometimes rather different from the 2-connected case. This behaviour largely depends on the nature of the dominant singularity of the generating function $C(z)$ that counts connected graphs of $\mathcal{A}$. We classify our examples accordingly, thus taking a first step towards a classification of minor-closed classes of graphs. Furthermore, we investigate a parameter that has not received any attention in this context yet: the size of the root component. This follows non-gaussian limit laws (beta and gamma), and clearly deserves a systematic investigation.

Keywords: Labelled graphs – Excluded minors – Enumeration – Asymptotic properties

1 Introduction

We consider simple graphs on the vertex set $\{1, \ldots, n\}$. A set of graphs is a *class* if it is closed under isomorphisms. A class of graphs $\mathcal{A}$ is *minor-closed* if any minor of a graph of $\mathcal{A}$ is in $\mathcal{A}$. To each such class one can associate a set $\mathcal{E}$ of *excluded minors*: an (unlabelled) graph is excluded if its labelled versions do not belong to $\mathcal{A}$, but the labelled versions of each of its proper minors belong to $\mathcal{A}$. A remarkable result of Robertson and Seymour states that $\mathcal{E}$ is always finite [19].

For a minor-closed class $\mathcal{A}$, we study the asymptotic properties of a random graph $\mathcal{G}_n$ taken uniformly in $\mathcal{A}_n$, the set of graphs of $\mathcal{A}$ having n vertices: what is the probability p_n that $\mathcal{G}_n$ is connected? More generally, what is the number N_n of connected components? What is the size S_n of the *root component*, that is, the component containing the vertex 1? Or the size L_n of the largest component?

Thanks to the work of McDiarmid and his collaborators, a lot is known if all excluded graphs are 2-connected: then p_n converges to constant larger than $1/\sqrt{e}$, N_n converges in law to a Poisson distribution, $n - S_n$ and $n - L_n$ converge in law to the same discrete distribution. Details are given in Section 3. If some

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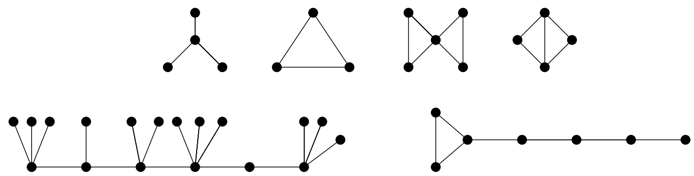
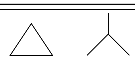
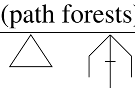
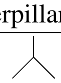
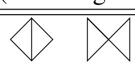
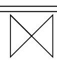


Fig. 1: Top: the 3-star, the triangle K_3, the bowtie and the diamond. Bottom: A caterpillar and the 4-spoon.

excluded minors are *not* 2-connected, the properties of $\mathcal{G}_n$ may be rather different (imagine we exclude the one edge graph...). This paper takes a preliminary step towards a classification of the possible behaviours by presenting an organized catalogue of examples. We refer to [7] for more examples, complete proofs and Boltzmann samplers for our classes of graphs.

For each class $\mathcal{A}$ that we study, we first determine the generating functions $C(z)$ and $A(z)$ that count connected and general graphs of $\mathcal{A}$, respectively. The minors that we exclude are always connected, which implies that $\mathcal{A}$ is *decomposable*: a graph belongs to $\mathcal{A}$ if and only if all its connected components belong to $\mathcal{A}$. In particular, $A(z) = \exp(C(z))$. Our exact and asymptotic results make extensive use of the

Excluded minors	$C(\rho)$	Sing. of $C(z)$	$\lim p_n$	number N_n of comp.	root comp. S_n	largest comp. L_n	Refs. and methods
2-connected	$< \infty$	?	$\geq e^{-\frac{1}{2}}$ < 1	$O(1)$ Poisson	$n - S_n$ $\to$ disc.	$n - L_n$ $\to$ disc.	[1, 16] [17]
a spoon, but no tree	$< \infty$	$(1 - ze)^{3/2}$	> 0 $\leq e^{-\frac{1}{2}}$	id.	id.	id.	Sec. 8 sing. an.
 (path forests)	∞	simple pole	0	$\sqrt{n}$ gaussian	$\sqrt{n}$ xe^{-x}	$\sqrt{n}\log n$ Gumbel	Sec. 5 saddle
 (caterpillar for.)	∞	id.	0	id.	id.	?	Sec. 5 saddle
 (max. deg. 2)	∞	id. (+ log)	0	id.	id.	?	Sec. 5 saddle
	∞	log $(+\sqrt{\ })$	0	$\log n$ gaussian	n $\frac{1}{4}(1-x)^{-\frac{3}{4}}$	$\mathrm{PD}^{(1)}(1/4)$	Sec. 6 sing. an.
	∞	$1/\sqrt{\ }$	0	$n^{1/3}$ gaussian	$n^{2/3}$ $2\sqrt{x/\pi}e^{-x}$	?	Sec. 7 saddle

Tab. 1: Summary of the results: for each quantity N_n, S_n and L_n, we give the limit law (or its density) and an estimate of the expected value when it diverges (up to a constant). The symbol $\mathrm{PD}^{(1)}(1/4)$ stands for the first component of a Poisson-Dirichlet distribution of parameter $1/4$.

techniques of Flajolet and Sedgewick's book [11]: symbolic combinatorics, singularity analysis, saddle point method, and their application to the derivation of limit laws. We recall a few basic principles in Section 2, but then we only sketch the proofs, at best. We also need and prove two general results of independent interest related to the saddle point method (Theorems 3 and 4).

Our results are summarized in Table 1. A first dichotomy emerges: when $C(z)$ is finite at its radius of convergence ρ, the properties of $\mathcal{A}$ are qualitatively the same as in the 2-connected case (for which $C(\rho)$ is known to converge), except that the limit of p_n can be arbitrarily small (Section 8). When $C(\rho)$ diverges, a whole variety of behaviours can be observed, depending on the nature of the singularity of $C(z)$ at ρ (Sections 5 to 7): the probability p_n tends to 0 at various speeds; the number N_n of components goes to infinity at various speeds (but is invariably gaussian after normalization); the size S_n of the root component follows, after normalization, non-gaussian limit laws (gamma or beta). We only study the size L_n of the largest component in a few cases. Much remains to be done in this direction.

Let us conclude with a few words on the size of the root component. It appears that this parameter, which can be defined for any exponential family of objects, has not been studied systematically yet, and follows interesting (*i.e.*, non-gaussian!) continuous limit laws, after normalization. We are currently working on such a systematic study, in the spirit of what Bell *et al*. [4] and Gourdon [13] did for the number of components and the largest component, respectively. This project is also reminiscent of the study of the 2-connected root component in a planar map [3], which also leads to non-gaussian limit laws.

2 Generatingfunctionology for graphs

Let $\mathcal{E}$ be a finite set of (unlabelled) *connected* graphs that forms an antichain with respect to the minor order. Let $\mathcal{A}$ be the set of labelled graphs that do not contain any element of $\mathcal{E}$ as a minor. By $\mathcal{A}_n$ we denote the subset of $\mathcal{A}$ formed of graphs having n vertices and by a_n the cardinality of $\mathcal{A}_n$. The associated exponential generating function (g.f.) is $A(z) = \sum_{n\geq 0} a_n z^n/n!$. We use similar notation (c_n and $C(z)$) for the subset $\mathcal{C}$ of $\mathcal{A}$ consisting of (non-empty) connected graphs. Since the excluded minors are connected, $\mathcal{A}$ is decomposable, and $A(z) = \exp(C(z))$. Several refinements of this series are of interest, for instance the g.f. that keeps track of the number of (connected) components as well:

$$A(z,u) = \sum_{G\in\mathcal{A}} u^{c(G)} \frac{z^{|G|}}{|G|!} = \exp(uC(z)),$$

where $|G|$ is the *size* of G (the number of its vertices) and $c(G)$ the number of its components. We denote by N_n the number of components in a (uniform) random graph $\mathcal{G}_n$ of $\mathcal{A}_n$. Clearly,

$$\mathbb{P}(N_n = i) = \frac{[z^n]C(z)^i}{i![z^n]A(z)} \qquad \text{and} \qquad \mathbb{E}(N_n(N_n-1)\cdots(N_n-i+1)) = \frac{[z^n]C(z)^i A(z)}{[z^n]A(z)}. \tag{1}$$

Several general results provide a limit law for N_n if $C(z)$ satisfies certain conditions: for instance the results of Bell *et al*. [4] that require $C(z)$ to converge at its radius; or the *exp-log schema* of [11, Prop. IX.14, p. 670], which requires $C(z)$ to have a logarithmic singularity. We use them when applicable, and prove a new result of this type, which applies when $C(z)$ diverges with an algebraic singularity (Theorem 4).

We also study the size $r(G)$ of the *root component* (the component containing the vertex 1). Let

$$\bar{A}(z,v) = \sum_{G\in\mathcal{A}, G\neq\emptyset} v^{r(G)-1} \frac{z^{|G|-1}}{(|G|-1)!} = C'(zv)A(z). \tag{2}$$

The choice of $|G|-1$ instead of $|G|$ slightly simplifies some calculations. Note that $\bar{A}(z,1) = A'(z)$. Denoting by S_n the size of the root component in $\mathcal{G}_n$, we have

$$\mathbb{P}(S_n = k) = \frac{c_k a_{n-k}\binom{n-1}{k-1}}{a_n} \qquad \text{and} \qquad \mathbb{E}((S_n-1)\cdots(S_n-i)) = \frac{[z^{n-i-1}]C^{(i+1)}(z)A(z)}{n[z^n]A(z)}. \quad (3)$$

Surprisingly, this parameter has not been studied before. Our examples lead to non-gaussian limit laws (gamma or beta, cf. Propositions 7 or 12). In fact, the form (2) of the generating function shows that this parameter is bound to give rise to interesting limit laws, as both the location and nature of the singularity change as v moves to $1-\varepsilon$ to $1+\varepsilon$. Using the terminology of [11, Sec. IX.11], a *phase transition* occurs. We are currently working on a systematic study of this parameter for exponential objects.

3 Classes defined by 2-connected excluded minors

We assume here that the class $\mathcal{A}$ excludes at least one minor, and that all excluded minors are 2-connected. This includes the class of forests, series-parallel graphs, planar graphs, and many more. Many results are known in this case. The general picture is that $\mathcal{A}$ shares many properties with the class of forests.

Proposition 1 *The series $C(z)$ and $A(z) = e^{C(z)}$ are finite at their (positive) radius of convergence ρ. Moreover, the sequence $(a_n/n!)_n$ is* smooth*, meaning that na_{n-1}/a_n tends to ρ as n grows.*

The probability that $\mathcal{G}_n$ is connected tends to $1/A(\rho)$, which is clearly in $(0,1)$. In fact, this limit is also larger than or equal to $1/\sqrt{e}$. This value is reached when $\mathcal{A}$ is the class of forests.

The fact that $\rho > 0$ is due to Norine *et al.* [18], and holds for any proper minor-closed class. The next results are due to McDiarmid [16]. The fact that $1/A(\rho) \geq 1/\sqrt{e}$, or equivalently, that $C(\rho) \leq 1/2$, was proved independently in [1] and [14].

For forests, all results are well-know (see for instance [11, p. 132]). We have $C(z) = T(z) - T(z)^2/2$, where $T(z) = ze^{T(z)}$ counts rooted trees. The series T, C and A have radius $\rho = 1/e$, and $A(\rho) = \sqrt{e}$.

The nature of the singularity of $C(z)$ at ρ depends on the class: $(1-z/\rho)^{3/2}$ for forests (and more generally, for *subcritical* minor-closed classes [8]), but $(1-z/\rho)^{5/2}$ for planar graphs. We refer to [12] for a more detailed discussion that applies to classes that exclude 3-connected minors.

Proposition 2 *The random variable $N_n - 1$ converges in law to a Poisson distribution of parameter $C(\rho)$:*

$$\mathbb{P}(N_n = i+1) \to \frac{C(\rho)^i}{i! e^{C(\rho)}}.$$

The random variables $n - L_n$ and $n - S_n$ both converge to a discrete limit distribution X given by

$$\mathbb{P}(X = k) = \frac{1}{A(\rho)} \frac{a_k \rho^k}{k!}. \quad (4)$$

Proof: The first two results (on N_n and L_n) are due to McDiarmid [16, Cor. 1.6]. The third one is in fact equivalent to the second (the root component is, with high probability, the largest one), but we give here an independent proof, as we will recycle its ingredients later for some classes of graphs that avoid non-2-connected minors. Let $k \geq 0$ be fixed. By (3),

$$\mathbb{P}(S_n = n-k) = \frac{c_{n-k} a_k \binom{n-1}{k}}{a_n} = \frac{a_k}{k!} \frac{c_{n-k}}{a_{n-k}} \frac{(n-1)! a_{n-k}}{(n-k-1)! a_n}.$$

By Proposition 1, the term c_{n-k}/a_{n-k}, which is the probability that $\mathcal{G}_{n-k}$ is connected, converges to $1/A(\rho)$. Moreover, the sequence $a_n/n!$ is smooth, so that $\frac{(n-1)!a_{n-k}}{(n-k-1)!a_n}$ converges to ρ^k. The result follows. $\square$

4 General tools: Hayman admissibility and extensions

We consider in Sections 5 to 7 minor-closed classes of graphs such that $C(z)$ diverges at its radius of convergence ρ. This often results in $A(z)$ diverging rapidly at ρ, and leads us to estimate a_n using the saddle point method, or rather, a black box that applies to *Hayman-admissible* (or H-admissible) series: see [11, Thm. VIII.4, p. 565]. These series have useful closure properties [ibid., p. 568]. Here is one that we did not find in the literature.

Theorem 3 *Let $A(z) = F(z)G(z)$ where $F(z)$ and $G(z)$ are power series with real coefficients and radii of convergence $0 < \rho_F < \rho_G \leq \infty$. Assume that $F(z)$ has non-negative coefficients and is H-admissible, and that $G(\rho_F) > 0$. Then $A(z)$ is H-admissible.*

We will also need a *uniform* version of Hayman-admissibility for series of the form $e^{uC(z)}$.

Theorem 4 *Let $C(z)$ be a power series with non-negative coefficients and radius of convergence ρ. Assume $A(z) = e^{C(z)}$ has radius ρ and is H-admissible. Define*

$$b(r) = rC'(r) + r^2C''(r) \quad \textit{and} \quad V(r) = C(r) - \frac{(rC'(r))^2}{rC'(r) + r^2C''(r)}.$$

Assume that, as $r \to \rho$,

$$V(r) \to +\infty, \qquad \frac{C(r)}{V(r)^{3/2}} \to 0, \qquad b(r)^{1/\sqrt{V(r)}} = O(1). \tag{5}$$

Then $A(z, u) := e^{uC(z)}$ satisfies Conditions (1)–(6), (8) *and* (9) *of* [10, Def. 1]. *If N_n is a sequence of random variables such that*

$$\mathbb{P}(N_n = i) = \frac{[z^n]C(z)^i}{i![z^n]e^{C(z)}},$$

then the mean and variance of N_n satisfy:

$$\mathbb{E}(N_n) \sim C(\zeta_n), \qquad \mathbb{V}(N_n) \sim V(\zeta_n), \tag{6}$$

where $\zeta \equiv \zeta_n$ is the unique solution in $(0, \rho)$ of the saddle point equation $\zeta C'(\zeta) = n$. Moreover, the normalized random variable $\frac{N_n - \mathbb{E}(N_n)}{\sqrt{\mathbb{V}(N_n)}}$ converges in law to a standard normal distribution.

Proof: We carefully check the eight conditions (the only that do not come for free are (2) and (3)). As explained in [10] just below Theorem 2, they give the estimates (6) of $\mathbb{E}(N_n)$ and $\mathbb{V}(N_n)$ and imply the existence of a gaussian limit law. $\square$

We finish with a simple but useful result on products of series [5, Thm. 2].

Proposition 5 *Let $F(z) = \sum_n f_n z^n$ and $G(z) = \sum_n g_n z^n$ be power series with radii of convergence $0 \leq \rho_F < \rho_G \leq \infty$, respectively. Suppose that $G(\rho_F) \neq 0$ and f_{n-1}/f_n approaches a limit (which is necessarily ρ_F) as $n \to \infty$. Then $[z^n]F(z)G(z) \sim G(\rho_F)f_n$.*

5 Forests of paths or caterpillars: a simple pole

Let $\mathcal{A}$ be a decomposable class (for instance defined by excluding connected minors), with g.f. $A(z) = \exp(C(z))$. Assume that $C(z)$ has a unique dominant singularity ρ, which is an isolated simple pole:

$$C(z) = \frac{\alpha}{1 - z/\rho} + D(z), \quad \text{where} \quad D(\rho) = \beta \tag{7}$$

and D has radius of convergence strictly larger than ρ. Of course, we assume $\alpha > 0$.

Proposition 6 *If the above conditions hold, then, as $n \to \infty$,*

$$c_n \sim n!\, \alpha \rho^{-n} \quad \textit{and} \quad a_n \sim n!\, \frac{\alpha^{1/4} e^{\alpha/2+\beta}}{2\sqrt{\pi} n^{3/4}} \rho^{-n} e^{2\sqrt{\alpha n}}. \tag{8}$$

In particular, the probability that $\mathcal{G}_n$ is connected tends to 0 at speed $n^{3/4} e^{-2\sqrt{\alpha n}}$.

Proof: The asymptotic behaviour of c_n follows from [11, Thm. IV.10, p. 258]. For a_n, we first write

$$A(z) = F(z)G(z) \qquad \text{with} \qquad F(z) = \exp\left(\frac{\alpha}{1 - z/\rho}\right) \qquad \text{and} \qquad G(z) = e^{D(z)}, \tag{9}$$

where $G(z)$ has radius strictly larger than ρ. To estimate the coefficients of F, we apply the ready-to-use results of Macintyre and Wilson [15, Eqs. (10)–(14)], according to which, for $\alpha > 0$ and $\gamma \geq 0$,

$$[z^n] \frac{1}{(1-z)^\gamma} \exp\left(\frac{\alpha}{1-z}\right) \sim \frac{\alpha^{1/4} e^{\alpha/2}}{2\sqrt{\pi} n^{3/4}} \left(\frac{n}{\alpha}\right)^{\gamma/2} e^{2\sqrt{\alpha n}}. \tag{10}$$

This gives

$$f_n := [z^n]F(z) \sim \frac{\alpha^{1/4} e^{\alpha/2}}{2\sqrt{\pi} n^{3/4}} \rho^{-n} e^{2\sqrt{\alpha n}}.$$

In particular, f_{n-1}/f_n tends to ρ, so that we can apply Proposition 5 to (9) and conclude. □

Proposition 7 *Assume* (7) *holds.*
1. *The mean and variance of N_n satisfy:*

$$\mathbb{E}(N_n) \sim \sqrt{\alpha n}, \qquad \mathbb{V}(N_n) \sim \sqrt{\alpha n}/2, \tag{11}$$

and the random variable $\frac{N_n - \sqrt{\alpha n}}{(\alpha n/4)^{1/4}}$ converges in law to a standard normal distribution.
2. *For $i \geq 0$, the i^{th} moment of S_n satisfies, as $n \to \infty$,*

$$\mathbb{E}(S_n^i) \sim (i+1)!(n/\alpha)^{i/2}.$$

Consequently, the normalized variable $S_n/\sqrt{n/\alpha}$ converges in distribution to a gamma(2) *law, of density xe^{-x} on $[0, \infty)$. A local limit law also holds: for $x > 0$ and $k = \lfloor x\sqrt{n/\alpha} \rfloor$,*

$$\sqrt{n/\alpha}\, \mathbb{P}(S_n = k) \sim xe^{-x}.$$

Proof: 1. We apply Theorem 4. The H-admissibility of $A(z)$ follows from Theorem 3, using (9) and the H-admissibility of $\exp(\alpha/(1-z/\rho))$ (see [11, p. 562]). Conditions (5) are then readily checked, using

$$C(r) \sim \frac{\alpha}{1-z/\rho}, \quad b(r) \sim \frac{2\alpha}{(1-z/\rho)^3} \quad \text{and} \quad V(r) \sim \frac{\alpha}{2(1-z/\rho)}.$$

We thus conclude that the normalized version of N_n converges in law to a standard normal distribution. For (11), we use (6) with the saddle point estimate $\zeta_n = \rho - \rho\sqrt{\alpha/n} + O(1/n)$.

2. We apply (3), with

$$C^{(i+1)}(z) = \frac{\alpha(i+1)!}{\rho^{i+1}(1-z/\rho)^{i+2}} + D^{(i+1)}(z). \tag{12}$$

As in the proof of Proposition 6, we combine Proposition 5, (9) and (10) to obtain

$$\frac{a_n}{(n-1)!}\mathbb{E}((S_n-1)\cdots(S_n-i)) \quad = \quad \alpha(i+1)!\,\frac{\alpha^{1/4}e^{\alpha/2+\beta}}{2\sqrt{\pi}n^{3/4}}\left(\frac{n}{\alpha}\right)^{i/2+1}\rho^{-n}e^{2\sqrt{\alpha n}}.$$

Combined with (8), this gives the limiting i^{th} moment of S_n. Since these moments characterize the above gamma distribution, we conclude [11, Thm. C.2] that $S_n/\sqrt{n/\alpha}$ converges in law to this distribution.

For the local limit law, we simply combine the first part of (3) with (8). □

We now apply these results to two classes for which $C(z)$ has a simple pole: forests of paths, and forests of *caterpillars* (a caterpillar is a tree made of a simple path to which leaves are attached; see Figure 1).

Proposition 8 *The generating functions of paths and of caterpillars are respectively*

$$C_p(z) = \frac{z(2-z)}{2(1-z)} \qquad \textit{and} \qquad C_c(z) = \frac{z^2(e^z-1)^2}{2(1-ze^z)} + ze^z - \frac{z^2}{2}.$$

For both series, Condition (7) *is satisfied and Propositions* 6 and 7 *hold. For paths we have* $\rho = 1$, $\alpha = 1/2$ *and* $\beta = 0$. *For caterpillars,* $\rho \simeq 0.567$ *is the only real such that* $\rho e^\rho = 1$,

$$\alpha = \frac{(1-\rho)^2}{2(1+\rho)} \simeq 0.06 \quad \textit{and} \quad \beta = \frac{\rho\,\left(10+3\,\rho-4\,\rho^2-\rho^3\right)}{4\,(1+\rho)^2} \simeq 0.59.$$

For forests of paths, we have also studied the size L_n of the largest component.

Proposition 9 *In forests of paths, the size of the largest component converges to a Gumbel distribution:*

$$\mathbb{P}\left(\frac{L_n - \sqrt{n/2}\log n}{\sqrt{n/2}} < x\right) \to \exp\left(-\frac{e^{-x/2}}{\sqrt{2}}\right).$$

Proof: The generating function of paths of size less than k is $C^{[k]}(z) = z/2 + (z-z^k)/(2(1-z))$. We then use Cauchy's formula and a saddle point approach. □

Graphs with maximal degree 2: a simple pole plus a logarithm. Let $\mathcal{A}$ be the class of graphs avoiding the 3-star. The connected components of such graphs are paths and cycles. The series $C(z)$ has now, in addition to a simple pole, a logarithmic singularity at its radius. The logarithm changes the asymptotic behaviour of the numbers a_n, but the other results remain unaffected. The proofs are very similar to the above ones.

Proposition 10 *The generating function of connected graphs of $\mathcal{A}$ is*

$$C(z) = \frac{z(2-z+z^2)}{4(1-z)} + \frac{1}{2}\log\frac{1}{1-z}.$$

The generating function of graphs of $\mathcal{A}$ is $A(z) = e^{C(z)}$. We have, for large n,

$$c_n = \frac{n!}{2} + \frac{(n-1)!}{2} \qquad \textit{and} \qquad a_n \sim n!\,\frac{1}{2\sqrt{e\pi}n^{1/2}}e^{\sqrt{2n}}.$$

In particular, the probability that $\mathcal{G}_n$ is connected tends to 0 at speed $n^{1/2}e^{-\sqrt{2n}}$.
The number of components and the size of the root component behave as in Proposition 7, with $\alpha = 1/2$.

6 Excluding the diamond and the bowtie: a logarithm dominates

Let $\mathcal{A}$ be the class of graphs avoiding the diamond and the bowtie (shown in Figure 1). The connected components are trees or unicyclic graphs, and have been counted a long time ago by Wright [20].

Proposition 11 *Let $T(z) = ze^{T(z)}$ be the g.f. of rooted trees. The g.f. of connected graphs of $\mathcal{A}$ is*

$$C(z) = \frac{T}{2} - \frac{3T^2}{4} + \frac{1}{2}\log\frac{1}{1-T}.$$

The generating function of graphs of $\mathcal{A}$ is $A(z) = e^{C(z)}$. As $n \to \infty$,

$$c_n \sim n!\frac{e^n}{4n} \quad \textit{and} \quad a_n \sim n!\,\frac{1}{(2e)^{1/4}\Gamma(1/4)}\frac{e^n}{n^{3/4}}. \tag{13}$$

In particular, the probability that $\mathcal{G}_n$ is connected tends to 0 at speed $n^{-1/4}$ as $n \to \infty$.

Proof: A connected graph of $\mathcal{A}$ is either an (unrooted) tree (with g.f. $T - T^2/2$), or consists of a cycle, in which each vertex is replaced by a rooted tree. The generating function of cycles is

$$Cyc(z) = \frac{1}{2}\sum_{n\geq 3}\frac{z^n}{n} = \frac{1}{2}\left(\log\frac{1}{1-z} - z - \frac{z^2}{2}\right), \tag{14}$$

and this gives the expression of $C(z)$. We then estimate c_n and a_n via *singularity analysis* [11, VI.4]. □

Proposition 12 1. *The mean and variance of N_n satisfy $\mathbb{E}(N_n) \sim \mathbb{V}(N_n) \sim \log n/4$, and the random variable $\frac{N_n - \log n/4}{\sqrt{\log n/4}}$ converges in law to a standard normal distribution.*
2. *For $i \geq 0$, the i^{th} moment of S_n satisfies, as $n \to \infty$,*

$$\mathbb{E}(S_n^i) \sim \frac{\Gamma(5/4)i!}{\Gamma(i+5/4)}n^i.$$

Consequently, the normalized variable S_n/n converges in distribution to a beta law, of density $(1-x)^{-3/4}/4$ on $[0,1]$. A local limit law also holds: for $x \in (0,1)$ and $k = \lfloor xn \rfloor$,

$$n\,\mathbb{P}(S_n = k) \sim \frac{1}{4}(1-x)^{-3/4}.$$

3. *The normalized variable L_n/n converges in law to the first component of a Poisson-Dirichlet distribution of parameter $1/4$.*

Proof: Once the singular expansion of $C(z)$ is obtained, the first result follows from [11, Prop. IX.14, p. 670]. To study the moments of S_n, we apply (3). Using $T(z) = ze^{T(z)}$, we obtain, for $i \geq 1$,

$$C^{(i+1)}(z) \sim \frac{i!}{4}\left(\frac{e}{1-ze}\right)^{i+1}.$$

The estimate of the i^{th} moment of S_n then follows again from singularity analysis. Since these moments characterize the above beta distribution, we conclude [11, Thm. C.2] that S_n/n converges in law to this distribution. For the local limit law, we start from (3), and use (13).

Finally, the third result follows from general results on *logarithmic structures* [2]. □

Remark. A subdominant term in $\sqrt{1-ze}$ occurs in the expansion of $C(z)$, but has no influence on the asymptotic results. They would be the same (with possibly different constants) for any $C(z)$ having a purely logarithmic singularity.

7 Excluding the bowtie: a singularity in $(1-z/\rho)^{-1/2}$

We now denote by $\mathcal{A}$ the class of graphs avoiding the bowtie (shown in Figure 1).

Proposition 13 *Let $T(z) = ze^{T(z)}$ be the g.f. of rooted trees. The g.f. of connected graphs in $\mathcal{A}$ is*

$$C(z) = \frac{T^2\left(1-T+T^2\right)\mathrm{e}^T}{1-T} + \frac{1}{2}\log\left(\frac{1}{1-T}\right) + \frac{T\left(12-54\,T+18\,T^2-5\,T^3-T^4\right)}{24(1-T)}.$$

As $n \to \infty$,

$$c_n \sim n!\,\frac{e-5/4}{\sqrt{2\pi}}\frac{e^n}{\sqrt{n}} \quad \textit{and} \quad a_n \sim n!\,\frac{(e-5/4)^{1/6}e^{19/8-11e/3}}{\sqrt{6\pi}}\frac{e^n}{n^{2/3}}\,e^{3(e-5/4)^{2/3}n^{1/3}/2}.$$

Proof: This is the most delicate enumeration result of the paper. We have

$$C(z) = T(z) - T(z)^2/2 + \bar{C}(T(z)),$$

where $\bar{C}(z)$ counts graphs with minimal degree 2 avoiding the bowtie. After studying the properties of these graphs, we conclude that they are either cycles, or K_4 with one edge possibly replaced by a chain of vertices of degree 2, or the graphs of Figure 2. Counting these various classes gives $\bar{C}(z)$, and thus $C(z)$. □

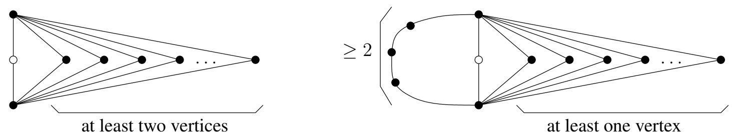


Fig. 2: Some graphs avoiding the bowtie. The white vertex is optional.

Proposition 14 1. *The mean and variance of N_n satisfy:*

$$\mathbb{E}(N_n) \sim (e-5/4)^{2/3} n^{1/3}, \qquad \mathbb{V}(N_n) \sim \frac{2}{3}(e-5/4)^{2/3} n^{1/3},$$

and the random variable $\frac{N_n - \mathbb{E}(N_n)}{\sqrt{\mathbb{V}(N_n)}}$ *converges in law to a standard normal distribution.*
2. *For $i \geq 0$, the i^{th} moment of S_n satisfies, as $n \to \infty$,*

$$\mathbb{E}(S_n^i) \sim \frac{\Gamma(i+3/2)}{\Gamma(3/2)} \left(\frac{2n^{2/3}}{(e-5/4)^{2/3}} \right)^i .$$

Consequently, the normalized variable $(e-5/4)^{2/3} S_n/(2n^{2/3})$ converges in distribution to a gamma$(3/2)$ *law, of density $2\sqrt{x}e^{-x}/\sqrt{\pi}$ on $[0,\infty)$. A local limit law also holds: for $x > 0$ and $k = \lfloor x \frac{2n^{2/3}}{(e-5/4)^{2/3}} \rfloor$,*

$$\frac{2n^{2/3}}{(e-5/4)^{2/3}}\, \mathbb{P}(S_n = k) \sim 2\sqrt{\frac{x}{\pi}} e^{-x}.$$

8 When trees dominate: $C(z)$ converges at ρ

Let $\mathcal{A}$ be a decomposable class of graphs (for instance, a class defined by excluding *connected* minors) with set of components $\mathcal{C}$. Assume that $\mathcal{C}$ contains all trees (counted by $T - T^2/2$), and that

$$C(z) = T(z) - T(z)^2/2 + D(z), \tag{15}$$

where $D(z)$ has radius strictly larger than $1/e$ (the radius of T). We say that $\mathcal{A}$ is *dominated by trees*. Some examples are presented below. In this case, the properties that hold for forests (Section 3) still hold, except that the limit of c_n/a_n is now smaller than $1/\sqrt{e}$.

Proposition 15 *Assume $\mathcal{A}$ is dominated by trees. As $n \to \infty$,*

$$c_n \sim n! \frac{e^n}{\sqrt{2\pi} n^{5/2}} \quad \text{and} \quad a_n \sim A(1/e) c_n.$$

In particular, the probability that $\mathcal{G}_n$ is connected tends to $1/A(1/e) = e^{-1/2-D(1/e)}$.
The asymptotic behaviours of N_n, L_n and S_n are described by Proposition 2, with $\rho = 1/e$.

Proof: The asymptotic behaviours of c_n and a_n are obtained via singularity analysis. For N_n, we can either start from (1) and apply singularity analysis, or use directly [4, Thm. 2]. For S_n, the two ingredients used in the proof of Proposition 2, namely smoothness of $a_n/n!$ and convergence of c_n/a_n, still hold here. For L_n, we use the fact that the root component is with high probability the biggest one. □

We now give examples where trees dominate.

Proposition 16 *Let $k \geq 1$. Let $\mathcal{A}$ be a minor-closed class of graphs containing all trees, but not the k-spoon (shown in Figure 1). Then $\mathcal{A}$ is dominated by trees, and the results of Proposition 15 apply.*

Proof: We partition the set $\mathcal{C}$ of connected graphs of $\mathcal{A}$ into three subsets: the set of trees, counted by $T - T^2/2$, the set $\mathcal{C}_1$ of unicyclic graphs (counted by C_1), and finally the set $\mathcal{C}_2$ containing graphs with at least two cycles (counted by C_2). We prove that C_1 has radius strictly larger than $1/e$, and that C_2 is entire. □

Proposition 17 *Let T_k be the g.f. of rooted trees of height less than k. That is, $T_k = ze^{T_{k-1}}$ with $T_1 = z$. Let $\mathcal{A}^{(k)}$ be the class of graphs avoiding the diamond, the bowtie and the k-spoon. Then (15) holds with*

$$D(z) \equiv D^{(k)}(z) = \frac{1}{2}\left(\log \frac{1}{1 - T_k(z)} - T_k(z) - \frac{T_k(z)^2}{2}\right).$$

The class $\mathcal{A}^{(k)}$ is dominated by trees, and Proposition 15 applies. In particular, the probability that a random graph of $\mathcal{A}_n^{(k)}$ is connected tends to $1/A^{(k)}(1/e)$ as $n \to \infty$. This value tends to 0 as k increases.

By specializing the proof of Proposition 13, we have also counted graphs avoiding the 2-spoon.

Proposition 18 *The g.f. of connected graphs avoiding the 2-spoon satisfies (15) with*

$$D(z) = \frac{1}{2}\left(\log \frac{1}{1 - ze^z} - ze^z - \frac{z^2 e^{2z}}{2}\right) + \frac{z^4}{4!} + z^2 e^{2z}\left(e^z - 1 - z - \frac{z^2}{4}\right).$$

9 Final comments

The nature of the dominant singularities of $C(z)$ is clearly a central parameter of the class, as the quantities N_n and S_n seem to depend largely on it (see Table 1). Is it possible to determine the nature of this singularity from the properties of the excluded minors? For instance, $C(\rho)$ is finite when all excluded minors are 2-connected, but Section 8 shows that this happens as well with non-2-connected minors. Which excluded minors give rise to a simple pole in $C(z)$? to a logarithmic singularity? to a singularity in $(1 - z/\rho)^{-1/2}$?

Other parameters. We have focussed on certain parameters that are well understood for 2-connected excluded minors. But others have been investigated in different contexts: the number of edges, the size of the largest 2-connected component, the distribution of vertex degrees [6, 8, 9, 12]. When specialized to the theory of minor-closed classes, these papers generally assume that all excluded minors are (at least) 2-connected. Including (some of) these parameters in our results may be the topic of future work.

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EL-labelings and canonical spanning trees for subword complexes

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Abstract. We describe edge labelings of the increasing flip graph of a subword complex on a finite Coxeter group, and study applications thereof. On the one hand, we show that they provide canonical spanning trees of the facet-ridge graph of the subword complex, describe inductively these trees, and present their close relations to greedy facets. Searching these trees yields an efficient algorithm to generate all facets of the subword complex, which extends the greedy flip algorithm for pointed pseudotriangulations. On the other hand, when the increasing flip graph is a Hasse diagram, we show that the edge labeling is indeed an EL-labeling and derive further combinatorial properties of paths in the increasing flip graph. These results apply in particular to Cambrian lattices, in which case a similar EL-labeling was recently studied by M. Kallipoliti and H. Mühle.

Résumé. Nous décrivons des étiquetages d'arêtes du graphe des flips croissants d'un complexe de sous-mots sur un groupe de Coxeter fini, et nous en étudions certaines applications. D'une part, nous montrons qu'ils fournissent des arbres couvrants canoniques du graphe des flips du complexe de sous-mots, nous décrivons inductivement ces arbres, et nous présentons leurs liens étroits avec les facettes gloutonnes du complexe. Le parcours de ces arbres permet d'engendrer efficacement les facettes du complexe des sous-mots, généralisant ainsi l'algorithme de flips gloutons pour les pseudotriangulations. D'autre part, lorsque le graphe des flips croissants est un diagramme de Hasse, nous montrons que notre étiquetage d'arêtes est lexicographique et nous en déduisons des propriétés supplémentaires du graphe des flips croissants. Ces résultats s'appliquent en particulier aux treillis Cambriens pour lesquels un étiquetage lexicographique similaire a été récemment étudié par M. Kallipoliti and H. Mühle.

Keywords: EL-labelings, subword complexes, spanning trees, exhaustive generation, Möbius function.

Subword complexes on Coxeter groups were defined and studied by A. Knutson and E. Miller in the context of Gröbner geometry of Schubert varieties [KM04, KM05]. Type A spherical subword complexes can be visually interpreted using pseudoline arrangements on primitive sorting networks. These were studied by V. Pilaud and M. Pocchiola [PP12] as combinatorial models for pointed pseudotriangulations of planar point sets [RSS08] and for multitriangulations of convex polygons [PS09]. These two families of geometric graphs extend in two different ways the family of triangulations of a convex polygon.

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The greedy flip algorithm was initially designed to generate all pointed pseudotriangulations of a given set of points or convex bodies in general position in the plane [PV96, BKPS06]. It was then extended in [PP12] to generate all pseudoline arrangements supported by a given primitive sorting network. The key step in this algorithm is to construct a spanning tree of the flip graph on the combinatorial objects, which has to be sufficiently canonical to be visited in polynomial time per node and polynomial working space.

In the present paper, we study natural edge lexicographic labelings of the increasing flip graph of a subword complex on any finite Coxeter group. As a first line of applications of these EL-labelings, we obtain canonical spanning trees of the flip graph of any subword complex. We provide alternative descriptions of these trees based on their close relations to greedy facets, which are defined and studied in this paper. Moreover, searching these trees provides an efficient algorithm to generate all facets of the subword complex. For type A spherical subword complexes, the resulting algorithm is precisely that of [PP12], although the presentation is quite different.

The second line of applications of the EL-labelings concerns combinatorial properties ensuing from EL-shellability [Bjö80, BW96]. Indeed, when the increasing flip graph is the Hasse diagram of the increasing flip poset, this poset is EL-shellable, and we can compute its Möbius function. These results extend recent work of M. Kallipoliti and H. Mühle [KM12] on EL-shellability of N. Reading's Cambrian lattices [Rea06, Rea07], which are, for finite Coxeter groups, increasing flip posets of specific subword complexes studied by C. Ceballos, J.-P. Labbé and C. Stump [CLS13] and by the authors in [PS11].

This extended abstract presents the results and the main ideas of the paper [PS12]. We refer the reader to this paper for further details, examples, and all proofs which we omit here due to limited space.

1 Edge labelings of graphs and posets

In [Bjö80], A. Björner introduced EL-labelings of partially ordered sets to study topological properties of their order complexes. These labelings are edge labelings of the Hasse diagrams of the posets with certain combinatorial properties. In this paper, we consider edge labelings of finite, acyclic, directed graphs which might differ from the Hasse diagrams of their transitive closures.

1.1 ER-labelings of graphs and associated spanning trees

Let $G := (V, E)$ be a finite, acyclic, directed graph. For $u, v \in V$, we write $u \to v$ if there is an edge from u to v in G, and $u \dashrightarrow v$ if there is a *path* $u = x_1 \to x_2 \to \cdots \to x_\ell \to x_{\ell+1} = v$ from u to v in G (this path has *length* ℓ). The *interval* $[u, v]$ in G is the set of vertices $w \in V$ such that $u \dashrightarrow w \dashrightarrow v$.

An *edge labeling* of G is a map $\lambda : E \to \mathbb{N}$. It induces a labeling of any path $p : x_1 \to x_2 \to \cdots \to x_\ell \to x_{\ell+1}$ given by $\lambda(p) := \lambda(x_1 \to x_2) \cdots \lambda(x_\ell \to x_{\ell+1})$. The path p is *λ-rising* (resp. *λ-falling*) if $\lambda(p)$ is strictly increasing (resp. weakly decreasing). The labeling λ is an *edge rising labeling* of G (or *ER-labeling* for short) if there is a unique λ-rising path p between any vertices $u, v \in V$ with $u \dashrightarrow v$.

Remark 1.1 (Spanning trees) *Let $u, v \in V$, and $\lambda : E \to \mathbb{N}$ be an ER-labeling of G. Then the union of all λ-rising paths from u to any other vertex of the interval $[u, v]$ forms a spanning tree of $[u, v]$, rooted at and directed away from u. We call it the* λ-source tree *of $[u, v]$ and denote it by $\mathcal{T}_\lambda([u, v])$. Similarly, the union of all λ-rising paths from any vertex of the interval $[u, v]$ to v forms a spanning tree of $[u, v]$, rooted at and directed towards v. We call it the* λ-sink tree *of $[u, v]$ and denote it by $\mathcal{T}^*_\lambda([u, v])$. In particular, if G has a unique source and a unique sink, this provides two canonical spanning trees $\mathcal{T}_\lambda(G)$ and $\mathcal{T}^*_\lambda(G)$ for the graph G itself.*

Example 1.2 (Cube) *Consider the* 1*-skeleton* $\square_d$ *of the* d*-dimensional cube* $[0,1]^d$*, directed from vertex* $\mathbf{0} := (0,\dots,0)$ *to vertex* $\mathbf{1} := (1,\dots,1)$*. Its vertices are the elements of* $\{0,1\}^d$ *and its edges are the pairs of vertices which differ in a unique position. Note that* $\varepsilon := (\varepsilon_1,\dots,\varepsilon_d) \dashrightarrow \varepsilon' := (\varepsilon'_1,\dots,\varepsilon'_d)$ *if and only if* $\varepsilon_k \le \varepsilon'_k$ *for all* $k \in [d]$*.*

For any edge $\varepsilon \to \varepsilon'$ *of* $\square_d$*, let* $\lambda(\varepsilon \to \varepsilon')$ *denote the unique position in* $[d]$ *where* ε *and* ε' *differ. Then the map* λ *is an ER-labeling of* $\square_d$*. If* $\varepsilon \in \{0,1\}^d \smallsetminus \mathbf{0}$*, then the father of* ε *in* $\mathcal{T}_\lambda(\square_d)$ *is obtained from* ε *by changing its last* 1 *into a* 0*. Similarly, if* $\varepsilon \in \{0,1\}^d \smallsetminus \mathbf{1}$*, then the father of* ε *in* $\mathcal{T}^*_\lambda(\square_d)$ *is obtained from* ε *by changing its first* 0 *into a* 1*. See Figure 1.*

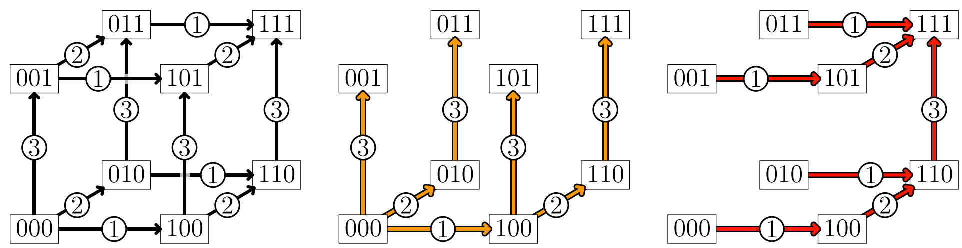

Fig. 1: An ER-labeling λ of the 1-skeleton $\square_3$ of the 3-cube, the λ-source tree $\mathcal{T}_\lambda(\square_3)$ and the λ-sink tree $\mathcal{T}^*_\lambda(\square_3)$.

1.2 EL-labelings of graphs and posets

Although ER-labelings of graphs are sufficient to produce canonical spanning trees, we need the following extension for further combinatorial properties. The labeling $\lambda : E \to \mathbb{N}$ is an *edge lexicographic labeling* of G (or *EL-labeling* for short) if for any vertices $u, v \in V$ with $u \dashrightarrow v$,

(i) there is a unique λ-rising path p from u to v, and

(ii) its labeling $\lambda(p)$ is lexicographically first among the labelings $\lambda(p')$ of all paths p' from u to v.

For example, the ER-labeling of the 1-skeleton of the cube $\square_d$ presented in Example 1.2 is in fact an EL-labeling. Remember now that one can associate a finite poset to a finite acyclic directed graph and *vice versa*. Namely,

(i) the *transitive closure* of a finite acyclic directed graph $G = (V, E)$ is the finite poset $(V, \dashrightarrow)$;

(ii) the *Hasse diagram* of a finite poset P is the finite acyclic directed graph whose vertices are the elements of P and whose edges are the *cover relations* in P, *i.e.* $u \to v$ if $u <_P v$ and there is no $w \in P$ such that $u <_P w <_P v$.

The transitive closure of the Hasse diagram of P always coincides with P, but the Hasse diagram of the transitive closure of G might as well be only a subgraph of G. An *EL-labeling* of the poset P is an EL-labeling of the Hasse diagram of P. If such a labeling exists, then the poset is called *EL-shellable*.

As already mentioned, A. Björner [Bjö80] originally introduced EL-labelings of finite posets to study topological properties of their order complex. In particular, they provide a tool to compute the Möbius function of the poset. Recall that the *Möbius function* of the poset P is the map $\mu : P \times P \to \mathbb{Z}$ defined recursively by

$$\mu(u,v) := \begin{cases} 1 & \text{if } u = v, \\ -\sum_{u \leq_P w <_P v} \mu(u,w) & \text{if } u <_P v, \\ 0 & \text{otherwise.} \end{cases}$$

When the poset is EL-shellable, this function can be computed as follows.

Proposition 1.3 ([BW96, Proposition 5.7]) *Let λ be an EL-labeling of the poset P. For every $u, v \in P$ with $u \leq_P v$, we have*

$$\mu(u,v) = \mathrm{even}_\lambda(u,v) - \mathrm{odd}_\lambda(u,v),$$

where $\mathrm{even}_\lambda(u,v)$ (resp. $\mathrm{odd}_\lambda(u,v)$) denotes the number of even (resp. odd) length λ-falling paths from u to v in the Hasse diagram of P.

Example 1.4 (Cube) *The directed 1-skeleton $\square_d$ of the d-dimensional cube $[0,1]^d$ is the Hasse diagram of the boolean poset. The edge labeling λ of $\square_d$ of Example 1.2 is thus an EL-labeling of the boolean poset. Moreover, for any two vertices $\varepsilon \dashrightarrow \varepsilon'$ of $\square_d$, there is a unique λ-falling path between ε and ε', whose length is the* Hamming distance *$\delta(\varepsilon, \varepsilon') := |\{k \in [d] \mid \varepsilon_k \neq \varepsilon'_k\}|$. The Möbius function is thus given by $\mu(\varepsilon, \varepsilon') = (-1)^{\delta(\varepsilon,\varepsilon')}$ if $\varepsilon \dashrightarrow \varepsilon'$ and $\mu(\varepsilon, \varepsilon') = 0$ otherwise. In particular, $\mu(\mathbf{0}, \mathbf{1}) = (-1)^d$.*

2 Subword complexes on Coxeter groups

2.1 Subword complexes

We consider a finite Coxeter system (W, S), with root system Φ and simple roots $\{\alpha_1, \dots, \alpha_n\}$. We fix a word $\mathrm{Q} := \mathrm{q}_1\mathrm{q}_2 \cdots \mathrm{q}_m$ on the generators of S, and an element $\rho \in W$. Background on Coxeter groups can be found in [Hum90].

A. Knutson and E. Miller [KM04] define the *subword complex* $\mathcal{SC}(\mathrm{Q}, \rho)$ to be the simplicial complex of those subwords of Q whose complements contain a reduced expression for ρ as a subword. A vertex of $\mathcal{SC}(\mathrm{Q}, \rho)$ is a position of a letter in Q. We denote by $[m] := \{1, 2, \dots, m\}$ the set of positions in Q. A facet of $\mathcal{SC}(\mathrm{Q}, \rho)$ is the complement of a set of positions which forms a reduced expression for ρ in Q. We denote by $\mathcal{F}(\mathrm{Q}, \rho)$ the set of facets of $\mathcal{SC}(\mathrm{Q}, \rho)$.

Example 2.1 *Consider the type A Coxeter group $\mathfrak{S}_4$ generated by $\{\tau_1, \tau_2, \tau_3\}$, where $\tau_i := (i \;\; i+1)$. Consider $\mathrm{Q}^{\mathrm{ex}} := \tau_2\tau_3\tau_1\tau_3\tau_2\tau_1\tau_2\tau_3\tau_1$ and $\rho^{\mathrm{ex}} := [4,1,3,2]$. The reduced expressions of ρ^{ex} are $\tau_2\tau_3\tau_2\tau_1$, $\tau_3\tau_2\tau_3\tau_1$, and $\tau_3\tau_2\tau_1\tau_3$. Therefore, the facets of the subword complex $\mathcal{SC}(\mathrm{Q}^{\mathrm{ex}}, \rho^{\mathrm{ex}})$ are $\{1,2,3,5,6\}$, $\{1,2,3,6,7\}$, $\{1,2,3,7,9\}$, $\{1,3,4,5,6\}$, $\{1,3,4,6,7\}$, $\{1,3,4,7,9\}$, $\{2,3,5,6,8\}$, $\{2,3,6,7,8\}$, $\{2,3,7,8,9\}$, $\{3,4,5,6,8\}$, $\{3,4,6,7,8\}$, and $\{3,4,7,8,9\}$. We will use this example throughout this paper to illustrate further notions.*

Remark 2.2 *There is a natural reversal operation on subword complexes. Namely,*

$$\mathcal{SC}(\mathrm{q}_m \cdots \mathrm{q}_1, \rho^{-1}) = \{\{m+1-i \mid i \in I\} \mid I \in \mathcal{SC}(\mathrm{q}_1 \cdots \mathrm{q}_m, \rho)\}.$$

We will use this operation to relate positive and negative labelings, facets and trees.

2.2 Inductive structure

We denote by $\mathrm{Q}_{\vdash} := \mathrm{q}_2 \cdots \mathrm{q}_m$ and $\mathrm{Q}_{\dashv} := \mathrm{q}_1 \cdots \mathrm{q}_{m-1}$ the words on S obtained from $\mathrm{Q} := \mathrm{q}_1 \cdots \mathrm{q}_m$ by deleting its first and last letters respectively. We denote by $\mathcal{X} \star z := \{X \cup z \mid X \in \mathcal{X}\}$ the join of a collection $\mathcal{X}$ of subsets of $\mathbb{Z}$ with an element $z \in \mathbb{Z}$. We let $\ell(\rho)$ denote the length of $\rho \in W$ and we write $\rho \prec \mathrm{Q}$ when Q contains a reduced expression of ρ, ***i.e.*** when $\mathcal{SC}(\mathrm{Q}, \rho)$ is non-empty.

We can decompose inductively the facets of the subword complex $\mathcal{SC}(\mathrm{Q}, \rho)$ according on whether or not they contain the last letter of Q. Denoting by ε the empty word and by e the identity of W, we have $\mathcal{F}(\varepsilon, e) = \{\emptyset\}$ and $\mathcal{F}(\varepsilon, \rho) = \emptyset$ if $\rho \neq e$. For a non-empty word Q on S, the set $\mathcal{F}(\mathrm{Q}, \rho)$ is given by

(i) $\mathcal{F}(\mathrm{Q}_{\dashv}, \rho q_m)$ if m appears in none of the facets of $\mathcal{SC}(\mathrm{Q}, \rho)$ (equivalently if $\rho \not\prec \mathrm{Q}_{\dashv}$);

(ii) $\mathcal{F}(\mathrm{Q}_{\dashv}, \rho) \star m$ if m appears in all the facets of $\mathcal{SC}(\mathrm{Q}, \rho)$ (equivalently if $\ell(\rho q_m) > \ell(\rho)$);

(iii) $\mathcal{F}(\mathrm{Q}_{\dashv}, \rho q_m) \sqcup \big(\mathcal{F}(\mathrm{Q}_{\dashv}, \rho) \star m\big)$ otherwise.

By reversal (see Remark 2.2), there is also a similar inductive decomposition of the facets of the subword complex $\mathcal{SC}(\mathrm{Q}, \rho)$ according on whether or not they contain the first letter of Q. Although we will only use these decompositions for the facets $\mathcal{F}(\mathrm{Q}, \rho)$, they extend to the whole subword complex $\mathcal{SC}(\mathrm{Q}, \rho)$ and are used to obtain the following result.

Theorem 2.3 ([KM04, Corollary 3.8]) *The subword complex $\mathcal{SC}(\mathrm{Q}, \rho)$ is either a simplicial sphere or a simplicial ball.*

2.3 Flips and roots

Let I be a facet of $\mathcal{SC}(\mathrm{Q}, \rho)$ and i be a position in I. If there exists a facet J of $\mathcal{SC}(\mathrm{Q}, \rho)$ and a position $j \in J$ such that $I \smallsetminus i = J \smallsetminus j$, we say that I and J are *adjacent* facets, that i is *flippable* in I, and that J is obtained from I by *flipping* i. Note that, if i is flippable, then J and j are unique by Theorem 2.3. We denote by $\mathcal{G}(\mathrm{Q}, \rho)$ the graph of flips, whose vertices are the facets of $\mathcal{SC}(\mathrm{Q}, \rho)$ and whose edges are pairs of adjacent facets. That is, $\mathcal{G}(\mathrm{Q}, \rho)$ is the ridge graph of the simplicial complex $\mathcal{SC}(\mathrm{Q}, \rho)$. This graph is connected by Theorem 2.3. It can moreover be naturally oriented by the direction of the flips as follows. Let I and J be two adjacent facets of $\mathcal{SC}(\mathrm{Q}, \rho)$ with $I \smallsetminus i = J \smallsetminus j$. We say that the flip from I to J is *increasing* if $i < j$. We then orient the corresponding edge of $\mathcal{G}(\mathrm{Q}, \rho)$ from I to J.

Example 2.4 *Figure 2 shows the increasing flip graph $\mathcal{G}(\mathrm{Q}^{\mathrm{ex}}, \rho^{\mathrm{ex}})$ for the subword complex $\mathcal{SC}(\mathrm{Q}^{\mathrm{ex}}, \rho^{\mathrm{ex}})$ of Example 2.1. The facets of $\mathcal{SC}(\mathrm{Q}^{\mathrm{ex}}, \rho^{\mathrm{ex}})$ appear in lexicographic order from left to right.*

Remark 2.5 *The increasing flip graph of $\mathcal{SC}(\mathrm{Q}, \rho)$ was already considered by A. Knutson and E. Miller [KM04, Remark 4.5]. It carries various combinatorial informations about the subword complex $\mathcal{SC}(\mathrm{Q}, \rho)$. In particular, since the lexicographic order on the facets of $\mathcal{SC}(\mathrm{Q}, \rho)$ is a shelling order for $\mathcal{SC}(\mathrm{Q}, \rho)$, the h-vector of the subword complex $\mathcal{SC}(\mathrm{Q}, \rho)$ is the in-degree sequence of the increasing flip graph $\mathcal{G}(\mathrm{Q}, \rho)$.*

We consider flips as elementary operations on subword complexes. In practice, the necessary information to perform flips in a facet I of $\mathcal{SC}(\mathrm{Q}, \rho)$ is encoded in its *root function* $\mathsf{r}(I, \cdot) : [m] \to \Phi$ defined by

$$\mathsf{r}(I, k) := \Pi\mathrm{Q}_{[k-1] \smallsetminus I}(\alpha_{q_k}),$$

where $\Pi\mathrm{Q}_X$ denotes the product of the reflections $q_x \in \mathrm{Q}$ for $x \in X$. The root function was introduced by C. Ceballos, J.-P. Labbé and C. Stump [CLS13] and its main properties can be found in [CLS13, Lemmas

3.3 and 3.6]. Essentially, an element i of a facet I is flippable if and only if $\mathsf{r}(I,i) \in \{\pm\beta \mid \beta \in \mathrm{inv}(\rho)\}$, and then i flips to the unique position $j \notin I$ such that $\mathsf{r}(I,j) \in \{\pm\mathsf{r}(I,i)\}$. Moreover, $\mathsf{r}(I,i) = \mathsf{r}(I,j) \in \Phi^+$ if $i < j$ (increasing flip), while $\mathsf{r}(I,i) = -\mathsf{r}(I,j) \in \Phi^-$ if $i > j$ (decreasing flip). The *root configuration* of the facet I is the multiset $\mathsf{R}(I) := \{\!\{\mathsf{r}(I,i) \mid i \text{ flippable in } I\}\!\}$. We extensively studied root configurations in [PS11] in the construction of brick polytopes for spherical subword complexes.

3 EL-labelings and spanning trees for the subword complex

3.1 EL-labelings of the increasing flip graph

We now define two natural edge labelings of the increasing flip graph $\mathcal{G}(\mathrm{Q},\rho)$. Let I and J be two adjacent facets of $\mathcal{SC}(\mathrm{Q},\rho)$, with $I \smallsetminus i = J \smallsetminus j$ and $i < j$. We label the edge $I \to J$ of $\mathcal{G}(\mathrm{Q},\rho)$ with the positive edge label $\mathsf{p}(I \to J) := i$ and with the negative edge label $\mathsf{n}(I \to J) := j$. We call $\mathsf{p} : E(\mathcal{G}(\mathrm{Q},\rho)) \to [m]$ the *positive edge labeling* and $\mathsf{n} : E(\mathcal{G}(\mathrm{Q},\rho)) \to [m]$ the *negative edge labeling* of the increasing flip graph $\mathcal{G}(\mathrm{Q},\rho)$. The terms “positive” and “negative” emphasize the fact that the roots $\mathsf{r}(I,\mathsf{p}(I \to J))$ and $\mathsf{r}(J,\mathsf{n}(I \to J))$ are always positive and negative roots respectively. The positive and negative edge labelings are clearly reverse to one another (see Remark 2.2).

Example 3.1 *Consider the subword complex $\mathcal{SC}(\mathrm{Q}^{\mathrm{ex}},\rho^{\mathrm{ex}})$ of Example 2.1. We have represented on Figure 2 the positive and negative edge labelings* p *and* n. *Since we have represented the graph $\mathcal{G}(\mathrm{Q}^{\mathrm{ex}},\rho^{\mathrm{ex}})$ such that the flips are increasing from left to right, each edge has its positive label on the left and its negative label on the right.*

Our main result concerns the positive and negative edge labelings of the increasing flip graph.

Theorem 3.2 *The positive edge labeling* p *and the negative edge labeling* n *are both EL-labelings of the increasing flip graph.*

For Cambrian lattices, whose Hasse diagrams were shown to be particular cases of increasing flip graphs in [PS11, Section 6], a similar result was recently obtained by M. Kallipoliti and H. Mühle in [KM12].

In Sections 3.2 and 3.3, we present applications of Theorem 3.2 to the construction of canonical spanning trees and to the generation of the facets of the subword complex. Further combinatorial applications of this theorem are also discussed in Section 4.

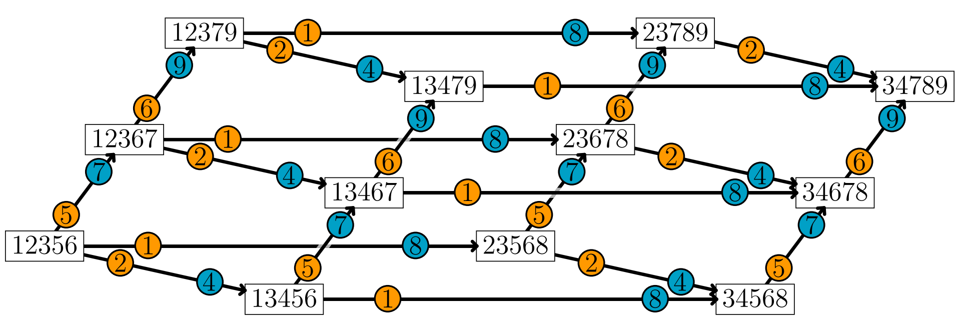

Fig. 2: The positive and negative edge labelings p and n of $\mathcal{G}(\mathrm{Q}^{\mathrm{ex}},\rho^{\mathrm{ex}})$. Each edge has its positive label on the left (orange) and its negative label on the right (blue).

Before going on, we want to give a very brief idea of the proof of Theorem 3.2. We refer the interested reader to the complete proof in [PS12]. To prove the existence of a p-rising path between any two comparable facets of $\mathcal{SC}(\mathrm{Q}, \rho)$, we use a procedure which improves locally a path by restriction of $\mathcal{SC}(\mathrm{Q}, \rho)$ to a dihedral parabolic subgroup. The uniqueness and lexicographic property of the p-rising are then obtained from the following proposition.

Proposition 3.3 *Let $I_1 \to \cdots \to I_{\ell+1}$ be a path of increasing flips, and define $\mathsf{p}_k := \mathsf{p}(I_k \to I_{k+1})$ and $\mathsf{n}_k := \mathsf{n}(I_k \to I_{k+1})$. Then, for all $k \in [\ell]$, we have*

$$\min\{\mathsf{p}_k, \dots, \mathsf{p}_\ell\} = \min(I_k \smallsetminus I_{\ell+1}) \quad \textit{and} \quad \max\{\mathsf{n}_1, \dots, \mathsf{n}_k\} = \max(I_{k+1} \smallsetminus I_1).$$

Moreover, the path is p-rising if and only if $\mathsf{p}_k = \min(I_k \smallsetminus I_{\ell+1})$ for all $k \in [\ell]$, while the path is n-rising if and only if $\mathsf{n}_k = \max(I_{k+1} \smallsetminus I_1)$ for all $k \in [\ell]$.

3.2 Greedy facets

We now characterize the unique source and sink of the increasing flip graph $\mathcal{G}(\mathrm{Q}, \rho)$.

Proposition 3.4 *The lexicographically smallest (resp. largest) facet of $\mathcal{SC}(\mathrm{Q}, \rho)$ is the unique source (resp. sink) of $\mathcal{G}(\mathrm{Q}, \rho)$.*

We call *positive* (resp. *negative*) *greedy facet* and denote by $\mathsf{P}(\mathrm{Q}, \rho)$ (resp. $\mathsf{N}(\mathrm{Q}, \rho)$) the unique source (resp. sink) of the graph $\mathcal{G}(\mathrm{Q}, \rho)$ of increasing flips. The term "positive" (resp. "negative") emphasizes the fact that $\mathsf{P}(\mathrm{Q}, \rho)$ (resp. $\mathsf{N}(\mathrm{Q}, \rho)$) is the unique facet of $\mathcal{SC}(\mathrm{Q}, \rho)$ whose root configuration is a subset of positive (resp. negative) roots, while the term "greedy" refers to the greedy properties of these facets. The greedy facets $\mathsf{P}(\mathrm{Q}, \rho)$ and $\mathsf{N}(\mathrm{Q}, \rho)$ are reverse to one another (see Remark 2.2). Namely, $\mathsf{N}(\mathrm{q}_m \cdots \mathrm{q}_1, \rho^{-1}) = \{m + 1 - p \mid p \in \mathsf{P}(\mathrm{q}_1 \cdots \mathrm{q}_m, \rho)\}$.

Example 3.5 *The positive and negative greedy facets of the subword complex $\mathcal{SC}(\mathrm{Q}^{\mathrm{ex}}, \rho^{\mathrm{ex}})$ presented in Example 2.1 are respectively $\mathsf{P}(\mathrm{Q}^{\mathrm{ex}}, \rho^{\mathrm{ex}}) = \{1, 2, 3, 5, 6\}$ and $\mathsf{N}(\mathrm{Q}^{\mathrm{ex}}, \rho^{\mathrm{ex}}) = \{3, 4, 7, 8, 9\}$. They appear respectively as the leftmost and rightmost facets in Figure 2.*

We have seen in Theorem 3.2 that for any two facets $I, J \in \mathcal{F}(\mathrm{Q}, \rho)$ such that $I \dashrightarrow J$, there is a p-rising (resp. n-rising) path from I to J. In particular, there is always a p-rising (resp. n-rising) path from $\mathsf{P}(\mathrm{Q}, \rho)$ to $\mathsf{N}(\mathrm{Q}, \rho)$. It turns out that there is also at least one p-falling (resp. n-falling) path from $\mathsf{P}(\mathrm{Q}, \rho)$ to $\mathsf{N}(\mathrm{Q}, \rho)$ if the subword complex $\mathcal{SC}(\mathrm{Q}, \rho)$ is spherical.

Proposition 3.6 *For any spherical subword complex $\mathcal{SC}(\mathrm{Q}, \rho)$, there is always a p-falling and an n-falling path from $\mathsf{P}(\mathrm{Q}, \rho)$ to $\mathsf{N}(\mathrm{Q}, \rho)$.*

Note that this proposition fails if we drop the condition that $\mathcal{SC}(\mathrm{Q}, \rho)$ is spherical, as illustrated in the subword complex $\mathcal{SC}(\mathrm{Q}^{\mathrm{ex}}, \rho^{\mathrm{ex}})$ of Example 2.1.

3.3 Spanning trees

As discussed in Remark 1.1, the edge labelings p and n automatically produce canonical spanning trees of any interval of the increasing flip graph $\mathcal{G}(\mathrm{Q}, \rho)$. Since $\mathcal{G}(\mathrm{Q}, \rho)$ has a unique source $\mathsf{P}(\mathrm{Q}, \rho)$ and a unique sink $\mathsf{N}(\mathrm{Q}, \rho)$, we obtain in particular four spanning trees of the graph $\mathcal{G}(\mathrm{Q}, \rho)$ itself. The goal of this section is to give alternative descriptions of these four spanning trees.

We call respectively *positive source tree*, *positive sink tree*, *negative source tree*, and *negative sink tree*, and denote respectively by $\mathcal{P}(\mathrm{Q},\rho)$, $\mathcal{P}^*(\mathrm{Q},\rho)$, $\mathcal{N}(\mathrm{Q},\rho)$, and $\mathcal{N}^*(\mathrm{Q},\rho)$, the p-source, p-sink, n-source, and n-sink trees of $\mathcal{G}(\mathrm{Q},\rho)$. The tree $\mathcal{P}(\mathrm{Q},\rho)$ (resp. $\mathcal{N}(\mathrm{Q},\rho)$) is formed by all p-rising (resp. n-rising) paths from the positive greedy facet $\mathsf{P}(\mathrm{Q},\rho)$ to all the facets of $\mathcal{SC}(\mathrm{Q},\rho)$. Both $\mathcal{P}(\mathrm{Q},\rho)$ and $\mathcal{N}(\mathrm{Q},\rho)$ are rooted at and directed away from the positive greedy facet $\mathsf{P}(\mathrm{Q},\rho)$. The tree $\mathcal{P}^*(\mathrm{Q},\rho)$ (resp. $\mathcal{N}^*(\mathrm{Q},\rho)$) is formed by all p-rising (resp. n-rising) paths from all the facets of $\mathcal{SC}(\mathrm{Q},\rho)$ to the negative greedy facet $\mathsf{N}(\mathrm{Q},\rho)$. Both $\mathcal{P}^*(\mathrm{Q},\rho)$ and $\mathcal{N}^*(\mathrm{Q},\rho)$ are rooted at and directed towards the negative greedy facet $\mathsf{N}(\mathrm{Q},\rho)$. Note that the positive source and negative sink trees (resp. the positive sink and the negative source trees) are reverse to one another (see Remark 2.2).

Example 3.7 *Consider the subword complex $\mathcal{SC}(\mathrm{Q}^{\mathrm{ex}},\rho^{\mathrm{ex}})$ of Example 2.1. Figure 3 represents the trees $\mathcal{P}(\mathrm{Q}^{\mathrm{ex}},\rho^{\mathrm{ex}})$, $\mathcal{P}^*(\mathrm{Q}^{\mathrm{ex}},\rho^{\mathrm{ex}})$, $\mathcal{N}(\mathrm{Q}^{\mathrm{ex}},\rho^{\mathrm{ex}})$, and $\mathcal{N}^*(\mathrm{Q}^{\mathrm{ex}},\rho^{\mathrm{ex}})$. Observe that these four canonical spanning trees of $\mathcal{G}(\mathrm{Q},\rho)$ are all different in general.*

We now give a direct description of the father of a facet I in $\mathcal{P}^*(\mathrm{Q},\rho)$ and $\mathcal{N}(\mathrm{Q},\rho)$.

Proposition 3.8 *Let I be a facet of $\mathcal{SC}(\mathrm{Q},\rho)$. If $I \neq \mathsf{N}(\mathrm{Q},\rho)$, then the father of I in $\mathcal{P}^*(\mathrm{Q},\rho)$ is obtained from I by flipping the smallest position in $I \smallsetminus \mathsf{N}(\mathrm{Q},\rho)$. Similarly, if $I \neq \mathsf{P}(\mathrm{Q},\rho)$, then the father of I in $\mathcal{N}(\mathrm{Q},\rho)$ is obtained from I by flipping the largest position in $I \smallsetminus \mathsf{P}(\mathrm{Q},\rho)$.*

We now focus on the positive source tree $\mathcal{P}(\mathrm{Q},\rho)$ and on the negative sink tree $\mathcal{N}^*(\mathrm{Q},\rho)$, and provide two different descriptions of them. The first is an inductive description of $\mathcal{P}(\mathrm{Q},\rho)$ and $\mathcal{N}^*(\mathrm{Q},\rho)$ (see Proposition 3.10). The second is a direct description of the father of a facet I in $\mathcal{P}(\mathrm{Q},\rho)$ and $\mathcal{N}^*(\mathrm{Q},\rho)$ in terms of greedy prefixes and suffixes of I (see Proposition 3.11). These descriptions mainly rely on the following property of the greedy facets.

Proposition 3.9 *If m is a flippable position of $\mathsf{N}(\mathrm{Q},\rho)$, then $\mathsf{N}(\mathrm{Q}_{\dashv},\rho q_m)$ is obtained from $\mathsf{N}(\mathrm{Q},\rho)$ by flipping m. Similarly, if 1 is a flippable position of $\mathsf{P}(\mathrm{Q},\rho)$, then $\mathsf{P}(\mathrm{Q}_{\vdash},q_1\rho)$ is obtained from $\mathsf{P}(\mathrm{Q},\rho)$ by flipping 1 and shifting to the left.*

Using Proposition 3.9, we can describe inductively the two trees $\mathcal{P}(\mathrm{Q},\rho)$ and $\mathcal{N}^*(\mathrm{Q},\rho)$. The induction follows the induction formulas for the facets $\mathcal{F}(\mathrm{Q},\rho)$ presented in Section 2.2. Remember that we denote the deletion of the first or last letter in $\mathrm{Q} := \mathrm{q}_1\cdots\mathrm{q}_m$ by $\mathrm{Q}_{\vdash} := \mathrm{q}_2\cdots\mathrm{q}_m$ and $\mathrm{Q}_{\dashv} := \mathrm{q}_1\cdots\mathrm{q}_{m-1}$ respectively. For a tree $\mathcal{T}$ whose vertices are subsets of $\mathbb{Z}$ and for an element $z\in\mathbb{Z}$, we denote by $\mathcal{T}\star z$ the tree with a vertex $X\cup z$ for each vertex X of $\mathcal{T}$ and an edge $X\cup z \to Y\cup z$ for each edge $X\to Y$ of $\mathcal{T}$.

The inductive description of the negative sink tree $\mathcal{N}^*(\mathrm{Q},\rho)$ is based on the right induction formula. For the empty word ε, the tree $\mathcal{N}^*(\varepsilon,e)$ is formed by the unique facet $\emptyset$ of $\mathcal{SC}(\varepsilon,e)$, and the tree $\mathcal{N}^*(\varepsilon,\rho)$ is empty if $\rho\neq e$. Otherwise, $\mathcal{N}^*(\mathrm{Q},\rho)$ is obtained as follows.

Proposition 3.10 *For a non-empty word Q, the tree $\mathcal{N}^*(\mathrm{Q},\rho)$ equals*

(i) $\mathcal{N}^*(\mathrm{Q}_{\dashv},\rho q_m)$ *if m appears in none of the facets of $\mathcal{SC}(\mathrm{Q},\rho)$;*

(ii) $\mathcal{N}^*(\mathrm{Q}_{\dashv},\rho)\star m$ *if m appears in all the facets of $\mathcal{SC}(\mathrm{Q},\rho)$;*

(iii) the disjoint union of $\mathcal{N}^*(\mathrm{Q}_{\dashv},\rho q_m)$ and $\mathcal{N}^*(\mathrm{Q}_{\dashv},\rho)\star m$, with an additional edge from $\mathsf{N}(\mathrm{Q}_{\dashv},\rho q_m)$ to $\mathsf{N}(\mathrm{Q},\rho) = \mathsf{N}(\mathrm{Q}_{\dashv},\rho)\cup m$, otherwise.

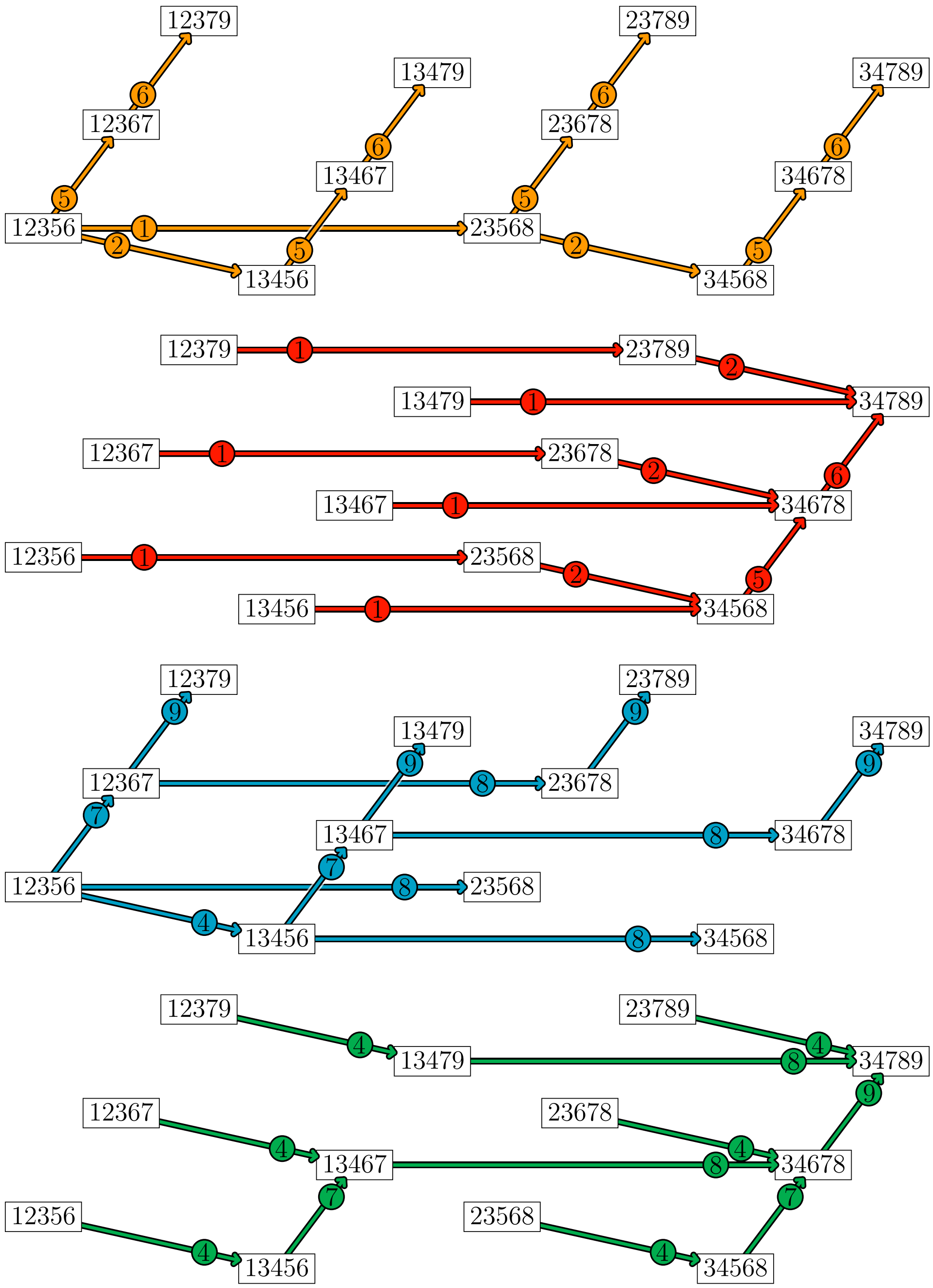

Fig. 3: The positive source tree $\mathcal{P}(\mathrm{Q}^{\mathrm{ex}}, \rho^{\mathrm{ex}})$, the positive sink tree $\mathcal{P}^*(\mathrm{Q}^{\mathrm{ex}}, \rho^{\mathrm{ex}})$, the negative source tree $\mathcal{N}(\mathrm{Q}^{\mathrm{ex}}, \rho^{\mathrm{ex}})$, and the negative sink tree $\mathcal{N}^*(\mathrm{Q}^{\mathrm{ex}}, \rho^{\mathrm{ex}})$ of the subword complex $\mathcal{SC}(\mathrm{Q}^{\mathrm{ex}}, \rho^{\mathrm{ex}})$ of Example 2.1.

A similar inductive description of the positive source tree $\mathcal{P}(\mathrm{Q},\rho)$ can be obtained from the left induction formula. See [PS12].

We now give a direct characterization of the father of a facet I of $\mathcal{SC}(\mathrm{Q},\rho)$ in the positive source and negative sink trees $\mathcal{P}(\mathrm{Q},\rho)$ and $\mathcal{N}^*(\mathrm{Q},\rho)$. This description can be understood in terms of the longest greedy prefix or suffix of I.

Proposition 3.11 *Let I be a facet of $\mathcal{SC}(\mathrm{Q},\rho)$. If $I \neq \mathsf{N}(\mathrm{Q},\rho)$, then the father of I in $\mathcal{N}^*(\mathrm{Q},\rho)$ is obtained from I by flipping the smallest position $x \in [m]$ such that $I \cap [x] \neq \mathsf{N}(\mathrm{q}_1 \cdots \mathrm{q}_x, \Pi\mathrm{Q}_{[x]\smallsetminus I})$. Similarly, if $I \neq \mathsf{P}(\mathrm{Q},\rho)$, then the father of I in $\mathcal{P}(\mathrm{Q},\rho)$ is obtained from I by flipping the largest position $x \in [m]$ such that $\{i - x \mid i \in I \smallsetminus [x]\} \neq \mathsf{P}(\mathrm{q}_{x+1} \cdots \mathrm{q}_m, \Pi\mathrm{Q}_{[x+1,m]\smallsetminus I})$.*

3.4 Greedy flip algorithm

The initial motivation of this paper was to find efficient algorithms for the exhaustive generation of the set $\mathcal{F}(\mathrm{Q},\rho)$ of facets of the subword complex $\mathcal{SC}(\mathrm{Q},\rho)$. The properties of the subword complex described in Sections 2.2 and 2.3 already provide two immediate enumeration algorithms. First, the inductive structure of $\mathcal{F}(\mathrm{Q},\rho)$ yields an *inductive algorithm* whose running time per facet is polynomial. The second option is an *exploration of the flip graph* $\mathcal{G}(\mathrm{Q},\rho)$, whose running time is still polynomial per facet. The problem of a naive exploration is that we would need to store all facets of $\mathcal{F}(\mathrm{Q},\rho)$ during the algorithm, which may require an exponential working space. Using the canonical spanning trees constructed in this paper, we can bypass this difficulty: we avoid to store all visited facets while preserving the same running time. The *greedy flip algorithm* generates all facets of the subword complex $\mathcal{SC}(\mathrm{Q},\rho)$ by a depth first search procedure on one of the four canonical spanning trees described in Section 3.3. The preorder traversal of the tree also provides an iterator on the facets of $\mathcal{SC}(\mathrm{Q},\rho)$. We refer to [PS12] for a discussion on the complexity and on an implementation of this algorithm. This algorithm is similar to that of [BKPS06] for pointed triangulations and that of [PP12] for primitive sorting networks.

4 Further combinatorial properties of the EL-labelings

In this section, we discuss some implications of the EL-labelings of the increasing flip graph presented in Section 3.1. These results concern combinatorial properties of the *increasing flip poset* $\Gamma(\mathrm{Q},\rho)$, defined as the transitive closure of the increasing flip graph $\mathcal{G}(\mathrm{Q},\rho)$. The key property for the validity of these results is that the increasing flip graph $\mathcal{G}(\mathrm{Q},\rho)$ coincides with the Hasse diagram of the increasing flip poset $\Gamma(\mathrm{Q},\rho)$ (see the discussion in the beginning of Section 1.2). We first characterize and study the subword complexes which fulfill this property.

We say that the subword complex $\mathcal{SC}(\mathrm{Q},\rho)$ has a *double root* if there is a facet $I \in \mathcal{SC}(\mathrm{Q},\rho)$ and two distinct positions $i \neq j \in [m]$ both flippable in I such that $\mathsf{r}(I,i) = \mathsf{r}(I,j)$. Otherwise, we say that the subword complex $\mathcal{SC}(\mathrm{Q},\rho)$ is *double root free*. We focus on double root free subword complexes due to the following characterization.

Proposition 4.1 *The subword complex $\mathcal{SC}(\mathrm{Q},\rho)$ is double root free if and only if its increasing flip graph $\mathcal{G}(\mathrm{Q},\rho)$ coincides with the Hasse diagram of its increasing flip poset $\Gamma(\mathrm{Q},\rho)$.*

Intervals in the increasing flip graph of a double root free subword complex have the following property.

Proposition 4.2 *Let I and J be two facets of a double root free subword complex $\mathcal{SC}(\mathrm{Q},\rho)$. Then the intersection $I \cap J$ is contained in all facets of the interval $[I,J]$ in the increasing flip graph $\mathcal{G}(\mathrm{Q},\rho)$.*

Corollary 4.3 *There is at most one* p*-falling (resp.* n*-falling) path between any two facets I and J of a double root free subword complex $\mathcal{SC}(\mathrm{Q}, \rho)$. If it exists, its length is given by $|I \smallsetminus J| = |J \smallsetminus I|$.*

Corollary 4.4 *Let I and J be two facets of a double root free subword complex such that $I \dashrightarrow J$. The unique* p*-rising (resp.* n*-rising) path from I to J has maximal length among all path from I to J. Moreover, if there is a* p*-falling (resp.* n*-falling) path from I to J, it has minimal length.*

Remark 4.5 *Note that the conclusions of Proposition 4.2, Corollary 4.3, and Corollary 4.4 do indeed not hold if $\mathcal{SC}(\mathrm{Q}, \rho)$ has double roots. This situation reduces to the situation of type A_1 with generator s for the word $\mathrm{Q} = \mathrm{sss}$ and the element $\rho = s$, which contradicts the three statements.*

Corollary 4.6 *The Möbius function on the increasing flip poset $\Gamma(\mathrm{Q}, \rho)$ of a double root free subword complex $\mathcal{SC}(\mathrm{Q}, \rho)$ is given by*

$$\mu(I, J) = \begin{cases} (-1)^{|J \smallsetminus I|} & \text{if there is a } \mathsf{p}\text{-falling (resp. } \mathsf{n}\text{-falling) path from } I \text{ to } J, \\ 0 & \text{otherwise.} \end{cases}$$

By this corollary, we can compute the Möbius function of an interval $[I, J]$ of the increasing flip poset as soon as we can decide whether or not there is a p-falling path from I to J. According to Proposition 3.6, there is always a p-falling path from the positive greedy facet to the negative greedy facet of a spherical subword complex. We therefore obtain the following statement.

Corollary 4.7 *In a spherical double root free subword complex $\mathcal{SC}(\mathrm{Q}, \rho)$, we have*

$$\mu\big(\mathsf{P}(\mathrm{Q}, \rho), \mathsf{N}(\mathrm{Q}, \rho)\big) = (-1)^{|\mathrm{Q}| - \ell(\rho)}.$$

Observe again that this result fails if we drop the condition that $\mathcal{SC}(\mathrm{Q}, \rho)$ is spherical. The subword complex $\mathcal{SC}(\mathrm{Q}^{\mathrm{ex}}, \rho^{\mathrm{ex}})$ of Example 2.1 provides a counter-example.

Example 4.8 (Cambrian lattices) *We finally want to recall that cluster complexes of finite types are particular examples of subword complexes, see [CLS13]. This implies that Cambrian lattices of finite types are indeed increasing flip graphs, see [PS11]. Our construction thus proves that Cambrian lattices of finite types are EL-shellable. This result was as well obtained by M. Kallipoliti and H. Mühle in [KM12]. We want to emphasize that the two resulting labelings differ, as do the two resulting spanning trees. We refer to the long version of this paper [PS12] for further details.*

Example 4.9 (Duplicated words) *Fix an element $\rho \in W$ and a reduced expression of it. Consider a word $\mathrm{Q}^{\mathrm{dup}}$ obtained by duplicating $d \leq \ell(\rho)$ letters in this reduced expression. Any facet of the subword complex $\mathcal{SC}(\mathrm{Q}^{\mathrm{dup}}, \rho)$ contains precisely one position among each pair of duplicated letters, and no other position. Therefore, the subword complex $\mathcal{SC}(\mathrm{Q}^{\mathrm{dup}}, \rho)$ is the boundary complex of a d-dimensional cross-polytope, its increasing flip graph $\mathcal{G}(\mathrm{Q}^{\mathrm{dup}}, \rho)$ is the directed 1-skeleton $\square_d$ of a d-dimensional cube, and the increasing flip poset $\Gamma(\mathrm{Q}^{\mathrm{dup}}, \rho)$ is a boolean poset. Let $\phi : \square_d \to \Gamma(\mathrm{Q}^{\mathrm{dup}}, \rho)$ be the natural graph isomorphism which sends $\mathbf{0}$ to $\mathsf{P}(\mathrm{Q}^{\mathrm{dup}}, \rho)$ and $\mathbf{1}$ to $\mathsf{N}(\mathrm{Q}^{\mathrm{dup}}, \rho)$. It sends the edge labeling λ of $\square_d$ (see Example 1.2) to the positive and negative edge labelings* p *and* n *of the subword complex $\mathcal{SC}(\mathrm{Q}^{\mathrm{dup}}, \rho)$. More precisely, $\lambda(\varepsilon \to \varepsilon') = \mathsf{p}(\phi(\varepsilon) \to \phi(\varepsilon')) = \mathsf{n}(\phi(\varepsilon) \to \phi(\varepsilon')) - 1$. Thus, ϕ sends the λ-source tree of $\square_d$ to the source trees $\mathcal{P}(\mathrm{Q}^{\mathrm{dup}}, \rho) = \mathcal{N}^*(\mathrm{Q}^{\mathrm{dup}}, \rho)$, and the λ-sink tree of $\square_d$ to the sink trees $\mathcal{P}^*(\mathrm{Q}^{\mathrm{dup}}, \rho) = \mathcal{N}^*(\mathrm{Q}^{\mathrm{dup}}, \rho)$. See Example 1.2 and Figure 1. Finally, the Möbius function on the increasing flip poset $\Gamma(\mathrm{Q}^{\mathrm{dup}}, \rho)$ is given by $\mu(\phi(\varepsilon), \phi(\varepsilon')) = (-1)^{\delta(\varepsilon, \varepsilon')}$ if $\varepsilon \dashrightarrow \varepsilon'$ (where δ denotes the Hamming distance on the vertices of the cube) and $\mu(\phi(\varepsilon), \phi(\varepsilon')) = 0$ otherwise. See Example 1.4.*

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A q, t-analogue of Narayana numbers

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Abstract. We study the statistics area, bounce and dinv associated to polyominoes in a rectangular box m times n. We show that the bi-statistics $(\mathsf{area}, \mathsf{bounce})$ and $(\mathsf{area}, \mathsf{dinv})$ give rise to the same q, t-analogue of Narayana numbers, which was introduced by two of these authors in a recent paper. We prove the main conjectures of that same work, i.e. the symmetries in q and t, and in m and n of these polynomials, by providing a symmetric functions interpretation which relates them to the famous diagonal harmonics.

Résumé. Nous étudions les statistiques area, bounce et dinv associées aux polyominos dans un rectangle m par n. Nous montrons que les bi-statistiques $(\mathsf{area}, \mathsf{bounce})$ et $(\mathsf{area}, \mathsf{dinv})$ donnent lieu au même q, t-analogue des nombres de Narayana, qui a été introduit par deux de ces auteurs dans un article récent. Noous démontrons les conjectures principales du même article, c'est-à-dire la symétrie dans q et t, et dans m et n de ces polynômes, en donnant une interprétation en termes de fonctions symétriques qui les connecte aux célèbre diagonales harmoniques.

Keywords: q, t-Narayana, rectangular polyominoes, parking functions.

1 Introduction

Given two natural numbers m and n, the set of $m \times n$ *rectangular polyominoes* $\mathrm{Polyo}_{m,n}$ is known to have cardinality equal to $N(m+n-1, m)$, where for positive integers $a, b \in \mathbb{N}$,

$$N(a,b) := \frac{1}{a}\binom{a}{b}\binom{a}{b-1}$$

are the famous *Narayana numbers*.

In [3] two authors of this work introduced two statistics on these combinatorial objects, area and bounce, which led to a q, t-analogue of the Narayana numbers $N(m+n-1, m)$, namely

$$\mathsf{Nara}_{m,n}(q,t) := \sum_{P \in \mathrm{Polyo}_{m,n}} q^{\mathsf{area}(P)} t^{\mathsf{bounce}(P)}.$$

In that same work it was conjectured that these polynomials were symmetric both in q and t, and in m and n.

In this work we introduce a new statistic dinv, which gives a new q,t-analogue of the same numbers

$$\widetilde{\mathsf{Nara}}_{m,n}(q,t) := \sum_{P \in \mathrm{Polyo}_{m,n}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)}.$$

The following theorem establish a relation between these two polynomials.

Theorem 1.1 *For all $m \geq 1$ and $n \geq 1$, we have*

$$\mathsf{Nara}_{m,n}(q,t) = \widetilde{\mathsf{Nara}}_{n,m}(q,t).$$

The main result of this paper is the proof of the symmetries conjectured in [3].

Theorem 1.2 *For all $m \geq 1$ and $n \geq 1$, we have*

$$\mathsf{Nara}_{m,n}(q,t) = \mathsf{Nara}_{m,n}(t,q)$$

and

$$\mathsf{Nara}_{m,n}(q,t) = \mathsf{Nara}_{n,m}(q,t).$$

In order to prove this result, we use a symmetric functions interpretation of our q,t-Narayana numbers:

Theorem 1.3 *For all $m \geq 1$ and $n \geq 1$ we have*

$$\mathsf{Nara}_{m,n}(q,t) = (qt)^{m+n-1} \cdot \langle \nabla e_{m+n-2}, h_{m-1} h_{n-1} \rangle,$$

where e_k and h_k are the elementary and the homogeneous symmetric functions of degree k respectively, ∇ is the well known nabla operator introduced by Bergeron and Garsia (see [2, Section 9.6]), and the scalar product is the usual Hall inner product on symmetric functions.

This result brings a surprising link with the famous *diagonal harmonics* DH_n, since ∇e_n is the Frobenius characteristic of this important module of the symmetric group $\mathfrak{S}_n$, as it was shown by Haiman in [6].

Haglund in [4] gave a combinatorial interpretation of the polynomial $\langle \nabla e_{m+n-2}, h_{m-1} h_{n-1} \rangle$ in terms of parking functions. In fact Haglund's result would be an easy consequence of the famous *shuffle conjecture*, which predicts a combinatorial interpretation of ∇e_n in terms of parking functions (see [5, Chapter 6]).

In order to prove our main result, we used the recursion already established for $\mathsf{Nara}_{m,n}(q,t)$, proving that the combinatorial polynomials in Haglund's result satisfy the same recursion.

This paper is organized in the following way:

- In Section 2 we give the basic definitions, introducing our statistics and our q,t-analogues of Narayana numbers.
- In Section 3 we provide a bijection between $\mathrm{Polyo}_{m,n}$ and $\mathrm{Polyo}_{n,m}$ sending the bistatistic $(\mathsf{area}, \mathsf{bounce})$ in the bistatistic $(\mathsf{dinv}, \mathsf{area})$, so establishing Theorem 1.1.
- In Section 4 we provide a recursion satisfied by both our q,t-Narayana, which gives another proof of Theorem 1.1.
- In Section 5 we provide the necessary background to state Theorem 1.3, we see some of its consequences, and we explain the strategy of its proof.

2 The statistics

In these paper we consider polyominoes in rectangular boxes. More precisely, consider a square grid in $\mathbb{Z}^2$ of width m and height n. On this grid consider two paths, both starting from the South-West corner and arriving at the North-East corner, travelling on the grid, performing only North or East steps, with the further restriction that they touch each other only at the starting point and at the ending point. The region between the two paths is called the *interior* of the polyomino.

In Figure 1 there is an example where $m = 12$ and $n = 7$, and the interior is shadowed.

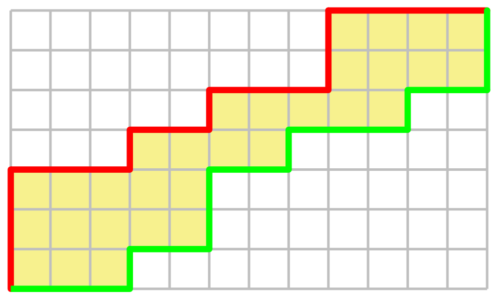

Fig. 1: A parallelogram polyomino having a 12 times 7 bounding box.

We encode the polyomino in an *area word* consisting of natural numbers and natural numbers with a bar on top, in the following way.

We will label each North step of the upper (red) path with a number with a bar, and each East step of the lower (green) path with a number without a bar. We do this in two stages.

First, for each East step of the lower path we draw a line starting with the East endpoint and going North-West until reaching the upper path: we label this step with the number of squares crossed by this line.

Second, we label each North step of the upper path by the number of squares in the interior of the polyomino to the East of it which were not crossed by any of the lines that we drew at the previous stage.

An example of this labelling is shown in Figure 2, where we put a black dot in the non-crossed squares.

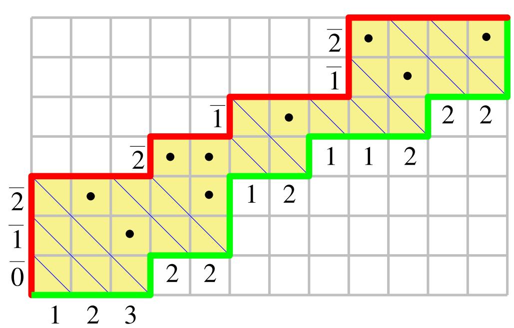

Fig. 2: The parallelogram polyomino of Figure 1 with its perimeter labelled.

Once we have done this labelling, we read the labels in the following order: starting from South-West and going to North-East imagine to span the polyomino with a straight line oriented North-West to South-East, and when we encounter vertical steps of the upper path or horizontal steps of the lower path we write the corresponding labels, recalling that if we encounter both types of steps at the same time we write the label of the upper path first.

For example, the area word of the example in Figure 2 is $\overline{0}1\overline{1}2\overline{2}322\overline{2}1\overline{1}211\overline{1}2\overline{2}22$.

Notice that the sum of these numbers (disregarding the bars) gives the area of the polyomino, which is the first of the statistics that are relevant to us. In the example the area is 30.

The next statistic that we want to consider is the bounce. Consider the following path in a given polyomino: we start with a single East step from the South-West corner, and then we move North until we reach the East endpoint of a horizontal step of the upper path; at this point we "bounce", i.e. we start moving East, until we reach the North endpoint of a vertical step of the lower path; at this point we "bounce" again, starting moving North, and we repeat this procedure until we reach the North-East corner.

Once we have the *bounce path*, starting from South-West, we label each step of the first sequence of vertical steps with 1, then each step of the second of such sequences with 2, and so on; while we label each step of the first sequence of horizontal steps with $\overline{0}$, then each step of the second of such sequences with $\overline{1}$, and so on. See Figure 3 for an example.

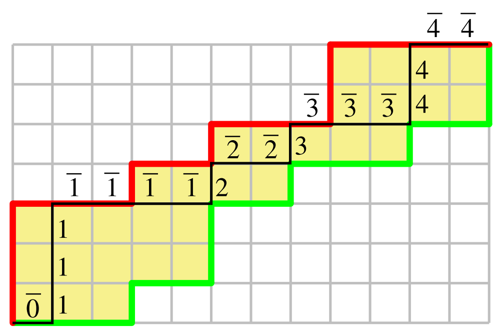

Fig. 3: The labelled bounce path.

We define the bounce of the polyomino to be the sum of the labels of the bounce path, disregarding the bars. For example the bounce of the example in Figure 3 is 41.

Consider now the total order on the labels

$$\overline{0} < 1 < \overline{1} < 2 < \overline{2} < 3 < \overline{3} < 4 < \overline{4} < \cdots .$$

Given a polyomino with area word $a_1 a_2 \dots a_k$, we define the dinv as the number of pairs a_i, a_j with $i < j$ and a_j is the successor of a_i in the fixed order. For example the dinv of the polyomino in Figure 2 is 35.

We fix the following notations: let $\mathrm{Polyo}_{m,n}$ be the set of polyominoes in a rectangle m times n, and let

$$\mathsf{Nara}_{m,n}(q,t) := \sum_{P \in \mathrm{Polyo}_{m,n}} t^{\mathsf{area}(P)} q^{\mathsf{dinv}(P)}$$

and

$$\widetilde{\mathsf{Nara}}_{m,n}(q,t) := \sum_{P \in \mathrm{Polyo}_{m,n}} t^{\mathsf{bounce}(P)} q^{\mathsf{area}(P)}.$$

The polynomials $\mathsf{Nara}_{m,n}(q,t)$ where first introduced in [3] by two of the authors of the present work. In the same paper, it was conjectured that these polynomials where symmetric both in q and t, and in m and n.

3 Bijection sending $(\mathsf{area}, \mathsf{bounce})$ in $(\mathsf{dinv}, \mathsf{area})$

This section is dedicated to prove Theorem 1.1, which we restate here for convenience.

Theorem 3.1 *For all m and n,*

$$\mathsf{Nara}_{m,n}(q,t) = \widetilde{\mathsf{Nara}}_{n,m}(q,t).$$

In order to prove it, we now describe a bijection between $\mathrm{Polyo}_{m,n}$ and $\mathrm{Polyo}_{n,m}$ which sends the bi-statistic $(\mathsf{area}, \mathsf{bounce})$ in the bi-statistic $(\mathsf{dinv}, \mathsf{area})$. Clearly this implies the theorem.

Starting from the polyomino, we read the labels of its bounce path, getting a word in integers and integers with a bar on top. Then, starting from the bottom-left corner, for each turn of the bounce path, we look at the part of the path (upper or lower) that includes it. For example in the polyomino of Figure 3 or 6, the first turn of the bounce path is between $\overline{0}$ and the next 1 in the labelling of the bounce path. The including path consists of the first 4 steps (counted from the South-West corner) of the upper path. We label the vertical steps of the including path with the labels used for the vertical steps in that part of the bounce path, and the horizontal steps of the including path with the labels used for the horizontal steps in that part of the bounce path. See Figure 6 for an example.

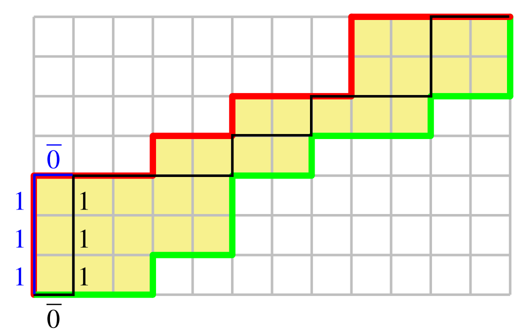

Fig. 4: The containing path and the new labels are blue.

Then we read the new labels by following the including path from North-East down to South-West. In the example we read $\overline{0}111$.

The rule is to preserve the relative positions of these labels along the rest of the construction.

We then repeat the algorithm with the second turn. In the example this occurs between the last 1 and the first $\overline{1}$ in the bounce path. This time the including path consists of the steps of the lower path between the second and the eighth. We repeat the procedure that we used before, and the word that we get reading the new labels will prescribe the relative positions of the 1's and the $\bar{1}$'s. In the example (see Figure 5) we get the prescriptions $11\bar{1}\bar{1}1\bar{1}\bar{1}$. This together with the other prescription gives a partial word $\bar{0}11\bar{1}\bar{1}1\bar{1}\bar{1}$. In general we will construct this partial word in a way that it can be the word of a polyomino and respecting all the prescriptions. This will always be possible since the first step of the including path that we read will always be labelled by the smallest of the two types of labels that we are considering: this is due to the definition of the bounce path.

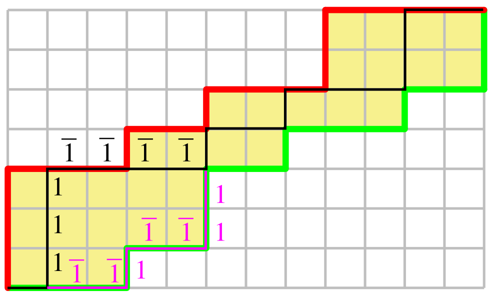

Fig. 5: The containing path and the new labels are violet.

We keep doing this until all the labels of the bounce path are included. At the end we will get a word of a polyomino. In the example, at the next step we get the prescriptions $\bar{1}\bar{1}2\bar{1}\bar{1}$, which gives the partial word $\bar{0}11\bar{1}\bar{1}121\bar{1}\bar{1}$; then we get the prescriptions $2\bar{2}\bar{2}$, which gives the partial word $\bar{0}11\bar{1}\bar{1}12\bar{2}\bar{2}1\bar{1}\bar{1}$; then we get the prescriptions $\bar{2}\bar{2}3$, which gives the partial word $\bar{0}11\bar{1}\bar{1}12\bar{2}\bar{2}31\bar{1}\bar{1}$; then we get the prescriptions $3\bar{3}\bar{3}\bar{3}$, which gives the partial word $\bar{0}11\bar{1}\bar{1}12\bar{2}\bar{2}3\bar{3}\bar{3}\bar{3}1\bar{1}\bar{1}$; then we get the prescriptions $\bar{3}\bar{3}44\bar{3}$, which gives the partial word $\bar{0}11\bar{1}\bar{1}12\bar{2}\bar{2}3\bar{3}\bar{3}44\bar{3}1\bar{1}\bar{1}$; and finally we get the prescriptions $44\bar{4}\bar{4}$, which gives the final word $\bar{0}11\bar{1}\bar{1}12\bar{2}\bar{2}3\bar{3}\bar{3}44\bar{4}\bar{4}\bar{3}1\bar{1}\bar{1}$.

With this construction we get a polyomino, which is clearly in a rectangle n times m, since the number of integers without a bar is n and the number of integers with the bar is m by construction. Moreover it has clearly area equal to the bounce of the original polyomino, again by construction. See Figure 6 for a picture of the image polyomino of the example.

We need to show that the dinv of the new polyomino is equal to the area of the original one.

To see this, recall how we constructed the word of the new polyomino: for consecutive types of labels, we prescribed the relative positions by reading the corresponding including path. But each pair of a vertical step and an horizontal step in the including path contributing to the dinv corresponds to a square in the area of the polyomino.

It remains to see that this is a bijection. To see this, we can consider the inverse function: given a polyomino, write in weakly increasing order its word, and draw it as a bounce path with labels. Then reading the relative positions of consecutive types of labels you can reconstruct piecewise both the upper and lower path. This completes the proof.

Let us observe some consequence of this result.

First of all, notice that iterating this bijection a second time, we get a bijection of the polyominoes in a rectangle m times n into themselves which sends the bounce in the area. Moreover, applying the inverse and composing it with the flip along the South-West to North-East line that pass through the South-West

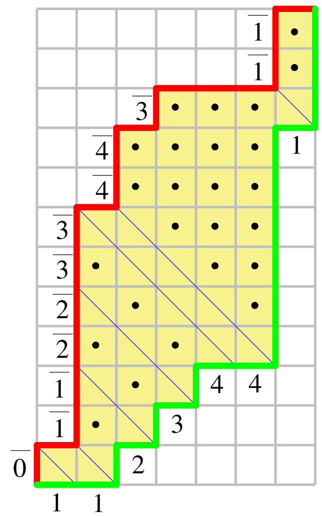

Fig. 6: The image polyomino.

corner (which obviously preserves the area) we get a bijection of the polyominoes in a rectangle m times n into themselves which sends the dinv in the area.

In conclusion we see that all our three statistics are equidistributed both inside the same rectangle m times n and with the polyominoes in the flipped rectangle n times m.

4 A recursion

In this section we prove that both $\mathsf{Nara}_{m,n}(q,t)$ and $\widetilde{\mathsf{Nara}}_{n,m}(q,t)$ satisfy a certain recursion. As an immediate byproduct we get another proof of the identity $\mathsf{Nara}_{m,n}(q,t) = \widetilde{\mathsf{Nara}}_{n,m}(q,t)$ stated in Theorem 1.1.

We call $\widetilde{\text{Polyo}}^{(r,s)}_{m,n}$ the set of polyominoes in a rectangle $m \times n$ whose labelled bounce path has r many 1's and s many $\overline{1}$'s. In other words, r is the number of steps between the first and the second bounce of the bounce path, while s is the number of steps between the second and the third bounce.

We fix the notation

$$\widetilde{\mathsf{Nara}}^{(r,s)}_{m,n}(q,t) := \sum_{P \in \widetilde{\text{Polyo}}^{(r,s)}_{m,n}} t^{\mathsf{bounce}(P)} q^{\mathsf{area}(P)},$$

so that $\widetilde{\mathsf{Nara}}_{m,n}$ is the sum over all r and s of $\widetilde{\mathsf{Nara}}^{(r,s)}_{m,n}(q,t)$. Also, for all positive integers n,

$$[n]_q := \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{-1}$$

denotes the q-analogue of n (by convention we set $[0]_q := 1$),

$$[n]_q! := \prod_{i=0}^{n} [i]_q,$$

denotes the q-analogue of the factorial $n!$, and finally for $n \geq k \geq 0$,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[n-k]_q![k]_q!}$$

denotes the q-analogue of the binomial $\binom{n}{k}$.

Theorem 4.1 *For all $m \geq 1$ and $n \geq 1$, and for $1 \leq r \leq n$ and $0 \leq s \leq m-1$ we have the recursion*

$$\widetilde{\mathsf{Nara}}_{m,n}^{(r,s)}(q,t) = t^{m+n-1} q^{r+s} \sum_{h=1}^{n-r} \sum_{k=0}^{m-s-1} \begin{bmatrix} s+r-1 \\ s \end{bmatrix}_q \begin{bmatrix} s+h-1 \\ h \end{bmatrix}_q \widetilde{\mathsf{Nara}}_{m-s,n-r}^{(h,k)}(q,t),$$

with initial conditions

$$\widetilde{\mathsf{Nara}}_{m,n}^{(n,s)}(q,t) = \begin{cases} (qt)^{m+n-1} \begin{bmatrix} m+n-2 \\ m-1 \end{bmatrix}_q & \text{if } s = m-1 \\ 0 & \text{if } s < m-1 \end{cases},$$

and

$$\widetilde{\mathsf{Nara}}_{1,n}^{(r,0)}(q,t) = 0 \text{ for } r < n.$$

For a proof see [1].

Let us denote by $\mathrm{Polyo}_{n,m}^{(r,s)}$ the set of polyominoes in a rectangle $n \times m$ whose area word has r many 1's and s many $\bar{1}$'s.

We fix the notation

$$\mathsf{Nara}_{n,m}^{(r,s)}(q,t) := \sum_{P \in \widetilde{\mathrm{Polyo}}_{n,m}^{(r,s)}} t^{\mathsf{area}(P)} q^{\mathsf{dinv}(P)},$$

so that $\widetilde{\mathsf{Nara}}_{n,m}$ is the sum over all r and s of $\widetilde{\mathsf{Nara}}_{n,m}^{(r,s)}(q,t)$.

These polynomials satisfy the same recursion satisfied by the $\widetilde{\mathsf{Nara}}_{m,n}^{(r,s)}(q,t)$'s:

Theorem 4.2 *For all $m \geq 1$ and $n \geq 1$, and for $1 \leq r \leq n$ and $0 \leq s \leq m-1$ we have the recursion*

$$\mathsf{Nara}_{n,m}^{(r,s)}(q,t) = t^{m+n-1} q^{r} \sum_{h=1}^{n-r} \sum_{k=0}^{m-s-1} q^s \begin{bmatrix} s+r-1 \\ s \end{bmatrix}_q \begin{bmatrix} s+h-1 \\ h \end{bmatrix}_q \mathsf{Nara}_{n-r,m-s}^{(h,k)}(q,t),$$

with initial conditions

$$\mathsf{Nara}_{n,m}^{(n,s)}(q,t) = \begin{cases} (qt)^{m+n-1} \begin{bmatrix} m+n-2 \\ m-1 \end{bmatrix}_q & \text{if } s = m-1 \\ 0 & \text{if } s < m-1 \end{cases},$$

and

$$\mathsf{Nara}_{n,1}^{(r,0)}(q,t) = 0 \textit{ for } r < n.$$

For a proof see [1].

These recursions give immediately $\mathsf{Nara}_{n,m}^{(r,s)}(q,t) = \widetilde{\mathsf{Nara}}_{m,n}^{(r,s)}(q,t)$, and hence another proof of the identity $\mathsf{Nara}_{m,n}(q,t) = \widetilde{\mathsf{Nara}}_{n,m}(q,t)$.

5 Symmetric functions interpretation

In this section we will use some tools from the theory of Macdonald polynomials. For a quick survey of what we need (and more), we refer to the book [2], in particular Chapters 3 and 9.

Here we will recall only some basic facts, mostly to fix the notation.

Let $\Lambda = \bigoplus_{n\geq 0} \Lambda^n$ be the space of symmetric functions with coefficients in $\mathbb{C}(q,t)$, where q and t are variables, with its natural decomposition in components of homogeneous degree.

Recall the fundamental bases of symmetric functions: *elementary* $\{e_\mu\}_\mu$, *homogeneous* $\{h_\mu\}_\mu$, *power* $\{p_\mu\}_\mu$, *monomial* $\{m_\mu\}_\mu$ and *Schur* $\{s_\mu\}_\mu$, where the indices μ are partitions.

A scalar product is defined on Λ by declaring the Schur basis to be orthonormal:

$$\langle s_\lambda, s_\mu \rangle = \chi(\lambda = \mu),$$

where χ is the indicator function, which is 1 when its argument is true, and 0 otherwise.

Another fundamental basis of Λ $\{\widetilde{H}_\mu\}_\mu$ has been introduced by Macdonald, and its elements are called *Macdonald polynomials*.

The fundamental ingredient of the theory is the nabla operator ∇ acting on Λ. This is an homogeneous invertible operator introduced by F. Bergeron and A. Garsia in the study of the famous diagonal harmonics DH_n of S_n. In fact, it turns out that ∇e_n gives precisely the bigraded Frobenius characteristic of DH_n.

The so called *shuffle conjecture* predicts a combinatorial interpretation of ∇e_n in terms of parking functions. Special cases of this conjecture have been proven by several authors. In particular J. Haglund proved the combinatorial interpretation of

$$\langle \nabla e_n, h_j h_{n-j} \rangle$$

for $1 \leq j \leq n$ predicted by the shuffle conjecture.

Surprisingly, this same polynomial provides the symmetric functions interpretation of our q,t-Narayana numbers.

More precisely, we have Theorem 1.3, which is the main result of this paper. For a proof see [1].

Theorem 5.1

$$\mathsf{Nara}_{m,n}(q,t) = (qt)^{m+n-1} \cdot \langle \nabla e_{m+n-2}, h_{m-1} h_{n-1} \rangle.$$

We give here an immediate corollary.

Corollary 5.2 *The polynomials $\mathsf{Nara}_{m,n}(q,t)$ are symmetric both in q and t, and in m and n. Moreover, we have*

$$\mathsf{Nara}_{m,n}(q,t) = \widetilde{\mathsf{Nara}}_{m,n}(q,t)$$

Proof of the Corollary: The symmetry in q and t comes from a general property of the nabla operator, which is easy to show: nabla applied to any Schur function is symmetric in q and t.

The symmetry in m and n is obvious from the formula.

Now the fact that $\mathsf{Nara}_{m,n}(q,t) = \widetilde{\mathsf{Nara}}_{m,n}(q,t)$ is a direct consequence of the symmetries and of Theorem 1.1. $\square$

In order to prove Theorem 1.3, we used the combinatorial interpretation given by Haglund for $\langle \nabla e_{m+n-2}, h_{m-1}h_{n-1}\rangle$. In order to state it, we need some definitions.

For us a *Dyck path* of order k will be given by an *area word*, i.e. a sequence of non-negative integers $b_1 b_2 \cdots b_k$ such that $b_1 = 0$, and $b_{i+1} \leq b_i + 1$ for all $i = 1, 2, ..., k-1$.

A *parking function* of order k will be given by a *domino sequence*, i.e. sequence of dominoes $\begin{array}{|c|}\hline a \\ b \\ \hline\end{array}$ like $PF = \begin{array}{|c|c|c|c|}\hline a_1 & a_2 & \cdots & a_k \\ b_1 & b_2 & \cdots & b_k \\ \hline\end{array}$, where $b_1 b_2 \cdots b_k$ is the area word of a Dyck path, and the a_i's are the integers from 1 to k, and they satisfy $a_i < a_{i+1}$ if $b_i < b_{i+1}$.

For example $PF = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline 5 & 11 & 1 & 9 & 6 & 8 & 3 & 4 & 7 & 10 & 2 \\ 0 & 1 & 1 & 2 & 0 & 1 & 0 & 1 & 2 & 3 & 3 \\ \hline\end{array}$ is a parking function of order 11.

The set of parking functions of order k is denoted by PF_k.

Given a parking function, we can reorder its dominoes by comparing first the bottom numbers, from the biggest to the smallest, and then, we place the dominoes with the same bottom number in order as we read them from right to left in the parking function. The *reading word* $\sigma(PF)$ associated to the parking function PF is the permutation that we obtain by reading the upper entries of this reordered sequence of dominoes.

For example, the parking function that we have seen before get reordered as

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline 2 & 10 & 7 & 9 & 4 & 8 & 1 & 11 & 3 & 6 & 5 \\ 3 & 3 & 2 & 2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ \hline\end{array},$$

so the corresponding reading word is

$$\sigma(PF) = 2\ 10\ 7\ 9\ 4\ 8\ 1\ 11\ 3\ 6\ 5.$$

Given a parking function $PF = \begin{array}{|c|c|c|c|}\hline a_1 & a_2 & \cdots & a_k \\ b_1 & b_2 & \cdots & b_k \\ \hline\end{array}$, we define its *area* $\mathsf{area}(PF)$ as the sum $\sum_{i=1}^{k} b_i$ of the bottom numbers of the dominoes, and its *dinv* $\mathsf{dinv}(PF)$ as the number of pairs of dominoes $\begin{array}{|c|}\hline a_i \\ b_i \\ \hline\end{array}, \begin{array}{|c|}\hline a_j \\ b_j \\ \hline\end{array}$ of PF with $i < j$, where $b_i = b_j$ and $a_i < a_j$, or $b_i = b_j + 1$ and $a_i > a_j$.

For example the area of the parking function of our previous example is 14, while its dinv is 8.

Given two disjoint sequences of numbers A and B, we denote by $A \cup\!\!\cup B$ the set of *shuffles* of A and B, i.e. the sequences consisting of the numbers from $A \cup B$ in which all the elements of A and B appear in their original order, so that $|A \cup\!\!\cup B| = \binom{|A|+|B|}{|A|}$.

For any a and b in $\mathbb{N}$, we call $\mathrm{Park}_{a,b}$ the set of parking functions PF of order $a+b$ such that $\sigma(PF) \in (1, 2, \ldots, a) \cup\!\!\cup (a+1, a+2, \ldots, a+b)$.

We finally set

$$\mathsf{Para}_{a,b}(q,t) := \sum_{PF\in \mathrm{Park}_{a,b}} t^{\mathsf{area}(PF)} q^{\mathsf{dinv}(PF)}.$$

We can state now the result of Haglund (see [4] for a proof, and [5] for the needed background).

Theorem 5.3 (Haglund) *For all $m \geq 1$ and $n \geq 1$, we have*

$$\langle \nabla e_{m+n-2}, h_{m-1}h_{n-1}\rangle = \mathsf{Para}_{n-1,m-1}(q,t).$$

Hence in order to prove Theorem 1.3, it remains to show that

$$\mathsf{Nara}_{m,n}(q,t) = (qt)^{m+n-1}\mathsf{Para}_{n-1,m-1}(q,t).$$

We proved this by showing that they both satisfy the recurrence given in Section 4. See [1] for the details.

Acknowledgements

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The critical surface fugacity for self-avoiding walks on a rotated honeycomb lattice

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Abstract. In a recent paper with Bousquet-Mélou, de Gier, Duminil-Copin and Guttmann (2012), we proved that a model of self-avoiding walks on the honeycomb lattice, interacting with an impenetrable surface, undergoes an adsorption phase transition when the surface fugacity is $1 + \sqrt{2}$. Our proof used a generalisation of an identity obtained by Duminil-Copin and Smirnov (2012), and confirmed a conjecture of Batchelor and Yung (1995). Here we consider a similar model of self-avoiding walk adsorption on the honeycomb lattice, but with the impenetrable surface placed at a right angle to the previous orientation. For this model there also exists a conjecture for the critical surface fugacity, made by Batchelor, Bennett-Wood and Owczarek (1998). We adapt the methods of the earlier paper to this setting in order to prove the critical surface fugacity, but have to deal with several subtle complications which arise.

This article is an abbreviated version of a paper of the same title, currently being prepared for submission.

Résumé. Dans un article récent avec Bousquet-Mélou, de Gier, Duminil-Copin et Guttmann (2012), nous avons prouvé qu'un modèle de marches auto-évitantes sur le réseau hexagonal, interagissant avec une surface impénétrable, subit une transition de phase absorbante quand la fugacité de la surface est $1 + \sqrt{2}$. Notre preuve utilisait une généralisation d'une identité obtenue par Duminil-Copin et Smirnov (2012), et permettait d'établir une conjecture de Batchelor et Yung (1995). Ici nous considérons un modèle similaire d'absorption de marches aléatoires auto-évitantes sur le réseau hexagonal, mais avec une surface impénétrable placée à angle droit par rapport à l'orientation précédente. Pour ce modèle il existe aussi une conjecture concernant la fugacité critique de la surface, formulée par Batchelor, Bennett-Wood et Owczarek (1998). Nous adaptons les méthodes de l'article précédent à ce cadre afin de prouver la fugacité critique de la surface, mais devons faire face à plusieurs complications subtiles qui apparaissent.

Cet article est la version courte d'une article ayant le même titre et actuellement en préparation.

Keywords: self-avoiding walks, polymer adsorption, honeycomb lattice, discrete holomorphicity

1 Introduction

Self-avoiding walks (SAWs) have been considered a model of long-chain polymers in solution for a number of decades – see for example early works by Orr (1947) and Flory (1949). In the simplest model one associates a weight (or fugacity) x with each step (or monomer, in the context of polymers) of a walk, and then (for a given lattice) considers the generating function

$$C(x) = \sum_{n \geq 0} c_n x^n,$$

where c_n is the number of SAWs starting at a fixed origin and comprising n steps.

It is straightforward to show (see e.g. Madras and Slade (1993)) that the limit

$$\mu := \lim_{n\to\infty} c_n^{1/n}$$

exists and is finite. The lattice-dependent value μ is known as the *growth constant*, and is the reciprocal of the radius of convergence of the generating function $C(x)$. The honeycomb lattice is the only regular lattice in two or more dimensions for which the value of the growth constant is known; its value $\mu = \sqrt{2+\sqrt{2}}$ was conjectured in 1982 by Nienhuis (1982) and proved by Duminil-Copin and Smirnov (2012).

The interaction of long-chain polymers with an impenetrable surface can be modelled by restricting SAWs to a half-space, and associating another fugacity y with vertices (or edges) in the boundary of the half-space which are visited by a walk. It is standard practice to place the origin on the boundary. This naturally leads to the definition of a partition function

$$C_n^+(y) = \sum_{m\geq 0} c_n^+(m) y^m,$$

where $c_n^+(m)$ is the number of n-step SAWs starting on the boundary of the half-space and occupying m vertices in the boundary.

The limit

$$\mu(y) := \lim_{n\to\infty} C_n^+(y)^{1/n}$$

has been shown to exist for the d-dimensional hypercubic lattice for $y > 0$ (see e.g. Hammersley et al. (1982)). It is a finite, log-convex and non-decreasing function of y, and is thus continuous and almost everywhere differentiable. The adaptation of the proof to other regular lattices (in particular, to the honeycomb lattice) is elementary – see Beaton (2012) for details.

It can also be shown that for $0 < y \leq 1$,

$$\mu(y) = \mu(1) = \mu,$$

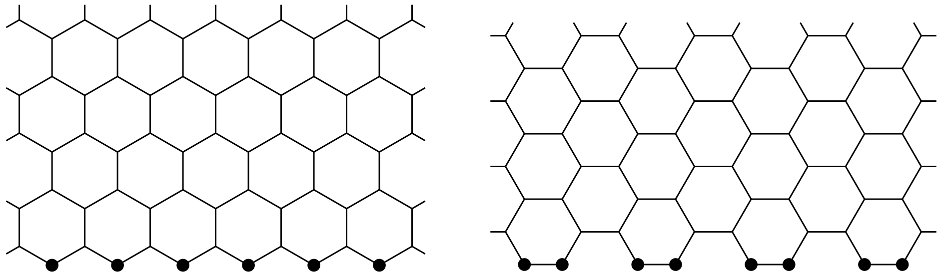
and that $\mu(y) \geq \max\{\mu, \sqrt{y}\}$. (The lower bound $\sqrt{y}$ applies to the honeycomb lattice as discussed in this paper, but this bound varies depending on the lattice and orientation of the surface.[i]) This implies the existence of a *critical fugacity* $y_\mathrm{c} \geq 1$ satisfying

$$\mu(y) \begin{cases} = \mu & \text{if } y \leq y_\mathrm{c}, \\ > \mu & \text{if } y > y_\mathrm{c}. \end{cases}$$

This critical fugacity signifies an *adsorption phase transition*, and demarcates the *desorbed* phase $y < y_\mathrm{c}$ and the *adsorbed* phase $y > y_\mathrm{c}$.

Just as the honeycomb lattice is the only regular lattice whose growth constant is known exactly, it is also the only lattice for which an exact value for y_c is known. In fact, because there are two different ways to orient the surface (see Figure 1) for the honeycomb lattice, there are two different values of y_c. When the

[i] In general, it is straightforward to show $\mu(y) \geq y^{1/k}$, where k is the minimum number of steps required to walk from one weighted vertex to another.



(a) (b)

Fig. 1: The two orientations of an impenetrable surface on the honeycomb lattice, with the surface vertices indicated.

surface is oriented so that there are lattice edges perpendicular to the surface (i.e. Figure 1(a)), the critical fugacity is $y_c = 1+\sqrt{2}$. This value was conjectured by Batchelor and Yung (1995), using the integrability of the model and comparison with a more general solvable loop model on the square lattice. The critical boundary weight was obtained by finding reflection matrices which satisfy the boundary Yang-Baxter equation. A proof was discovered by Beaton et al. (2012); it used a generalisation of an identity obtained by Duminil-Copin and Smirnov (2012), as well as an adaptation of some results of Duminil-Copin and Hammond (2012).

It is the other orientation of an impenetrable surface on the honeycomb lattice (i.e. Figure 1(b)) that is the focus of this article. For this model of polymer adsorption there is also a conjecture regarding the critical surface fugacity, due to Batchelor et al. (1998) and obtained using the same methods as for the first orientation. In this extended abstract we sketch the proof of that result:

Theorem 1 *For the self-avoiding walk model on the semi-infinite honeycomb lattice with the boundary oriented as per Figure 1(b), the critical surface fugacity is*

$$y = y_c = \sqrt{\frac{2+\sqrt{2}}{1+\sqrt{2}-\sqrt{2+\sqrt{2}}}} = 2.455\ldots$$

This paper is an overview of Beaton (2012), which in turn largely follows the same structure as Beaton et al. (2012). In the interest of brevity we omit most proofs. We first present an identity relating several different generating functions of SAWs in a finite domain, evaluated at the critical step fugacity $x = x_c = \mu^{-1}$. We then give adaptations of some existing results for the hypercubic lattice to the honeycomb lattice, and show how the critical fugacity relates to an appropriate limiting case of our identity. This relationship enables us to derive a proof of Theorem 1, subject to a certain generating function in a restricted geometry (specifically, the generating function of self-avoiding bridges which span a strip of height T) disappearing in a limit. We omit the proof of that result here; it is given in the appendix of Beaton (2012). The proof there is very similar to that of the appendix in Beaton et al. (2012), which was in turn based on arguments featured in Duminil-Copin and Hammond (2012).

In Beaton et al. (2012), we also established identities for a generalisation of the self-avoiding walk model, namely the $O(n)$ loop model. The equivalent generalisation for the rotated lattice is discussed in Beaton (2012), and we refer the reader to that article for further details.

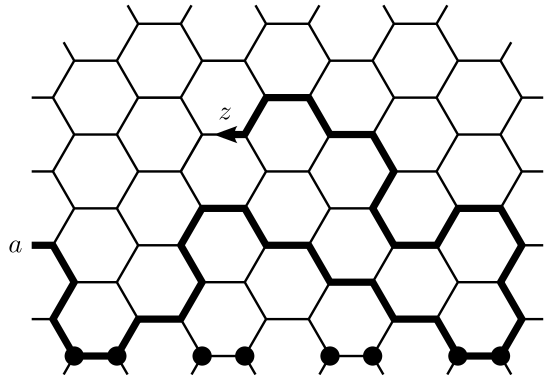


Fig. 2: A SAW on the honeycomb lattice. The contribution of this SAW to $F(z)$ is $\mathrm{e}^{-\mathrm{i}\sigma\pi}x^{33}y^4$.

2 The identities

2.1 The local identity for bulk vertices

We consider the semi-infinite honeycomb lattice, oriented as in Figure 1(b), embedded in the complex plane in such a way that the edges have unit length. We follow the examples of Duminil-Copin and Smirnov (2012) and Beaton et al. (2012) and consider self-avoiding walks which start and end at the *mid-points* of edges on the lattice. Note that this means the length of a walk is the same as the number of vertices it occupies. We define a *domain* Ω to be a finite connected collection of mid-edges with the property that for every vertex v adjacent to a mid-edge of Ω, all three mid-edges adjacent to v must be in Ω. We denote by $V(\Omega)$ the set of vertices adjacent to mid-edges of Ω, and by $\partial\Omega$ the set of mid-edges of Ω adjacent to only one vertex of $V(\Omega)$. Let γ be a self-avoiding walk. We denote by $|\gamma|$ the number of vertices occupied by γ and by $c(\gamma)$ the number of contacts with the surface (i.e. vertices on the surface occupied by γ).

Now define the following so-called *parafermionic observable*: for $a \in \partial\Omega$ and $z \in \Omega$, set

$$F(\Omega, a, p; x, y, \sigma) \equiv F(p) := \sum_{\gamma:a\to p} x^{|\gamma|}y^{c(\gamma)}\mathrm{e}^{-\mathrm{i}\sigma W(\gamma)},$$

where the sum is over all SAWs $\gamma \subset \Omega$ which run from a to p, and $W(\gamma)$ is the *winding angle* of γ, that is, $\pi/3$ times the difference between the number of left turns and right turns. See Figure 2 for an example.

The following lemma appears as part of Lemma 3 in Beaton et al. (2012); the case $y = 1$ is due to Smirnov (2010).

Lemma 2 *Let*

$$\sigma = -\frac{1}{8}, \qquad x_{\mathrm{c}}^{-1} = 2\cos\left(\frac{3\pi}{8}\right) = \sqrt{2-\sqrt{2}}, \qquad \textit{or} \tag{1}$$

$$\sigma = \frac{5}{8}, \qquad x_{\mathrm{c}}^{-1} = 2\cos\left(\frac{\pi}{8}\right) = \sqrt{2+\sqrt{2}}. \tag{2}$$

Then for a vertex $v \in V(\Omega)$ not belonging to the weighted surface, the observable F satisfies

$$(p-v)F(p)+(q-v)F(q)+(r-v)F(r)=0, \tag{3}$$

where p, q, r are the three mid-edges adjacent to v, and the variable x is set to x_{c}.

Equation (1) corresponds to the larger of the two critical values of the step weight x and hence to the dense regime critical point, while (2) corresponds to the line of critical points separating the dense and dilute phases. In what follows we refer to (1) and (2) as the dense and dilute regimes respectively.

2.2 The local identity for surface vertices

We now wish to generalise Lemma 2 to include vertices lying on the weighted boundary. To do this, we have to be more particular about the domain being used. We work in the special domain $D_{T,L}$, as illustrated in Figure 3. The height T of the domain is the length of the shortest walk starting at a and ending at the top boundary; the width $2L+1$ is the number of columns of cells. Walks start at the mid-edge a. We choose this mid-edge in order to preserve the reflective symmetry of the domain, which greatly simplifies an important identity. However, the fact that a is not an external mid-edge does introduce some complications:

- A walk which ends at a particular external mid-edge could have two different winding angles, depending on whether it started from a in the left or right direction. This is undesirable, but easily corrected. Define

$$W^*(\gamma) := \begin{cases} W(\gamma)+\pi/2 & \text{if } \gamma \text{ starts in the left direction,} \\ W(\gamma)-\pi/2 & \text{if } \gamma \text{ starts in the right direction,} \\ 0 & \text{if } \gamma \text{ is the empty walk.} \end{cases}$$

  Then, define $F^*(p)$ in the same way as $F(p)$, but now using W^* instead of W.

- This new observable F^* will satisfy the same identity (3) as F on all non-boundary vertices of $D_{T,L}$, except for the vertices a^- and a^+ adjacent to a. To deal with this, we define $V'(D_{T,L}) := D_{T,L} \backslash \{a^-, a^+\}$, and will end up evaluating (3) only on the vertices of $V'(D_{T,L})$.

Proposition 3 *Let σ and x_{c} be as defined in* (2). *Define $\mathbf{1}_{\beta^+}(v)$ to be 1 if the vertex v is adjacent to a mid-edge in β^+ and 0 otherwise, and similarly define $\mathbf{1}_{\beta^-}(v)$. Then for every vertex v in $V'(D_{T,L})$ with adjacent mid-edges p, q, r,*

$$\begin{aligned}
&(p-v)F^*(p)+(q-v)F^*(q)+(r-v)F^*(r) \\
&= \mathbf{1}_{\beta^+}(v)(1-y)\mathrm{e}^{-\mathrm{i}\sigma(-\pi/6)}(x_{\mathrm{c}}y)^{-1}\left((r-v)\bar{\lambda}\sum_{\gamma:a\to r\to p} x_{\mathrm{c}}^{|\gamma|}y^{c(\gamma)}+(q-v)\lambda\sum_{\gamma:a\to q\to p} x_{\mathrm{c}}^{|\gamma|}y^{c(\gamma)}\right) \\
&+ \mathbf{1}_{\beta^-}(v)(1-y)\mathrm{e}^{-\mathrm{i}\sigma(\pi/6)}(x_{\mathrm{c}}y)^{-1}\left((r-v)\bar{\lambda}\sum_{\gamma:a\to r\to p} x_{\mathrm{c}}^{|\gamma|}y^{c(\gamma)}+(q-v)\lambda\sum_{\gamma:a\to q\to p} x_{\mathrm{c}}^{|\gamma|}y^{c(\gamma)}\right),
\end{aligned} \tag{4}$$

where for vertices adjacent to mid-edges in β^+ or β^-, the surrounding mid-edges p, q, r are in clockwise order from the external mid-edge, and the sums are over walks which visit the indicated mid-edges in the prescribed order.

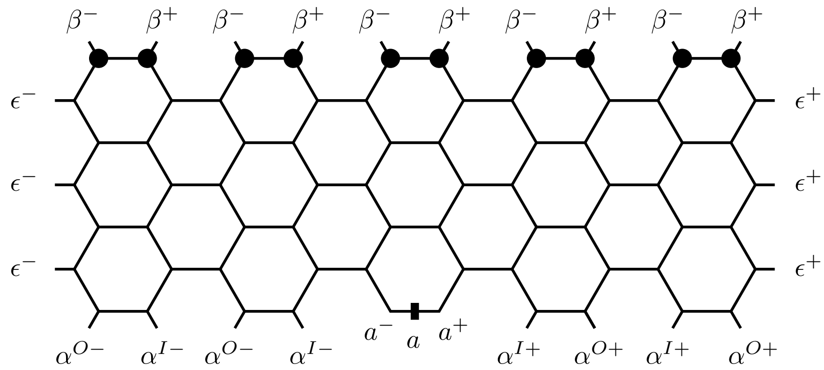


Fig. 3: The domain $D_{T,L}$ of height $T = 7$ and width $2L + 1 = 9$, with the weighted vertices on the β boundary indicated. The external mid-edges attached to a^- and a^+ are present in the domain but will not play a part in the identity, and are thus not illustrated.

It is clear that if $y = 1$ or if v is not a weighted vertex, then the RHS of (4) disappears and thus (4) reduces to something very similar to (3) – the differences being that here a is not an external mid-edge, and $V'(D_{T,L})$ does not quite include all vertices in the domain. The factors $\mathrm{e}^{-\mathrm{i}\sigma(-\pi/6)}$ and $\mathrm{e}^{-\mathrm{i}\sigma(\pi/6)}$ are the contributions of the modified winding angles of walks to β^+ and β^- mid-edges respectively.

2.3 The domain identity

In Duminil-Copin and Smirnov (2012), the authors use Lemma 2 to prove that the growth constant of self-avoiding walks (the dilute regime) is $x_{\mathrm{c}}^{-1} = \sqrt{2+\sqrt{2}}$. They do so by considering a special trapezoidal domain, and using the local identity (3) to derive a domain identity satisfied by generating functions of SAWs which end on different sides of the domain. In Beaton et al. (2012), that identity is generalised to one which relates generating functions of the $O(n)$ loop model and takes into account the surface fugacity y.

Here, we construct a similar identity to the one used in Beaton et al. (2012). We take σ and x_{c} to be the values given in (2).

Define

$$A^O_{T,L}(x,y) = \sum_{\gamma:a\to\alpha^{O+}\bigcup\alpha^{O-}} x^{|\gamma|}y^{c(\gamma)} \qquad A^I_{T,L}(x,y) = \sum_{\gamma:a\to\alpha^{I+}\bigcup\alpha^{I-}} x^{|\gamma|}y^{c(\gamma)}$$
$$E_{T,L}(x,y) = \sum_{\gamma:a\to\epsilon^{+}\bigcup\epsilon^{-}} x^{|\gamma|}y^{c(\gamma)} \qquad B_{T,L}(x,y) = \sum_{\gamma:a\to\beta^{+}\bigcup\beta^{-}} x^{|\gamma|}y^{c(\gamma)},$$

where each sum runs over SAWs which start at a and end in the indicated set of external mid-edges of $D_{T,L}$. Also, define

$$P_{T,L}(x,y) = \sum_{\rho\ni a} x^{|\rho|}y^{c(\rho)},$$

which sums over all undirected (non-empty) self-avoiding *polygons* in $D_{T,L}$ which contain a. That is, ρ is a simple closed loop on the edges of $D_{T,L}$ which passes through a, $|\rho|$ is the number of edges (or, equivalently, vertices) occupied by ρ and $c(\rho)$ is the number of boundary vertices occupied by ρ.

Proposition 4 *Let $T + L \equiv 1(mod\ 2)$. Then the generating functions $A^O_{T,L}, A^I_{T,L}, E_{T,L}, B_{T,L}$ and $P_{T,L}$, evaluated at $x = x_c$, satisfy the identity*

$$c^O_A A^O_{T,L}(x_c, y) + c^I_A A^I_{T,L}(x_c, y) + c_E E_{T,L}(x_c, y) + c_P P_{T,L}(x_c, y) + c_B(y) B_{T,L}(x_c, y) = c_G, \quad (5)$$

where

$$c^O_A := 2\cos\left(\frac{5\pi}{16}\right) = \sqrt{2 - \sqrt{2 - \sqrt{2}}}, \qquad c^I_A := 2\cos\left(\frac{7\pi}{16}\right) = \sqrt{2 - \sqrt{2 + \sqrt{2}}},$$

$$c_E := 2\cos\left(\frac{3\pi}{16}\right) = \sqrt{2 + \sqrt{2 - \sqrt{2}}},$$

$$c_P := \frac{4}{x_c}\cos\left(\frac{7\pi}{16}\right) = 2\sqrt{4 + 2\sqrt{2} - \sqrt{2\left(10 + 7\sqrt{2}\right)}},$$

$$c_G := 4x_c\cos\left(\frac{\pi}{16}\right) = \sqrt{2\left(4 - 2\sqrt{2} + \sqrt{2\left(2 - \sqrt{2}\right)}\right)}, \text{ and}$$

$$c_B(y) := 2\cos\left(\frac{\pi}{16}\right) - \frac{2(1 - x_c y - x_c^2 y^2)\cos\left(\frac{15\pi}{16}\right) + 2x_c^2 y^2 \cos\left(\frac{5\pi}{16}\right)}{x_c y(1 + x_c y)}$$

$$= \frac{c_B}{x_c y(1 + x_c y)} - \frac{x_c y c^O_A}{1 + x_c y}, \qquad \text{and } c_B := c_B(1) = 2\cos\left(\frac{\pi}{16}\right) = \sqrt{2 + \sqrt{2 + \sqrt{2}}}.$$

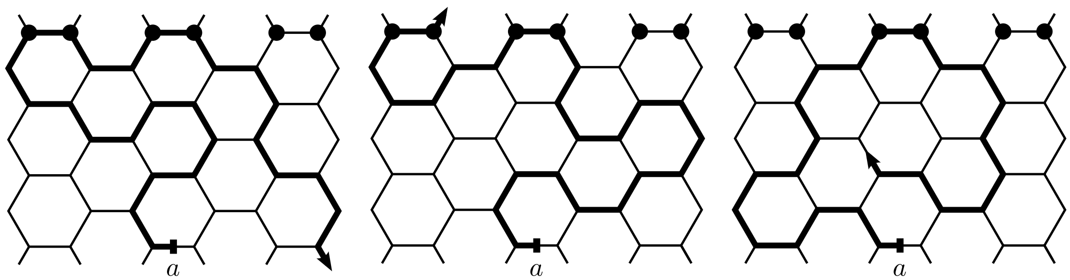
The proof follows by computing the sum

$$S = \sum_{\substack{v \in V'(D_{T,L}) \\ p,q,r \sim v}} (p - v)F^*(p) + (q - v)F^*(q) + (r - v)F^*(r), \quad (6)$$

where p, q, r are the three mid-edges adjacent to vertex v, in two ways. One one hand, the contribution to S of any "internal" mid-edge (i.e. any mid-edge adjacent to two vertices in $V'(D_{T,L})$) will be 0, and thus we only need to consider the contributions of external mid-edges. On the other hand, (4) guarantees that the contribution of any unweighted vertex is 0, and so we can compute S by calculating the contributions of the vertices on the β boundary. We require $T + L \equiv 1(\text{mod } 2)$ so that we can pair up vertices on the β boundary.

3 The critical surface fugacity

In this extended abstract we omit most of the technical results which enable us to adapt known results for the hypercubic lattice (see Hammersley et al. (1982) and van Rensburg et al. (2006)) to the honeycomb lattice, and instead only present the main result that we need. In a strip of height T, we set the mid-edge a



(a) (b) (c)

Fig. 4: Sections of the strip of height T, with (a) an arch, (b) a bridge and (c) a general walk. The dark circles indicate the weighted vertices on the top of the strip.

to be a horizontal mid-edge on the bottom of the strip (similar to its placement in the finite domain $D_{T,L}$). We then define the following three types of SAWs (see Figure 4): *bridges*, which start at a and end at the top of the strip; *arches*, which start at a and end on the bottom of the strip; and *general walks*, which start at a and may end anywhere in the strip. We then define the generating function

$$B_T(x,y) = \sum_{n,m\geq 0} b_{T,n}(m)x^n y^m$$

where $B_{T,n}(m)$ is the number of length n bridges in the strip of width T which contain m vertices at the top of the strip. We likewise define $A_T(x,y)$ and $C_T(x,y)$ for arches and general walks respectively.

The following proposition will allow us to relate the generating functions we considered in the previous section with the critical surface fugacity y_{c}. Recall from Section 1 the definition of $\mu(y)$.

Proposition 5 *For $y > 0$, the generating functions $A_T(x,y)$, $B_T(x,y)$ and $C_T(x,y)$ all have the same radius of convergence, $\rho_T(y)$. The sequence $\rho_T(y)$ decreases to $\rho(y) := \mu(y)^{-1}$ as $T \to \infty$. In particular, $\rho_T(y)$ decreases to $\rho := \mu^{-1}$ for $y \leq y_{\mathrm{c}}$.*

There exists a unique $y_T > 0$ such that $\rho_T(y_T) = x_{\mathrm{c}} := \mu^{-1}$. The series (in y) $A_T(x_{\mathrm{c}},y)$, $B_T(x_{\mathrm{c}},y)$ and $C_T(x_{\mathrm{c}},y)$ have radius of convergence y_T, and y_T decreases to the critical fugacity y_{c} as $T \to \infty$.

We now return to the identity (5) relating the generating functions in the domain $D_{T,L}$. Note that $c_B(y)$ is a continuous and monotone decreasing function of y for $y > 0$, and that $c_B(y^\dagger) = 0$ where

$$y^\dagger = \sqrt{\frac{2+\sqrt{2}}{1+\sqrt{2}-\sqrt{2+\sqrt{2}}}}.$$

For $0 < y < y^\dagger$, every term in (5) is non-negative. Observe that $A^O_{T,L}$, $A^I_{T,L}$, $B_{T,L}$ and $P_{T,L}$ are increasing with L. (As L increases these generating functions just count more and more objects.) We then see that for those values of L satisfing $T + L \equiv 1 \pmod 2$, $E_{T,L}$ must decrease as L increases. It is thus valid to take the limit $L \to \infty$ of (5) over the values of L with $T + L \equiv 1 \pmod 2$. But now $A^O_{T,L}$, $A^I_{T,L}$, $B_{T,L}$ and $P_{T,L}$ actually increase with L regardless of whether $T + L \equiv 1 \pmod 2$ or not, and so they have the same limits as $L \to \infty$ over any subsequence of L values. Hence, we can in fact take the limit

$L \to \infty$ of (5) over all values of L. If we define

$$A_T^O(x_c, y) := \lim_{L\to\infty} A_{T,L}^O(x_c, y),$$

and similar limits for the other generating functions (we also have $\lim_{L\to\infty} B_{T,L}(x, y) = B_T(x, y)$ as defined earlier), then we obtain

$$c_A^O A_T^O(x_c, y) + c_A^I A_T^I(x_c, y) + c_E E_T(x_c, y) + c_P P_T(x_c, y) + c_B(y) B_T(x_c, y) = c_G. \tag{7}$$

In this rest of this section, we will prove the following:

Proposition 6 *If it can be shown that*

$$B(x_c, 1) := \lim_{T\to\infty} B_T(x_c, 1) = 0$$

then $y_c = y^\dagger$.

The proof that $B(x_c, 1) = 0$ is quite involved and will thus be omitted from this extended abstract; see the appendix of Beaton (2012).

We begin by establishing a lower bound on y_c with a straightforward corollary to Proposition 5.

Corollary 7 *The critical surface fugacity* y_c *satisfies*

$$y_c \geq y^\dagger.$$

Proof: For $y < y^\dagger$ the identity (7) establishes the finiteness of $B_T(x_c, y)$, and thus we see $y_T \geq y^\dagger$. By Proposition 5 it then follows that $y_c \geq y^\dagger$. □

We now show that one of the generating functions in (7) has disappeared in the limit $L \to \infty$.

Corollary 8 *For* $0 \leq y < y^\dagger$,

$$E_T(x_c, y) := \lim_{L\to\infty} E_{T,L}(x_c, y) = 0,$$

and hence

$$c_A^O A_T^O(x_c, y) + c_A^I A_T^I(x_c, y) + c_P P_T(x_c, y) + c_B(y) B_T(x_c, y) = c_G. \tag{8}$$

Proof: By Proposition 5, y_T is the radius of convergence of $C_T(x_c, y)$. Since $y_T \geq y_c \geq y^\dagger$, it follows that $C_T(x_c, y)$ is convergent for $0 \leq y < y^\dagger$. Now

$$\sum_L E_{T,L}(x_c, y) \leq C_T(x_c, y) < \infty,$$

as each walk counted by $E_{T,L}$, for every value of L, will also be counted by C_T. The corollary follows immediately. □

We note here that $A_T^O(x_c, y) \leq C_T(x_c, y)$ (since any walk counted by A_T^O is also counted by C_T), and likewise for A_T^O and P_T. Hence all the generating functions featured in (8) have radius of convergence at least y_T.

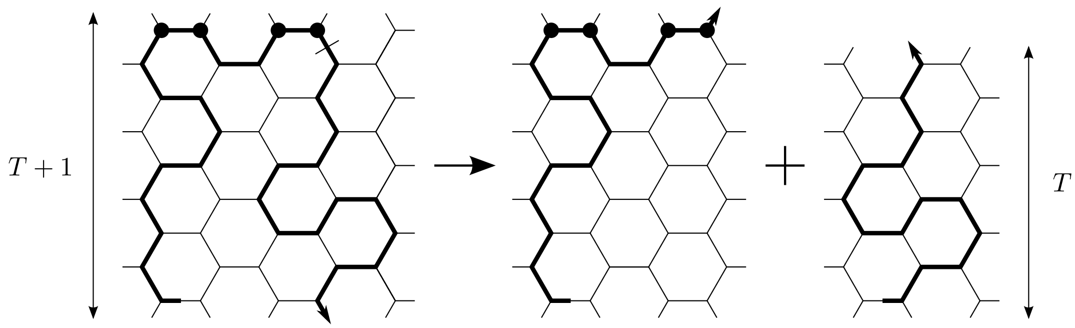


Fig. 5: Factorisation of a walk counted by A^O_{T+1} into two bridges.

Now consider the $y = 1$ case of (8):

$$c^O_A A^O_T(x_c, 1) + c^I_A A^I_T(x_c, 1) + c_P P_T(x_c, 1) + c_B B_T(x_c, 1) = c_G.$$

Since $A^O_T(x_c, 1)$, $A^I_T(x_c, 1)$ and $P_T(x_c, 1)$ all increase with T (as T increases these generating functions count more and more objects), and since they are all bounded by this identity, it follows that they all have limits as $T \to \infty$. Then $B_T(x_c, 1)$ must decrease as T increases, and it too has a limit as $T \to \infty$. As indicated in Proposition 6, we denote this limit

$$B(x_c, 1) := \lim_{T\to\infty} B_T(x_c, 1).$$

Proof of Proposition 6: Assume now that $B(x_c, 1) = 0$. Any walk counted by $A^O_{T+1}(x_c, y)$ which has contacts with the top boundary can be factored into two pieces by cutting it at the mid-edge immediately following its last surface contact. (See Figure 5.) The first piece, after reflecting the last step, is an object counted by $B_{T+1}(x_c, y)$, while the second piece (with its direction reversed) will be counted by $(1 + x_c)B_T(x_c, 1)/2$. Thus we obtain

$$\begin{aligned} A^O_{T+1}(x_c, y) - A^O_T(x_c, 1) &\le \frac{1 + x_c}{2} \cdot B_{T+1}(x_c, y)B_T(x_c, 1) \\ &\le B_{T+1}(x_c, y)B_T(x_c, 1). \end{aligned}$$

This inequality is valid in the domain of convergence of the series it involves, that is, for $y < y_{T+1}$. Using similar arguments we can obtain the equivalent inequality for $A^I_{T+1}(x_c, y)$ and $P_{T+1}(x_c, y)$.

Combining this decomposition for A^O_{T+1}, A^I_{T+1} and P_{T+1}, we find for $0 \le y < y_{T+1}$,

$$\begin{aligned} c^O_A[A^O_{T+1}(x_c, y) - A^O_T(x_c, 1)] + c^I_A[A^I_{T+1}(x_c, y) - A^I_T(x_c, 1)] + c_P[P_{T+1}(x_c, y) - P_T(x_c, 1)] \\ \le (c^O_A + c^I_A + c_P)B_{T+1}(x_c, y)B_T(x_c, 1). \quad (9) \end{aligned}$$

Using (8) to eliminate the A^O, A^I and P terms, we obtain

$$c_B B_T(x_c, 1) - c_B(y)B_{T+1}(x_c, y) \le (c^O_A + c^I_A + c_P)B_{T+1}(x_c, y)B_T(x_c, 1),$$

and hence

$$0 \leq \frac{1}{B_{T+1}(x_c, y)} \leq \frac{(c_A^O + c_A^I + c_P)}{c_B} + \frac{c_B(y)}{c_B B_T(x_c, 1)}. \tag{10}$$

In particular, for $0 \leq y < y_c = \lim_{T\to\infty} y_T$ and for any T,

$$0 \leq \frac{x_c(c_A^O + c_A^I + c_P)}{c_B} + \frac{c_B(y)}{c_B B_T(x_c, 1)}. \tag{11}$$

Now consider what happens as $T \to \infty$. By assumption, $\lim_{T\to\infty} B_T(x_c, 1) = 0$. Suppose (for a contradiction) that $y_c > y^\dagger$. Then for any $y^\dagger < y < y_c$ and sufficiently large T, the RHS of (11) will be negative, because $c_B(y) < 0$ for $y > y^\dagger$ and $B_T(x_c, 1)^{-1}$ will become arbitrarily large. This contradicts the inequality, and we are forced to conclude $y_c \leq y^\dagger$, and hence $y_c = y^\dagger$. □

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Generalized monotone triangles: an extended combinatorial reciprocity theorem

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Abstract. In a recent work, the combinatorial interpretation of the polynomial $\alpha(n; k_1, k_2, \dots, k_n)$ counting the number of Monotone Triangles with bottom row $k_1 < k_2 < \cdots < k_n$ was extended to weakly decreasing sequences $k_1 \geq k_2 \geq \cdots \geq k_n$. In this case the evaluation of the polynomial is equal to a signed enumeration of objects called Decreasing Monotone Triangles. In this paper we define Generalized Monotone Triangles – a joint generalization of both ordinary Monotone Triangles and Decreasing Monotone Triangles. As main result of the paper we prove that the evaluation of $\alpha(n; k_1, k_2, \dots, k_n)$ at arbitrary $(k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$ is a signed enumeration of Generalized Monotone Triangles with bottom row $(k_1, k_2, \dots, k_n)$. Computational experiments indicate that certain evaluations of the polynomial at integral sequences yield well-known round numbers related to Alternating Sign Matrices. The main result provides a combinatorial interpretation of the conjectured identities and could turn out useful in giving bijective proofs.

Résumé. Dans un travail récent, l'interprétation combinatoire du polynôme $\alpha(n; k_1, k_2, \dots, k_n)$ comptant le nombre de triangles monotones avec dernière ligne $k_1 < k_2 < \cdots < k_n$ a été étendue aux suites faiblement décroissantes $k_1 \geq k_2 \geq \cdots \geq k_n$. Dans ce cas l'évaluation du polynôme est égale à l'énumération signée d'objets appelés triangles monotones décroissants. Dans ce papier nous définissons des triangles monotones généralisés – une généralisation commune des triangles monotones ordinaires et décroissants. Notre résultat principal est que l'évaluation de $\alpha(n; k_1, k_2, \dots, k_n)$ en un quelconque $(k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$ est une énumération signée de triangles monotones généralisés avec dernière ligne $(k_1, k_2, \dots, k_n)$. Des calculs par ordinateur indiquent que certaines valeurs du polynôme sont des nombres bien connus liés aux matrices à signe alternant. Le résultat principal fournit une interprétation combinatoire des identités conjecturales et pourrait être utile dans l'obtention de preuves bijectives.

Keywords: Combinatorial Reciprocity, Monotone Triangle, Generalized Monotone Triangle, Alternating Sign Matrix

1 Introduction

A *Monotone Triangle* of size n is a triangular array of integers $(a_{i,j})_{1 \leq j \leq i \leq n}$

$$
\begin{array}{ccccccc}
 & & & a_{1,1} & & & \\
 & & a_{2,1} & & a_{2,2} & & \\
 & \iddots & & & & \ddots & \\
a_{n,1} & & \cdots & & \cdots & & a_{n,n}
\end{array}
$$

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with strictly increasing rows, i.e. $a_{i,j} < a_{i,j+1}$, and weakly increasing North-East- and South-East-diagonals, i.e. $a_{i+1,j} \leq a_{i,j} \leq a_{i+1,j+1}$. An example of a Monotone Triangle of size 5 is given in Fig.1.

$$\begin{array}{ccccccccc} & & & & 4 & & & & \\ & & & 4 & & 5 & & & \\ & & 3 & & 5 & & 7 & & \\ & 2 & & 5 & & 6 & & 8 & \\ 2 & & 4 & & 5 & & 8 & & 9 \end{array}$$

Fig. 1: One of the 16939 Monotone Triangles with bottom row $(2, 4, 5, 8, 9)$.

For each $n \geq 1$, there exists a unique polynomial $\alpha(n; k_1, k_2, \ldots, k_n)$ of degree $n-1$ in each of the n variables such that the evaluation of this polynomial at strictly increasing sequences $k_1 < k_2 < \cdots < k_n$ is equal to the number of Monotone Triangles with prescribed bottom row $(k_1, k_2, \ldots, k_n)$ – for example $\alpha(5; 2, 4, 5, 8, 9) = 16939$. This result was derived in [Fis06], where the polynomials are given explicitly in terms of an operator formula.

In [FR13] we studied the evaluation of $\alpha(n; k_1, \ldots, k_n)$ at weakly decreasing sequences $k_1 \geq k_2 \geq \cdots \geq k_n$. As it turned out, the evaluation can be interpreted as signed enumeration of the following combinatorial objects:

A *Decreasing Monotone Triangle* (DMT) of size n is a triangular array of integers $(a_{i,j})_{1 \leq j \leq i \leq n}$ having the following properties:

- The entries along NE- and SE-diagonals are weakly decreasing.
- Each integer appears at most twice in a row.
- Two consecutive rows do not both contain the same integer exactly once.

One of the motivations for considering evaluations of $\alpha(n; k_1, \ldots, k_n)$ at non-increasing $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ stems from the connection to Alternating Sign Matrices. An *Alternating Sign Matrix* (ASM) of size n is a $n \times n$-matrix with entries in $\{0, 1, -1\}$ such that in each row and column the non-zero entries alternate in sign and sum up to 1. It is well-known ([MRR83]) that the set of ASMs is in bijection with the set of Monotone Triangles with bottom row $(1, 2, \ldots, n)$. Counting the number of ASMs of size n had been an open problem for more than a decade until the first two proofs were given by D. Zeilberger ([Zei96]) and G. Kuperberg ([Kup96]) in 1996 (see [Bre99] for more details). The Refined ASM Theorem – i.e. the refined enumeration with respect to the unique 1 in the first row – was reproven by I. Fischer in 2007 ([Fis07]). The identity

$$\alpha(n; k_1, \ldots, k_n) = (-1)^{n-1} \alpha(n; k_2, \ldots, k_n, k_1 - n) \tag{1}$$

plays one of the key roles in this algebraic proof. A bijective proof of (1) could give more combinatorial insight to the theorem. However, note that if $k_1 < k_2 < \cdots < k_n$, then $k_n > k_1 - n$, i.e. (1) can per se only be understood as identity satisfied by the polynomial.

The objective of this paper is to give a combinatorial interpretation to the evaluation of $\alpha(n; k_1, \ldots, k_n)$ at arbitrary $(k_1, \ldots, k_n) \in \mathbb{Z}^n$. For this, we define triangular arrays of integers which locally combine the restrictions of ordinary Monotone Triangles and Decreasing Monotone Triangles:

A *Generalized Monotone Triangle* (GMT) is a triangular array $(a_{i,j})_{1\leq j\leq i\leq n}$ of integers satisfying the following conditions:

(1) Each entry is weakly bounded by its SW- and SE-neighbour, i.e.

$$\min\{a_{i+1,j}, a_{i+1,j+1}\} \leq a_{i,j} \leq \max\{a_{i+1,j}, a_{i+1,j+1}\}.$$

(2) If three consecutive entries in a row are weakly increasing, then their two interlaced neighbours in the row above are strictly increasing, i.e.

$$a_{i+1,j} \leq a_{i+1,j+1} \leq a_{i+1,j+2} \rightarrow a_{i,j} < a_{i,j+1}.$$

(3) If two consecutive entries in a row are strictly decreasing and their interlaced neighbour in the row above is equal to its SW-/SE-neighbour, then the interlaced neighbour has a left/right neighbour and is equal to it, i.e.

$$a_{i,j} = a_{i+1,j} > a_{i+1,j+1} \rightarrow a_{i,j-1} = a_{i,j},$$

$$a_{i+1,j} > a_{i+1,j+1} = a_{i,j} \rightarrow a_{i,j+1} = a_{i,j}.$$

Note that by Condition (1) and (2) three consecutive entries in a row of a GMT can not coincide. By way of illustration, let us find all GMTs with bottom row $(4,2,1,3)$: first, construct all possible penultimate rows (l_1,l_2,l_3). Condition (1) implies that $l_1 \in \{2,3,4\}$, Condition (3) further restricts it to $l_1 \in \{2,3\}$. If on the one hand $l_1 = 2$, then Condition (3) forces $l_2 = 2$. The right-most entry l_3 is bounded by 1 and 3, but since $l_1 = l_2 = l_3 = 2$ can not occur, we have $l_3 \in \{1,3\}$. If on the other hand $l_1 = 3$, then Condition (3) implies that $l_2 = l_3 = 1$. Continuing in the same way with all penultimate rows yields the four GMTs depicted in Fig.2.

$$\boxed{\begin{array}{ccccccc} & & & 2 & & & \\ & & 2 & & 2 & & \\ & 2 & & 2 & & 1 & \\ 4 & & 2 & & 1 & & 3 \end{array}} \quad \boxed{\begin{array}{ccccccc} & & & 2 & & & \\ & & 2 & & 3 & & \\ & 2 & & 2 & & 3 & \\ 4 & & 2 & & 1 & & 3 \end{array}} \quad \boxed{\begin{array}{ccccccc} & & & 3 & & & \\ & & 2 & & 3 & & \\ & 2 & & 2 & & 3 & \\ 4 & & 2 & & 1 & & 3 \end{array}} \quad \boxed{\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 3 & & 1 & & 1 & \\ 4 & & 2 & & 1 & & 3 \end{array}}$$

Fig. 2: The four GMTs with bottom row $(4,2,1,3)$.

For $k_1 < k_2 < \cdots < k_n$, the set of GMTs with bottom row $(k_1,\ldots,k_n)$ is equal to the set of Monotone Triangles with this bottom row: Every GMT with strictly increasing bottom row is by Conditions (1) and (2) a Monotone Triangle. Conversely, the weak increase along NE- and SE-diagonals of Monotone Triangles implies Condition (1) of GMTs, the strict increase along rows Condition (2), and the premise of (3) can not hold.

For $k_1 \geq k_2 \geq \cdots \geq k_n$, the set of GMTs with bottom row $(k_1,\ldots,k_n)$ is equal to the set of Decreasing Monotone Triangles with this bottom row: The NE- and SE-diagonals of every GMT with weakly decreasing bottom row are by Condition (1) weakly decreasing. This also implies a weak decrease along rows, and since three consecutive equal entries can not occur, each integer appears at most twice in a row. Furthermore, two consecutive rows can not both contain an integer exactly once due to Condition (3). Conversely, the weak decrease of DMTs along NE- and SE-diagonals implies Condition (1) and weak decrease along rows. Thus, the premise of (2) can only hold if three consecutive entries coincide, which is not admissible in DMTs. Finally, Condition (3) follows from the weak decrease along rows together with the condition that two consecutive rows do not both contain the same integer exactly once.

Therefore, Generalized Monotone Triangles are indeed a joint generalization of ordinary Monotone Triangles and Decreasing Monotone Triangles. The main result of the paper is that the evaluation of $\alpha(n; k_1, k_2, \dots, k_n)$ at integral values is a signed enumeration of the GMTs with bottom row $(k_1, k_2, \dots, k_n)$. The sign of a GMT is determined by the following two statistics:

1. An entry $a_{i,j}$ is called *newcomer* if $a_{i+1,j} > a_{i,j} > a_{i+1,j+1}$.
2. A pair (x, x) of two consecutive equal entries in a row is called *sign-changing*, if their interlaced neighbour in the row below is also equal to x.

In the following, let $\mathcal{G}_n(k_1, k_2, \dots, k_n)$ denote the set of GMTs with bottom row $(k_1, k_2, \dots, k_n)$.

Theorem 1 *Let $n \geq 1$ and $(k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$. Then*

$$\alpha(n; k_1, k_2, \dots, k_n) = \sum_{A \in \mathcal{G}_n(k_1, \dots, k_n)} (-1)^{\mathrm{sc}(A)},$$

where $\mathrm{sc}(A)$ is the total number of newcomers and sign-changing pairs in A.

Applying Theorem 1 to our example in Fig.2 gives $\alpha(4; 4, 2, 1, 3) = -2$, since only the left-most GMT has an even number of sign-changes.

Theorem 1 is known to be true for strictly increasing sequences $k_1 < k_2 < \cdots < k_n$, as in this case the set $\mathcal{G}_n(k_1, \dots, k_n)$ is equal to the set of Monotone Triangles with bottom row $(k_1, k_2, \dots, k_n)$ and $\mathrm{sc}(A) = 0$ for every Monotone Triangle.

Lemma 3 of [FR13] implies the correctness of Theorem 1 for weakly decreasing bottom rows: In this case $\mathcal{G}_n(k_1, \dots, k_n)$ is equal to the set of DMTs with bottom row $(k_1, \dots, k_n)$ and the sc-functions coincide. K. Jochemko and R. Sanyal recently gave a proof of the theorem in this case from a geometric point of view ([JS12]).

In Section 2 we sketch a straight-forward inductive proof of Theorem 1 using a recursion satisfied by $\alpha(n; k_1, \dots, k_n)$ and case distinctions (more details in [Rie12]). In Section 3 a connection with a known generalization ([Fis11]) is established, which enables us to give a shorter, more subtle proof of Theorem 1. Apart from being a joint generalization of Monotone Triangles and DMTs, the newly introduced generalization is more reduced in the sense that fewer cancellations occur in the signed enumeration than in previously known generalizations. In Section 4 we apply the theorem to give a combinatorial proof of an identity satisfied by $\alpha(n; k_1, \dots, k_n)$ and provide a collection of open problems.

2 Summation Operator & Proof of Theorem 1

The number of Monotone Triangles with bottom row $(k_1, \dots, k_n)$ can be counted recursively by determining all admissible penultimate rows $(l_1, \dots, l_{n-1})$ and summing over the number of Monotone Triangles with these bottom rows. The polynomial $\alpha(n; k_1, \dots, k_n)$ hence satisfies

$$\alpha(n; k_1, \dots, k_n) = \sum_{\substack{(l_1, \dots, l_{n-1}) \in \mathbb{Z}^{n-1}, \\ k_1 \leq l_1 \leq k_2 \leq l_2 \leq \cdots \leq k_{n-1} \leq l_{n-1} \leq k_n, \\ l_i < l_{i+1}}} \alpha(n-1; l_1, \dots, l_{n-1}) \tag{2}$$

for all $k_1 < k_2 < \cdots < k_n$, $k_i \in \mathbb{Z}$. In fact ([Fis06]), one can define a summation operator $\sum\limits_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)}$ for arbitrary $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ such that

$$\alpha(n; k_1, \ldots, k_n) = \sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} \alpha(n-1; l_1, \ldots, l_{n-1}) \tag{3}$$

holds. This summation operator is defined recursively by

$$\begin{aligned}\sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} A(l_1, \ldots, l_{n-1}) := &\sum_{(l_1,\ldots,l_{n-2})}^{(k_1,\ldots,k_{n-1})} \sum_{l_{n-1}=k_{n-1}+1}^{k_n} A(l_1, \ldots, l_{n-2}, l_{n-1}) \\ &+ \sum_{(l_1,\ldots,l_{n-2})}^{(k_1,\ldots,k_{n-2},k_{n-1}-1)} A(l_1, \ldots, l_{n-2}, k_{n-1}), \quad n \geq 2,\end{aligned} \tag{4}$$

with $\sum\limits_{()}^{(k_1)} := \mathrm{id}$. Using induction, it is clear that the summation operators in (2) and (3) coincide for increasing sequences $k_1 < k_2 < \cdots < k_n$. In order to give a meaning to (4) for arbitrary $(k_1, \ldots, k_n) \in \mathbb{Z}^n$, we have to extend the definition of simple sums. Motivated by the formal identity $\sum_{i=a}^{b} f(i) = \sum_{i=a}^{\infty} f(i) - \sum_{i=b+1}^{\infty} f(i)$ for $a \leq b$, we define

$$\sum_{i=a}^{b} f(i) := \begin{cases} 0, & b = a-1, \\ -\sum\limits_{i=b+1}^{a-1} f(i), & b+1 \leq a-1. \end{cases} \tag{5}$$

To prove (3) for arbitrary $(k_1, \ldots, k_n) \in \mathbb{Z}^n$, let us first note that applying $\sum\limits_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)}$ to a polynomial in $(l_1, \ldots, l_{n-1})$ yields a polynomial in $(k_1, \ldots, k_n)$: In the base case $n = 2$, write the polynomial $p(l_1)$ in terms of the binomial basis $p(l_1) = \sum_{i=0}^{m-1} c_i \binom{l_1}{i}$. The polynomial $q(x) := \sum_{i=0}^{m-1} c_i \binom{x}{i+1}$ then satisfies $q(x+1) - q(x) = p(x)$. For integers $k_1 \leq k_2$, it follows that $\sum_{l_1=k_1}^{k_2} p(l_1) = q(k_2+1) - q(k_1)$, but this is by definition (5) true for arbitrary $k_1, k_2 \in \mathbb{Z}$. The inductive step is immediate using (4). Thus, we know that the right-hand side of (3) is a polynomial in $(k_1, \ldots, k_n)$ coinciding with the polynomial on the left-hand side whenever $k_1 < k_2 < \cdots < k_n$. Since a polynomial in n variables is uniquely determined by these values, it follows that (3) indeed holds. The same is true for the alternative recursive description

$$\begin{aligned}\sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} A(l_1, \ldots, l_{n-1}) = &\sum_{(l_1,\ldots,l_{n-2})}^{(k_1,\ldots,k_{n-1})} \sum_{l_{n-1}=k_{n-1}}^{k_n} A(l_1, \ldots, l_{n-2}, l_{n-1}) \\ &- \sum_{(l_1,\ldots,l_{n-3})}^{(k_1,\ldots,k_{n-2})} A(l_1, \ldots, l_{n-3}, k_{n-1}, k_{n-1}), \quad n \geq 3.\end{aligned} \tag{6}$$

Lemma 1 *For $(k_1, \dots, k_n) \in \mathbb{Z}^n$ let $\mathcal{P}(k_1, \dots, k_n)$ denote the set of $(n-1)$-st rows of elements in $\mathcal{G}_n(k_1, k_2, \dots, k_n)$. Then every function $A(l_1, \dots, l_{n-1})$ satisfies*

$$\sum_{(l_1,\dots,l_{n-1})}^{(k_1,\dots,k_n)} A(l_1, \dots, l_{n-1}) = \sum_{(l_1,\dots,l_{n-1}) \in \mathcal{P}(k_1,\dots,k_n)} (-1)^{\mathrm{sc}(\boldsymbol{k};\boldsymbol{l})} A(l_1, \dots, l_{n-1}), \quad n \geq 2,$$

where $\mathrm{sc}(\boldsymbol{k}; \boldsymbol{l}) := \mathrm{sc}(k_1, \dots, k_n; l_1, \dots, l_{n-1})$ is the total number of newcomers and sign-changing pairs in $(l_1, \dots, l_{n-1})$.

It is instructive to see how the base case $n = 2$ follows from (4), (5) and the definition of GMTs. In general, the set of admissible values for an entry l_i depends on its neighbours l_{i-1} and l_{i+1} as well as the four adjacent entries k_{i-1}, k_i, k_{i+1} and k_{i+2} in the row below – ordered

$$\begin{array}{ccccccc} & l_{i-1} & & l_i & & l_{i+1} & \\ k_{i-1} & & k_i & & k_{i+1} & & k_{i+2} \end{array}$$

– in the following way: If $k_{i-1} > l_{i-1} = k_i$, then the only admissible value is $l_i = k_i$. Symmetrically, if $k_{i+1} = l_{i+1} > k_{i+2}$, then $l_i = k_{i+1}$. Otherwise, l_i can take any value strictly between k_i and k_{i+1}. To determine whether $l_i = k_i$ is allowed, distinguish between $k_i > k_{i+1}$, $k_{i-1} > k_i \leq k_{i+1}$ and $k_{i-1} \leq k_i \leq k_{i+1}$. If $k_i > k_{i+1}$, then $l_i = k_i$ is admissible, if and only if $l_{i-1} = k_i$. If $k_{i-1} > k_i \leq k_{i+1}$, then $l_i = k_i$ is admissible. If $k_{i-1} \leq k_i \leq k_{i+1}$, then $l_i = k_i$ is admissible, if and only if $l_{i-1} < k_i$. Determining whether $l_i = k_{i+1}$ is admissible works symmetrically.

In order to prove Lemma 1 inductively, we hence have to distinguish between the cases $k_{n-1} \leq k_n$ (Case 1) and $k_{n-1} > k_n$ (Case 2). Since a different behaviour occurs depending on whether l_{n-1} – the rightmost entry of the penultimate row – is equal to k_{n-1} or not, we have to consider sub-cases 1.1, 1.2 and 2.1, 2.2 respectively. Using Recursion (4) in Case 1 and Recursion (6) in Case 2, one can now give a straight-forward proof of the Lemma. The proof in full length can be found in [Rie12]. Theorem 1 is then an immediate consequence of (3) and Lemma 1:

$$\begin{aligned} \alpha(n; k_1, \dots, k_n) &= \sum_{(l_1,\dots,l_{n-1})}^{(k_1,\dots,k_n)} \alpha(n-1; l_1, \dots, l_{n-1}) \\ &= \sum_{(l_1,\dots,l_{n-1}) \in \mathcal{P}(k_1,\dots,k_n)} (-1)^{\mathrm{sc}(\boldsymbol{k};\boldsymbol{l})} \alpha(n-1; l_1, \dots, l_{n-1}) \\ &= \sum_{(l_1,\dots,l_{n-1}) \in \mathcal{P}(k_1,\dots,k_n)} (-1)^{\mathrm{sc}(\boldsymbol{k};\boldsymbol{l})} \sum_{A \in \mathcal{G}_{n-1}(l_1,\dots,l_{n-1})} (-1)^{\mathrm{sc}(A)} = \sum_{A \in \mathcal{G}_n(k_1,\dots,k_n)} (-1)^{\mathrm{sc}(A)}. \end{aligned}$$

3 Connection with different extension & Alternative proof

In [Fis11] four different combinatorial extensions of $\alpha(n; k_1, \dots, k_n)$ to all $(k_1, \dots, k_n) \in \mathbb{Z}^n$ are described. The idea behind all of them is to write the sum in (2) in terms of simple summations, i.e. summations as defined in (5). In the third extension this is based on the inclusion-exclusion principle: For

$k_1 < k_2 < \cdots < k_n$ let

$$\begin{aligned} M &:= \{(l_1, \ldots, l_{n-1}) \in \mathbb{Z}^{n-1} \mid \forall j : k_j \le l_j \le k_{j+1} \ \wedge \ l_j < l_{j+1}\}, \\ A &:= \{(l_1, \ldots, l_{n-1}) \in \mathbb{Z}^{n-1} \mid \forall j : k_j \le l_j \le k_{j+1}\}, \\ A_i &:= \{(l_1, \ldots, l_{n-1}) \in \mathbb{Z}^{n-1} \mid \forall j : k_j \le l_j \le k_{j+1} \ \wedge \ l_{i-1} = k_i = l_i\}, \quad i = 2, \ldots, n-1. \end{aligned}$$

From $k_i < k_{i+1}$ it follows that $A_i \cap A_{i+1} = \emptyset$, and thus we have for any function $f(\boldsymbol{l}) := f(l_1, \ldots, l_{n-1})$ that

$$\sum_{\boldsymbol{l} \in M} f(\boldsymbol{l}) = \sum_{\boldsymbol{l} \in A} f(\boldsymbol{l}) - \sum_{i=2}^{n-1} \sum_{\boldsymbol{l} \in A_i} f(\boldsymbol{l}) + \sum_{\substack{2 \le i_1 < i_2 \le n-1 \\ i_2 \ne i_1 + 1}} \sum_{\boldsymbol{l} \in A_{i_1} \cap A_{i_2}} f(\boldsymbol{l}) - \sum_{\substack{2 \le i_1 < i_2 < i_3 \le n-1 \\ i_{j+1} \ne i_j + 1}} \sum_{\boldsymbol{l} \in A_{i_1} \cap A_{i_2} \cap A_{i_3}} f(\boldsymbol{l}) \cdots, \quad (7)$$

which can be written in terms of simple sums as

$$\sum_{p \ge 0} (-1)^p \sum_{\substack{2 \le i_1 < i_2 < \cdots < i_p \le n-1 \\ i_{j+1} \ne i_j + 1}} \sum_{l_1 = k_1}^{k_2} \sum_{l_2 = k_2}^{k_3} \cdots \sum_{l_{i_1} - 1 = k_{i_1}}^{k_{i_1}} \sum_{l_{i_1} = k_{i_1}}^{k_{i_1}} \cdots \sum_{l_{i_p} - 1 = k_{i_p}}^{k_{i_p}} \sum_{l_{i_p} = k_{i_p}}^{k_{i_p}} \cdots \sum_{l_{n-1} = k_{n-1}}^{k_n} f(\boldsymbol{l}). \quad (8)$$

Applying (5), we can interpret (8) for arbitrary $(k_1, \ldots, k_n) \in \mathbb{Z}^n$. Hence, let us show that

$$\alpha(n; k_1, \ldots, k_n) = \sum_{p \ge 0} (-1)^p \sum_{\substack{2 \le i_1 < i_2 < \cdots < i_p \le n-1 \\ i_{j+1} \ne i_j + 1}} \sum_{l_1 = k_1}^{k_2} \sum_{l_2 = k_2}^{k_3} \cdots \sum_{l_{i_1} - 1 = k_{i_1}}^{k_{i_1}} \sum_{l_{i_1} = k_{i_1}}^{k_{i_1}} \cdots \sum_{l_{i_p} - 1 = k_{i_p}}^{k_{i_p}} \sum_{l_{i_p} = k_{i_p}}^{k_{i_p}} \cdots \sum_{l_{n-1} = k_{n-1}}^{k_n} \alpha(n-1; l_1, \ldots, l_{n-1}) \quad (9)$$

holds for $(k_1, \ldots, k_n) \in \mathbb{Z}^n$: The correctness for $k_1 < k_2 < \cdots < k_n$ is ensured by (2), (7) and (8). Similar to the proof of (3) for arbitrary $(k_1, \ldots, k_n) \in \mathbb{Z}^n$, it suffices to note that the right-hand side of (9) is a polynomial in $k_1, \ldots, k_n$ and thus uniquely determined by its evaluations at $k_1 < \ldots < k_n$.

As pointed out in [Fis11], we can give (9) a combinatorial meaning by interpreting $\alpha(n; k_1, \ldots, k_n)$ as signed enumeration of the following combinatorial objects: In a triangular array $(a_{i,j})_{1 \le j \le i \le n}$ of integers, let us call the entries $a_{i-1,j-1}$ and $a_{i-1,j}$ the parents of $a_{i,j}$. Among the entries $(a_{i,j})_{1 < j < i \le n}$, there may be *special entries*. Special entries in the same row must not be adjacent (choosing these special entries corresponds to fixing the i_l's in (9)). The requirements for the entries are

- If $a_{i,j}$ is special, then $a_{i-1,j-1} = a_{i,j} = a_{i-1,j}$.
- If $a_{i,j}$ is not the parent of a special entry and $a_{i+1,j} \le a_{i+1,j+1}$, then $a_{i+1,j} \le a_{i,j} \le a_{i+1,j+1}$.
- If $a_{i,j}$ is not the parent of a special entry and $a_{i+1,j} > a_{i+1,j+1}$, then $a_{i+1,j+1} > a_{i,j} > a_{i+1,j}$. In this case $a_{i,j}$ is called *inversion*.

Let us denote by $\mathcal{T}_n(k_1,\ldots,k_n)$ the set of these objects with bottom row $(a_{n,1},\ldots,a_{n,n}) = (k_1,\ldots,k_n)$. For $A \in \mathcal{T}_n(k_1,\ldots,k_n)$ let $s(A)$ be the total number of special entries and inversions. Using induction and (9) yields

$$\alpha(n;k_1,\ldots,k_n) = \sum_{A\in\mathcal{T}_n(k_1,\ldots,k_n)} (-1)^{s(A)}. \tag{10}$$

In the following, we give an alternative proof of Theorem 1 by finding cancellations occurring in (10). An advantage of removing these cancellations is that the notion of special entries will no longer be required. In fact, what we obtain after this reduction are exactly the GMTs. To be more concrete, we can eliminate those arrays $(a_{i,j})_{1\le j\le i\le n}$ violating the condition

$$a_{i,j-1} \le a_{i,j} \le a_{i,j+1} \;\rightarrow\; a_{i-1,j-1} < a_{i-1,j}$$

by using the following sign-reversing involution: find the minimal index i, and under those the minimal index j such that $a_{i,j-1} \le a_{i,j} \le a_{i,j+1}$ and $a_{i-1,j-1} = a_{i,j} = a_{i-1,j}$. If $a_{i,j}$ is special, then turn it non-special, and vice-versa. Note that the minimality ensures that if $a_{i,j}$ is not special, then the neighbours of $a_{i,j}$ are not special, i.e. turning $a_{i,j}$ special is admissible. It follows that

$$\alpha(n;k_1,\ldots,k_n) = \sum_{\substack{A\in\mathcal{T}_n(k_1,\ldots,k_n)\\ a_{i,j-1}\le a_{i,j}\le a_{i,j+1}\rightarrow a_{i-1,j-1}<a_{i-1,j}}} (-1)^{s(A)}.$$

Note that in this reduced set an entry $a_{i,j}$ is special if and only if $a_{i-1,j-1} = a_{i,j} = a_{i-1,j}$. Since special entries now correspond to sign-changing pairs and inversions to newcomers, the only remaining part for proving Theorem 1 is to show that

$$\mathcal{G}_n(k_1,\ldots,k_n) = \{A \in \mathcal{T}_n(k_1,\ldots,k_n) : a_{i,j-1} \le a_{i,j} \le a_{i,j+1} \;\rightarrow\; a_{i-1,j-1} < a_{i-1,j}\},$$

where an entry $a_{i,j}$ is special if and only if $a_{i-1,j-1} = a_{i,j} = a_{i-1,j}$.

Let $A \in \mathcal{G}_n(k_1,\ldots,k_n)$. Then two adjacent special entries in a row would imply three consecutive equal entries in a row, in contradiction to Condition (2) of GMTs. If $a_{i,j}$ is special, then $a_{i-1,j-1} = a_{i,j} = a_{i-1,j}$ by definition. If $a_{i+1,j} \le a_{i+1,j+1}$, then $a_{i+1,j} \le a_{i,j} \le a_{i+1,j+1}$ by Condition (1) of GMTs. If $a_{i+1,j} > a_{i+1,j+1}$, then $a_{i+1,j} \ge a_{i,j} \ge a_{i+1,j+1}$ by Condition (1) of GMTs, and if $a_{i+1,j}$ and $a_{i+1,j+1}$ are neither special, Condition (3) of GMTs implies that $a_{i+1,j} > a_{i,j} > a_{i+1,j+1}$. We thus have $A \in \mathcal{T}_n(k_1,\ldots,k_n)$, and the additional property is exactly Condition (2) of GMTs.

Conversely, let $A \in \mathcal{T}_n(k_1,\ldots,k_n)$ such that $a_{i,j-1} \le a_{i,j} \le a_{i,j+1}$ implies $a_{i-1,j-1} < a_{i-1,j}$. Conditions (1) and (2) of GMTs are then trivially satisfied. If $a_{i,j} = a_{i+1,j} > a_{i+1,j+1}$, then $a_{i+1,j}$ is a special entry, and thus $a_{i,j} = a_{i+1,j} = a_{i,j-1}$. Symmetrically, if $a_{i+1,j} > a_{i+1,j+1} = a_{i,j}$, then $a_{i+1,j+1}$ is special, and thus $a_{i,j} = a_{i+1,j+1} = a_{i,j+1}$. In total, we have $A \in \mathcal{G}_n(k_1,\ldots,k_n)$.

4 Applications & Open Problems

With this generalization at hand, we can try to give a combinatorial interpretation to identities satisfied by $\alpha(n;k_1,\ldots,k_n)$. By way of illustration, take the identity

$$\begin{aligned}&\alpha(n;k_1,\ldots,k_{i-1},k_i,k_i+1,k_{i+2},\ldots,k_n)\\ &= \alpha(n;k_1,\ldots,k_{i-1},k_i,k_i,k_{i+2},\ldots,k_n) + \alpha(n;k_1,\ldots,k_{i-1},k_i+1,k_i+1,k_{i+2},\ldots,k_n).\end{aligned} \tag{11}$$

A combinatorial proof of this identity in the case that $k_1 < k_2 < \cdots < k_i$ and $k_i + 1 < k_{i+2} < \cdots < k_n$ was given in [Fis11]. Using Theorem 1, we can now give a combinatorial proof for arbitrary $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ by showing that there exists a sign-preserving bijection

$$\begin{aligned}&\mathcal{G}_n(k_1, \ldots, k_{i-1}, k_i, k_i + 1, k_{i+2}, \ldots, k_n)\\ &\qquad \leftrightarrow \mathcal{G}_n(k_1, \ldots, k_{i-1}, k_i, k_i, k_{i+2}, \ldots, k_n) \,\dot{\cup}\, \mathcal{G}_n(k_1, \ldots, k_{i-1}, k_i + 1, k_i + 1, k_{i+2}, \ldots, k_n).\end{aligned}$$

If $\mathcal{P}(k_1, \ldots, k_n)$ denotes the set of penultimate rows of GMTs with bottom row $(k_1, \ldots, k_n)$, it suffices to show that

$$\begin{aligned}&\mathcal{P}(k_1, \ldots, k_{i-1}, k_i, k_i + 1, k_{i+2}, \ldots, k_n)\\ &\qquad = \mathcal{P}(k_1, \ldots, k_{i-1}, k_i, k_i, k_{i+2}, \ldots, k_n) \,\dot{\cup}\, \mathcal{P}(k_1, \ldots, k_{i-1}, k_i + 1, k_i + 1, k_{i+2}, \ldots, k_n),\end{aligned} \tag{12}$$

where each fixed row has the same total number of sign-changes on both sides.

Each $(l_1, \ldots, l_{n-1}) \in \mathcal{P}(k_1, \ldots, k_{i-1}, k_i, k_i + 1, k_{i+2}, \ldots, k_n)$ satisfies $l_i \in \{k_i, k_i + 1\}$. Let us show that the set of penultimate rows with $l_i = k_i$ is equal to $\mathcal{P}(k_1, \ldots, k_{i-1}, k_i, k_i, k_{i+2}, \ldots, k_n)$. With $l_i = k_i$ it is clear that the restrictions for $(l_1, \ldots, l_{i-1})$ and $(l_{i+3}, \ldots, l_{n-1})$ are identical for both $\mathcal{P}(k_1, \ldots, k_{i-1}, k_i, k_i + 1, k_{i+2}, \ldots, k_n)$ and $\mathcal{P}(k_1, \ldots, k_{i-1}, k_i, k_i, k_{i+2}, \ldots, k_n)$. For the restrictions of (l_{i+1}, l_{i+2}) distinguish between $k_i + 1 \leq k_{i+2}$, $k_i = k_{i+2}$ and $k_i > k_{i+2}$:

- If $k_i + 1 \leq k_{i+2}$, then $k_i + 1 \leq l_{i+1} \leq k_{i+2}$ on both sides and the restrictions for l_{i+2} are the same:

Left-hand side of (12)	Right-hand side of (12)
$\begin{array}{ccccccccc} & & k_i & & & & l_{i+1} & & \\ & \rotatebox{45}{$=$} & & \rotatebox{-45}{$<$} & & \rotatebox{45}{$\leq$} & & \rotatebox{-45}{$\leq$} & \\ k_i & & < & & k_i+1 & & \leq & & k_{i+2} \end{array}$	$\begin{array}{ccccccccc} & & k_i & & & & l_{i+1} & & \\ & \rotatebox{45}{$=$} & & \rotatebox{-45}{$=$} & & \rotatebox{45}{$<$} & & \rotatebox{-45}{$\leq$} & \\ k_i & & = & & k_i & & \leq & & k_{i+2} \end{array}$

- If $k_i = k_{i+2}$, then $\mathcal{P}(k_1, \ldots, k_{i-1}, k_i, k_i, k_{i+2}, \ldots, k_n)$ is empty, and each element of $\mathcal{P}(k_1, \ldots, k_{i-1}, k_i, k_i + 1, k_{i+2}, \ldots, k_n)$ with $l_i = k_i$ would have to satisfy $l_i = l_{i+1} = l_{i+2} = k_i$:

Left-hand side of (12)	Right-hand side of (12)
$\begin{array}{cccccccccc} & & k_i & & & & k_i & & & k_i \\ & \rotatebox{45}{$=$} & & \rotatebox{-45}{$<$} & & \rotatebox{45}{$>$} & & \rotatebox{-45}{$=$} & & \\ k_i & & < & & k_i+1 & & > & & k_i & \end{array}$	$\begin{array}{ccccccccc} & & k_i & & & & \lightning & & \\ & \rotatebox{45}{$=$} & & \rotatebox{-45}{$=$} & & & & & \\ k_i & & = & & k_i & & = & & k_i \end{array}$

  But, since a GMT can not contain three consecutive equal entries, there is also no element in $\mathcal{P}(k_1, \ldots, k_{i-1}, k_i, k_i + 1, k_i, \ldots, k_n)$ with $l_i = k_i$.

- If $k_i > k_{i+2}$, then $k_i \geq l_{i+1} \geq k_{i+2}$ on both sides and the restrictions for l_{i+2} are the same:

Left-hand side of (12)	Right-hand side of (12)
$\begin{array}{ccccccccc} & & k_i & & & & l_{i+1} & & \\ & \rotatebox{45}{$=$} & & \rotatebox{-45}{$<$} & & \rotatebox{45}{$>$} & & \rotatebox{-45}{$\geq$} & \\ k_i & & < & & k_i+1 & & > & & k_{i+2} \end{array}$	$\begin{array}{ccccccccc} & & k_i & & & & l_{i+1} & & \\ & \rotatebox{45}{$=$} & & \rotatebox{-45}{$=$} & & \rotatebox{45}{$\geq$} & & \rotatebox{-45}{$\geq$} & \\ k_i & & = & & k_i & & > & & k_{i+2} \end{array}$

The entry l_{i+1} is involved in a sign-change on both sides (note the special case $l_{i+1} = k_i$, where l_{i+1} is a newcomer on the left-hand side and in a sign-changing pair on the right-hand side).

Symmetrically, one can also see that the set $\mathcal{P}(k_1, \ldots, k_{i-1}, k_i, k_i + 1, k_{i+2}, \ldots, k_n)$ restricted to $l_i = k_i + 1$ is the same as $\mathcal{P}(k_1, \ldots, k_{i-1}, k_i + 1, k_i + 1, k_{i+2}, \ldots, k_n)$, concluding the combinatorial proof of (11) for arbitrary $(k_1, \ldots, k_n) \in \mathbb{Z}^n$.

A natural question is now whether similar identities hold if the difference between k_{i+1} and k_i is larger. For fixed integers $k_1, \ldots, k_{i-1}, k_{i+2}, \ldots, k_n$, let

$$t_n(k_i, k_{i+1}) := \alpha(n; k_1, \ldots, k_{i-1}, k_i, k_{i+1}, k_{i+2}, \ldots, k_n).$$

Similarly - with a bit more patience - one can also show the identity

$$t_n(k_i, k_i+2) = t_n(k_i, k_i)+t_n(k_i+1, k_i+1)+t_n(k_i+2, k_i+2)+t_n(k_i+2, k_i+1)+t_n(k_i+1, k_i) \quad (13)$$

combinatorially. Both (11) and (13) are special cases of the following algebraic identity: Let $V_{x,y}$ be the operator defined as $V_{x,y} f(x, y) := f(x - 1, y) + f(x, y + 1) - f(x - 1, y + 1)$. The function $f_i(k_1, \ldots, k_n) := V_{k_i,k_{i+1}} \alpha(n; k_1, \ldots, k_n)$ then satisfies

$$f_i(k_1, \ldots, k_n) = -f_i(k_1, \ldots, k_{i-1}, k_{i+1} + 1, k_i - 1, k_{i+2}, \ldots, k_n). \quad (14)$$

Setting $k_{i+1} = k_i - 1$ in (14) immediately implies (11). Equation (13) is then the special case $k_{i+1} = k_i - 2$ in (14). A similar shift-antisymmetry property for *Gelfand-Tsetlin Patterns* (Monotone Triangles without the condition of strict increase along rows) was shown bijectively in a recent work ([Fis11]). It would be interesting to give a bijective proof of (14) in the general case (an algebraic proof was given in [Fis06]).

In [FR13] we showed the surprising identity

$$A_n := \alpha(n; 1, 2, \ldots, n) = \alpha(2n; n, n, n - 1, n - 1, \ldots, 1, 1) \quad (15)$$

algebraically and gave initial thoughts on how a bijective proof could succeed. Let us conclude with a list of related identities – all of them are up to this point conjectured using mathematical computing software. As Theorem 1 provides a combinatorial interpretation of these identities, bijective proofs are of high interest.

Conjecture 1 ([FR13]) *Let* $n \geq 1$. *Then*

$$\alpha(n; 2, 4, \ldots, 2n) = (-1)^n \alpha(2n + 1; 2n + 1, 2n, \ldots, 1) \quad (16)$$

holds, whereby the left-hand side is known to be the number of Vertically Symmetric ASMs of size $2n + 1$. *By Theorem 1, the right-hand side is further equal to* $\alpha(2n; 2n, 2n, 2n - 2, 2n - 2, \ldots, 2, 2)$.

Conjecture 2 *Let* $n \geq 1$. *Then*

$$A_n = \alpha(n + i; 1, 2, \ldots, i, 1, 2, \ldots, n), \quad i = 0, \ldots, n, \quad (17)$$

$$A_n = (-1)^n \alpha(2n + 1; 1, 2, \ldots, n + 1, 1, 2, \ldots, n) \quad (18)$$

holds. Furthermore, the numbers

$$W_{n,i} = \alpha(2n + 1; i, 2, \ldots, n + 1, 1, 2, \ldots, n), \quad i = 1, \ldots, 3n + 2$$

satisfy the symmetry $W_{n,i} = W_{n,3n+3-i}$.

Conjecture 3 *Let $n \geq 2$. Then*

$$A_n = \alpha(n+2; 1, 2, \ldots, i+1, i, i+1, \ldots, n), \quad i = 1, \ldots, n-1 \tag{19}$$

holds.

Further computational experiments led to the conjecture that (15) and (19) have the following joint generalization:

Conjecture 4 *Let $n \geq 1$. Then*

$$A_n = \alpha(n+k; 1, \ldots, i-1, i+k-1, i+k-1, i+k-2, i+k-2, \ldots, i, i, i+k, i+k+1, \ldots, n) \tag{20}$$

holds for $i = 1, \ldots, n-k+1$, $k = 1, \ldots, n$.

In words, the last identity takes a subsequence $(i, i+1, \ldots, i+k-1)$ of length k of $(1, 2, \ldots, n)$, reverses the order, duplicates each entry and puts the subsequence back. Identity (15) is thus the special case of (20) where $k = n$. Applying (11) and the fact that a GMT can not contain three consecutive equal entries, shows that (19) is the special case of (20) with $k = 2$:

$$\begin{aligned}
&\alpha(n+2; 1, 2, \ldots, i-1, i, i+1, i, i+1, i+2, \ldots, n) \\
&= \alpha(n+2; 1, 2, \ldots, i-1, i, i, i, i+1, i+2, \ldots, n) + \alpha(n+2; 1, 2, \ldots, i-1, i+1, i+1, i, i+1, i+2, \ldots, n) \\
&= \alpha(n+2; 1, 2, \ldots, i-1, i+1, i+1, i, i, i+2, \ldots, n) + \alpha(n+2; 1, 2, \ldots, i-1, i+1, i+1, i+1, i+1, i+2, \ldots, n) \\
&= \alpha(n+2; 1, 2, \ldots, i-1, i+1, i+1, i, i, i+2, \ldots, n).
\end{aligned}$$

From the correspondence between ASMs of size n and Monotone Triangles with bottom row $(1, 2, \ldots, n)$, it follows that $\alpha(n-1; 1, 2, \ldots, i-1, i+1, \ldots, n)$ is equal to the number of ASMs of size n with the first row's unique 1 in column i – denoted $A_{n,i}$. In the following conjecture we analogously remove the i-th argument of the right-hand side in (19):

Conjecture 5 *Let $n \geq 1$. Then*

$$\alpha(n+1; 1, 2, \ldots, i-1, i+1, i, i+1, \ldots, n) = -\sum_{j=1}^{n} (j-i) A_{n,j}, \quad i = 1, \ldots, n-1 \tag{21}$$

holds.

As a note on how we found (21), let us prove the case $i = 1$: Each penultimate row $(l_1, \ldots, l_n)$ of a GMT with bottom row $(2, 1, 2, \ldots, n)$ satisfies $l_1 = l_2 = 1$ by Condition (3) of GMTs. Taking Conditions (1) and (2) into account, Lemma 1 implies that

$$\alpha(n+1; 2, 1, 2, \ldots, n) = -\sum_{p=2}^{n} \alpha(n; 1, 1, 2, \ldots, p-1, p+1, \ldots, n).$$

Each penultimate row $(m_1, \ldots, m_{n-1})$ of a GMT with bottom row $(1, 1, 2, \ldots, p-1, p+1, \ldots, n)$ satisfies $m_1 = 1, m_2 = 2, \ldots, m_{p-1} = p-1$. Applying Lemma 1 again yields the claimed equation:

$$\alpha(n+1; 2, 1, 2, \ldots, n) = -\sum_{p=2}^{n} \sum_{j=p}^{n} A_{n,j} = -\sum_{j=2}^{n} (j-1) A_{n,j}.$$

For general i, the set of GMTs with bottom row $(1, 2, \dots, i-1, i+1, i, i+1, \dots, n)$ can be written as disjoint union of those with structure

$$S_1: \begin{array}{ccccccccccccccccccc} & l_1 & & \cdots & & l_{i-2} & & i+1 & & i+1 & & i & & l_{i+2} & & \cdots & & l_n & \\ 1 & & \cdots & & i-2 & & i-1 & & i+1 & & i & & i+1 & & i+2 & & \cdots & & n, \end{array}$$

$$S_2: \begin{array}{ccccccccccccccccccc} & l_1 & & \cdots & & l_{i-2} & & i+1 & & i & & i & & l_{i+2} & & \cdots & & l_n & \\ 1 & & \cdots & & i-2 & & i-1 & & i+1 & & i & & i+1 & & i+2 & & \cdots & & n, \end{array}$$

$$S_3: \begin{array}{ccccccccccccccccccc} & l_1 & & \cdots & & l_{i-2} & & i-1 & & i & & i & & l_{i+2} & & \cdots & & l_n & \\ 1 & & \cdots & & i-2 & & i-1 & & i+1 & & i & & i+1 & & i+2 & & \cdots & & n. \end{array}$$

Similar to the case $i = 1$, one can see that the signed enumeration of GMTs with structure S_3 is equal to $-\sum_{j=i+1}^{n} (j-i) A_{n,j}$. Proving that the signed enumeration of GMTs with structure S_1 and S_2 yields $-\sum_{j=1}^{i-1} (j-i) A_{n,j}$ remains an open problem. A list of more conjectures can be found in [Rie12].

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On the ranks of configurations on the complete graph

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Abstract. We consider the parameter rank introduced for graph configurations by M. Baker and S. Norine. We focus on complete graphs and obtain an efficient algorithm to determine the rank for these graphs. The analysis of this algorithm leads to the definition of a parameter on Dyck words, which we call prerank. We prove that the distribution of area and prerank on Dyck words of given length $2n$ leads to a polynomial with variables q, t which is symmetric in these variables. This polynomial is different from the q, t-Catalan polynomial studied by A. Garsia, J. Haglund and M. Haiman.

Résumé. Nous considérons le paramètre rang sur les configurations d'un graphes introduit par Baker et Norine . Nous nous intéressons plus particulièrement aux graphes complets et obtenons un algorithme efficace de déxtermination du rang d'une configuration pour ceux-ci. L'analyse de la complexité de cet algorithme conduit à définir un paramètre sur les mots de Dyck que nous appelons pré-rang. Nous démontrons que la distribution des aires et pré-rangs des mots de Dyck donne lieu à un polynôme à deux variables qui est symétrique en celles-ci. Il est différent du polynôme q, t-Catalan étudié par A. Garsia, J. Haglund et par M. Haiman.

Keywords: Rank, Riemann-Roch for graphs, Complete graphs, Dyck Words

We consider the following solitary game on an undirected connected graph with no loops: at the beginning a configuration u is given, meaning that integer values u_i are attributed to the n vertices $x_1, x_2, \ldots x_n$ of the graph. These values can be positive or negative. At each step a toppling can be performed by the player on a vertex x_i: it consists in subtracting d_i (the number of neighbors of x_i) to the amount u_i and adding 1 to all the amounts u_j of the neighbors x_j of x_i. In this operation the amount of vertex x_i may become negative. The aim of the player is to find a sequence of toppling operations which will end with a configuration where all the u_i are non negative. Since the sum $deg(u)$ of the u_i is invariant by the toppling, a necessary condition to succeed is that in the initial configuration $deg(u)$ should be non negative.

This game has much to do with the chip firing game (see Björner et al. (1991), Biggs (1999)) and the sandpile model (see Bak et al. (1988), Dhar (1990), Dhar and Majumdar (1992)), for which recurrent configurations were defined and proved to be canonical representatives of the classes of configurations equivalent by a sequence of topplings.

The game was introduced and studied in detail by Baker and Norine (Baker and Norine (2007)) who introduced a new parameter on graph configurations: the rank. The rank $\rho(u)$ of a configuration u is non negative if and only if one can get from u a positive configuration by performing a sequence of topplings.

For this parameter they obtain a simple formula expressing a symmetry similar to the Riemann-Roch formula for surfaces (a classical reference to this formula is the book by Farkas and Kra (1992)).

Our aim here is to study the values of this parameter when G is the complete graph on n vertices, for these graphs it was noticed (see Proposition 2.8. in Cori and Rossin (2000)) that the recurrent configurations correspond to the parking functions which play a central role in combinatorics. We obtain a simple greedy algorithm to compute the rank in that case, expected to be of linear complexity after optimisation, while there is no known polynomial time algorithm to compute that rank for arbitrary graphs.

The distribution of rank and degree on a natural subset of configurations over a graph G, the parking ones, is a bivariate power series $P_G(x,r)$ which has a symmetry inherited from the Riemann-Roch theorem. We show that some coefficients of these series are related to an evaluation of Tutte polynomial. In the case of complete graphs, we prove that our greedy algorithm to compute the rank has a linear complexity when assuming that arithmetic operations on the u_i may be performed in constant time. Up to the classical action of symmetric group S_n on configurations our algorithm may be described in terms of Dyck words. The analysis of this algorithm leads to the definition of a parameter on Dyck words, which will call prerank. We prove that the distribution of area and prerank on Dyck words of length $2n$ leads to a polynomial in two variables which is symmetric in these. This polynomial has some values in common with the q,t-Catalan polynomial studied in Garsia and Haiman (1996); Haglund (2008). We provide a bijective proof of the symmetry of our polynomial and propose an expression for it using Tchebychev polynomials. Moreover the bistatistic prerank and dinv leads to the q,t-Catalan polynomial.

1 Configurations on a graph

1.1 The Laplacian configurations

Let $G = (X, E)$ be a multi-graph, where $X = \{x_1, x_2, \ldots, x_n\}$ is the vertex set and E is a symmetric matrix such that $e_{i,j}$ is the number of edges with endpoints x_i, x_j, hence $e_{i,j} = e_{j,i}$. In all this paper n denotes the number of vertices of the graph G and m the number of its edges. Moreover we suppose that G is connected and has no loops, so that $e_{i,i} = 0$ for all i.

We will consider *configurations* on this graph, which are elements of the discrete lattice $\mathbb{Z}^n$. Each configuration u may be considered as assigning (positive or negative) tokens to the vertices. When there is no possibility of confusion the symbol x_i will also denote the configuration in which the value 1 is assigned to vertex x_i is and the value 0 is assigned to all others. *Laplacian configurations* $\Delta^{(i)}$ given by: $\Delta^{(i)} = d_i x_i - \sum_{i=1}^{n} e_{i,j} x_j$, where $d_i = \sum_{i=1}^{n} e_{i,j}$ is the degree of the vertex x_i, play a central role througout this paper.

The *degree* of the configuration u is the sum of the u_i's and is denoted $deg(u)$. We denote by L_G the subgroup of $\mathbb{Z}^n$ generated by the $\Delta^{(i)}$, and two configurations u and v will be said *toppling equivalent* if $u - v \in L_G$, which will also be written as $u \sim_{L_G} v$.

1.2 Parking configurations

In each class of $\sim_{L_G}$ one configuration may be considered as a canonical representative. We call such configurations *parking configurations* since in the case of complete graphs, these are exactly the parking functions, a central object in combinatorics.

Definition 1 *A configuration u on a graph G is a parking configuration if $u_i \geq 0$ for $i < n$ and for any subset Y of $\{x_1, x_2, \ldots, x_{n-1}\}$ there is a vertex x_i in Y such that u_i is less than the number of edges* *which have as endpoints x_i and an x_j not in Y. More precisely if there exists i such that $x_i \in Y$ and $u_i < \sum_{x_j \notin Y} e_{i,j}$.*

In other words a configuration u is a parking configuration if and only if there is no toppling of all the vertices in a subset Y of $\{x_1, x_2, \dots x_{n-1}\}$ leaving all the $u_i \geq 0$.

Proposition 1 *For any configuration u there exists a unique parking configuration denoted $parking(u)$ such that $u - parking(u) \in L_G$*

The proof of this Proposition is based on the notion of recurrent configurations which was considered and characterized by D. Dhar, a simple proof of the the uniqueness of a recurrent configuration is given in Cori and Rossin (2000).

1.3 Parking configurations and acyclic orientations

An orientation of G is a directed graph obtained from G by orienting each edge, so that one end vertex is called the head and the other vertex is called the tail. A directed path in such a graph consists of a sequence of edges such that the head of an edge is equal to the tail of the subsequent one.

The orientation is acyclic if there is no directed circuit, i.e. a directed path starting and ending at the same vertex. We associate to any parking configuration u an acyclic orientation by:

Proposition 2 *For any parking configuration u on $G = (X, E)$ there exists an acyclic orientation $\overrightarrow{G}$ such that for any vertex x_i, $i \neq n$, u_i is strictly less than its indegree d_i^-.*

Proof: We orient the edges using an algorithm that terminates after n steps. Consider $Y = \{x_1, x_2, \dots, x_{n-1}\}$. From the condition for parking configurations given above, there is one vertex x_i such that $u_i < e_{i,n}$ then orient all these $e_{i,n}$ edges from x_n to x_i, and remove x_i from Y. Repeat the following operation until Y is empty:

- Find x_k in Y such that $u_k < \sum_{x_j \notin Y} e_{k,j}$; orient all the edges joining any vertex j outside Y to x_k from x_j to x_k and remove x_k from Y.

□

2 Effective configurations and rank

In this section we give the main results of Baker and Norine (2007).

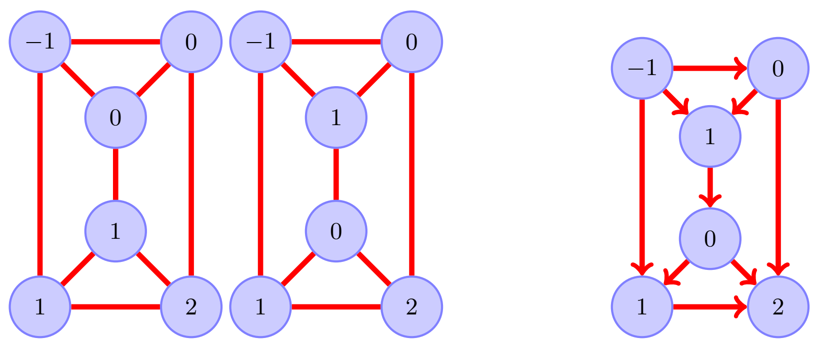
Definition 2 *A configuration u is positive if $u_i \geq 0$ for all i. A configuration u is effective if there exists a positive configuration v such that $u - v \in L_G$.*

Since two equivalent configurations by $\sim_{L_G}$ have the same degree, it is clear that a configuration with negative degree is not effective. However we will prove that configurations with positive degree are not necessarily effective as the two examples in Figure 1(a) show.

2.1 Configuration associated to an acyclic orientation of G

Let $\overrightarrow{G}$ be an acyclic orientation of G, we define the configuration $u_{\overrightarrow{G}}$ by: $(u_{\overrightarrow{G}})_i = d_i^- - 1$, where d_i^- is the number of edges which have head x_i. The configuration represented in Figure 1(b) is equal to $u_{\overrightarrow{G}}$ for the represented orientation of G.

Proposition 3 *The configuration associated to an acyclic orientation of G is non effective.*



(a) An effective configuration (left) and a non effective one (right)

(b) An orientation of G and the corresponding configuration

Fig. 1: Examples of effective, non-effective configurations and orientations.

2.2 Characterisation of effective configurations

Theorem 1 *For any configuration u, one and only one of the following assertions is satisfied:*
(1) u is effective.
(2) There exists an acyclic orientation $\overrightarrow{G}$ such that $u_{\overrightarrow{G}} - u$ is effective.
Moreover u is effective if and only if the parking configuration v such that $u \sim_{L_G} v$ satisfies $v_n \geq 0$.

Corollary 1 *Any configuration u with degree greater than $m - n$ is effective.*

Proof: If u such that $deg(u) > m - n$ is not effective, by the above theorem there exists an acyclic orientation $\overrightarrow{G}$ of G such that $u_{\overrightarrow{G}} - u$ is. But the degree of this configuration is negative, giving a contradiction. □

2.3 The rank of configurations

From now on it will be convenient to denote positive configurations by using the letters $f, g \cdots$ and configurations with no particular assumptions on them by the letters u, v, w.

Definition 3 *The rank $\rho(u)$ of a configuration is the integer defined by:*

- *If u is non effective it is equal to -1*
- *If u is effective, it is the largest integer r such that for any positive configuration f of degree r the configuration $u - f$ is effective.*

Denoting $\mathbb{P}$ as the set of positive configurations and $\mathbb{E}$ as the set of effective configurations, this definition can given by the following formula which is valid in both cases:

$$\rho(u) + 1 = \min_{f \in \mathbb{P}, u-f \notin \mathbb{E}} \quad deg(f).$$

An immediate consequence of this definition is that if $deg(u) \geq -1$ then $\rho(u) \leq deg(u)$, and for any acyclic orientation $\overrightarrow{G}$ the rank of $u_{\overrightarrow{G}}$ is -1. Moreover if two configurations u and v are such that $u_i \leq v_i$ for all i then $\rho(u) \leq \rho(v)$.

Definition 4 *A positive configuration f is a proof for the rank $\rho(u)$ of an effective configuration u if $u-f$ is non effective and $u-h$ is effective for any positive configuration h such that $deg(h) < deg(f)$.*

Notice that if f is a proof for $\rho(u)$ then $\rho(u) = deg(f) - 1 = deg(f) + \rho(u-f)$.

Proposition 4 *A configuration u of degree greater than $2m-2n$ has rank r such that*

$$r + 1 = deg(u) - (m - n).$$

Proof: We first show that for any positive configuration f such that $deg(f) = r$, the configuration $u - f$ is effective. This follows from $deg(u - f) = deg(u) - r = m - n + 1$ by Corollary 1.

We now build a positive configuration f of degree $r + 1$ such that $u - f$ is not effective. Consider any acyclic orientation $\overrightarrow{G}$ of G and let $v = u - u_{\overrightarrow{G}}$. Then v is effective since its degree is equal to $deg(u) - m + n$ and is therefore greater than $m - n$. Let f be the positive configuration such that $v \sim_{L_G} f$, then $u - f$ is such that $u_{\overrightarrow{G}} \sim_{L_G} u - v \sim_{L_G} u - f$ so that $u - f$ is not effective by Theorem 1. □

This result can be generalized into the following theorem which was given in Baker and Norine (2007) and called the Riemann-Roch theorem for graphs. A geometric interpretation of it is given in Amini and Manjunath (2010) and used in Manjunath (2011).

Theorem 2 *Let κ be the configuration such that $\kappa_i = d_i - 2$ where for $i = 1, \ldots, n$, the value d_i is the degree of the vertex x_i. Then we have for any configuration u:*

$$\rho(u) - \rho(\kappa - u) = deg(u) - (m - n).$$

3 A greedy algorithm computing the rank for configurations on complete graphs

Configurations on the complete graph may be sorted in such a way that the first $n-1$ components form a weakly decreasing sequence. Clearly any configuration and its sorted version have equal ranks. The algorithm for determining the rank of u that we will describe proceeds in a certain number of steps. Each of these steps consists in replacing u by a u', and it will be convenient to work on their sorted versions. From an algebraic point of view this consists in considering orbits of the action of the symmetric group S_{n-1} on the first $n-1$ components instead of mere configurations; the correctness of the computation is validated by the fact that all configurations in the same orbit have the same rank.

3.1 Greedy algorithm on parking functions

Any configuration u is toppling equivalent to a single parking configuration $parking(u)$. In the case of the complete graph K_n there is a linear time algorithm to compute it. It will be given below after developing the link between Dyck words and parking configurations. We first examine how to determine the rank of a parking configuration. On K_n, a configuration u is a parking one if and only if after sorting the first $n-1$ entries one obtains $v = (v_1, \ldots v_{n-1}, u_n)$, satisfying $0 \le v_i < n - i$ for any $1 \le i < n$. In particular, $v_{n-1} = 0$; so in any parking configuration at least one of the u_i's is equal to 0. Our greedy algorithm determines the rank of a configuration u on K_n by iteratively computing the parking configuration v equivalent to u and subtracting 1 on one of the v_i such that [i] $v_i = 0$ until the resulting parking configuration is such that $u_n < 0$. The rank is then equal to the number of iterations done, the algorithm is given in the left part of Figure 2.The fact that this algorithm correctly computes the rank is a consequence of the lemma below.

$u \leftarrow parking(u)$
$rank \leftarrow -1$
while $u_n \geq 0$ **do**
  $u \leftarrow$ subtract 1 in one of a u_i such that $u_i = 0$ and $i < n$
  $u \leftarrow parking(u)$
  $rank \leftarrow rank + 1$
end while
Return $rank$

$u \leftarrow parking(u)$
$(d, s) \leftarrow (d(u), s(u))$
$rank \leftarrow -1$
while $s \geq 0$ **do**
  match d **with** $afbg$
  $d \leftarrow gabf$
  $rank \leftarrow rank + 1$
  $s \leftarrow s - |afb|_a$
end while
Return $rank$

Fig. 2: Two versions of a greedy algorithm computing rank on K_n: on configurations and Dyck words.

Lemma 1 *Any positive configuration u where $u_i = 0$ admits a proof g for its rank such that $g_i > 0$.*

Proof: Denote by $\epsilon^{(i)}$ the configuration where $\epsilon^{(i)}_i = 1$ and for $j \neq i$, $\epsilon^{(i)}_j = 0$. Let $f \geq 0$ be a proof of $\rho(u)$ and assume $f_i = 0$, otherwise $g = f$ satisfies the lemma. Let $j \neq i$ such that $u_j - f_j = -a < 0$. Let $v = u-(f-a\epsilon^{(j)})$. Then $0 \leq f-a\epsilon^{(j)} \leq f$ and $v_i = 0 = v_j$. Let τ be the transposition which exchanges i and j. Since $v = \tau v$, we have $g = f - a\epsilon^{(j)} + a\epsilon^{(i)}$ satisfies $g_i > 0$, hence it is positive and has the same degree as f. Moreover $u - g$ is also non-effective since $u - g = v - a\epsilon^{(i)} = \tau.[v - a\epsilon^{(j)}] = \tau(u - f)$, hence g is the proof of $\rho(u)$ as required. □

To prove the correctness of the algorithm it suffices to remark that it determines a proof g of the rank of u such that $g_i > 0$.

3.2 Greedy algorithm on Dyck words

Let A be the alphabet with two letters $\{a, b\}$. For a word w on the alphabet A and for a letter $x \in A$, $|w|_x$ denotes the number of occurrences of x in w. The function δ on words is defined by: $\delta(w) = |w|_a - |w|_b$. A *Dyck word* w is a word on the alphabet $\{a, b\}$ such that $\delta(w) = 0$, and for any of its prefixes w' one has $\delta(w') \geq 0$. The size of a Dyck word w is $|w|_a = |w|/2$. The height $h(w')$ of a prefix w' ending by an a of a Dyck word w is given by: $h(w') = \delta(w') - 1$. The maximal height $H(w)$ of a Dyck word w is $h(w) = \max_{w'} h(w')$ where w' runs through all prefixes of w ending with a.

To any (sorted) configuration u of K_n such that

$$n - 1 \geq u_1 \geq u_2 \geq \cdots u_{n-1} \geq 0 \tag{1}$$

we associate a word $w = D(u)$ with $n - 1$ occurrences of a and n occurrences of b the following way: the ith occurrence of a in w has exactly u_{n-i} occurrences of b before it; notice that $D(u)$ ends

[i] We recall that configurations may have a negative number of tokens.

with an occurrence of b. Moreover $D(u)$ is a Dyck word followed by a b, if and only if u is a parking configuration. This leads to a reformulation of the preceding greedy algorithm in terms of Dyck words. When u is a sorted parking configuration it is convenient to write $D(u) = d(u)b$ such that $d(u)$ is a Dyck word.

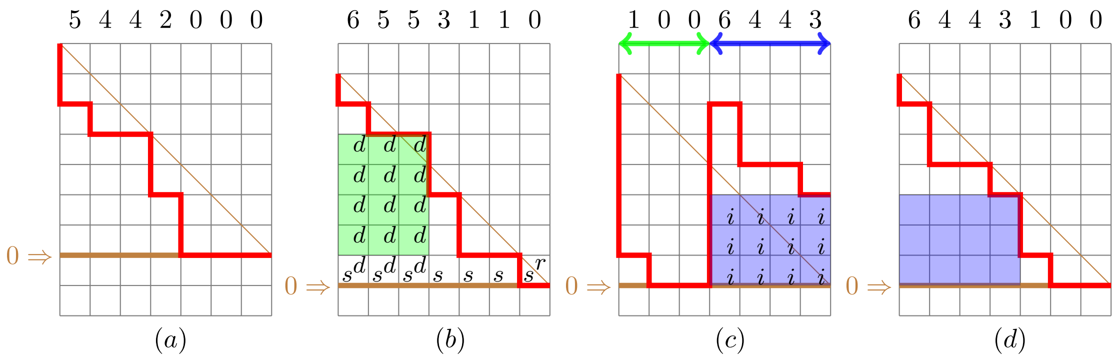


Fig. 3: An example of four steps of a loop iteration of algorithm computing rank

Any non-empty Dyck word w admits the non-ambiguous classical *first return decomposition* $w = afbg$ where f and g are Dyck words. As announced at the beginning of this Section, we consider the algorithm computing the rank in terms of sorted parking configurations toppling equivalent to it and its image via the preceding map $u \longrightarrow D(u)$. The algorithm may be described in terms of Dyck words due to:

Proposition 5 *For any sorted parking configuration u, one step of the algorithm computing the rank consists in the subtraction of 1 on u_{n-1} and then computing the sorted parking configuration u' toppling equivalent to it. In terms of words, this translates to the following: if $w = d(u) = afbg$ is the first return decomposition of u then the new value of w is $d(u') = gabf$.*

The algorithm is described in detail in the right part of Figure 2. We do not provide a detailed proof of Proposition 5 in this extended abstract, however we give details on an example of a loop iteration.

Assume that the algorithm reaches the sorted parking configuration $u = (5, 4, 4, 2, 0, 0, 0, s)$ for some $s \geq 0$, also described by $(d(u), s(u)) = (aaabbabbaababb, s)$. We draw $d(u)$ in red from south-east to north-west in part (a) of Figure 3 above. This red path and the brown horizontal axis pointed by $\Rightarrow 0$ define the diagram of the partition $(u_1, \ldots u_{n-1})$ in which u_n is omitted . We observe the following iteration step: we subtract 1 to u_{n-1} and to recover positivity the vertex x_n is toppled to reach $v = (6, 5, 5, 3, 1, 1, 0, s-7)$. These two steps are represented in part (b) of Figure 3. The cell labeled by r describes the removed token and then the brown horizontal axis is lowered by one unit, adding one cell labeled by s on each column of the partition which is the token coming from the toppling of the sink. This configuration v is not parking since the three first vertices may topple together, preserving positivity. On (b), observe that it corresponds to the rightmost vertical cross of the red path with the brown diagonal, this should not be crossed if the configuration was a parking one. The toppling of the three first vertices leads to $w = (1, 0, 0, 6, 4, 4, 3, s-4)$ is illustrated in part (c) of Figure 3. The tokens transmitted from these three toppled vertices to the four untoppled vertices different from x_n may be interpreted as those in cells labeled by d in (b) (before toppling) and by cells on (c) labeled by i (after toppling). The configuration w is sorted to get $w' = (6, 4, 4, 3, 1, 0, 0, s-4)$ described in part (d) , and this sorting may be interpreted as taking a conjugate of the word $d(u)$. This sorting operation may also be also described by the exchange of f and g in the rewriting of $afbg$ into $gabf$. In this example we have $d(u) = afbg$ with $f = aabbab$ and $g = aababb$ giving $gabf = aababb.ab.aabbab = d(w')$.

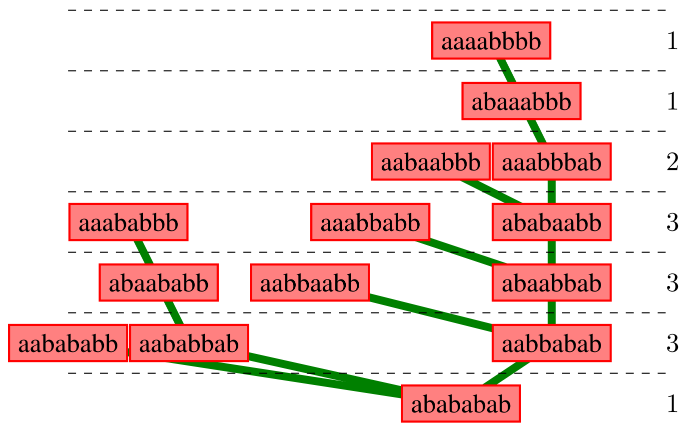


Fig. 4: The tree of Dyck words of size 4 describing the function R.

The rewriting $R(afbg) = gabf$ is a function on Dyck words of same size n that may be described by a tree T_n as in Figure 4 where edges $(w, R(w))$ are oriented downward. There is a loop not drawn at the root of the tree related to the single fixed point $R((ab)^n) = (ab)^n$. We define *prerank* $p(w)$ of any Dyck word as its distance to the root $(ab)^n$ or in other words $p(w) = \min\{k | k \geq 0 \; and \; R^k(w) = (ab)^n\}$. This is motivated by a count of the iterations required in the loop of the algorithm.

3.3 Computing a parking configuration equivalent to u

Lemma 2 *Two configurations u and v are toppling equivalent in K_n if and only if the following holds:*

$$deg(u) = deg(v) \text{ and for any } 1 \leq i, j \leq n\text{: } u_i - u_j = v_i - v_j \pmod{n} \tag{2}$$

Proof: It suffices to show that the configuration u is toppling equivalent to 0 if and only if $deg(u) = 0$ and $u_i - u_j = 0 (\bmod\, n)$. But this follows from the fact that these relations are not modified by any toppling and are satisfied by the parking configuration equivalent to 0 which is equal to $(0, 0, \ldots, 0)$.

□

Given a configuration u one can find a configuration v toppling equivalent to u and such that $0 \leq v_i < n$ for any $1 \leq i \leq n-1$ by setting $v_1 = 0$, then $v_i = u_i - u_1 \pmod{n}$ and $v_n = deg(u) - \sum_{i=1}^{n-1} v_i$. From such a v one builds the parking configuration using the following:

Proposition 6 *Let u be a configuration satisfying equation (1) and let $w = D(u)$. The classical Cyclic Lemma states that there exists a unique conjugate w' of w which is equal to a Dyck word followed by a* *letter b. Consider the configuration v such that $D(v) = w'$ and such that v_n is such that $deg(u) = deg(v)$, then v is the sorted version of the parking configuration equivalent to u.*

4 Symmetry of area and prerank distribution on Dyck words

4.1 A symmetry and a bijective proof of it

The area of a Dyck word w is defined by $area(w) = \sum_{w'} h(w')$ where w' runs over all prefixes of w ending with the letter a. We also consider for a Dyck word w the largest prefix u of it among those whose height is $H(w)$, and define the coheight $h^c(w')$ for any prefix w' of w ending with an a, this coheight is $H(w) - h(w')$ if w' is not larger than u and it is $H(w) - h(w') - 1$ if w' is larger that u. Using Proposition 5 it is possible to prove that $prerank(w) = \sum_{w'} h^c(w')$ where w' runs over all prefixes of w ending with the letter a.

We consider the generating function on Dyck words of size n counted according to the statistics area and prerank:

$$D_n^{area,prerank}(q,t) = \sum_w q^{area(w)} t^{prerank(w)}.$$

Theorem 3 *For any $n \geq 1$, we have the symmetry $D_n^{area,prerank}(q,t) = D_n^{area,prerank}(t,q)$.*

The proof follows from an involution Φ on Dyck words that exchanges areas and preranks, and is defined as follows:

A non-empty Dyck word w admits a non-ambiguous last maximum decomposition $w = ubv$ where u is the largest prefix of w among those whose height is $H(w)$. The mirror image $\tilde{w}$ of the word w whose letters are $w_1 w_2 \ldots w_{k-1} w_k$ is the word $\tilde{w} = w_k w_{k-1} \ldots w_2 w_1$; notice that we do not exchange letters a and b. The involution Φ is defined from the last maximum decomposition $w = ubv$ by: $\Phi(ubv) = \tilde{u} b \tilde{v}$.

This symmetry can be refined at the level of occurrences of the letter a in a Dyck word.

Lemma 3 *For any Dyck word w of size n there is a bijection from the occurrences of the letter a in w into those of the letter a in $\Phi(w)$ that exchanges heights and coheights. This bijection associates to an occurrence of a in w its image by the involution Φ.*

The involution Φ has another property with respect to the $dinv$ parameter introduced by Haiman (see Haglund (2008) for the definition of $dinv$).

Proposition 7 *For any Dyck word w, $dinv(\Phi(w)) = dinv(w)$.*

An immediate corollary is that the bistatistic $(prerank, dinv)$ is the image by ϕ of the bistatistic $(area, dinv)$ which defines the q,t-Catalan numbers studied by A. Garsia, M. Haiman, J. Haglund.

Our definition of Φ may be seen, using mirror image, in the classical cyclic lemma attributed to Dvoretsky and Motzkin (1947). A word w on the alphabet $\{a, b\}$ is called a quasi-balanced word of size n if $|w|_a = n$ and $|w|_b = n + 1$. The cyclic lemma states that for any quasi-balanced word w, among the $2n + 1$ conjugates of the bi-infinite periodic word $w^{\mathbb{Z}}$ exactly one may be written $(w'b)^{\mathbb{Z}}$ where w' is a Dyck word of size n. The image of this via the mirror mapping is related to our definition of Φ: among the $2n+1$ conjugates of $(\tilde{w})^{\mathbb{Z}}$ exactly one may be written $(w''b)^{\mathbb{Z}}$ where w'' is a Dyck word and $w'' = \Phi(w')$.

It is also possible to prove that the involution Φ on Dyck paths satisfies a commutativity relation with the function ζ introduced in Haglund (2008) (page 50). More precisely : $Flip.\zeta = \zeta\Phi$, where $Flip$ is the map that reflects a Dyck word and exchanges occurrences of a's and b's [(ii)]

4.2 Another description of the rank algorithm

The conjugate $\Phi R\Phi$ of function R with this bijection Φ is described by the following lemma which leads to another description of the rank algorithm.

Lemma 4 *For any non-empty Dyck word w, let $\Phi(w) = ubv = (u'a)bv$ be the last maximum decomposition of $\Phi(w)$ then $\Phi(R(w)) = u'bav$.*

The building of the tree in Fig. 4 becomes obvious from this viewpoint, when the nodes of T_n are labeled by $\Phi(w)$ instead of w since the rewriting described by the edge $(\Phi(d), \Phi(R(d)))$ corresponds to a flip of the last highest peak ab into a valley ba.

4.3 Computing the area, prerank distribution

We currently have two ways to describe the distribution of the bistatistic $(area, prerank)$ on Dyck words of given size n. First, we have a non-ambiguous shuffle of any possible distribution of pairs heights and coheights on occurences of letter a leading to all Dyck words with this distribution:

Proposition 8 *For any $n \geq 0$ and k such that $1 \leq k \leq n$, let $c = (c_0, c_1, \dots c_{2k-2})$ be a composition of $n-k$ into $2k-1$ parts. The number $N_{n,k,c}$ of Dyck words such that $1 + c_{2i}$ is the number of letters a of height i and coheight $k-i$ and c_{2i+1} is the number of letters a of height i and coheight $k-1-i$ is*

$$N_{n,k,c} = \prod_{i=0} \binom{c_{2i} + c_{2i+2}}{c_{2i}} \binom{c_{2i+1} + c_{2i+3}}{c_{2i+1}}$$

consequently,

$$D_n^{area,prerank}(q,t) = \sum_{k=1}^{n} \sum_{c \text{ composition of } n-k} N_{n,k,c} \prod_{i=0}^{k} (q^i t^{k-i})^{1+c_{2i}} (q^i t^{k-1-i})^{c_{2i+1}}.$$

Using an interpretation of the decomposition at last maximum of the Dyck word in terms of heaps of dimers in the framework of Viennot's theory of heaps (see Krattenthaler (2006)) we also have:

Lemma 5 *Let $(T_n(y,z))_{n\geq 0}$ the polynomials recursively defined by $T_0(y,z) = 1 = T_1(y,z)$ and for $n \geq 2$,*

$$T_n(y,z) = T_{n-1}(y,z) + y^{n-2} z T_{n-2}(y,z)$$

then

$$\sum_{n\geq 1} D_n^{area,prerank}(q,t) z^n = \sum_{k\geq 2} \frac{(qt)^{\binom{k-1}{2}} z^{k-1}}{T_k(q/t, t^{k-2}z) T_{k-1}(q/t, -t^{k-3}z)}.$$

(ii) We thank one of the anonymous referees of FPSAC 2013 to have suggested the existence of this link

5 On degree and rank distribution

5.1 On any graph G

Given a sink, labeled by n in our notation, the toppling classes of configurations may be indexed by G-parking configurations (π, s) where π belongs to Π_G the finite set of restrictions of G-parking configurations outside the sink and $s \in \mathbb{Z}$ is a number of tokens on the sink. These indices are used to define the Laurent series related to the distribution of degree and rank by

$$P_G^{degree,rank}(x,r) = \sum_{\pi \in \Pi_G, s \in \mathbb{Z}} x^{degree((\pi,s))} r^{rank((\pi,s))}.$$

Since a negative degree implies a rank equal to -1, using Proposition 4 for higher degrees we can consider that the relevant part of this series is a ("Laurent") polynomial $P_{G,[0,2m-2n]}^{degree,rank}(x,r)$ defined on configurations with intermediate degree, that is belonging to the interval $[0, 2m-2n]$. Hence we write:

$$P_G^{degree,rank}(x,r) = \frac{(rx)^{-1}|\Pi_G|}{1-x^{-1}} + P_{G,[0,2m-2n]}^{degree,rank}(x,r) + \frac{x(x^2r)^{m-n}|\Pi_G|}{1-xr}.$$

Theorem 2 uses configuration κ of degree $2m-2n$ to give a relation between the rank and degree of two configurations u and $\kappa - u$, it implies the following formula expressing symmetry of degree and rank distribution:

$$P_G^{degree,rank}(x,r) = (rx^2)^{m-n} P_G^{degree,rank}\left(\frac{1}{xr}, r\right).$$

The non-effective configurations are exactly those of rank -1 and the degree distribution on these configurations may be related to an evaluation of the Tutte polynomial $T_G(x,y)$ of the graph G (see Lopez (1997)) where x (respectively y) counts internal (respectively external) activity:

$$[r^{-1}]P_G^{degree,rank}(x,r) = \frac{1}{1-x^{-1}} T_G(1,x).$$

5.2 On complete graphs

In the particular case the complete graph K_n, $m = \binom{n}{2}$, we define the distribution of degree and rank at the level of orbits under the action of S_{n-1} leading to the " Laurent " polynomial:

$$D_n^{degree,rank}(x,r) = \sum_u x^{degree(u)} r^{rank(u)},$$

where u runs over sorted parking configurations such that $degree(u) \in [0, n(n-3)]$.

Baker and Norine's theorem is compatible with the action of S_{n-1} so we also have the symmetry

$$D_n^{degree,rank}(x,r) = (rx^2)^{n(n-3)/2} D_n^{degree,rank}\left(\frac{1}{xr}, r\right).$$

We conclude this extended abstract by the partial announcement of an enumerative result we obtained recently via combinatorial considerations on the analysis of our algorithm computing the rank. This can be stated as follows:

$$D_n^{degree,rank}(x,r) = x^{\binom{n-1}{2}-1} r^{-1}([z^n]F(q_1,q_2;z))$$

where $[z^n]f(z)$ denotes the coefficient of z^n in power sum $f(z)$, and $F(q_1, q_2; z)$ is an explicit rational function in $q_1 = x^{-1}$, $q_2 = xr$, z, $C(q_1; z)$,$C(q_1; q_1 z)$, $C(q_2; z)$ and $C(q_2; q_2 z)$ where

$$C(q; z) = \sum_{w\ dyck} q^{area(w)} z^{size(z)}$$

is the well known Carlitz q-analogue of Catalan numbers.

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Operators of equivalent sorting power and related Wilf-equivalences

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Abstract. We study sorting operators $\mathbf{A}$ on permutations that are obtained composing Knuth's stack sorting operator $\mathbf{S}$ and the reverse operator $\mathbf{R}$, as many times as desired. For any such operator $\mathbf{A}$, we provide a bijection between the set of permutations sorted by $\mathbf{S} \circ \mathbf{A}$ and the set of those sorted by $\mathbf{S} \circ \mathbf{R} \circ \mathbf{A}$, proving that these sets are enumerated by the same sequence, but also that many classical permutation statistics are equidistributed across these two sets. The description of this family of bijections is based on an apparently novel bijection between the set of permutations avoiding the pattern 231 and the set of those avoiding 132 which preserves many permutation statistics. We also present other properties of this bijection, in particular for finding families of Wilf-equivalent permutation classes.

Résumé. On étudie les opérateurs $\mathbf{A}$ de tri de permutations obtenus en composant l'opérateur $\mathbf{S}$ de tri par une pile de Knuth et l'opérateur $\mathbf{R}$ de miroir, un certain nombre de fois. Pour tout opérateur $\mathbf{A}$ de cette forme, on donne une bijection entre l'ensemble des permutations triées par $\mathbf{S} \circ \mathbf{A}$ et l'ensemble de celles triées par $\mathbf{S} \circ \mathbf{R} \circ \mathbf{A}$, démontrant ainsi que ces ensembles ont la même séquence d'énumération, mais aussi que de nombreuses statistiques classiques sur les permutations ont la même distribution sur ces deux ensembles. La description de cette famille de bijections repose sur une bijection apparemment nouvelle entre l'ensemble des permutations qui évitent le motif 231 et l'ensemble de celles qui évitent 132, qui préserve de nombreuses statistiques. On présente aussi d'autres propriétés de cette bijection, en particulier pour trouver des familles de classes de permutations équivalentes au sens de Wilf.

Keywords: permutation, stack, sorting, enumeration, Wilf-equivalence

1 Introduction

Partial sorting algorithms were one of the early motivations for the study of permutation patterns. For instance, Knuth (1975) considered the problem of sorting a permutation of length n, *i.e.* of the set $[n] = \{1, 2, \dots, n\}$, using only a stack. If such a permutation, π, is written in one line notation as $\alpha n \beta$, then π is sortable if and only if: each of α and β is sortable (thought of as permutations of the values they contain); and each value in α is less than any value in β (or simply $\alpha < \beta$). The first condition is clearly necessary – the second condition is also necessary as, when n is the first element remaining to be added to the stack, the entire stack must be emptied to have any hope of success, otherwise n will precede some other element in the output, and the output will not be sorted. In the same fashion, the stack must at all times obey the *Hanoi* condition that it never has a greater element lying on top of a lesser one. That the conditions are sufficient is also clear – the requisite operations are: sort and output α; add n to the stack; sort and output β; remove n from the stack. Figure 1 shows an example of performing stack sorting on a permutation. This simple behavior prompted many other investigations of stack sorting and its variations and extensions beginning with works by Pratt (1973) and Tarjan (1972).

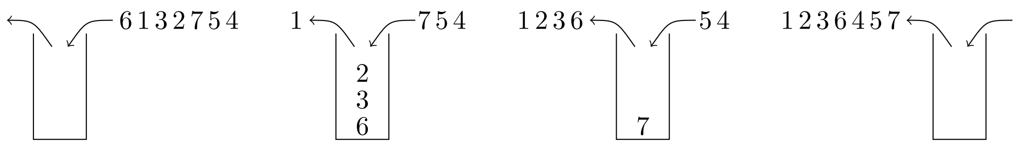


Fig. 1: Some steps of the stack sorting procedure applied to $\pi = 6\,1\,3\,2\,7\,5\,4$. Thus, $\mathbf{S}(\pi) = 1\,2\,3\,6\,4\,5\,7$.

Stack sorting can be considered as an operator or procedure, $\mathbf{S}$, applied to permutations. It is defined recursively as: $\mathbf{S}(\alpha n \beta) = \mathbf{S}(\alpha)\mathbf{S}(\beta)n$. With this definition $\mathbf{S}(\pi)$ is the result of attempting to sort π using a stack, maintaining the condition that the items in the stack must always be ordered from least to greatest when read from top to bottom. We adopt the viewpoint throughout that any sequence of distinct values can be interpreted as a permutation and "n" always denotes the maximum element of such a sequence. West (1993) described the permutations that can be sorted using $\mathbf{S}\circ\mathbf{S}$, and Zeilberger (1992) subsequently confirmed a conjecture of West's on their enumeration.

Bousquet-Mélou (2000) also considered the operator $\mathbf{S}$ and characterized, given π, the set $\mathbf{S}^{-1}(\pi)$. We shall be extending her results, and will discuss them in more detail later. Central to her analysis is the observation that the operator $\mathbf{S}$ can be described in the following terms: given a permutation π form the unique decreasing binary tree $\mathrm{T}_{\mathrm{in}}(\pi)$ whose in-order reading is π, then $\mathbf{S}(\pi)$ is the post-order reading of this tree.

A second operator on permutations is the reversal operator, that reads permutations from right to left – it can also be modeled by using a stack where we are obliged to input the entire permutation to the stack before performing any output. The reversal operator, $\mathbf{R}$, is one of eight natural symmetries on the collection of permutations. Bouvel and Guibert (2012) considered the enumeration of permutations sorted by $\mathbf{S}\circ\mathbf{R}\circ\mathbf{S}$ as well as the sets defined similarly with other symmetries in place of $\mathbf{R}$. In experimental investigations aimed at providing extensions to their results they noticed an interesting phenomenon that can be expressed as:

Conjecture 1 *For any composition,* $\mathbf{A}$*, of the operators* $\mathbf{S}$ *and* $\mathbf{R}$ *the number of permutations sorted by* $\mathbf{S}\circ\mathbf{A}$ *and by* $\mathbf{S}\circ\mathbf{R}\circ\mathbf{A}$ *is the same. Moreover, many permutation statistics are equidistributed across these two sets.*

It is the primary purpose of this article to prove that this is indeed the case. To do so, we make use of another classical description of stack sortable permutations. It is simply derived from their description by Knuth (1975) that we reported at the beginning of this section. Stack sortable permutations are those that may not contain subwords (not necessarily consecutive) of the form bca where $a < b < c$. Such permutations are said to *avoid the pattern* 231, and the collection of all such is denoted $\mathrm{Av}(231)$. More generally and more formally, a permutation $\pi = \pi(1)\pi(2)\cdots\pi(k)$ is a *pattern* of a permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ when there exist $1 \le i_1 < i_2 < \cdots < i_k \le n$ such that π is order isomorphic to $\sigma(i_1)\sigma(i_2)\cdots\sigma(i_k)$. If π is not a pattern of σ then we say that σ *avoids* π. We denote by

$\mathrm{Av}(\pi, \pi', \cdots, \pi'')$ the set of all permutations that avoid simultaneously the patterns $\pi, \pi', \cdots, \pi''$. Such a collection of permutations defined by the avoidance of a given set of permutations is also called a *permutation class*.

With the characterization of stack sortable permutations as $\mathrm{Av}(231)$, proving Conjecture 1 is equivalent to showing that there is a bijection between the elements of $\mathrm{Av}(231)$ belonging to the image of $\mathbf{A}$, and the elements of $\mathrm{Av}(231)$ belonging to the image of $\mathbf{R} \circ \mathbf{A}$, with the additional condition that the bijection preserves the number of preimages under $\mathbf{A}$ (resp. $\mathbf{R} \circ \mathbf{A}$). Equivalently, we can replace this latter set by the elements of $\mathrm{Av}(132)$ belonging to the image of $\mathbf{A}$, since the self-inverse operator $\mathbf{R}$ immediately provides a bijection between $\mathrm{Av}(231)$ and $\mathrm{Av}(132)$.

In establishing this result we demonstrate an apparently novel bijection between $\mathrm{Av}(231)$ and $\mathrm{Av}(132)$ which preserves many permutation statistics. We also present some other properties of this bijection.

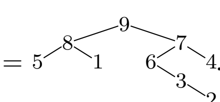
2 Preimages of permutations in the image of S

As noted earlier, the description of the elements of $\mathbf{S}^{-1}(\pi)$ for π in the image of $\mathbf{S}$ was carried out by Bousquet-Mélou (2000). This description is central to our work, so we review it here.

There exists for any permutation σ a unique decreasing binary tree, $\mathrm{T_{in}}(\sigma)$ whose in-order reading is σ. As usual, $\mathrm{T_{in}}(\sigma)$ is recursively defined: if $\sigma = \alpha n \beta$ then the root of $\mathrm{T_{in}}(\sigma)$ is n and its left (resp. right) subtrees are $\mathrm{T_{in}}(\alpha)$ (resp. $\mathrm{T_{in}}(\beta)$). The recursive description of $\mathbf{S}$ given above ($\mathbf{S}(\alpha n \beta) = \mathbf{S}(\alpha)\mathbf{S}(\beta)n$) then shows that $\mathbf{S}$ converts in-order reading of decreasing binary trees to post-order reading. Therefore, describing $\mathbf{S}^{-1}(\pi)$ is equivalent to describing the decreasing binary trees, T, with post-order reading π. For convenience we denote the post-order reading of a tree T by $\mathbf{Post}(T)$.

Definition 2 *A decreasing binary tree is* canonical *if it has the following property: any node, z, that has a left child, x, also has a right child, and the leftmost value y in the subtree of the right child of z is less than x.*

From (Bousquet-Mélou, 2000, Proposition 2.6), we know that for π in the image of $\mathbf{S}$ there is a unique canonical tree T_π with $\mathbf{Post}(T_\pi) = \pi$. In fact, the permutation σ obtained from the in-order reading of T_π is the element of $\mathbf{S}^{-1}(\pi)$ having the greatest number of inversions. Moreover, any decreasing binary tree whose post-order reading is π (and only such trees) can be obtained from T_π by a sequence of operations of the following type: take a node z with no left child, and one of its descendants y on the leftmost branch of its right subtree; remove the subtree rooted at y and make it the left subtree of z. It follows that $|\mathbf{S}^{-1}(\pi)|$ depends only on the structure of the tree T_π and not on its labeling.

Example 3 *The canonical tree associated with $\pi = 5\,1\,8\,2\,3\,6\,4\,7\,9$ is $T_\pi =$* 

. Its in-order reading, $\sigma = 5\,8\,1\,9\,6\,3\,2\,7\,4$ gives the permutation with the largest number of inversions subject to $\mathbf{S}(\sigma) = \pi$. The four other decreasing binary trees with the same post-order reading are shown in Figure 2. Thus $|\mathbf{S}^{-1}(\pi)| = 5$. If the labels 8 and 7, and 5 and 4, were exchanged in the original tree, corresponding to $\pi' = 4\,1\,7\,2\,3\,6\,5\,8\,9$ then, because the tree is still canonical, the method for constructing permutations in $\mathbf{S}^{-1}(\pi')$ is still the same, and in particular $|\mathbf{S}^{-1}(\pi')| = |\mathbf{S}^{-1}(\pi)|$.

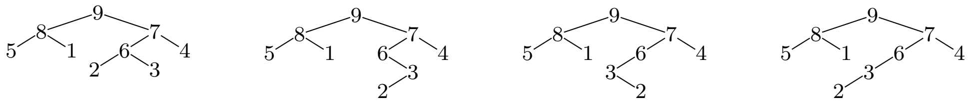
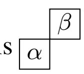
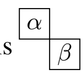


Fig. 2: The four non canonical decreasing trees whose post-order reading is $\pi = 5\,1\,8\,2\,3\,6\,4\,7\,9$.

3 A recursive bijection between $\mathrm{Av}(231)$ and $\mathrm{Av}(132)$

In this section we introduce a bijection, P, between permutations in $\mathrm{Av}(231)$ and those in $\mathrm{Av}(132)$. It is very easy to describe P recursively using the *sum*, $\oplus$, and *skew sum*, $\ominus$, operations on permutations. These operations are easily understood on the *diagrams* corresponding to permutations. The diagram of any permutation σ of length n is the set of n points in the plane at coordinates $(i, \sigma(i))$. If α is a permutation of $[a]$ and β of $[b]$ we define:

$\alpha \oplus \beta = \alpha\,(\beta + a)$ whose diagram is 

$\alpha \ominus \beta = (\alpha + b)\,\beta$ whose diagram is  .

Here for example $\beta + a$ is just the sequence obtained by adding a to every element of the sequence β and $\boxed{\alpha}$ represents the diagram of permutation α.

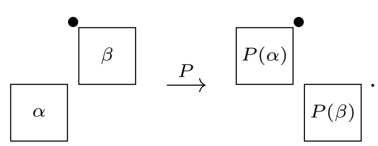
Example 4 *Let* $\alpha = 2\,3\,1$ *and* $\beta = 3\,1\,4\,2$*. Then* $\alpha \oplus \beta = 2\,3\,1\,6\,4\,7\,5$*, while* $\alpha \ominus \beta = 6\,7\,5\,3\,1\,4\,2$*.*

Any permutation σ that can be written as a sum $\alpha \oplus \beta$ (resp. skew sum $\alpha \ominus \beta$) is said $\oplus$-*decomposable* (resp. $\ominus$-*decomposable*). Otherwise, we say that σ is $\oplus$-*indecomposable* (resp. $\ominus$-*indecomposable*).

Any $\pi \in \mathrm{Av}(231)$ is either the empty permutation ε or has a unique decomposition in the form $\alpha \oplus (1 \ominus \beta)$ where $\alpha, \beta \in \mathrm{Av}(231)$ (and are possibly empty), and conversely any permutation of this latter form lies in $\mathrm{Av}(231)$. This is simply because the elements preceding the maximum in a 231-avoiding permutation must all be less than those following the maximum, and the prefix before and suffix after the maximum must also avoid 231. Conversely, if a permutation has this structure it cannot involve 231. This decomposition makes it easy to define the bijection P recursively: $P(\varepsilon) = \varepsilon$ and

$$\text{if } \pi = \alpha \oplus (1 \ominus \beta) \text{ then } P(\pi) = (P(\alpha) \oplus 1) \ominus P(\beta).$$

Alternatively, with diagrams:



As the 132-avoiding permutations have a generic decomposition of the form shown on the right above, and since $P(1) = 1$ maps the unique 231-avoiding permutation of length 1 to the unique 132-avoiding permutation of length 1, induction immediately implies that $P : \mathrm{Av}(231) \to \mathrm{Av}(132)$ is a bijection. Notice that the restriction of P to the set $\mathrm{Av}(231, 132)$ is the identity map.

Example 5 *For* $\pi = 1\,5\,3\,2\,4\,9\,8\,6\,7 \in \mathrm{Av}(231)$*, we have* $P(\pi) = 7\,8\,5\,4\,6\,9\,3\,1\,2$*.*

We recall a definition from the introduction:

Definition 6 *For any permutation π, $\mathrm{T_{in}}(\pi)$ is the decreasing binary tree whose in-order reading is π.*

It follows immediately by induction from the recursive description of P that:

Observation 7 *Both $\mathrm{T_{in}}(\pi)$ and $\mathrm{T_{in}}(P(\pi))$ have the same underlying unlabeled tree, or briefly "P preserves the shape of in-order trees". An example is provided in Figure 3.*

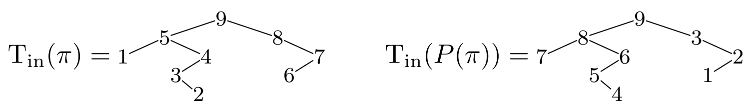


Fig. 3: $\mathrm{T_{in}}(\pi)$ and $\mathrm{T_{in}}(P(\pi))$ for the permutation $\pi = 1\,5\,3\,2\,4\,9\,8\,6\,7$ of Example 5.

It is for this reason that P preserves many permutation statistics. Recall that, for π a permutation of length n, a *left-to-right* (resp. *right-to-left*) *maximum* of π is an element $\pi(i)$ such that for all $j < i$ (resp. $j > i$), $\pi(j) < \pi(i)$, and that the *up-down word* of π is $w_\pi \in \{u, d\}^{n-1}$ with $w_\pi(i) = u$ (resp. d) if $\pi(i) < \pi(i+1)$ (resp. $\pi(i) > \pi(i+1)$).

Observation 8 *P preserves the following statistics: the number and positions of the right-to-left maxima, the number and positions of the left-to-right maxima and the up-down word.*

Proof: All of these follow from Observation 7, since the value of each statistic mentioned for a permutation π is determined by the shape of $\mathrm{T_{in}}(\pi)$. $\square$

Among all the statistics reported in (Claesson and Kitaev, 2008/09, Section 2), the only ones that are preserved by P are the ones that depend only on the shape of in-order trees.

4 Proof of Conjecture 1

4.1 Preparation

In addition to the results of Section 2, the principal ingredients in the proof to follow are a pair of observations concerning P and operators $\mathbf{A}$ which are compositions of $\mathbf{S}$ and $\mathbf{R}$.

Observation 9 *Let τ be any permutation, and $\mathbf{A}$ be any composition of the operators $\mathbf{S}$ and $\mathbf{R}$. Suppose that $x, y \in [n]$ and that in τ there are no values larger than $\max(x, y)$ occurring between x and y. Then the same holds in $\mathbf{A}(\tau)$.*

Proof: It suffices to prove the result for $\mathbf{S}$ and $\mathbf{R}$ individually. For $\mathbf{R}$ it is trivial and for $\mathbf{S}$ it is not hard to prove that it follows by induction from the recursive description: $\mathbf{S}(\alpha n \beta) = \mathbf{S}(\alpha)\mathbf{S}(\beta)n$. $\square$

For the second observation we introduce a notational convention that we shall continue to use throughout. Let $\pi \in \mathrm{Av}(231)$ be given. We think of the sequence $P(\pi)$ as describing a relabeling of the values that occur in π according to a certain permutation λ_π, specifically $P(\pi) = \lambda_\pi \circ \pi$.

Observation 10 *Let $\pi \in \mathrm{Av}(231)$ be given and suppose that $x, y \in [n]$, $x < y$, and in π there are no values larger than $\max(x, y)$ occurring between x and y. Then $\lambda_\pi(x) < \lambda_\pi(y)$.*

Proof: The proof shall not be detailed here. Observation 10 simply says that λ_π preserves the ordering among elements of π which do not contain a larger element between them. This follows from the construction of P since the only way that one element can be moved above another one is to (at some point in the recursion) have a larger element in between. $\square$

4.2 The main argument

In this section we prove the main result. Recall that $\mathbf{A}$ is an operator formed by some composition of $\mathbf{S}$ and $\mathbf{R}$. For any such operator, we shall write $\pi \in \mathbf{A}$ to denote that π is in the image of $\mathbf{A}$.

As above we consider λ_π as a relabeling of the elements of $[n]$. We extend its effect to permutations, trees etc. that carry labels from $[n]$: applying λ_π to such an object will simply mean to apply λ_π to each of its labels.

Definition 11 *We define a function $\Phi_\mathbf{A}$ from the set of permutations sorted by $\mathbf{S} \circ \mathbf{A}$ to the set of all permutations as follows. For θ a permutation sorted by $\mathbf{S} \circ \mathbf{A}$, since $\mathbf{A}(\theta) \in \mathrm{Av}(231)$, we have $\lambda_{\mathbf{A}(\theta)}$ defined by $P(\mathbf{A}(\theta)) = \lambda_{\mathbf{A}(\theta)} \circ \mathbf{A}(\theta)$ and we then set $\Phi_\mathbf{A}(\theta) = \lambda_{\mathbf{A}(\theta)} \circ \theta$.*

In other words $\Phi_\mathbf{A}$ relabels a permutation θ sorted by $\mathbf{S} \circ \mathbf{A}$ in the same way that $\mathbf{A}(\theta)$ is relabeled to produce $P(\mathbf{A}(\theta))$. We will prove (see Corollary 15) that $\Phi_\mathbf{A}$ is a bijection from the set of permutations sorted by $\mathbf{S} \circ \mathbf{A}$ to the set of those sorted by $\mathbf{S} \circ \mathbf{R} \circ \mathbf{A}$. The key to this argument of course is to establish that $\mathbf{A}(\Phi_\mathbf{A}(\theta)) = P(\mathbf{A}(\theta))$.

We are concerned with operators $\mathbf{A}$ which are compositions of $\mathbf{S}$ and $\mathbf{R}$. We say that such an operator *respects P* if it has the following property:

For each $\pi \in \mathrm{Av}(231) \cap \mathbf{A}$,

- For each θ such that $\mathbf{A}(\theta) = \pi$, we have $\mathbf{A}(\Phi_\mathbf{A}(\theta)) = P(\pi) = \lambda_\pi \circ \pi$ and $\mathrm{T_{in}}(\Phi_\mathbf{A}(\theta)) = \lambda_\pi(\mathrm{T_{in}}(\theta))$, and
- the correspondence $\Phi_\mathbf{A} : \theta \mapsto \Phi_\mathbf{A}(\theta)$ is a bijection between $\mathbf{A}^{-1}(\pi)$ and $\mathbf{A}^{-1}(P(\pi))$.

In the above, notice that because $\mathbf{A}(\theta) = \pi$ we actually have $\Phi_\mathbf{A}(\theta) = \lambda_\pi \circ \theta$.

Proposition 12 *If $\mathbf{A}$ respects P then so does $\mathbf{A} \circ \mathbf{R}$.*

Proof: We shall only give the main arguments of the proof.

Let $\pi \in \mathrm{Av}(231) \cap (\mathbf{A} \circ \mathbf{R})$ and θ be such that $(\mathbf{A} \circ \mathbf{R})(\theta) = \pi$. Let $\tau = \mathbf{R}(\theta)$. Then $\mathbf{A}(\tau) = \pi$ and since $\mathbf{A}$ respects P, $\mathbf{A}(\Phi_\mathbf{A}(\tau)) = P(\pi)$ and $\mathrm{T_{in}}(\Phi_\mathbf{A}(\tau)) = \lambda_\pi(\mathrm{T_{in}}(\tau))$.

Because $\mathbf{R}$ is an involution on permutations that acts only on positions whereas λ_π acts on values only, it can be proved that $\mathbf{R}\big(\Phi_{\mathbf{A}\circ\mathbf{R}}(\theta)\big) = \Phi_\mathbf{A}(\tau)$. It follows that $(\mathbf{A} \circ \mathbf{R})\,(\Phi_{\mathbf{A}\circ\mathbf{R}}(\theta)) = \mathbf{A}(\Phi_\mathbf{A}(\tau)) = P(\pi)$. Moreover, applying $\mathbf{R}$ to a permutation is equivalent to recursively exchanging left and right subtrees in its in-order tree. This is how we deduce $\mathrm{T_{in}}(\Phi_{\mathbf{A}\circ\mathbf{R}}(\theta)) = \lambda_\pi(\mathrm{T_{in}}(\theta))$ from $\mathrm{T_{in}}(\Phi_\mathbf{A}(\tau)) = \lambda_\pi(\mathrm{T_{in}}(\tau))$. Finally, the correspondence $\Phi_{\mathbf{A}\circ\mathbf{R}}$ is the composition of three bijections: $\mathbf{R}$, $\Phi_\mathbf{A}$ and $\mathbf{R}^{-1} = \mathbf{R}$, and so is also a bijection. $\square$

Proposition 13 *If $\mathbf{A}$ respects P then so does $\mathbf{A} \circ \mathbf{S}$.*

Proof: For brevity, we only sketch the proof and omit the details.

Let $\pi \in \mathrm{Av}(231) \cap (\mathbf{A} \circ \mathbf{S})$ and θ be such that $(\mathbf{A} \circ \mathbf{S})(\theta) = \pi$. Let $\tau = \mathbf{S}(\theta)$. Then $\mathbf{A}(\tau) = \pi$ and since $\mathbf{A}$ respects P, $\mathbf{A}(\Phi_{\mathbf{A}}(\tau)) = P(\pi)$ and $\mathrm{T}_{\mathrm{in}}(\Phi_{\mathbf{A}}(\tau)) = \lambda_\pi(\mathrm{T}_{\mathrm{in}}(\tau))$.

We first define $\tau' = \lambda_\pi \circ \tau = \Phi_{\mathbf{A}}(\tau)$ and show that $\tau' \in \mathbf{S}$. From Bousquet-Mélou (2000), we know that it is enough to prove that τ' is the post-order reading of some decreasing binary tree. Denoting T the unique canonical tree such that $\mathbf{Post}(T) = \tau$, and defining $T' = \lambda_\pi(T)$, we remark that $\mathbf{Post}(T') = \tau'$, hence $\tau' \in \mathbf{S}$. Moreover, it can be proved that the tree T' is canonical, so that T' is the unique canonical tree such that $\mathbf{Post}(T') = \tau'$.

Defining furthermore $\theta' = \lambda_\pi \circ \theta$, we next prove that $\mathrm{T}_{\mathrm{in}}(\theta') = \mathrm{T}_{\mathrm{in}}(\theta)'$ (i.e. the result of applying λ_π to the labels of $\mathrm{T}_{\mathrm{in}}(\theta)$) and $\mathbf{S}(\theta') = \tau'$. From Bousquet-Mélou (2000) again, because $\mathbf{S}(\theta) = \tau$, we know that $\mathrm{T}_{\mathrm{in}}(\theta)$ has been obtained from T by a series of moves of the following form:

> Take a node z with no left child, and one of its descendants y on the leftmost branch of its right subtree. Remove the subtree rooted at y and make it the left subtree of z.

Applying the same sequence of operations to T', that is, creating a tree with the same underlying structure as $\mathrm{T}_{\mathrm{in}}(\theta)$, but with the labels arising from T', we obtain a decreasing tree (because the operations cannot create an increasing pair) whose in-order reading is θ', and whose post-order reading is τ', and hence $\mathrm{T}_{\mathrm{in}}(\theta') = \mathrm{T}_{\mathrm{in}}(\theta)'$ and $\mathbf{S}(\theta') = \tau'$.

This implies that

- $\mathrm{T}_{\mathrm{in}}(\Phi_{\mathbf{A}\circ\mathbf{S}}(\theta)) = \mathrm{T}_{\mathrm{in}}(\lambda_\pi \circ \theta) = \lambda_\pi(\mathrm{T}_{\mathrm{in}}(\theta))$;
- $\mathbf{A} \circ \mathbf{S}(\Phi_{\mathbf{A}\circ\mathbf{S}}(\theta)) = \mathbf{A} \circ \mathbf{S}(\theta') = \mathbf{A}(\mathbf{S}(\theta')) = \mathbf{A}(\tau') = \mathbf{A}(\Phi_{\mathbf{A}}(\tau)) = P(\pi)$.

The correspondence $\theta \mapsto \theta'$ is a bijective map between $\mathbf{S}^{-1}(\tau)$ and $\mathbf{S}^{-1}(\tau')$ (a consequence of Proposition 2.7 of Bousquet-Mélou (2000)), and the correspondence $\Phi_{\mathbf{A}\circ\mathbf{S}}$ between $(\mathbf{A} \circ \mathbf{S})^{-1}(\pi)$ and $(\mathbf{A}\circ\mathbf{S})^{-1}(P(\pi))$ is just the union of all these correspondences on the disjoint sets $\mathbf{S}^{-1}(\tau)$ for $\tau \in \mathbf{A}^{-1}(\pi)$ and to the disjoint sets $\mathbf{S}^{-1}(\tau')$ for $\tau' \in \mathbf{A}^{-1}(P(\pi))$. So it is a bijection, and $\mathbf{A} \circ \mathbf{S}$ respects P. $\square$

Combining the two preceding propositions with the fact that from Observation 7 the identity operator respects P we obtain our main theorem:

Theorem 14 *Every operator that is formed by composition from* $\{\mathbf{S}, \mathbf{R}\}$ *respects* P.

Corollary 15 *For any composition* $\mathbf{A}$ *of operators from* $\{\mathbf{S}, \mathbf{R}\}$*,* $\Phi_{\mathbf{A}}$ *is a bijection between the set of permutations sorted by* $\mathbf{S} \circ \mathbf{A}$ *and those sorted by* $\mathbf{S} \circ \mathbf{R} \circ \mathbf{A}$.

Corollary 15 proves the first part of Conjecture 1, namely that the number of permutations sorted by $\mathbf{S} \circ \mathbf{A}$ and by $\mathbf{S} \circ \mathbf{R} \circ \mathbf{A}$ is the same.

We now study the properties of bijections $\Phi_{\mathbf{A}}$ in somewhat greater detail. This will prove the second part of Conjecture 1, that deals with permutation statistics equidistributed over the set of permutations sorted by $\mathbf{S} \circ \mathbf{A}$ and the set of those sorted by $\mathbf{S} \circ \mathbf{R} \circ \mathbf{A}$.

4.3 Statistics preserved by the bijections $\Phi_{\mathbf{A}}$

In this section, $\mathbf{A}$ denotes any composition of operators from $\{\mathbf{S}, \mathbf{R}\}$.

Theorem 16 *The shape of the in-order tree is preserved by* $\Phi_{\mathbf{A}}$.

Proof: For θ a permutation sorted by $\mathbf{S} \circ \mathbf{A}$, writing $\pi = \mathbf{A}(\theta) \in \mathrm{Av}(231)$ we have $\Phi_{\mathbf{A}}(\theta) = \lambda_\pi \circ \theta$. From Theorem 14, $\mathbf{A}$ respects P, so that $\mathrm{T_{in}}(\lambda_\pi \circ \theta)$ and $\mathrm{T_{in}}(\theta)$ have the same shape. □

Because the shape of the in-order tree determines many permutation statistics, we have:

Corollary 17 $\Phi_{\mathbf{A}}$ *preserves the following statistics: the number and positions of the right-to-left maxima, the number and positions of the left-to-right maxima and the up-down word (and hence also the many classical permutation statistics determined by the up-down word).*

Zeilberger (1992) introduced a statistic to aid in the enumeration of the permutations sorted by $\mathbf{S} \circ \mathbf{S}$. Unsurprisingly, this statistic and one of its close relatives is also preserved by $\Phi_{\mathbf{A}}$:

Theorem 18 *If* $\mathbf{A} = \mathbf{A}_0 \circ \mathbf{S}$ *for some arbitrary composition* $\mathbf{A}_0$ *of operators from* $\{\mathbf{S}, \mathbf{R}\}$*, then* $\Phi_{\mathbf{A}}$ *preserves the Zeilberger statistic, defined as:* $\mathrm{zeil}(\theta) = \max\{k \mid n(n-1)\cdots(n-k+1)$ *is a subword of* $\theta\}$*. In addition, if there is at least one operator* $\mathbf{S} \circ \mathbf{R}$ *in the composition that defines* $\mathbf{A}_0$*, then* $\Phi_{\mathbf{A}}$ *also preserves the reverse of the above statistics:* $\mathrm{Rzeil}(\theta) = \max\{k \mid (n-k+1)\cdots(n-1)n$ *is a subword of* $\theta\}$.

Proof: We only provide a sketch of the proof.

Consider θ a permutation sorted by $\mathbf{S} \circ \mathbf{A}$, and set $\pi = \mathbf{A}(\theta)$. Then $\Phi_{\mathbf{A}}(\theta) = \lambda_\pi \circ \theta$, and we may interpret this identity as $\Phi_{\mathbf{A}}(\theta)$ being obtained relabeling the elements of θ according to λ_π. As before, we extend the effect of relabeling by λ_π to any object that carries labels from $[n]$.

For the first statement, let $c \leq n$ be the smallest value of $[n]$ such that all $d \geq c$ are unaffected by the relabeling λ_π. Because $\mathrm{T_{in}}(\lambda_\pi \circ \theta) = \lambda_\pi(\mathrm{T_{in}}(\theta))$, it is not hard to see that it is enough to prove that $c \leq n - k$, where $k = \mathrm{zeil}(\theta)$. This is proved by contradiction, using the fact that $\mathbf{S}(\theta)$ is the post-order reading of $\mathrm{T_{in}}(\theta)$, together with Observations 9 and 10.

For the second statement, we may write $\mathbf{A} = \mathbf{B}_0 \circ \mathbf{S} \circ \mathbf{R} \circ \mathbf{S}^k$, with $k \geq 1$. Then, we apply the first statement to $\mathbf{B}_0 \circ \mathbf{S}$, and we notice that $\mathbf{R}$ maps the zeil statistics to Rzeil. To conclude the proof, the most important fact is that applying operator $\mathbf{S}$ may only increase the value of the Rzeil statistics. □

5 More properties of the bijection P

5.1 Bijection P and Wilf-equivalences

Two permutation classes are said to be *Wilf-equivalent* if they contain the same number of permutations of length n for every n. One common form of Wilf-equivalence arises from symmetries of the avoidance relationship. For example, the reverse symmetry $\mathbf{R}$ provides a bijection between $\mathrm{Av}(231)$ and $\mathrm{Av}(132)$, proving that these classes are Wilf-equivalent. More generally, for any symmetry $\mathbf{Z}$ obtained composing reverse, complement and inverse, $\mathrm{Av}(\pi, \pi', \cdots, \pi'')$ and $\mathrm{Av}(\mathbf{Z}(\pi), \mathbf{Z}(\pi'), \cdots, \mathbf{Z}(\pi''))$ are Wilf-equivalent classes, and we say that they are *trivially* Wilf-equivalent. However, more interesting Wilf-equivalences are also somewhat common, and in this section we show how the bijection P from Section 3 furnishes a supply of such Wilf-equivalences.

We say that a permutation $\pi \in \mathrm{Av}(231)$ respects P when P restricted to $\mathrm{Av}(231, \pi)$ is a bijection with $\mathrm{Av}(132, P(\pi))$. We define two families of permutations (λ_n) and (ρ_n) recursively by $\lambda_1 = \rho_1 = 1$ and for all $n \geq 1$, $\lambda_{n+1} = 1 \ominus \rho_n$ and $\rho_{n+1} = \lambda_n \oplus 1$ (see Figure 4). We also take the convention that λ_0 and ρ_0 denote the empty permutation. Notice that for any n, λ_n and ρ_n are fixed by P, since they avoid both 231 and 132. Notice also that for any n, λ_n is $\oplus$-indecomposable and ρ_n is $\ominus$-indecomposable.

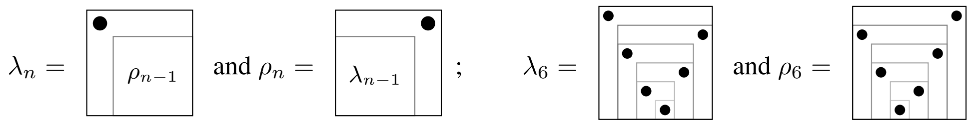


Fig. 4: Diagrams of λ_n and ρ_n, for general n and for $n = 6$.

Lemma 19 *For every $n \geq 0$, and every $0 \leq k \leq n$, the permutation $\lambda_k \oplus \lambda_{n-k}$ respects P.*

Proof: The proof of this result is based on an induction on n, and simply requires a careful analysis of the way in which a pattern such as $\lambda_k \oplus \lambda_{n-k}$ can occur in a 231-avoiding permutation, and dually how $P(\lambda_k \oplus \lambda_{n-k})$ can occur in a 132-avoiding permutation. □

Since λ_n and ρ_n are fixed by P, and because $\lambda_k \oplus \lambda_{n-k} = \lambda_k \oplus (1 \ominus \rho_{n-k-1})$, a consequence of Lemma 19 is:

Theorem 20 *For every $n \geq 0$, and every $0 \leq k \leq n-1$, the permutation classes $\mathrm{Av}(231, \lambda_k \oplus (1 \ominus \rho_{n-k-1}))$ and $\mathrm{Av}(132, (\lambda_k \oplus 1) \ominus \rho_{n-k-1})$ are Wilf-equivalent. Moreover, P provides a bijection from one to the other, that preserves the shape of the in-order trees.*

Even though there are more classes $\mathrm{Av}(231, \pi)$ and $\mathrm{Av}(132, P(\pi))$ that are Wilf-equivalent, we are able to show that except when π of the form of Lemma 19, P will not provide a bijection between $\mathrm{Av}(231, \pi)$ and $\mathrm{Av}(132, P(\pi))$. This is obtained proving the converse of Lemma 19, i.e. proving that all permutations that respect P are of the form $\lambda_k \oplus \lambda_{n-k}$. The proof is omitted for brevity.

Theorem 21 *The permutations that respect P are exactly those of the form $\lambda_k \oplus \lambda_{n-k} = \lambda_k \oplus (1 \ominus \rho_{n-k-1})$, for $n \geq 0$ and $0 \leq k \leq n-1$.*

Table 1 shows all patterns that respect P of length 3 to 8. To each such pattern corresponds a Wilf-equivalence between $\mathrm{Av}(231, \pi)$ and $\mathrm{Av}(132, P(\pi))$. They are non trivial, except for three of them that correspond to the reverse symmetry – those are indicated in *italics*. Due to symmetries, some Wilf-equivalences may however correspond to several rows in Table 1. For instance, $\pi = 42135$ and $\pi' = 53124$ yield the same Wilf-equivalence up to a reverse symmetry.

For π of length 3 or 4, the Wilf-equivalences obtained from Table 1 may be compared to those reported in Wikipedia (2013). Among the Wilf-equivalences reported therein that we may hope to recover (i.e. when one of the excluded pattern is 231 or one of its symmetries), we find three of them, while five are left aside. These three are:

- because $P(312) = 312$, $\mathrm{Av}(231, 312)$ is Wilf-equivalent to $\mathrm{Av}(132, 312)$;
- because $P(3124) = 3124$, $\mathrm{Av}(231, 3124)$ is Wilf-equivalent to $\mathrm{Av}(132, 3124)$ which is up to reverse symmetry the same as $\mathrm{Av}(132, 4213)$ being Wilf-equivalent to $\mathrm{Av}(132, 3124)$;
- because $P(1423) = 3412$, $\mathrm{Av}(231, 1423)$ is Wilf-equivalent to $\mathrm{Av}(132, 3412)$ which is up to inverse-complement symmetry the same as $\mathrm{Av}(132, 4213)$ being Wilf-equivalent to $\mathrm{Av}(132, 3412)$.

Computer experiments have shown that there are (conjecturally) other Wilf-equivalences between classes $\mathrm{Av}(231, \pi)$ and $\mathrm{Av}(132, P(\pi))$, where π does not respect P. These are shown in Table 2.

π	$P(\pi)$
213	213
132	*231*
312	312

π	$P(\pi)$
2143	3241
1423	3412
4213	4213
3124	3124

π	$P(\pi)$
42135	42135
21534	*43512*
53124	53124
31254	42351
15324	45213

π	$P(\pi)$
216435	546213
531246	531246
312645	534612
642135	642135
421365	532461
164235	563124

π	$P(\pi)$
6421357	6421357
3127546	*6457213*
7531246	7531246
4213756	6435712
1753246	6742135
5312476	6423571
2175346	6573124

π	$P(\pi)$
31286457	75683124
75312468	75312468
64213587	75324681
53124867	75346812
86421357	86421357
21864357	76842135
42138657	75468213
18642357	78531246

Tab. 1: Pairs of patterns $(\pi, P(\pi))$ such that π respects P, *i.e.* such that P provides a bijection between $\mathrm{Av}(231, \pi)$ and $\mathrm{Av}(132, P(\pi))$. In particular, these classes are Wilf-equivalent.

π	$P(\pi)$
2137465	5467231
1327645	5647312

π	$P(\pi)$
63125478	64235178
87153246	87452136
65312478	65312478
87421356	87421356

Tab. 2: The other patterns π up to length 8 such that $\mathrm{Av}(231, \pi)$ and $\mathrm{Av}(132, P(\pi))$ are (conjecturally) Wilf-equivalent.

5.2 Enumeration of $\mathrm{Av}(231, \pi)$, for π respecting P

Theorem 20 shows that for any n, there are n permutations $\pi \in \mathrm{Av}_n(231)$ such that the two classes $\mathrm{Av}(231, \pi)$ and $\mathrm{Av}(132, P(\pi))$ are Wilf-equivalent. We can actually prove that these $2n$ permutation classes we obtain (as exemplified in Table 1) are *all* Wilf-equivalent. Notice that for both $n = 7$ and 8, all classes $\mathrm{Av}(231, \pi)$ with π of length n in Table 2 are not in the same Wilf-equivalence class.

The above Wilf-equivalence result follows immediately from Theorem 24 below. We first define a family of generating function $F_n(t)$ recursively as follows: $F_1(t) = 1$, and for $n \geq 1$

$$F_{n+1}(t) = \frac{1}{1 - tF_n(t)} \text{ for } n \geq 1.$$

This family satisfies a property that we shall use in the proof of Theorem 24:

Lemma 22 *Define* $g(x, y) = \frac{1-txy}{1-tx-ty}$. *For any* $n \geq 3$, *and any* $j, k \geq 1$ *such that* $j + k = n - 1$, $F_n = g(F_j, F_k)$.

Proof: Fix some $n \geq 3$. Let us remark that $g\left(\frac{1}{1-tx}, \frac{y-1}{ty}\right) = g(x,y)$.
Consequently, for any $j > 1$ and $k = n - j - 1$, we have $g(F_j, F_k) = g(F_{j-1}, F_{k+1})$. So it is enough to prove that $g(F_1, F_{n-2}) = F_n$. It is easily derived from the definition of the family (F_n). □

Based on the decompositions $\lambda_n = 1 \ominus \rho_{n-1}$ and $\rho_n = \lambda_{n-1} \oplus 1$ it is relatively easy to prove inductively that:

Lemma 23 *The generating functions of* $\mathrm{Av}(231, \lambda_n)$ *and* $\mathrm{Av}(231, \rho_n)$ *respectively are both equal to* F_n.

Finally we can also establish using the preceding two results:

Theorem 24 *Let* $\pi \in \mathrm{Av}_n(231)$ *be a permutation that respects* P. *The generating function of* $\mathrm{Av}(231, \pi)$ *is* F_n.

Proof: This follows immediately from Lemma 23 if π is of the form λ_n or ρ_n for any $n \geq 1$. Otherwise, by Theorem 21, we have $\pi = \lambda_j \oplus (1 \ominus \rho_k)$ for some $j \geq 1$ and $k \geq 1$. Let $\mathcal{C} = \mathrm{Av}(231, \pi)$ and let C be the corresponding generating function. When decomposing permutations of $\mathcal{C}$ as $\alpha \oplus (1 \ominus \beta)$, the subsequent constraints on α and β, together with Lemma 23, allow us to write that

$$C = 1 + tF_jC + t(C - F_j)F_k, \ \ \textit{i.e.}\ C = \frac{1 - tF_jF_k}{1 - tF_j - tF_k}.$$

Lemma 22 then ensures that $C = F_n$. □

6 Conclusions

Many other permutation classes have recursive descriptions similar to those of $\mathrm{Av}(231)$ and $\mathrm{Av}(132)$. In such cases it may well be possible to define analogous bijections to P which could lead to a unified framework for understanding Wilf-equivalences between their subclasses. Indeed, even for these two classes it is possible to combine the bijections P and $\mathbf{R}$ into various hybrid forms, and some of these may be useful in characterising the additional Wilf-equivalences that seem to exist in this context.

Of course our results provide some bijections between collections of permutations sorted by some combinations of $\mathbf{S}$ and $\mathbf{R}$. However, they do not provide enumerations of these collections – this seems to remain a difficult problem in general (and even more so if symmetries other than $\mathbf{R}$ are included) as suggested by the relative difficulty of enumerating the permutations sorted by $\mathbf{S} \circ \mathbf{S}$ compared to those sorted by $\mathbf{S}$. Another point is to determine whether or not the bijection here between specifically the permutations sorted by $\mathbf{S} \circ \mathbf{S}$ and those sorted by $\mathbf{S} \circ \mathbf{R} \circ \mathbf{S}$ is the same as the one described implicitly in Bouvel and Guibert (2012).

There are other relatively natural sorting operators. For instance bubble sort can be defined by $\mathsf{B}(\alpha n \beta) = \mathsf{B}(\alpha)\beta n$. Albert et al. (2011) considered the inverse images of permutation classes under B and some investigations of composites of B and related operators have been reported by Ferrari (2012). Combining such operators with $\mathbf{S}$ (and other possibilities) offers further scope for the discovery (or explanation) of Wilf-equivalences among permutation classes.

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Pattern-avoiding Dyck paths†

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Abstract. We introduce the notion of *pattern* in the context of lattice paths, and investigate it in the specific case of Dyck paths. Similarly to the case of permutations, the pattern-containment relation defines a poset structure on the set of all Dyck paths, which we call the *Dyck pattern poset*. Given a Dyck path P, we determine a formula for the number of Dyck paths covered by P, as well as for the number of Dyck paths covering P. We then address some typical pattern-avoidance issues, enumerating some classes of pattern-avoiding Dyck paths. Finally, we offer a conjecture concerning the asymptotic behavior of the sequence counting Dyck paths avoiding a generic pattern and we pose a series of open problems regarding the structure of the Dyck pattern poset.

Résumé. Nous proposons la notion d'un motif dans le contexte de chemins de treillis, et étudions le cas spécifique des chemins de Dyck. Comme dans le cas des permutations, on obtient une structure de poset sur l'ensemble de tous les chemins de Dyck, que nous appelons l'ensemble des chemins de Dyck partiellement ordonné selon le motif. Étant donné un chemin de Dyck P, nous déterminons une formule pour le nombre de chemins de Dyck couverts par P, ainsi que pour le nombre de chemins de Dyck couvrant P. Nous énumérons ensuite les chemins de Dyck évitant certaines catégories de motif. Enfin, nous proposons une conjecture asymptotique concernant le nombre de chemins de Dyck évitant un motif générique et nous posons quelques problèmes ouverts concernants la structure du poset etudié.

Keywords: Dyck path, pattern containment relation, enumeration

1 Introduction

One of the most investigated and fruitful notions in contemporary combinatorics is that of a *pattern*. Historically it was first considered for permutations [Kn], then analogous definitions were provided in the context of many other structures, such as set partitions [Go, Kl, Sa], words [Bj, Bu], and trees [DPTW, Gi, R]. Perhaps all of these examples have been motivated or informed by the more classical notion of graphs and subgraphs. Informally speaking, given a specific class of combinatorial objects, a pattern can be thought of as an occurrence of a small object inside a larger one; the word "inside" means that the pattern is suitably embedded into the larger object, depending on the specific combinatorial class of objects. The main aim of the present work is to introduce the notion of pattern in the context of lattice paths and to begin its systematic study in the special case of Dyck paths.

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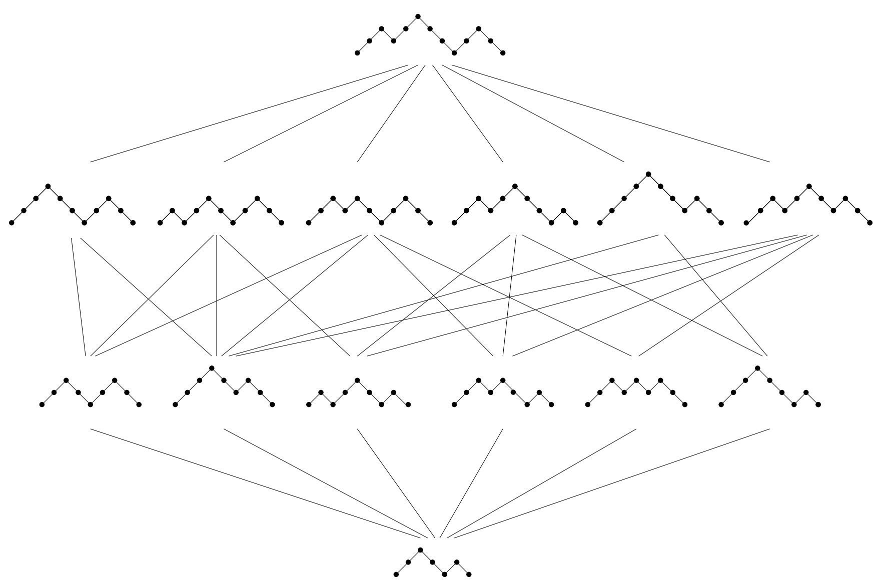
For our purposes, a *lattice path* is a path in the discrete plane starting at the origin of a fixed Cartesian coordinate system, ending somewhere on the x-axis, never going below the x-axis and using only a prescribed set of steps Γ. We will refer to such paths as Γ-*paths*. This definition is extremely restrictive if compared to what is called a lattice path in the literature, but it will be enough for our purposes. Observe that a Γ-path can be alternatively described as a finite word on the alphabet Γ obeying certain conditions. Using this language, we say that the *length* of a Γ-path is simply the length of the word which encodes such a path. Among the classical classes of lattice paths, the most common are those using only steps $U(p) = (1,1)$, $D(own) = (1,-1)$ and $H(orizontal) = (1,0)$; with these definitions, Dyck, Motzkin and Schröder paths correspond respectively to the set of steps $\{U, D\}$, $\{U, H, D\}$ and $\{U, H^2, D\}$.

Consider the class $\mathcal{P}_\Gamma$ of all Γ-paths, for some choice of the set of steps Γ. Given $P, Q \in \mathcal{P}_\Gamma$ having length k and n respectively, we say that Q *contains (an occurrence of) the pattern* P whenever P occurs as a subword of Q. So, for instance, in the class of Dyck paths, $UUDUDDUDUUDD$ contains the pattern $UUDDUD$, whereas in the class of Motzkin paths, $UUHDUUDHDDUDHUD$ contains the pattern $UHUDDHUD$. When Q does not contain any occurrence of P we will say that Q *avoids* P. In the Dyck case, the previously considered path $UUDUDDUDUUDD$ avoids the pattern $UUUUDDDD$.

This notion of pattern gives rise to a partial order in a very natural way, by declaring $P \leq Q$ when P occurs as a pattern in Q. In the case of Dyck paths, the resulting poset will be denoted by $\mathcal{D}$. It is immediate to notice that $\mathcal{D}$ has a minimum (the empty path), does not have a maximum, is locally finite and is ranked (the rank of a Dyck path is given by its semilength). As an example, we provide the Hasse diagram of an interval in the Dyck pattern poset:



Observe that this notion of pattern for paths is very close to the analogous notion for words (considered, for instance, in [Bj], where the author determines the Möbius function of the associated pattern poset). Formally, instead of considering the set of *all* words of the alphabet $\{U, D\}$, we restrict ourselves to the set of Dyck words (so what we actually do is to consider a subposet of Björner's poset). However, the conditions a word has to obey in order to belong to this subposet (which translate into the fact of being a Dyck word) make this subposet highly nontrivial, and fully justify our approach, consisting of the study of its properties independently of its relationship with the full word pattern poset.

2 The Dyck pattern poset

In the Dyck pattern poset $\mathcal{D}$, following the usual notation for covering relation, we write $P \prec Q$ (Q *covers* P) to indicate that $P \leq Q$ and the rank of P is one less than the rank of Q (i.e., $rank(P) = rank(Q) - 1$). Our first result concerns the enumeration of Dyck paths covered by a given Dyck path Q. We need some notation before stating it. Let $k+1$ be the number of points of Q lying on the x-axis (call such points $p_0, p_1, \dots, p_k$). Then Q can be factorized into k Dyck factors $F_1, \dots, F_k$, each F_i starting at p_{i-1} and ending at p_i. Let n_i be the number of ascents in F_i (an ascent being a consecutive run of U steps; n_i also counts both the number of descents and the number of peaks in F_i). Moreover, we denote by $|UDU|$ and $|DUD|$ the number of occurrences in a Dyck path of a consecutive factor UDU and DUD, respectively. In the path Q of Figure 1, we have $n_1 = 2$, $n_2 = 1$, $n_3 = 2$, $|UDU| = 3$, and $|DUD| = 2$.

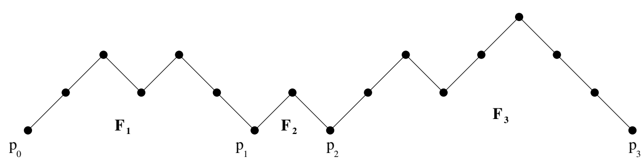


Fig. 1: A Dyck path having three factors.

Proposition 2.1 *If Q is a Dyck path with k factors $F_1, \dots F_k$, with F_i having n_i ascents, then the number of Dyck paths covered by Q is given by*

$$\frac{\sum_{i=1}^{k} {n_i}^2 + (\sum_{i=1}^{k} n_i)^2}{2} - |UDU| - |DUD| \quad . \tag{1}$$

In a similar fashion, we are also able to find a formula for the number of all Dyck paths which cover a given path.

Proposition 2.2 *If Q is a Dyck path of semilength n with k factors $F_1, \dots F_k$, with F_i having semilength f_i, then the number of Dyck paths covering Q is given by*

$$1 + \sum_{i} f_i^2 + \sum_{i<j} f_i f_j. \tag{2}$$

3 Enumerative results on pattern-avoiding Dyck paths

In the present section we will be concerned with the enumeration of some classes of pattern-avoiding Dyck paths. Similarly to what has been done for other combinatorial structures, we are going to consider classes of Dyck paths avoiding a single pattern, and we will examine the cases of short patterns. Specifically, we will count Dyck paths avoiding any single path of length ≤ 3; each case will arise as a special case of a more general result concerning a certain class of patterns.

Given a pattern P, we denote by $D_n(P)$ the set of all Dyck paths of semilength n avoiding the pattern P, and by $d_n(P)$ the cardinality of $D_n(P)$.

3.1 The pattern $(UD)^k$

This is one of the easiest cases.

Proposition 3.1 *For any $k \in \mathbf{N}$, $Q \in D_n((UD)^k)$ if and only if Q has at most $k-1$ peaks.*

Since it is well known that the number of Dyck paths of semilength n and having k peaks is given by the Narayana number $N_{n,k}$ (sequence A001263 in [Sl]), we have that $d_n((UD)^k) = \sum_{i=0}^{k-1} N_{n,i}$ (partial sums of Narayana numbers). Thus, in particular:

- $d_n(UD) = 0$;
- $d_n(UDUD) = 1$;
- $d_n(UDUDUD) = 1 + \binom{n}{2}$.

3.2 The pattern $U^{k-1}DUD^{k-1}$

Let Q be a Dyck path of length $2n$ and $P = U^{k-1}DUD^{k-1}$. Clearly if $n < k$, then Q avoids P, and if $n = k$, then all Dyck paths of length $2n$ except one (Q itself) avoid Q. Therefore:

- $d_n(P) = C_n$ if $n < k$, and
- $d_n(P) = C_n - 1$ if $n = k$,

where C_n is the n-th Catalan number.

Now suppose $n > k$. Denote by A the end point of the $(k-1)$-th U step of Q. It is easy to verify that A belongs to the line r having equation $y = -x + 2k - 2$. Denote with B the starting point of the $(k-1)$-th-to-last D step of Q. An analogous computation shows that B belongs to the line s having equation $y = x - (2n - 2k + 2)$.

Depending on how the two lines r and s intersect, it is convenient to distinguish two cases.

1. If $2n - 2k + 2 \geq 2k - 4$ (i.e. $n \geq 2k - 3$), then r and s intersect at height ≤ 1, whence $x_A \leq x_B$ (where x_A and x_B denote the abscissas of A and B, respectively). The path Q can be split into three parts (see Figure 2): a prefix Q_A from the origin $(0,0)$ to A, a path X from A to B, and a suffix Q_B from B to the last point $(2n, 0)$.

   We point out that Q_A has exactly $k-1$ U steps and its last step is a U step. Analogously, Q_B has exactly $k-1$ D steps and its first step is a D step. Notice that there is a clear bijection between the set $\mathcal{A}$ of Dyck prefixes having $k-1$ U steps and ending with a U and the set $\mathcal{B}$ of Dyck suffixes having $k-1$ D steps and starting with a D, since each element of $\mathcal{B}$ can be read from right to left thus obtaining an element of $\mathcal{A}$. Moreover, $\mathcal{A}$ is in bijection with the set of Dyck paths of semilength $k-1$ (just complete each element of $\mathcal{A}$ with the correct sequence of D steps), hence $|\mathcal{A}| = C_{k-1}$.

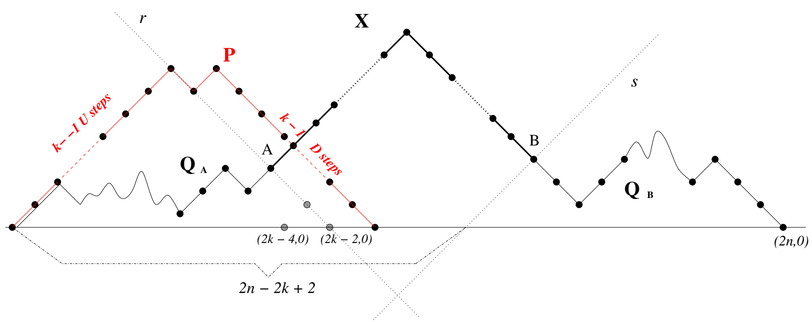


Fig. 2: Avoiding $U^{k-1}DUD^{k-1}$, with $n \geq 2k-3$

If we require Q to avoid P, then necessarily $X = U^iD^j$, for suitable i, j (for, if a valley DU occurred in X, then Q would contain P since U^{k-1} and D^{k-1} already occur in Q_A and Q_B, respectively). In other words, A and B can be connected only in one way, using a certain number (possibly zero) of U steps followed by a certain number (possibly zero) of D steps. Therefore, a path Q avoiding P is essentially constructed by choosing a prefix Q_A from $\mathcal{A}$ and a suffix Q_B from $\mathcal{B}$, whence:

$$d_n(P) = C_{k-1}^2, \quad (\text{if} \quad n \geq 2k-3). \tag{3}$$

2. Suppose now $k+1 \leq n < 2k-3$ (which means that r and s intersect at height > 1). Then it can be either $x_A \leq x_B$ or $x_A > x_B$.

   a) If $x_A \leq x_B$, then we can count all Dyck paths Q avoiding P using an argument analogous to the previous one. However, in this case the set of allowable prefixes of each such Q is a proper subset of $\mathcal{A}$. More specifically, we have to consider only those for which $x_A = k-1, k, k+1, \ldots, n$ (see Figure 3). In other words, an allowable prefix has $k-1$ U steps

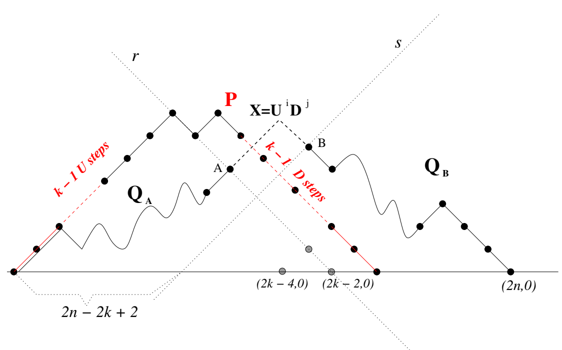


Fig. 3: Avoiding $U^{k-1}DUD^{k-1}$, with $x_A \leq x_B$

and $0, 1, 2, \ldots$ or $n-k+1$ D steps. If $b_{i,j}$ denotes the numbers of Dyck prefixes with i U

steps and j D steps ($i \geq j$), then the contribution to $d_n(P)$ in this case is

$$d_n^{(1)}(P) = \left(\sum_{j=0}^{n-k+1} b_{k-2,j} \right)^2 .$$

The coefficients $b_{i,j}$ are the well-known *ballot numbers* (sequence A009766 in [Sl]), whose first values are reported in Table 1.

b) If $x_A > x_B$, then it is easy to see that Q necessarily avoids P, since A clearly occurs after B, and so there are strictly less than $k-1$ D steps from A to $(2n, 0)$. Observe that, in this case, the path Q lies below the profile drawn by the four lines $y = x$, r, s and $y = -x + 2n$. In order to count these paths, referring to Figure 4, just split each of them into a prefix and a suffix of equal length n and call C the point having abscissa n.

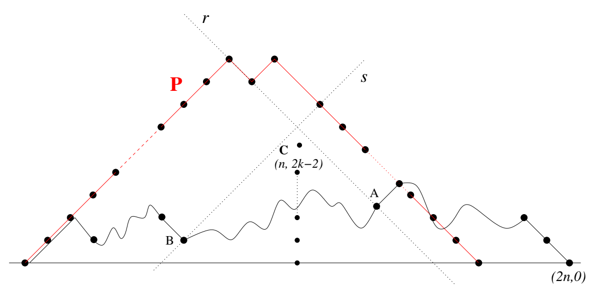


Fig. 4: Avoiding $U^{k-1}DUD^{k-1}$, with $x_A > x_B$

Since C must lie under the point where r and s intersect, then its ordinate y_C equals $-n + 2k - 2 - 2t$ with $t \geq 1$ (and also recalling that $y_C = -n + 2k - 2 - 2t \geq 0$). A prefix whose final point is C has $k - j$ U steps and $n - k + j$ D steps, with $j \geq 2$. Since, in this case, a path Q avoiding P is constructed by gluing a prefix and a suffix chosen among $b_{k-j,n-k+j}$ possibilities ($j \geq 2$), we deduce that the contribution to $d_n(P)$ in this case is:

$$d_n^{(2)}(P) = \sum_{j \geq 2} b^2_{k-j,n-k+j}.$$

Summing up the two contributions we have obtained in a) and b), we get:

$$\begin{aligned} d_n(P) &= d_n^{(1)}(P) + d_n^{(2)}(P) \\ &= \left(\sum_{j=0}^{n-k+1} b_{k-2,j} \right)^2 + \sum_{j \geq 2} b^2_{k-j,n-k+j}, \quad \text{if} \quad k+1 \leq n < 2k-3. \end{aligned} \tag{4}$$

i \ j	0	1	2	3	4	5	6	7	8	9
0	1									
1	1	1								
2	1	2	2							
3	1	3	5	5						
4	1	4	9	14	14					
5	1	5	14	28	**42**	42				
6	*1*	*6*	*20*	**48**	90	132	132			
7	1	7	**27**	75	165	297	429	429		
8	1	8	35	110	275	572	1001	1430	1430	
9	1	9	44	154	429	1001	2002	3432	4862	4862

Tab. 1: The sum of the gray entries gives the bold entry in the line below. The sum of the squares of the bold entries gives an appropriate element of Table 2.

Notice that formula (4) reduces to the first sum if $n \geq 2k-3$, since in that case $n-k+j > k-j$, for $j \geq 2$. We then have a single formula including both cases 1. and 2.:

$$d_n(P) = \left(\sum_{j=0}^{n-k+1} b_{k-2,j}\right)^2 + \sum_{j\geq 2} b^2_{k-j,n-k+j}, \quad \text{if} \quad n \geq k+1 \quad . \tag{5}$$

Formula (5) can be further simplified by recalling a well known recurrence for ballot numbers, namely that, when $j \leq i+1$,

$$b_{i+1,j} = \sum_{s=0}^{j} b_{i,s}.$$

Therefore, we get the following interesting expression for $d_n(P)$ (when $n \geq k+1$) in terms of sums of squares of ballot numbers along a skew diagonal (see also Tables 1 and 2):

$$d_n(P) = \begin{cases} C^2_{k-1} & \text{if } n \geq 2k-3; \\ \sum_{j\geq 1} b^2_{k-j,n-k+j} & \text{otherwise.} \end{cases} \tag{6}$$

Therefore we obtain in particular:

$$d_n(UUDUDD) = 4, \text{ when } n \geq 3.$$

3.3 The pattern U^kD^k

The case $P = U^kD^k$ is very similar to the previous one. We just observe that, when $x_A \leq x_B$, the two points A and B can be connected only using a sequence of D steps followed by a sequence of U steps. This is possible only if $n \leq 2k-2$, which means that r and s do not intersect below the x-axis. Instead, if $n \geq 2k-1$, Q cannot avoid P. Therefore we get (see also Table 3):

$$d_n(P) = \begin{cases} 0 & \text{if } n \geq 2k-1; \\ \sum_{j\geq 1} b^2_{k-j,n-k+j} & \text{otherwise.} \end{cases}$$

k \ n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	…
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	…
2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	…
3	1	1	2	4	4	4	4	4	4	4	4	4	4	4	…
4	1	1	2	5	13	25	25	25	25	25	25	25	25	25	…
5	1	1	2	5	14	41	**106**	196	196	196	196	196	196	196	…
6	1	1	2	5	14	42	131	**392**	**980**	1764	1764	1764	1764	1764	…
7	1	1	2	5	14	42	132	428	**1380**	**4068**	**9864**	17424	17424	17424	…
8	1	1	2	5	14	42	132	429	1429	**4797**	**15489**	**44649**	**105633**	184041	…
9	1	1	2	5	14	42	132	429	1430	4861	**16714**	**56749**	**181258**	**511225**	…

Tab. 2: Number of Dyck paths of semilength n avoiding $U^{k-1}DUD^{k-1}$. Entries in boldface are the nontrivial ones ($k+1 \leq n < 2k-3$).

k \ n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	…
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	…
2	1	1	1	0	0	0	0	0	0	0	0	0	0	0	…
3	1	1	2	4	4	0	0	0	0	0	0	0	0	0	…
4	1	1	2	5	13	25	25	0	0	0	0	0	0	0	…
5	1	1	2	5	14	41	**106**	196	196	0	0	0	0	0	…
6	1	1	2	5	14	42	131	**392**	**980**	1764	1764	0	0	0	…
7	1	1	2	5	14	42	132	428	**1380**	**4068**	**9864**	17424	17424	0	…
8	1	1	2	5	14	42	132	429	1429	**4797**	**15489**	**44649**	**105633**	184041	…
9	1	1	2	5	14	42	132	429	1430	4861	**16714**	**56749**	**181258**	**511225**	…

Tab. 3: Number of Dyck paths of semilength n avoiding U^kD^k . Entries in boldface are the nontrivial ones ($k+1 \leq n < 2k-3$).

In particular, we then find:

- $d_n(UUDD) = 0$, when $n \geq 3$;
- $d_n(UUUDDD) = 0$, when $n \geq 5$.

3.4 The pattern $U^{k-1}D^{k-1}UD$

This is by far the most challenging case.

Let Q be a Dyck path of length $2n$ and $P = U^{k-1}D^{k-1}UD$. If Q avoids P, then there are two distinct options: either Q avoids $U^{k-1}D^{k-1}$ or Q contains such a pattern. In the first case, we already know that $d_n(U^{k-1}D^{k-1})$ is eventually equal to zero. So, for the sake of simplicity, we will just find a formula for $d_n(P)$ when n is sufficiently large, i.e. $n \geq 2k-3$. Therefore, for the rest of this section, we will suppose that Q contains $U^{k-1}D^{k-1}$.

The $(k-1)$-th D step of the first occurrence of $U^{k-1}D^{k-1}$ in Q lies on the line having equation $y = -x+2n$. This is due to the fact that Q has length $2n$ and there cannot be any occurrence of UD after the first occurrence of $U^{k-1}D^{k-1}$. The path Q touches the line of equation $y = -x + 2k - 2$ for the first time with the end point A of its $(k-1)$-th U step. After that, the path Q must reach the starting point B of the $(k-1)$-th D step occurring after A. Finally, a sequence of consecutive D steps terminates Q (see Figure 5).

Therefore, Q can be split into three parts: the first part, from the beginning to A, is a Dyck prefix having $k-1$ U steps and ending with a U step; the second part, from A to B, is a path using $n-k+1$ U steps and $k-2$ D steps; and the third part, from B to the end, is a sequence of D steps (whose length depends on the coordinates of A). However, both the first and the second part of Q have to obey some additional constraints.

The height of the point A (where the first part of Q ends) must allow Q to have at least $k-1$ D steps after A. Thus, the height of A plus the number of U steps from A to B minus the number of D steps from A to B must be greater than or equal to 1 (to ensure that the pattern $U^{k-1}D^{k-1}$ occurs in Q). Hence, denoting with x the maximum number of D steps which can occur before A, either $x = k-2$ or the following equality must be satisfied:

$$(k-1) - x + (n-k+1) - (k-2) = 1.$$

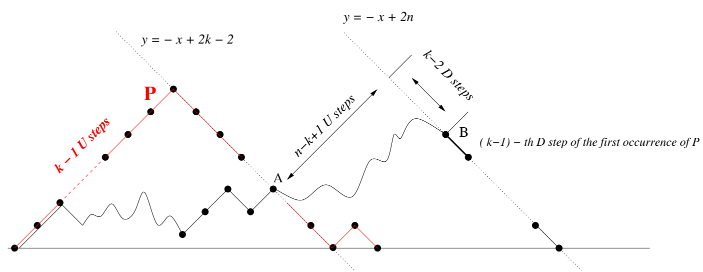


Fig. 5: A path Q avoiding $P = U^{k-1}D^{k-1}UD$

Therefore, $x = \min\{n-k+1, k-2\}$. Observe however that, since we are supposing that $n \geq 2k-3$, we always have $x = k-2$.

Concerning the part of Q between A and B, since we have to use $n-k+1$ U steps and $k-2$ D steps, there are $\binom{n-1}{k-2}$ distinct paths connecting A and B. However, some of them must be discarded, since they fall below the x-axis. In order to count these "bad" paths, we split each of them into two parts. Namely, if A' and B' are the starting and ending points of the first (necessarily D) step below the x-axis, the part going from A to A', and the remaining part (see Fig. 6).

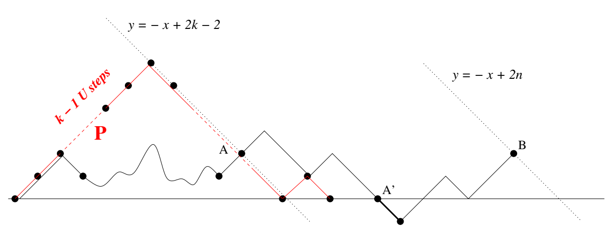


Fig. 6: A forbidden subpath from A to B.

It is not too hard to realize that the number of possibilities we have to choose the first part is given

k \ n	0	1	2	3	4	5	6	7	8	9
1	1	1	0	0	0	0	0	0	0	0
2	1	1	1	1	1	1	1	1	1	1
3	1	1	2	4	6	8	10	12	14	16
4	1	1	2	5	13	28	48	73	103	138
5	1	1	2	5	14	41	110	245	450	739
6	1	1	2	5	14	42	131	397	1069	2427

Tab. 4: Avoiding $U^{k-1}D^{k-1}UD$

by a ballot number (essentially because, reading the path from right to left, we have to choose a Dyck prefix from A' to A), whereas the number of possibilities we have to choose the second part is given by a binomial coefficient (essentially because, after having discarded the step starting at A', we have to choose an unrestricted path from B' to B). After a careful inspection, we thus get to the following expression for the total number $d_n(P)$ of Dyck paths of semilength $n \geq 2k-3$ avoiding P:

$$\begin{aligned} d_n(P) = & \binom{n-1}{k-2} C_{k-1} \\ & - \sum_{s=2}^{k-2} b_{k-2,s} \cdot \left(\sum_{i=0}^{s-2} b_{k-3-i,s-2-i} \binom{n-k-s+3+2i}{i} \right). \end{aligned} \tag{7}$$

Formula (7) specializes to the following expressions for low values of k (see also Table 4):

- when $k = 3$, $d_n(P) = 2n - 2$ for $n \geq 3$;
- when $k = 4$, $d_n(P) = \frac{5n^2-15n+6}{2}$ for $n \geq 5$;
- when $k = 5$, $d_n(P) = \frac{14n^3-84n^2+124n-84}{6}$ for $n \geq 7$.

4 Some remarks on the asymptotics of pattern-avoiding Dyck paths

In this final section we collect some thoughts concerning the asymptotic behavior of integer sequences counting pattern-avoiding Dyck paths. Unlike the case of permutations, for Dyck paths it seems plausible that a sort of "master theorem" exists, at least in the case of single avoidance. This means that all the sequences which count Dyck paths avoiding a single pattern P have the same asymptotic behavior (with some parameters, such as the leading coefficient, depending on the specific path P). We have some computational evidence which leads us to formulate a conjecture, whose proof we have not been able to complete, and so we leave it as an open problem.

Let P denote a fixed Dyck path of semilength x. We are interested in the behavior of $d_n(P)$ when $n \to \infty$. Our conjecture is the following:

Conjecture. *Suppose that P starts with a U steps and ends with b D steps. Then, setting $k = 2x - 2 - a - b$, we have that $d_n(P)$ is asymptotic to*

$$\frac{\alpha_P \cdot C_a \cdot C_b}{k!} n^k,$$

where C_m denotes the m-th Catalan numbers and α_P is the number of saturated chains in the Dyck lattice of order x (see [FP]) from P to the maximum $U^x D^x$.

Equivalently, α_P is the number of standard Young tableaux whose Ferrers shape is determined by the region delimited by the path P and the path $U^x D^x$, as shown in Figure 7.

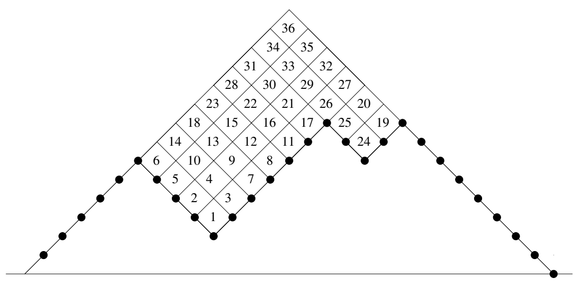


Fig. 7: The standard Young tableau determined by a Dyck path.

We close our paper with some further conjectures concerning the order structure of the Dyck pattern poset.

- What is the Möbius function of the Dyck pattern poset (from the bottom element to a given path? Of a generic interval?)?
- How many (saturated) chains are there up to a given path? Or in a general interval?
- Does there exist an infinite antichain in the Dyck pattern poset?

The last conjecture has been suggested by an analogous one for the permutation pattern poset which has been solved in the affirmative (see [SB] and the accompanying comment). In the present context we have no intuition on what could be the answer, though we are a little bit less optimistic than in the permutation case.

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The unreasonable ubiquitousness of quasi-polynomials†

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Abstract. A function g, with domain the natural numbers, is a quasi-polynomial if there exists a period m and polynomials $p_0, p_1, \dots, p_{m-1}$ such that $g(t) = p_i(t)$ for $t \equiv i \bmod m$. Quasi-polynomials classically – and "reasonably" – appear in Ehrhart theory and in other contexts where one examines a family of polyhedra, parametrized by a variable t, and defined by linear inequalities of the form $a_1x_1 + \cdots + a_dx_d \leq b(t)$.

Recent results of Chen, Li, Sam; Calegari, Walker; and Roune, Woods show a quasi-polynomial structure in several problems where the a_i are also allowed to vary with t. We discuss these "unreasonable" results and conjecture a general class of sets that exhibit various (eventual) quasi-polynomial behaviors: sets S_t that are defined with quantifiers ($\forall$, $\exists$), boolean operations (and, or, not), and statements of the form $a_1(t)x_1 + \cdots + a_d(t)x_d \leq b(t)$, where $a_i(t)$ and $b(t)$ are polynomials in t. These sets are a generalization of sets defined in the Presburger arithmetic. We prove several relationships between our conjectures, and we prove several special cases of the conjectures.

Résumé. Une fonction g, ayant les entiers naturels pour domaine, est un quasi-polynôme si il existe un entier m et des ploynômes $p_0, p_1, \dots, p_{m-1}$ tels que $g(t) = p_i(t)$ pour $t \equiv i \bmod m$. Les quasi-polynômes apparaissent dans la théorie d'Erhart, ainsi que dans d'autres contextes où l'on s'intéresse à des familles de polyhèdres paramétrisées par une variable t, et définies par des inégalités linéaires de la forme $a_1x_1 + \cdots + a_dx_d \leq b(t)$.

Des résultats récents de Chen, Li, Sam; Calegari, Walker; et Roune, Woods exhibent une structure de quasi-polynôme dans plusieurs problèmes où les a_i peuvent aussi varier en fonction de t. Nous nous intéressons à ces cas "non-raisonnables" et nous conjecturons l'existence d'une classe générale d'ensembles qui exhibent divers (possiblement) comportement de type quasi-polynômes : il s'agit des ensembles S_t qui sont définis en termes de quantifieurs ($\forall$, $\exists$), d'opérateurs booléens (conjonction, disjonction, négation), et d'énoncés de la forme $a_1(t)x_1 + \cdots + a_d(t)x_d \leq b(t)$, où $a_i(t)$ et $b(t)$ sont des polynômes en la variable t. Ces ensembles généralisent des ensembles définis dans l'arithmétique de Presburger. Nous démontrons plusieurs relations entre nos conjectures, ainsi que plusieurs cas spéciaux de ces mêmes conjectures.

Keywords: Quasi-polynomials, Ehrhart theory, Presburger arithmetic, rational generating functions

†With apologies to Wigner (1960) and Hamming (1980). A full version of this paper is available at `http://www.oberlin.edu/faculty/kwoods/papers.html`.

1 Reasonable Ubiquitousness

In this section, we survey classical appearances of quasi-polynomials (though Section 1.3 might be new even to readers already familiar with Ehrhart theory). In Section 2, we survey some recent results where the appearance of quasi-polynomials is more surprising. In Section 3, we make several conjectures generalizing these "unreasonable" results. We state theorems relating these conjectures and state theorems proving certain cases. In particular, we conjecture that any family of sets S_t – defined with quantifiers ($\forall$, $\exists$), boolean operations (and, or, not), and statements of the form $\mathbf{a}(t)\cdot\mathbf{x}\leq b(t)$ (where $\mathbf{a}(t)\in\mathbb{Q}[t]^d, b(t)\in\mathbb{Q}[t]$, and $\cdot$ is the standard dot product) – exhibits eventual quasi-polynomial behavior, as well as rational generating function behavior. Of course, reasonable people may disagree on what is unreasonable; the title is a play on "The unreasonable effectiveness of mathematics in the natural sciences" Wigner (1960).

For reasons of space, proofs are omitted here; they are in the full version of this paper, available on the author's website. We use bold letters such as $\mathbf{x}$ to indicate multi-dimensional vectors.

Definition 1 *A function $g:\mathbb{N}\to\mathbb{Q}$ is a* quasi-polynomial *if there exists a period m and polynomials $p_0, p_1, \ldots, p_{m-1}\in\mathbb{Q}[t]$ such that*

$$g(t) = p_i(t), \textit{ for } t\equiv i \bmod m.$$

Example 2

$$g(t) = \left\lfloor \frac{t+1}{2} \right\rfloor = \begin{cases} \frac{t}{2} & \textit{if } t \textit{ even},\\ \frac{t+1}{2} & \textit{if } t \textit{ odd},\end{cases}$$

is a quasi-polynomial with period 2.

This example makes it clear that the ubiquitousness of quasi-polynomials shouldn't be too surprising: anywhere there are floor functions, quasi-polynomials are likely to appear. We will generally be concerned with *integer-valued* quasi-polynomials, those quasi-polynomials whose range lies in $\mathbb{Z}$. Note that Example 2 demonstrates that such quasi-polynomials may still require rational coefficients.

1.1 Ehrhart theory

Perhaps the most well-studied quasi-polynomials are the *Ehrhart quasi-polynomials*:

Theorem 3 (Ehrhart, 1962) *Suppose P is a polytope (bounded polyhedron) whose vertices have rational coordinates. Let $g(t)$ be the number of integer points in tP, the dilation of P by a factor of t. Then $g(t)$ is a quasi-polynomial, with period the smallest m such that mP has integer coordinates.*

Example 4 *Let P be the triangle with vertices $(0,0)$, $\left(\frac{1}{2},0\right)$, and $\left(\frac{1}{2},\frac{1}{2}\right)$. Then*

$$g(t) = \#(tP\cap\mathbb{Z}^2) = \frac{\big(\lfloor t/2\rfloor+1\big)\big(\lfloor t/2\rfloor+2\big)}{2} = \begin{cases}(t+2)(t+4)/8 & \textit{if } t \textit{ even,}\\ (t+1)(t+3)/8 & \textit{if } t \textit{ odd,}\end{cases} \tag{1}$$

is a quasi-polynomial with period 2.

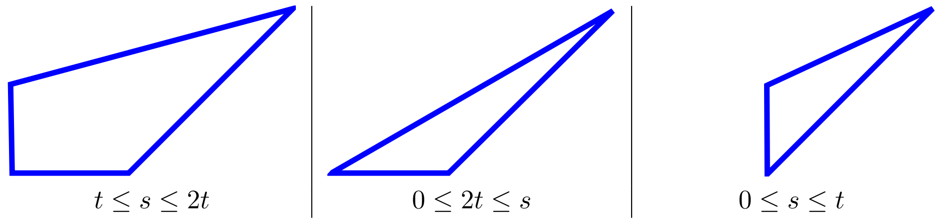

Fig. 1: Polyhedra defined in Example 5 for various $(s,t) \in \mathbb{N}^2$.

Writing tP from this example as

$$\{(x,y) \in \mathbb{R}^2 :\ 2x \le t,\ y - x \le 0,\ -y \le 0\}$$

suggests a way to generalize this result: for $\mathbf{t} \in \mathbb{N}^n$, let $S_\mathbf{t}$ be the set of integer points, $\mathbf{x} \in \mathbb{Z}^d$, in a polyhedron defined with linear inequalities of the form $\mathbf{a} \cdot \mathbf{x} \le b(\mathbf{t})$, where $\mathbf{a} \in \mathbb{Z}^d$ and $b(\mathbf{t})$ is a degree 1 polynomial in $\mathbf{t}$.

Example 5 *Let*

$$S_{s,t} = \left\{(x,y) \in \mathbb{Z}^2 :\ 2y - x \le 2t - s,\ x - y \le s - t,\ x, y \ge 0\right\}.$$

For a fixed (s,t), $S_{s,t}$ is the set of integer points in a polyhedron in $\mathbb{R}^2$. As (s,t) varies, the "constant" term of these inequalities change, but the coefficients of x and y do not; in other words, the normal vectors to the facets of the polyhedron do not change, but the facets move "in and out". In fact, they can move in and out so much that the combinatorial structure of the polyhedron changes. Figure 1 shows the combinatorial structure for different $(s,t) \in \mathbb{N}^2$. Using various methods, Beck (2004) and Verdoolaege and Woods (2008) compute that

$$g(s,t) = |S_{s,t}| = \begin{cases} \frac{s^2}{2} - \lfloor \frac{s}{2} \rfloor s + \frac{s}{2} + \lfloor \frac{s}{2} \rfloor^2 + \lfloor \frac{s}{2} \rfloor + 1 & \text{if } t \le s \le 2t, \\ st - \lfloor \frac{s}{2} \rfloor s - \frac{t^2}{2} + \frac{t}{2} + \lfloor \frac{s}{2} \rfloor^2 + \lfloor \frac{s}{2} \rfloor + 1 & \text{if } 0 \le 2t \le s, \\ \frac{t^2}{2} + \frac{3t}{2} + 1 & \text{if } 0 \le s \le t. \end{cases}$$

In this example, the function $g(s,t)$ is a quasi-polynomial (in this multivariate case, one must consider both s and t modulo some periods), at least *piecewise*. Sturmfels (1995) effectively proved this generalization of Ehrhart theory:

Theorem 6 *Let $S_\mathbf{t}$ be the set of integer points, $\mathbf{x} \in \mathbb{Z}^d$, in a polyhedron defined with linear inequalities $\mathbf{a} \cdot \mathbf{x} \le b(\mathbf{t})$, where $\mathbf{a} \in \mathbb{Z}^d$ and $b(\mathbf{t})$ is a degree 1 polynomial in $\mathbb{Z}[\mathbf{t}]$. Then $g(\mathbf{t}) = |S_\mathbf{t}|$ is a piecewise-defined quasi-polynomial, where the finite number of pieces are polyhedral regions of parameter space.*

Sections 2 and further will predominantly be concerned with *univariate* functions. Being a univariate piecewise quasi-polynomial $g : \mathbb{N} \to \mathbb{Q}$ is equivalent to *eventually* being a quasi-polynomial; that is, there exists a T such that for all $t \ge T$, $g(t)$ agrees with a quasi-polynomial.

1.2 Generating functions

Classic proofs of Ehrhart's Theorem (Theorem 3) use generating functions. To prove that a function $g(t)$ is a quasi-polynomial of period m, it suffices (see Section 4.4 of Stanley, 2012) to prove that the Hilbert series $\sum_{t\in\mathbb{N}} g(t)y^t$ can be written as a rational function of the form

$$\frac{p(y)}{(1-y^m)^d},$$

where $p(y)$ is a polynomial of degree less than md. For $g(t) = |tP|$ with P the triangle in Example 4, we can see that

$$\sum_{t\in\mathbb{N}} g(t)y^t = 1 + y + 3y^2 + 3y^3 + 6y^4 + \cdots = \frac{1+y}{(1-y^2)^3}. \tag{2}$$

Indeed, these proofs of Ehrhart's Theorem start by considering the generating function $\sum_{t\in\mathbb{N},\mathbf{s}\in tP\cap\mathbb{Z}^d} \mathbf{x}^{\mathbf{s}}y^t$ (where $\mathbf{x}^{\mathbf{s}} = x_1^{s_1}\cdots x_d^{s_d}$) and substituting in $\mathbf{x} = (1,\ldots,1)$ to get the Hilbert series. For P in Example 4,

$$\begin{aligned}\sum_{t\in\mathbb{N},\mathbf{s}\in tP\cap\mathbb{Z}^d} \mathbf{x}^{\mathbf{s}}y^t &= 1 + y + (1 + x_1 + x_1x_2)y^2 + (1 + x_1 + x_1x_2)y^3 + (1 + \cdots + x_1^2x_2^2)y^4 + \cdots \\ &= \frac{1+y}{(1-y^2)(1-x_1y^2)(1-x_1x_2y^2)},\end{aligned}$$

as can be checked by expanding as a product of infinite geometric series. Substituting $x_1 = x_2 = 1$ yields the Hilbert series in Equation 2.

Definition 7 *We call any generating function or Hilbert series a* rational generating function *if it can be written in the form*

$$\frac{p(\mathbf{x})}{(1-\mathbf{x}^{\mathbf{b}_1})\cdots(1-\mathbf{x}^{\mathbf{b}_k})},$$

where p is a Laurent polynomial over $\mathbb{Q}$ and $\mathbf{b}_i \in \mathbb{Z}^d$ are lexicographically positive (first nonzero entry is positive), .

While we will generally be assuming that the generating functions are for subsets of $\mathbb{N}^d$, we need $\mathbf{b}_i$ to be lexicographically positive rather than simply in $\mathbb{N}^d \setminus \{0\}$ for examples like the following:

Example 8 *Let $S = \{(x,y) \in \mathbb{N}^2 :\ x + y = 1000\}$. While $y^{1000} + xy^{999} + \cdots + x^{1000}$ is a legitimate generating function, it makes more sense to write it as*

$$\frac{y^{1000} - x^{1001}y^{-1}}{1 - xy^{-1}}.$$

If $\mathbf{b}$ is lexicographically negative, then

$$\frac{1}{1-\mathbf{x}^{\mathbf{b}}} = \frac{-\mathbf{x}^{-\mathbf{b}}}{1-\mathbf{x}^{-\mathbf{b}}}$$

with $-\mathbf{b}$ is lexicographically positive. Having $\mathbf{b}$ lexicographically positive guarantees that $1/(1-\mathbf{x}^{\mathbf{b}}) = 1 + \mathbf{x}^{\mathbf{b}} + \mathbf{x}^{2\mathbf{b}} + \cdots$ is the Laurent series convergent in a neighborhood of $\mathbf{x} = (e^{-\varepsilon}, e^{-\varepsilon^2}, \ldots, e^{-\varepsilon^d})$ for sufficiently small ε.

In Section 3, we will use a different generating function: for *fixed* t, examine the generating function $\sum_{\mathbf{s}\in tP\cap\mathbb{Z}^d}\mathbf{x}^{\mathbf{s}}$. In the triangle from Example 4, this gives us

$$\left(1+x_1+x_1^2+\cdots+x_1^{\lfloor t/2\rfloor}\right)+\left(x_1+x_1^2+\cdot+x_1^{\lfloor t/2\rfloor}\right)x_2+\cdots+\left(x_1^{\lfloor t/2\rfloor}\right)x_2^{\lfloor t/2\rfloor}.$$

In general, powerful tools such as Brion's Theorem (Brion, 1988) help us compute a compact form for this generating function; see Verdoolaege and Woods (2008) for more details. In this example, we can verify directly, by expanding the fractions as products of geometric series, that

$$\sum_{\mathbf{s}\in tP\cap\mathbb{Z}^d}\mathbf{x}^{\mathbf{s}}=\frac{1}{(1-x_1)(1-x_1x_2)}-\frac{x_1^{\lfloor t/2\rfloor+1}}{(1-x_1)(1-x_2)}+\frac{x_1^{\lfloor t/2\rfloor+1}x_2^{\lfloor t/2\rfloor+2}}{(1-x_2)(1-x_1x_2)}. \tag{3}$$

Given this generating function, we can count the number of integer points in tP by substituting in $\mathbf{x}=(1,\ldots,1)$. Substituting $x_1=x_2=1$ into Equation 3, we see that $(1,1)$ is a pole of these fractions. Fortunately, getting a common denominator and applying L'Hôpital's rule to find the limit as x_1 and x_2 approach 1 will work, and it is evident that the differentiation involved in L'Hôpital's rule will yield a quasi-polynomial in t as the result; careful calculation will show that it matches Equation 1.

1.3 Presburger arithmetic

So far, our examples have been integer points in polyhedra. A key property of such sets is that they can be defined without quantifiers. However, even for sets defined with quantifiers, we end up with reasonable appearances of quasi-polynomials.

Definition 9 *A* Presburger formula *is a boolean formula with variables in $\mathbb{N}$ that can be written using quantifiers ($\exists,\forall$), boolean operations (and, or, not), and linear (in)equalities in the variables. We write a Presburger formula as $F(\mathbf{u})$ to indicate the the free variables $\mathbf{u}$ (those not associated with a quantifier).*

Presburger (1929) (see Presburger, 1991, for a translation) examined this first order theory and proved it is decidable.

Example 10 *Given $t\in\mathbb{N}$, let*

$$S_t=\left\{x\in\mathbb{N}:\ \exists y\in\mathbb{N}, 2x+2y+3=5t \textit{ and } t<x\le y\right\}.$$

We can compute that

$$S_t=\begin{cases}\left\{t+1,t+2,\ldots,\left\lfloor\frac{5t-3}{4}\right\rfloor\right\} & \textit{if } t \textit{ odd, } t\ge 3,\\ \emptyset & \textit{else.}\end{cases}$$

This set has several properties, *cf.* Section 3:

1. The set of t such that S_t is nonempty is $\{3,5,7,\ldots\}$. This set is eventually periodic.
2. The cardinality of S_t is
$$S_t=\begin{cases}\left\lfloor\frac{5t-3}{4}\right\rfloor-t & \text{if } t \text{ odd, } t\ge 3,\\ 0 & \text{else,}\end{cases}$$
which is eventually a quasi-polynomial of period 4.

3. When S_t is nonempty, we can obtain an element of S_t with the function $x(t) = t + 1$, and $x(t)$ is eventually a quasi-polynomial.

3a. More strongly, when S_t is nonempty, we can obtain the maximum element of S_t with the function $x(t) = \lfloor (5t-3)/4 \rfloor$, and $x(t)$ is eventually a quasi-polynomial.

4. We can compute the generating function

$$\sum_{s \in S_t} x^s = \begin{cases} x^{t+1} + x^{t+2} + \cdots + x^{\lfloor (5t-3)/4 \rfloor} & \text{if } t \text{ odd, } t \geq 3, \\ 0 & \text{else,} \end{cases}$$
$$= \begin{cases} \dfrac{x^{t+1} - x^{\lfloor (5t-3)/4) \rfloor + 1}}{1-x} & \text{if } t \text{ odd, } t \geq 3, \\ 0 & \text{else.} \end{cases}$$

We see that, for fixed t, this generating function is a rational function. Considering each residue class of $t \bmod 4$ separately, the exponents in the rational function can eventually be written as polynomials in t.

Versions of these properties always hold for sets defined in Presburger arithmetic. For example, Woods (2012) gave several properties of Presburger formulas that hold even for sets defined with multivariate parameters, $\mathbf{t} \in \mathbb{N}^n$:

Theorem 11 (from Theorems 1 and 2 of Woods, 2012) *Suppose $F(\mathbf{s}, \mathbf{t})$ is a Presburger formula, with $\mathbf{s}$ and $\mathbf{t}$ collections of free variables. Then*

- $g(\mathbf{t}) = \#\big\{\mathbf{s} \in \mathbb{N}^d : \ F(\mathbf{s}, \mathbf{t})\big\}$ *is a piecewise quasi-polynomial,*
- $\sum_{\mathbf{s},\mathbf{t}: F(\mathbf{s},\mathbf{t})} \mathbf{x}^{\mathbf{s}} \mathbf{y}^{\mathbf{t}}$ *is a rational generating function, and*
- $\sum_{\mathbf{t} \in \mathbb{N}^n} g(\mathbf{t}) \mathbf{y}^{\mathbf{t}}$ *is a rational generating function.*

Property 4 from Example 10 can be proved in general by using Theorem 11 to write $\sum_{\mathbf{s},\mathbf{t}: F(\mathbf{s},\mathbf{t})} \mathbf{x}^{\mathbf{s}} \mathbf{y}^{\mathbf{t}}$ as a rational generating function and applying Theorem 29. The proof of Theorem 26 then shows that all of the other properties follow, though the exact definitions of these properties are only stated in Section 3 for a univariate parameter, $t \in \mathbb{N}$.

2 Unreasonable Ubiquitousness

We now turn to the inspiration for this paper. Three recent results exhibit quasi-polynomial behavior, in situations that seem "unreasonable". In particular, all three involve sets S_t defined by inequalities $\mathbf{a}(t) \cdot \mathbf{x} \leq b(t)$, where $\mathbf{a}(t)$ is a polynomial in t; that is, the normal vectors to the facets change as t changes. First we give an example showing that, unlike in Section 1, it is now important that we restrict to only one parameter, t.

Example 12 *Define $S_{s,t} = \big\{(x, y) \in \mathbb{N}^2 : \ sx + ty = st\big\}$. Then $S_{s,t}$ is an interval in $\mathbb{Z}^2$ with endpoints $(t, 0)$ and $(0, s)$, and*

$$|S_{s,t}| = \gcd(s, t) + 1.$$

There is no hope for simple quasi-polynomial behavior here, as the cardinality depends on the arithmetic relationship of s and t.

2.1 Three results

This first result most directly generalizes Ehrhart Theory. Chen, Li, and Sam (2012) prove that, if S_t is the set of integer points in a polytope defined by inequalities of the form $\mathbf{a}(t) \cdot \mathbf{x} \le b(t)$, then $|S_t|$ is eventually a quasi-polynomial.

Theorem 13 (Theorem 2.1 of Chen et al., 2012) *Let $A(t)$ be an $r \times d$ matrix, and $b(t)$ be a column vector of length r, all of whose entries are in $\mathbb{Z}[t]$. Assume $P_t = \{\mathbf{x} \in \mathbb{R}^d :\ A(t)\mathbf{x} \le b(t)\}$ is eventually a bounded set (a polytope). Then $|P_t \cap \mathbb{Z}^d|$ is eventually a quasi-polynomial.*

Note that this can be equivalently phrased (Theorem 1.1 of Chen et al., 2012) using equalities $A(t)\mathbf{x} = b(t)$, where $\mathbf{x}$ is constrained to be nonnegative, or it can be phrased (Theorem 1.4 of Chen et al., 2012) by listing the vertices of P_t as rational functions of t.

Calegari and Walker (2011) were similarly concerned with the integer points in polyhedra defined by $A(t)\mathbf{x} \le b(t)$. Rather than counting $|P_t \cap \mathbb{Z}^d|$, they wanted to find the integer hull of P_t, that is, the set of vertices of the convex hull of $P_t \cap \mathbb{Z}^d$.

Theorem 14 (Theorem 3.5 of Calegari and Walker, 2011) *Let $\mathbf{v}_i(t)$ be vectors in $\mathbb{Q}^d$ whose coordinates are rational functions of size $O(t)$, and let P_t be the convex hull of the $\mathbf{v}_i(t)$. Then there exists a modulus m and functions $\mathbf{p}_{ij} : \mathbb{N} \to \mathbb{Z}^d$ with polynomial coordinates such that, for $0 \le i < m$ and for sufficiently large $t \equiv i \bmod m$, the integer hull of P_t is $\{\mathbf{p}_{i1}(t), \mathbf{p}_{i2}(t), \ldots, \mathbf{p}_{ik_i}(t)\}$.*

This theorem could be similarly phrased using facet definitions of the polyhedra, rather than vertex definitions. That the vertices are $\mathrm{O}(t)$ (grow no faster that ct for some constant c) is important for the proof, though Calegari and Walker conjecture that the theorem still holds without this restriction.

A third recent result concerns the Frobenius number.

Definition 15 *Given $a_1, \ldots, a_d \in \mathbb{N}$, let S be the semigroup generated by the a_i, that is,*

$$S = \{a \in \mathbb{N} :\ \exists \lambda_1, \ldots, \lambda_d \in \mathbb{N}, a = \lambda_1 a_1 + \cdots + \lambda_d a_d\}.$$

If the a_i are relatively prime, then S contains all sufficiently large integers, and the Frobenius number *is defined to be the largest integer not in S.*

Now we let $a_i = a_i(t)$ vary with t. Roune and Woods (2012) prove that, if the $a_i(t)$ are linear functions of t, then the Frobenius number is eventually a quasi-polynomial, and they conjecture that this is true if the $a_i(t)$ are any polynomial functions of t:

Theorem 16 *Let $a_i(t) \in \mathbb{Z}[t]$ have degree at most one and be eventually positive. Then the set of t such that the $a_i(t)$ are relatively prime is eventually periodic, and, for such t, the Frobenius number is eventually given by a quasi-polynomial.*

Example 17 *Consider $a_1(t) = t$, $a_2(t) = t + 3$. These are relatively prime exactly when $t \equiv 1, 2 \bmod 3$. Since there are only two generators, a well-known formula (seemingly due to Sylvester, 1884) gives that the Frobenius number is*

$$a_1 a_2 - a_1 - a_2 = t^2 + t - 3.$$

Note that Theorem 16 utilizes sets defined with quantifiers; Presburger arithmetic seems a good place to look for generalizations encompassing these three results.

2.2 Common tools

Each of these three results has their own method for proving quasi-polynomial behavior, but there are several common tools needed. Chen et al. (2012) and Calegari and Walker (2011) independently prove Theorems 18 through 22, and Calegari and Walker (2011) prove Theorem 23.

Theorem 18 (Division Algorithm) *Given $f(t), g(t)$ integer-valued polynomials,*

1. *if $\deg g > 0$, there exist integer-valued quasi-polynomials $q_1(t)$ and $r_1(t)$ such that $f(t) = q_1(t)g(t) + r_1(t)$, with $\deg r_1 < \deg g$, and*

2. *if $g \neq 0$, there exist integer-valued quasi-polynomials $q_2(t)$ and $r_2(t)$ such that $f(t) = q_2(t)g(t) + r_2(t)$, with eventually $0 \leq r_2(t) < |g(t)|$.*

These are both useful results, and only slightly different. For example, suppose $f(t) = 2t - 3$ and $g(t) = t$. Then Statement 1 is a traditional polynomial division algorithm: $f = 2g + -3$. Statement 2, however, is a numerical division algorithm: $f = 1g + (t - 3)$, and the remainder $t - 3$ is between 0 and g as long as $t \geq 3$. In other words, if we have found q_1 and r_1, but we eventually have $r_1(t) < 0$, then we should use quotient $q_2 = q_1 - \operatorname{sgn}(g)$ and remainder $r_2 = |g| + r_1$ instead, as eventually $0 \leq |g(t)| + r_1(t) < |g(t)|$.

The main subtlety in proving Statement 1 of this theorem is the following: Suppose $f(t) = t^2 + 3t$ and $g(t) = 2t + 1$. Then the leading coefficient of g does not divide the leading coefficient of f, and the traditional polynomial division algorithm would produce quotients that are not integer-valued. Instead, we look at t modulo the leading coefficient of g; for example, if $t = 2s + 1$, substituting gives $f(2s+1) = 4s^2 + 10s + 3$ and $g(2s + 1) = 4s + 3$, and now the leading term does divide evenly.

The division algorithm in hand, one can prove some stronger results:

Theorem 19 (Euclidean Algorithm and gcds) *Let f and g be integer-valued quasi-polynomials. Then there exists integer-valued quasi-polynomials $p(t)$, $q(t)$, and $d(t)$ such that $\gcd\big(f(t), g(t)\big) = d(t)$ and $d(t) = p(t)f(t) + q(t)g(t)$.*

This is obtained by repeated applications of the division algorithm.

Example 20

$$\gcd(2t + 1, 5t + 6) = \gcd(t + 4, 2t + 1) = \gcd(7, t + 4) = \begin{cases} 7 & \text{if } t \equiv 3 \bmod 7, \\ 1 & \text{else.} \end{cases}$$

Similarly, repeated application of the Euclidean algorithm can produce the Hermite or Smith normal forms of matrices. We won't define those here, but they are important, for example, in producing a basis for lower-dimensional sublattices of $\mathbb{Z}^d$ (see Newman, 1972).

Theorem 21 (Hermite/Smith Normal Forms) *Given a matrix $A(t)$ with integer-valued quasi-polynomial entries, the Hermite and the Smith Normal forms, as well as their associated change-of-basis matrices, also have quasi-polynomial entries.*

The following theorem is obvious, but is repeatedly used.

Theorem 22 (Dominance) *Suppose $f, g \in \mathbb{Q}[t]$ with $f \neq g$. Then either eventually $f(t) > g(t)$ or eventually $g(t) > f(t)$.*

Repeated use of this property, for example, shows that the combinatorial structure of a polyhedron P_t eventually stabilizes, when P_t is defined by $A(t)\mathbf{x} \leq b(t)$.

Rational functions commonly appear in these results. For example, if a polyhedron is defined by $A(t)\mathbf{x} \leq b(t)$, a vertex will be a point where several of these inequalities are equalities, i.e., the solution to some $A'(t)\mathbf{x} = b'(t)$, where $A'(t)$ is a full-rank $d \times d$ matrix of polynomials in t. Solving for $\mathbf{x}$ using the adjunct matrix of A' will result in $\mathbf{x}(t)$ given as a rational function of t. For large t, the behavior of a rational function is predictable:

Theorem 23 (Rounding) *Let $f(t), g(t) \in \mathbb{Z}[t]$. Then $f(t)/g(t)$ converges to a polynomial, and $\lfloor f(t)/g(t) \rfloor$ is eventually a quasi-polynomial.*

3 Conjectures

Let $S_t \subseteq \mathbb{N}^d$ be a family of subsets of natural numbers. We now discuss some properties that it would be nice (though unreasonable!) for such sets to have; *cf.* Example 10.

Property 1: The set of t such that S_t is nonempty is eventually periodic.

This is the weakest of the properties we will discuss, but an important one, as it is related to the decision problem – "Is there a solution?"

Property 2: There exists a function $g : \mathbb{N} \to \mathbb{N}$ such that, if S_t has finite cardinality, then $g(t) = |S_t|$, and $g(t)$ is eventually a quasi-polynomial. The set of t such that S_t has finite cardinality is eventually periodic.

This is the property found in Theorem 13, where S_t is the set of integer points in a polytope defined by inequalities $\mathbf{a}(t) \cdot \mathbf{x} \leq b(t)$. Theorems 14 and 16, on the other hand, are not about counting points but about finding points:

Property 3: There exists a function $\mathbf{x} : \mathbb{N} \to \mathbb{N}^d$ such that, if S_t is nonempty, then $\mathbf{x}(t) \in S_t$, and the coordinate functions of $\mathbf{x}$ are eventually quasi-polynomials. The set of t such that S_t is nonempty is eventually periodic.

This function $\mathbf{x}(t)$ acts as a certificate that the set is nonempty. But we may want to go further and pick out particular elements of S_t:

Property 3a: Given $\mathbf{c} \in \mathbb{Z}^d$, there exists a function $\mathbf{x} : \mathbb{N} \to \mathbb{N}^d$ such that, if $\max_{\mathbf{y} \in S_t} \mathbf{c} \cdot \mathbf{y}$ exists, then it is attained at $\mathbf{x}(t) \in S_t$, and the coordinate functions of $\mathbf{x}$ are eventually quasi-polynomials. The set of t such that the maximum exists is eventually periodic.

This corresponds to Theorem 16, where we want to find the Frobenius number, the maximum element of the complement of the semigroup. On the other hand, we may want to list *multiple* elements of the set:

Property 3b: Fix $k \in \mathbb{N}$. There exist functions $\mathbf{x}_1, \ldots, \mathbf{x}_k : \mathbb{N} \to \mathbb{N}^d$ such that, if $|S_t| \geq k$, then $\mathbf{x}_1(t), \ldots, \mathbf{x}_k(t)$ are distinct elements of S_t, and the coordinate functions of $\mathbf{x}_i$ are eventually quasi-polynomials. The set of t such that $|S_t| \geq k$ is eventually periodic.

If there is a uniform bound on $|S_t|$, then this property can be used to enumerate all elements of S_t, for all t. This is the content of Theorem 14. Property 2 is about counting all solutions and Properties 3/3a/3b are about obtaining specific solutions, and so they seem somewhat orthogonal to each other. The following property, we shall see, unifies them:

Property 4: There exists a period m such that, for $t \equiv i \bmod m$,

$$\sum_{\mathbf{s}\in S_t} \mathbf{x}^{\mathbf{s}} = \frac{\sum_{j=1}^{n_i} \alpha_{ij}\mathbf{x}^{\mathbf{q}_{ij}(t)}}{\left(1-\mathbf{x}^{\mathbf{b}_{i1}(t)}\right)\cdots\left(1-\mathbf{x}^{\mathbf{b}_{ik_i}(t)}\right)},$$

where $\alpha_{ij} \in \mathbb{Q}$, and the coordinate functions of $\mathbf{q}_{ij}, \mathbf{b}_{ij} : \mathbb{N} \to \mathbb{Z}^d$ are polynomials with the $\mathbf{b}_{ij}(t)$ lexicographically positive.

For what sets S_t can we hope for these properties to hold? Here is a candidate:

Definition 24 *A family of sets S_t is a* parametric Presburger family *if they can be defined over the natural numbers using quantifiers, boolean operations, and inequalities of the form $\mathbf{a}(t)\cdot\mathbf{x} \le b(t)$, where $b \in \mathbb{Z}[t]$ and $\mathbf{a} \in \mathbb{Z}[t]^d$.*

We conjecture that these properties do, in fact, hold for any parametric Presburger family:

Conjecture 25 *Let S_t be a parametric Presburger family. Then Properties 1, 2, 3, 3a, 3b, and 4 all hold.*

Note that one can define a family S_t of subsets of $\mathbb{Z}^d$ rather than of $\mathbb{N}^d$, though one must be more careful when talking about generating functions. For example, the set $\mathbb{Z}$ has generating function

$$\cdots + x^{-1} + 1 + x^1 + x^2 + \cdots = \frac{x^{-1}}{1-x^{-1}} + \frac{1}{1-x} = -\frac{1}{1-x} + \frac{1}{1-x} = 0.$$

See, for example, Barvinok (2008) for more details.

As evidence that Property 4 is interesting, we will show that it generalizes both 2 and 3/3a/3b:

Theorem 26 *Let S_t be any family of subsets of $\mathbb{N}^d$. We have the following implications among possible properties of S_t.*

$$\begin{tikzcd}
& 2 \arrow[drr, Rightarrow, bend left] & & \\
4 \arrow[ur, Rightarrow, bend left] \arrow[r, Rightarrow] \arrow[dr, Rightarrow, bend right] & 3a \arrow[r, Rightarrow] & 3 \arrow[r, Rightarrow] & 1 \\
& 3b \arrow[ur, Rightarrow, bend right] & &
\end{tikzcd}$$

As a final relationship between these properties, we note that, for the class of parametric Presburger families, 3, 3a, and 3b are equivalent:

Theorem 27 *Suppose all parametric Presburger families have Property 3. Then all parametric Presburger families have Properties 3a and 3b.*

Theorem 27 is a weaker implication than Theorem 26, which holds for a single family S_t in isolation. To prove that 3 "implies" 3a and 3b, on the other hand, we will need to create new families S_t' using additional quantifiers or boolean operators, and we need to know that these new families still have Property 3.

Finally, we give evidence that these properties might actually hold. We can show that they all hold for two broad classes of parametric Presburger families:

Theorem 28 *Suppose S_t is a parametric Presburger family such that either*

(a) *S_t is defined without using any quantifiers,* or

(b) *S_t is defined using only inequalities of the form $\mathbf{a} \cdot \mathbf{x} \le b(t)$, where $b(t)$ is a polynomial (that is, the normal vector, $\mathbf{a}$, to the hyperplane must be fixed).*

Then Properties 1, 2, 3, 3a, 3b, and 4 all hold.

We isolate a piece of the proof of Part (b), in order to point out that Property 4 is a weaker property than we might hope for, but seems to be as strong a property as we can get. Indeed, we might hope that $\sum_{t\in\mathbb{N},\mathbf{s}\in S_t} \mathbf{x}^{\mathbf{s}} y^t$ is a rational generating function. Theorem 11 shows that this is true for sets defined in the normal Presburger arithmetic, and the following theorem shows that this implies Property 4.

Theorem 29 *Suppose $S_{\mathbf{p}}$, for $\mathbf{p} \in \mathbb{N}^n$, is a family of subsets of $\mathbb{N}^d$. If $\sum_{\mathbf{p}\in\mathbb{N}^n,\mathbf{s}\in S_{\mathbf{p}}} \mathbf{x}^{\mathbf{s}}\mathbf{y}^{\mathbf{p}}$ is a rational generating function, then there is a finite decomposition of $\mathbb{N}^n$ into pieces of the form $P \cap \mathbb{Z}^n$ (with P a polyhedron) such that, considering the $\mathbf{p}$ in each piece separately,*

$$\sum_{\mathbf{s}\in S_{\mathbf{p}}} \mathbf{x}^{\mathbf{s}} = \sum_i \epsilon_i \frac{\mathbf{x}^{\mathbf{q}_i(\mathbf{p})}}{(1-\mathbf{x}^{\mathbf{b}_{i1}})\cdots(1-\mathbf{x}^{\mathbf{b}_{ik_i}})},$$

where $\epsilon_i = \pm 1$, $\mathbf{b}_{ij} \in \mathbb{Z}^d$ are lexicographically positive, and the coordinate functions of $\mathbf{q}_i : \mathbb{N}^n \to \mathbb{Z}^d$ are degree 1 quasi-polynomials in $\mathbf{p}$.

In general, however, $\sum_{t\in\mathbb{N},\mathbf{s}\in S_t} \mathbf{x}^{\mathbf{s}} y^t$ will not be a rational generating function:

Example 30 *Let S_t be the set $\{(s_1, s_2) \in \mathbb{N}^2 : \ ts_1 = s_2\}$. Then*

$$\sum_{\mathbf{s}\in S_t} \mathbf{x}^{\mathbf{s}} = 1 + x_1 x_2^t + x_1^2 x_2^{2t} + \cdots = \frac{1}{1 - x_1 x_2^t}$$

is a rational generating function with exponents depending on t, so Property 4 is satisfied. Nevertheless,

$$\sum_{t\in\mathbb{N},\mathbf{s}\in S_t} \mathbf{x}^{\mathbf{s}} y^t = \frac{1}{1-x_1} + \frac{y}{1-x_1x_2} + \frac{y^2}{1-x_1x_2^2} + \cdots$$

cannot be written as a rational function.

To prove that it cannot be so written, note that the set $\{(s_1, s_2, t) : \ \mathbf{s} \in S_t\}$ cannot be written as a finite union of sets of the form $P \cap (\lambda + \Lambda)$, where P is a polyhedron, $\lambda \in \mathbb{Z}^3$ and $\Lambda \subseteq \mathbb{Z}^3$ is a lattice; Theorem 1 of Woods (2012) then implies that $\sum_{t\in\mathbb{N},\mathbf{s}\in S_t} \mathbf{x}^{\mathbf{s}} y^t$ is not a rational generating function.

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Gale-Robinson Sequences and Brane Tilings

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We study variants of Gale-Robinson sequences, as motivated by cluster algebras with principal coefficients. For such cases, we give combinatorial interpretations of cluster variables using brane tilings, as from the physics literature.

Résumé.

On étudie des variantes des suites de Gale-Robinson motivées par les algèbres amassées à coefficients principaux. Pour ces cas, on donne des interprétations combinatoires des variables d'amas en termes de pavages branes, interprétations qui ressemblent à celles qu'on trouve dans des articles de physique.

Keywords: cluster algebras, principal coefficients, F-polynomials, Aztec diamonds, Gale-Robinson recurrence, perfect matchings, brane tilings, Seiberg dualities

1 Introduction

This article is concerned with a variant of the Gale-Robinson integer sequence [Gal91], i.e. $\{x_n\}$ satisfying $x_n x_{n-N} = x_{n-r}x_{n-N+r} + x_{n-s}x_{n-N+s}$, where we include a second alphabet of variables, $\{y_1, y_2, \ldots, y_n\}$, that breaks the symmetry of this recurrence. This deformation is motivated by the theory of cluster algebras with *principal coefficients*.

The undeformed version of this sequence has been studied by several authors [BPW09, FZ02b, S07]. For example, Bousquet-Mélou, Propp, and West [BPW09] describe sequences of graphs, termed *pinecones*, such that the nth term in the associated Gale-Robinson sequence enumerates perfect matchings in the nth pinecone graph. Such pinecones can also be constructed by using Speyer's "crosses-and-wrenches" method [S07], which provides graph theoretical formulas for Laurent expansions of expressions satisfying the *Octahedron recurrence*. In particular, if one chooses the appropriate plane of initial conditions, then one can build graphs that are known by experts to be isomorphic (modulo elementary transformations) to the pinecones. We now further investigate pinecone graphs with the following goals in mind:

1) Develop a more natural way to obtain pinecone graphs from cluster algebra theory directly. This will take us on a detour through the physics literature of *brane tilings* which motivates further families

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of examples for future study. Though most of these details are omitted in this extended abstract, the interested reader may turn to [J], [JMZ], or [Z] for further details. We also turn the reader's attention to Eager's work [E11] which discusses these examples in terms of terminology from physics and geometry.

2) Explain how to generalize results of [BPW09] and [S07] to include principal coefficients. Our main result to this effect is the following Theorem.

Theorem 1 *Let $\widehat{\mathcal{A}}_{Q_N^{(r,s)}} \subset \mathbb{Q}[y_1, y_2, \ldots, y_N][x_1^{\pm}, x_2^{\pm}, \ldots, x_N^{\pm}]$ denote the cluster algebra with principal coefficients associated to the Gale-Robinson quiver of type (r, s, N). For $n \in \{N+1, N+2, \ldots\}$, define the cluster variables $\widehat{x_n}$ by mutating the initial seed $(\widehat{Q}_N^{(r,s)}, \{x_1, x_2, \ldots, x_N, y_1, y_2, \ldots, y_N\})$ periodically by the sequence $1, 2, 3, \ldots, N, 1, 2, \ldots$. Let $G_n^{(r,s,N)}$ be the graph as in Definition 16. Then for $n \geq N+1$, the Laurent expansion of $\widehat{x_n}$ is given by the combinatorial formula: $\widehat{x_n} = w(G_n^{(r,s,N)})$.*

The Gale-Robinson quivers are defined in Section 3, the graphs $G_n^{(r,s,N)}$ are defined in Section 5, and the weights appearing in the combinatorial formula appear in Section 6. We conjecture that formulas for a large class of examples from [S07] and the physics literature [E11, EF, DHP10, FHKVW, HS12] can be generalized similarly, but we leave their study for the future.

2 Preliminaries: Periodic Quivers and Cluster Mutation

In this section, we review the necessary background material on cluster mutation and periodic quivers from [FZ02a, FZ02b] and [FM11]. A *quiver* $Q = (Q_0, Q_1)$ is a directed finite graph with vertex set Q_0 and edge set Q_1 (also known as the set of arrows). We will usually assume that quivers have no 1-cycles nor 2-cycles, and state when this restriction is relaxed. Let $|Q_0| = N$.

Definition 2 (Quiver Mutation) *The mutation of Q at vertex k, denoted by $\mu_k Q$, is constructed (from Q) by the following three steps: (1) For every 2-path $i \to k \to j$ in Q, add an arrow $i \to j$. (2) Reverse the direction of all arrows incident to vertex k. (3) Remove any 2-cycles created by steps (1) and (2).*

To any quiver, we can associate a *cluster algebra* defined as follows. First, we associate a variable, which we denote as x_i, to each vertex of Q. This yields an *initial cluster*, $\{x_1, x_2, \ldots, x_N\}$, associated to Q. We then define a cluster mutation that proceeds alongside the aforementioned quiver mutation.

Remark 3 *Later on, we will discuss how to associate brane tilings, i.e. bipartite graphs on a torus, to the quivers we study. In this context, quiver mutation corresponds to* Urban Renewal *or* Seiberg Duality.

Definition 4 (Cluster Mutation) *Given a quiver Q and a cluster $X = \{X_1, X_2, \ldots, X_N\}$, the mutation of the* cluster seed *(Q, X) in the direction k is defined as $\mu_k(Q, X) = (\mu_k Q, X')$, where X' equals $X \setminus \{X_k\} \cup \{X'_k\}$ and X'_k is defined below. If there is an arrow from vertex i to vertex k in Q, we let b_{ik} denote the number of such arrows, and $b_{ki} = -b_{ik}$, yielding a skew-symmetric matrix B. We define X'_k as*

$$\frac{\prod_{i \to k \text{ in } Q} X_i^{b_{ik}} + \prod_{k \to i \text{ in } Q} X_i^{b_{ki}}}{X_k}.$$

A cluster seed $(Q, \{X_1, X_2, \ldots, X_N\})$ can be mutated in N directions. Then, these newly constructed seeds can then be again mutated in N directions, noting that $\mu_k^2 = id$. There will possibly be cycles in this mutation graph, but we generally get an infinite tree where each vertex has degree N.

Definition 5 (Cluster variables and algebras) *The set of* cluster variables *is the union of all clusters obtained via all finite sequences of mutations. The* cluster algebra $\mathcal{A}_Q$ *associated with the initial seed* $(Q, \{x_1, \ldots, x_N\})$ *is the subalgebra of* $\mathbb{Q}(x_1, \ldots, x_N)$*, the field of rational functions in* N *variables, generated by the set of cluster variables.*

Please see [FZ02a, GSV10] for more details about cluster algebras in general. We now introduce Fordy and Marsh's notion of periodic quivers [FM11]. For convenience, we draw such quivers by arranging the vertices on a regular N-gon in clockwise order. Let ρ denote $(1, N, N-1, N-2, \ldots, 3, 2)$, the permutation which rotates the vertices of the quiver Q clockwise while keeping the arrows fixed.

Definition 6 (Periodic Quiver) *We say that a quiver* Q *is* periodic*, of period* m*, if the mutated quiver* $Q^{(m)} = \mu_m \circ \cdots \circ \mu_2 \circ \mu_1(Q)$ *equals* $\rho^m(Q)$*. In other words, the quiver obtained by mutating by* $1, 2, \ldots, m$ *in sequence is equal to the quiver obtained by cyclically permuting the vertex labels of* Q.

In particular, a quiver Q is of *period* 1 if and only if mutating at vertex 1 and then applying ρ^{-1} (sending $2 \to 1, 3 \to 2, \ldots, N \to N-1, 1 \to N$) yields back the original quiver Q. The importance of period 1 quivers is that as long as we mutate at $1, 2, 3, \ldots$ in sequence and periodically, the quivers obtained by mutation are equivalent to one another, up to cyclic permutation.

Definition 7 (Primitive Period 1 Quiver) *Following [FM11], for* $1 \leq k \leq N/2$*, we define the* primitive period 1 quiver $+P_N^{(k)}$ *(resp.* $-P_N^{(k)}$*) as the* N *vertex quiver with* N *arrows (See Figure 1 for examples):*

- *For all* $1 \leq i \leq k$*, draw an arrow* $i + N - k \to i$ *(resp.* $i \to i + N - k$ *),*
- *For all* $1 \leq j \leq N - k$*, draw an arrow* $j + k \to j$ *(resp.* $j \to j + k$ *).*

We also let $\pm P^{(k)}_{\{i_1, i_2, \ldots, i_{N'}\}}$ *denote the quiver* $\pm P^{(k)}_{N'}$ *where the vertices are relabeled using* $i_1, \ldots, i_{N'}$.

For a periodic quiver Q and an initial cluster $\{x_1, x_2, \ldots, x_N\}$, we may define x_n, for all $n \geq 1$, by mutating periodically at $1, 2, 3, \ldots$. For example, we denote the new clusters $\mu_1(\{x_1, x_2, \ldots, x_N\}, Q)$ and $\mu_2 \circ \mu_1(\{x_1, x_2, \ldots, x_N\}, Q)$ as $\{x_{N+1}, x_2, \ldots, x_N\}$ and $\{x_{N+1}, x_{N+2}, \ldots, x_N\}$, respectively. More generally, for $n = mq + r$, we define x_n to be the rth element of the cluster obtained by the mutation $\mu_r \circ \mu_{r-1} \circ \cdots \circ \mu_1 \circ (\mu_m \circ \mu_{m-1} \circ \cdots \circ \mu_1)^q(\{x_1, x_2, \ldots, x_N\}, Q)$. We obtain a one-parameter infinite subsequence of cluster variables indexed by the positive integers. If Q is of period 1, then there is a single recurrence relation

$$x_n x_{n-N} = \prod_{i \to 1 \text{ in } Q} x_{n-i}^{b_{i1}} + \prod_{1 \to i \text{ in } Q} x_{n-i}^{b_{1i}}$$

satisfied by all $n \geq N + 1$. For higher periods, there are m interlaced recurrence relations instead.

3 Gale-Robinson Sequences

Using the constructions of the previous section, we now focus on a certain two-parameter family of period 1 quivers. These quivers correspond to the Gale-Robinson sequence [Gal91] and were studied, implicitly, in work by Bousquet-Mélou, Propp, and West [BPW09]. The Somos 4 and Somos 5 sequences (due to M. Somos as described in [Gal91]) appear as special cases. Any Gale-Robinson sequence can also be shown to be a specialization of the Octahedron Recurrence [S07]. See Remark 10 for details.

Definition 8 (Gale Robinson Sequences) *For* $1 \leq r < s \leq N/2$*, the* Gale Robinson sequence *of type* (r, s, N) *is defined to be the sequence* $\{x_n : n \geq 1\}$ *satisfying the recurrence relation (for* $n \geq N + 1$*):*

$$x_n x_{n-N} = x_{n-r} x_{n-N+r} + x_{n-s} x_{n-N+s}. \tag{1}$$

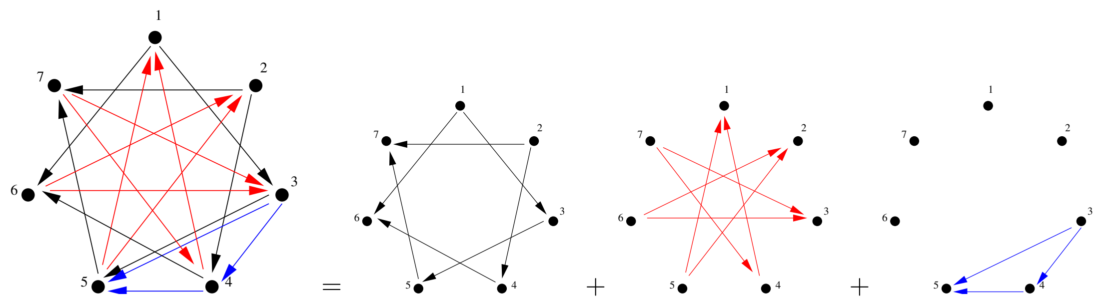


Fig. 1: The Gale-Robinson Quiver $Q_7^{(2,3)}$ as a sum of the primitive period 1 quivers $-P_7^{(2)}$, $+P_7^{(3)}$, and $-P_{\{4,5,6\}}^{(1)}$.

As explained in Example 8.7 of [FM11], for each triple of positive integers (r, s, N) with $r < s \leq N/2$, there is a unique period 1 quiver whose mutations yield the sequence of x_n's satisfying recurrence (1).

Definition 9 (The Gale-Robinson Quiver) *For $1 \leq r \leq s < N/2$, we let $Q_N^{(r,s)}$ denote the quiver constructed by the following four step process, starting with the edge-less quiver on N vertices:*

1. *For all $1 \leq i \leq N - r$, draw an arrow $i \to i + r$, and for all $1 \leq j \leq r$, draw an arrow $j \to N - r + j$, i.e. adjoin the primitive period 1 quiver $-P_N^{(r)}$.*
2. *For all $1 \leq i \leq N - s$, draw an arrow $s + i \to i$, and for all $1 \leq j \leq s$, draw an arrow $N - s + j \to j$, i.e. adjoin the primitive period 1 quiver $+P_N^{(s)}$ (resp. $+2P_N^{(N/2)}$ if $s = N/2$).*
3. *For all $1 \leq i \leq N - r - s$, draw an arrow from $r + i \to s + i$ and for all $1 \leq j \leq s - r$, draw an arrow $r + j \to N - s + j$, i.e. adjoin $-P_{r+1,r+2,\ldots,N-r}^{(s-r)}$ (resp. $-2P_{r+1,r+2,\ldots,N-r}^{(N/2-r)}$ if $s = N/2$).*
4. *Erase any 2-cycles created in $Q_N^{(r,s)}$.*

Note that there might be multiple arrows between vertices i and j. See Figure 1 for the example of $Q_7^{(2,3)}$.

In [BPW09], the authors provide a combinatorial interpretation for the Gale-Robinson sequence, given by $\{x_n : n \geq 1\}$, with the initial conditions $x_1 = x_2 = \cdots = x_N = 1$. In particular, each x_n is an integer, which is a non-trivial fact since the recurrence relation (1) involves division. This was proven directly in [Gal91], and also follows from Fomin and Zelevinsky's Laurent Phenomenon [FZ02b], which states that every cluster variable is a *Laurent* polynomial in terms of the initial cluster.

More specifically, in [BPW09], they introduce a family of graphs, known as *pinecones*. For each quadruple of positive integers (n, r, s, N) such that $r < s \leq N/2$ and $n > N$, they define the pinecone $P(n; r, N - r, s, N - s)$ so that the specialized cluster variable $x_n(x_1 = x_2 = \cdots = x_N = 1)$ counts the number of perfect matchings in $P(n; r, N - r, s, N - s)$. In the next section, we provide an alternate construction of pinecones that is motivated by recent literature on supersymmetric quiver gauge theories.

Remark 10 *While it has not been written down explicitly in print, the pinecone graphs constructed in [BPW09] are equivalent to the subgraphs obtained in [S07] by David Speyer using his method of "crosses* *and wrenches". More generally, for any sequence of cluster variables $\{f(i,j,k)\}$ coming from a specialization of the* Octahedron Recurrence*:*

$$f(i,j,k)f(i-2,j,k) = f(i-1,j-1,k)f(i-1,j+1,k) - f(i-1,j,k-1)f(i-1,j,k+1),$$

Speyer's method constructs families of graphs $\{\mathcal{G}_{i,j,k}\}$ and a weighting $w(M)$ for the perfect matchings of $\mathcal{G}_{i,j,k}$ such that the Laurent polynomial (equiv. cluster variable) $f(i,j,k)$ equals the generating function $\sum_{M \text{ a perfect matching of } \mathcal{G}_{a,b,c}} w(M)$. A variant of this weighting was also presented in [GK], and we will elaborate on and utilize this in Section 6. As explained in [S07, Section 1.3]), by choosing an appropriate plane of initial conditions, namely, (i,j,k) such that $-N < \frac{Ni+(2r-N)j+(2s-N)k}{2} \leq 0$, a subset of the $f(i,j,k)$'s satisfy the Gale-Robinson recurrence relation of type (r,s,N).

4 From Gale-Robinson Quivers to Brane Tilings

We now describe how to use techniques from Supersymmetric Quiver Gauge Theories [FHH01, FHKVW, FHMSVW] to obtain the pinecones more directly. By letting $r = a$ and $s = c$, the Gale-Robinson sequence $\{x_n\}$ defined above agrees with the $\{Z_n\}$'s appearing in [EF, Section 9.1]. In the quiver gauge theory and brane tiling literature, Z_n denotes a *Pyramid Partition Function* (cluster variable) associated to a certain *cascade of Seiberg dualities* (mutation sequence). The example highlighted in Section 9.1 of [EF] is inspired by a $L^{a,b,c}$-geometry which comes from a toric Calabi-Yau 3-manifold. See [FHMSVW] for more on the construction of the $L^{a,b,c}$-geometry and how to obtain a corresponding brane tiling. Further details also appear in [E11], which describes connections to [S07], as in Remark 10, in this language.

A *brane tiling* is a tiling of the torus, which we visualize as a doubly-periodic tiling of its universal cover, the infinite plane. We now summarize how to go from a Gale-Robinson quiver, $Q_N^{(r,s)}$, to an associated brane tiling, denoted as $\mathcal{T}_N^{(r,s)}$. Towards this end, we must now allow quivers with 2-cycles. Let $\overline{Q_N^{(r,s)}}$ denote the quiver obtained by following steps (1)-(3) of Definition 9. By abuse of notation, we will also refer to $\overline{Q_N^{(r,s)}}$ as a Gale-Robinson quiver, since 2-cycles do not affect the associated recurrence.

1. Firstly, since $\overline{Q_N^{(r,s)}}$ is highly symmetric, we can unfold it onto the plane, obtaining an infinite quiver $\widetilde{Q}_N^{(r,s)}$ that is straightforward to describe:

   a) Start with the $\mathbb{Z}^2$ lattice as an undirected graph, connecting (a,b) with $(a \pm 1, b)$ and $(a, b \pm 1)$.

   b) Label the vertex at the origin $(0,0)$ as 1. For all integer points (A,B), we label the corresponding vertex as $(1 + Ar + Bs)(\text{mod } N) \in \{1, 2, \dots, N\}$.

   c) We now turn this lattice into a directed graph. For all horizontal edges, we orient $i \to j$ if and only if $i < j$. For all vertical edges, we do the opposite (orient $i \to j$ if and only if $i > j$).

   d) Lastly, we add diagonal arrows as needed so that all triangles or squares in this planar directed graph are cyclically oriented. Proposition 11 ensures that this process is well-defined.

2. Secondly, we take the planar dual of $\widetilde{Q}_N^{(r,s)}$, and label its faces using the labels of vertices of $\widetilde{Q}_N^{(r,s)}$.

The resulting doubly-periodic tiling of the plane is the *brane tiling* $\mathcal{T}_N^{(r,s)}$. See Figure 3 for an example.

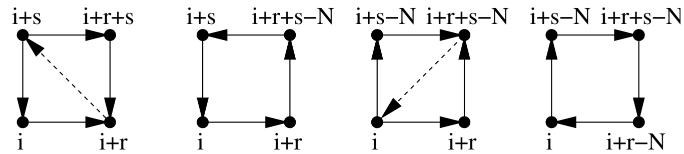


Fig. 2: The four possible local configurations.

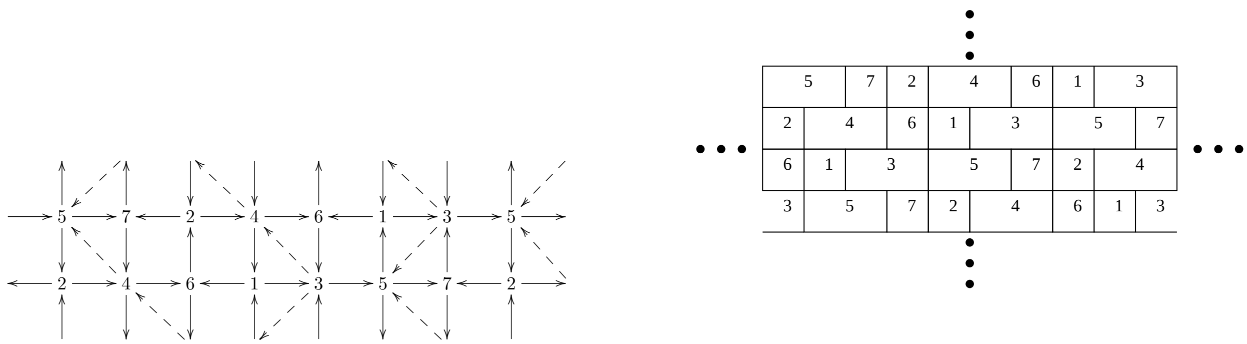


Fig. 3: The unfolded quiver $\widetilde{Q}_7^{(2,3)}$ and brane tiling $\mathcal{T}_7^{(2,3)}$.

Proposition 11 *Consider a square S with vertices corresponding to $i, i{+}s, i{+}r{+}s, i{+}r \in \{1, 2, 3, \ldots, N\}$, taken modulo N and in clockwise order starting from the lower-left. Orient the four edges of the square using the convention of (1c). Then, as in Figure 2, either the edges of S form an oriented 4-cycle, or can be split into to two cyclically oriented triangles by adding a single oriented diagonal.*

Remark 12 *These four local configurations also appear in the* square-ice *or* six-vertex *models.*

Proposition 13 *Construct $\widetilde{Q}_N^{(r,s)}$ as above and then identify vertices with the same labels. The resulting folded-up quiver exactly agrees with the Gale-Robinson quiver $\overline{Q_N^{(r,s)}}$ (possibly with 2-cycles).*

Remark 14 *If we attempted to unfold the 2-cycle-less $Q_N^{(r,s)}$ instead of unfolding $\overline{Q_N^{(r,s)}}$, we would be missing some of the diagonal edges which are relevant for obtaining a regular pattern of hexagons.*

Corollary 15 *For $1 \leq i \leq r$, and $N - r \leq i \leq N$, the faces labeled with an i are squares. On the other hand, the faces labeled with an i for $r + 1 \leq i \leq N - r - 1$ are hexagons.*

Note: When drawing brane tilings or their subgraphs, we will depict hexagonal faces as horizontal rectangles of height one and width two.

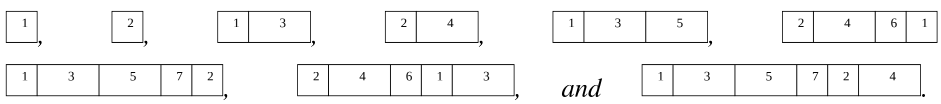
5 From Brane Tilings to Pinecones

We now describe how to obtain the pinecone graphs, $P(n; r, N - r, s, N - s)$, constructed in [BPW09], directly from brane tilings. Given a Gale-Robinson sequence and quiver $\overline{Q_N^{(r,s)}}$, we described in the last section how to construct the associated brane tiling $\mathcal{T}_N^{(r,s)}$. We now describe how to construct a family of finite subgraphs of $\mathcal{T}_N^{(r,s)}$, each of which we denote as $G_n^{(r,s,N)}$ for $n \geq N+1$.

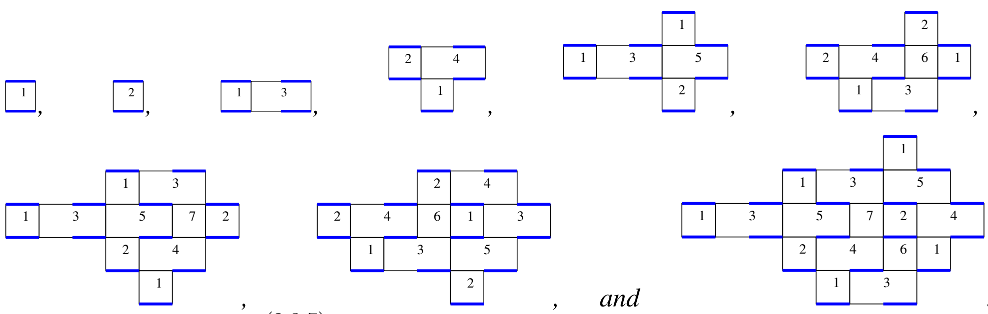
Definition 16 (Gale-Robinson Brane Subgraphs) *For $N+1 \leq n \leq N+r$, we define $G_n^{(r,s,N)}$ as the subgraph of $\mathcal{T}_N^{(r,s)}$ consisting of the square face labeled $n-N$. If $n > N+r$, we instead build $G_n^{(r,s,N)}$ layer-by-layer. For this construction, we need some notation. For $n > N+r$, let $\overline{n} \in \{1,2,\ldots,r\}$ denote the integer such that $n \equiv N + \overline{n} \pmod r$. Define the horizontal strip $H_n^{(r,N)}$ to be the induced subgraph of $\mathcal{T}_N^{(r,s)}$ obtained by taking the grid graph of unit height and width equal to $2\lfloor \frac{n-N-1}{r} \rfloor + 1$ starting with the square face labeled as $\overline{n}$ as the left-most face. In particular, $H_n^{(r,N)}$ is defined to be empty if $n \leq N$.*

For $n > N+r$, we then construct a graph by using $H_n^{(r,N)}$ as a central horizontal strip, and then gluing to its top (resp. bottom) the strips $H_{n-(N-s)}^{(r,N)}$, $H_{n-2(N-s)}^{(r,N)}$, $\ldots$ (resp. $H_{n-s}^{(r,N)}$, $H_{n-2s}^{(r,N)}$, $\ldots$) until the strips added above and below are empty. We glue these together in the unique way so that successive strips, emanating out from the center, are contained in the interior of the more central strip. This defines an induced subgraph of $\mathcal{T}_N^{(r,s)}$, that we denote as $G_n^{(r,s,N)}$.

Example 17 *Consider the case $r=2$, $s=3$, and $N=7$. The corresponding quiver $Q_7^{(2,3)}$ appears in Figure 1 and its brane tiling $\mathcal{T}_7^{(2,3)}$ appears in Figure 3. Then for $8 \leq n \leq 16$, the strips $H_n^{(2,7)}$ are:*



Gluing these strips together, we obtain the Gale-Robinson brane subgraphs $\{G_n^{(2,3,7)}\}$ for $8 \leq n \leq 16$:



For example, the graph $G_{16}^{(2,3,7)}$ is obtained by gluing together the horizontal strips (from top to bottom) $H_8^{(2,7)}$, $H_{12}^{(2,7)}$, $H_{16}^{(2,7)}$, $H_{13}^{(2,7)}$, and $H_{10}^{(2,7)}$. (The highlighted edges are minimal matchings which are discussed further in Definition 23.)

Remark 18 *As will be described in [JMZ], the graphs $G_n^{(r,s,N)}$ can also be constructed by superimposing Aztec Diamonds of increasing sizes centered on top of a face (of the center row) of the brane tiling $\mathcal{T}_N^{(r,s)}$. In particular, the first r graphs are squares labeled with $1 \leq i \leq r$. Subsequently, we have r subsequences of Aztec Diamonds. In particular, for $N' \geq 0$, the graph $G_{rN'+i}^{(r,s,N)}$ can be constructed by the following:*

(i) locate a face of $\mathcal{T}_N^{(r,s)}$ labeled as i. If it is a square, let (a,b) denote this face, as viewed in the $\mathbb{Z}^2$ lattice. If instead it is a hexagon, let (a,b) denote the left-hand-side of this face.

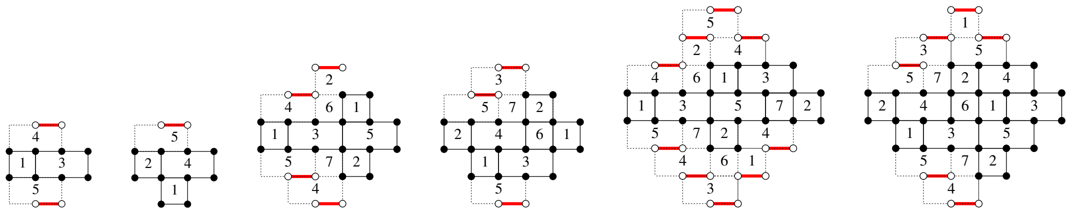


Fig. 4: Recovering the Gale-Robinson subgraph for $(r, s, N) = (2, 3, 7)$ from the associated Aztec Diamond for $10 \leq n \leq 15$. Called the *core* of a pinecone in [BPW09]. Compare with Example 17 and $\mathcal{T}_7^{(2,3)}$ of Figure 3.

(ii) Take the Aztec Diamond of size $(N' + 1)$ (which has a central row of size $2N' + 1$) and center it on top of the cell $(a + N', b)$.

(iii) This superposition will usually result in a graph containing vertices of degree one. By removing these, one-by-one, we obtain the desired subgraph $G_n^{(r,s,N)}$.

See Figure 4 for an example. Note that this procedure is equivalent to taking the core *of a pinecone, as described in [BPW09, Section 2.4].*

Proposition 19 *For each choice of integers $1 \leq r < s \leq N/2$ and $n \geq N + 1$, the graphs $G_n^{(r,s,N)}$ and pinecones $P(n; r, N - r, s, N - s)$ from [BPW09] are equal (up to a vertical reflection).*

Remark 20 *A method for constructing subgraphs of brane tilings also appears in the string theory literature. For instance, in [EF, Sections 6, 7.3], they discuss a construction for the "shadow of a pyramid".*

6 Principal Coefficients and Combinatorial Formulas

We now generalize Theorem 9 of [BPW09] by enriching the cluster algebra $\mathcal{A}_{Q_N^{(r,s)}}$ with *principal coefficients*. More generally, a coefficient system for a cluster algebra can be constructed by enlarging the set of initial cluster variables by including so called *frozen variables*. These variables correspond to new vertices at which mutation is disallowed. A system of *principal coefficients* is a special case where the arrows incident to the new vertices are particularly simple. By Theorem 3.7 of [FZ07], it follows that any coefficient system of geometric type can be algebraically deduced from a system of principal coefficients.

Definition 21 (Quiver with Principal Coefficients) *Given a quiver Q with N vertices, we let $\widehat{Q}$ denote the quiver on $2N$ vertices that (i) contains Q as an induced subgraph on the vertices $\{1, 2, \ldots, N\}$, and (ii) contains a single arrow $v \to v - N$ for each vertex $v \in \{N + 1, N + 2, \ldots, 2N\}$.*

We then let $\widehat{\mathcal{A}_Q}$ denote the cluster algebra $\mathcal{A}_{\widehat{Q}}$, which we refer to as the *cluster algebra for Q with principal coefficients*. Just as in Section 2, we obtain an infinite sequence of cluster variables by mutating the enlarged quiver $\widehat{Q_N^{(r,s)}}$ periodically by $1, 2, \ldots$. We let $\{x_1, x_2, \ldots, x_N, y_1, y_2, \ldots, y_N\}$ denote the corresponding initial cluster, and denote the next two clusters as $\{\widehat{x_{N+1}}, x_2, \ldots, x_N, y_1, y_2, \ldots, y_N\}$ and $\{\widehat{x_{N+1}}, \widehat{x_{N+2}}, \ldots, x_N, y_1, y_2, \ldots, y_N\}$. Continuing in this way, we let $\{\widehat{x_n} : n \geq N + 1\}$ denote the infinite sequence of non-initial cluster variables obtained by this periodic mutation sequence. Since we never mutate at vertex v for $v \in \{N + 1, N + 2, \ldots, 2N\}$, it follows that all of the $\widehat{x_n}$'s are Laurent polynomials whose denominators are free of y_i's. We now discuss how to generalize the numerical result of [BPW09] to obtain a combinatorial interpretation of the $\widehat{x_n}$'s.

Given a graph G, a set of edges M which covers all vertices in G exactly once is called a (perfect) *matching* of G. We say that G is a *weighted graph* if there is a real number or a formal variable $w(e)$ associated to each edge e. When G is a subgraph of a brane tiling, we now define a weighting scheme inspired by the *Conductance Coordinates*[(i)] appearing in [GK, Section 5.3] and Speyer's weighting [S07].

Definition 22 (Weight of a perfect matching) *Given a subgraph G of a brane tiling (with face labels F_i), we define the weight $x(e)$ of an edge e (straddling faces F_i and F_j) to be $x(e) = \frac{1}{x_i x_j}$. Given a perfect matching M of G, we define $x(M) = \prod_{e \in M} x(e)$.*

We additionally utilize height functions, as appearing in the literature [CKP01, CY, MSW11, P, Th90].

Definition 23 (Height of a perfect matching) *For a pinecone graph $G = G_n^{(r,s,N)}$ of the brane tiling $\mathcal{T}_N^{(r,s)}$, let M_- denote the unique perfect matching of G using only horizontal edges. See for instance the highlighted edges in Example 17. Given another perfect matching M, we let $M \oplus M_-$ denote the superposition of these two perfect matchings. We then define the height, $y(M)$, as the monomial*

$$y(M) = \prod_{i=1}^{N} \prod_{\text{Face } F \text{ of graph } G \text{ labeled as } i} y_i^{\#\text{ cycles of } M \oplus M_- \text{ enclosing the face } F}.$$

We also have to define a certain monomial that is given by the labels of the faces appearing in G and its boundary in the ambient tiling $\mathcal{T}$.

Definition 24 (Covering monomial of a subgraph) *Given a subgraph G of a brane tiling $\mathcal{T}$ (with face labels F_i), let $\overline{G}$ denote the subgraph of $\mathcal{T}$ consisting of all faces that are incident to an edge appearing in G. In particular, $\overline{G}$ contains G as a proper subgraph, as well as a "ring" of exterior faces. (These are referred to as "open faces" in [S07].) Recall that by definition, each face of $\mathcal{T}$ is a $2k$-gon where $k \geq 2$. Then for any face F of G, with label i, we define $m(F) = x_i^{\frac{\#\text{ edges in } F}{2} - 1}$. For any of the open faces $F \in \overline{G} \setminus G$, with label i, we define $m(F) = x_i^{\left\lceil \frac{\#\text{ edges in } F \text{ incident to } G}{2} \right\rceil}$. Then the* covering monomial *of G is defined to be $cm(G) = \prod_{F \in \overline{G}} m(F)$.*

Remark 25 *A more general definition of covering monomials appears in [J].*

Given the above definition, the *weight* of a graph G is defined as $w(G) = cm(G) \cdot \sum_M x(M) y(M)$, where the sum is taken over all perfect matchings M of G.

We now give a sketch of our main result, Theorem 1, stated in Section 1. See [JMZ] for the full proof.

Proof:

Step 1: We show that as we mutate $Q_N^{(r,s)}$ periodically at $1, 2, 3, \ldots, N, 1, 2, \ldots$, we get a Gale-Robinson recurrence relation (with coefficients) of the following form:

$$x_n x_{n-N} = x_{n-r} x_{n-N+r} \;+\; \prod_{i=1}^{N} y_i^{d(n-N-i, r, N-r)} \; x_{n-s} x_{n-N+s}$$

where $d(n-N-i, r, N-r)$ denotes $\#\{(A, B) \in \mathbb{Z}_{\geq 0}^2$ such that $(n-N-i) = A \cdot r + B \cdot (N-r)\}$.

(i) Unlike Goncharov and Kenyon, we have weights only on faces, i.e. trivial weights on vertices.

Step 2: For $i \in \{1, 2, \ldots, N\}$ and integer n such that $n > N$, we show that $d(n - N - i, r, N - r)$ equals the number of faces labeled i in the central strip $H_n^{(r,N)}$ of $G_n^{(r,s,N)}$.

Step 3: Finally, we apply Kuo's technique of graphical condensation [K04]. A superposition of a perfect matching of $G_{n-N}^{(r,s,N)}$ centered on top of a perfect matching $G_n^{(r,s,N)}$ can be decomposed uniquely (up to cycles) as exactly one of the following: (i) Into an east-west superposition of perfect matchings of $G_{n-r}^{(r,s,N)}$ and $G_{n-N+r}^{(r,s,N)}$; or (ii) a north-south superposition of perfect matchings of $G_{n-s}^{(r,s,N)}$ and $G_{n-N+s}^{(r,s,N)}$. This method is also detailed for this case in [BPW09] and follows from Speyer's more general proof in [S07]. In particular, this decomposition is weight-preserving with respect to edge-weights and covering monomials.

For this part of the proof, what is new relative to [BPW09] and [S07] is that an east-west superposition of minimal matchings again decomposes into a decomposition of minimal matchings of $G_{n-N}^{(r,s,N)}$ and $G_n^{(r,s,N)}$. However a north-south superposition of minimal matchings does not. Instead, such a superposition decomposes into a minimal matching of $G_{n-N}^{(r,s,N)}$ and a perfect matching of $G_n^{(r,s,N)}$ where every face in the central strip $H_n^{(r,N)}$ has been *twisted down* [P] exactly once. In the twisting down operation, one perfect matching of the square or rectangle is exchanged for the other. This is also referred to as a *plaquette flip* in [CY] and elsewhere. The proof of the Theorem then follows from Steps 1 and 2. $\square$

Remark 26 *Related formulas also appeared in [EF, Appendix B] where all the x_i's are set to be one so that F-polynomials [FZ07] are recovered. In the physics literature, these are referred to as* pyramid partition functions*. F-polynomials for the related case of Aztec Diamonds also appear in [G11].*

7 Further Topics

The authors have already started investigating other families of brane tilings and their connections to cluster algebras. See [J] and [Z] for the related REU reports. Further details will appear in [JMZ].

In particular, In-Jee Jeong investigated the cluster algebras associated to the four-vertex quiver where the vertices are arranged clockwise around a square, and there are two clockwise arrows between any pair of adjacent vertices. If one mutates this quiver periodically, one obtains cluster variables whose Laurent polynomials can be expressed in terms of $w(AD_n)$ where AD_n is the nth Aztec Diamond, with a certain face label. However, for certain non-periodic mutation sequences, Jeong also obtained graph theoretical interpretations for the cluster variables here too. Jeong also initiated an investigation for a more general framework that was part of the motivation for further study of the brane tiling literature.

Sicong Zhang studied a certain six vertex quiver, known as the dP3 (del-Pezzo 3 quiver) in the physics literature. Certain subgraphs of the associated brane tiling were previously studied by C. Cottrell-B.Young [CY] and M. Ciucu [C03] after being introduced by J. Propp [P99] under the name Aztec Dragons. Zhang proved that like above, a certain infinite sequence of cluster variables associated to this quiver, obtained by periodic mutation, has the property that their Laurent polynomial expansions can be expressed, under a suitable weighting scheme, in terms of perfect matchings of these subgraphs.

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A generalization of Mehta-Wang determinant and Askey-Wilson polynomials

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Abstract. Motivated by the Gaussian symplectic ensemble, Mehta and Wang evaluated the $n\times n$ determinant $\det((a+j-i)\Gamma(b+j+i))$ in 2000. When $a=0$, Ciucu and Krattenthaler computed the associated Pfaffian $\mathrm{Pf}((j-i)\Gamma(b+j+i))$ with an application to the two dimensional dimer system in 2011. Recently we have generalized the latter Pfaffian formula with a q-analogue by replacing the Gamma function by the moment sequence of the little q-Jacobi polynomials. On the other hand, Nishizawa has found a q-analogue of the Mehta–Wang formula. Our purpose is to generalize both the Mehta-Wang and Nishizawa formulae by using the moment sequence of the little q-Jacobi polynomials. It turns out that the corresponding determinant can be evaluated explicitly in terms of the Askey-Wilson polynomials.

Résumé. Motivés par des travaux sur ensemble Gaussien symplectique, Mehta et Wang sont amenés à calculer le déterminant $\det((a+j-i)\Gamma(b+j+i))$ de taille $n\times n$ en 2000. Lorsque $a=0$, Ciucu et Krattenthaler ont calculé le Pfaffien $\mathrm{Pf}((j-i)\Gamma(b+j+i))$ avec une application au système à deux dimènsions dimeres en 2011. Récemment nous avons généralisé le dernier Pfaffien avec un q-analogue en remplacant la fonction Gamma par les moments de petits q-polynômes de Jacobi. Par ailleurs, Nishizawa a trouvé un q-analogue de la formule de Mehta-Wang. Dans cet article nous démontrons une formule qui généralise à la fois la formule de Mehta-Wang et celle de Nishizawa en utilisant les moments de petits q-polynômes de Jacobi. Il en resulte que le determinant correspondant peut s'écrire de facon explicite à l'aide des polynômes de Askey-Wilson.

Keywords: The Mehta-Wang determinants, the moments of the little q-Jacobi polynomials, the Askey-Wilson polynomials.

1 Introduction

Motivated by the Gaussian symplectic ensemble, [13] obtain the determinant identity

$$\det\big((a+j-i)\Gamma(b+i+j)\big)_{0\le i,j\le n-1} = D_n \prod_{i=0}^{n-1} i!\Gamma(b+i), \tag{1.1}$$

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where (N.B. the binomial coefficient $\binom{n}{k}$ is missing in [13, (7)])

$$D_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(\frac{b-a}{2}\right)_k \left(\frac{a+b}{2}\right)_{n-k}, \tag{1.2}$$

where $(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$ is known as the *rising factorial*. This D_n satisfies the three term recurrence relation

$$D_{-1} = 0, \quad D_0 = 1, \quad D_{n+1} = aD_n + n(b+n-1)D_{n-1}, \tag{1.3}$$

which can be considered as the recurrence relation for a special case of the Meixner-Pollaczek polynomials (see [13, 14]), and one may notice that the sequence $\{\Gamma(b+n)\}_{n\geq 0}$ of the Gamma functions in the left-hand side can be considered as the moment sequence of the Laguerre polynomials (see, for example, [7, 8, 15]). [14] obtains a q-analogue of (1.1), which will be stated below. In this article we replace the Gamma functions by the moments of the little q-Jacobi polynomials and show that we obtain a special case of the Askey-Wilson polynomials as D_n, which also generalize the two results in our previous papers [4, 5]. Before we describe our results we need more notation.

Throughout this paper we use the standard notation for q-series (see [3, 7, 8]):

$$(a;q)_\infty = \prod_{k=0}^{\infty} (1-aq^k), \qquad (a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}$$

for any integer n. Usually $(a;q)_n$ is called the q-*shifted factorial*, and we frequently use the compact notation:

$$(a_1, a_2, \ldots, a_r; q)_n = (a_1;q)_n (a_2;q)_n \cdots (a_r;q)_n.$$

The ${}_{r+1}\phi_r$ *basic hypergeometric series* is defined by

$${}_{r+1}\phi_r \left(\begin{matrix} a_1, a_2, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{matrix} ; q, z \right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q)_n}{(q, b_1, \ldots, b_r; q)_n} z^n. \tag{1.4}$$

Here we also use the q-Gamma function

$$\Gamma_q(z) = (1-q)^{1-z} \frac{(q;q)_\infty}{(q^z;q)_\infty},$$

the q-integer $[n]_q = \frac{1-q^n}{1-q}$ and the q-factorial $[n]_q! = \prod_{k=1}^{n} [k]_q$. The *Askey-Wilson polynomials* $p_n(x)$ (see [3, 7, 8]) satisfy the well-known recurrence relation

$$2xp_n(x) = A_n p_{n+1}(x) + B_n p_n(x) + C_n p_{n-1}(x), \quad n \geq 0, \tag{1.5}$$

with $p_{-1}(x) = 0$, $p_0(x) = 1$, where

$$\begin{aligned} A_n &= \frac{1-abcdq^{n-1}}{(1-abcdq^{2n-1})(1-abcdq^{2n})}, \\ C_n &= \frac{(1-q^n)\left(1-abq^{n-1}\right)\left(1-acq^{n-1}\right)\left(1-adq^{n-1}\right)}{(1-abcdq^{2n-2})(1-abcdq^{2n-1})} \left(1-bcq^{n-1}\right)\left(1-bdq^{n-1}\right)\left(1-cdq^{n-1}\right), \end{aligned}$$

and

$$\begin{aligned} B_n = a + a^{-1} &- A_n a^{-1} (1-abq^n)(1-acq^n)(1-adq^n) \\ &- C_n a / \left(1-abq^{n-1}\right)\left(1-acq^{n-1}\right)\left(1-adq^{n-1}\right). \end{aligned}$$

They have the basic hypergeometric expression

$$p_n(x;a,b,c,d;q)=\frac{(ab,ac,ad;q)_n}{a^n}\,{}_4\phi_3\left(\begin{matrix}q^{-n},abcdq^{n-1},ae^{\imath\theta},ae^{-\imath\theta}\\ ab,ac,ad\end{matrix};q,q\right) \tag{1.6}$$

with $x=\cos\theta$, where $\imath=\sqrt{-1}$. We also use the symbol

$$\chi(A)=\begin{cases}1 & \text{if } A \text{ is true,}\\ 0 & \text{if } A \text{ is false.}\end{cases}$$

In [4] we have proven the Hankel determinant identity

$$\det\left(\frac{(aq;q)_{i+j+r-2}}{(abq^2;q)_{i+j+r-2}}\right)_{1\le i,j\le n}=a^{\frac{n(n-1)}{2}}q^{\frac{n(n-1)(2n-1)}{6}+\frac{n(n-1)r}{2}}\prod_{k=1}^{n}\frac{(q,bq;q)_{k-1}(aq;q)_{k+r-1}}{(abq^2;q)_{k+n+r-2}} \tag{1.7}$$

for a positive integer n. Here

$$\mu_n=\frac{(aq;q)_n}{(abq^2;q)_n}\quad(n=0,1,2,\dots)$$

is the moments of the little q-Jacobi polynomials. In our previous paper [5], we have exploited the Pfaffian identity

$$\begin{aligned}&\operatorname{Pf}\left((q^{i-1}-q^{j-1})\frac{(aq;q)_{i+j+r-2}}{(abq^2;q)_{i+j+r-2}}\right)_{1\le i,j\le 2n}\\&=a^{n(n-1)}q^{\frac{n(n-1)(4n+1)}{3}+n(n-1)r}\prod_{k=1}^{n-1}(bq;q)_{2k}\prod_{k=1}^{n}\frac{(q;q)_{2k-1}(aq;q)_{2k+r-1}}{(abq^2;q)_{2(k+n)+r-3}}\end{aligned} \tag{1.8}$$

for a positive integer n (see also [11, 12]).

In [14], Nishizawa has proven the q-analogue of the Mehta-Wang result:

$$\begin{aligned}&\det\big([a+j-i]_q\Gamma_q(b+i+j)\big)_{0\le i,j\le n-1}\\&=q^{na+n(n-1)b/2+n(n-1)(2n-7)/6}\,D_{n,q}\prod_{k=0}^{n-1}[k]_q!\cdot\Gamma_q(b+k),\end{aligned} \tag{1.9}$$

where $D_{n,q}$ satisfies the recurrence relation

$$D_{-1,q}=0,\quad D_{0,q}=1,\quad D_{n+1,q}=q^{-a+n}[a]_qD_{n,q}+q^{-a-b}[n]_q[b+n-1]_qD_{n-1,q}. \tag{1.10}$$

Comparing this recurrence relation with the recurrence equation

$$2xQ_n(x)=Q_{n+1}(x)+(A+B)q^nQ_n(x)+(1-q^n)(1-ABq^{n-1})Q_{n-1}(x) \tag{1.11}$$

of the Al-Salam–Chihara polynomials

$$Q_n(x)=Q_n(x;A,B;q)=\frac{(AB;q)_n}{A^n}{}_3\phi_2\left(\begin{matrix}q^{-n},Ae^{\imath\theta},Ae^{-\imath\theta}\\ AB,0\end{matrix};q;q\right) \tag{1.12}$$

with $x=\cos\theta$ (see [7, 8]), we may remark that $D_{n,q}$ can be considered as a special case of the Al-Salam–Chihara polynomials because

$$D_{n,q}=(-\imath)^nq^{-\frac{a+b}{2}n}(1-q)^{-n}Q_n\left(0;q^{\frac{a+b}{2}}\imath,-q^{\frac{b-a}{2}}\imath;q\right). \tag{1.13}$$

By this observation, we can write $D_{n,q}$ explicitly as

$$D_{n,q} = \frac{(q^b;q)_n}{q^{n(a+b)}(q-1)^n} \sum_{k=0}^{n} q^k \frac{(q^{-n};q)_k}{(q;q)_k} \prod_{j=0}^{k-1} \frac{1-q^{a+b+2j}}{1-q^{b+j}}. \tag{1.14}$$

One natural question we may ask is what can we obtain if we replace the q-Gamma function in the determinant of (1.9) by the moment of the little q-Jacobi polynomials. The aim of this paper is to answer this question, and we can express the determinant by the Askey-Wilson polynomials.

Theorem 1.1 *Let a, b and c be parameters, and let $n \geq 1$ and r be integers. Then we have*

$$\det\left((q^{i-1}-cq^{j-1})\frac{(aq;q)_{i+j+r-2}}{(abq^2;q)_{i+j+r-2}}\right)_{1\leq i,j\leq n}$$

$$= (-1)^n a^{\frac{n(n-3)}{2}} q^{\frac{n(n+1)(2n-5)}{6}+\frac{n(n-3)r}{2}} (abcq^{r+1};q^2)_n \prod_{k=1}^{n} \frac{(q;q)_{k-1}(aq;q)_{k+r}(bq;q)_{k-2}}{(abq^2;q)_{k+n+r-2}}$$

$$\times {}_4\phi_3\left(\begin{matrix} q^{-n}, a^{\frac12}c^{\frac12}q^{\frac{r+1}{2}}, -a^{\frac12}c^{\frac12}q^{\frac{r+1}{2}}, abq^{n+r} \\ aq^{r+1}, a^{\frac12}b^{\frac12}c^{\frac12}q^{\frac{r+1}{2}}, -a^{\frac12}b^{\frac12}c^{\frac12}q^{\frac{r+1}{2}} \end{matrix}; q, q\right) \tag{1.15}$$

$$= (-\imath)^n a^{\frac{n(n-2)}{2}} c^{\frac{n}{2}} q^{\frac{n(n-2)(2n+1)}{6}+\frac{n(n-2)r}{2}} \prod_{k=1}^{n} \frac{(q;q)_{k-1}(aq;q)_{k+r-1}(bq;q)_{k-2}}{(abq^2;q)_{k+n+r-2}}$$

$$\times p_n\left(0; a^{\frac12}c^{\frac12}q^{\frac{r+1}{2}}\imath, -a^{\frac12}c^{-\frac12}q^{\frac{r+1}{2}}\imath, b^{\frac12}\imath, -b^{\frac12}\imath; q\right). \tag{1.16}$$

Remark 1.2 *If we put $c = 0$ in (1.15), then we recover our previous result (1.7) easily by using the q-Chu-Vandermonde formula [3, (1.5.3)]*

$${}_2\phi_1\left(\begin{matrix} a, q^{-n} \\ c \end{matrix}; q, q\right) = \frac{(c/a;q)_n}{(c;q)_n} a^n. \tag{1.17}$$

If we put $a = q^{\alpha-1}$, $b = 0$, $c = q^{\gamma}$ and $r = 0$ in (1.15), then the left-hand side equals

$$\frac{q^{\frac{n(n-1)}{2}}(1-q)^{n^2}}{\{\Gamma_q(\alpha)\}^n} \det\left([\gamma+j-i]_q \Gamma_q(\alpha+i+j-2)\right)_{1\leq i,j\leq n}$$

because of $(q^{\alpha};q)_n = (1-q)^n \cdot \frac{\Gamma_q(\alpha+n)}{\Gamma_q(\alpha)}$, and the right-hand side equals

$$(-\imath)^n q^{\frac{n(n-2)}{2}\alpha+\frac{n}{2}\gamma+\frac{n(n-1)(n-2)}{3}} \prod_{k=1}^{n} (q;q)_{k-1}(q^{\alpha};q)_{k-1} \cdot Q_n(0; q^{\frac{\alpha+\gamma}{2}}\imath, -q^{\frac{\alpha-\gamma}{2}}\imath; q)$$

because of the relation $Q_n(x;A,B;q) = p_n(x;A,B,0,0;q)$ between the Al-Salam–Chihara polynomials and the Askey-Wilson polynomials. Hence we obtain Nishizawa's formula (1.9) as a corollary.

Corollary 1.3 *Let a, b and c be parameters, and let $n \geq 1$ and r be integers.*

(i) If the size $n = 2m$ of the matrix is even, then we have

$$\det\left((q^{i-1}-cq^{j-1})\frac{(aq;q)_{i+j+r-2}}{(abq^2;q)_{i+j+r-2}}\right)_{1\leq i,j\leq 2m}$$

$$
\begin{aligned}
&= a^{2m(m-1)} c^m q^{\frac{2m(m-1)(4m+1)}{3}+2m(m-1)r} \prod_{k=1}^{m} \left\{ \frac{(q;q)_{2k-1}(aq;q)_{2k+r-1}(bq;q)_{2k-2}}{(abq^2;q)_{2(k+m)+r-3}} \right\}^2 \\
&\quad \times {}_4\phi_3 \left({q^{-2m}, b^{-1}q^{-2m+1}, c, c^{-1} \atop q, aq^{r+1}, a^{-1}b^{-1}q^{1-4m-r}} ; q^2, q^2 \right) \qquad (1.18) \\
&= (-1)^m a^{m(2m-1)} b^m c^m q^{\frac{m(8m^2+3m-2)}{3}+m(2m-1)r} \prod_{k=1}^{2m} \frac{(q;q)_{k-1}(aq;q)_{k+r-1}}{(abq^2;q)_{k+2m+r-2}} \\
&\quad \times \prod_{k=1}^{m} \{(bq;q)_{2k-2}\}^2 \cdot p_m \left(\frac{c+c^{-1}}{2} ; 1, q, aq^{r+1}, a^{-1}b^{-1}q^{1-4m-r}; q^2 \right). \qquad (1.19)
\end{aligned}
$$

(ii) If the size $n = 2m+1$ of the matrix is odd, then we have

$$
\begin{aligned}
&\det \left((q^{i-1} - cq^{j-1}) \frac{(aq;q)_{i+j+r-2}}{(abq^2;q)_{i+j+r-2}} \right)_{1 \le i,j \le 2m+1} \\
&= a^{2m^2} c^m q^{\frac{2m(m+1)(4m-1)}{3}+2m^2 r} \cdot \frac{1-c}{1-q} \cdot \prod_{k=1}^{m+1} \frac{(q;q)_{2k-1}(aq;q)_{2k+r-2}(bq;q)_{2k-2}}{(abq^2;q)_{2(k+m-1)+r}} \\
&\quad \times \prod_{k=1}^{m} \frac{(q;q)_{2k-1}(aq;q)_{2k+r}(bq;q)_{2k-2}}{(abq^2;q)_{2(k+m-1)+r}} \cdot {}_4\phi_3 \left({q^{-2m}, b^{-1}q^{-2m+1}, cq, c^{-1}q \atop q^3, aq^{r+2}, a^{-1}b^{-1}q^{-4m-r}} ; q^2, q^2 \right) \qquad (1.20) \\
&= (-1)^m a^{m(2m+1)} b^m c^m (1-c) q^{\frac{m(8m^2+15m+4)}{3}+m(2m+1)r} \prod_{k=1}^{2m+1} \frac{(q;q)_{k-1}(aq;q)_{k+r-1}}{(abq^2;q)_{k+2m+r-1}} \\
&\quad \times \prod_{k=1}^{m+1} (bq;q)_{2k-2} \cdot \prod_{k=1}^{m} (bq;q)_{2k-2} \cdot p_m \left(\frac{c+c^{-1}}{2} ; q, q^2, aq^{r+1}, a^{-1}b^{-1}q^{-4m-r-1}; q^2 \right). \qquad (1.21)
\end{aligned}
$$

Remark 1.4 *If we put $c = 1$ in (1.18) for the even case, then it is clear that the ${}_4\phi_3$ sum reduces to 1, so that the determinant becomes the product which equals the square of the Pfaffian (1.8) obtained in [5]. Meanwhile, it does not suffice to prove (1.8) since it is not so trivial to take the square root of the determinant and determine the sign (see [2, 5]). If we put $c = 1$ in (1.20) for the odd case, then the factor $(1-c)$ reduces the right-hand side to 0.*

2 Determinant formula for arbitrary rows

In our previous paper [4], we prove the following formula in which the rows are arbitrary chosen. Let n be a positive integer, and $k_1, \ldots, k_n$ be arbitrary positive integers. Then we have

$$
\begin{aligned}
\det \left(\frac{(aq;q)_{k_i+j-2}}{(abq^2;q)_{k_i+j-2}} \right)_{1 \le i,j \le n} &= a^{\frac{n(n-1)}{2}} q^{\frac{(n+1)n(n-1)}{6}} \\
&\quad \times \prod_{i=1}^{n} \frac{(aq;q)_{k_i-1}}{(abq^2;q)_{k_i+n-2}} \prod_{1 \le i < j \le n} (q^{k_i-1} - q^{k_j-1}) \prod_{j=1}^{n} (bq;q)_{j-1}. \qquad (2.1)
\end{aligned}
$$

This formula is a generalization of (1.7) and a special case is obtained in [10, Theorem 3]. In this section we give this type formula, i.e., Theorem 2.1, which is crucial to prove Theorem 1.1.

First we fix some notation. If a and b are integers, we write $[a,b] = \{\, x \in \mathbb{Z} \,|\, a \le x \le b \,\}$. We also write $[n] = [1,n]$ for short. If S is a finite set and r a nonnegative integer, let $\binom{S}{r}$ denote the set of all r-element subsets of S. Let A be an $m \times n$ matrix. If $\mathbf{i} = (i_1, \dots, i_r)$ is an r-tuple of positive integers and $\mathbf{j} = (j_1, \dots, j_s)$ is an s-tuple of positive integers, then let $A^{\mathbf{i}}_{\mathbf{j}} = A^{i_1,\dots,i_r}_{j_1,\dots,j_s}$ denote the submatrix formed by selecting the row $\mathbf{i}$ and the column $\mathbf{j}$ from A. Then the following theorem generalize (2.1).

Theorem 2.1 *Let a, b and c be parameters. Let n be a positive integer, and $\mathbf{k} = (k_1, \dots, k_n)$ be an n-tuple of positive integers. Then we have*

$$\begin{aligned}
&\det\left((q^{k_i-1} - cq^{j-1}) \frac{(aq;q)_{k_i+j-2}}{(abq^2;q)_{k_i+j-2}} \right)_{1\le i,j\le n} \\
&= a^{\frac{n(n-3)}{2}} q^{\frac{n(n+1)(n-4)}{6}} \prod_{i=1}^{n} \frac{(aq;q)_{k_i-1}(bq;q)_{i-2}}{(abq^2;q)_{k_i+n-2}} \prod_{1\le i<j\le n} (q^{k_i-1} - q^{k_j-1}) \\
&\quad \times \sum_{\nu=0}^{n} (-1)^{n-\nu} (abcq^{2\nu+1};q^2)_{n-\nu} (acq;q^2)_\nu R_{n,\nu}(\mathbf{k},a,b;q), \qquad (2.2)
\end{aligned}$$

where

$$R_{n,\nu}(\mathbf{k},a,b;q) = \sum_{(\mathbf{i},\mathbf{j})} q^{\sum_{l=1}^{n-\nu} i_l - n + \nu} \prod_{l=1}^{n-\nu} (1 - aq^{k_{i_l} - i_l + l + \nu}) \prod_{l=1}^{\nu} (1 - abq^{k_{j_l} + j_l - l + \nu - 1}). \qquad (2.3)$$

Here the sum on the right-hand side runs over all pairs $(\mathbf{i}, \mathbf{j})$ such that $[n]$ is a disjoint union of $\mathbf{i} = \{i_1, \dots, i_{n-\nu}\} \in \binom{[n]}{n-\nu}$ and $\mathbf{j} = \{j_1, \dots, j_\nu\} \in \binom{[n]}{\nu}$ (i.e., $\mathbf{i} \cup \mathbf{j} = [n]$ and $\mathbf{i} \cap \mathbf{j} = \emptyset$).

For example, if $n = 3$ and $\nu = 2$, then the pairs $(\mathbf{i}, \mathbf{j})$ runs over

$$\{(\{1\},\{2,3\}), (\{2\},\{1,3\}), (\{3\},\{1,2\})\}.$$

Hence we have

$$\begin{aligned}
R_{3,2}(\{k_1,k_2,k_3\},a,b;q) &= (1-aq^{k_1+2})(1-abq^{k_2+2})(1-abq^{k_3+2}) \\
&\quad + q(1-aq^{k_2+1})(1-abq^{k_1+1})(1-abq^{k_3+2}) \\
&\quad + q^2(1-aq^{k_3})(1-abq^{k_1+1})(1-abq^{k_2+1}).
\end{aligned}$$

Here we have no space to describe the proof of Theorem 2.1. We need some intensive use of linear algebra for the proof. The interested reader should consult [6]. Here we describe only the sketch of the proof.

Let n be a positive integer, and let a, b, c and q be parameters. For an index set $\mathbf{k} = \{k_1, \dots, k_n\}$ of positive integers, let $M_n(\mathbf{k},a,b,c;q) = (M_n(\mathbf{k},a,b,c;q)_{i,j})_{1\le i,j\le n}$ denote the matrix whose (i,j) entry is given by

$$M_n(\mathbf{k},a,b,c;q)_{i,j} = (q^{k_i-1} - cq^{j-1})\,(aq^{k_i};q)_{j-1}(abq^{k_i+j};q)_{n-j}. \qquad (2.4)$$

Then we have

$$\det\left((q^{k_i-1} - cq^{j-1}) \frac{(aq;q)_{k_i+j-2}}{(abq^2;q)_{k_i+j-2}} \right)_{1\le i,j\le n} = \prod_{i=1}^{n} \frac{(aq;q)_{k_i-1}}{(abq^2;q)_{k_i+n-2}} \cdot \det M_n(\mathbf{k},a,b,c;q). \qquad (2.5)$$

Hence it is enough to evaluate $\det M_n(\mathbf{k},a,b,c;q)$ to prove Theorem 2.1. The main task of this evaluation is to show the following recurrence equation:

$$\begin{aligned}&\frac{\det M_n(\mathbf{k},a,b,c;q)}{a^{n-2}(bq;q)_{n-2}\prod_{i=1}^{n-1}(q^{k_i}-q^{k_n})}\\&=q^{-1}(1-acq)(1-abq^{k_n+n-1})\det M_{n-1}(\mathbf{k}',aq,b,cq;q)\\&\qquad -q^{n(n-3)/2}(1-abcq^{2n-1})(1-aq^{k_n})\det M_{n-1}(\mathbf{k}',a,b,c;q),\end{aligned}\tag{2.6}$$

where $\mathbf{k}'=\{k_1,\dots,k_{n-1}\}$ denote the subset of the first $(n-1)$ indices of $\mathbf{k}=\{k_1,\dots,k_{n-1},k_n\}$. This identity enable us to prove (2.2) by induction.

We introduce four triangular matrices $X_n(\mathbf{k},a;q)$, $Y_n(q)$, $L_n(\mathbf{k},a,b;q)$ and $U_n(q)$ which play an important role to manipulate $M_n(\mathbf{k},a,b,c;q)$ in (2.5). Let $X_n(\mathbf{k},a;q)=(X(\mathbf{k},a;q)_{i,j})_{1\le i,j\le n}$ and $Y_n(q)=(Y_n(q)_{i,j})_{1\le i,j\le n}$ be the $n\times n$ lower triangular matrices whose (i,j)-entry is, respectively, given by

$$X(\mathbf{k},a;q)_{i,j}=-\frac{\chi(i\ge j)}{q^{k_j}(1-aq^{k_j})\prod_{\substack{l=1\\l\ne j}}^{i}(q^{k_l}-q^{k_j})},\tag{2.7}$$

$$Y_n(q)_{i,j}=(-1)^{i+j}q^{-\frac{(i-j)(2n+1-i-j)}{2}}\begin{bmatrix}n-j\\i-j\end{bmatrix}_q.\tag{2.8}$$

Similarly, let $L_n(\mathbf{k},a,b;q)=(L_n(\mathbf{k},a,b;q)_{i,j})_{1\le i,j\le n}$ (resp. $U_n(q)=(U(q)_{i,j})_{1\le i,j\le n}$) be the $n\times n$ lower (resp. upper) triangular matrix whose (i,j)-entry is, respectively, given by

$$L_n(\mathbf{k},a,b;q)_{i,j}=-\frac{\chi(i\ge j)}{q^{k_j}(1-abq^{k_j+n-1})\prod_{\substack{l=1\\l\ne j}}^{i}(q^{k_l}-q^{k_j})},\tag{2.9}$$

$$U(q)_{i,j}=(-1)^{i+j}q^{\frac{(j-i)(j-i+1)}{2}}\begin{bmatrix}j-1\\j-i\end{bmatrix}_q.\tag{2.10}$$

We define the $n\times n$ matrices $P_n(\mathbf{k},a,b,c;q)$ and $Q_n(\mathbf{k},a,b,c;q)$ by

$$\begin{aligned}P_n(\mathbf{k},a,b,c;q)&=X_n(\mathbf{k},a;q)M_n(\mathbf{k},a,b,c;q)Y_n(q),\\Q_n(\mathbf{k},a,b,c;q)&=L_n(\mathbf{k},a,b;q)M_n(\mathbf{k},a,b,c;q)U_n(q).\end{aligned}$$

Since $X_n(\mathbf{k},a;q)$, $L_n(\mathbf{k},a,b;q)$ are triangular and $Y_n(q)$, $U_n(q)$ are unitriangular, we easily obtain

$$\det P_n(\mathbf{k},a,b,c;q)=\frac{(-1)^n\det M_n(\mathbf{k},a,b,c;q)}{q^{\sum_{i=1}^n k_i}\prod_{i=1}^n(1-aq^{k_i})\prod_{1\le i<j\le n}(q^{k_i}-q^{k_j})},\tag{2.11}$$

$$\det Q_n(\mathbf{k},a,b,c;q)=\frac{(-1)^n\det M_n(\mathbf{k},a,b,c;q)}{q^{\sum_{i=1}^n k_i}\prod_{i=1}^n(1-abq^{k_i+n-1})\prod_{1\le i<j\le n}(q^{k_i}-q^{k_j})}.\tag{2.12}$$

The key to prove (2.6) is the following lemma:

Lemma 2.2 *Let n be a positive integer, and let a, b, c and q be parameters. Let $P_n(\mathbf{k},a,b,c;q)$ and $Q_n(\mathbf{k},a,b,c;q)$ be as defined above. When $\mathbf{k}=\{k_1,\dots,k_{n-1},k_n\}$ is a row index set, let $\mathbf{k}'=$*

$\{k_1, \ldots, k_{n-1}\}$ *denote the subset of the first* $(n-1)$ *indices of* $\mathbf{k}$. *Then we have*

$$\det P_n(\mathbf{k}, a, b, c; q)_{[2,n]}^{[1,n-1]} = \frac{(-1)^{n-1} \det M_{n-1}(\mathbf{k}', aq, b, cq; q)}{q^{\sum_{i=1}^{n-1} k_i} \prod_{1 \le i < j < n} (q^{k_i} - q^{k_j})}, \tag{2.13}$$

$$\det Q_n(\mathbf{k}, a, b, c; q)_{[1,n-1]}^{[1,n-1]} = \frac{(-1)^{n-1} \det M_{n-1}(\mathbf{k}', a, b, c; q)}{q^{\sum_{i=1}^{n-1} k_i} \prod_{1 \le i < j < n} (q^{k_i} - q^{k_j})}, \tag{2.14}$$

$$\frac{\det P_n(\mathbf{k}, a, b, c; q)_{[1,n-1]}^{[1,n-1]}}{\prod_{\nu=1}^{n-1} (1 - abq^{k_\nu + n-1})} = (-q)^{-n+1} \frac{\det Q_n(\mathbf{k}, a, b, c; q)_{[2,n]}^{[1,n-1]}}{\prod_{\nu=1}^{n-1} (1 - aq^{k_\nu})}. \tag{2.15}$$

Now we are in position to prove Theorem 2.1. In fact the proof is straightforward by induction.

Proof of Theorem 2.1. First, we note that, for any integers n and ν, it holds

$$\begin{aligned} R_{n,\nu}(\mathbf{k}, a, b; q) \quad = \quad & (1 - abq^{k_n + n-1}) R_{n-1,\nu-1}(\mathbf{k}', aq, b; q) \\ & + q^{n-1}(1 - aq^{k_n}) R_{n-1,\nu}(\mathbf{k}', a, b; q), \end{aligned} \tag{2.16}$$

where $\mathbf{k} = \{k_1, \ldots, k_{n-1}, k_n\}$ and $\mathbf{k}' = \{k_1, \ldots, k_{n-1}\}$ are as before. (2.16) follows from the definition (2.3) of $R_{n,\nu}(\mathbf{k}, a, b; q)$ by considering two exclusive cases, $j_\nu = n$ or $i_{n-\nu} = n$. Now we prove the identity

$$\begin{aligned} \det M_n(\mathbf{k}, a, b, c; q) &= (-1)^n a^{\frac{n(n-3)}{2}} q^{\frac{n(n+1)(n-4)}{6}} \prod_{i=1}^{n} (bq; q)_{i-2} \\ \times \prod_{1 \le i < j \le n} (q^{k_i - 1} - q^{k_j - 1}) & \sum_{\nu=0}^{n} (-1)^\nu (abcq^{2\nu+1}; q^2)_{n-\nu} (acq; q^2)_\nu \, R_{n,\nu}(\mathbf{k}, a, b; q), \end{aligned} \tag{2.17}$$

by induction on n. If $n = 1$, then the left-hand side of (2.17) is trivially $q^{k_1 - 1} - c$ from (2.4). It is straightforward computation to check the right-hand side equals $q^{k_1 - 1} - c$. Assume $n > 1$ and (2.17) holds up to $(n-1)$. Using (2.6) and the induction hypothesis, we obtain

$$\begin{aligned} & \frac{\det M_n(\mathbf{k}, a, b, c; q)}{(-1)^n a^{\frac{n(n-3)}{2}} q^{\frac{n(n^2-6n-1)}{6}} \prod_{i=1}^{n} (bq; q)_{i-2} \prod_{1 \le i < j \le n} (q^{k_i} - q^{k_j})} \\ & = (1 - abq^{k_n + n-1}) \sum_{\nu=0}^{n-1} (-1)^{\nu+1} (abcq^{2\nu+3}; q^2)_{n-\nu-1} (acq; q^2)_{\nu+1} R_{n-1,\nu}(\mathbf{k}', aq, b; q) \\ & + q^{n-1}(1 - aq^{k_n}) \sum_{\nu=0}^{n-1} (-1)^\nu (abcq^{2\nu+1}; q^2)_{n-\nu} (acq; q^2)_\nu R_{n-1,\nu}(\mathbf{k}', a, b; q). \end{aligned}$$

Replacing $\nu + 1$ by ν in the first sum and applying (2.16), we establish (2.17) for n. Hence (2.17) holds for an arbitrary positive integer n. Finally, (2.5) and (2.17) immediately implies (2.2). This completes the proof of Theorem 2.1. $\square$

For the detail of the proofs of the lemmas in this section, the reader can consult [6]

3 Proof of the main theorems

The aim of this section is to describe the outline of the proofs of the theorems in Section 1, i.e., Theorem 1.1 from Theorem 2.1, and then prove Corollary 1.3 from Theorem 1.1. Once we prove Theorem 2.1, then it is easy and straightforward to prove the main theorems mainly by induction. In fact, to prove Theorem 1.1, we need to set $\mathbf{k} = [n] = \{1, 2, \ldots, n\}$ in (2.2). Hence, the following lemma is essential to prove (1.16).

Lemma 3.1 *If we put $\mathbf{k} = [n]$ in (2.2), then we obtain*

$$R_{n,\nu}([n], a, b; q) = q^{\frac{(n-\nu)(n-\nu-1)}{2}} \begin{bmatrix} n \\ \nu \end{bmatrix}_q (aq^{\nu+1}; q)_{n-\nu} (abq^n; q)_\nu. \tag{3.1}$$

In fact the proof of Theorem 1.1 is quite straightforward by substitution $\mathbf{k} = [n]$ into (2.2) using (3.1). We use some well-known q-series identities. The details are described in [6].

To derive Corollary 1.3 from Theorem 1.1 the following proposition plays a crucial role:

Proposition 3.2 *Let n be an integer, a, b and c be arbitrary parameters. Then we have*

$$\begin{aligned} p_n(0; a, b, c, -c; q) &= (-1)^m a^m b^m c^{2m} q^{m(3m-1)} (-c^2; q^2)_m \\ &\qquad \times p_m\left(x_0; 1, q, ab, -a^{-1}b^{-1}c^{-2}q^{-4m+2}; q^2\right), \end{aligned} \tag{3.2}$$

if $n = 2m$ is even, and

$$\begin{aligned} p_n(0; a, b, c, -c; q) &= (-1)^{m+1} a^m b^{m+1} c^{2m} (1 + ab^{-1}) q^{m(3m+1)} (-c^2; q^2)_{m+1} \\ &\qquad \times p_m\left(x_0; q, q^2, ab, -a^{-1}b^{-1}c^{-2}q^{-4m}; q^2\right), \end{aligned} \tag{3.3}$$

if $n = 2m + 1$ is odd, where $x_0 = -\frac{ab^{-1} + a^{-1}b}{2}$.

When $b = -a$, replacing c by b, one gets incidentally the following known result due to Andrews (see [3, (II.17)]).

Corollary 3.3

$$p_n(0; a, -a, b, -b; q) = \begin{cases} (-1)^m (q, -a^2, -b^2, a^2b^2q^{2m}; q^2)_m & \textit{if } n = 2m, \\ 0 & \textit{if } n = 2m + 1. \end{cases} \tag{3.4}$$

To prove of Proposition 3.2, we use the following contiguous relations for ${}_4\phi_3$.

Proposition 3.4 *Let z, a, b, c, d, e, f, g and q be arbitrary parameters. Then we have*

$$\begin{aligned} &{}_4\phi_3\left(\begin{matrix} a, bq, c, d \\ e, f, g \end{matrix}; q, z\right) - {}_4\phi_3\left(\begin{matrix} aq, b, c, d \\ e, f, g \end{matrix}; q, z\right) \\ &\quad = \frac{z\,(b-a)\,(1-c)(1-d)}{(1-e)(1-f)(1-g)}\,{}_4\phi_3\left(\begin{matrix} aq, bq, cq, dq \\ eq, fq, gq \end{matrix}; q, z\right), \end{aligned} \tag{3.5}$$

$$\begin{aligned} &(1-f)(a-e)\,{}_4\phi_3\left(\begin{matrix} a, b, c, d \\ eq, f, g \end{matrix}; q, z\right) - (1-e)(a-f)\,{}_4\phi_3\left(\begin{matrix} a, b, c, d \\ e, fq, g \end{matrix}; q, z\right) \\ &\quad = (1-a)(f-e)\,{}_4\phi_3\left(\begin{matrix} aq, b, c, d \\ eq, fq, g \end{matrix}; q, z\right), \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} &(1-e)(1-f)(1-g)\,{}_4\phi_3\left(\begin{matrix} a, b, c, d \\ e, f, g \end{matrix}; q, q\right) \\ &\quad = c(1-e)\left(1 - \frac{f}{c}\right)\left(1 - \frac{g}{c}\right)\,{}_4\phi_3\left(\begin{matrix} aq, bq, c, d \\ e, fq, gq \end{matrix}; q, q\right) \\ &\quad +d(1-c)\left(1 - \frac{e}{d}\right)\left(1 - \frac{fg}{cd}\right)\,{}_4\phi_3\left(\begin{matrix} aq, bq, cq, d \\ eq, fq, gq \end{matrix}; q, q\right), \end{aligned} \tag{3.7}$$

where, in the last identity, we assume $abcdq = efg$ and $a = q^{-n}$ for some nonnegative integer n.

Remark 3.5 *The contiguous relations (3.5) (resp. (3.6)) correspond to (3.2) (resp. (3.10)) in [9], meanwhile (3.6) can be written as a contiguous relation for ${}_8W_7$. In fact, if one uses Watson's transformation formula [3, (2.5.1)]*

$$ {}_8W_7\left(a; b, c, d, e, q^{-n}; q, \tfrac{a^2q^{n+2}}{bcde}\right) = \frac{\left(aq, \frac{aq}{de}; q\right)_n}{\left(\frac{aq}{d}, \frac{aq}{e}; q\right)_n} {}_4\phi_3\left(\begin{matrix} q^{-n}, d, e, \frac{aq}{bc} \\ \frac{aq}{b}, \frac{aq}{c}, \frac{deq^{-n}}{a} \end{matrix}; q, q\right) \tag{3.8}$$

for a terminating very-well-poised ${}_8\phi_7$ series, where

$$ {}_{r+1}W_r(a_1; a_4, \dots, a_{r+1}; q, z) = {}_{r+1}\phi_r\left(\begin{matrix} a_1, qa_1^{\frac12}, -qa_1^{\frac12}, a_4, \dots, a_{r+1} \\ a_1^{\frac12}, -a_1^{\frac12}, \frac{qa_1}{a_4}, \dots, \frac{qa_1}{a_{r+1}} \end{matrix}; q, z\right), \tag{3.9}$$

then (3.7) is equivalent to

$$\begin{aligned} &(c-a)(d-aq)(e-aq)(b-aq^n)\, {}_8W_7\left(a; b, cq, d, e, q^{-n}; q, \tfrac{a^2q^{n+1}}{bcde}\right) \\ &= a(1-b)(1-aq)(de-aq)(1-cq^n)\, {}_8W_7\left(aq; bq, cq, d, e, q^{-n+1}; q, \tfrac{a^2q^{n+1}}{bcde}\right) \\ &\quad + (bc-a)(d-aq)(e-aq)(1-aq^n)\, {}_8W_7\left(a; b, c, d, e, q^{-n+1}; q, \tfrac{a^2q^{n+1}}{bcde}\right). \end{aligned} \tag{3.10}$$

4 A quadratic relation

First we recall the reader a well-known theorem for determinants. The following identity is known as the Desnanot-Jacobi adjoint matrix theorem [1, Theorem 3.12]

$$\det A^{[2,n-1]}_{[2,n-1]} \det A^{[n]}_{[n]} = \det A^{[n-1]}_{[n-1]} \det A^{[2,n]}_{[2,n]} - \det A^{[n-1]}_{[2,n]} \det A^{[2,n]}_{[n-1]}. \tag{4.1}$$

Let

$$D_n(a, b, c; q) = \det\left((q^{i-1} - cq^{j-1})\frac{(aq; q)_{i+j-2}}{(abq^2; q)_{i+j-2}}\right)_{1\le i,j\le n}$$

and apply (4.1) to this determinant. Then we obtain

$$\begin{aligned} D_n(a, b, c; q) D_{n-2}(aq^2, b, c; q) &= \frac{q(aq; q)_2}{(abq^2; q)_2} \cdot D_{n-1}(a, b, c; q) D_{n-1}(aq^2, b, c; q) \\ &- \frac{q(1-aq)^n(1-abq^3)^{n-2}}{(1-aq^2)^{n-2}(1-abq^2)^n} \cdot D_{n-1}(aq, b, cq; q) D_{n-1}(aq, b, cq^{-1}; q). \end{aligned} \tag{4.2}$$

Hence we can substitute (1.16) into (4.2), then replacing $a^{\frac12}c^{\frac12}q^{\frac12}\imath$, $-a^{\frac12}c^{-\frac12}q^{\frac12}\imath$ and $b^{\frac12}\imath$ by a, b and c, respectively, we obtain the following corollary.

Corollary 4.1 *Let n be a positive integer and a, b, c and q parameters. Then we have*

$$\begin{aligned} &ab(1-q^{n-1})(1+c^2q^{n-2}) p_n(0; a, b, c, -c; q) p_{n-2}(0; aq, bq, c, -c; q) \\ &= (1-abq^{n-1})(1+abc^2q^{n-1}) p_{n-1}(0; a, b, c, -c; q) p_{n-1}(0; aq, bq, c, -c; q) \\ &\quad - (1-ab)(1+abc^2q^{2n-2}) p_{n-1}(0; aq, b, c, -c; q) p_{n-1}(0; a, bq, c, -c; q). \end{aligned} \tag{4.3}$$

Here we derive Corollary 4.1 as a corollary of Theorem 1.1.

In fact a more general formula holds. Recently one of the authors has proven that the following quadratic equation in a different method.

Theorem 4.2 *For $r \geq 1$, there holds*

$$(1-a_0^{-1})(a_1-b_1)\,{}_{r+1}\phi_r\left[\begin{matrix}a_0/q, a_1, a_2, \dots, a_r\\ b_1/q, b_2, \dots, b_r\end{matrix}; q, z\right]{}_{r+1}\phi_r\left[\begin{matrix}a_0q, a_1, a_2q \dots, a_rq\\ b_1q, b_2q, \dots, b_rq\end{matrix}; q, z\right]$$
$$= (1-a_1/a_0)(1-b_1)\,{}_{r+1}\phi_r\left[\begin{matrix}a_0, a_1, a_2, \dots, a_r\\ b_1, b_2, \dots, b_r\end{matrix}; q, z\right]{}_{r+1}\phi_r\left[\begin{matrix}a_0, a_1, a_2q \dots, a_rq\\ b_1, b_2q, \dots, b_rq\end{matrix}; q, z\right]$$
$$- (1-a_1)(1-b_1/a_0)\,{}_{r+1}\phi_r\left[\begin{matrix}a_0, a_1/q, a_2, \dots, a_r\\ b_1/q, b_2, \dots, b_r\end{matrix}; q, z\right]{}_{r+1}\phi_r\left[\begin{matrix}a_0, a_1q, a_2q \dots, a_rq\\ b_1q, b_2q, \dots, b_rq\end{matrix}; q, z\right] \quad (4.4)$$

This formula gives a simple proof of Theorem 1.1 using the Desnanot-Jacobi adjoint matrix theorem (4.1). But, note that Theorem 2.1 is more general and cannot be derived from the quadratic equation. This thorem may also hint us there could exist a more general formula than Theorem 1.1. But it is not an easy task to find the appropriate entry of the determinant which gives this quadratic relation.

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A referee suggested us that Theorem 1.1 can be proven by another method, i.e., using Wilson's result [16] on the Gram determinants for the Askey-Wilson polynomials. As our second proof, this third proof can also avoid our lengthy proof of Theorem 2.1, which is used in our first proof in [6]. But analyzing the referre's suggestion we've discovered Theorem 2.1 type formula not only for the Gram determinants for the Askey-Wilson polynomials but also for the biorthogonal rational function appearing in the paper [16]. This means that we can choose arbitrary rows of the most general Gram determinants in [16] and give an explicit formula. Further we've found that the proof can be much simplified from our original proof in [6] by analizing the referee's idea. Hence our new result will generalize [16], but we don't have enough space to state it here. So we will present our new version in our poster of FPSAC 2013. We slso would like to express our appreciation to the referee for giving us a hint for our new progresses.

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Cycles and sorting index for matchings and restricted permutations

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Abstract. We prove that the Mahonian-Stirling pairs of permutation statistics $(\mathrm{sor}, \mathrm{cyc})$ and $(\mathrm{inv}, \mathrm{rlmin})$ are equidistributed on the set of permutations that correspond to arrangements of n non-atacking rooks on a fixed Ferrers board with n rows and n columns. The proofs are combinatorial and use bijections between matchings and Dyck paths and a new statistic, sorting index for matchings, that we define. We also prove a refinement of this equidistribution result which describes the minimal elements in the permutation cycles and the right-to-left minimum letters.

Résumé. Nous prouvons que les paires de statistiques de Mahonian-Stirling $(\mathrm{sor}, \mathrm{cyc})$ et $(\mathrm{inv}, \mathrm{rlmin})$ suivent la même distribution pour des permutations correspondant à des placements de n tours sur un tableau de Ferrer fixé avec n lignes et n colonnes. Les preuves sont combinatoires et utilisent des bijections entre les couplages et les chemins de Dyck. Nous définissons une nouvelle statistique, l'indice de tri pour les couplages. Nous prouvons également un résultat plus fin qui décrit les éléments minimaux dans les cycles des permutations et les lettres minimum droite á gauche.

Keywords: sorting index, cycle, matching, Ferrers board

1 Introduction

An inversion in a permutation σ is a pair $\sigma(i) > \sigma(j)$ such that $i < j$. The number of inversions in σ is denoted by $\mathrm{inv}(\sigma)$. The distribution of inv over the symmetric group S_n was first found by Rodriguez [9] in 1837 and is well known to be

$$\sum_{\sigma \in S_n} q^{\mathrm{inv}(\sigma)} = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}).$$

Much later, MacMahon [6] defined the major index maj and proved that it has the same distribution as inv. In his honor, all permutation statistics that are equally distributed with inv are called Mahonian. MacMahon's remarkable result initiated a systematic research of permutation statistics and in particular many more Mahonian statistics have been described in the literature since then.

Another classical permutation statistic is the number of cycles, cyc. Its distribution is given by

$$\sum_{\sigma \in S_n} t^{\mathrm{cyc}(\sigma)} = t(t+1)(t+2)\cdots(t+n-1)$$

and the coefficients of this polynomial are known as the unsigned Stirling numbers of the first kind.

Given these two distributions, it is natural then to ask which "Mahonian-Stirling" pairs of statistics $(\mathrm{stat}_1, \mathrm{stat}_2)$ have the distribution

$$\sum_{\sigma \in S_n} q^{\mathrm{stat}_1(\sigma)} t^{\mathrm{stat}_2(\sigma)} = t(t+q)(t+q+q^2)\cdots(t+q+\cdots+q^{n-1}). \tag{1}$$

As proved by Björner and Wachs [1], $(\mathrm{inv}, \mathrm{rlmin})$ and $(\mathrm{maj}, \mathrm{rlmin})$ are two such pairs, where rlmin is the number of right-to-left minimum letters. A right-to-left minimum letter of a permutation σ is a letter $\sigma(i)$ such that $\sigma(i) < \sigma(j)$ for all $j > i$. The set of all right-to-left minimum letters in σ will be denoted by $\mathrm{Rlminl}(\sigma)$. In fact, Björner and Wachs proved the following stronger result

$$\sum_{\sigma \in S_n} q^{\mathrm{inv}(\sigma)} \prod_{i \in \mathrm{Rlminl}(\sigma)} t_i = \sum_{\sigma \in S_n} q^{\mathrm{maj}(\sigma)} \prod_{i \in \mathrm{Rlminl}(\sigma)} t_i = t_1(t_2+q)(t_3+q+q^2)\cdots(t_n+q+\cdots+q^{n-1}). \tag{2}$$

A natural Mahonian partner for cyc was found by Petersen [7]. For a given permutation $\sigma \in S_n$ there is a unique expression

$$\sigma = (i_1 j_1)(i_2 j_2)\cdots(i_k j_k)$$

as a product of transpositions such that $i_s < j_s$ for $1 \le s \le k$ and $j_1 < \cdots < j_k$. The sorting index of σ is defined to be

$$\mathrm{sor}(\sigma) = \sum_{s=1}^{k} (j_s - i_s).$$

The sorting index can also be described as the total distance the elements in σ travel when σ is sorted using the Straight Selection Sort algorithm [5] in which, using a transposition, we move the largest number to its proper place, then the second largest to its proper place, etc. For example, the steps for sorting $\sigma = 6571342$ are

$$65\mathbf{7}1342 \xrightarrow{(37)} \mathbf{6}521347 \xrightarrow{(16)} 4\mathbf{5}21367 \xrightarrow{(25)} \mathbf{4}321567 \xrightarrow{(14)} 1\mathbf{3}24567 \xrightarrow{(23)} 1234567$$

and therefore $\sigma = (2\ 3)(1\ 4)(2\ 5)(1\ 6)(3\ 7)$ and $\mathrm{sor}(\sigma) = (3-2)+(4-1)+(5-2)+(6-1)+(7-3) = 16$. The relationship to other Mahonian statistics and the Eulerian partner for sor were studied by Wilson [10] who called the sorting index DIS.

Petersen showed that

$$\sum_{\sigma \in S_n} q^{\mathrm{sor}(\sigma)} t^{\mathrm{cyc}(\sigma)} = t(t+q)(t+q+q^2)\cdots(t+q+\cdots+q^{n-1}),$$

which implies equidistribution of the pairs $(\mathrm{inv}, \mathrm{rlmin})$ and $(\mathrm{sor}, \mathrm{cyc})$.

In this article we show that the pairs $(\mathrm{inv}, \mathrm{rlmin})$ and $(\mathrm{sor}, \mathrm{cyc})$ have the same distribution on the set of restricted permutations

$$S_{\mathbf{r}} = \{\sigma \in S_n : \sigma(k) \le r_k, 1 \le k \le n\}$$

for a nondecreasing sequence of integers $\mathbf{r} = (r_1, \ldots, r_n)$ with $1 \le r_1 \le r_2 \le \cdots \le r_n \le n$. These can be described as permutations that correspond to arrangements of n non-atacking rooks on a Ferrers board with rows of length $r_1, \dots, r_n$. To obtain the results, in Section 2 we define a sorting index and cycles for perfect matchings and study the distributions of these statistics over matchings of fixed type. We use bijections between matchings and weighted Dyck paths which enable us to keep track of set-valued statistics and obtain more refined results similar to (2) for restricted permutations.

Analogously to sor, Petersen defined the sorting index for signed permutations of type B_n and D_n. Using algebraic methods he proved that

$$\sum_{\sigma\in B_n} q^{\mathrm{sor}_B(\sigma)} t^{\ell'_B(\sigma)} = \sum_{\sigma\in B_n} q^{\mathrm{inv}_B(\sigma)} t^{\mathrm{nmin}_B(\sigma)} = \prod_{i=1}^{n} (1 + t[2i]_q - t), \tag{3}$$

where for an element $\sigma \in B_n$, $\ell'_B(\sigma)$ denotes its reflection length, $\mathrm{inv}_B(\sigma)$ denotes the type B_n inversion number, and nmin is a signed permutation statistic similar to rlmin. Petersen also defined sor_D, a sorting index for type D_n permutations and showed that it is equidistributed with the number of type D_n inversions:

$$\sum_{\sigma\in D_n} q^{\mathrm{sor}_D(\sigma)} = \sum_{\sigma\in D_n} q^{\mathrm{inv}_D(\sigma)} = [n]_q \cdot \prod_{i=1}^{n-1} [2i]_q. \tag{4}$$

While space constraints prevent us from providing details in this extended abstract, we mention that in [8] we define a sorting index and cycle number for bicolored matchings in a fashion analogous to what we will show for ordinary matchings. In particular, this gives a combinatorial proof that the pairs $(\mathrm{sor}_B, \ell'_B)$ and $(\mathrm{inv}_B, \mathrm{nmin}_B)$ are equidistributed on the set of restricted signed permutations

$$B_{\mathbf{r}} = \{\sigma \in B_n : |\sigma(k)| \le r_k, 1 \le k \le n\}$$

for a nondecreasing sequence of integers $1 \le r_1 \le r_2 \le \cdots \le r_n \le n$. Using bijections between bicolored matchings and weighted Dyck paths with bicolored rises, we in fact prove equidistribution of set-valued statistics and their generating functions. Moreover, we find natural Stirling partners for sor_D and inv_D and prove equidistribution of the two Mahonian-Stirling pairs on sets of restricted permutations of type D_n:

$$D_{\mathbf{r}} = \{\sigma \in D_n : |\sigma(k)| \le r_k, 1 \le k \le n\}.$$

2 Statistics on perfect matchings

A matching is a partition of a set in blocks of size at most two and if it has no single-element blocks the matching is said to be perfect. The set of all perfect matchings with n blocks is denoted by $\mathcal{M}_n$. All matchings in this work will be perfect and henceforth we will omit this adjective.

2.1 Statistics based on crossings and nestings

A matching in $\mathcal{M}_n$ can be represented by a graph with $2n$ labeled vertices and n edges in which each vertex has a degree 1. The vertices $1, 2, \dots, 2n$ are drawn on a horizontal line in natural order and two vertices that are in a same block are connected by a semicircular arc in the upper half-plane. We will use $i \cdot j$ to denote an arc with vertices $i < j$. The vertex i is said to be the opener while j is said to be the closer of the arc. For a vertex i, we will denote by $M(i)$ the other vertex which is in the same block in the matching M as i. Two arcs $i \cdot j$ and $k \cdot l$ with $i < k$ can be in three different relative positions. We say that they form a crossing if $i < k < j < l$, they form a nesting if $i < k < l < j$, and they form an alignment if $i < j < k < l$. The arc with the smaller opener will be called the left arc of the crossing, nesting, or the alignment, respectively, while the arc with the larger opener will be called the right arc. The numbers of crossings, nestings, and alignements in a matching M are denoted by $\mathrm{cr}(M)$, $\mathrm{ne}(M)$, and $\mathrm{al}(M)$, respectively.

If $o_1 < \cdots < o_n$ and $c_1 < \cdots < c_n$ are the openers and the closers in M, respectively, let

$$\mathrm{Long}(M) = \{k : o_k \cdot M(o_k) \text{ is not a right arc in a nesting}\}$$

and

$$\mathrm{Short}(M) = \{k : M(c_k) \cdot c_k \text{ is not a left arc in a nesting}\}.$$

Similarly, let

$$\mathrm{Left}(M) = \{k : o_k \cdot M(o_k) \text{ is not a right arc in a crossing}\}.$$

We will use lower-case letters to denote the cardinalities of the sets. For example, $\mathrm{long}(M) = |\mathrm{Long}(M)|$.

Example 2.1. *For the matching M in Figure 1 we have* $\mathrm{ne}(M) = \mathrm{cr}(M) = \mathrm{al}(M) = 5$*,* $\mathrm{Long}(M) = \{1, 2\}$*,* $\mathrm{Short}(M) = \{1, 2, 3, 5\}$*, and* $\mathrm{Left}(M) = \{1, 5\}$.

The pair of sets $(\{o_1, \ldots, o_n\}, \{c_1, \ldots, c_n\})$ of openers and closers of a matching M is called the type of M. There is a natural one-to-one correspondence between types of matchings in $\mathcal{M}_n$ and Dyck paths of semilength n, i.e., lattice paths that start at $(0, 0)$, end at $(2n, 0)$, use steps $(1, 1)$ (rises) and $(1, -1)$ (falls), and never go below the x-axis. The set of all such Dyck paths will be denoted by $\mathcal{D}_n$. Namely, the openers in the type correspond to the rises in the Dyck path while the closers correspond to the falls. Therefore, for convenience, we will say that a matching in $\mathcal{M}_n$ is of type D, for some Dyck path $D \in \mathcal{D}_n$, and we will denote the set of all matchings of type D by $\mathcal{M}_n(D)$.

The height of a rise of a Dyck path is the y-coordinate of the right endpoint of the corresponding $(1, 1)$ segment. The sequence $(h_1, \ldots, h_n)$ of the heights of the rises of a $D \in \mathcal{D}_n$ when read from left to right will be called shortly the height sequence of D. For example, the height sequence of the Dyck path in Figure 1 is $(1, 2, 3, 3, 3, 4)$. A weighted Dyck path is a pair $(D, (w_1, \ldots, w_n))$ where $D \in \mathcal{D}_n$ with height sequence $(h_1, \ldots, h_n)$ and $w_i \in \mathbb{Z}$ with $1 \leq w_i \leq h_i$. There is a well-known bijection φ from the set $\mathcal{WD}_n$ of weighted Dyck paths of semilength n to $\mathcal{M}_n$ [2]. Namely, the openers $o_1 < o_2 < \cdots < o_n$ of the matching that corresponds to a given $(D, (w_1, \ldots, w_n)) \in \mathcal{WD}_n$ are determined according to the type D. To construct the corresponding matching M, we connect the openers from right to left, starting from o_n. After $o_n, o_{n-1}, \ldots, o_{k+1}$ are connected to a closer, there are exactly h_k unconnected closers that are larger than o_k. We connect o_k to the w_k-th of the available closers, when they are listed in decreasing order (see Figure 1).

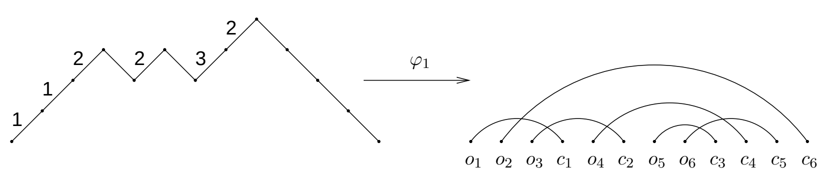

Fig. 1: The bijection φ between weighted Dyck paths and matchings.

Via the bijection φ we immediately get the following generating function.

Theorem 2.2. *If $D \in \mathcal{D}_n$ has a height sequence $(h_1, \ldots, h_n)$, then*

$$\sum_{M \in \mathcal{M}_n(D)} p^{\mathrm{cr}(M)} q^{\mathrm{ne}(M)} \prod_{i \in \mathrm{Left}(M)} s_i \prod_{i \in \mathrm{Long}(M)} t_i = \prod_{k=1}^{n} (t_k p^{h_k - 1} + p^{h_k - 2} q + \cdots + p q^{h_k - 2} + s_k q^{h_k - 1}). \tag{5}$$

Proof: The edge $o_k \cdot M(o_k)$ will be a right arc in exactly $w_k - 1$ nestings and exactly $h_k - w_k$ crossings in $M = \varphi(D, (w_1, \ldots, w_n))$. So, $k \in \mathrm{Long}(M)$ if and only if $w_k = 1$ while the closer that is connected to o_k is in $\mathrm{Left}(M)$ if and only if $w_k = h_k$. □

The map φ also has the following property. The definition of Rlminl was given for permutations but it extends to words in a straightforward way.

Proposition 2.3. *Let $(D, (w_1, \ldots, w_n)) \in \mathcal{WD}_n$ and $M = \varphi(D, (w_1, \ldots, w_n))$. Then*

$$\mathrm{Short}(M) = \mathrm{Rlminl}(2 - w_1, 3 - w_2, \ldots, n + 1 - w_n). \tag{6}$$

2.2 Cycles and sorting index for matchings

Let M_0 be a matching in $\mathcal{M}_n(D)$. For $M \in \mathcal{M}_n(D)$ define $\mathrm{cyc}(M, M_0)$ as the number of cycles in the graph $G = (M, M_0)$ on $2n$ vertices in which the arcs from M are drawn in the upper half-plane as usual and the arcs of M_0 are drawn in the lower half-plane, reflected about the number axis. If the openers of M are $o_1 < \cdots < o_n$, we define

$$\mathrm{Cyc}(M, M_0) = \{k : o_k \text{ is a minimal vertex in a cycle in the graph } (M, M_0)\}.$$

Figure 2 shows the calculation of cyc and Cyc for all matchings of type with respect to the nonnesting matching of that type.

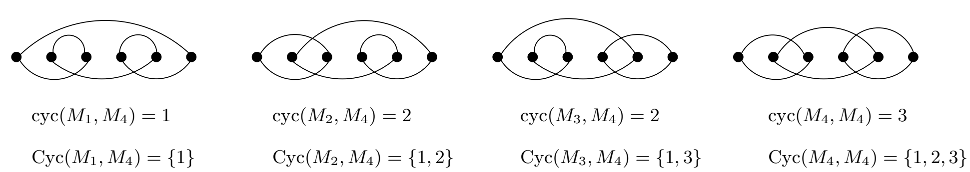

Fig. 2: Counting cycles in matchings.

For $M, M_0 \in \mathcal{M}_n(D)$, we define the sorting index of M with respect to M_0, denoted by $\mathrm{sor}(M, M_0)$, in the following way. Let $o_1 < o_2 < \cdots < o_n$ be the openers in M and M_0. We construct a sequence of matchings $M_n, M_{n-1}, \ldots, M_2, M_1$ as follows. First, set $M_n = M$. Then, if $M_k(o_k) = M_0(o_k)$, set $M_{k-1} = M_k$. Otherwise, set M_{k-1} to be the matching obtained by replacing the edges $o_k \cdot M_k(o_k)$ and $M_k(M_0(o_k)) \cdot M_0(o_k)$ in the matching M_k by the edges $o_k \cdot M_0(o_k)$ and $M_k(M_0(o_k)) \cdot M_k(o_k)$. It follows from the definition that $M_1 = M_0$. In other words, we gradually sort the matching M by reconnecting the openers to the closers as "prescribed" by M_0. Note that when swapping of edges takes place, it is always true that $M_k(M_0(o_k)) < o_k$ and therefore all the intermediary matchings we get in the process are of type

D. Define

$$\operatorname{sor}_k(M, M_0) = \begin{cases} |\{c : c > o_k, c \in [M_k(o_k), M_0(o_k)] \text{ and } M_0(c) < o_k\}|, & \text{if } M_k(o_k) \le M_0(o_k) \\ |\{c : c > o_k, c \notin (M_0(o_k), M_k(o_k)) \text{ and } M_0(c) < o_k\}|, & \text{if } M_0(o_k) < M_k(o_k) \end{cases}$$

and

$$\operatorname{sor}(M, M_0) = \sum_{k=1}^{n} \operatorname{sor}_k(M, M_0).$$

Example 2.4. *Figure 3 shows the intermediate matchings that are obtained when $M = M_6$ is sorted to $M_0 = M_1$. So,*

$$\begin{array}{lll} \operatorname{sor}_6(M, M_0) = |\{c_3, c_5, c_6\}| = 3, & \operatorname{sor}_5(M, M_0) = |\{c_3, c_5\}| = 2, & \operatorname{sor}_4(M, M_0) = |\{c_2, c_5\}| = 2, \\ \operatorname{sor}_3(M, M_0) = |\emptyset| = 0, & \operatorname{sor}_2(M, M_0) = |\{c_5\}| = 1, & \operatorname{sor}_1(M, M_0) = |\emptyset| = 0, \end{array}$$

and $\operatorname{sor}(M, M_0) = 0 + 1 + 0 + 2 + 2 + 3 = 8$.

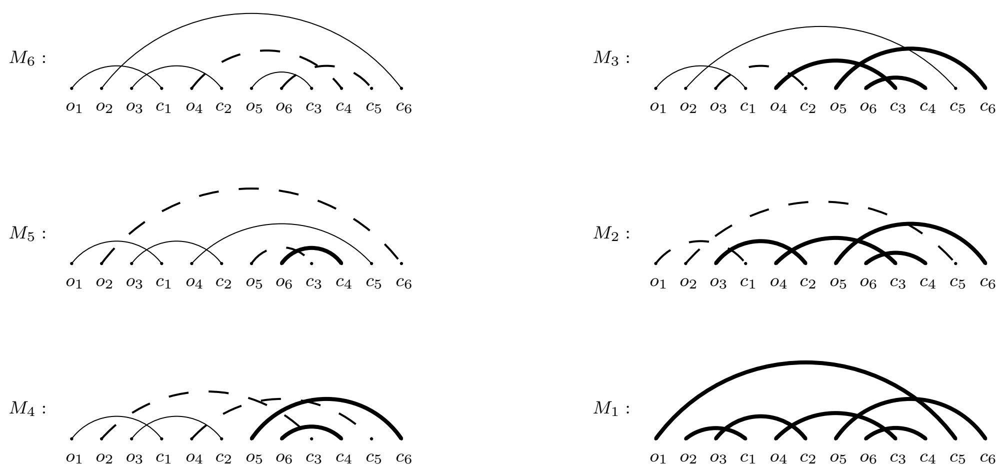

Fig. 3: Sorting of the matching $M = M_6$ to the matching $M_0 = M_1$. The dashed lines indicate arcs that are about to be swapped while the bold lines represent arcs that have been placed in correct position.

Theorem 2.5. *Let D be a Dyck path with height sequence $(h_1, \ldots, h_n)$. For each $M_0 \in \mathcal{M}_n(D)$, there is a bijection*

$$\phi : \{(w_1, w_2, \ldots, w_n) : 1 \le w_i \le h_i\} \to \mathcal{M}_n(D)$$

which depends on M_0 such that

(a) $\operatorname{sor}(\phi(w_1, \ldots, w_n), M_0) = \sum_{i=1}^{n}(w_i - 1)$,

(b) $\operatorname{Cyc}(\phi(w_1, \ldots, w_n), M_0) = \{k : w_k = 1\}$.

Additionally, if M_0 is the unique nonnesting matching of type D, then

(c) $\mathrm{Short}(\phi(w_1, \ldots, w_n)) = \mathrm{Rlminl}(2 - w_1, 3 - w_2, \ldots, n + 1 - w_n)$.

Proof: Fix $M_0 \in \mathcal{M}_n(D)$. We construct the bijection ϕ in the following way. Draw the matching M_0 with arcs in the lower half-plane. Suppose $o_1 < \cdots < o_n$ are the openers of M_0. To construct $M = \phi(w_1, \ldots, w_n)$, we draw arcs in the upper half plane by connecting the openers from right to left to closers as follows.

Suppose that the openers $o_n, o_{n-1}, \ldots, o_{k+1}$ are already connected to a closer and denote the partial matching in the upper half-plane by N_k. To connect o_k, we consider all the closers c with the property $c > o_k$ and $M_0(c) \leq o_k$. There are exactly h_k such closers, call them candidates for o_k.

Let c_{k_0} be the closer which is w_k-th on the list when all those h_k candidates are listed starting from $M_0(o_k)$ and then going cyclically to left. If c_{k_0} is not connected to an opener by an arc in the upper half-plane, draw the arc $o_k \cdot c_{k_0}$. Otherwise, there is a maximal path in the graph of the type: $c_{k_0}, N_k(c_{k_0}), M_0(N_k(c_{k_0})), N_k(M_0(N_k(c_{k_0}))), \ldots, c^*$ which starts with c_{k_0}, follows arcs in N_k and M_0 alternately and ends with a closer c^* which has not been connected to an opener yet (see Figure 4). Due to the order in which we have been drawing the arcs in the upper half-plane, all vertices in the aforementioned path are to the right of o_k. In particular, c^* is to the right of o_k and is not one of the candidates for o_k. Draw an arc in the upper half-plane connecting o_k to c^*. After all openers are connected in this manner, the resulting matching in the upper half-plane is $M = \phi(w_1, \ldots, w_n)$.

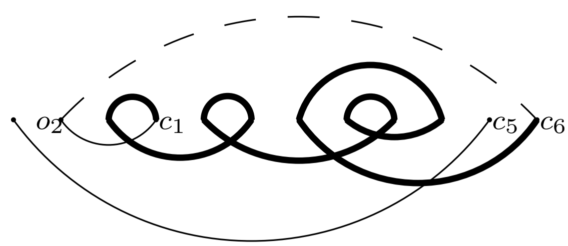

Fig. 4: The solid arcs in the top half-plane represent the partial matching N_2. The candidates for o_2 are c_1 and c_5. If $w_2 = 1$, o_2 will try to connect to c_1, but since it is already connected to an opener, we follow the bold path that starts with c_1 to reach $c^* = c_6$ and connect it to o_2.

Let $M_n = M, M_{n-1}, \ldots, M_2, M_1 = M_0$ be the intermediary sequence of matchings constructed when sorting M to M_0. Then $M_k(o_k)$ is exactly the closer c_{k_0} defined above. This means that $\mathrm{sor}_k(M, M_0) = w_k - 1$ and therefore $\mathrm{sor}(M, M_0) = \sum_{k=1}^{n}(w_k - 1)$. This property also gives us a way of finding the sequence $(w_1, \ldots, w_n)$ which corresponds to a given $M \in \mathcal{M}_n(D)$. Namely, $w_k = \mathrm{sor}_k(M, M_0) + 1$.

To prove the second property of ϕ, we analyze when connecting o_k by an arc will close a cycle. There are two cases.

1. The closer c_{k_0} which was w_k-th on the list of candidates for o_k was not incident to an arc in the partial matching N_k and we drew the arc $o_k \cdot c_{k_0}$. If $w_k = 1$, then $c_{k_0} = M_0(o_k)$ and the arcs connecting o_k and c_{k_0} in the upper and lower half-planes close a cycle. Otherwise, $M_0(c_{k_0}) < o_k$ and therefore $M_0(c_{k_0})$ is not incident to an arc in N_k and the arc $o_k \cdot c_{k_0}$ will not close a cycle.

2. The closer c_{k_0} which was w_k-th on the list of candidates for o_k was incident to an arc in the partial matching N_k and we drew the arc $o_k \cdot c^*$. If $w_k = 1$, the path traced from c_{k_0} to c^*, the arc $o_k \cdot c_{k_0}$ in M_0, and the newly added arc $o_k \cdot c^*$ form a cycle. Otherwise, connecting o_k to c^* does not close a cycle since the opener $M_0(c_{k_0})$ is in the same connected component of the graph (M, M_0) as o_k, but is not connected to a closer yet, since $M_0(c_{k_0}) < o_k$.

We conclude that a cycle is closed exactly when $w_k = 1$ and therefore

$$\mathrm{Cyc}(\phi(w_1, \dots, w_n), M_0) = \{k : w_k = 1\}.$$

Finally, we prove the third property of ϕ. If M_0 is a nonnesting matching, its edges are $o_k \cdot c_k$ where the openers and closers are indexed in ascending order. Let $M = \phi(w_1, \dots, w_n)$. The following observations are helpful. When connecting o_k in the construction of M, the first choice for o_k, i.e., the w_k-th candidate for o_k is exactly c_{k+1-w_k}. Also, $M(o_k) \geq c_{k+1-w_k}$. Furthermore, if c_k was not a candidate for $M(c_k)$, i.e. if the edge c_k was chosen as a partner for $M(c_k)$ by following a path in the graph as described above, then $k \notin \mathrm{Short}(M)$. Namely the edge $M(c_{k_0}) \cdot c_{k_0}$, where c_{k_0} was the first choice when the opener $M(c_k)$ was connected in the construction of M, is nested below it.

For a number $k \in [n]$ there are three possibilities:

1. $k \notin \{2 - w_1, 3 - w_2, \dots, n + 1 - w_n\}$

   In this case, c_k was not a first choice for any of the openers and therefore must have been connected to an opener by following a path in the graph (M, M_0). It follows from the observation above that $k \notin \mathrm{Short}(M)$.

2. $k \in \{2 - w_1, 3 - w_2, \dots, n + 1 - w_n\}$ and $k \in \mathrm{Rlminl}(2 - w_1, 3 - w_2, \dots, n + 1 - w_n)$

   Then c_k was a first choice for at least one opener. Let o be the largest one. Then all openers to the right of o got connected to a closer which is greater than c_k, so no edge is nested below $o \cdot c_k \in M$. Consequently, $k \in \mathrm{Short}(M)$.

3. $k \in \{2 - w_1, 3 - w_2, \dots, n + 1 - w_n\}$ but $k \notin \mathrm{Rlminl}(2 - w_1, 3 - w_2, \dots, n + 1 - w_n)$

   In this case, let $m + 1 - w_m$ be the rightmost number in the sequence $(2 - w_1, \dots, n + 1 - w_n)$ which is smaller than k. It is necessarily to the right of k in this sequence and belongs to $\mathrm{Rlminl}(2 - w_1, \dots, n + 1 - w_n)$. This implies that the edge $o_m \cdot c_{m+1-w_m}$ is in M, while $M(o_l) > c_k$ for all $l > m$. So, $M(c_k) < o_m$ and therefore the edge $o_m \cdot c_{m+1-w_m}$ is nested below $M(c_k) \cdot c_k$, which means that $k \notin \mathrm{Rlminl}(2 - w_1, 3 - w_2, \dots, n + 1 - w_n)$.

□

As a consequence, we get the following generating functions. Note that their explicit formulas imply that in fact the distributions are independent of the choice of M_0.

Corollary 2.6. *Let $M_0 \in \mathcal{M}_n(D)$ and let $(h_1, \dots, h_n)$ be the height sequence of D. Then*

$$\sum_{M \in \mathcal{M}_n(D)} q^{\mathrm{sor}(M, M_0)} \prod_{i \in \mathrm{Cyc}(M, M_0)} t_i = \prod_{k=1}^{n} (t_k + q + \cdots + q^{h_k - 1}). \tag{7}$$

Combining Theorem 2.2 and Corollary 2.6 we get the following corollary.

Corollary 2.7. *Let $M_0 \in \mathcal{M}_n(D)$ and let $(h_1, \ldots, h_n)$ be the height sequence of D. Then*

$$\sum_{M \in \mathcal{M}_n(D)} q^{\mathrm{sor}(M,M_0)} \prod_{i \in \mathrm{Cyc}(M,M_0)} t_i = \sum_{M \in \mathcal{M}_n(D)} q^{\mathrm{ne}(M)} \prod_{i \in \mathrm{Long}(M)} t_i.$$

Corollary 2.8. *If M_0 is the unique nonnesting matching of type D then the multisets*

$$\{(\mathrm{sor}(M, M_0), \mathrm{Cyc}(M, M_0), \mathrm{Short}(M)) : M \in \mathcal{M}_n(D)\}$$

and

$$\{(\mathrm{ne}(M), \mathrm{Long}(M), \mathrm{Short}(M)) : M \in \mathcal{M}_n(D)\}$$

are equal.

2.3 Connections with restricted permutations

For a fixed n, let $\mathbf{r}$ denote the non-decreasing sequence of integers $1 \le r_1 \le r_2 \le \cdots \le r_n \le n$. Let

$$S_{\mathbf{r}} = \{\sigma \in S_n : \sigma(k) \le r_k, 1 \le k \le n\}.$$

Note that $S_{\mathbf{r}} \neq \emptyset$ precisely when $r_k \ge k$, for all k, so we will consider only the sequences that satisfy this condition without explicitly mentioning it. Let $D(\mathbf{r})$ be the unique Dyck path whose k-th fall is preceded by exactly r_k rises. Consider the following bijection $f_{\mathbf{r}} : S_{\mathbf{r}} \to \mathcal{M}_n(D(\mathbf{r}))$. If $\sigma \in S_{\mathbf{r}}$, then $f_{\mathbf{r}}(\sigma)$ is the matching in $\mathcal{M}_n(D(\mathbf{r}))$ with edges $o_{\sigma(k)} \cdot c_k$, where $o_1 < \cdots < o_n$ are the openers and $c_1 < \cdots < c_n$ are the closers. It is not difficult to see that $f_{\mathbf{r}}$ is well defined and that it is a bijection.

Two arcs $o_{\sigma(j)} \cdot c_j$ and $o_{\sigma(k)} \cdot c_k$ in $f_{\mathbf{r}}(\sigma)$ with $j < k$ form a nesting if and only if $\sigma(j) > \sigma(k)$. So, $\mathrm{ne}(f_{\mathbf{r}}(\sigma)) = \mathrm{inv}(\sigma)$. Moreover, $\sigma(j) \in \mathrm{Rlminl}(\sigma)$ if and only if $\sigma(j)$ does not form an inversion with a $\sigma(k)$ for any $k > j$, which means if and only if $o_{\sigma(j)} \cdot c_j$ is not nested within anything in $f_{\mathbf{r}}(\sigma)$, i.e., $\sigma(j) \in \mathrm{Long}(f_{\mathbf{r}}(\sigma))$. From Theorem 2.2 we get the following corollary.

Corollary 2.9. *Let $\mathbf{r}$ be a non-decreasing sequence of integers $1 \le r_1 \le r_2 \le \cdots \le r_n \le n$ with $r_k \ge k$, for all k. Then*

$$\sum_{\sigma \in S_{\mathbf{r}}} q^{\mathrm{inv}(\sigma)} \prod_{i \in \mathrm{Rlminl}(\sigma)} t_i = \prod_{k=1}^{n} (t_k + q + q^2 + \cdots + q^{h_k - 1})$$

where $(h_1, \ldots, h_n)$ is the height sequence of $D(\mathbf{r})$. In particular,

$$\sum_{\sigma \in S_{\mathbf{r}}} q^{\mathrm{inv}(\sigma)} t^{\mathrm{rlminl}(\sigma)} = \prod_{k=1}^{n} (t + q + q^2 + \cdots + q^{r_k - k}).$$

Proof: The first result follows directly from the discussion above and Theorem 2.2. For the second equality, note that the height sequence $(h_1, \ldots, h_n)$ of the Dyck path $D(\mathbf{r})$ is a permutation of the sequence of the heights of the falls in $D(\mathbf{r})$, where the height of a fall is the y-coordinate of the higher end of the corresponding $(1, -1)$ step. The height of the k-th fall is easily seen to be $r_k - k + 1$. $\square$

In particular, when $r_1 = r_2 = \cdots = r_n = n$, we have $S_{\mathbf{r}} = S_n$. The height sequence of $D(\mathbf{r})$ is $(1, 2, \ldots, n)$ and we recover the result of Björner and Wachs about the distribution of $(\mathrm{inv}, \mathrm{Rlmin})$ given in (2).

If $M_0 \in \mathcal{M}(D(\mathbf{r}))$ the sorting index $\mathrm{sor}(\ \cdot\ , M_0)$ induces a permutation statistic on $S_\mathbf{r}$. Namely, if $\sigma, \sigma_0 \in S_\mathbf{r}$, define
$$\mathrm{sor}_\mathbf{r}(\sigma, \sigma_0) = \mathrm{sor}(f_\mathbf{r}^{-1}(\sigma), f_\mathbf{r}^{-1}(\sigma_0)).$$

Equivalently, the statistic $\mathrm{sor}_\mathbf{r}(\sigma, \sigma_0)$ on $S_\mathbf{r}$ can be defined directly via a sorting algorithm similar to Straight Selection Sort. Namely, permute the elements in $\sigma \in S_\mathbf{r}$ by applying transpositions which place the largest element n in position $\sigma_0^{-1}(n)$, then the element $n-1$ in position $\sigma_0^{-1}(n-1)$, etc. Let $\sigma_n = \sigma, \sigma_{n-1}, \ldots, \sigma_1 = \sigma_0$, be the sequence of permutations obtained in this way. Specifically, $\sigma_k^{-1}(i) = \sigma_0^{-1}(i)$ for $i > k$, and σ_{k-1} is obtained by swapping k and $\sigma_k(\sigma_0^{-1}(k))$ in σ_k.

Let $l = \sigma_k^{-1}(k)$ and $m = \sigma_0^{-1}(k)$. Define

$$a_k = \begin{cases} |\{i : l \le i \le m, \sigma_0(i) < k\}|, & l < m \\ 0, & l = m \\ |\{i : r_i \ge k, i \notin (m, l), \sigma_0(i) < k\}|, & l > m. \end{cases} \tag{8}$$

Then
$$\mathrm{sor}_\mathbf{r}(\sigma, \sigma_0) = \sum_{k=1}^{n} a_k.$$

Note that, $\mathrm{sor}_\mathbf{r}(\sigma, \sigma_0)$ in general depends on $\mathbf{r}$. However, the case when σ_0 is the identity permutation is an exception.

Lemma 2.10. *Let* $\mathbf{r}$ *be a non-decreasing sequence of integers* $1 \le r_1 \le r_2 \le \cdots \le r_n \le n$ *with* $r_k \ge k$, *for all* k. *Let* $\sigma \in S_\mathbf{r}$. *Then*
$$\mathrm{sor}_\mathbf{r}(\sigma, \mathbf{id}) = \mathrm{sor}(\sigma).$$

Proof: First note that the case $l > m$ in (8) cannot occur. Namely, in the case when $\sigma_0 = \mathbf{id}$, we have $m = k$ and if $l > k$, $\sigma_k^{-1}(l) = \sigma_0^{-1}(l) = l$. This contradicts $l = \sigma_k^{-1}(k)$. Therefore, the definition of a_k simplifies to
$$a_k = |\{i : l \le i < k\}|.$$
This is precisely the "distance" that k travels when being placed in its correct position with the Straight Selection Sort algorithm. □

Corollary 2.11. *Let* $\mathbf{r}$ *be a non-decreasing sequence of integers* $1 \le r_1 \le r_2 \le \cdots \le r_n \le n$ *with* $r_k \ge k$, *for all* k. *Let* $\sigma_0 \in S_\mathbf{r}$. *Then*

$$\sum_{\sigma \in S_\mathbf{r}} q^{\mathrm{sor}_\mathbf{r}(\sigma, \sigma_0)} \prod_{i \in \mathrm{Cyc}(\sigma\sigma_0^{-1})} t_i = \prod_{i=1}^{n} (t_i + q + \cdots + q^{h_i - 1}), \tag{9}$$

where $(h_1, \ldots, h_n)$ *is the height sequence of* $D(\mathbf{r})$ *and* $\mathrm{Cyc}(\sigma)$ *is the set of the minimal elements in the cycles of* σ. *In particular,*

$$\sum_{\sigma \in S_\mathbf{r}} q^{\mathrm{sor}(\sigma)} \prod_{i \in \mathrm{Cyc}(\sigma)} t_i = \prod_{i=1}^{n} (t_k + q + \cdots + q^{h_k - 1}) \tag{10}$$

and

$$\sum_{\sigma \in S_{\mathbf{r}}} q^{\mathrm{sor}(\sigma)} t^{\mathrm{cyc}(\sigma)} = \sum_{\sigma \in S_{\mathbf{r}}} q^{\mathrm{inv}(\sigma)} t^{\mathrm{rlminl}(\sigma)} \tag{11}$$

Proof: Let $f_{\mathbf{r}}(\sigma_0) = M_0$ and $f_{\mathbf{r}}(\sigma) = M$. The cycle $k \to \sigma_0\sigma^{-1}(k) \to \cdots \to (\sigma_0\sigma^{-1})^s(k) = k$ of the permutation $\sigma_0\sigma^{-1}$ corresponds to the cycle $o_k \frown M(o_k) \smile M_0(M(o_k)) \frown \cdots \smile o_k$ in the graph (M, M_0). So, $k \in \mathrm{Cyc}(\sigma_0\sigma^{-1})$ if and only if $k \in \mathrm{Cyc}(M, M_0)$. Now, (9) follows from (7) and the fact that the cycles of $\sigma\sigma_0^{-1}$ are equal to the cycles of $\sigma_0\sigma^{-1}$ reversed. Since **id** $\in S_{\mathbf{r}}$ for every sequence $\mathbf{r}$, we get (10) as a corollary of Lemma 2.10. $\square$

Let $\mathrm{Lrmaxp}(\sigma)$ denote the set of left-to-right maximum places in the permutation σ, i.e,

$$\mathrm{Lrmaxp}(\sigma) = \{k : \sigma(k) > \sigma(j) \text{ for all } j < k\}.$$

From Corollary 2.8 we get the following result for restricted permutations.

Corollary 2.12. *The triples* $(\mathrm{inv}, \mathrm{Rlminl}, \mathrm{Lrmaxp})$ *and* $(\mathrm{sor}, \mathrm{Cyc}, \mathrm{Lrmaxp})$ *are equidistributed on* $S_{\mathbf{r}}$. *That is, the multisets*

$$\{(\mathrm{inv}(\sigma), \mathrm{Rlminl}(\sigma), \mathrm{Lrmaxp}(\sigma)) : \sigma \in S_{\mathbf{r}}\}$$

and

$$\{(\mathrm{sor}(\sigma), \mathrm{Cyc}(\sigma), \mathrm{Lrmaxp}(\sigma)) : \sigma \in S_{\mathbf{r}}\}$$

are equal.

The equidistribution of the pairs $(\mathrm{Rlminl}, \mathrm{Lrmaxp})$ and $(\mathrm{Cyc}, \mathrm{Lrmaxp})$ on $S_{\mathbf{r}}$ for the special case when the corresponding Dyck path $D(\mathbf{r})$ is of the form $u^{k_1}d^{k_1}u^{k_2}d^{k_2}\cdots u^{k_s}d^{k_s}$ was shown by Foata and Han [3] .

Corollary 2.13. *Let* $\sigma_0 \in S_{\mathbf{r}}$. *Then*

$$\sum_{\sigma \in S_{\mathbf{r}}} t^{\mathrm{cyc}(\sigma\sigma_0^{-1})} = \prod_{k=1}^{n} (t + r_k - k). \tag{12}$$

In particular, the left-hand side of (12) *does not depend on* σ_0.

We remark that the sets $\{\sigma\sigma_0^{-1} : \sigma \in S_{\mathbf{r}}\}$ and $S_{\mathbf{r}}$ are in general not equal. For example, let $\sigma_0 = 143265 \in S_{[4,4,4,6,6,6]}$. Then $\sigma = 231546 \in S_{[4,4,4,6,6,6]}$ but $\sigma\sigma_0^{-1} = 251364 \notin S_{[4,4,4,6,6,6]}$.

The polynomial $\prod_{k=1}^{n}(t + r_k - k)$ is well-known in rook theory. It is equal [4] to the polynomial

$$\sum_{k=0}^{n} r_{n-k}(t-1)(t-2)\cdots(t-k)$$

where r_k is the number of placements of k non-atacking rooks on a Ferrers board with rows of length $r_1, r_2, \ldots, r_n$.

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Crossings and Nestings for Arc-Coloured Permutations

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Abstract The equidistribution of many crossing and nesting statistics exists in several combinatorial objects like matchings, set partitions, permutations, and embedded labelled graphs. The involutions switching nesting and crossing numbers for set partitions given by Krattenthaler, also by Chen, Deng, Du, Stanley, and Yan, and for permutations given by Burrill, Mishna, and Post involved passing through tableau-like objects. Recently, Chen and Guo for matchings, and Marberg for set partitions extended the result to coloured arc annotated diagrams. We prove that symmetric joint distribution continues to hold for arc-coloured permutations. As in Marberg's recent work, but through a different interpretation, we also conclude that the ordinary generating functions for all j-noncrossing, k-nonnesting, r-coloured permutations according to size n are rational functions. We use the interpretation to automate the generation of these rational series for both noncrossing and nonnesting coloured set partitions and permutations.

L'équidistribution de plusieurs statistiques décrites en termes d'emboitements et de chevauchements d'arcs s'observes dans plusieurs familles d'objects combinatoires, tels que les couplages, partitions d'ensembles, permutations et graphes étiquetés. L'involution échangeant le nombre d'emboitements et de chevauchements dans les partitions d'ensemble due à Krattenthaler, et aussi Chen, Deng, Du, Stanley et Yan, et l'involution similaire dans les permutations due à Burrill, Mishna et Post, requièrent d'utiliser des objets de type tableaux. Récemment, Chen et Guo pour les couplages, et Marberg pour les partitions d'ensembles, ont étendu ces résultats au cas de diagrammes arc-annotés coloriés. Nous démontrons que la propriété d'équidistribution s'observe est aussi vraie dans le cas de permutations aux arcs coloriés. Tout comme dans le travail résent de Marberg, mais via un autre chemin, nous montrons que les séries génératrices ordinaires des permutations r-coloriées ayant au plus j chevauchements et k emboitements, comptées selon la taille n, sont des fonctions rationnelles. Nous décrivons aussi des algorithmes permettant de calculer ces fonctions rationnelles pour les partitions d'ensembles et les permutations coloriées sans emboitement ou sans chevauchement.

Keywords: arc-coloured permutation, crossing, nesting, bijection, enumeration, tableau, generating tree, finite state automaton, transfer matrix, automation

1 Introduction

Crossing and nesting statistics have intrigued combinatorialists for many decades. For example, it is well known that Catalan numbers, $c_n = \frac{1}{n+1}\binom{2n}{n}$, count the number of noncrossing matchings on $[2n]$ which is also the number of nonnesting matchings of the same size. The concept of crossing and nesting

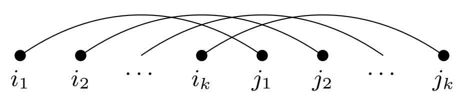


Figure 1: The arc diagram of a k-crossing

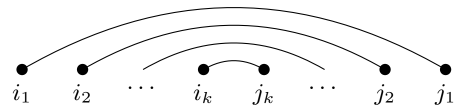


Figure 2: The arc diagram of a k-nesting

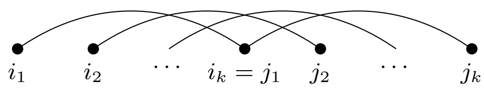


Figure 3: The arc diagram of an *enhanced* k-crossing

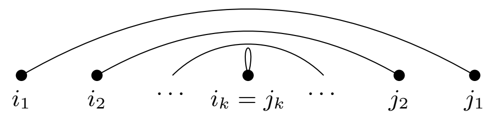


Figure 4: The arc diagram of an *enhanced* k-nesting

was then extended to higher numbers where symmetric joint distribution continues to hold not only for matchings (8), but also for set partitions (4, 9), labelled graphs (7), set partitions of classical types (13), and permutations (2). In all cases, bijective proofs were given; and for some, generating functions were found.

Inspired by recent works of Chen and Guo (3) on coloured matchings and Marberg (10) on coloured set partitions, we give a bijection to establish symmetric joint distribution of crossing and nesting statistics for *arc-coloured permutations*. We also show that the ordinary generating functions for j-noncrossing, k-nonnesting, r-coloured permutations according to size n are rational functions.

1.1 Definitions and Terminology

A permutation S of the set $[n] := \{1, 2, \ldots, n\}$ is a bijection from $[n]$ to itself, $\sigma : [n] \to [n]$. Using two-line notation, we can write $S = \left(\begin{smallmatrix} 1 & 2 & 3 & \cdots & n \\ \sigma(1) & \sigma(2) & \sigma(3) & \cdots & \sigma(n) \end{smallmatrix}\right)$. An *arc annotated diagram* is a labelled graph on n vertices drawn horizontally, labelled left to right consecutively such that $\mathrm{Arc}(i, j)$ joins vertex i to vertex j. A permutation has a representation as an arc annotated diagram where $\mathrm{Arc}(i, \sigma(i))$ is drawn as an upper arc for $\sigma(i) \geq i$, and a lower arc for $\sigma(i) < i$. Note that the dissymmetry draws a fixed point in S as an upper loop. When this diagram is restricted to only the upper arcs (or lower arcs) with all n vertices, then it also represents a set partition of $[n]$. Separately, we call these *upper* and *lower arc diagrams* of a permutation. From this diagram, we define a *k-crossing* (resp. *k-nesting*) as k arcs $\{(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)\}$ all mutually cross, or $i_1 < i_2 < \cdots < i_k < j_1 < j_2 < \cdots < j_k$ (resp. nest, i. e. $i_1 < i_2 < \cdots < i_k < j_k < j_{k-1} < \cdots < j_1$) as shown in Figure 1 (resp. Figure 2). We also need a variant: *enhanced k-crossing* (resp. *enhanced k-nesting*) where $i_1 < i_2 < \cdots < i_k \leq j_1 < j_2 < \cdots < j_k$ (resp. $i_1 < i_2 < \cdots < i_k \leq j_k < j_{k-1} < \cdots < j_1$) as shown in Figure 3 (resp. Figure 4).

We need both notions of crossings and nestings for permutations because the *enhanced* definitions are used for upper arc diagrams whereas the other definitions (without *enhanced*), for lower arc diagrams. This is in accordance with the literature (6) on permutation statistics for weak exceedances and pattern avoidance. We define the *crossing number*, $\mathrm{cr}(S) = j$ (resp. *nesting number*, $\mathrm{ne}(S) = k$) of a permutation S as the maximum j (resp. k) such that S has a j-enhanced crossing (resp. k-enhanced nesting) in the upper arc diagram or a j-crossing (resp. k-nesting) in the lower arc diagram. When a permutation S does not have a j-(enhanced)-crossing (resp. k-(enhanced)-nesting), then we say S is j-noncrossing (resp. k-nonnesting). Burrill, Mishna, and Post (2) gave an involution mapping between the set of permutations of

$[n]$ with $\mathrm{cr}(S) = j$ and $\mathrm{ne}(S) = k$ and those with $\mathrm{cr}(S) = k$ and $\mathrm{ne}(S) = j$, thus extending the result of symmetric joint distribution for matchings and set partitions of Chen, Deng, Du, Stanley, and Yan (4) and Krattenthaler (9) to permutations.

Next, Chen and Guo (3) generalized symmetric equidistribution of crossing and nesting statistics to *coloured* complete matchings. Most recently, Marberg (10) extended the result to coloured set partitions with a novel way of proving that the ordinary generating functions of j-noncrossing, k-nonnesting, r-coloured partitions according to size n are rational functions. We extend their results to r-arc-coloured permutations, or r-coloured permutations in short.

Coloured permutations are generalizations of permutations represented as arc annotated diagrams. Once the arcs are coloured to satisfy j-noncrossing and k-nonnesting conditions for each colour class, the resulting arc diagrams can be represented in the topological graph theoretic book embedding setting (11), each colour on a separate page while the vertices are on the spine of the book. The differences are two fold: each page satisfies the crossing/nesting conditions instead of finding a minimum number of pages, noncrossing on each page, to represent a given (non-planar) graph, and the number of pages is not necessarily minimal with respect to the crossing/nesting conditions. Secondary RNA structures with different bonding energies have been analysed in the book embedding setting, naturally represented as coloured set partitions (5); however, arc-coloured permutations have yet to find a natural application.

Some caution on terminology is in order here. Group properties of coloured permutations have been widely studied since the 1990's (1, 15), but there the colours are assigned to *vertices* instead of *arcs*.

1.2 Main Theorem

Since crossing and nesting statistics involves arcs, we define an r*-coloured permutation* parallel to (10) as a pair, (S, ϕ) consisting of a permutation of $[n]$ and an arc-colour assigning map $\phi : \mathrm{Arc}(S) \to [r]$, and use a capital Greek letter, Σ, to denote these objects. We say Σ has a k-crossing (resp. k-nesting) if k arcs of the *same* colour cross (resp. nest). As always throughout this paper, *enhanced* statistics is applied to *upper* arc diagrams while *non-enhanced* for *lower* arc diagrams of permutations. Define $\mathrm{cr}(\Sigma)$ (resp. $\mathrm{ne}(\Sigma)$) as the maximum integer k such that Σ has a k-crossing (resp. k-nesting). The bijection of (2) can be extended to establish symmetric joint distribution of the numbers $\mathrm{cr}(\Sigma)$ and $\mathrm{ne}(\Sigma)$ over r-coloured permutations preserving opener and closer sequences (equivalently, sets of minimal and maximal elements of each block when upper arc and lower arc diagrams are viewed separately as set partitions).

More formally, vertices of a permutation are of five types, an opener ( ), a closer ( ), a fixed point ( ) , an upper transitory ( ), and a lower transitory( ). For a particular Σ, restricting to only one colour, both upper arc and lower arc diagrams can be seen as set partitions whose minimal block elements are the openers, and maximal block elements are the closers. For upper arc diagrams, both a fixed point and an upper transitory contribute to the set of minimal (opener) and the set of maximal (closer) elements over blocks of the set partition. Lower arc diagrams are set partitions in Marberg's partition setting, thus Theorem1.1 and Corollary 1.2 of (10) apply exactly here.

Given an r-coloured permutation $\Sigma = (S, \phi)$, let the set of openers (resp. the set of closers) be $\mathcal{O}(\Sigma)$ (resp. $\mathcal{C}(\Sigma)$) of the uncoloured permutation, S. For all positive integers, j and k, and subsets $O, C \subseteq [n]$, define $\mathrm{NCN}_{j,k}^{O,C}(n, r)$ to be the number of r-coloured permutations Σ of $[n]$ with $\mathrm{cr}(\Sigma) < j$, $\mathrm{ne}(\Sigma) < k$, $\mathcal{O}(\Sigma) = O$, and $\mathcal{C}(\Sigma) = C$. Then Theorem 1 is analogous to Theorem 1.1 in (4, 10) for r-coloured permutations.

Theorem 1 *For all positive integers, j and k, and subsets O, $C \subseteq [n]$,* $\mathrm{NCN}_{j,k}^{O,C}(n,r) = \mathrm{NCN}_{k,j}^{O,C}(n,r)$.

As customary in the literature, we let $\mathrm{NCN}_{j,k}(n,r)$ denote the number of all r-coloured, j-noncrossing, k-nonnesting permutations of $[n]$. Summing both sides of Theorem 1 over all O, $C \subseteq [n]$ gives the generalization of (4, 10) for Corollary 1. We also let $\mathrm{NC}_k(n,r)$ (resp. $\mathrm{NN}_k(n,r)$) denote the number of k-noncrossing (resp. k-nonnesting) r-coloured permutations on $[n]$.

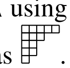
Corollary 1 *For all integers, j, k, n, r,* $\mathrm{NCN}_{j,k}(n,r) = \mathrm{NCN}_{k,j}(n,r)$ *and* $\mathrm{NC}_k(n,r) = \mathrm{NN}_k(n,r)$.

1.3 Plan

The tools needed for the proof of Theorem 1 are given in Section 2. Section 3 gives the proof of Theorem 1 combining essential ingredients of both (2, 10) with the added care of managing both upper and lower arc diagrams simultaneously where both notions of crossing and nesting are applied. The transfer matrix approach Marberg used to establish the rationality of the ordinary generating function, $\sum_{n\geq 0} \mathrm{NCN}_{j,k}(n+1,r)x^n$ for set partitions of size $n+1$ is through translating the original problem to counting all closed walks of n-steps with certain column and row length restrictions (according to j,k) for each component from $\emptyset \in \mathbf{Y}^r$, that is, r copies of the Hasse diagram of the Young lattice. This idea cannot be extended to permutations on $(\mathbf{Y}^r, \mathbf{Y}^r)$ because upper arc diagrams are dependent on lower arc diagrams. However, another interpretation of Marberg's multigraphs $\mathcal{G}_{j,k,r}$ in terms of the types of vertices and colours of edges leads to the multigraphs for r-coloured permutations which permits the application of transfer matrix method to draw the same conclusion: The ordinary generating function, $\sum_{n\geq 0} \mathrm{NCN}_{j,k}(n,r)x^n$ for j-noncrossing, k-nonnesting, r-coloured permutations is rational. The combination of the method of generating trees and finite state automata in the interpretation can be extended to other combinatorial objects where both crossing and nesting statistics are bounded, thus leading to the same conclusion that the corresponding generating functions are rational.

2 Background

The proof of Theorem 1 requires working knowledge of the theory of integer partition, especially its representation as Young diagrams, the Hasse diagram of the Young lattice, and the RSK-algorithm for filling positive integers to obtain the beginning of some standard Young tableau. We refer the reader to Volume 2 of Stanley's Enumerative Combinatorics (14) for more details.

Define a partition of $n \in \mathbf{N}$ to be a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbf{N}^k$ such that $\sum_{i=1}^{k} \lambda_i = n$, and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$. If λ is a partition of n, we write $\lambda \vdash n$ or $|\lambda| = n$. The non-zero terms λ_i are called the parts of λ, and we say λ has k parts if $\lambda_k > 0$. We can draw λ using a left-justified array of boxes with λ_i boxes in row i. For example, $\lambda = (5,3,2,2,1)$ is drawn as . This representation is the Young diagram of a partition. To "add a box" to a partition λ means to obtain a partition μ such that $|\lambda| + 1 = |\mu|$, and λ's Young diagram is included in that of μ. This inclusion induces a partial order on the set of partitions of non-negative integers, denoted by $\mathbf{Y}$, or the Young lattice. When we place integers $1, 2, \ldots, n$ in all n boxes of a Young diagram so that entries increase in each row and column, we produce a standard Young tableau, abbreviated as SYT. As one builds an SYT from the empty set through the process of adding a box at a time, a sequence of integer partitions, $(\lambda^0 = \emptyset, \lambda^1, \lambda^2, \ldots, \lambda^n)$ emerges where $\lambda^{i-1} \subset \lambda^i$, and $|\lambda^i| = |\lambda^{i-1}| + 1$. In addition to adding a box, we include "deleting a box" and "doing nothing" for the following four types in Definition 1.

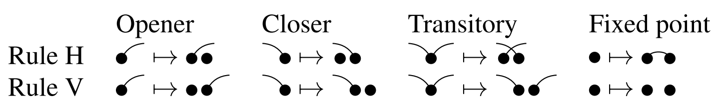
Definition 1 *We define four types of sequences of tableaux, $T = (\lambda^0 = \emptyset, \lambda^1, \lambda^2, \ldots, \lambda^n)$, where $\lambda^0 = \lambda^n = \emptyset$ such that λ^i is obtained from λ^{i-1} for each $i \in [n]$ by one of the three actions: adding a box, deleting a box, or doing nothing.*

1. *A* semi-oscillating tableau *is any such sequence T.*
2. *An* oscillating tableau *has distinct neighbouring λ^i's.*
3. *A* vacillating tableau *is any such sequence T which has $\lambda^{i-1} \subseteq \lambda^i$ when i is even, and $\lambda^{i-1} \supseteq \lambda^i$ when i is odd.*
4. *A* hesitating tableau *is any such sequence T which has $\lambda^{i-1} \subseteq \lambda^i$ when i is odd, and $\lambda^{i-1} \supseteq \lambda^i$ when i is even.*

In the uncoloured case, Marberg (10) links the sequence T to an n-step walk on the Hasse diagram of the Young lattice, $\mathbf{Y}$ where "doing nothing" is also counted as a step. For his enumeration purposes, Marberg's definitions differ slightly from (4) to achieve that these n-step walks are closed walks from $\emptyset$. Though we will not walk on an ordered pair of r-tuple Hasse diagrams, we will keep the requirement that each sequence T begins and ends with $\emptyset$.

3 Proof of Main Theorem

The proof of Theorem 1 needs two local rules for changing set partitions to involutions: Rule H for hesitating tableaux tracking *enhanced* statistics in upper arcs and Rule V for vacillating tableaux.



3.1 Proof of Theorem 1

Proof: We show an involution between the set of r-coloured permutations of $[n]$ with maximal crossing number j, nesting number k and those with maximal crossing number k and nesting number j.

Given an r-coloured permutation of $[n]$, say $\Sigma = (S, \phi)$, first consider its corresponding uncoloured permutation S. Let O be $\mathcal{O}(S)$, the set of openers and C be $\mathcal{C}(S)$, the set of closers. Applying the involution of (2) results in another permutation with the same O and C while switching maximal crossing and nesting numbers.

Now for each colour class, the resulting arc diagram is no longer a permutation, but two set partitions: *enhanced* for the upper arc diagram, and *non-enhanced* for the lower arc diagram. We employ the same encoding techniques from (2):

Step 1 Translate the upper arc diagram into a hesitating tableau sequence, and the lower arc diagram into a vacillating tableau sequence.

Step 2 Perform a component-wise transpose to each tableau sequence.

Step 3 Apply reverse RSK to fill each tableau in the sequence from the right to the left.

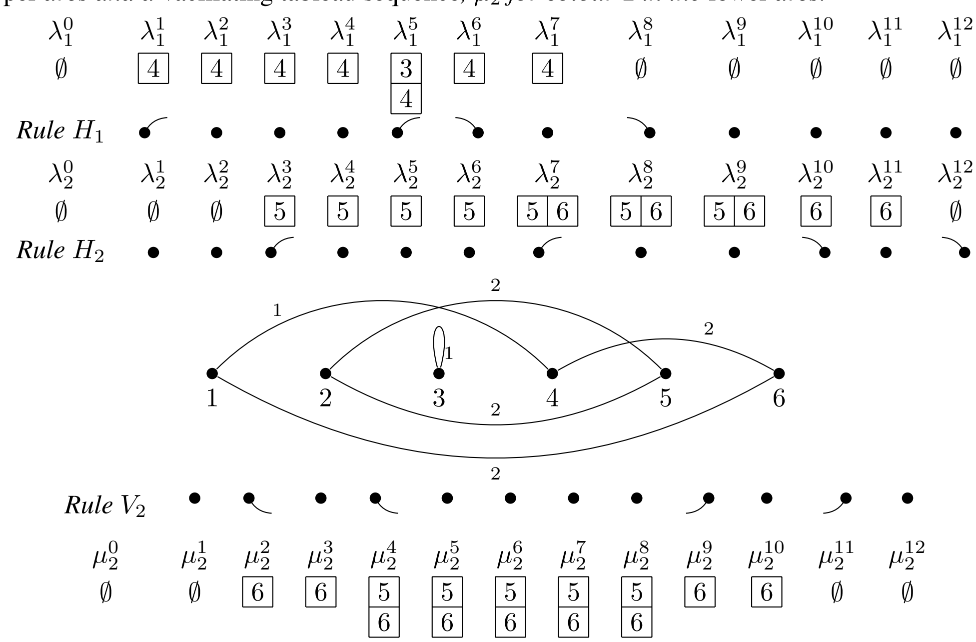
Step 4 Translate the newly filled sequence of tableaux back to arc diagrams according to its own rule.

Thus, we obtain the resulting arc diagram with its upper and lower arc components where maximal crossing and nesting numbers are switched because the bijections of (2, 4, 9) interchange maximal column length with maximal row length while preserving sets of maximal and minimal block elements. This interchange achieved through taking the conjugate (transpose) of each tableau translates to the switching of maximal nesting and crossing numbers while preserving the sets of openers and closers. The preservation of these sets when restricted to one colour of arcs permits the involution to be applied separately to all arcs of the same colour, one colour at a time, without interfering with the sets of openers and closers from other colour classes. Finally, the combination of all r involutions, one for each colour, produces the desired r-coloured permutation such that for each colour, crossing number and nesting number are switched. If the original r-coloured Σ is j-noncrossing and k-nonnesting, then its image after the r-fold involution is j-nonnesting and k-noncrossing. □

3.2 An example of a 2-coloured permutation

We show a 2-coloured permutation where we apply the involution of the proof of Theorem 1 to find its image.

Example 1 *A permutation encoded by a* hesitating tableau sequence*, λ_1 for colour 1, λ_2 for colour 2 in the* upper *arcs and a* vacillating tableau sequence*, μ_2 for colour 2 in the* lower *arcs.*



The result of transposing every tableau in each sequence λ_1, λ_2, and μ_2, and filling the tableau from the right is the following 2-coloured permutation in Figure 5.

4 Enumeration of r-coloured permutations

Before we enumerate r-coloured permutations, a quick overview of Marberg's approach for the enumeration of coloured set partitions helps set the stage for a new interpretation.

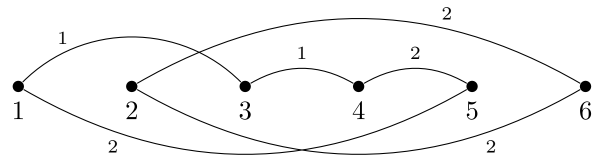


Figure 5: The image of Example 1 under the involution in the proof of Theorem 1

4.1 Another interpretation of $\mathcal{G}_{j,k,r}$ for set partitions

Marberg viewed r sequences of vacillating tableaux, one for each colour, as $r \times (k-1)$ matrices $A = [A_{i,l}]$ encoding λ_i^l in a vacillating tableau sequence T for colour i. If the set partition is j-noncrossing and k-nonnesting, then this tableau has a maximum of $j-1$ columns and $k-1$ rows. For colour i, the ith row of matrix A just lists parts of λ^l, thus at most $k-1$ non-zero parts. The multigraph $\mathcal{G}_{j,k,r}$ is drawn using all such allowable A's as vertices, and edges and loops connecting vertices corresponding to adding a box, deleting a box, or doing nothing in the construction of vacillating tableaux so that the resulting sequence contains only tableaux of at most $j-1$ columns and $k-1$ rows. Once completed, the multigraph $\mathcal{G}_{j,k,r}$ gives rise to an adjacency matrix. To find the number $\mathrm{NCN}_{j,k}(n, r)$ which is also the number of $(n-1)$-step walks on $\mathcal{G}_{j,k,r}$ from the zero matrix to itself, the method of transfer matrix gives a quotient of two polynomials (determinants actually), thus concluding that the ordinary generating function $\sum_{n \geq 0} \mathrm{NCN}_{j,k}(n+1, r)x^n$ is rational.

4.2 Examples of $\mathcal{G}_{2,2,1}$ and $\mathcal{G}_{2,2,2}$ for set partitions

To illustrate the construction of $\mathcal{G}_{j,k,r}$, we first reconstruct Marberg's $\mathcal{G}_{2,2,1}$ and $\mathcal{G}_{2,2,2}$ by naming each vertex and edge as it becomes necessary.

The arc annotated diagram of a set partition on $[n]$ has $n-1$ consecutive gaps, i. e. between each pair of adjacent points. Let the set of non-crossing, non-nesting, uncoloured set partitions on $[n]$ be denoted by $\mathcal{P}_{2,2,1}(n)$. For each $P \in \mathcal{P}_{2,2,1}(n)$, a snap shot of each gap belongs to one of the first four types in Table 1 where the matching steps in $\mathcal{G}_{2,2,1}$ are also given. Since $r = 1$, only two vertices exist in $\mathcal{G}_{2,2,1}$: v_0, the initial state for no opener, and v_1, for one opener. No other vertices accounting for other states are present because any state v_i where $i \geq 2$ would mean two or more openers which will form at least a 2-nesting or 2-crossing when closed. Incident at v_0 are three types of edges: two loops, ${}^{\times}$ for no arc in the consecutive gap, and ^{1} for a distance 1-arc both of which do not change the number of openers present as the set partition is scanned from the left to the right; the last type is a directed edge from v_0 to v_1 to indicate that an opener is present in the consecutive gap. Once at v_1, only the loop, ${}^{\times}$, is allowed because a 1-arc ^{1} will create a 2-nesting in P with the existing opener. A directed edge from v_1 to v_0 means that an opener is closed. To simplify drawing, an edge without arrows is bidirectional. The result is shown in Figure 6.

To construct $\mathcal{G}_{2,2,2}$, we require four vertices: still v_0 as the initial state for no opener, but also two states indicating one r-coloured ($r \in [2]$) opener, v_{1_1} and v_{1_2}. Since two arcs of different colours do not create a crossing or nesting, one more state is needed, $v_{2_{12}}$, for two openers, one of each colour. As in $\mathcal{G}_{2,2,1}$,

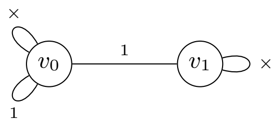


Figure 6: An uncoloured set partition graph, $\mathcal{G}_{2,2,1}$.

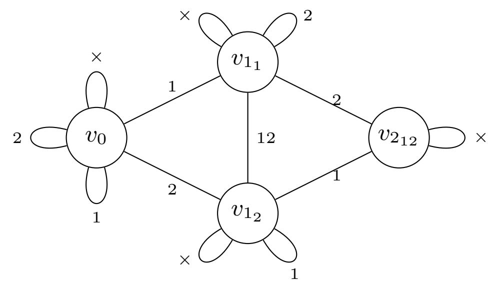


Figure 7: A 2-coloured set partition graph, $\mathcal{G}_{2,2,2}$.

the loops and edges are placed according to what is allowed in P, but a new edge between v_{1_1} and v_{1_2} is added in the last row of Table 1 for the closing of one colour on point m while an opener is present at point $m-1$ in P. The result is shown in Figure 7.

For details on how the adjacency matrices for Figures 6 and 7 give rise to generating functions, please see (16).

In general, we obtain $\mathcal{G}_{j,k,r}$ directly through labelling the edges and vertices of $\mathcal{G}_{j,k,r}$ similar to generating such set partitions through the method of generating trees except that each vertex v_i (considered as a state) in $\mathcal{G}_{j,k,r}$ indicates that i openers are pending to close. When drawn from the left to the right where all vertices of the same first subscript line up vertically, we get edges either between v_i and v_{i+1} for each $i \geq 0$ for openers or closers as in Figure 8, or between vertices of the same first subscript for the presence of both (drawn as vertical edges, not shown in Figure 8). Care needs to be taken when many arcs of the same colour are open because the order in which they are closed relates to how crossing and nesting are formed.

We list the first few series for $\mathcal{G}_{2,2,r}$, $r = \{3, 4\}$. The first two series, $r = 1, 2$ were found by Marberg (10) where A216949 in (12) is for $r = 2$. Our series mark the number of consecutive gaps, namely, x^k counts the number of such coloured set partitions on $k+1$ elements. For more terms and the rational functions, please consult A225029–A225033 in (12) for $r = 3$ to 7.

$$\sum_{n\geq 0} \mathrm{NCN}_{2,2}(n,3)x^n = \frac{1-10x+22x^2-x^3}{1-14x+59x^2-74x^3+x^4} = 1+4x+19x^2+103x^3+616x^4+3949x^5+\dots$$

$$\sum_{n\geq 0} \mathrm{NCN}_{2,2}(n,4)x^n = \frac{1-20x+122x^2-224x^3+x^4}{1-25x+218x^2-782x^3+973x^4-x^5} = 1+5x+29x^2+193x^3+1441x^4+\dots$$

Using an average personal computer, Maple15 can generate up to 7 colours. The next case, $r = 8$, with a matrix size of 256×256, computation would take too long to find the determinants.

4.3 Multigraphs, $\mathcal{G}_{2,2,1}$ and $\mathcal{G}_{2,2,2}$ for permutations

Instead of translating consecutive gaps from set partitions into steps in the multigraph $\mathcal{G}$, we examine each vertex in the arc diagram of a coloured permutation and assign each type of vertex to a step in $\mathcal{G}$. As for set partitions, we first construct the multigraph $\mathcal{G}_{2,2,1}$ for non-crossing, non-nesting, uncoloured permutations. Let us denote the set of all such permutations on $[n]$ by $\mathcal{S}_{2,2,1}(n)$. If $S \in \mathcal{S}_{2,2,1}(n)$, then a

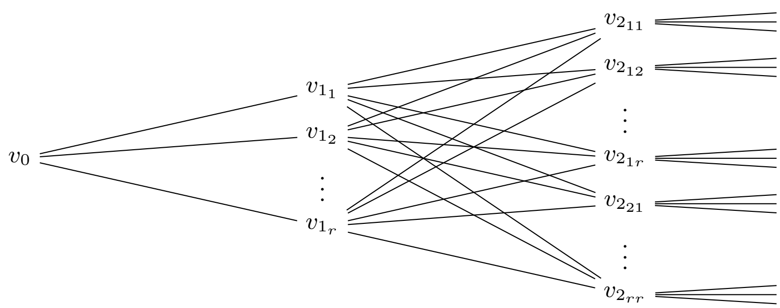


Figure 8: The line-up for states of the same number of openers

Domain	Set partition	Types of arcs	Types of steps in $\mathcal{G}_{2,2,1}$ and $\mathcal{G}_{2,2,2}$
$m \geq 2$	$m-1$ m	no arc	$\times$
$m \geq 2$	r $m-1$ m	a 1-arc coloured r	r, $r \in [2]$
$m \geq 2$	r $m-1$ m	an opener	r v_i v_{i+1}
$m \geq 3$	r $m-1$ m	a closer	v_i r v_{i+1}
$m \geq 3$	1 2 $m-1$ m	a closer and an opener	v_{i_1} 12 v_{i_2}

Table 1: Five situations between point $m-1$ and point m for set partitions and the matching steps in $\mathcal{G}$.

vertex is either a fixed point ( ) , an opener ( ), a closer ( ), or a lower transitory( ). We can't have an upper transitory which contributes to a 2-(enhanced) crossing.

In Figure 9, v_0 still indicates the initial state with 0 opener; v_1 indicates the state with 1 opener. The loop labelled 1 is the step taken when a fixed point coloured 1 is encountered in the permutation scanned from the left. The loop labelled 1_t is the presence of a lower transitory with coloured 1 arcs on both sides; this is possible only when an opener coloured 1 is present, thus at v_1. Note that a lower transitory does not alter the state. The directed edge (v_0, v_1) indicates the presence of an opener, and the edge traversed in reverse indicates that of a closer. An edge drawn without arrows still means a bidirectional edge.

The construction of $\mathcal{G}_{2,2,2}$ involves more types of vertices and edges which we summarize in Table 2.

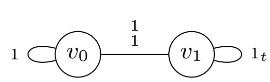


Figure 9: An uncoloured permutation graph, $\mathcal{G}_{2,2,1}$.

Each state with one opener has the colours of the openers as subscripts. When a state has two openers, both colours are used, thus only one such vertex in v_2. The method of transfer matrix gives the following generating function. Here x marks the size of the permutation.

$$\sum_{n\geq 0} \mathrm{NCN}_{2,2}(n,2)x^n = \frac{1-6x+4x^2}{(1-2x)(1-6x)} = 1+2x+8x^2+40x^3+224x^4+1312x^5+7808x^6+O(x^7).$$

This series, A092807 in (12), counts (with interpolated zeros) the number of closed walks of length n at a vertex of the edge-vertex incidence graph of K_4, the complete graph on 4 vertices associated with the edges of K_4. The next two series, A224992 and A224993 in (12), however, are new. For 5 colours, the matrix size, 252×252, hinders fast computation of determinants.

$$\sum_{n\geq 0} \mathrm{NCN}_{2,2}(n,3)x^n = \frac{1-17x+66x^2-36x^3}{(1-2x)(1-6x)(1-12x)} = 1+3x+18x^2+144x^3+1368x^4+O(x^5),$$

$$\sum_{n\geq 0} \mathrm{NCN}_{2,2}(n,4)x^n = \frac{1-36x+380x^2-1200x^3+576x^4}{(1-2x)(1-6x)(1-12x)(1-20x)} = 1+4x+32x^2+352x^3+4736x^4+O(x^5).$$

Domain	Permutation Vertex	Types of arcs	Types of steps in $\mathcal{G}_{2,2,2}$
all vertices	l	a fixed point	l, $l \in [2]$
all except the last	r s	an opener	$v_i \xrightarrow{r\,s} v_{i+1}$, $r, s \in [2]$
all except the first	r s	a closer	$v_i \xleftarrow{r\,s} v_{i+1}$, $r, s \in [2]$
no first, no last	r r	a lower transitory	rt, $r \in [2]$
no first, no last	r s	an upper transitory	v_{i_r} $\underline{rs}$ v_{i_s}, $r, s \in [2]$
no first, no last	s r	a lower transitory	v_{i_r} $\overline{sr}$ v_{i_s}, $r, s \in [2]$

Table 2: Vertices in permutations and the matching steps in $\mathcal{G}_{2,2,2}$.

4.4 Proof of Rationality through Multigraphs for r-coloured permutations

In general, drawing $\mathcal{G}_{j,k,r}$ for coloured permutations is a tedious task. As the j, k, and r increase, types of edges and vertices increase. Not only does one need to track the order in which coloured arcs are closed, one also needs to create unidirectional edges which go to the right states. Regardless of the complexity of

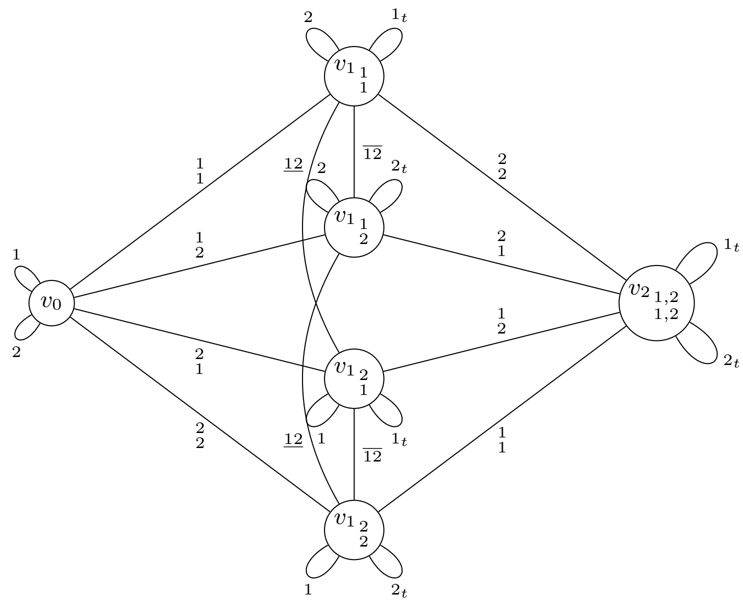


Figure 10: A 2-coloured permutation multigraph, $\mathcal{G}_{2,2,2}$

the multigraph, $\mathcal{G}_{j,k,r}$, only a finite number of vertices and edges are present because both crossing and nesting numbers are bounded for the set of r-coloured permutations. Furthermore, the number of such permutations on $[n]$ is the number of n-step paths from v_0 to v_0 in $\mathcal{G}_{j,k,r}$ because all openers must be closed. Using the method of transfer matrix then yields a rational function for the ordinary generating function, $\sum_{n\geq 0} \mathrm{NCN}_{j,k}(n,r)x^n$.

5 Concluding Remarks

When both nesting and crossing numbers are bounded, a finite multigraph can be constructed. This method of transfer matrix may be extended to the enumeration of set partitions of classical types as in the works of Rubey and Stump (13), even their coloured counterparts. The challenge lies in finding the generating function when only one of the bounds is present. For instance, Marberg (10) showed that the ordinary generating function for noncrossing 2-coloured set partitions is D-finite, but conjectured non-D-finite series for noncrossing r-coloured set partitions when $r \geq 3$.

6 Acknowledgements

The author would like to thank Marni Mishna for recommending Marberg's paper (10), Eric Marberg for clarifying the construction of the matrices which act as vertex labels of the multigraph $\mathcal{G}_{j,k,r}$, and two anonymous referees for their constructive comments which led to the automation of generating the rational series for coloured noncrossing nonnesting set partitions and permutations. Mogens Lemvig Hansen helped with figures and Maple code (16) for the automation.

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Results and conjectures on the number of standard strong marked tableaux

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Abstract. Many results involving Schur functions have analogues involving k-Schur functions. Standard strong marked tableaux play a role for k-Schur functions similar to the role standard Young tableaux play for Schur functions. We discuss results and conjectures toward an analogue of the hook length formula.

Résumé. De nombreux résultats impliquant les fonctions de Schur possèdent des analogues pour les fonctions de k-Schur. Les tableaux standard fortement marqués jouent un rôle pour les fonctions de k-Schur semblable á celui joué par les tableaux de Young pour les fonctions de Schur. Nous proposons ici des résultats et conjectures vers un analogue de la formule des équerres.

Keywords: k-Schur functions, strong marked tableaux, enumeration

1 Introduction

In 1988, Macdonald introduced a new class of polynomials and conjectured that they expand positively in terms of Schur functions. This conjecture, verified in Haiman (2001), has led to an enormous amount of work, including the development of the k-Schur functions. The k-Schur functions were defined in Lapointe et al. (2003). Lapointe, Lascoux, and Morse conjectured that they form a basis for a certain subspace of the space of symmetric functions and that the Macdonald polynomials indexed by partitions whose first part is not larger than k expand positively in terms of the k-Schur functions, leading to a refinement of the Macdonald conjecture. The k-Schur functions have since been found to arise in other contexts; for example, as the Schubert cells of the cohomology of affine Grassmannian permutations Lam (2006), and they are related to the quantum cohomology of the affine permutations Lapointe and Morse (2008).

One of the intriguing features of standard Young tableaux is the Frame-Thrall-Robinson hook-length formula, which enumerates them. It has many different proofs and many generalizations, see e.g. (Stanley, 1999, Chapter 7), Greene et al. (1979), Ciocan-Fontanine et al. (2011) and the references therein.

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In this extended abstract, we partially succeed in finding an analogue of the hook-length formula for standard strong marked tableaux (or starred tableaux for short), which are a natural generalization of standard Young tableaux in the context of k-Schur functions. For a fixed n, the shape of a starred tableau (see Subsection 2.5 for a definition) is necessarily an n-core, a partition for which all hook-lengths are different from n. In Lam et al. (2010), a formula is given for the number of starred tableaux for $n = 3$.

Proposition 1.1 (Lam et al. (2010), Proposition 9.17) *For a 3-core λ, the number of starred tableaux of shape λ equals*

$$\frac{m!}{2^{\lfloor \frac{m}{2} \rfloor}},$$

where m is the number of boxes of λ with hook-length < 3. □

The number of 2-hooks is $\lfloor \frac{m}{2} \rfloor$. Therefore we can rewrite the result as

$$\frac{m!}{\prod\limits_{\substack{i,j \in \lambda \\ h_{ij} < 3}} h_{ij}}.$$

Note that this is reminiscent of the classical hook-length formula.

The authors left the enumeration for $n > 3$ as an open problem. The main result (Theorem 3.1) of this extended abstract implies the existence, for each n, of $(n-1)!$ rational numbers which we call *correction factors*. Once the corrections factors have been calculated by enumerating all starred tableaux for certain shapes, the number of starred tableaux of shape λ for any n-core λ can be easily computed. In fact, Theorem 3.1 is a t-analogue of the hook formula. The theorem is "incomplete" in the sense that we were not able to find explicit formulas for the (weighted) correction factors. We have, however, been able to state some of their properties (some conjecturally), the most interesting of these properties being unimodality (Conjecture 3.7).

Another result of interest is a new, alternative description of strong marked covers via simple triangular arrays of integers which we call *residue tables* and *quotient tables* (Theorem 4.2).

The extended abstract is structured as follows. In Section 2, we give the requisite background, notation, definitions, and results. In Section 3, we state the main results and conjectures. In Section 4, we give an alternative description of strong covers directly in terms of bounded partitions (instead of via cores, abacuses or affine permutations). We envision this description as the first steps toward an inductive proof of the main formula We finish with some remarks and open questions in Section 5.

2 Preliminaries

Here we introduce notation and review some constructions. Please see Macdonald for the definitions of integer partitions, ribbons, hook lengths, etc., which we omit in this extended abstract.

2.1 Cores and bounded partitions

Let n be a positive integer. An *n-core* is a partition λ such that $h_{ij}^{\lambda} \neq n$ for all $(i,j) \in \lambda$. Core partitions were introduced by Nakayama to describe when two ordinary irreducible representations of the symmetric group belong to the same block. There is a close connection between $(k+1)$-cores and *k-bounded partitions*, which are partitions whose first part (and hence every part) is $\leq k$. Indeed, in Lapointe and Morse (2005), a simple bijection between $(k+1)$-cores and k-bounded partitions is presented. Given a $(k+1)$-core λ, whose diagram has λ_i boxes in row i, let π_i be the number of boxes in row i of λ with hook-length $\leq k$. The resulting $\pi = (\pi_1, \pi_2, \ldots, \pi_\ell)$ is a k-bounded partition, we denote it $\mathfrak{b}(\lambda)$. Conversely, given a k-bounded partition π, move from the last row of π upwards, and in row i, shift the π_i boxes of the diagram of π to the right until their hook-lengths are at most k. The resulting $(k+1)$-core is denoted $\mathfrak{c}(\pi)$. In this extended abstract, we will always use n as shorthand for $k+1$.

Example 2.1 *On the left-hand side of Figure 1, the hook-lengths of the boxes of the* 5*-core* $\lambda = 953211$ *are shown, with the ones that are* < 5 *in bold. That means that* $\mathfrak{b}(\lambda) = 432211$.

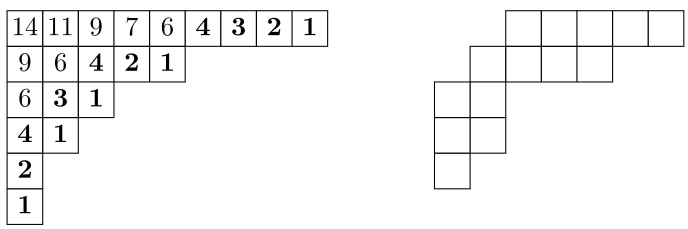

Fig. 1: Bijections $\mathfrak{b}$ and $\mathfrak{c}$.

The right-hand side shows the construction of $\mathfrak{c}(\pi) = 75221$ *for the* 6*-bounded partition* $\pi = 54221$. $\diamond$

Of particular importance are k-bounded partitions π that satisfy $m_i(\pi) \leq k - i$ for all $i = 1, \ldots, k$. We call such partitions *k-irreducible partitions*, see Lapointe et al. (2003). The number of k-irreducible partitions is $k!$.

2.2 Young tableaux and the hook-length formula

Young's lattice $\mathcal{Y}$ takes as its vertices all integer partitions, and the relation is containment. If λ and μ are partitions, then μ covers λ if and only if $\lambda \subseteq \mu$ and $|\mu| = |\lambda| + 1$. The rank of a partition is given by its size.

A *semistandard Young tableau* T of shape λ is a Young diagram of shape λ whose boxes have been filled with positive integers satisfying the following: the integers must be nondecreasing as we read a row from left to right, and increasing as we read a column from top to bottom. The *weight* of T is the composition $(\alpha_1, \alpha_2, \ldots)$, where α_i is the number of i's in T. The tableau T is a *standard Young tableau* if the entries are $1, \ldots, |\lambda|$ in some order, i.e. if the weight is $(1, \ldots, 1)$. A standard Young tableau of shape λ represents a saturated chain in the interval $[\emptyset, \lambda]$ of the Young's lattice. Let $(\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(m)})$, $\lambda^{(0)} = \emptyset$, $\lambda^{(m)} = \lambda$, be such a chain. Then in the tableau corresponding to this chain, i is the entry in the box added in moving from $\lambda^{(i-1)}$ to $\lambda^{(i)}$.

The Frame-Thrall-Robinson hook-length formula shows how to compute f_λ, the number of standard Young tableaux of shape λ. We have:

$$f_\lambda = \frac{|\lambda|!}{\prod_{i,j\in\lambda} h_{ij}^\lambda}. \tag{2.1}$$

This formula has a well-known weighted version, see (Stanley, 1999, Corollary 7.21.5). For a standard Young tableau T, define a *descent* to be an integer i such that $i+1$ appears in a lower row of T than i, and define the *descent set* $D(T)$ to be the set of all descents of T. Define the *major index* of T as $\mathrm{maj}(T) = \sum_{i\in D(T)} i$, and the polynomial

$$f_\lambda(t) = \sum t^{\mathrm{maj}(T)},$$

where the sum is over all standard Young tableaux of shape λ. Then

$$f_\lambda(t) = \frac{t^{b(\lambda)}(\boldsymbol{|\lambda|})\boldsymbol{!}}{\prod_{i,j\in\lambda}(\boldsymbol{h_{ij}^\lambda})}. \tag{2.2}$$

Here $b(\lambda) = \sum_i (i-1)\lambda_i = \sum_i \binom{\lambda_i'}{2}$, $(\boldsymbol{i}) = 1 + t + \ldots + t^{i-1}$ and $(\boldsymbol{i})\boldsymbol{!} = (\boldsymbol{1})\cdot(\boldsymbol{2})\cdots(\boldsymbol{i})$.

2.3 Strong marked and starred tableaux

The *strong n-core poset* $\mathcal{C}_n$ is the subposet of $\mathcal{Y}$ induced by the set of all n-core partitions. That is, its vertices are n-core partitions and $\lambda \leq \mu$ in $\mathcal{C}_n$ if $\lambda \subseteq \mu$. The cover relations are trickier to describe in $\mathcal{C}_n$ than in $\mathcal{Y}$.

Proposition 2.2 (Lam et al. (2010), Proposition 9.5) *Suppose $\lambda \leq \mu$ in $\mathcal{C}_n$, and let $C_1, \ldots, C_m$ be the connected components of μ/λ. Then μ covers λ (denoted $\lambda \lessdot \mu$) if and only if each C_i is a ribbon, and all the components are translates of each other with heads on consecutive diagonals with the same residue.*

The rank of an n-core is the number of boxes of its diagram with hook-length $< n$. If $\lambda \lessdot \mu$ and μ/λ consists of m ribbons, we say that μ *covers* λ *in the strong order with multiplicity* m. Figure 2 shows the strong marked covers for 4-cores with rank at most 6. Only multiplicities $\neq 1$ are marked.

A *strong marked cover* is a triple (λ, μ, c) such that $\lambda \lessdot \mu$ and that c is the content of the head of one of the ribbons. We call c the *marking* of the strong marked cover. A *strong marked horizontal strip of size r and shape μ/λ* is a sequence $(\nu^{(i)}, \nu^{(i+1)}, c_i)_{i=0}^{r-1}$ of strong marked covers such that $c_i < c_{i+1}$, $\nu^{(0)} = \lambda$, $\nu^{(r)} = \mu$. If λ is an n-core, a *strong marked tableau* T of shape λ is a sequence of strong marked horizontal strips of shapes $\mu^{(i+1)}/\mu^{(i)}$, $i = 0, \ldots, m-1$, such that $\mu^{(0)} = \emptyset$ and $\mu^{(m)} = \lambda$. The *weight* of T is the composition $(r_1, \ldots, r_m)$, where r_i is the size of the strong marked horizontal strip $\mu^{(i)}/\mu^{(i-1)}$. If all strong marked horizontal strips are of size 1, we call T a *standard strong marked tableau* or a *starred tableau* for short. For a k-bounded partition π (recall that $n = k+1$), denote the number of starred tableaux of shape $\mathfrak{c}(\pi)$ by $F_\pi^{(k)}$.

Figure 3 illustrates $F_{211}^{(3)} = 6$.

If λ is a k-bounded partition that is also an n-core (i.e., if $\lambda_1 + \ell(\lambda) \leq k+1$), then strong marked covers on the interval $[\emptyset, \lambda]$ are equivalent to the covers in the Young lattice, strong marked tableaux of shape λ are equivalent to semistandard Young tableaux of shape λ, and starred tableaux of shape λ are equivalent to standard Young tableaux of shape λ.

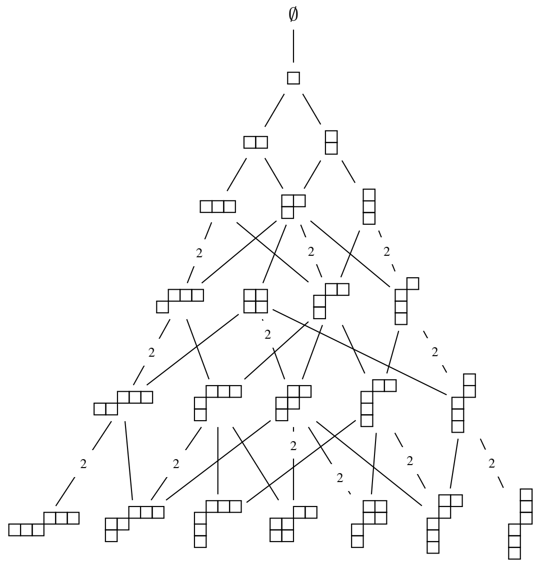

Fig. 2: The 4-core lattice up to rank 6. Only boxes with hook-lengths ¡ 4 are drawn.

1^*	2^*	3^*
4^*		
4		

1^*	2^*	4^*
3^*		
4		

1^*	2^*	4
3^*		
4^*		

1^*	3^*	4^*
2^*		
4		

1^*	3^*	4
2^*		
4^*		

1^*	4	4^*
2^*		
3^*		

Fig. 3: All starred tableaux of shape 311.

2.4 Schur functions

For the definition of Λ, the ring of symmetric functions, see Macdonald or Stanley (1999). For a partition λ, define the *monomial symmetric function*

$$m_\lambda = m_\lambda(x_1, x_2, \ldots) = \sum_\alpha x^\alpha,$$

where the sum is over all weak compositions α that are a permutation of λ, and $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$. For partitions λ and μ of the same size, define the *Kostka number* $K_{\lambda\mu}$ as the number of semistandard Young tableaux of shape λ and weight μ. Define the *Schur function*

$$s_\lambda = \sum K_{\lambda\mu} m_\mu$$

with the sum over all partitions μ. The Schur functions form the most important basis of Λ and have numerous beautiful properties. See for example (Stanley, 1999, Chapter 7) and (Macdonald, Chapter 1).

2.5 k-Schur functions

There are at least three conjecturally equivalent definitions of k-Schur functions. Here, we give the definition from Lam et al. (2010) via strong marked tableaux. For k-bounded partitions π and τ, define the *k-Kostka number* $K_{\pi\tau}^{(k)}$ as the number of strong marked tableaux of shape $\mathfrak{c}(\pi)$ and weight τ. Then we define the *k-Schur function*

$$s_\pi^{(k)} = \sum_\tau K_{\pi\tau}^{(k)} m_\tau, \tag{2.3}$$

where the sum is over all k-bounded partitions τ.

If π is also a $(k+1)$-core, then strong marked tableaux of shape π are equivalent to semistandard Young tableaux of shape π, and therefore in this case $s_\pi^{(k)} = s_\pi$.

The original definition of k-Schur functions was via *atoms* Lapointe et al. (2003), which we will not use here (but see **5.2**). Note that in full generality, the k-Schur functions (in any definition) have a parameter t. In this extended abstract, $t = 1$.

3 Main results and conjectures

For a starred tableau T, define the *descent set of* T, $D(T)$, as the set of all i for which the marked box at i is strictly above the marked box at $i+1$. Define the *major index of* T, $\mathrm{maj}(T)$, by $\sum_{i \in D(T)} i$. For a k-bounded partition π, define the polynomial

$$F_\pi^{(k)}(t) = \sum_T t^{\mathrm{maj}(T)}, \tag{3.1}$$

where the sum is over all starred tableaux of shape $\mathfrak{c}(\pi)$. Recall that $F_\pi^{(k)}$ denotes the number of such starred tableaux, i.e. $F_\pi^{(k)} = F_\pi^{(k)}(1)$.

Our main result is the following theorem.

Theorem 3.1 *Let π be a k-bounded partition, and write*

$$\pi = \langle k^{a_1+1\cdot w_1}, (k-1)^{a_2+2\cdot w_2}, \ldots, 1^{a_k+k\cdot w_k} \rangle,$$

for $0 \le a_i < i$. Then

$$F_\pi^{(k)}(t) = \frac{t^{\sum_{i=1}^k w_i \binom{i}{2}(k-i+1)} (|\boldsymbol{\pi}|)!\, F_\sigma^{(k)}(t)}{(|\boldsymbol{\sigma}|)! \prod_{j=1}^k (\boldsymbol{j})^{\sum_{i=1}^k w_i \min\{i,j,k+1-i,k+1-j\}}},$$

where $\sigma = \langle k^{a_1}, (k-1)^{a_2}, \ldots, 1^{a_k} \rangle$.

By plugging in $t = 1$, we get the following.

Corollary 3.2 *Let π be a k-bounded partition, and write*

$$\pi = \langle k^{a_1+1\cdot w_1}, (k-1)^{a_2+2\cdot w_2}, \ldots, 1^{a_k+k\cdot w_k} \rangle,$$

for $0 \le a_i < i$. *Then*

$$F_\pi^{(k)} = \frac{|\pi|!\, F_\sigma^{(k)}}{|\sigma|!\, \prod_{j=1}^{k} j^{\sum_{i=1}^{k} w_i \min\{i,j,k+1-i,k+1-j\}}},$$

where $\sigma = \langle k^{a_1}, (k-1)^{a_2}, \ldots, 1^{a_k} \rangle$. □

The theorem (respectively, corollary) implies that in order to compute $F_\pi^{(k)}(t)$ (resp., $F_\pi^{(k)}$) for all k-bounded partitions π, it suffices to compute $F_\sigma^{(k)}(t)$ (resp., $F_\sigma^{(k)}$) only for k-irreducible partitions σ; recall that there are $k!$ such partitions.

The proof is omitted in the extended abstract. Let us just mention that it uses the expansion of k-Schur functions in terms of fundamental quasisymmetric functions, and the stable principal specialization (i.e., evaluation at $1, t, t^2, \ldots$) of fundamental quasisymetric functions.

Example 3.3 *The following gives the formulas for* $k \le 3$.

1. *For* $k = 1$, *we have* $F_{1^0}^{(1)}(t) = 1$ *and therefore*

$$F_{1^{w_1}}^{(1)}(t) = \frac{\boldsymbol{(w_1)!} \cdot 1}{\boldsymbol{(0)!} \cdot \boldsymbol{(1)}^{w_1}} = \boldsymbol{(w_1)!}.$$

This is consistent with (Lam et al., 2010, §9.4.1), which states that $F_{1^{w_1}}^{(1)} = w_1!$.

2. *For* $k = 2$, *we have* $F_{2^0 1^0}^{(2)}(t) = 1$ *and* $F_{2^0 1^1}^{(2)}(t) = 1$. *Therefore,*

$$F_{2^{w_1} 1^{2w_2}}^{(2)}(t) = \frac{t^{w_2} \boldsymbol{(2w_1 + 2w_2)!} \cdot 1}{\boldsymbol{(0)!} \cdot \boldsymbol{(2)}^{w_1+w_2}} = \frac{t^{w_2} \boldsymbol{(2w_1 + 2w_2)!}}{\boldsymbol{(2)}^{w_1+w_2}}.$$

$$F_{2^{w_1} 1^{1+2w_2}}^{(2)}(t) = \frac{t^{w_2} \boldsymbol{(2w_1 + 2w_2 + 1)!} \cdot 1}{\boldsymbol{(0)!} \cdot \boldsymbol{(2)}^{w_1+w_2}} = \frac{t^{w_2} \boldsymbol{(2w_1 + 2w_2 + 1)!}}{\boldsymbol{(2)}^{w_1+w_2}}.$$

This is consistent with (Lam et al., 2010, Proposition 9.17), reprinted here as Proposition 1.1.

3. *For* $k = 3$, *we have* $F_{3^0 2^0 1^0}^{(3)} = F_{3^0 2^0 1^1}^{(3)} = F_{3^0 2^1 1^0}^{(3)} = 1$, $F_{3^0 2^0 1^2}^{(3)} = t$, $F_{3^0 2^1 1^1}^{(3)} = t(1+t)$ *and,* $F_{3^0 2^1 1^2}^{(3)} = t\left(t^2+1\right)\left(t^2+t+1\right)$. *So, among other formulas, we have*

$$F_{3^{w_1} 2^{1+2w_2} 1^{1+3w_3}}^{(3)}(t) = \frac{t^{2w_2+3w_3+1} \cdot \boldsymbol{(3w_3 + 4w_2 + 3w_1 + 3)!}}{\boldsymbol{(2)}^{w_1+2w_2+w_3} \cdot \boldsymbol{(3)}^{w_1+w_2+w_3+1}}.$$

Using a computer, it is easy to obtain formulas for larger k. ◇

We now introduce weighted correction factors. For a k-bounded partition π, let $H_\pi^{(k)}(t) = \prod \boldsymbol{(h_{ij})}$, where the product is over all boxes (i,j) of the $(k+1)$-core $\mathfrak{c}(\pi)$ with hook-lengths at most k, and let $H_\pi^{(k)} = H_\pi^{(k)}(1)$ be the product of all hook-lengths $\le k$ of $\mathfrak{c}(\pi)$. Furthermore, if b_j is the number of boxes in the j-column of $\mathfrak{c}(\pi)$ with hook-length at most k, write $b_\pi^{(k)} = \sum_j \binom{b_j}{2}$.

Example 3.4 *For the* 6*-bounded partition* $\pi = 54211$ *from Example 2.1, we have* $H_\pi^{(6)}(t) = (\mathbf{1})^4(\mathbf{2})^3(\mathbf{3})^2(\mathbf{4})^2(\mathbf{5})(\mathbf{6})^2$, $H_\pi^{(6)} = 207360$ *and* $b_\pi^{(6)} = 2\binom{3}{2} + 3\binom{2}{2} + 2\binom{1}{2} = 9$. $\diamond$

By introducing *weighted correction factors* $C_\sigma^{(k)}(t)$ for a k-irreducible partition σ, we can, by Theorem 3.1, express $F_\pi^{(k)}(t)$ (for all k-bounded partitions π) in another way which is reminiscent of the classical hook-length formula. More precisely, define a rational function $C_\sigma^{(k)}(t)$ so that

$$F_\sigma^{(k)}(t) = \frac{t^{b_\sigma^{(k)}}(|\boldsymbol{\sigma}|)!C_\sigma^{(k)}(t)}{H_\sigma^{(k)}(t)}. \tag{3.2}$$

Note that this implies, in the notation of Theorem 3.1, that

$$F_\pi^{(k)}(t) = \frac{t^{b_\sigma^{(k)} + \sum_{i=1}^k w_i \binom{i}{2}(k+1-i)}(|\boldsymbol{\pi}|)!\,C_\sigma^{(k)}(t)}{H_\sigma^{(k)}(t) \cdot \prod_{j=1}^k (\boldsymbol{j})^{\sum_{i=1}^k w_i \min\{i,j,k+1-i,k+1-j\}}}.$$

The *correction factor* $C_\sigma^{(k)}$ is defined as $C_\sigma^{(k)}(1)$.

For $k \leq 3$, all weighted correction factors are 1. For $k = 4$, all but four of the 24 weighted correction factors–for 4-bounded partitions 2211, 321, 3211 and 32211–are 1, and the ones different from 1 are

$$\frac{1+2t+t^2+t^3}{(\mathbf{2})(\mathbf{3})}, \frac{1+t+2t^2+t^3}{(\mathbf{2})(\mathbf{3})}, \frac{1+2t+2t^2+2t^3+t^4}{(\mathbf{3})^2}, \frac{1+t+3t^2+t^3+t^4}{(\mathbf{3})^2},$$

respectively.

We state some results and conjectures about the weighted correction factors. For a k-bounded partition π, denote by $\partial_k(\pi)$ the boxes of $\mathfrak{c}(\pi)$ with hook-length $\leq k$. If $\partial_k(\pi)$ is not connected, we say that π *splits*. Each of the connected components of $\partial_k(\pi)$ is a horizontal translate of $\partial_k(\pi^i)$ for some k-bounded partition π^i. Call $\pi^1, \pi^2, \ldots$ the *components* of π.

Proposition 3.5 *The weighted correction factors are multiplicative in the following sense. If a* k*-irreducible partition* σ *splits into* $\sigma^1, \sigma^2, \ldots, \sigma^m$*, then* $C_\sigma^{(k)}(t) = \prod_{i=1}^m C_{\sigma^i}^{(k)}(t)$.

Conjecture 3.6 *For a* k*-irreducible partition* σ*, the weighted correction factor is* 1 *if and only if* σ *splits into* $\sigma^1, \sigma^2, \ldots, \sigma^l$*, where each* σ^i *is a* k*-bounded partition that is also a* $(k+1)$*-core.*

The "if" direction is easy: if a k-bounded partition σ is also a $(k+1)$-core, then strong covers on the interval $[0, \sigma]$ are precisely the regular covers in the Young lattice, the starred tableaux of shape σ are standard Young tableaux of shape σ, and the major index of a starred tableau of shape σ is the classical major index for standard Young tableaux; the fact that the weighted correction factor is 1 then follows from the classical weighted version of the hook-length formula (2.2). If σ splits into cores, we can use (Denton, 2012, Theorem 1.1).

The most interesting conjecture about the weighted correction factors is the following. Recall that a sequence $(\alpha_i)_i$ is *unimodal* if there exists I so that $\alpha_i \leq \alpha_{i+1}$ for $i < I$ and $\alpha_i \geq \alpha_{i+1}$ for $i \geq I$, and a *unimodal polynomial* is a polynomial whose sequence of coefficients is unimodal.

Conjecture 3.7 *For a k-irreducible partition σ, we can write*

$$1 - C_\sigma^{(k)}(t) = \frac{P_1(t)}{P_2(t)},$$

where $P_1(t)$ is a unimodal polynomial with non-negative integer coefficients and $P_2(t)$ is a polynomial of the form $\prod_{i=1}^{k-1} (j)^{w_j}$ for some non-negative integers w_j.
In particular, we have $0 < C_\sigma^{(k)} \leq 1$ for all σ.

4 Strong covers and k-bounded partitions

Our proof of Theorem 3.1, omitted in the extended abstract, closely follows one of the possible proofs of the classical (non-weighted and weighted) hook-length formula, see e.g. (Stanley, 1999, §7.21). Note, however, that the truly elegant proofs (for example, the celebrated probabilistic proof due to Greene, Nijenhuis and Wilf Greene et al. (1979)) are via induction. In this section, we show the first steps toward such a proof.

In the process, we present a new description of strong marked covers in terms of bounded partitions (previous descriptions included cores – at least implicitly, via k-conjugation – affine permutations and abacuses). See the definition of residue and quotient tables below, and Theorem 4.2.

We identify a bounded partition $\pi = \langle k^{p_1}, (k-1)^{p_2}, \ldots, 1^{p_k} \rangle$ with the sequence $p = (p_1, \ldots, p_k)$. Given i, j, m, $0 \leq m < i \leq j \leq k$, define $p^{i,j,m}$ as follows.

For $i < j$, $$p_h^{i,j,m} = \begin{cases} p_h + m & \text{if } h = i-1 \\ p_h - m & \text{if } h = i \\ p_h - m - 1 & \text{if } h = j \\ p_h + m + 1 & \text{if } h = j+1 \\ P_h & \text{otherwise.} \end{cases}$$ For $i = j$, $$p_h^{i,i,m} = \begin{cases} p_h + m & \text{if } h = i-1 \\ p_h - 2m - 1 & \text{if } h = i \\ p_h + m + 1 & \text{if } h = i+1 \\ P_h & \text{otherwise.} \end{cases}$$

If $j = k$, then we are adding $m+1$ copies of $k - j = 0$, which does not change the partition. If $i = 1$, we have $m = 0$, so adding m copies of $k + 2 - i = k + 1$ also does not change the partition. To put it another way: to get $p^{i,j,m}$ from p, increase the first m copies of $k+1-i$ by 1, and decrease the last $m+1$ copies of $k+1-j$ by 1. See Example 4.3.

Define upper-triangular arrays $\mathcal{R} = (r_{ij})_{1 \leq i \leq j \leq k}$, $\mathcal{Q} = (q_{ij})_{1 \leq i \leq j \leq k}$ by

- $r_{jj} = p_j \bmod j$, $r_{ij} = (p_i + r_{i+1,j}) \bmod i$ for $i < j$,
- $q_{jj} = p_j \operatorname{div} j$, $q_{ij} = (p_i + r_{i+1,j}) \operatorname{div} i$ for $i < j$.

We call $\mathcal{R}$ the *residue table* and $\mathcal{Q}$ the *quotient table*.

Example 4.1 *Take $k = 4$ and $p = (1, 3, 2, 5)$. Then the residue and quotient tables are given by*

$$\begin{array}{|cccc|} 0 & 0 & 0 & 0 \\ & 1 & 1 & 1 \\ & & 2 & 0 \\ & & & 1 \end{array} \qquad \begin{array}{|cccc|} 1 & 2 & 2 & 2 \\ & 1 & 2 & 1 \\ & & 0 & 1 \\ & & & 1 \end{array}$$

It is easy to reconstruct p *from the diagonals of* $\mathcal{R}$ *and* $\mathcal{Q}$*:* $p_1 = 0 + 1 \cdot 1$, $p_2 = 1 + 1 \cdot 2$, $p_3 = 2 + 0 \cdot 3$, $p_4 = 1 + 1 \cdot 4$. ◇

It turns out that the residue and quotient tables determine strong marked covers (and probably other important relations as well, see **5.5**).

Theorem 4.2 *Take* $p = (p_1, \ldots, p_k)$ *and* $1 \leq i \leq j \leq k$*. If* $r_{ij} < r_{i+1,j}, \ldots, r_{jj}$*, then* p *covers* $p^{i,j,r_{ij}}$ *in the strong order with multiplicity* $q_{ij} + \ldots + q_{jj}$*. Furthermore, these are precisely all strong covers. In particular, an element of the* $(k+1)$*-core lattice covers at most* $\binom{k+1}{2}$ *elements.*

Example 4.3 *Take* $k = 4$ *and* $p = (1, 3, 2, 5)$ *as before. Let us underline the entries* r_{ij} *in the residue table* $\mathcal{R}$ *for which* $r_{ij} < r_{i+1,j}, \ldots, r_{jj}$*.*

$$\boxed{\begin{array}{cccc} \underline{0} & \underline{0} & \underline{0} & 0 \\ & \underline{1} & \underline{1} & 1 \\ & & \underline{2} & \underline{0} \\ & & & \underline{1} \end{array}}$$

By Theorem 4.2, p *covers (exactly) the following elements in the strong order:*
$p^{1,1,0} = (0, 4, 2, 5)$ *with multiplicity* 1, $p^{1,2,0} = (1, 2, 3, 5)$ *with multiplicity* $2{+}1 = 3$, $p^{2,2,1} = (2, 0, 4, 5)$ *with multiplicity* 1, $p^{1,3,0} = (1, 3, 1, 6)$ *with multiplicity* $2{+}2{+}0 = 4$, $p^{2,3,1} = (2, 2, 0, 7)$ *with multiplicity* $2 + 0 = 2$, $p^{3,3,2} = (1, 5, -3, 8)$ *with multiplicity* 0, $p^{3,4,0} = (1, 3, 2, 4)$ *with multiplicity* $1 + 1 = 2$*, and* $p^{4,4,1} = (1, 3, 3, 2)$ *with multiplicity* 1.

Note that while $(1, 5, -3, 8)$ *does not represent a valid partition, the multiplicity of the cover is* 0*, so we can ignore this cover relation.* ◇

For a k-bounded partition π, we clearly have

$$F_\pi^{(k)} = \sum_\tau m_{\tau\pi} F_\tau^{(k)},$$

where the sum is over all k-bounded τ that are covered by π, and $m_{\tau\pi}$ is the multiplicity of the cover. Therefore Theorem 4.2 can be used to prove Corollary 3.2 for small values of k by induction. First, we need the following corollary.

Corollary 4.4 *Let* $p = (p_1, \ldots, p_k)$*,* $p_i < i$*, with corresponding residue and quotient tables* $\mathcal{R}$ *and* $\mathcal{Q}$*. Assume that for* $1 \leq i \leq j \leq k$*, we have* $r_{ij} < r_{i+1,j}, \ldots, r_{jj}$*. For* $s_i \in \mathbb{N}$*, write* $s = (s_1, 2s_2, \ldots, ks_k)$*. Then* $p + s$ *covers* $p^{i,j,r_{ij}} + s$ *with multiplicity* $q_{ij} + \ldots + q_{jj} + s_i + \ldots + s_j$*.*

The corollary implies that in order to prove Corollary 3.2, all we have to do is check $k!$ equalities. The authors did all such calculations with a computer for small k ($k \leq 8$).

5 Final remarks

5.1

There are also notions of *weak horizontal strips* and *weak tableaux*. For n-cores λ and μ, $\lambda \subseteq \mu$, we say that μ/λ is a weak horizontal strip if $\mathfrak{b}(\mu)/\mathfrak{b}(\lambda)$ is a horizontal strip and $\mathfrak{b}(\mu')/\mathfrak{b}(\lambda')$ is a vertical strip. If in addition $|\mathfrak{b}(\mu)| = |\mathfrak{b}(\lambda)| + 1$, we say that μ *covers* λ *in the weak order*. A weak tableau of shape λ is a sequence of weak horizontal strips $\mu^{(i+1)}/\mu^{(i)}$, $i = 0, \ldots, m-1$, such that $\mu^{(0)} = \emptyset$ and $\mu^{(m)} = \lambda$.

Define $f_\pi^{(k)}$ to be the number of weak tableaux of shape $\mathfrak{c}(\pi)$. In Lam et al. (2010), it was proved that $f^{(2)}_{2^{w_1}1^{2w_2}} = f^{(2)}_{2^{w_1}1^{1+2w_2}} = \frac{(w_1+w_2)!}{w_1!w_2!}$.

It is not hard to prove by induction that

$$f^{(3)}_{3^{w_1}2^{2w_2}1^{3w_3}} = \frac{2^{2w_2}(w_1+w_2)!(w_2+w_3)!(w_1+2w_2+w_3-1)!(2w_1+2w_2+2w_3)!}{w_1!w_2!w_3!(w_1+w_2+w_3-1)!(2w_1+2w_2)!(2w_2+2w_3)!};$$

similar formulas exist for

$$f^{(3)}_{3^{w_1}2^{2w_2}1^{1+3w_3}}, f^{(3)}_{3^{w_1}2^{2w_2}1^{2+3w_3}}, f^{(3)}_{3^{w_1}2^{1+2w_2}1^{3w_3}}, f^{(3)}_{3^{w_1}2^{1+2w_2}1^{1+3w_3}}, f^{(3)}_{3^{w_1}2^{1+2w_2}1^{2+3w_3}}.$$

We were unable to find formulas for $k \geq 4$, and it seems unlikely that simple formulas exist. For example, the simplest recurrence relation that $g(i,j) = f^{(4)}_{2^{3i}1^{4j}}$ seem to satisfy is

$$a(i,j)g(i,j) + b(i,j)g(i,j+1) - c(i,j)g(i+1,j) = 0,$$

where a and b are fourth degree polynomials in i and j with rational coefficients and c, also fourth degree, is a polynomial with integer coefficients.

5.2

Our work has led us to consider (weighted) correction factors. They seem to be mysterious objects that deserve further study. The unimodality conjecture (Conjecture 3.7) is certainly intriguing and could hint that the factors have some geometric meaning.

Let us give another perspective on these factors. Since k-Schur functions are symmetric, they can be expanded in terms of Schur functions; in fact, the original definition of k-Schur functions via atoms gives precisely such an expansion. For example, $s^{(4)}_{2211} = s_{2211} + s_{321}$. Take the stable principal specialization and multiply by $\mathbf{(6)!}(1-t)^6$. By calculations done in our proof of Theorem3.1 and (Stanley, 1999, Proposition 7.19.11), we have

$$F^{(4)}_{2211}(t) = f_{2211}(t) + f_{321}(t).$$

Then, by (3.2) and (Stanley, 1999, Corollary 7.21.5),

$$C^{(k)}_{2211}(t) = \mathbf{(2)(3)(4)}\left(\frac{t^3}{\mathbf{(2)}^2\mathbf{(4)(5)}} + \frac{1}{\mathbf{(3)}^2\mathbf{(5)}}\right) = \frac{1+2t+t^2+t^3}{\mathbf{(2)(3)}}.$$

5.3

There is also a formula for the principal specialization of s_λ of order i (i.e. evaluation at $1, t, \ldots, t^{i-1}$, see e.g. (Stanley, 1999, Theorem 7.21.2)), in which both hook-lengths and contents of boxes appear. By imitating **5.2**, we can get rational functions (which depend on i) which converge to the weighted correction factors as $i \to \infty$. These rational functions also seem interesting and worthy of further study.

5.4

As we already mentioned, it would be preferable to prove Corollary 3.2 by induction, using the cover relations in Section 4 for a general k and in a way that would make apparent the meaning of hook-lengths and correction factors (the ideal being a variant of the probabilistic proof from Greene et al. (1979)). It seems likely that one would need to know a formula for the correction factors before such a proof would be feasible.

5.5

We showed (in Theorem 4.2) how to interpret the residue and quotient table to find strong covers. We feel that residue (and quotient) tables could prove important in other aspects of the k-Schur function theory. These tables can also be used to describe weak covers, weak horizontal and vertical strips and at least one of the possible cases of LLMS insertion for standard strong marked tableaux (see Lam et al. (2010)).

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Generating tuples of integers modulo the action of a permutation group and applications

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Abstract. Originally motivated by algebraic invariant theory, we present an algorithm to enumerate integer vectors modulo the action of a permutation group. This problem generalizes the generation of unlabeled graph up to an isomorphism. In this paper, we present the full development of a generation engine by describing the related theory, establishing a mathematical and practical complexity, and exposing some benchmarks. We next show two applications to effective invariant theory and effective Galois theory.

Résumé. Initialement motivé par la théorie algébrique des invariants, nous présentons une stratégie algorithmique pour énumérer les vecteurs d'entiers modulo l'action d'un groupe de permutations. Ce problème généralise le problème d'énumération des graphes non étiquetés. Dans cet article, nous développons un moteur complet d'énumération en expliquant la théorie sous-jacente, nous établissons des bornes de complexité pratiques et théoriques et exposons quelques bancs d'essais. Nous détaillons ensuite deux applications théoriques en théorie effective des invariants et en théorie de Galois effective.

Keywords: Generation up to an Isomorphism, Enumerative Combinatorics, Computational Invariant Theory, Effective Galois Theory

1 Introduction

Let G be a group of permutations, that is, a subgroup of some symmetric group $\mathfrak{S}_n$. Several problems in effective Galois theory (see [Girstmair(1987), Abdeljaouad(2000)]), computational commutative algebra (see [Faugère and Rahmany(2009), Borie and Thiéry(2011), Borie(2011)]) and generation of unlabeled with repetitions species of structures rely on the following computational building block.

Let $\mathbb{N}$ be the set of non-negative integers. An *integer vector* of length n is an element of $\mathbb{N}^n$. The symmetric group $\mathfrak{S}_n$ acts on positions on integer vectors in $\mathbb{N}^n$: for σ a permutation and $(v_1, \dots, v_n)$ an integer vector,

$$\sigma.(v_1, \dots, v_n) := (v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(n)}) .$$

This action coincides with the usual action of $\mathfrak{S}_n$ on monomials in the multivariate polynomial ring $\mathbb{K}[\mathbf{x}]$ with $\mathbb{K}$ a field and $\mathbf{x} := x_1, \dots, x_n$ indeterminates.

Problem 1.1 *Let $G \subset \mathfrak{S}_n$ be a permutation group. Enumerate the integer vectors of length n modulo the action of G.*

Note that there are infinitely many such vectors; in practice one usually wants to enumerate the vectors with a given sum or content.

For example, the Problem 1.1 contains the listing non-negative integer matrices with fixed sum up to the permutations of rows or columns appearing in the theory of multisymmetric functions [Gessel(1987), MacMahon(2004)] and in the more recent investigations of multidiagonal coinvariant [Bergeron(2009), Bergeron et al.(2011)Bergeron, Borie, and Thiéry].

Define the following equivalence relation over elements of $\mathbb{N}^n$: two vectors $\mathbf{u} := (a_1, \ldots, a_n)$ and $\mathbf{v} := (b_1, \ldots, b_n)$ are equivalent if there exists a permutation $\sigma \in G$ such that

$$\sigma \cdot \mathbf{u} = (a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(n)}) = (b_1, \ldots, b_n) = \mathbf{v}.$$

Problem 1.1 consists in enumerating all $\mathbb{N}^n/G$ equivalence classes.

This problem is not well solved in the literature. Some applications present a greedy strategy searching and deleting all pairs of vectors such that the second part can be obtained from the first part. The most famous sub-problem is the unlabeled graph generation which consists in enumerate tuples over 0 and 1 of length $\binom{n}{2}$ enumerated up to the action of the symmetric groups acting on pair on nodes. This example has a very efficient implementation in Nauty which is able to enumerate all graphs over a small number of nodes.

The algorithms presented in this paper have been implemented, optimized, and intensively tested in `Sage` [Stein et al.(2009)]; most features are integrated in `Sage` since release 4.7 (2011-05-26, ticket #6812, 1303 lines of code including documentation).

2 Orderly generation and tree structure over integer vectors

The orderly strategy consists in setting a total order on objects before quotienting by the equivalence relation. This allows us to define a single representative by orbit. Using the lexicographic order on integer vectors, we will call a vector $\mathbf{v}$ *canonical under the action of G* or just *canonical* if $\mathbf{v}$ is maximum in its orbit under G for the lexicographic order:

$$\mathbf{v} \text{ is canonical } \Leftrightarrow \mathbf{v} = \max_{lex}\{\sigma \cdot \mathbf{v} \mid \sigma \in G\}.$$

Now, the goal being to avoid to test systematically if vectors are canonical, we decided to use a tree structure on the objects in which we will get properties relaying the *canonical* vectors. Any result relating fathers, sons and the property of being canonical in the tree may allow us to skip some canonical test.

2.1 Tree Structure over integer vectors

Let $\mathbf{r}$ be the vector $\mathbf{r} := (0, \ldots, 0)$ called *root*, we build a tree with the following function *father*.

Definition 2.1 *Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ be a tuple of integers of length n which is not the* root*. Let $1 \leqslant i \leqslant n$ be the position of the last non-zero entry of $\mathbf{a}$. We define the* father *of $\mathbf{a}$*

$$father(a_1, a_2, \ldots, a_i, 0, 0, \ldots, 0) := (a_1, a_2, \ldots, a_i - 1, 0, 0, \ldots, 0)$$

For any integer vector $\mathbf{v} = (a_1, \ldots, a_n)$, we can go back to the generation root $(0, \ldots, 0)$ by $sum(\mathbf{v}) := a_1 + \cdots + a_n$ steps. The corresponding application giving the children of an integer vector is thus:

Definition 2.2 *Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ be a tuple of integers of length n. Let $1 \leqslant i \leqslant n$ be the position of the last non-zero entry of $\mathbf{a}$ ($i = 1$ if all entries are null). The set of* children *of $\mathbf{a}$ is obtained as:*

$$\textit{children: } (a_1, a_2, \ldots, a_i, 0, 0, \ldots, 0) \longmapsto \left\{ \begin{array}{c} (a_1, a_2, \ldots, a_i + 1, 0, 0, \ldots, 0) \\ (a_1, a_2, \ldots, a_i, 1, 0, \ldots, 0) \\ (a_1, a_2, \ldots, a_i, 0, 1, \ldots, 0) \\ \ldots \\ (a_1, a_2, \ldots, a_i, 0, 0, \ldots, 1) \end{array} \right\}$$

Proposition 2.3 *For any permutation group $G \subset \mathfrak{S}_n$, for any integer vector $\mathbf{v}$, if $\mathbf{v}$ is not canonical under G, all children of $\mathbf{v}$ are not canonical. Therefore, the* canonicals *form a "prefix tree" in the tree of all integer vectors.*

Sketch of proof: *When a father is not canonical, there exists a permutation such that the permuted vector is greater. Applying the same permutation on the children shows also it cannot be canonical.*

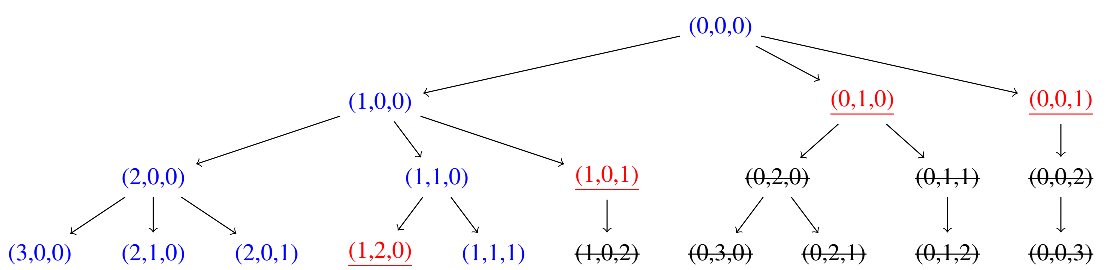


Figure 1: Enumeration tree of integer vectors modulo the action of $G = \langle(1, 2, 3)\rangle \subset \mathfrak{S}_3$, the cyclic group of degree 3.

Figure 1 displays integer vectors of length 3 whose sum is at most 3 and shows the tree relations between them. Choosing the cyclic group of order 3 and using the generation strategy, underlined integer vectors are tested but are recognized to be not *canonical*. Using Proposition 2.3, crossed-out integer vectors are not tested as they cannot be *canonical* as children of non canonical vectors.

Our strategy consists now in making a breath first search over the sub-tree of *canonicals*. This is done lazily using `Python` iterators.

2.2 Testing whether an integer vector is canonical

As we have seen, the fundamental operation for orderly generation is to test whether an integer vector is canonical; it is thus vital to optimize this operation. To this end, we use the work horse of computational group theory for permutation groups: stabilizer chains and strong generating sets.

Following the needs required by applications, we want to test massively if vectors are canonical or not. For this reason, we will use a *strong generating system* of the group G. We can compute this last item in almost linear time [Seress(2003)] using `GAP` [GAP(1997)].

Let n a positive integer and G a permutation group $G \subset \mathfrak{S}_n$. Recall that its *stabilizer chain* is $G_n = \{e\} \subset G_{n-1} \subset \cdots \subset G_1 \subset G_0 = G$, where

$$\forall i, 1 \leqslant i \leqslant n : G_i := \{g \in G | \forall j \leqslant i : g(j) = j\} .$$

From this chain, we build a *strong generating system* $T = \{T_1, T_2, \ldots, T_n\}$ where T_i is a transversal of G_{i-1}/G_i. This set of strong generators is particularly adapted to the partial lexicographic order as stabilizers are defined with positions $1, 2, \ldots, n$ from left to right.

Let n and i be two positive integers such that $1 \leqslant i \leqslant n$. For $\mathbf{v} = (v_1, \ldots, v_n)$ and $\mathbf{w} = (w_1, \ldots, w_n)$ two integer vectors of length n, let us define the following binary relations

$$\begin{array}{lcl} \mathbf{v} <_i \mathbf{w} & \iff & (v_1, \ldots, v_i) <_{lex} (w_1, \ldots, w_i) \\ \mathbf{v} \leqslant_i \mathbf{w} & \iff & (v_1, \ldots, v_i) \leqslant_{lex} (w_1, \ldots, w_i) \\ \mathbf{v} =_i \mathbf{w} & \iff & \forall j, 1 \leqslant j \leqslant i : v_j = w_j \end{array}$$

where $<_{lex}$ and $\leqslant_{lex}$ represent regular strict and large lexicographic comparison.

Algorithm 1 is a natural extension of McKay's canonical graph labeling algorithm as it is explained in [Hartke and Radcliffe(2009)].

Algorithm 1 Testing whether an integer vector is canonical

Arguments

- $\mathbf{v}$: An integer vector of length n;
- $sgs(G)$: A strong generating set for G, as a list $\{T_1, \ldots, T_n\}$ of transversals.

```
def is_canonical(v, sgs(G)) :
    todo ← {v}
    for i ∈ {1, 2, ..., n} :
        new_todo ← { }
        for w ∈ todo :
            children ← {g · w | g ∈ T_i}
            for child ∈ children :
                if v <_i child :
                    return False
                else :
                    if v =_i child and child ∉ new_todo :
                        new_todo ← new_todo ∪ {child}
        todo ← new_todo
    return True
```

Algorithm 1 takes advantage of partial lexicographic orders and the *strong generating system* of the group G. It tries to explore only a small part of the orbit of the vector $\mathbf{v}$; the worst case complexity of this step is bounded by the size of the orbit, and not by $|G|$. In this sense, it does take into account the automorphism group of the vector $\mathbf{v}$.

Proposition 2.4 *Let n be a positive integer and G a subgroup of $\mathfrak{S}_n$. Let $\mathbf{v}$ be an integer vector of length n. Algorithm 1 returns $True$ if $\mathbf{v}$ is* canonical *under the action of G and returns $False$ otherwise.*

Sketch of proof: *It is based on the properties of a* strong generating system.

3 Complexity

3.1 Theoretical complexity

3.1.1 Efficiency of the tree structure

Let n be a positive integer and $G \subset \mathfrak{S}_n$ a permutation group. For any non negative integer d, let $C(d)$ (resp. $\overline{C}(d)$) be the number of canonical (resp. non canonical) integer vectors of degree d. Based on the tree structure presented in Section 2.1, let $T(n)$ (resp. $\overline{T}(n)$) the number of tested (resp. non tested) integer vectors.

Proposition 3.1 *Generating all canonical integer vectors up to degree $d \geqslant 0$ using the generation strategy presented in Section 2 presents an* absolute error *bounded by $\overline{C}(d)$. Equivalently, regarding the series, we have*

$$\sum_{i=0}^{d} T(i) - \sum_{i=0}^{d} C(i) \leqslant \overline{C}(d)$$

Sketch of proof: *Using Lemma 2.3, we get this bound noticing two tested but non canonical vectors cannot have a paternity relation.*

This *absolute error* is not very explicit (directly usable), but it can be used to get a *relative error* at the price of a rough approximation.

Corollary 3.2 *Let n and b be two positive integers and $G \subset \mathfrak{S}_n$ a permutation group. Generating all canonical monomials under the action of G up to degree d using the generation strategy presented in Section 2 presents a* relative error *bounded by* $\min\{\frac{n(|G|-1)}{n+d}, n-1\}$.

Sketch of proof: *We use the previous proposition with the fact that any integer vector has at least one child but no more than $n-1$ children (the generation root is the only one having n children).*

The bound is optimal for trivial groups ($\{e\} \subset \mathfrak{S}_n$), and seems to be better as the permutation group is of small cardinality. This relative error becomes better as we go up along the degree and tends to become optimal when the degree goes to infinity.

3.1.2 Complexity of testing if a vector is canonical

We now investigate the complexity of Algorithm 1. We need first to select a reasonable statistic to collect, which will define the complexity of this algorithm.

The explosion appearing in the algorithm is conditioned by the size of the set new_todo. For $\mathbf{v}$ an integer vector and $\{T_1, \ldots, T_n\}$ a *strong generating system* of a permutation G, when i runs over $\{1, 2, \ldots, n\}$ in the main loop, the set new_todo_i contains at step i:

$$new\_todo_i = \{g_1 \cdots g_i \cdot \mathbf{v} | g_1 \cdots g_i \cdot \mathbf{v} =_i \mathbf{v}, \forall j \leqslant i : g_j \in T_j\}$$

The right statistic to record is the size of the union of the new_todo_i for all i such that the algorithm is still running: that corresponds to the part of the orbit explored by the algorithm. This statistic appears to be very difficult to evaluate by a theoretical way. However, collecting it with a computer is a simple task.

3.1.3 Parallelization and memory complexity

Let us note that this generation engine is trivially amenable for parallelism: one can devote the study of each branch to a different processor. Our implementation uses a little framework `SearchForest`, co-developed by the author, for exploration trees and map-reduce operations on them. To get a parallel implementation, it is sufficient to use the drop-in parallel replacement for `SearchForest` under development by Jean-Baptiste Priez and Florent Hivert.

The memory complexity of the generation engine is reasonable, bounded by the size of the answer. Indeed, we keep in the cache only the *Canonical* vectors of degree $d-1$ when we search for those in degree d. In case one wants to only *iterate* through the elements of a given degree d, then this can be achieved with memory complexity $O(nd)$.

3.2 Benchmarks design

To benchmark our implementation, we chose the following problem as test-case.

Problem 3.3 *Let n be a positive integer and $G \subset \mathfrak{S}_n$ a permutation group. Iterate through all the canonical integer vectors v under the staircase (i.e. $v_i \leq n-i$).*

A vector $\mathbf{v}$ of length n is said to be *under the staircase* when it is componentwise smaller than the vector $(n-1, n-2, \dots 1, 0)$.

This problem contains essentially all difficulties that can appear. The family of $n!$ integer vectors under the staircase contains vectors with trivial automorphism group as well as vectors with a lot of symmetries. Applications also require to deal with this problem as the corresponding family of monomials plays a crucial role in algebra.

3.2.1 Benchmarks for transitive permutation groups

We now need a good family of permutation groups, representative of the practical use cases. We chose to use the database of all transitive groups of degree ≤ 30 [Hulpke(2005)] available in `Sage` through the system `GAP` [GAP(1997)].

The benchmarks have been run on an off-the-shelf `2.40 GHz` dual core Mac Book laptop running `Ubuntu 12.4` and `Sage` version 5.3.

3.3 Benchmarks

3.3.1 Tree Structure over integer vectors

This first benchmark investigates the efficiency of the tree structure presented in Section 2.1. As we don't test children of non canonical integer vectors, one wants to take measures of the part of tested non canonical vectors (which corresponds to the useless part of computations). For that, we solve Problem 3.3 for each group of the database and we collect the following information as follows.

Transitive Groups of degree 5				
Database Id.	$\|G\|$	Index in $\mathfrak{S}_n$	*Canonicals*	number of tests
1	5	24	71	81
2	10	12	68	81
3	20	6	46	67
4	60	2	41	67
5	120	1	41	67

This table displays the statistics for transitive groups of degree 5. *Database Id.* is the integer indexing the group, $|G|$ and *Index in* $\mathfrak{S}_n$ are respectively the cardinality and the index of the group G in the symmetric group $\mathfrak{S}_n$. *Canonicals* denotes the number of canonical vectors under the staircase and *number of tests* is the number of times the algorithm testing if an integer vector is canonical is called.

From this information, we set a quantity Err defined as follows:

$$Err := \frac{\text{number of tests} - \text{Canonicals}}{\text{Canonicals}}.$$

The following figure shows Err depending on the index $\frac{n!}{|G|}$. The figure contains 166 crosses, one for each transitive group over at most 10 variables. We use a logarithmic scale on the x axis.

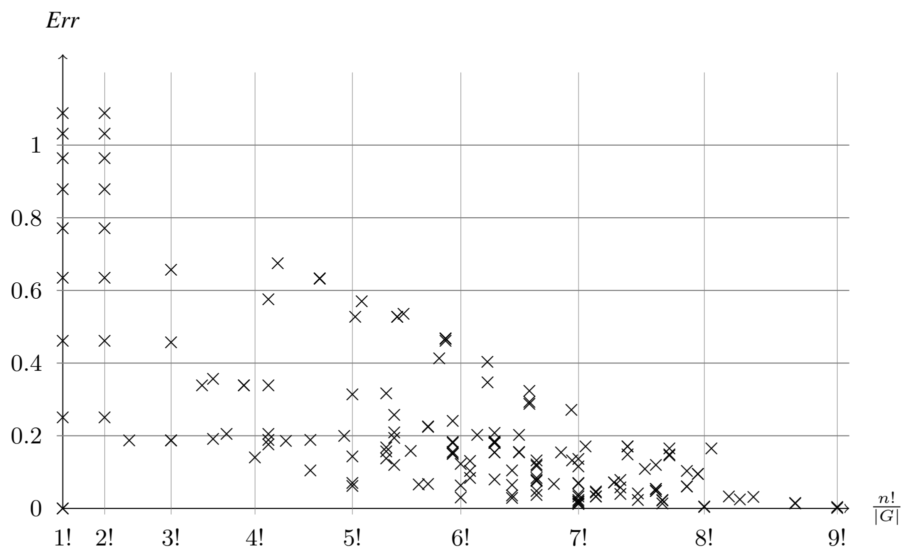


Figure 2: Relative Error between number of tested vectors and number of *canonicals* vectors.

3.3.2 Empirical complexity of testing if a vector is canonical

Algorithm 1 needs to explore a part of the orbit of the tested integer vectors. The following table displays for each transitive group over 5 variables, the number of elements of all orbits of tested vectors solving

Problem 3.3 compared to the total number of integer vectors explored.

Transitive Groups of degree 5				
Database Id.	$\|G\|$	Index in $\mathfrak{S}_n$	total orbits	total explored
1	5	24	401	351
2	10	12	691	393
3	20	6	1091	365
4	60	2	1891	328
5	120	1	1891	326

Now we define $Ratio$ to be the average size of the orbit needed to be explored to know if an integer vector is canonical:

$$Ratio := \frac{\text{total explored}}{\text{total orbits}}.$$

The following figure plots $Ratio$ in terms of $|G|$ for transitive groups on at most 9 variables.

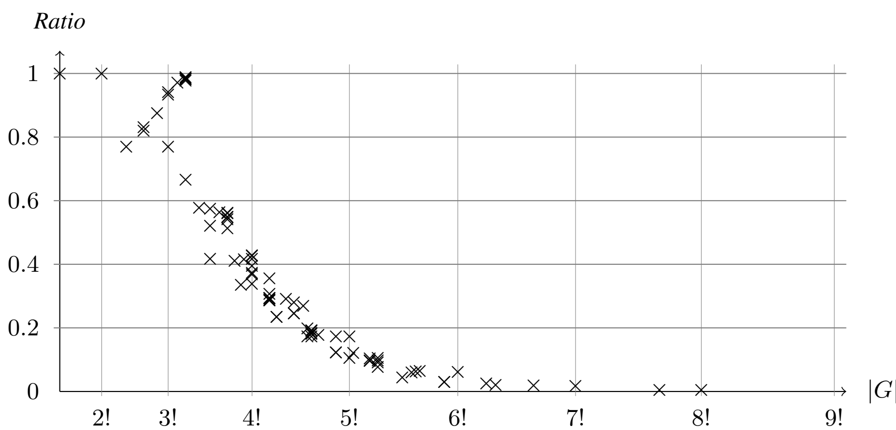


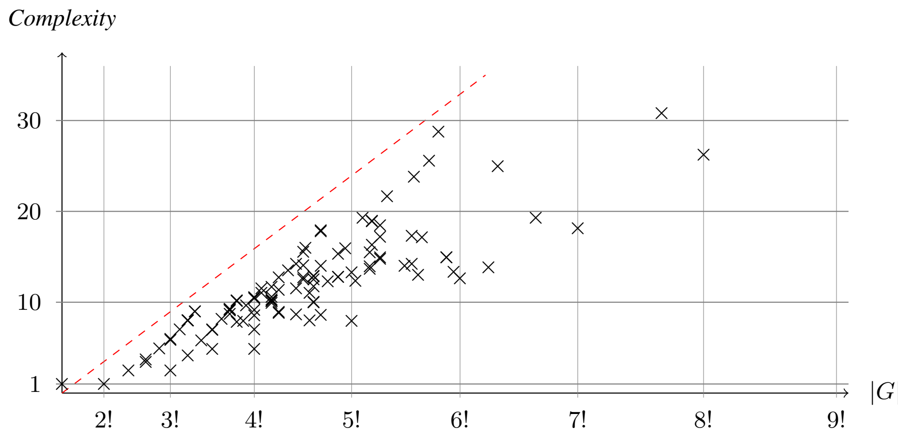
Figure 3: Average, over all integer vectors v under the stair case, of the number of vectors in the orbit of v explored by `is_canonical(v)`.

3.3.3 Overall empirical complexity of the generation engine

We now evaluate the overall complexity by comparing the ratio between the computations and the size of the output. We define the measure *Complexity* as follows:

$$Complexity := \frac{\text{total explored}}{\text{Canonicals}}.$$

The following graph displays *Complexity* in terms of the size of the group $|G|$ for transitive Groups on up to 9 variables (and excluding the alternate and symmetric group of degree 9).



The dashed line has as equation $y = 5ln(|G|)$. Therefore, we get the following empirical overall complexity:

$$\text{Computations} = O(ln(|G|) \times \text{Output size})$$

3.3.4 Tests around the unlabeled graph generation problem

Although the generation engine is not optimized for the unlabeled graph generation problem, we can apply our strategy on it.

Fix n, and consider the set E of pairs of elements of n. The symmetric group $\mathfrak{S}_n$ acts on pairs by $\sigma \cdot (i,j) = (\sigma(i), \sigma(j))$ for $\sigma \in \mathfrak{S}_n$ and $(i,j) \in E$. Let G be the induced group of permutations of E. A labeled graph can be identified with the integer vector with parts in $0, 1$. Then, two graphs are isomorphic if and only if the corresponding vectors are in the same G-orbit.

Now, one needs just to know which are these permutation groups acting on pairs of integers. In the following example, we retrieve the number of graphs on n unlabeled nodes is, for small values of n is given by: 1, 1, 2, 4, 11, 34, 156, 1044, 12346, 274668, 12005168, ...

```
sage: L = [TransitiveGroup(1,1), TransitiveGroup(3,2),
TransitiveGroup(6,6), TransitiveGroup(10,12), TransitiveGroup(15,28),
TransitiveGroup(21,38), TransitiveGroup(28,502)]
sage: [IntegerVectorsModPermutationGroup(G,max_part=1).cardinality() for G in
L]

[2, 4, 11, 34, 156, 1044, 12346]
```

Notice that our generation engine generalizes the graph generation problem in two directions. Removing the option `max_part`, one enumerates multigraphs (graphs with multiple edges between nodes). On the other hand, graphs correspond to special cases of permutation groups. From an algebraic point of view, we saw graphs as monomials whose exponents are 0 or 1, canonical for the action of the symmetric group on pairs of nodes.

4 Computing the invariants ring of a permutation group

Let us explain how the generation engine from Section 2 is plugged into effective invariant theory (see [Derksen and Kemper(2002)] and [King(2007)]).

A well-known application to build an *invariant polynomial* under the action of a permutation group G is the Reynolds operator R. From any polynomial P in n variables $\mathbf{x} := x_1, x_2, \ldots, x_n$, the invariant is

$$R(P) := \frac{1}{|G|} \sum_{\sigma \in G} \sigma \cdot P,$$

where $\sigma \cdot P$ is the polynomial built from P for which σ has permuted by position the tuple of variables $(x_1, x_2, \ldots, x_n)$. Formally, for any $\sigma \in G$

$$(\sigma \cdot P)(x_1, x_2, ..., x_n) := P(x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \ldots, x_{\sigma^{-1}(n)}).$$

For large groups, the Reynolds operator is not very convenient to build invariant polynomials. If P is a monomial $\mathbf{x}^{\mathbf{a}} := x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ where $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, the minimal invariant one can build in number of terms is the orbit sum $\sum_{\mathrm{Orb}(\mathfrak{S})} (\mathbf{x}^{\mathbf{a}})$ of $\mathbf{x}$.

Let $\mathbb{K}$ a field, we denote by $\mathbb{K}[\mathbf{x}]^G$ the ring formed by all polynomials invariant under the action of G.

$$\mathbb{K}[\mathbf{x}]^G := \{P \in \mathbb{K}[\mathbf{x}] | \forall \sigma \in G : \sigma \cdot P = P\}.$$

For any subgroups G of $\mathfrak{S}_n$ and $\mathbb{K}$ a field of characteristic 0, a result due to Hilbert and Noether state that the ring of invariant $\mathbb{K}[\mathbf{x}]^G$ is a free module of rank $\frac{n!}{|G|}$ over the symmetric polynomials in the variable $\mathbf{x}$. Computing the invariant ring $\mathbb{K}[\mathbf{x}]^G$ consists essentially in building algorithmically an explicit family (called *secondary invariant polynomials*) of generators of this free module.

Searching the secondary invariant polynomials from orbit sum of monomials whose vector of exponents is *canonical* (instead of all monomials) produces a gain of complexity of $|G|$ if we assume that all orbits are of cardinality $|G|$. This assumption is obviously false; however, in practice, it seems to hold in average and up to a constant factor [Borie(2011)]).

In [Borie and Thiéry(2011)], the authors calculate the secondary invariants of the 61^{st} transitive group over 14 variables whose cardinality is 50803200. Using the *canonical* monomials, they managed to build a family of 28 irreducible secondary invariants deploying a set of 1716 secondary invariants. This computation is unreachable by Gröbner basis techniques.

5 Computing primitive invariants for a permutation group

5.1 Introduction

We now apply our generation strategy to this problem concerning effective Galois theory.

Problem 5.1 *Let n a positive integer and G a permutation group, subgroup of $\mathfrak{S}_n$. Let $\mathbb{K}$ be a field and $\mathbf{x} := x_1, \ldots, x_n$ be n formal variables. Find a polynomial $P \in \mathbb{K}[x_1, \ldots, x_n]$ such that*

$$\{\sigma \in \mathfrak{S}_n | \sigma \cdot P = P\} = G.$$

A such polynomial is called a primitive invariant *for G.*

Problem 5.1 (exposed in [Girstmair(1987)] and [Abdeljaouad(2000)]) consists in finding an invariant P under the action of G such that its stabilizer $Stab_{\mathfrak{S}_n}(P)$ in $\mathfrak{S}_n$ is equal to G itself. Solving this problem becomes difficult when we want to construct a *primitive invariant* of minimal degree or a *primitive invariant* with a minimal number of terms.

5.2 Primitive invariant of minimal degree

Algorithm 2 Primitive invariant using stabilizer refinement

Prerequisites :
- $IntegerVectorsModPermgroup$: module to enumerate orbit representatives;
- $stabilizer\_of\_orbit\_of(G, v)$: a function returning the permutation group which stabilizes the orbit of v under the action of the permutation group G.

Arguments:
- G: A permutation group, subgroup of $\mathfrak{S}_n$.

```
def minimal_primitive_invariant(G) :
    cumulateStab ← SymmetricGroup(degree(G))
    chain ← [[(0, 0, . . . , 0), cumulateStab, cumulateStab]]
    if Cardinality(cumulateStab) == Cardinality(G) :
        return chain
    for v ∈ IntegerVectorsModPermgroup(G) :
        AutV ← stabilizer_of_orbit_of(G, v)
        Intersect ← cumulateStab ∩ AutV
        if Cardinality(Intersect) < Cardinality(cumulateStab) :
            chain ← chain ∪ [v, AutV, Intersect]
            cumulateStab ← Intersect
            if Cardinality(cumulateStab) == Cardinality(G) :
                return chain
```

5.3 Benchmarks

Algorithm 2 terminates in less than an hour for any subgroup of $\mathfrak{S}_{10}$. Even, it can calculate some primitive invariants for a lot of subgroups with degree between 10 and 20 while the literature only provides examples up to degree 7 or 8. Using the same computer, this benchmark just collects the average time in seconds of execution of Algorithm 2 by executing systematically the algorithm on transitive groups of degree n.

Degree of Groups	1	2	3	4	5	6	7	8	9
Computations time	0.008	0.064	0.104	0.160	0.208	0.393	0.537	2.364	27.093

This research was driven by computer exploration using the open-source mathematical software `Sage` [Stein et al.(2009)]. In particular, we perused its algebraic combinatorics features developed by the `Sage-Combinat` community [Sage-Combinat community(2008)], as well as its group theoretical features provided by `GAP` [GAP(1997)].

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Diagrams of affine permutations, balanced labellings, and affine Stanley symmetric functions (Extended Abstract)

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Abstract. We study the *diagrams* of affine permutations and their *balanced labellings*. As in the finite case, which was investigated by Fomin, Greene, Reiner, and Shimozono, the balanced labellings give a natural encoding of reduced decompositions of affine permutations. In fact, we show that the sum of weight monomials of the *column strict* balanced labellings is the affine Stanley symmetric function defined by Lam and we give a simple algorithm to recover reduced words from balanced labellings. Applying this theory, we give a necessary and sufficient condition for a diagram to be an affine permutation diagram. Finally, we conjecture that if two affine permutations are *diagram equivalent* then their affine Stanley symmetric functions coincide.

Résumé. Nous étudions les schémas de permutations affines et de leurs étiquetages équilibrés. Comme ce fut le cas fini, qui a été étudiée par Fomin, Greene, Reiner, et Shimozono, les étiquetages équilibrés donner un codage naturel des décompositions réduites de permutations affines. En fait, nous montrons que l'addition des monômes poids de la colonne strictes étiquetages équilibrés est le symétrique affine de Stanley fonction définie par Lam, et nous donnons un algorithme simple pour récupérer des mots réduits étiquetages équilibrés. Sur l'application de cette théorie, nous donnons une condition nécessaire et suffisante pour qu'un diagramme soit un schéma affine permutation. Enfin, nous supposons que si deux permutations affines sont les schémas équivalents puis leurs fonctions symétriques affines Stanley coïncident.

Keywords: affine permutations, permutation diagrams, balanced labellings, reduced words, Stanley symmetric functions

1 Introduction

The *diagram*, or the *Rothe diagram* of a permutation is a widely used technique to visualize the inversions of the permutation on the plane. It is well known that there is a one-to-one correspondence between the permutations and the set of their inversions.

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Balanced labellings are labellings of the diagram $D(w)$ of a permutation $w \in \Sigma_n$ such that each cell of the diagram is *balanced*. They are defined in [FGRS97] to encode reduced decompositions of the permutation w. There is a notion of *injective labellings* which generalize both standard Young tableaux and Edelman-Greene's balanced tableaux [EG87], and *column strict labellings* which generalize semi-standard Young tableaux. Column strict labellings yield symmetric functions in the same way semi-standard Young tableaux yield Schur functions. In fact, these symmetric functions $F_w(x)$ are the *Stanley symmetric functions*, which were introduced to calculate the number of reduced decompositions of $w \in \Sigma_n$ [Sta84]. The Stanley symmetric function coincides with the Schur function when w is a *Grassmannian* permutation. Furthermore, if one imposes flag conditions on column strict labellings, they yield Schubert polynomial of Lascoux and Schützenberger [LS85]. One can directly observe the limiting behaviour of Schubert polynomials (e.g. stability, convergence to $F_w(x)$, etc.) in this context. In [FGRS97] it was also shown that the balanced flagged labellings form a basis of the Schubert modules whose character is the Schubert polynomial.

The main purpose of this paper is to extend the idea of diagrams and balanced labellings to affine permutations. We first define the diagrams of affine permutations and balanced labellings on them. Following the footsteps of [FGRS97], we show that the column strict labellings on affine permutation diagrams yield the affine Stanley symmetric function defined by Lam in [Lam06]. When a permutation is 321-*avoiding affine Grassmannian*, the balanced labellings coincide with semi-standard cylindric tableaux, and they yield the cylindric Schur function of Postnikov [Pos05].

Also as a byproduct of balanced labellings, we give the complete characterization of diagrams of affine permutations using the notions of *content*. We will introduce the notion of a *wiring diagram* of an affine permutation diagram in the process, which generalizes Postnikov's wiring diagram of Grassmannian permutations [Pos06].

2 Balanced labellings and reduced words

In this section we study balanced labellings and their relations with reduced words and affine Stanley symmetric functions. Our terms, lemmas, and theorems will be in parallel with [FGRS97], extending them from finite to affine permutations. Although most of the definitions in [FGRS97] will remain the same with slight modifications, we state them here for the sake of completeness.

Let $\widetilde{\Sigma}_n$ denote the affine symmetric group generated by $s_0, s_1, \ldots, s_{n-1}$ satisfying the relations

$$\begin{aligned} s_i^2 &= 1 && \text{for all } i \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} && \text{for all } i \\ s_i s_j &= s_j s_i && \text{for } |i-j| \geq 2 \end{aligned}$$

where the indices are taken modulo n. An element w of $\widetilde{\Sigma}_n$ is called an *affine permutation (of period n)*. A *reduced decomposition* of w is a decomposition $w = s_{i_1} \cdots s_{i_\ell}$ where ℓ is the minimal number for which such a decomposition exists. In this case, ℓ is called the *length* of w and denoted by $\ell(w)$. The word $i_1 i_2 \cdots i_\ell$ is called a *reduced word* of w.

Another way to realize $\widetilde{\Sigma}_n$ is as the set of bijections $w : \mathbb{Z} \to \mathbb{Z}$ such that $w(i+n) = w(i) + n$ and $\sum_{i=1}^{n} w(i) = n(n+1)/2$. In this alternative realization, one can write each element of $\widetilde{\Sigma}_n$ in the *window notation*: $w = [w(1), \cdots, w(n)]$, since these n numbers are sufficient for identifying w. The (finite) symmetric group Σ_n generated by $s_1, \cdots, s_{n-1}$ can be naturally embedded in $\widetilde{\Sigma}_n$ and the window notation for elements of Σ_n is the usual one-line notation for finite permutations. We will call w *finite* if it is contained in this subgroup.

The *diagram*, or *affine permutation diagram*, of $w \in \widetilde{\Sigma}_n$ is the set

$$D(w) = \{(i, w(j)) \mid i < j, w(i) > w(j)\} \subseteq \mathbb{Z} \times \mathbb{Z}.$$

This is a natural generalization of the *Rothe diagram* for finite permutations. When w is finite, $D(w)$ consists of infinite number of identical copies of the Rothe diagram of w diagonally.

Throughout this paper, we will use a matrix-like coordinate system on $\mathbb{Z} \times \mathbb{Z}$: The vertical axis corresponds to the first coordinate increasing as one moves toward south, and the horizontal axis corresponds to the second coordinate increasing as one moves toward east. We will visualize $D(w)$ as the collection of unit square lattice boxes on $\mathbb{Z} \times \mathbb{Z}$ whose positions are given by $D(w)$.

From the construction it is clear that $(i, j) \in D(w) \Leftrightarrow (i+n, j+n) \in D(w)$. We will call a collection D of unit square lattice boxes on $\mathbb{Z}\times\mathbb{Z}$ an *affine diagram* (*of period* n) if there are finite number of cells on each row and column, and $(i, j) \in D \Leftrightarrow (i+n, j+n) \in D$. Obviously $D(w)$ is an affine diagram of period n. For an affine diagram D, we will call the collection of boxes $\{(i + rn, j + rn) \mid r \in \mathbb{Z}\}$ a *cell* of D, and denote it by $\overline{(i, j)}$. From the periodicity, we can take the representatives of each cell $\overline{(i, j)}$ in the first n rows $\{1, 2, \ldots, n\} \times \mathbb{Z}$, called the *fundamental window*. Each horizontal strip $\{1 + rn, \cdots, n + rn\} \times \mathbb{Z}$ for some $r \in \mathbb{Z}$ will be called a *window*. The intersection of D and the fundamental window will be denoted by $[D]$. The boxes in $[D]$ are the natural representatives of the cells of D. An affine diagram D is said to be of size ℓ if the number of boxes in $[D]$ is ℓ. Note that the size of $D(w)$ for $w \in \widetilde{\Sigma}_n$ is the length of w.

2.1 Balanced labellings: basic definitions and results

In this section, we define the notion of balanced labellings of affine diagrams.

To each cell (i, j) of an affine diagram D, we associate the *hook* $H_{i,j} := H_{i,j}(D)$ consisting of the cells (i', j') of D such that either $i' = i$ and $j' \geq j$ or $i' \geq i$ and $j' = j$. The cell (i, j) is called the *corner* of $H_{i,j}$.

Definition 2.1 (Balanced hooks) *A labelling of the cells of $H_{i,j}$ with positive integers is called* balanced *if it satisfies the following condition: if one rearranges the labels in the hook so that they weakly increase from right to left and from top to bottom, then the corner label remains unchanged.*

A labelling of an affine diagram is a map $T : D \to \mathbb{Z}_{>0}$ from the boxes of D to the positive integers such that $T(i, j) = T(i + n, j + n)$ for all $(i, j) \in D$. In other words, it sends each cell $\overline{(i, j)}$ to some positive integer. Therefore if D has size ℓ, there can be at most ℓ different numbers for the labels of the boxes in D.

Definition 2.2 (Balanced labellings) *Let D be an affine diagram of size ℓ.*

1. *A labelling of D is* balanced *if each hook $H_{i,j}$ is balanced for all $(i, j) \in D$.*

2. *A balanced labelling is* injective *if each of the labels* $1, \cdots, \ell$ *appears exactly once in* $[D]$.

3. *A balanced labelling is* column strict *if no column contains two equal labels.*

Given $w \in \widetilde{\Sigma}_n$ and its reduced decomposition $w = s_{a_1} \cdots s_{a_\ell}$, we read from left to right and interpret s_k as adjacent transpositions switching the numbers at $(k + rn)$-th and $(k + 1 + rn)$-th positions, for all $r \in \mathbb{Z}$. In other words, w can be obtained from applying the sequence of transpositions $s_{a_1}, s_{a_2}, \ldots, s_{a_\ell}$ to the identity permutation. It is clear that each s_i corresponds to a unique inversion of w. Here, an *inversion* of w is a family of pairs $\{(w(i+rn), w(j+rn)) \mid r \in \mathbb{Z}\}$ where $i < j$ and $w(i) > w(j)$. Note that $w(i + rn) > w(j + rn) \Leftrightarrow w(i) > w(j)$. Often we will ignore r and use a representative of pairs when we talk about the inversions. On the other hand, each cell of $D(w)$ also corresponds to a unique inversion of w. In fact, $(i, j) \in D(w)$ if and only if $(w(i), j)$ is an inversion of w.

Definition 2.3 (Canonical labelling) *Let $w \in \widetilde{\Sigma}_n$ be of length ℓ, and $a = a_1 a_2 \cdots a_\ell$ be a reduced word of w. Let $T_a : D \to \{1, \cdots, \ell\}$ be the injective labelling defined by setting $T_a(i, w(j)) = k$ if s_{a_k} transposes $w(i)$ and $w(j)$ in the partial product $s_{a_1} \cdots s_{a_k}$ where $w(i) > w(j)$. Then T_a is called the* canonical labelling *of $D(w)$ induced by a.*

Theorem 2.4 *Let $\mathcal{R}(w)$ denote the set of reduced words of $w \in \widetilde{\Sigma}_n$, and $\mathcal{B}(D)$ denote the set of injective balanced labellings of the affine diagram D. The correspondence $a \mapsto T_a$ is a bijection between $\mathcal{R}(w)$ and $\mathcal{B}(D(w))$.*

Theorem 2.4 follows as a corollary from more general results in next sections, namely Lemma 2.11 and Theorem 2.12. Algorithm to decode the reduced word from a balanced labelling will be given in Section 2.3. The following lemma will be useful in the proof.

Lemma 2.5 (Localization) *Let $w \in \widetilde{\Sigma}_n$ and let T be an injective labelling of $D(w)$. Then T is balanced if and only if for all integers $i < j < k$ the restriction of T to the sub-diagram of $D(w)$ determined by the intersections of rows i, j, k and columns $w(i), w(j), w(k)$ is balanced.*

2.2 Column strict balanced labellings and affine Stanley symmetric functions

In this section we consider column strict balanced labellings of affine permutation diagrams. We show that they give us the affine Stanley symmetric function in the same way the semi-standard Young tableaux give us the Schur function.

Affine Stanley symmetric functions are symmetric functions parametrized by affine permutations. They are defined in [Lam06] as an affine counterpart of the Stanley symmetric function [Sta84]. Like Stanley symmetric functions, they play an important role in combinatorics of reduced words. The affine Stanley symmetric functions also have a natural geometric interpretation [Lam08], namely they are pullbacks of the cohomology Schubert classes of the affine flag variety $LSU(n)/T$ to the affine Grassmannian $\Omega SU(n)$ under the natural map $\Omega SU(n) \to LSU(n)/T$. There are various ways to define the affine Stanley symmetric function, including the geometric one above. For our purpose, we use one of the two combinatorial definitions in [Lam10].

A word $a_1 a_2 \cdots a_\ell$ with letters in $\mathbb{Z}/n\mathbb{Z}$ is called *cyclically decreasing* if (1) each letter appears at most once, and (2) whenever i and $i+1$ both appears in the word, $i+1$ precedes i. An affine permutation $w \in \widetilde{\Sigma}_n$

is called cyclically decreasing if it has a cyclically decreasing reduced word. We call $w = v_1 v_2 \cdots v_r$ *cyclically decreasing factorization* of w if each $v_i \in \widetilde{\Sigma}_n$ is cyclically decreasing, and $\ell(w) = \sum_{i=1}^{r} \ell(v_i)$.

Definition 2.6 ([Lam10]) *Let $w \in \widetilde{\Sigma}_n$ be an affine permutation. The affine Stanley symmetric function $\widetilde{F}_w(x)$ corresponding to w is defined by*

$$\widetilde{F}_w(x) := \widetilde{F}_w(x_1, x_2, \cdots) = \sum_{w = v_1 v_2 \cdots v_r} x_1^{\ell(v_1)} x_2^{\ell(v_2)} \cdots x_r^{\ell(v_r)},$$

where the sum is over all cyclically decreasing factorization of w.

Given an affine diagram D, let $\mathcal{CB}(D)$ denote the set of column strict balanced labellings of D. Now we can state our first main theorem.

Theorem 2.7 *Let $w \in \widetilde{\Sigma}_n$ be an affine permutation. Then*

$$\widetilde{F}_w(x) = \sum_{T \in \mathcal{CB}(D(w))} x^T$$

where x^T denotes the monomial $\prod_{(i,j)\in[D(w)]} x_{T(i,j)}$

Remark 2.8 *In the case of finite permutations, Fomin, Greene, Reiner, and Shimozono also showed that the generating function for the balanced labellings under certain flag condition is the Schubert polynomial $\mathfrak{S}_w$. In fact, if $\mathcal{CFB}(D(w))$ is the set of all column strict balanced labellings T such that $T(i,j) \le i$ for all $(i,j) \in D(w)$, then $\mathfrak{S}_w = \sum_{T \in \mathcal{CFB}(D(w))} x^T$. One may regard this formula as a direct translation of the result of Billey, Jockusch, and Stanley [BJS93] to the language of balanced labellings. It would be really interesting if we could extend this result to affine permutations.*

Comparing the coefficients of $x_1 x_2 \cdots x_{\ell(w)}$ in both sides of Theorem 2.7, we see that the set of reduced words of w and the set of injective balanced labellings of $D(w)$ have the same cardinality. See Theorem 2.4 and Section 2.3 for an explicit bijection between them.

Definition 2.9 (Border cell) *Let $w \in \widetilde{\Sigma}_n$ and $\overline{(i,j)}$ be a cell of $D(w)$. If $w(i+1) = j$ then the cell $\overline{(i,j)}$ is called a* border cell *of $D(w)$.*

The border cells correspond to the (right) descents of w, i.e. the simple reflections that can appear at the end of some reduced decomposition of w. When we multiply a descent of w to w from right side, we get an affine permutations whose length is $\ell(w) - 1$. It is immediate that this operation changes the diagram in the following manner:

Lemma 2.10 *Let s_i be a descent of w, and $\alpha = \overline{(i,j)}$ be the corresponding border cell of $D(w)$. Let $D(w) \setminus \alpha$ denote the diagram obtained from $D(w)$ by deleting every cell $(i + rn, j + rn)$ and exchanging rows $(i + rn)$ and $(i + 1 + rn)$, for all $r \in \mathbb{Z}$. Then the diagram $D(ws_i)$ is $D(w) \setminus \alpha$.*

Lemma 2.11 *Let T be a columns strict balanced labelling of $D(w)$ with largest label M. Then every row containing an M must contain an M in a border cell. In particular, if i is the index of such row, then i must be a descent of w.*

Theorem 2.12 *Let T be any labelling of $D(w)$, and assume some border cell α contains the largest label M in T. Then T is balanced if and only if $T \setminus \alpha$ is balanced.*

Note that Theorem 2.4 in the previous section follows directly from these results. We also obtain a recurrence relation on the number of injective balanced labellings.

Corollary 2.13 *Let $b_{D(w)}$ denote the number of injective balanced labellings of $D(w)$. Then,*

$$b_{D(w)} = \sum_{\alpha} b_{D(w)\backslash\alpha},$$

where the sum is over all border cells α of $D(w)$.

2.3 Encoding and decoding of reduced decompositions

In this section we present a direct combinatorial formula for decoding reduced words from injective balanced labellings of affine permutation diagrams. Again, the theorem in [FGRS97] extends to the affine case naturally.

Theorem 2.14 *Let $w \in \widetilde{\Sigma}_n$ and T be an injective balanced labelling of $D(w)$. Let α be the box in $[D]$ labelled by k. Let*

$$\begin{aligned} I(k) &:= \text{ the row index of } \alpha, \\ R^{+}(k) &:= \text{ the number of entries } k' > k \text{ in the same row of } \alpha, \\ U^{+}(k) &:= \text{ the number of entries } k' > k \text{ above } \alpha \text{ in the same column, and} \\ a_k &:= (I(k) + R^{+}(k) - U^{+}(k)) \text{ modulo } n. \end{aligned}$$

Then $a = a_1 a_2 \cdots a_{\ell(w)}$ is a reduced word of w, and T is the canonical labelling T_a induced by a.

3 Characterization of affine permutation diagrams

One unexpected application of balanced labellings is a nice characterization of affine permutation diagrams. We will introduce the notion of the *content map* of an affine diagram, which generalizes the classical notion of content of a Young diagram. We will conclude that the existence of such map, along with the *North-West property*, completely characterizes the affine permutation diagrams.

3.1 The content map

Given an affine diagram D of size n, the *oriental labelling* of D will denote the injective labelling of the diagram with numbers from 1 to n such that the numbers increases as we read the boxes in $[D]$ from top to bottom, and from right to left. See Figure 1. (This reading order reminds us of the traditional way to write and read a book in some East Asian countries such as Korea, China, or Japan, and hence the term "oriental".)

Lemma 3.1 *The oriental labelling of an affine (or finite) diagram is a balanced labelling.*

Now, suppose we start from an affine permutations and we construct the oriental labelling of the diagram of the permutation. For example, let $w = [2,6,1,4,3,7,8,5] \in \Sigma_8 \subset \widetilde{\Sigma}_8$. Figure 1 shows the oriental labelling of the diagram of w, where the box labelled by 7 is at the (1,1)-coordinate.

Following the spirit of Theorem 2.14, for each box with label k in the diagram, let us write down the integer a_k where $a_k = I(k) + R^+(k) - U^+(k)$. Recall that $I(k)$ is the row index, $R^+(k)$ the number of entries greater than k in the same row, $U^+(k)$ the number of entries greater than k and located above k in the same column. The formula is actually much simpler in the case of the oriental labelling, since $U^+(k)$ vanishes and $R^+(k)$ is simply the number of boxes to the left of the box labelled by k. Figure 2 illustrates the diagram filled with a_k instead of k. From Theorem 2.14, we already know that we can recover the affine permutation we started with by a_k's. For example, $w = [2,6,1,4,3,7,8,5] = s_5s_6s_7s_4s_3s_4s_1s_2$, where the right hand side comes from reading the Figure 2 "orientally" modulo 8.

Motivated by this example, we define a special way of assigning integers to each box of a diagram, which will take a crucial role in the rest of this section.

Definition 3.2 *Let D be an affine diagram with period n. A map $\mathcal{C} : D \to \mathbb{Z}$ is called a* content map *if it satisfies the following four conditions.*

(C1) If boxes b_1 and b_2 are in the same row (respectively, column), b_2 being to the east (resp., south) to b_1, and there are no boxes between b_1 and b_2, then $\mathcal{C}(b_2) - \mathcal{C}(b_1) = 1$.

(C2) If b_2 is strictly to the southeast of b_1, then $\mathcal{C}(b_2) - \mathcal{C}(b_1) \geq 2$.

(C3) If $b_1 = (i,j)$ and $b_2 = (i+n, j+n)$ coordinate-wise, then $\mathcal{C}(b_2) - \mathcal{C}(b_1) = n$.

(C4) For each row (resp., column), the content of the leftmost (resp., topmost) box is equal to the row (resp., column) index.

Proposition 3.3 *Let D be the diagram of an affine permutation $w \in \widetilde{\Sigma}_n$. Then, D has a unique content map.*

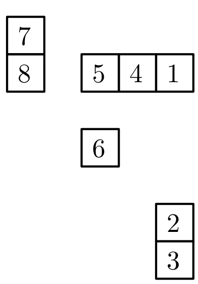

Fig. 1: oriental labelling of a finite diagram

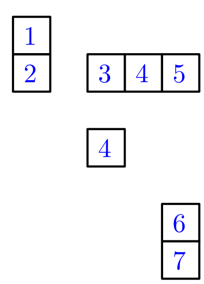

Fig. 2: a_k's of the oriental labelling

3.2 The wiring diagram and the bijection

We start this section by recalling a well-known property of (affine) permutation diagrams.

Definition 3.4 *An affine diagram is called* North-West *(or* NW*) if, whenever there is a box at* (i, j) *and at* (k, ℓ) *with the condition* $i < k$ *and* $j > \ell$*, there is a box at* (i, ℓ)*.*

It is easy to see that every affine permutation diagram is NW. In fact, if $(i, w^{-1}(j))$ and $(k, w^{-1}(\ell))$ is an inversion and $i < k$, $j > \ell$, then $(i, w^{-1}(\ell))$ is also an inversion since $i < k < w^{-1}(\ell)$ and $w(i) > j > \ell$. The main theorem of this section is that the content map and the NW property completely characterize the affine permutation diagrams.

Theorem 3.5 *An affine diagram is an affine permutation diagram if and only if it is NW and admits a content map.*

In fact, given a NW affine diagram D of period n with a content map, we will introduce a combinatorial algorithm to recover the affine permutation $w \in \widetilde{\Sigma}_n$ corresponding to D. This will turn out to be a generalization of the *wiring diagram* appeared in the section 19 of [Pos06], which gave a bijection between Grassmannian permutations and the partitions.

Let D be a NW affine diagram of period n with a content map. A northern edge of a box b in D will be called a *N-boundary* of D if

(1) b is the northeast-most box among all the boxes with the same content and

(2) there is no box above b on the same column.

Similarly, an eastern edge of a box b in D will be called a *E-boundary* of D if

(1) b is the northeast-most box among all the boxes with the same content and

(2) there is no box to the right of b on the same row.

A northern or eastern edge of a box in D will be called a *NE-boundary* if it is either a N-boundary or an E-boundary. We can define an *S-boundary*, *W-boundary*, and *SW-boundary* in the same manner by replacing "north" by "south", "east" by "west", "above" by "below", "right" by "left", etc.

Now, from the midpoint of each NE-boundary, we draw an infinite ray to NE-direction (red rays in Figure 3) and index the ray "i" if it is a N-boundary of a box of content i, and "$i + 1$" if it is an E-boundary of a box of content i. We call such rays *NE-rays*. Similarly, a *SW-ray* is an infinite ray from the midpoint of each SW-boundary to SW-direction (blue rays in Figure 3), indexed "w_i" if it is a W-boundary of a box of content i, and "w_{i+1}" if it is a S-boundary of a box of content i.

Lemma 3.6 *No two NE-rays (respectively, SW-rays) have the same index, and the indices increase as we read the rays from NW to SE direction.*

Lemma 3.7 *There is no NE-ray of index* k *if and only if there is no SW-ray of index* w_k*.*

Now, given a NW affine diagram D with a content map, we construct the *wiring diagram* of D through the following procedure.

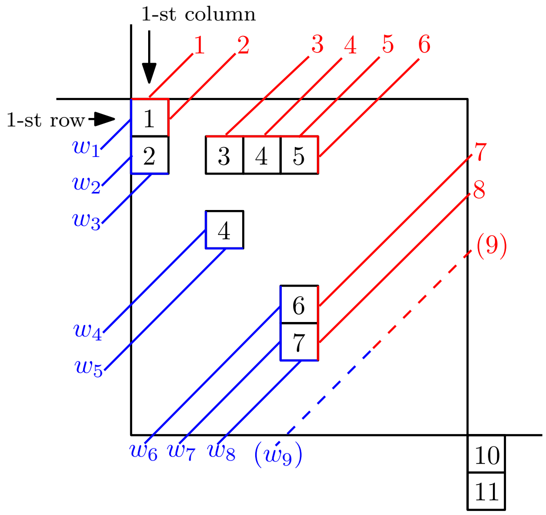

Fig. 3: content, (NE/SW-) boundaries, and rays

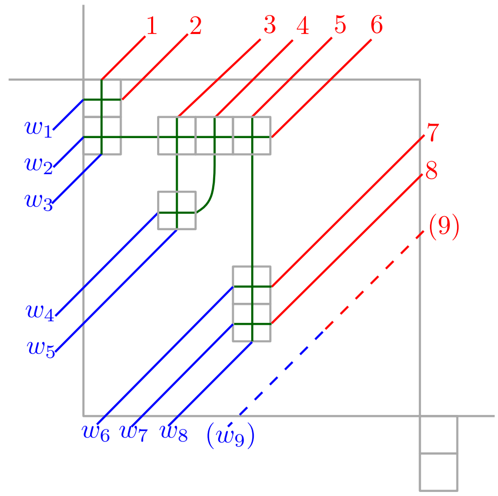

Fig. 4: wiring diagram

(a) (Rays) Draw NE- and SW-rays.

(b) (The "Crosses") Draw a "+" sign inside each box, i.e., connect the midpoint of the western edge to the midpoint of the eastern edge, and the midpoint of the northern edge to the midpoint of the southern edge of each box.

(c) (Horizontal Movement) If the box a and the box b are in the same row (a is to the left of b) and there are no boxes between them, then connect the midpoint of the eastern edge of a to the midpoint of the western edge of b.

(d) (Vertical Movement) If the box a and the box b are in the same column (a is above b) and there are no boxes between them, then connect the midpoint of the southern edge of a to the midpoint of the northern edge of b.

(e) (The "Tunnels") Suppose that the box a of content k is not the northeast-most box among all the boxes with content k and that there is no box on the same row to the right of a. Let b be the closest box to a, which is to the northeast of b and has content k. For every such pair a and b, connect the midpoint of the eastern edge of a to the midpoint of the southern edge of b.

Lemma 3.8 *Each midpoint of an edge of a box in D is connected to exactly two line segments of* (a), (b), (c), (d)*, and* (e).

Figure 4 illustrates the wiring diagram of the affine diagram of period 9 in Figure 2. Note that the curved line connecting two boxes of content 4 is a "tunnel". Once we draw this wiring diagram of a NW affine diagram with a content, it is very easy to recover the affine permutation corresponding to the diagram. From a NE-ray indexed by i, proceed to the southwest direction following the lines in the wiring diagram until we meet a SW-ray of index w_j. This translates to $w_j = i$ in the corresponding affine permutation. If there is no NE-ray of index i (equivalently, no SW-ray of index w_i), then let $w_i = i$. For instance, Figure 4 corresponds to the affine permutation $w = [w_1, w_2, \ldots, w_9] = [2, 6, 1, 4, 3, 7, 8, 5, 9] \in \Sigma_9 \subset \widetilde{\Sigma}_9$

Proposition 3.9 *The wiring diagram gives a bijection between the NW affine diagrams of period n with a content map, and the affine permutations in $\widetilde{\Sigma}_n$.*

Our main result of this section, Theorem 3.5, is a direct consequence of Proposition 3.9.

4 Diagram equivalence conjecture

In this section we consider some sufficient condition for affine permutations to have the same affine Stanley symmetric function.

For finite permutations, Stanley symmetric functions are the Frobenius character of the diagram Specht module of the permutation diagram [RS95]. By definition, the Specht module is invariant under permuting rows and columns of the diagram, hence so is the Stanley symmetric function. We call this invariance property *diagram equivalence*. We extend the notion to the affine permutations.

4.1 Diagram graphs and equivalence

To each affine diagram D, let us associate a bipartite graph G_D.

Definition 4.1 *The* diagram graph G_D *of D is defined as follows:*

1. *$G_D = (V, E_D)$ where $V = V_L \sqcup V_R$ is the vertex set, and each edge in E_D connects a vertex in V_L with a vertex in V_R. Here V_L and V_R are called the left and right vertices, respectively.*
2. *Both V_L and V_R are indexed by $\mathbb{Z}$, and $(i, j) \in E_D$ denotes the edge connecting $i \in V_L$ with $j \in V_R$.*
3. *$(i, j) \in E_D$ if and only if $(i, j) \in D$.*

It is immediate from the definition of affine diagram that G_D has the following properties: (1) $(i, j) \in E \Leftrightarrow (i+n, j+n) \in E$, (2) every vertex has a finite degree. We will call a graph that has these properties *n-periodic*.

Definition 4.2 *Two n-periodic graphs G and H are* isomorphic *if there is a pair of bijections (ϕ, ψ) between vertices of G and H such that*

1. *ϕ is a bijection between the left vertices, and ψ is between the right vertices,*
2. *$(i, j) \in E_G \Leftrightarrow (\phi(i), \psi(j)) \in E_H$.*

Definition 4.3 *Two affine permutations $u, v \in \widetilde{\Sigma}_n$ are said to be* diagram equivalent *if $G_{D(u)}$ and $G_{D(v)}$ are isomorphic.*

Conjecture 4.4 $\widetilde{F}_u(x) = \widetilde{F}_v(x)$ *if u and v are diagram equivalent.*

Remark 4.5 *Note that the converse is not true. The Stanley symmetric functions may coincide even if the permutations are not diagram equivalent. For finite skew diagrams, more precise conditions are studied in [RSvW07].*

There is a natural dihedral symmetry of the Dynkin diagram of affine type A. This symmetry gives us an obvious relation between the affine Stanley symmetric functions of certain permutations. The $*$-operator in [Lam06] is an example of this symmetry. Obviously w and $(w^*)^{-1}$ have the same affine Stanley symmetric function, and it is easy to check that they actually are diagram equivalent.

When $G_{D(w)}$ does not have any cycle, the number of reduced decompositions of w turns to be the same as the normalized volume of certain polytope called *periodic matching polytope*. Since the periodic matching polytope is invariant under the diagram equivalence, this result supports our conjecture. Precise definitions and the results on periodic matching polytopes will appear in a separate paper [Yoo13].

5 Further questions

1. Can we define the affine Schubert polynomial using balanced labellings? Geometrically, they have to form a basis of the cohomology ring of the affine flag variety, and combinatorially, they would have to satisfy an analogue of the transition equation of Schubert polynomials.

2. It is possible to define set-valued balanced labellings of affine diagrams and describe affine stable Grothen-dieck polynomials using them. Also we were able to use flagged column strict set-valued balanced labellings to describe Grothendieck polynomials. As in the previous question, we wonder if this result can be extended to affine Grothendieck polynomials.

3. One motivation of studying the diagram of affine permutations is to answer the question posed by Lam in [Lam06]. How can we characterize the "affine vexillary permutations"?

4. Can we prove Conjecture 4.4 by finding a bijection between column strict balanced labellings of two diagram equivalent permutations? For dihedral symmetry, it is easy to find such a bijection but it does not extend to the general situation.

5. Our Theorem 3.5 is not local since we have to find a global datum like content. Is there local criterion of affine permutation diagrams? More precisely, can we characterize the affine permutation diagrams by avoiding some diagram patterns?

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The probability of planarity of a random graph near the critical point

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Abstract. Erdős and Rényi conjectured in 1960 that the limiting probability p that a random graph with n vertices and $M = n/2$ edges is planar exists. It has been shown that indeed p exists and is a constant strictly between 0 and 1. In this paper we answer completely this long standing question by finding an exact expression for this probability, whose approximate value turns out to be $p \approx 0.99780$.

More generally, we compute the probability of planarity at the critical window of width $n^{2/3}$ around the critical point $M = n/2$. We extend these results to some classes of graphs closed under taking minors. As an example, we show that the probability of being series-parallel converges to 0.98003.

Our proofs rely on exploiting the structure of random graphs in the critical window, obtained previously by Janson, Łuczak and Wierman, by means of generating functions and analytic methods. This is a striking example of how analytic combinatorics can be applied to classical problems on random graphs.

Résumé. Erdős et Rényi ont conjecturé en 1960 que la probabilité limite p qu'un graphe aléatoire avec n sommets et $M = n/2$ arêtes soit planaire existe. Il a été prouvé qu'en fait p existe et est une constante comprise strictement entre 0 et 1. Dans ce travail nous fermons complètement cette question en trouvant l'expression exacte pour cette probabilité, dont la valeur approchée s'avère être $p \approx 0.99780$.

Plus genéralement, nous calculons la probabilité qu'un graphe soit planaire dans la fenêtre critique de largeur $n^{2/3}$ autour du point critique $M = n/2$. Nous étendons ces resultats à différentes classes de graphes closes par exclusion de mineurs. A titre d'exemple, nous montrons que la probabilité d'être série-parallèle converge vers 0.98003.

Nos preuves exploitent la structure des graphes aléatoires dans la fenêtre critique, décrite précedemment par Janson, Łuczak et Wierman, en utilisant les séries génératrices et des méthodes analytiques. Cet exemple notable montre que la combinatoire analytique peut être utilisée pour des problèmes classiques de graphes aléatoires.

Keywords: random graphs, planar cubic multigraphs, analytic combinatorics

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1 Introduction

The random graph model $G(n, M)$ assigns uniform probability to graphs on n labelled vertices with M edges. A fundamental result of Erdős and Rényi (1960) is that the random graph $G(n, M)$ undergoes an abrupt change when M is around $n/2$, the value for which the average vertex degree is equal to one. When $M = cn/2$ and $c < 1$, almost surely the connected components are all of order $O(\log n)$, and are either trees or unicyclic graphs. When $M = cn/2$ and $c > 1$, almost surely there is a unique giant component of size $\Theta(n)$. We direct to reader to the reference texts of Bollobás (1985) and Janson et al. (2000) for a detailed discussion of these facts.

We concentrate on the so-called *critical window* namely $M = \frac{n}{2}(1 + \lambda n^{-1/3})$, where λ is a real number, identified by Bollobás (1984a,b). Let us recall that the *excess* of a connected graph is the number of edges minus the number of vertices. A connected graph is *complex* if it has positive excess. As $\lambda \to -\infty$, complex components disappear and only trees and unicyclic components survive, and as $\lambda \to +\infty$, components with unbounded excess appear. A thorough analysis of the random graph in the critical window can be found in Janson et al. (1993) and Łuczak et al. (1994), which constitute our basic references.

For each fixed λ, we denote the random graph $G\left(n, \frac{n}{2}(1 + \lambda n^{-1/3})\right)$ by $G(\lambda)$. The *core* $C(\lambda)$ of $G(\lambda)$ is obtained by repeatedly removing all vertices of degree one from $G(\lambda)$. The *kernel* $K(\lambda)$ is obtained from $C(\lambda)$ by replacing all maximal paths of vertices of degree two by single edges. The parameter n is implicitly assumed in all the previous definitions. The graph $G(\lambda)$ satisfies asymptotically almost surely several fundamental properties, that were established by Łuczak et al. (1994) by a subtle simultaneous analysis of the $G(n, M)$ and the $G(n, p)$ models.

1. The number of complex components is bounded.
2. Each complex component has size of order $n^{2/3}$, and the largest suspended tree in each complex component has size of order $n^{2/3}$.
3. $C(\lambda)$ has size of order $n^{1/3}$ and maximum degree three, and the distance between two vertices of degree three in $C(\lambda)$ is of order $n^{1/3}$.
4. $K(\lambda)$ is a cubic (3-regular) multigraph of bounded size.

The key property for us is the last one. It implies that asymptotically almost surely the components of $G(\lambda)$ are trees, unicyclic graphs, and those obtained from a cubic multigraph K by attaching rooted trees to the vertices of K, and attaching ordered sequences of rooted trees to the edges of K. Some care is needed here, since the resulting graph may not be simple, but asymptotically this can be accounted for.

It is clear that $G(\lambda)$ is planar if and only if the kernel $K(\lambda)$ is planar. Then by counting planar cubic multigraphs it is possible to estimate the probability that $G(\lambda)$ is planar. To this end we use generating functions. The trees attached to $K(\lambda)$ are encoded by the generating function $T(z)$ of rooted trees, and complex analytic methods are used to estimate the coefficients of the corresponding series. This allows us to determine the exact probability

$$p(\lambda) = \lim_{n\to\infty} \Pr\left\{G\left(n, \tfrac{n}{2}(1 + \lambda n^{-1/3})\right) \text{ is planar}\right\}.$$

In particular, we obtain $p(0) \approx 0.99780$.

This approach was initiated in the seminal paper of Flajolet et al. (1989), where the authors determined the threshold for the appearance of the first cycles in $G(n, M)$. A basic feature in Flajolet et al. (1989)

is to estimate coefficients of large powers of generating functions using Cauchy integrals and the saddle point method. This path was followed by Janson et al. (1993), obtaining a wealth of results on $G(\lambda)$. Of particular importance for us is the determination in Janson et al. (1993) of the limiting probability that $G(\lambda)$ has given excess. The approach in Łuczak et al. (1994) is more probabilistic and has as starting point the classical estimates by Wright (1980) on the number of connected graphs with fixed excess. The range of these estimates was extended by Bollobás (1984a) and more recently the analysis was refined by Flajolet et al. (2004), by giving complete asymptotic expansions in terms of the Airy function. Very recently, the question of planarity has been analyzed in a wider setting of random subgraphs of a given graph by Frieze and Krivelevich.

The paper is organized as follows. In Section 2 we present the basic lemmas needed in the sequel. In Section 3 we compute the number of cubic planar multigraphs, suitably weighted, where we follow Kang and Łuczak (2012). In Section 4 we compute the exact probability that the random graph $G(\lambda)$ is planar as a function of λ. We generalize this result by determining the probability that $G(\lambda)$ belongs to a minor-closed class of graphs in several cases of interest.

We close this introduction with a remark. The problem of 2-satisfiability presents a striking analogy with the random graph process. Given n Boolean variables and a conjunctive formula of M clauses, each involving two literals, the problem is to determine the probability that the formula is satisfiable when M grows with n. The threshold has been established at $M = n$ and the critical window is also of width $n^{2/3}$; see Bollobás et al. (2001). However the exact probability of satisfiability when the number of clauses is $n(1 + \lambda n^{-1/3})$ has not been determined, and appears to be a more difficult problem.

2 Preliminaries

All graphs in this paper are labelled. The size of a graph is its number of vertices. A multigraph is a graph with loops and multiple edges allowed.

We recall that the exponential generating function $T(z)$ of rooted trees satisfies

$$T(z) = ze^{T(z)}.$$

Using Lagrange's inversion (see Flajolet and Sedgewick (2009)), one recovers the classical formula n^{n-1} for the number of rooted labelled trees.The generating function for unrooted trees is

$$U(z) = T(z) - \frac{T(z)^2}{2}.$$

This can be proved by integrating the relation $T(z) = zU'(z)$, or more combinatorially using the dissimilarity theorem for trees (see Otter (1948)).

A graph is unicyclic if it is connected and has a unique cycle. A unicyclic graph can be seen as an undirected cycle of length at least three to which we attach a sequence of rooted trees. Since the directed cycle construction corresponds algebraically to $\log(1/(1-z))$ (see Flajolet and Sedgewick (2009)), the generating function is

$$V(z) = \frac{1}{2}\left(\log\frac{1}{1-T(z)} - T(z) - \frac{T(z)^2}{2}\right).$$

Graphs whose components are unicyclic are given by the exponential formula:

$$e^{V(z)} = \frac{e^{-T(z)/2-T(z)^2/4}}{\sqrt{1-T(z)}}.$$

The following result, which is fundamental for us, is proved in Theorem 4 of Łuczak et al. (1994) by a careful analysis of the structure of complex components in $G(\lambda)$. We say that a property $\mathcal{P}$ holds asymptotically almost surely (a.a.s.) in $G(n, M)$ if the probability that $\mathcal{P}$ is satisfied tends to one as $n \to \infty$. Recall that $G(\lambda) = G\left(n, \frac{n}{2}(1 + \lambda n^{-1/3})\right)$.

Lemma 1 *For each λ, the kernel of $G(\lambda)$ is a.a.s. a cubic multigraph.*

Given a cubic multigraph M with a loops, b double edges and c triple edges, define its *weight* as

$$w(M) = 2^{-a}2^{-b}6^{-c}.$$

This weight (called the *compensation factor* in Janson et al. (1993)), has the following explanation. When we substitute edges of the kernel by sequences of rooted trees, a loop has two possible orientations that give rise to the same graph. A double (triple) edge can be permuted in two (six) ways, again producing the same graph. From now on, all multigraphs we consider are weighted, so that we omit the qualifier. The following lemma is proved in Janson et al. (1993) using a combination of guessing and recurrence relations. The proof we give appears in Chapter 2 of the book Bollobás (1985).

Lemma 2 *The number E_r of cubic multigraphs with $2r$ vertices is equal to*

$$E_r = \frac{(6r)!}{(3r)!2^{3r}6^{2r}}.$$

Proof: A cubic multigraph can be modeled as a pairing of darts (half-edges), 3 for each vertex, with a total of $6r$ darts. The number of such pairings is $(6r)!/((3r)!2^{3r})$. However, we have to divide by the number 6^{2r} of ways of permuting each of the $2r$ triples of darts. The weight takes care exactly of the number of times a cubic multigraph is produced in this process. $\square$

The next result is essentially proved in Janson et al. (1993). Here we present a concise proof. We denote by $[z^n]A(z)$ the coefficient of z^n in the power series $A(z)$.

Lemma 3 *The number $g(n, M, r)$ of simple graphs with n vertices, M edges and cubic kernel of size $2r$ satisfies*

$$g(n, M, r) \leq n!\,[z^n]\frac{U(z)^{n-M+r}}{(n-M+r)!}\,e^{V(z)}\,\frac{E_r}{(2r)!}\,\frac{T(z)^{2r}}{(1-T(z))^{3r}}$$

and

$$g(n, M, r) \geq n!\,[z^n]\frac{U(z)^{n-M+r}}{(n-M+r)!}\,e^{V(z)}\,\frac{E_r}{(2r)!}\,\frac{T(z)^{8r}}{(1-T(z))^{3r}}.$$

Proof: Such a graph is the union of a set of s unrooted trees, a set of unicyclic graphs, and a cubic multigraph K with a rooted tree attached to each vertex of K and a sequence (possibly empty) of rooted trees attached to each edge of K. Let us see first that $s = n - M + r$. Indeed, the final excess of edges over vertices must be $M - n$. Each tree component contributes with excess -1, each unicyclic component with excess 0, and K (together with the attached trees) with excess r. Hence $M - n = -s + r$.

The first two factors $U(z)^{n-M+r}/(n - M + r)!$ and $e^{V(z)}$ on the right-hand side of the inequalities encode the set of trees and unicyclic components. The last part encodes the kernel K. It has $2r$ vertices and is labelled, hence the factor $E_r/(2r)!$; the weighting guarantees that each graph contributing to $g(n,M,r)$ is counted exactly once. The trees attached to the $2r$ vertices give a factor $T(z)^{2r}$. The sequences of trees attached to the $3r$ edges give each a factor $1/(1-T(z))$. However, this allows for the empty sequence and the resulting graph may not be simple, so we get only an upper bound. To guarantee that the final graph is simple we take sequences of length at least two, encoded by $T(z)^2/(1-T(z))$ (length one is enough for multiple edges of K, but length two is needed for loops). Since this misses some graphs, we get a lower bound. $\square$

The following technical result is essentially Lemma 3 from Janson et al. (1993).

Lemma 4 *Let $M = \frac{n}{2}(1+\lambda n^{-1/3})$. Then for any fixed a and integer $r > 0$ we have*

$$\frac{n!}{\binom{\binom{n}{2}}{M}}[z^n]\frac{U(z)^{n-M+r}}{(n-M+r)!}\frac{T(z)^a}{(1-T(z))^{3r}}\,e^{V(z)} = \sqrt{2\pi}A\left(3r+\frac{1}{2},\lambda\right)\left(1+O\left(\frac{1+\lambda^4}{n^{1/3}}\right)\right) \quad (1)$$

uniformly for $|\lambda| \le n^{1/12}$, where

$$A(y,\lambda) = \frac{e^{-\lambda^3/6}}{3^{(y+1)/3}}\sum_{k\ge 0}\frac{\left(\frac{1}{2}3^{2/3}\lambda\right)^k}{k!\,\Gamma\big((y+1-2k)/3\big)}. \quad (2)$$

We omit the proof, which is based on relating the left-hand side of Equation (1) to the integral representation of $A(y,\lambda)$ defined in Equation (10.7) of Janson et al. (1993):

$$A(y,\lambda) = \frac{1}{2\pi i}\int_\Pi s^{1-y}e^{K(\lambda,s)}ds,$$

where $K(\lambda,s)$ is the polynomial

$$K(\lambda,s) = \frac{(s+\lambda)^2(2s-\lambda)}{6} = \frac{s^3}{3}+\frac{\lambda s^2}{2}-\frac{\lambda^3}{6}$$

and Π is a suitable path in the complex plane.

It is important to notice that in the previous lemma the final asymptotic estimate does *not* depend on the choice of a. The next result is a direct consequence and can be found as Formula (13.17) in Janson et al. (1993).

Lemma 5 *The limiting probability that the random graph $G(\lambda)$ has a cubic kernel of size $2r$ is equal to*

$$\sqrt{2\pi}\,e_r A\left(3r+\frac{1}{2},\lambda\right),$$

where $e_r = E_r/(2r)!$ (E_r is defined in Lemma 2) and $A(y,\lambda)$ is as in the previous lemma.

In particular, for $\lambda = 0$ the limiting probability is

$$\sqrt{\frac{2}{3}}\left(\frac{4}{3}\right)^r e_r\frac{r!}{(2r)!}.$$

Proof: Using the notation of Lemma 3, the probability for a given n is by definition

$$\frac{g(n, M, r)}{\binom{\binom{n}{2}}{M}}.$$

Lemma 3 gives upper and lower bounds for this probability, and using Lemma 4 we see that both bounds agree in the limit and are equal to

$$\frac{E_r}{(2r)!}\sqrt{2\pi}A\left(3r + \frac{1}{2}, \lambda\right),$$

thus proving the result. A key point is that the discrepancy between the factors $T(z)^{2r}$ and $T(z)^{8r}$ in the bounds for $g(n, M, r)$ does not affect the limiting value of the probability. □

Notice that if we replace the e_r by the numbers g_r arising by counting *planar* cubic multigraphs, we obtain immediately the probability that $G(\lambda)$ has a cubic planar kernel of size $2r$. Since $G(\lambda)$ is planar if and only if its kernel is planar, we can use this fact to compute the probability of $G(\lambda)$ being planar. But first we must compute g_r.

3 Planar cubic multigraphs

In this section we compute the numbers G_r of cubic weighted planar multigraphs of size $2r$. All multigraphs are labelled in vertices, hence the counting formulas are exponential in this parameter. The associated generating function has been obtained recently in Kang and Łuczak (2012) (generalizing the enumeration of simple cubic graphs in Bodirsky et al. (2007)), but their derivation contains some minor errors. They do not affect the correctness of Kang and Łuczak (2012), since the asymptotic estimates needed by the authors are still valid. However, for the computations that follow we need the exact values. The next result is from Kang and Łuczak (2012), the corrections are detailed below.

Lemma 6 *Let $G_1(z)$ be the generating function of connected cubic planar multigraphs. Then $G_1(z)$ is determined by the following system of equations:*

$$\begin{aligned}
3z\,\tfrac{dG_1(z)}{dz} &= D(z) + C(z)\\
B(z) &= \tfrac{z^2}{2}(D(z) + C(z)) + \tfrac{z^2}{2}\\
C(z) &= S(z) + P(z) + H(z) + B(z)\\
D(z) &= \tfrac{B(z)^2}{z^2}\\
S(z) &= C(z)^2 - C(z)S(z)\\
P(z) &= z^2C(z) + \tfrac{1}{2}z^2C(z)^2 + \tfrac{z^2}{2}\\
2(1 + C(z))H(z) &= u(z)(1 - 2u(z)) - u(z)(1 - u(z))^3\\
z^2(C(z) + 1)^3 &= u(z)(1 - u(z))^3.
\end{aligned}$$

The generating functions $B(z)$, $C(z)$, $D(z)$, $S(z)$, $P(z)$ and $H(z)$ correspond to distinct families of edge-rooted cubic planar graphs, and $u(z)$ is an algebraic function related to the enumeration of 3-connected cubic planar graphs (dually, 3-connected triangulations).

The corrections with respect to Kang and Łuczak (2012) are the following. In the first equation a term $-7z^2/24$ has been removed. In the second and sixth equations we have replaced a term $z^2/4$ by $z^2/2$. In the fourth equation we have removed a term $-z^2/16$. For the combinatorial interpretation of the various generating functions and the proof of the former equations we refer to Kang and Łuczak (2012). Notice that eliminating $u(z)$ from the last two equations we obtain a relation between $C(z)$ and $H(z)$. This relation can be used to obtain a single equation satisfied by $C(z)$, by eliminating $S(z)$, $P(z)$, $H(z)$, $D(z)$ and $B(z)$ from the first equations. We reproduce it here in case the reader wishes to check our computations.

$$\begin{aligned}
&1048576\,z^6 + 1034496\,z^4 - 55296\,z^2 + \\
&\left(9437184\,z^6 + 6731264\,z^4 - 1677312\,z^2 + 55296\right) C + \\
&\left(37748736\,z^6 + 18925312\,z^4 - 7913472\,z^2 + 470016\right) C^2 + \\
&\left(88080384\,z^6 + 30127104\,z^4 - 16687104\,z^2 + 1622016\right) C^3 + \\
&\left(132120576\,z^6 + 29935360\,z^4 - 19138560\,z^2 + 2928640\right) C^4 + \\
&\left(132120576\,z^6 + 19314176\,z^4 - 12429312\,z^2 + 2981888\right) C^5 + \\
&\left(88080384\,z^6 + 8112384\,z^4 - 4300800\,z^2 + 1720320\right) C^6 + \\
&\left(37748736\,z^6 + 2097152\,z^4 - 614400\,z^2 + 524288\right) C^7 + \\
&\left(9437184\,z^6 + 262144\,z^4 + 65536\right) C^8 + 1048576\,C^9 z^6 = 0.
\end{aligned}$$

The first terms are

$$C(z) = z^2 + \frac{25}{8}\,z^4 + \frac{59}{4}\,z^6 + \frac{11339}{128}\,z^8 + \cdots$$

This allows us to compute $B(z), D(z), S(z), P(z)$ and $H(z)$, hence also $G_1(z)$. The first coefficients of $G_1(z)$ are as follows.

$$G_1(z) = \frac{5}{24}z^2 + \frac{5}{16}z^4 + \frac{121}{128}z^6 + \frac{1591}{384}z^8 + \cdots$$

Using the set construction, the generating function $G(z)$ for cubic planar multigraphs is then

$$G(z) = e^{G_1(z)} = \sum_{r=0}^{\infty} G_r \frac{z^{2r}}{(2r)!} = 1 + \frac{5}{24}z^2 + \frac{385}{1152}z^4 + \frac{83933}{82944}z^6 + \frac{35002561}{7962624}z^8 + \cdots, \tag{3}$$

where G_r is the number of planar cubic multigraphs with $2r$ vertices. This coincides with the generating function for all cubic (non-necessarily planar) multigraphs up to the coefficient of z^4. The first discrepancy is in the coefficient of z^6. The difference between the coefficients is $1/72 = 10/6!$, corresponding to the 10 possible ways of labelling $K_{3,3}$, the unique non-planar cubic multigraph on six vertices.

4 Probability of planarity and generalizations

Let G be a graph with a cubic kernel K. Then clearly G is planar if and only if K is planar, and we can compute the probability that $G(n, M)$ is planar by counting over all possible planar kernels.

Theorem 7 *Let $g_r(2r)!$ be the number of cubic planar multigraphs with $2r$ vertices. Then the limiting probability that the random graph $G(n, M = \frac{n}{2}(1+\lambda n^{-1/3}))$ is planar is*

$$p(\lambda) = \sum_{r\geq 0} \sqrt{2\pi}\, g_r A\left(3r + \frac{1}{2}, \lambda\right).$$

In particular, the limiting probability that $G(n, \frac{n}{2})$ is planar is

$$p(0) = \sum_{r\geq 0} \sqrt{\frac{2}{3}} \left(\frac{4}{3}\right)^r g_r \frac{r!}{(2r)!} \approx 0.99780.$$

Proof: The same analysis as in Section 2 shows that $\sqrt{2\pi}\, g_r A(3r+\frac{1}{2}, \lambda)$ is the probability that the kernel is planar and has $2r$ vertices. Summing over all possible r, we get the desired result. $\square$

As already mentioned, in Erdős and Rényi (1960) it was conjectured that $p(0)$ exists and $0 < p(0) < 1$. This was proved in Łuczak et al. (1994), showing that $p(\lambda)$ exists for all λ and that $0 < p(\lambda) < 1$. The bounds in Janson et al. (1993) for $p(0)$ are

$$0.98707 < p(0) < 0.99977,$$

obtained by considering connected cubic multigraphs with at most six vertices. We remark that in Stepanov (1988) is shown that $p(\lambda) < 1$ for $\lambda \leq 0$ (without actually establishing the existence of the limiting probability). The function $p(\lambda)$ is plotted in Figure 1. As expected, $p(\lambda)$ is close to 1 when $\lambda \to -\infty$ and close to 0 when $\lambda \to \infty$. For instance, $p(-3) \approx 1 - 1.02 \cdot 10^{-7}$ and $p(5) \approx 4.9 \cdot 10^{-7}$.

Besides planar graphs, one can consider other classes of graphs. Let $\mathcal{G}$ be a class of graphs closed under taking minors, that is, if H is a minor of G and $G \in \mathcal{G}$, then $H \in \mathcal{G}$. If $H_1, \cdots, H_k$ are the excluded minors of $\mathcal{G}$, then we write $\mathcal{G} = \mathrm{Ex}(H_1, \ldots, H_k)$. (By the celebrated theorem of Robertson and Seymour, the number of excluded minors is finite, but we do not need this deep result here). The following result generalizes the previous theorem.

Theorem 8 *Let $\mathcal{G} = Ex(H_1, \ldots, H_k)$ and assume all the H_i are 3-connected. Let $h_r(2r)!$ be the number of cubic multigraphs in $\mathcal{G}$ with $2r$ vertices. Then the limiting probability that the random graph $G(n, M = \frac{n}{2}(1+\lambda n^{-1/3}))$ is in $\mathcal{G}$ is*

$$p_{\mathcal{G}}(\lambda) = \sum_{r\geq 0} \sqrt{2\pi}\, h_r A\left(3r + \frac{1}{2}, \lambda\right).$$

In particular, the limiting probability that $G(n, \frac{n}{2})$ is in $\mathcal{G}$ is

$$p_{\mathcal{G}}(0) = \sum_{r\geq 0} \sqrt{\frac{2}{3}} \left(\frac{4}{3}\right)^r h_r \frac{r!}{(2r)!}.$$

Moreover, for each λ we have

$$0 < p_{\mathcal{G}}(\lambda) < 1.$$

Proof: If all the H_i are 3-connected, then clearly a graph is in $\mathcal{G}$ if and only its kernel is in $\mathcal{G}$. The probability $p_{\mathcal{G}}(\lambda)$ is then computed as in Theorem 7. It is positive since $\mathcal{G}$ contains all trees and unicyclic graphs, which contribute with positive probability (although tending to 0 as $\lambda \to \infty$). To prove that it is less than one, let t be the smallest size of the excluded minors H_i. By splitting vertices it is easy to construct cubic graphs containing K_{t+1} as a minor, hence $G(\lambda)$ contains K_{t+1} as a minor with positive probability (alternatively, see the argument at the end of Łuczak et al. (1994)). It follows that $1 - p_{\mathcal{G}}(\lambda) > 0$. □

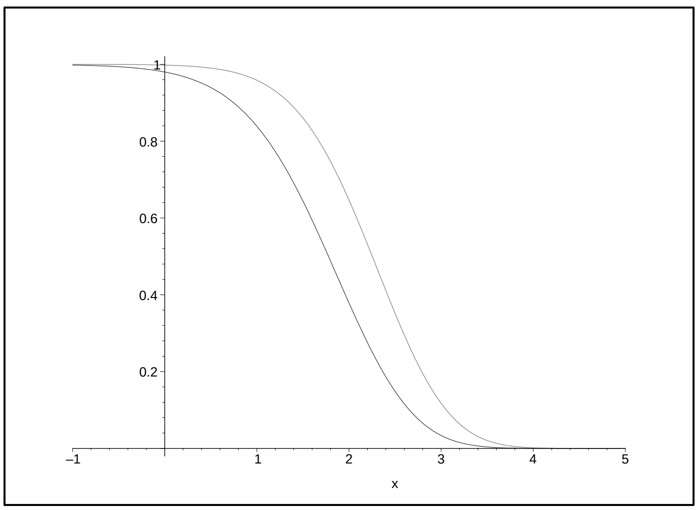

Fig. 1: The probability of $G(\lambda)$ being planar and of being series-parallel are both plotted for $\lambda \in [-1, 4]$. The function on top corresponds to the planar case.

In some cases of interest we are able to compute the numbers h_r explicitly. Let $\mathcal{G} = \mathrm{Ex}(K_4)$ be the class of series-parallel graphs. The same system of equations as in Lemma 6 holds for series-parallel graphs with the difference that now $H(z) = 0$ (this is due to the fact that there are no 3-connected series-parallel graphs). The generating function for cubic series-parallel multigraphs can be computed as

$$G_{\mathrm{sp}}(z) = 1 + \frac{5}{24}z^2 + \frac{337}{1152}z^4 + \frac{55565}{82944}z^6 + \frac{15517345}{7962624}z^8 + \cdots$$

For instance, $[z^4](G(z) - G_{\mathrm{sp}}(z)) = \frac{1}{24}$, corresponding to the fact that K_4 is the only cubic multigraph with 4 vertices which is not series-parallel. The limiting probability that $G(n, \frac{n}{2})$ is series-parallel is

$$p_{\mathrm{sp}}(0) \approx 0.98003.$$

See Figure 1 for a plot of $p_{\rm sp}(\lambda)$.

As another example, consider excluding $K_{3,3}$. Since the only 3-connected non-planar graph in $\mathrm{Ex}(K_{3,3})$ is K_5, which is not cubic, the values of h_r in this case are *exactly* the same as the ones in the planar case. Observe that 3-connectivity plays and important role in the equations of weighted cubic multigraphs in Section 3 (namely, the one related to the counting formula $H(z)$). Hence, the limiting probability of being in this class is *exactly* the same as of being planar, although $\mathrm{Ex}(K_{3,3})$ is exponentially larger than the class of planar graphs (see Gerke et al. (2008)). But excluding the graph $K_{3,3}^+$, obtained by adding one edge to $K_{3,3}$, does increase the probability, since $K_{3,3}$ is in the class and is cubic and non-planar (the probability is computable since the 3-connected graphs in $\mathrm{Ex}(K_{3,3}^+)$ are known, see Gerke et al. (2008)). Other classes such as $\mathrm{Ex}(K_5 - e)$ or $\mathrm{Ex}(K_3 \times K_2)$ can be analyzed too using the results from Giménez et al..

It would be interesting to compute the probability that $G(\lambda)$ has genus g. For this we need to count cubic multigraphs of genus g (orientable or not). We only know how to do this for $g = 0$, the reason being that a 3-connected planar graph has a unique embedding in the sphere. This is not at all true in positive genus. It is true though that almost all 3-connected graphs of genus g have a unique embedding in the surface of genus g (see Chapuy et al. (2011)). This could be the starting point for the enumeration, by counting first 3-connected maps of genus g (a map is a graph equipped with a 2-cell embedding). But this is not enough here, since we need the *exact* numbers of graphs.

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A direct bijection between permutations and a subclass of totally symmetric self-complementary plane partitions

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Abstract. We define a subclass of totally symmetric self-complementary plane partitions (TSSCPPs) which we show is in direct bijection with permutation matrices. This bijection maps the inversion number of the permutation, the position of the 1 in the last column, and the position of the 1 in the last row to natural statistics on these TSSCPPs. We also discuss the possible extension of this approach to finding a bijection between alternating sign matrices and all TSSCPPs. Finally, we remark on a new poset structure on TSSCPPs arising from this perspective which is a distributive lattice when restricted to permutation TSSCPPs.

Résumé. Nous définissons une sous-classe de partitions planes totalement symétriques autocomplémentaires (TSS-CPPs) que nous montrons est en bijection directe avec des matrices permutation. Cette bijection trace le numéro inverse de la permutation, la position du 1 dans la derniére colonne, et la position du 1 dans le dernier rayon aux statistiques naturelles sur cettes TSSCPPs. Aussi, nous discutons l'extension possible de cette approche pour trouver une bijection entre les matrices á signe alternat et toutes TSSCPPs. Finalement, nous remarquons sur une structure poset nouvelle sur les TSSCPPs se levant de cette perspective qui est une treillis distributif quand elle est limité aux TSSCPPs permutation.

Keywords: alternating sign matrix, plane partition, permutation, bijection

1 Introduction

Alternating sign matrices (ASMs) and their equinumerous friends, descending plane partitions (DPPs) and totally symmetric self-complementary plane partitions (TSSCPPs), have been bothering combinatorialists for decades by the lack of an explicit bijection between any two of the three sets of objects. (See [7] [8] [1] [12] [6] for these enumerations and bijective conjectures and [4] for the story behind these papers.) In [9], we gave a bijection between permutation matrices (which are a subclass of ASMs) and descending plane partitions with no special parts in such a way that the inversion number of the permutation matrix equals the number of parts of the DPP. In this paper, we complete the solution to this bijection problem in the special case of permutations by identifying the subclass of TSSCPPs corresponding to permutations and giving a bijection which yields a direct interpretation for the inversion number on these permutation TSSCPPs.

In Section 2, we define TSSCPPs and ASMs and give bijections within their respective families. We recall the standard bijection from ASMs to monotone triangles. We then outline a known bijection from TSSCPPs to non-intersecting lattice paths and then transform these to new objects we call *boolean triangles*.

In Section 3, we identify the permutation subclass of TSSCPPs in terms of the boolean triangles of Section 2. We use this characterization to present a direct bijection between this subclass of TSSCPPs and permutation matrices. This bijection gives a natural interpretation on the TSSCPP for the inversions of the permutation as well as the positions of the 1's in the bottom row and last column of the permutation matrix.

It is not obvious how to extend this bijection to all ASMs and TSSCPPs. No one knows statistics on TSSCPPs with distributions corresponding to the inversion number or the number of -1's in an ASM. In Section 4, we discuss the outlook of the general bijection problem and compare the bijection of this paper with another recent bijection of Biane and Cheballah [3].

Finally, in Section 5 we make some remarks about a new partial order on TSSCPPs obtained via boolean triangles, which reduces in the permutation case to the distributive lattice which is the product of chains of lengths $2, 3, \ldots, n$.

2 The objects and their alter egos: ASMs & monotone triangles, TSSCPPs & non-intersecting lattice paths / boolean triangles

We first define ASMs and recall the standard bijection to monotone triangles. We then define TSSCPPs and give bijections with non-intersecting lattice paths and new objects we call boolean triangles. Then in the next section, we give a bijection from permutation ASMs to permutation TSSCPPs via these intermediary objects.

Definition 1 *An* alternating sign matrix *(ASM) is a square matrix with entries 0, 1, or -1 whose rows and columns each sum to 1 and such that the nonzero entries in each row and column alternate in sign.*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Fig. 1: The seven 3×3 ASMs.

See Figure 1 for the seven 3×3 ASMs. It is clear that the alternating sign matrices with no -1 entries are the permutation matrices.

Alternating sign matrices are known to be in bijection with monotone triangles, which are certain semistandard Young tableaux (that are also strict Gelfand-Tsetlin patterns). See Figure 2.

Definition 2 *A* monotone triangle *of order n is a triangular arrays of integers with i integers in row i for all $1 \leq i \leq n$, bottom row $1\ 2\ 3\ \cdots\ n$, and integer entries $a_{i,j}$ for $1 \leq i \leq n$, $n-i \leq j \leq n-1$ such that $a_{i,j-1} \leq a_{i-1,j} \leq a_{i,j}$ and $a_{i,j} < a_{i,j+1}$.*

It is well-known that monotone triangles of order n are in bijection with $n \times n$ alternating sign matrices via the following map [4]. For each row of the ASM note which columns have a partial sum (from the top) of 1 in that row. Record the numbers of the columns in which this occurs in increasing order. This process

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 2 & \\ 1 & & 2 & & 3 \end{array} \quad \begin{array}{ccccc} & & 1 & & \\ & 1 & & 3 & \\ 1 & & 2 & & 3 \end{array} \quad \begin{array}{ccccc} & & 2 & & \\ & 1 & & 2 & \\ 1 & & 2 & & 3 \end{array} \quad \begin{array}{ccccc} & & 2 & & \\ & 1 & & 3 & \\ 1 & & 2 & & 3 \end{array} \quad \begin{array}{ccccc} & & 2 & & \\ & 2 & & 3 & \\ 1 & & 2 & & 3 \end{array} \quad \begin{array}{ccccc} & & 3 & & \\ & 1 & & 3 & \\ 1 & & 2 & & 3 \end{array} \quad \begin{array}{ccccc} & & 3 & & \\ & 2 & & 3 & \\ 1 & & 2 & & 3 \end{array}$$

Fig. 2: The seven monotone triangles of order 3, listed in order corresponding to Figure 1.

yields a monotone triangle of order n. Note that entries $a_{i,j}$ in the monotone triangle satisfying the strict diagonal inequalities $a_{i,j-1} < a_{i-1,j} < a_{i,j}$ are in bijection with the -1 entries of the corresponding ASM. Also, recall that the *inversion number* of an ASM A is defined as $I(A) = \sum A_{ij}A_{k\ell}$ where the sum is over all i, j, k, ℓ such that $i > k$ and $j < \ell$. This definition extends the usual notion of inversion in a permutation matrix.

We now define plane partitions.

Definition 3 *A* plane partition *is a two dimensional array of positive integers which weakly decreases across rows from left to right and down columns.*

We can visualize a plane partition as a stack of unit cubes pushed up against the corner of a room. If we identify the corner of the room with the origin and the room with the positive orthant, then denote each unit cube by its coordinates in $\mathbb{N}^3$, we obtain the following equivalent definition. A plane partition π is a finite set of positive integer lattice points (i, j, k) such that if $(i, j, k) \in \pi$ and $1 \leq i' \leq i$, $1 \leq j' \leq j$, and $1 \leq k' \leq k$ then $(i', j', k') \in \pi$. A plane partition is *totally symmetric* if whenever $(i, j, k) \in \pi$ then all six permutations of (i, j, k) are also in π.

Definition 4 *A* totally symmetric self–complementary plane partition *(TSSCPP) inside a* $2n \times 2n \times 2n$ *box is a totally symmetric plane partition which is equal to its complement, that is, the collection of empty cubes in the box is of the same shape as the collection of cubes in the plane partition itself.*

$$\begin{array}{cccccc} 6&6&6&3&3&3\\ 6&6&6&3&3&3\\ 6&6&6&3&3&3\\ 3&3&3&&&\\ 3&3&3&&&\\ 3&3&3&&& \end{array} \qquad \begin{array}{cccccc} 6&6&6&4&3&3\\ 6&6&5&3&3&2\\ 6&6&6&3&3&3\\ 4&3&3&1&&\\ 3&3&3&&&\\ 3&3&2&&& \end{array} \qquad \begin{array}{cccccc} 6&6&6&4&3&3\\ 6&6&5&4&3&3\\ 6&6&4&3&2&2\\ 4&4&3&2&&\\ 3&3&2&&&\\ 3&3&2&&& \end{array} \qquad \begin{array}{cccccc} 6&6&6&5&5&3\\ 6&5&5&4&3&1\\ 6&5&4&3&2&1\\ 5&4&3&2&1&\\ 5&3&2&1&1&\\ 3&1&1&&& \end{array}$$

$$\begin{array}{cccccc} 6&6&6&5&5&3\\ 6&5&5&3&3&1\\ 6&5&5&3&3&1\\ 5&3&3&1&1&\\ 5&3&3&1&1&\\ 3&1&1&&& \end{array} \qquad \begin{array}{cccccc} 6&6&6&5&4&3\\ 6&6&5&3&3&2\\ 6&5&5&3&3&1\\ 5&3&3&1&1&\\ 4&3&3&1&&\\ 3&2&1&&& \end{array} \qquad \begin{array}{cccccc} 6&6&6&5&4&3\\ 6&6&5&4&3&2\\ 6&5&4&3&2&1\\ 5&4&3&2&1&\\ 4&3&2&1&&\\ 3&2&1&&& \end{array}$$

Fig. 3: TSSCPPs inside a $6 \times 6 \times 6$ box

See Figure 3 for the seven TSSCPPs of order 3.

In [5], Di Francesco gives a bijection from TSSCPPs of order n to a collection of nonintersecting lattice paths. The bijection proceeds by taking a fundamental domain of the TSSCPP, and instead of reading the number of boxes in each stack, one looks at the paths going alongside those boxes. This yields a collection of nonintersecting paths with two types of steps. With a slight further deformation, he obtains that the following objects are in bijection with TSSCPPs. See Figure 4.

Proposition 5 (Di Francesco) *Totally symmetric self-complementary plane partitions inside a $2n \times 2n \times 2n$ box are in bijection with nonintersecting lattice paths (NILP) starting at $(i, -i)$, $i = 1, 2, \ldots, n-1$, and ending at positive integer points on the x-axis of the form $(r_i, 0)$, $i = 1, 2, \ldots, n-1$, making only vertical steps $(0, 1)$ or diagonal steps $(1, 1)$.*

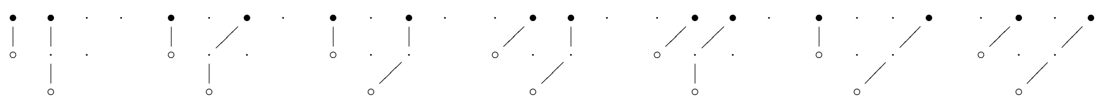


Fig. 4: The seven TSSCPP NILP of order 3.

In [5], Di Francesco uses the Lindström-Gessel-Viennot formula for counting nonintersecting lattice paths via a determinant evaluation to give an expression for the generating function of TSSCPPs with a weight of τ per vertical step. We will show that when restricted to permutation TSSCPPs, this weight corresponds to the inversion number of the permutation. Note that the distribution of the number of vertical steps in all TSSCPP NILPs does not correspond to the inversion number distribution on ASMs.

With another slight deformation, we obtain a tableaux version of these NILPs. See Figures 5 and 6.

Definition 6 *A* boolean triangle *of order n is a triangular integer array $\{b_{i,j}\}$ for $1 \le i \le n-1$, $n-i \le j \le n-1$ with entries in $\{0,1\}$ such that the diagonal partial sums satisfy*

$$1 + \sum_{i=j+1}^{i'} b_{i,n-j-1} \ge \sum_{i=j}^{i'} b_{i,n-j}. \tag{1}$$

$$\begin{array}{ccccccccc}
 & & & & b_{1,n-1} & & & & \\
 & & & b_{2,n-2} & & b_{2,n-1} & & & \\
 & & b_{3,n-3} & & b_{3,n-2} & & b_{3,n-1} & & \\
 & & & & \vdots & & & & \\
b_{n-1,1} & & & b_{n-1,2} & \cdots & b_{n-1,n-2} & & & b_{n-1,n-1}
\end{array}$$

Fig. 5: A generic boolean triangle

Proposition 7 *Boolean triangles of order n are in bijection with TSSCPPs inside a $2n \times 2n \times 2n$ box.*

Proof: The bijection proceeds by replacing each vertical step of the NILP with a 1 and each diagonal step with a 0 and vertically reflecting the array. The inequality on the partial sums is equivalent to the condition that the lattice paths are nonintersecting. $\square$

1	1	0	0	1	0	0
1 1	1 0	1 1	0 1	0 0	1 0	0 0

Fig. 6: The seven TSSCPP boolean triangles of order 3, listed in order corresponding to Figure 4 (and Figure 2 via the bijection of Theorem 9).

3 A bijection on permutations

In this section, we give a bijection between $n \times n$ permutation matrices and a subclass of totally symmetric self-complementary plane partitions inside a $2n \times 2n \times 2n$ box, preserving the inversion number statistic and two boundary statistics. First, we identify the permutation subclass of TSSCPPs.

Definition 8 *Let* permutation TSSCPPs *of order n be all TSSCPPs of order n whose corresponding boolean triangles have weakly decreasing rows. (In the NILP picture, each row has some number of vertical steps followed by some number of diagonal steps.)*

It is easy to see that there are $n!$ permutation TSSCPPs. The condition on the boolean triangle that the rows be weakly decreasing means that all the 1's must be left-justified, thus the defining partial sum inequality (1) is never violated. To construct a permutation TSSCPP, freely choose any number of left-justified 1's in each row of the boolean triangle and the rest zeros; there are $i+1$ choices for row i, and the choices are all independent.

We are now ready to state and prove our main theorem.

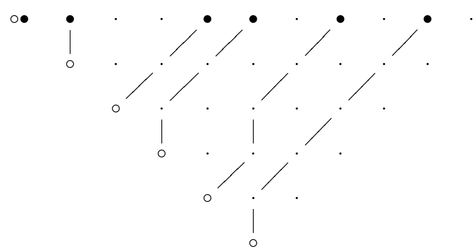
Theorem 9 *There is a natural, statistic-preserving bijection between $n \times n$ permutation matrices with inversion number p whose 1 in the last row is in column k and whose 1 in the last column is in row ℓ and permutation TSSCPPs of order n with p zeros in the boolean triangle, exactly $n-k$ of which are contained in the last row, and for which the lowest 1 in diagonal $n-1$ is in row $\ell - 1$.*

Proof: We first describe the bijection map. An example of this bijection is shown in Figure 7.

Begin with a permutation TSSCPP of order n. Consider its associated boolean triangle $b = \{b_{i,j}\}$ for $1 \le i \le n-1$, $n-i \le j \le n-1$. Define $a = \{a_{i,j}\}$ for $1 \le i \le n$, $n-i \le j \le n-1$ as follows: $a_{n,j} = j+1$ and for $i < n$, $a_{i,j} = a_{i+1,j}$ if $b_{i,j} = 0$ and $a_{i,j} = a_{i+1,j-1}$ if $b_{i,j} = 1$. We claim a is a monotone triangle. Clearly $a_{i,j-1} \le a_{i-1,j} \le a_{i,j}$. Also, $a_{i,j} < a_{i,j+1}$, since if $a_{i,j} = a_{i,j+1}$, then $a_{i,j} = a_{i+1,j}$ and $a_{i,j+1} = a_{i+1,j+1}$ so that we would need $b_{i,j} = 0$ and $b_{i,j+1} = 1$. This contradicts the fact that the rows of permutation boolean triangles must weakly decrease. Furthermore, a is a monotone triangle with no -1's in the corresponding ASM, since each entry is defined to be equal to one of it's diagonal neighbors in the row below. This process is clearly invertible.

We now show that this map takes a permutation TSSCPP boolean triangle with p zeros to a permutation matrix with p inversions. Recall that the inversion number of any ASM A (with the matrix entry in row i and column j denoted A_{ij}) is defined as $I(A) = \sum A_{ij} A_{k\ell}$ where the sum is over all i, j, k, ℓ such that $i > k$ and $j < \ell$. This definition extends the usual notion of inversion in a permutation matrix. In [10] we found that $I(A)$ satisfies $I(A) = E(A) + N(A)$, where $N(A)$ is the number of -1's in A and $E(A)$ is the number of entries in the monotone triangle equal to their southeast diagonal neighbor (entries $a_{i,j}$ satisfying $a_{i,j} = a_{i+1,j}$). Since in our case, $N(A) = 0$ and $E(A)$ equals the number of zeros in the corresponding TSSCPP boolean triangle, we have that $I(A)$ equals the number of zeros in b.

TSSCPP



$\Leftrightarrow$

Boolean triangle

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 1 & & 1 & & 0 & & \\ & 0 & & 0 & & 0 & & 0 & \\ 1 & & 0 & & 0 & & 0 & & 0 \end{array}$$

Monotone triangle

$$\Leftrightarrow \quad \begin{array}{ccccccccccc} & & & & & 4 & & & & & \\ & & & & \mathbf{4} & & \mathbf{6} & & & & \\ & & & 3 & & 4 & & \mathbf{6} & & & \\ & & \mathbf{3} & & \mathbf{4} & & \mathbf{5} & & \mathbf{6} & & \\ & 1 & & \mathbf{3} & & \mathbf{4} & & \mathbf{5} & & \mathbf{6} & \\ 1 & & 2 & & 3 & & 4 & & 5 & & 6 \end{array}$$

Permutation matrix

$$\Leftrightarrow \quad \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Fig. 7: An example of the bijection. The bold entries in the monotone triangle are the entries equal to their southeast diagonal neighbor. These are exactly the diagonal steps of the TSSCPP. Note that the matrix on the right represents the permutation 463512 which has 11 inversions. These inversions correspond to the 11 diagonal steps of the TSSCPP on the left.

We can see that the zeros of b correspond to permutation inversions directly by noting that to convert from the monotone triangle representation of a permutation to a usual permutation σ such that $i \to \sigma(i)$, we set $\sigma(i)$ equal to the unique new entry in row i of the monotone triangle. Thus for each entry of the monotone triangle $a_{i,j}$ such that $a_{i,j} = a_{i+1,j}$, there will be an inversion in the permutation between $a_{i,j}$ and $\sigma(i+1)$. This is because $a_{i,j} = \sigma(k)$ for some $k \leq i$ and $\sigma(k) = a_{i,j} > \sigma(i)$. These entries $a_{i,j}$ such that $a_{i,j} = a_{i+1,j}$ correspond exactly to zeros in row i of the boolean triangle b. Thus if a permutation TSSCPP has p zeros in its boolean triangle, its corresponding permutation will have p inversions.

Also, observe that if the number of zeros in the last row of the boolean triangle is k, then the 1 in the bottom row of the permutation matrix will be in column $n-k$. So the missing number in the penultimate monotone triangle row shows where the last row of the boolean triangle transitions from ones to zeros. So by the bijection between monotone triangles and ASMs, the 1 in the last row of A is in column $n-k$.

Finally, if the lowest 1 in diagonal $n-1$ of the boolean triangle is in row $\ell - 1$, this means that the entries $\{a_{i,n-1}\}$ for $\ell \leq i \leq n$ are all equal to n. So the 1 in the last column of the permutation matrix is in row ℓ. □

See Figure 7 for an example of this bijection.

4 Toward a bijection between all TSSCPPs and ASMs

In [9], we discussed the obstacles to turning the bijection between permutations and descending plane partitions presented there into a bijection between all ASMs and DPPs. Here we discuss some of the challenges to the ASM-TSSCPP bijection in full generality.

While DPPs have the property that the number of parts equals the inversion number of the ASM (this is now proved, though not bijectively [2]), TSSCPPs do not have such a statistic as of yet. We showed that the number of diagonal steps in a permutation-NILP gives the inversion number of the permutation matrix, but this is not true for general TSSCPPs and ASMs. Furthermore, while the number of special parts of a DPP corresponds to the number of -1's in the ASM, there is no such statistic on TSSCPP. It would seem reasonable to conjecture that the -1 of the ASM should correspond to all instances of a vertical step followed by a diagonal step as you go from left to right along a row of the NILP (or a 0 followed by a 1 as you go across a row of the boolean triangle). This holds up to $n = 4$, and it seems to hold for arbitrary n in the special cases of one -1 and the maximal number of -1's ($\lfloor \frac{n^2}{4} \rfloor$). But for the number of -1's between 1 and $\lfloor \frac{n^2}{4} \rfloor$, these statistics diverge.

Di Francesco has noted that the distribution of diagonal steps in the top row of the TSSCPP-NILP corresponds to the refined enumeration of ASMs. So one might hope to begin a general bijection by determining the $(n-1)$st row of the monotone triangle from the top row of the NILP (or the bottom row of the boolean triangle) by left-justifying all the vertical steps and then bijecting in the same way as in the permutation case. After that, though, it is unclear how to proceed. See Figure 4 for a summary of the various statistics which are preserved in the permutation case DPP-ASM-TSSCPP bijections and which should correspond in full generality. (See [9] for further explanation on the DPP case.)

DPP	ASM	TSSCPP boolean triangle
no special parts*	no -1's	rows weakly decrease
number of parts*	number of inversions	number of zeros
number of n's*	position of 1 in last column	position of lowest 1 in last diagonal
largest part value that does not appear	position of 1 in last row	number of zeros in last row*

Fig. 8: This table show the statistics preserved by the permutation case bijections of this paper and [9]. There is a star by the DPP and TSSCPP statistics that have the same distribution as the ASM statistic in the general case.

Finally, we compare this work with another recent bijection due to Biane and Cheballah. In [3], the authors give a bijection between *Gog* and *Magog trapezoids* of two diagonals. (*Gog triangles* are exactly monotone triangles. *Magog triangles* can be seen to be in bijection with the TSSCPP boolean triangles considered here. The term *trapezoid* indicates the truncation of the triangle to a fixed number of diagonals.) Their bijection is both more and less general than the one of this paper. It is more general in the sense that it includes configurations corresponding to the -1 in an ASM, where we consider only permutations. It is less general in that it uses only two diagonals of the triangle, where we are able to consider the full triangle.

Experimental evidence suggests the bijection of [3] and the bijection of this paper may coincide (up to slight deformation) in the case of permutation monotone triangles, truncated to two diagonals. Perhaps the combination of these two perspectives will provide insight on the full bijection.

5 Poset Structure

In [10], we examined a poset structure on TSSCPPs, which turned out to be a distributive lattice with poset of join irreducibles very similar to that of the ASM lattice. In this final section, we remark on a new partial order on TSSCPPs arising from this perspective which is not a distributive lattice, but which has nice distributive lattice structure when restricted to the permutation case.

Define the *boolean partial order* on TSSCPPs of order n as the boolean triangles of order n ordered by componentwise comparison of the entries. This is an induced subposet of the Boolean lattice on $\binom{n}{2}$ elements given by only taking the elements corresponding to TSSCPPs. This order on TSSCPPs is not a distributive lattice. But if we further restrict this order to the permutation TSSCPPs, the poset formed is $[2] \times [3] \times \cdots \times [n]$, that is, the product of chains of length $2, 3, 4, \ldots, n$, where the order ideal composed of k elements in the chain $[i]$ corresponds to row $i - 1$ of the boolean triangle containing k 1's. This permutation TSSCPP lattice is a partial order on permutations which sits between the weak and strong Bruhat orders on the symmetric group. It contains all of the ordering relations of the weak order plus some of the additional relations of the strong order. See Figure 9.

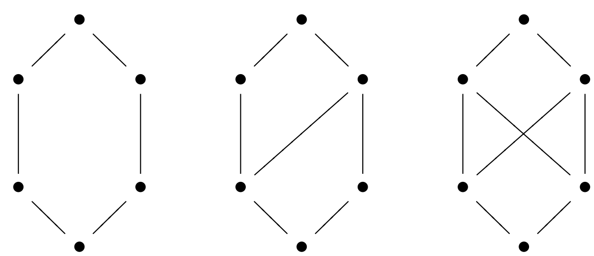


Fig. 9: From left to right: The weak order on S_3, the boolean partial order on permutation TSSCPPs of order 3, and the strong Bruhat order on S_3.

Conversely, the natural partial order on all ASMs is the distributive lattice of monotone triangles, but its restriction to permutations is the strong Bruhat order, which is not a lattice. In fact, the ASM lattice is the smallest lattice to contain the Bruhat order on the permutations as a subposet (i.e. it is the MacNeille completion of the Bruhat order [11]). See Figure 10 for a comparison of this order on ASMs with the TSSCPP boolean order.

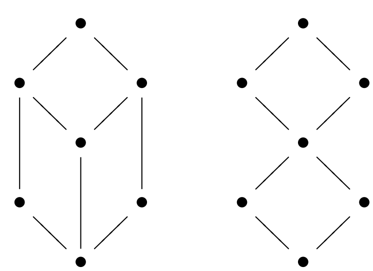


Fig. 10: Left: The boolean partial order on TSSCPPs of order 3. Right: The lattice of 3×3 ASMs.

We hope that the study of this new partial order on TSSCPPs will provide insight on the combinatorics of these objects and the associated outstanding bijection problems.

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Algebraic properties for some permutation statistics

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Abstract. In this article, we study some quotient sets on permutations built from peaks, valleys, double rises and double descents. One part is dedicated to the enumeration of the cosets using the bijection of Françon-Viennot which is a bijection between permutations and the so-called Laguerre histories. Then we study the algebraic properties of these quotient sets. After having shown that some of them give rise to quotient algebras of **FQSym**, we prove that they are also free.

Résumé. Dans cet article, on étudie certains quotients de l'ensemble des permutations construits à partir des pics, vallées, double montées, double descentes. Une des parties est consacrée au calcul des cardinaux des ensembles quotients. Pour cela, on utilisera la bijection de Françon-Viennot, qui transforme les permutations en histoires de Laguerre. Puis, nous nous intéressons aux propriétés algébriques de ces quotients. En particulier, après avoir montré que certains d'entre eux sont des algèbres quotients de **FQSym**, on montre qu'ils sont libres.

Keywords: Laguerre histories, Free quasi-symmetric functions, quotient algebra, increasing binary trees

1 Introduction

One of the goals of algebraic combinatorics is to study operations on discrete structures, and relations between these structures. The set of permutations has provided many recent results in combinatorics and in algebra. In particular, an algebra based on permutations, called **FQSym** ([11], [1]), is a non-commutative generalization of the algebra of symmetric functions. It contains the theory of the non-commutative symmetric functions ([4]) and many other algebras. Some of them are described thanks to equivalence relations on permutations, as **NCSF** (having the same descents [7]), $\mathbf{PBT}$ (having the same binary search tree [6]), or **FSym** (having the same P-symbol in the RSK algorithm [12]). The formalism of **FQSym** has enabled to find simple proofs of some results, as *e.g.*, the Littlewood-Richardson rule ([9]), the construction of the peak algebra ([7]), the representation theory of the 0-Hecke algebra ([8]).

On the enumerative part, in 1979, Françon and Viennot established a bijection between permutations and Laguerre histories ([3]). These objects encode in a natural way some statistics on permutations: peaks, valleys, double rises, and double descents. This bijection has been used in many occasions, and was one of the building blocks of the combinatorial theory of orthogonal polynomials along with [2]. Since this bijection concerns permutations, hence elements indexing the basis of **FQSym**, one natural question is how Laguerre histories can be understood in this algebraic framework.

A first approach is given here: indeed, some statistics have enabled us to build some subalgebras and/or quotient algebras of **FQSym**. So one can ask whether among the four statistics peaks, valleys, double rises, and double descents, some statistics allow one to build quotient algebras. In our case, we will see that some of them we can indeed construct some quotient algebras of **FQSym**.

We begin by recalling some classical definitions and the bijection of Françon-Viennot. Then we compute the cardinality of the different quotient sets. Finally, the algebraic aspect is tackled: we first study sets defining quotient algebras of **FQSym**, and then show that these algebras are free.

2 Definitions and bijection of Françon-Viennot

2.1 Some statistics on permutations

Let us denote by $\mathfrak{S}_n$ the set of permutations of size n and by $\mathfrak{S} = \cup_n \mathfrak{S}_n$. Let σ be in $\mathfrak{S}_n$, and i in $[1, n]$. Since we want to define the type of any value by comparing it to its left and right neighbours, we need to fix some conventions for $\sigma(0)$ and $\sigma(n+1)$. Up to simple transforms, there are only two distinct conventions: $\sigma(0) = 0,\ \sigma(n+1) = 0$, and $\sigma(0) = 0,\ \sigma(n+1) = \infty$. Then $\sigma(i)$ is called:

-a *peak*	if $\sigma(i-1) < \sigma(i) > \sigma(i+1)$,
-a *valley*	if $\sigma(i-1) > \sigma(i) < \sigma(i+1)$,
-a *double rise*	if $\sigma(i-1) < \sigma(i) < \sigma(i+1)$,
-a *double descent*	if $\sigma(i-1) > \sigma(i) > \sigma(i+1)$.

For example, with the convention 0-0, the permutation 27416358 has the peaks set $\{7,\ 6,\ 8\}$, the valleys set $\{1,\ 3\}$, the double rises set $\{2,\ 5\}$, and the double descents set $\{4\}$ (graphical representation of definitions on Figure 1). Note that these statistics extend directly to words over $\mathbb{N}$ with no repeating letters.

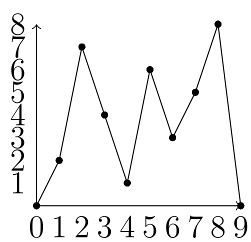


Figure 1: Graphical representation of the permutation 27416358.

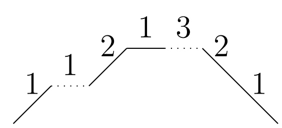


Figure 2: A Laguerre history of size 7.

From now on, we shall work with the convention 0-0, the case 0-∞ being essentially the same. Indeed, by embedding $\mathfrak{S}_n$ in $\mathfrak{S}_{n+1}$ by adding $n+1$ as fixed point for a permutation of size n, we come back to the case 0-0. By slightly adapting the proofs, all results of the case 0-0 transpose to the case 0-∞.

2.2 The algebra of free quasi-symmetric functions **FQSym**

Let w be a word over $\mathbb{N}$. Recall that the shifted word $w[k]$ is the word in which each letter of w is shifted by k: if $w = w_1 \cdots w_n$, then $w[k] = (w_1 + k) \cdots (w_n + k)$. The shuffle product is defined by induction as follows:

- $\epsilon \sqcup\!\sqcup a = a \sqcup\!\sqcup \epsilon = a$,
- $w = au,\ v = bt,\ w \sqcup\!\sqcup v = a(u \sqcup\!\sqcup v) + b(w \sqcup\!\sqcup t)$, where a and b are letters.

The shifted shuffle of $\sigma \in \mathfrak{S}_n$ and τ is denoted by $\sigma \,\overline{\sqcup\!\sqcup}\, \tau$, and is equal to $\sigma \sqcup\!\sqcup \tau[n]$. For example, the shifted shuffle product of $\sigma = 12$ and $\tau = \mathbf{21}$ is

$$\sigma \,\overline{\sqcup\!\sqcup}\, \tau = 12\mathbf{43} + 1\mathbf{4}2\mathbf{3} + 1\mathbf{43}2 + \mathbf{4}12\mathbf{3} + \mathbf{4}1\mathbf{3}2 + \mathbf{43}12. \tag{1}$$

The algebra **FQSym** is generated by the basis $\langle \mathbf{F}_\sigma \rangle_{\sigma \epsilon \mathfrak{S}}$ and has the following product formula:

$$\mathbf{F}_\sigma \mathbf{F}_\tau = \sum_{\nu \epsilon \sigma \overline{\sqcup\!\sqcup} \tau} \mathbf{F}_\nu. \tag{2}$$

More details about **FQSym** can be found in [1].

2.3 The bijection of Françon-Viennot

Let us now introduce the bijection of Françon-Viennot. It is a bijection between permutations and the so-called Laguerre histories. All details can be found in [3].

2.3.1 Laguerre histories

A Laguerre history of size n is a positive valued path, beginning at $(0, 0)$, ending at $(n, 0)$ staying above the horizontal axis, with four types of steps: $(1, 1)$, $(1, -1)$, $(1, 0)$, and an another type of horizontal step, $\overline{(1, 0)}$. The i-th step has an integer value between 1 and $\gamma(i)$, where γ is the following function:

- $\gamma(1) = 1$,
- $\gamma(i + 1) = \gamma(i)$ if the i-th step is $\overline{(1, 0)}$ or $(1, 0)$,
- $\gamma(i + 1) = \gamma(i) + 1$ if the i-th step is $(1, 1)$,
- $\gamma(i + 1) = \gamma(i) - 1$ if the i-th step is $(1, -1)$.

Figure 2 shows an example of a Laguerre history h of size 7, where $\overline{(1, 0)}$ is represented as a dotted line. On this example, the corresponding function γ takes the successive values $1,\ 2,\ 2,\ 3,\ 3,\ 3,\ 2$. Let $LH(n)$ be the set of Laguerre histories of size n.

2.3.2 The bijection of Françon-Viennot

Let h be a Laguerre history of size $n-1$ and let us build the corresponding permutation σ of size n. The algorithm starts with the word $w = \infty$, and we transform this word by applying some rules. One reads the Laguerre history from left to right and, for each step transforms w as follows: if t is the type of the i-th step of h, and j its weight, the j-th ∞ of w is replaced by:

$$\begin{cases} i & \text{if t is } (1, -1), \\ \infty i \infty & \text{if t is } (1, 1), \\ i\infty & \text{if t is } \overline{(1, 0)}, \\ \infty i & \text{if t is } (1, 0). \end{cases}$$

Finally, the last ∞ is changed into n.

For example, the Laguerre history shown Figure 2 gives the following steps:

$$\begin{gathered}\infty \xrightarrow{1} \infty 1 \infty \xrightarrow{2} 2\infty 1\infty \xrightarrow{3} 2\infty 1\infty 3\infty \xrightarrow{4} 2\infty 41\infty 3\infty \\ \xrightarrow{5} 2\infty 41\infty 35\infty \xrightarrow{6} 2\infty 41635\infty \xrightarrow{7} 2741635\infty \xrightarrow{8} 27416358.\end{gathered} \tag{3}$$

So its corresponding permutation is $\sigma = 27416358$. Since each step of the construction is reversible, it induces a bijection between Laguerre histories of size $n-1$ and $\mathfrak{S}_n$. Note that with the convention 0-0, the type of a step of a Laguerre history and the type of the corresponding value in σ in one-to-one correspondence. Indeed, the i-th step of h is respectively $(1,-1)$, $(1,1)$, $\overline{(1,0)}$, or $(1,0)$, when i is respectively a peak, a valley, a double rise, or a double descent in σ.

3 Enumeration of the quotient sets

Given the four types of statistics on $\mathfrak{S}_n$, we construct some quotient sets on $\mathfrak{S}_n$ as follows: consider a partition $(A_1, \cdots, A_p)$ of $\{$P, V, Dr, Dd$\}$. Then two permutations belong to the same class if and only if they have identical unions of sets of statistics for all the A_i. For example, if we choose (P, V, Dr, Dd), the equivalence relation consists in having the same peaks, valleys, double rises and double descents. At $n = 3$, there are five equivalence classes, 213 and 312 belonging to the same class. If we choose to regroup peaks, valleys and double descents, the equivalence relation consists in having the same double rises set, and we will denote it by (P $\cup$ V $\cup$ Dd, Dr). For example, 13245 and 14532 are in the same class for the second relation, but not for the first one. Indeed, the four sets respectively are ($\{3, 5\}$, $\{2\}$, $\{1, 4\}$, $\emptyset$) and ($\{5\}$, $\emptyset$, $\{1, 4\}$, $\{2, 3\}$). So the permutations do not belong to the same (P, V, Dr, Dd) class but they do belong to same (P $\cup$ V $\cup$ Dd, Dr) since both sets are ($\{2, 3, 5\}$, $\{1, 4\}$).

Thanks to the bijection of Françon-Viennot, each type of statistics is interpreted as a type of step. So two permutations of size n are in the same class for (P, V, Dr, Dd) if and only if they have the same unvalued Laguerre histories of size $n-1$, which are the Motzkin paths with two types of horizontal steps. Since each type of step represents one type of statistics, and since (P, V, Dr, Dd) is represented by the unvalued Laguerre histories, the other quotients are obtained by identifying the corresponding types of steps on the unvalued Laguerre histories. For example, if we identify the peaks with the valleys, we identify the steps $(1,-1)$ and $(1,1)$ in the Laguerre histories.

3.1 The quotient set (P, V, Dr, Dd)

Thanks to the bijection, (P, V, Dr, Dd) is in bijection with the unvalued Laguerre histories of size $n-1$, and hence is enumerated by the n-th Catalan number. The full proof is presented in [3].

3.2 The quotient set (P, V, Dr $\cup$ Dd)

In this case, the two types of horizontal steps are identified. Therefore, (P, V, Dr $\cup$ Dd) has a one-to-one correspondence with the Motzkin paths of size $n-1$. The representatives are given by the Laguerre histories with all horizontal steps equal to $(1,0)$.

3.3 The quotient set (P $\cup$ V, Dr, Dd)

In this case, the type of steps $(1,1)$ and $(1,-1)$ are identified. Note that in an unvalued Laguerre history, there is always the same number k of steps $(1,1)$ and $(1,-1)$. Let k be an integer between 0 and $\frac{n-1}{2}$. Since the horizontal steps can be anywhere, we have to select $n-1-2l$ steps in the set $\{1,\cdots,n-1\}$. These steps may be $\overline{(1,0)}$ or $(1,0)$, hence two choices. So we get the following formula for the number of classes:

$$\sum_{k=0}^{\frac{n-1}{2}} \binom{n-1}{n-1-2k} 2^{n-1-2k} = \sum_{k=0}^{\frac{n-1}{2}} \binom{n-1}{2k} 2^{n-1-2k} = \frac{(2+1)^{n-1}+(2-1)^{n-1}}{2}. \tag{4}$$

Thus, the number of classes is $\frac{3^{n-1}+1}{2}$. Let us give one representative of each class in term of unvalued Laguerre history. If there are $n-1-2l$ horizontal steps, there are $2k$ steps which are not. We take the first such k steps to be of type $(1,1)$, and the others to be of type $(1,-1)$.

3.4 The quotient sets (P $\cup$ V $\cup$ Dd, Dr) and (P $\cup$ V $\cup$ Dr, Dd)

Reversing dotted and not dotted horizontal steps is an involution on unvalued Laguerre histories. In terms of types of statistics, it sends two permutations having the same double rises into two permutations having the same double descents and vice versa, and preserves the peaks and the valleys. As a consequence, these two quotients are in bijection. Therefore, we only study the case of (P $\cup$ V $\cup$ Dd, Dr). Here, for each step, we can decide if it is or not a dotted horizontal step. So we have at most 2^{n-1} classes. Conversely, let us choose the dotted horizontal steps. Then the other steps can be taken as non dotted horizontal steps, which gives an unvalued Laguerre history. So there are 2^{n-1} classes.

3.5 The quotient sets (P, Dr $\cup$ V $\cup$ Dd) and (V, Dr $\cup$ P $\cup$ Dd)

First, we observe that reading the unvalued Laguerre histories right to left is an involution. In terms of types of statistics, it sends two permutations having the same peaks into two permutations having the same valleys, and vice versa, and preserves the double rises and double descents. So, the quotient sets (P, Dr $\cup$ V $\cup$ Dd) and (V, Dr $\cup$ P $\cup$ Dd) are in bijection. Now, for (P, Dr $\cup$ V $\cup$ Dd), identify the steps $\overline{(1,0)}$, $(1,0)$, and $(1,1)$. We then obtain a left factor of a Dyck path of length $n-1$. Conversely, given a left factor of a Dyck path, if there are k steps equal to $(1,-1)$, we keep the k first steps equal to $(1,1)$, and change the other steps equal to $(1,1)$ into $\overline{(1,0)}$. We then get an unvalued Laguerre history. Thus the equivalence classes are in bijection with the left Dyck factors, which are enumerated by the central binomial of $n-1$ [13]($A001405$).

3.6 The quotient sets (P, Dr, V $\cup$ Dd), (P, Dd, V $\cup$ Dr), (P $\cup$ Dd, V, Dr) and (P $\cup$ Dr, V, Dd)

With the involutions consisting in reading from right to left or reversing dotted and not dotted horizontal steps, these four quotient sets are in bijection. So, it is enough to study the case (P, Dr, V $\cup$ Dd). Here, the type of steps $(1,1)$ and $(1,0)$ are identified with the step $(1,1)$. So we obtain a left factor of a Motzkin path of size $n-1$. Conversely, if we take a left factor of a Motzkin path, if there are k steps of type $(1,-1)$, we keep the first k steps of type $(1,1)$, and change the others into $\overline{(1,0)}$. So we obtain an unvalued Laguerre history for each left factor of a Motzkin path. So (P, Dr, V $\cup$ Dd) is enumerated by the left factors of Motzkin paths of size $n-1$ which are in bijection with the directed animals of size $n-1$ [13]($A005773$).

3.7 The quotient set (P ∪ V, Dr ∪ Dd)

Here, the steps $(1,-1)$ and $(1,1)$ are identified. We also identify the step $\overline{(1,0)}$ with the step $(1,0)$. By the same arguments as in the case (P ∪ V, Dr, Dd), we show that the number of classes is 2^{n-2}.

3.8 The quotient sets (P ∪ Dr, V ∪ Dd) and (P ∪ Dd, V ∪ Dr)

By reading from right to left, these two quotient sets are in bijection. Here, we are identifying $(1,-1)$ with $\overline{(1,0)}$, and $(1,1)$ with $(1,0)$. So the number of classes is smaller than or equal to 2^{n-1}. Since the representatives are given by the paths without $(1,1)$ and $(1,-1)$, the number of classes is indeed 2^{n-1}.

4 The quotient algebras of **FQSym**

In recent papers, many quotients of **FQSym** have been constructed from statistics on permutations ([5], [12], [7]). For example, the algebra $QSym$ is the quotient algebra of **FQSym** where $\mathbf{F}_\sigma$ and $\mathbf{F}_\tau$ are identified if σ and τ have the same descents. Since we have built some quotient sets in $\mathfrak{S}_n$ for each n, it is natural to check if there are some quotient algebras in **FQSym** induced by these equivalence relations. To this aim, we first recall some notions on increasing binary trees. Indeed, our statistics interpret directly on these combinatorial structures. Then we will study all cases beginning with the quotient set (P, V, Dr, Dd).

4.1 Increasing binary trees

The notion of increasing binary trees appears in [2] as tournament trees. Let **A** be a totally ordered alphabet. Let w be a word on **A**, with no repeated letters. If w is the empty word (denoted by ϵ), then the corresponding tree (denoted by $T(w)$) is a leaf. Otherwise, we write $w = w_1 a w_2$, where a is the smallest letter in w. The corresponding increasing binary tree is recursively built as follows: the letter a is the root, the left subtree is the increasing binary tree associated with w_1, and the right subtree is the increasing binary tree associated with w_2. Note that here an increasing binary tree is a complete binary tree, where all internal nodes are labelled, and the leaves are not.

4.1.1 The grafting operation

Let T and T' be two trees, let l be a leaf of T. The *graft* of T' on T at position l is the substitution of l by T'.

$$\begin{array}{ccccccc} T\backslash\{l\} & & T' & & T\backslash\{l\} & & \\ & \diagdown & & \rightarrow & & \diagdown & \\ & & l & & & & T' \end{array}$$

4.1.2 Increasing binary trees and shifted shuffle

Let σ and τ be two permutations, and w in $\sigma \,\overline{\sqcup\!\sqcup}\, \tau$. We can write w as: $w = \tau^{(1)}\sigma_1 \cdots \sigma_n \tau^{(n+1)}$, where $\tau[n] = \tau^{(1)} \cdots \tau^{(n+1)}$, and $\tau^{(i)}$ may be an empty word. Then $T(w)$ is the tree built as follows: in the tree $T(\sigma)$, graft the tree of the word $\tau^{(i)}$ at its i-th leaf (in the infix order) .

Conversely, if we decompose $\tau[n] = \tau^{(1)} \cdots \tau^{(n+1)}$, with $\tau^{(i)}$ a factor of $\tau[n]$ (maybe empty), and graft at the i-th leaf (in the infix order) of $T(\sigma)$ the tree $T(\tau^{(i)})$, we obtain an increasing binary tree T'. By reading T' in infix order, we get a w which is in $\sigma \,\overline{\sqcup\!\sqcup}\, \tau$. or example, let $\overline{\sigma} = \overline{2413}$, $\tau = 3.2.14$. The word $w = 27416358$ decomposes as $\overline{2}7\overline{41}6\overline{3}58$ in $\sigma \,\overline{\sqcup\!\sqcup}\, \tau$ (the empty words are forgotten) and $T(w)$ is represented igure 3.

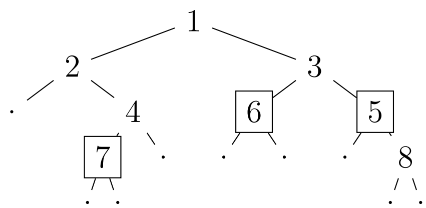


Figure 3: The increasing binary tree of 27416358.

4.1.3 *Increasing binary trees and permutations*

Let us recall that there is a classical bijection between the permutations of size n, and the complete increasing binary trees with n internal nodes labelled by $\{1, \cdots, n\}$. A permutation σ is sent to its increasing binary tree. Conversely, by reading an increasing binary tree in infix order, we obtain its corresponding permutation. Let us see how the types of statistics are interpreted in the increasing binary tree. Let σ be in $\mathfrak{S}_n$, i a letter in σ, and $T(\sigma)$ the corresponding increasing tree. Note that depending on i being a peak, a valley, a double rise, or a double descent in σ, the node i in $T(\sigma)$ has respectively zero labelled child, two labelled children, one labelled child to the left or to the right (see the exemple Figure 3).

4.2 *The quotient algebras of* ***FQSym***

Let us consider one equivalence relation $\sim$ in $\mathfrak{S}$. This relation induces a quotient of **FQSym**, by identifying $\mathbf{F}_\sigma$ and $\mathbf{F}_\tau$ if $\sigma \sim \tau$. Proving that this quotient is well-defined is equivalent to prove that if σ and τ are equivalent then for all s in $\mathfrak{S}$, there exists a bijection ϕ between $\sigma \,\overline{\sqcup\!\sqcup}\, s$ and $\tau \,\overline{\sqcup\!\sqcup}\, s$, a bijection ψ between $s \,\overline{\sqcup\!\sqcup}\, \sigma$ and $s \,\overline{\sqcup\!\sqcup}\, \tau$ such that each element in $\sigma \,\overline{\sqcup\!\sqcup}\, s$ or in $s \,\overline{\sqcup\!\sqcup}\, \sigma$ are respectively equivalent to their image by ϕ or ψ.

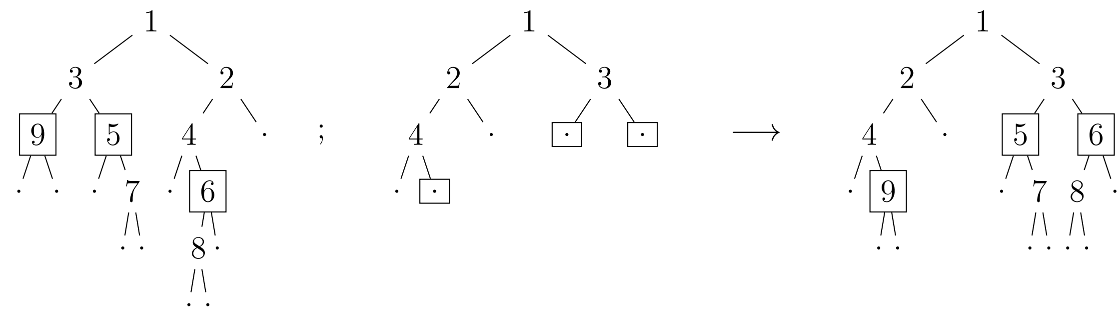
4.2.1 *The quotient by the four types of statistics*

In this Section, we write $\sigma \sim \tau$ if σ and τ belong to the same (P, V, Dr, Dd) class. In terms of trees, $\sigma \sim \tau$ if and only if for each labelled node of $T(\sigma)$ and $T(\tau)$, the left and right children are of same type (labelled or not). Now, let w be in $\sigma \,\overline{\sqcup\!\sqcup}\, s$.

Construction of ϕ: The word w decomposes in the following form: $w = s^{(1)}\sigma_1 \cdots \sigma_n s^{(n+1)}$, with $s[n] = s^{(1)} \cdots s^{(n+1)}$. Denote by $(f_1, \cdots, f_k)$ the non-empty factors among the $s^{(i)}$ in order. So we have, $s[n] = f_1 \cdots f_k$.

Since w is in $\sigma \,\overline{\sqcup\!\sqcup}\, s$, we know that $T(w)$ is obtained by grafting at the i-th leaf the tree $T(s^{(i)})$ (see Section 4.1.2). Let us mark the leaves where the $T(f_j)$ are grafted and denote by p_j the father of the leaf where $T(f_j)$ is grafted (two p_j may be equal). Note that $T(f_j)$ may graft at the left or right child of p_j.

Since σ and τ have the same four statistics, all nodes p_j in $T(\tau)$ are of the same type as in $T(\sigma)$. So the left and right children of p_j are the same type (labelled or not) in $T(\sigma)$ and $T(\tau)$. Let us then mark in $T(\tau)$ the children of p_j where there is a graft of a $T(f_j)$ in $T(\sigma)$. Now, we graft at the i-th marked



$T(w)$ Grafting locations on $T(\tau)$ $T' = T(\phi(w)))$

Figure 4: Construction of T' with $\sigma = 3142$, $\tau = 4213$, $s = 51342$, and $w = 935714862$.

leaf of $T(\tau)$ the tree $T(f_i)$. Let T' be this tree. By construction, the p_j have the same type in $T(w)$ and T'. Moreover, the other labelled nodes do not change type between type $T(\sigma)$, $T(s)$ and T'. So each labelled node of $T(w)$ and T' have same type. So the permutation $\phi(w)$ associated with T' is in the same equivalence class as w, and belongs to $\tau \,\overline{\sqcup\!\sqcup}\, s$.

If we apply the same algorithm on $\phi(w)$, exchanging the role of τ and σ, we find back w. So this operation is a bijection between $\sigma \,\overline{\sqcup\!\sqcup}\, s$ and $\tau \,\overline{\sqcup\!\sqcup}\, s$. Here is an example of computation of ϕ in Figure 4.

Construction of ψ: The key ingredient in building ψ relies on

Proposition 4.1 *let $\sigma \sim \tau$ be two words and write σ as a concatenation of k words: $\sigma = f_1 \cdots f_k$. Then there exists a decomposition of $\tau = g_1 \cdots g_k$ such that for each letter a in f_i, the word g_j containing a satisfies that a has same type in f_i and g_j.*

Proof:

Let a_i be the smallest letter between the last letter of f_i and the first letter of f_{i+1}. For each a_i, we cut τ at the right of a_i if a_i is in f_i and at the left of a_i otherwise. Thus, we obtain a decomposition of τ of the form $g_1 \cdots g_k$. Note that the letters that change type between τ and the g_j are exactly the a_i. So all but the a_i have same type in the f_i and g_j: their type is preserved from f_i to σ, from σ to τ, and from τ to g_j. Now, concerning the a_i, they have same type in σ and τ and it changes in the same way from σ to f_i than from τ to g_j.

□

For example, let $\sigma = 25.7.1.34.6$, and $\tau = 3612457$. Here are the different steps in order to find the decomposition of τ:

$$3612457 \xrightarrow{5<7} 361245.7 \xrightarrow{7>1} 36.1245.7 \xrightarrow{1<3} 36.1.245.7 \xrightarrow{4<6} 36.1.24.5.7.$$

So the decomposition associated with τ is 36.1.24.5.7. Note that the previous statement extends directly to words with no repeated letters of the same evaluation and same type.

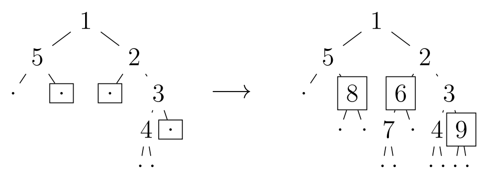
Let us now build the bijection ψ. For w in $s \shuffle \sigma$, we mark the k leaves in $T(s)$ where there is a graft in order to obtain $T(w)$. It corresponds to write $w = \sigma^{(1)} s_1 \cdots s_m \sigma^{(m+1)}$, with $\sigma[m] = \sigma^{(1)} \cdots \sigma^{(m+1)}$ and consider the sequence $(f_1, \cdots, f_k)$ of the non-empty factors among the $\sigma^{(i)}$ in order. We apply Proposition 4.1 to $\sigma[m] = f_1 \cdots f_k$ and $\tau[m]$ and obtain a decomposition of $\tau[m] = g_1 \cdots g_k$. Define T' as the tree obtained by grafting $T(g_i)$ at the i-th marked leaf of $T(s)$. By construction, the labelled nodes in $T(s)$ are of same type in $T(w)$ and T'. Thanks to Proposition 4.1, the other labelled nodes have also same type. So, by reading T' in infix order we obtain a word $\psi(w)$ in $s \shuffle \tau$, which has same type as w.

Note that if we apply this algorithm on $\psi(w)$, exchanging the role of σ and τ, we find back w. So ψ is a bijection.

Here is an example of computation of ψ, with $\sigma = 4132$, $\tau = 3214$, $s = 51243$, and $w = 591682437$ in $s \shuffle \sigma$. The word w gives the following factorization for σ: $4.13.2$. Then we factorize τ as:

$$3214 \xrightarrow{4>1} 321.4 \xrightarrow{3>2} 3.21.4.$$

So we have the following grafting locations in $T(s)$ and finally the tree T':



$T(s)$ $\qquad\qquad$ $T' = T(\psi(w))$

The definitions of both ϕ and ψ then proves

Theorem 4.1 *The quotient of **FQSym** by (P, V, Dr, Dd) is well-defined.*

4.2.2 The other cases

The case (P, V, Dr $\cup$ Dd) where we consider having the same peaks and valleys is also a well-defined quotient algebra of **FQSym**. Indeed, in terms of trees, the equivalence concerns nodes having the same number of children which are leaves. So in order to build ϕ, instead of considering how to graft to the left or to the right, we just graft where it is possible. In order to build ψ, we also adapt the factorization: we just cut at the only possible place in order to change the type of a letter.

The other cases studied in the previous part do not give any quotient algebra except (P $\cup$ Dr, V $\cup$ Dd) and (P $\cup$ Dd, V $\cup$ Dr). These cases give back an already known quotient algebra defined in [5].

Let us give some counter-examples for the other cases. In the cases where peaks and valleys are identified, shuffling by 1 on the right does not preserve the statistics. For example, if we take $\sigma = 2746351$ and $\tau = 2756341$, the words $w = 27563481$ and $w' = 27563841$ are in $\tau \shuffle 1$, and 4 is a double rise in w and a double descent in w', whereas 4 is always a valley in the elements of $\sigma \shuffle 1$. So the only case where peaks and valleys are identified and where the quotient is well-defined is where all four statistics are identified, that is the case where all elements of $\mathfrak{S}_n$ are in the same class, a rather uninteresting case. For (P $\cup$ Dr, V, Dd) and (V, P $\cup$ Dr $\cup$ Dd), consider $\sigma = 45312$ and $\tau = 53124$. They are in the same class for these two quotients. The permutation $w = 645312$ is in $\sigma \shuffle 1$, and 4 is a valley. But 4 cannot be a valley in the element of $\tau \shuffle 1$. All other cases are equivalent to the previous ones, up to reversal of the alphabet or mirror image of words.

5 The quotient algebras are all free

After proving that some quotients are well-defined algebras, let us now prove that these quotients are all free. To this aim, we give an isomorphism between a free subalgebra (denoted by **B**) of **FQSym** and a quotient (denoted by **C**) having the same sequence of dimensions. We denote by p the projection onto **C**. The strategy is the following: we consider a basis B of **B** and show that the family $(p(b))_{b\in\mathbf{B}}$ spans **C**. Since in our case we know a basis C of **C**, we then show that the matrix of $p(B)$ in the basis C is invertible. So **B** and **C** are isomorphic as algebras.

5.1 The quotient of **FQSym** by (P, V, Dr, Dd)

In this Section, we take $\mathbf{B} = \mathbf{PBT}$ (for more details about $\mathbf{PBT}$, see [10]), $\mathbf{C} = \mathbf{FQSym}/(\text{P, V, Dr, Dd})$. The projection p is the quotient from **FQSym** onto **C**. Let us denote by $\mathfrak{R} = \{\sigma \in \mathfrak{S} | \sigma \text{ avoiding } 312\}$. Thanks to [3], a basis of **C** is given by $(p(\mathbf{F}_\sigma))_{\sigma\in\mathfrak{R}}$. We order this family by the inverse lexicographic order, and denote it by C.

Recall ([6]) that the **E** basis of $\mathbf{PBT}$ is given by $\mathbf{E}_\sigma = \sum_{\tau \geq_P \sigma} \mathbf{F}_\tau$, for $\sigma \in \mathfrak{R}$, where $\geq_P$ is the order of the right permutohedron. We denote by B this family ordered by the inverse lexicographic order. Note that $\mathbf{PBT}$ and **C** have the same sequence of dimensions, which is the Catalan sequence. Moreover, the algebra $\mathbf{PBT}$ is free since a family of independent algebraic generators is given by the $\mathbf{E}_\sigma$ with σ in $\mathfrak{R}$, and ending by 1. Let σ be in $\mathfrak{R}$. We have:

$$p(\mathbf{E}_\sigma) = \sum_{\tau \geq_P \sigma} p(\mathbf{F}_\tau) = \sum_{s\in\mathfrak{R}} c_s^{(\sigma)} p(\mathbf{F}_s), \text{ where } \ c_s^{(\sigma)} = |\{\tau \geq_P \sigma | \tau \sim s\}|. \tag{5}$$

Let us admit temporarily the following Proposition.

Proposition 5.1 *Let σ be in $\mathfrak{R}$. If τ satisfies $\sigma \leq_P \tau$, then the element τ' in $\mathfrak{R}$ which has the same four statistics as τ satisfies $\sigma \leq_{lex} \tau'$.*

Thanks to Proposition 5.1, for s and σ in $\mathfrak{R}$, $c_s^{(\sigma)} = 0$ if $s <_{lex} \sigma$. Morever, $c_\sigma^{(\sigma)} \geq 1$. So the matrix of $p(B)$ in the basis C is upper triangular with non zero coefficients on the diagonal. So $p(B)$ is a basis of **C**, and $\mathbf{PBT}$ is isomorphic to **C**. In particular, **C** is free.

Let us prove a slightly more general case than Proposition 5.1. Before stating the result, we first need some notations. Denote by $\mathfrak{X}$ the set of words appearing during the execution of the Françon-Viennot algorithm. It is the set of words that are obtained by inserting some ∞ inside permutations, no two ∞ consecutive. Let $\mathfrak{X}_n$ be the subset of $\mathfrak{X}$ where the permutation is in $\mathfrak{S}_n$. Given such an element, working backwards, one easily checks that there is only one way to apply the Françon-Viennot algorithm to get it.

Therefore, given $\tau \in \mathfrak{X}_n$, we define two elements: $p(\tau) \in \mathfrak{S}_n$ obtained from τ by erasing the ∞ letters and $r(\tau)$ obtained by doing the same replacements in the Françon-Viennot algorithm as to obtain τ but always on the *first* ∞ sign. Note that in the case where there are no ∞ in τ, this algorithm produces the element of the class of τ avoiding 312 as already proven by Françon and Viennot. We have:

Proposition 5.2 *Let $\sigma \in \mathfrak{S}_n$ avoiding 312 and $w \in \mathfrak{X}_n$ containing at least one ∞, such that $\sigma \leq_P p(w)$. Then $\sigma\infty \leq_{lex} r(w)$.*

Proof: We make the proof by induction on n. For $n = 1$, it is obvious.

Assume that the property holds for all such elements of $\mathfrak{S}_k$ with $k \leq n-1$. Let $\sigma \in \mathfrak{S}_n$ and in $\mathfrak{R}$. Let w be a word satisfying the statement and let i be the position of 1 in σ.

Since σ avoids 312, $\sigma = \sigma_1 1 \sigma_2$ where σ_1 is a permutation of $\{2, \ldots, i\}$.

If $i = n$, σ ends with 1, so does $p(w)$, and so does $p(w')$. Then the property comes by induction on the longest prefixes of σ and w not containing 1.

Otherwise, let us consider the words $\sigma_{(i)}$ and $w_{(i)}$ obtained by stopping the algorithm after only i steps instead of n. Then $\sigma_{(i)} = \sigma_1 1 \infty$. Since $\sigma \leq_P p(w)$, we have $\sigma_1 1 \leq_P p(w_{(i)})$. So the induction hypothesis applies and $\sigma_{(i)} \leq_{\mathrm{lex}} r(w_{(i)})$.

If the inequality is strict, then since the ∞ in the prefix of size i of $r(w_{(i)})$ are only replaced by letters greater than i, we deduce that $\sigma_{(i)} <_{\mathrm{lex}} r(w_n) = r(w)$, and then $\sigma\infty <_{\mathrm{lex}} r(w)$. Otherwise, the induction applies on the longest suffixes of σ and w not containing 1 and the result follows. □

*5.2 The quotient of **FQSym** by (P, V, Dr ∪ Dd)*

In this Section, $\sim$ represents the equivalence of permutations having the same peaks and valleys. Let us denote by $\mathfrak{D}$ the permutations in $\mathfrak{R}$ without double rise. Then $(p(\mathbf{F}_\sigma))_{\sigma\in\mathfrak{D}}$ is a basis of $\mathbf{C} =$ **FQSym**$\diagup$(P, V, Dr ∪ Dd). Let us order this by the inverse lexicographic order, and denote this basis by C. In that case, let $\mathbf{B}$ be the subalgebra generated by the family $(\mathbf{E}_\sigma)_{\sigma\in\mathfrak{D}}$. Note that this algebra has the same sequence of dimensions as $\mathbf{C}$. Moreover, $\mathbf{B}$ is free, and a family of independent algebraic generators is given by the E_σ, with σ in $\mathfrak{D}$, and ending by 1.

In order to prove that the family $p(B)$ is free on $\mathbf{C}$, we will use similar arguments as in the case **FQSym**$\diagup$(P, V, Dr, Dd).

Proposition 5.3 *Let σ be in $\mathfrak{R}$. If τ is in $\mathfrak{R}$, equivalent to σ for (P, V, Dr ∪ Dd), and its set of double rises strictly contains the set of double rises of σ, then $\tau <_{lex} \sigma$.*

Proof:

Let σ and τ be two permutations satisfying the hypothesis. Let k be the smallest integer such that k is a double rise of τ, and a double descent of σ (since the inclusion of double rises set is strict, such a k exists). Since σ and τ are in $\mathfrak{R}$, we have: $\sigma_{(k-1)} = \tau_{(k-1)} = u\infty v$ where u is a word without ∞. So, if we apply the k-th step of the construction of Françon-Viennot, we have: $\sigma_{(k)} = u\infty k v$, and $\tau_{(k)} = uk\infty v$. Since the first ∞ in $\sigma_{(k)}$ will be replaced by a letter greater than k, we have $\tau <_{lex} \sigma$.

□

Let us denote by c_σ^τ the number $|\{s \geq_P \sigma | s \sim \tau\}|$. From Proposition 5.3, we have that if σ and τ are in $\mathfrak{D}$, and $\sigma >_{lex} \tau$, then c_σ^τ is equal to zero. The c_σ^σ are strictly positive, so $\mathbf{C}$ is a free algebra.

*5.3 The quotients of **FQSym** by (P ∪ Dr, V ∪ Dd) or (P ∪ Dd, V ∪ Dr)*

Thanks to the involutions, these two quotient sets are isomorphic. Note that the equivalence relation (P ∪ Dr, V ∪ Dd) amounts to having the same descent bottoms set, so, hence up to a simple involution, having the same Genocchi set which is the set (see [5]). Thus **FQSym**$\diagup$(P ∪ Dd, V ∪ Dr) is isomorphic to $\mathbf{NCSF}$, and so is free.

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Some simple varieties of trees arising in permutation analysis

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Abstract After extending classical results on simple varieties of trees to trees counted by their number of leaves, we describe a filtration of the set of permutations based on their strong interval trees. For each subclass we provide asymptotic formulas for number of trees (by leaves), average number of nodes of fixed arity, average subtree size sum, and average number of internal nodes. The filtration is motivated by genome comparison of related species.

Résumé Nous commençons par étendre les résultats classiques sur les variétés simples d'arbres aux arbres comptés selon leur nombre de feuilles, puis nous décrivons une filtration de l'ensemble des permutations qui repose sur leurs arbres des intervalles communs. Pour toute sous-classe, nous donnons des formules asymptotiques pour le nombre d'arbres (comptés selon les feuilles), le nombre moyen de nœuds d'arité fixée, la moyenne de la somme des tailles des sous-arbres, et le nombre moyen de nœuds internes. Cette filtration est motivée par des problématiques de comparaison de génomes.

Keywords: permutations, simple varieties of trees, random generation, tree parameters, asymptotic formulas

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This short paper is an extended abstract of [7], where details of the proofs are provided.

1 Introduction

The idea of viewing permutations as enriched trees has been around for several decades in different research communities. For example, the recent enumerative study [1] of pattern avoiding permutations, in which *(substitution) decomposition trees* play a crucial role. Also, the analysis of sorting algorithms is very linked to tree representations of permutations: *PQ trees* [5] appear in the context of graph algorithms and *strong interval trees* arise in comparative genomics [6, and references therein, for instance].

In each case it is of interest to understand the typical shape and structure of the trees that arise. For example, a cursory examination of permutations that arise in the comparison of mammalian genomes strongly suggests that not all permutations are equally likely, and in fact this is quite an understatement. Trees coming from permutations under the uniform distribution are somehow degenerate [6], and do not adequately represent the trees that arise in genomic comparisons. This has important consequences for

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algorithm analysis. Specifically, in [6], Bouvel *et al.* considered a subclass of strong interval trees – selected because they represent what is known as *commuting scenarios* [3]– that correspond to the class of *separable permutations*. This is a first step towards a more relevant model of permutations which arise in genome comparison. By studying asymptotic enumeration and parameter formulas for separable permutations, they proved that the complexity of the algorithm of [3] solving the *perfect sorting by reversals* problem is polynomial time on separable permutations, whereas this problem is NP-complete in general. Furthermore they were also able to describe some average-case properties of the perfect sorting scenarios for separable permutations.

Ultimately, a clear understanding of the properties possessed by the strong interval trees that represent the comparison of actual genomes might tell us something about the evolutionary process. Bouvel *et al.* [6] conclude their study on separable permutations with a suggestion for the next step: strong interval trees with degree restrictions on certain internal nodes. It is a very controlled way to introduce bias in the distribution of strong interval trees. This is precisely what we do in this work; namely, we study strong interval trees where the prime nodes have a bounded number of children. This is a class of trees that can be completely understood combinatorially and analytically, and so we have immediate access to enumeration and analysis of some tree parameters that are ultimately related to the complexity of computing perfect sorting scenarios, or to properties of these scenarios.

In this work, we focus on the combinatorial analysis of these restricted sets of trees. This study reveals a very lush substructure of permutations that is certainly of independent interest. We define nested simple varieties of trees whose limit is the set of all strong interval trees, recalling they form a class in a size preserving bijection with permutations. The components are families of trees, hence we are able to apply a very complete set of tools to all the components: asymptotic analysis, random generation– these tools are inaccessible to the full class without working through permutations. Thus, we decompose a transcendental and non-analytic class into neat, algebraic portions, each of which is easily understood.

The organization of this abstract is as follows: First, in Section 2 we present some very general theorems for asymptotic enumeration and parameter analysis that are widely applicable. Then in Section 3 we describe strong interval trees as a decomposable combinatorial class. Finally, we describe the class of prime-degree restricted trees in Section 4, and give tight bounds on values which control the asymptotic enumeration and the tree parameters.

2 When the size of a tree is the number of leaves

There are many works which consider the study of average case parameters of trees where the size is the number of internal nodes or of both internal nodes and leaves. The generating functions of these trees satisfy a functional equation of the form $T(z) = z \cdot \Phi(T(z))$, and when Φ satisfies certain conditions, such as analyticity, then there are formulas for inversion, resulting in explicit enumerative results. A class of trees amenable to this treatment is said to be a *simple variety of trees*. The subject is exhaustively treated in Section VII.3 of [10]. If, instead, we define size as the number of leaves, the generating function satisfies a relation of the form $T(z) = z + \Lambda(T(z))$. The same general theorems on inversion still work, and it suffices to apply them and unravel the results. Even though they are less frequent, these have also been well studied in the literature, and the applicability of the inversion lemmas is noted in Example VII.13 of [10]. In this section we do this explicitly.

Consider the analytic solutions $T(z)$ of the equation

$$T(z) = z + \Lambda(T(z)), \tag{1}$$

Asymptotic number of trees with n leaves	$\sqrt{\frac{\rho}{2\pi\Lambda''(\tau)}} \cdot \frac{\rho^{-n}}{n^{3/2}}$
The average number of nodes of arity κ in trees with n leaves	$\frac{\lambda_\kappa \tau^\kappa}{\rho} \cdot n$
The average number of internal nodes in trees with n leaves	$\frac{\Lambda(\tau)}{\rho} \cdot n = \frac{\tau-\rho}{\rho} \cdot n$
The average subtree size sum in trees with n leaves	$\sqrt{\frac{\pi}{2\rho\Lambda''(\tau)}} \cdot n^{3/2}$

Tab. 1: A summary of parameters of trees given by $T = z + \Lambda(T)$. The value τ is the unique solution to $\Lambda'(\tau) = 1$ between 0 and $R_\Lambda < 1$, and $\rho = \tau - \Lambda(\tau)$.

where $\Lambda(z) = \sum_{n\geq 2} \lambda_n z^n$ is analytic with radius of convergence R_Λ, and such that $\lambda_n \geq 0$ for any $n \geq 2$. Furthermore, assume that Λ is not the null function. Let $\Psi(z) := z - \Lambda(z)$. Equation (1) rewrites as $\Psi(T(z)) = z$, so what we are looking for is precisely an analytic inversion of Ψ.

The Table 1 summarizes the results of this section. We determine asymptotic formulas for number of trees, and several key parameters. The shape of the formulas are, unsurprisingly, not unlike those that arise in the study of trees counted by internal nodes.

2.1 Asymptotic number of trees

Our entire analysis is roughly a consequence of the analytic inversion lemma and transfer theorems. The version to which we appeal is given and proved in [10]. Citations to original sources may be found therein. The following theorem is a slight adaptation of Proposition IV.5 and Theorem VI.6 to combinatorial equations of the form $\mathcal{T} = \mathcal{Z} + \Lambda(\mathcal{T})$ instead of $\mathcal{T} = \mathcal{Z} \cdot \Lambda(\mathcal{T})$.

Theorem 1 *Let Λ be a function analytic at 0, with non-negative Taylor coefficients, and such that, near 0,*

$$\Lambda(z) = \sum_{n\geq 2} \lambda_n z^n.$$

Let R_Λ be the radius of convergence of this series. Under the condition $\lim_{x\to R_\Lambda^-} \Lambda'(x) > 1$, there exists a unique solution $\tau \in (0, R_\Lambda)$ of the equation $\Lambda'(\tau) = 1$.

Then, the formal solution $T(z)$ of the equation $T(z) = z + \Lambda(T(z))$ is analytic at 0, its unique dominant singularity is at $\rho = \tau - \Lambda(\tau)$ and its expansion near ρ is

$$T(z) = \tau - \sqrt{\frac{2\rho}{\Lambda''(\tau)}}\sqrt{1 - z/\rho} + O(1 - z/\rho). \tag{2}$$

Moreover, if T is aperiodic, then one has

$$[z^n]T(z) \sim \sqrt{\frac{\rho}{2\pi\Lambda''(\tau)}} \cdot \frac{\rho^{-n}}{n^{3/2}}. \tag{3}$$

2.2 Parameter Analysis

In the case of trees counted by internal nodes, the study of recursively defined parameters is very straightforward, starting from generating function equations. We can describe analogous versions for trees counted by leaves. In particular, we consider additive parameters, and describe a Modified Iteration Lemma, adapted to our notion of size. We illustrate the lemma on number of internal nodes, subtree size sum and number of nodes of a given arity.

Our focus is on tree parameters that can be computed additively by parameters of subtrees. More precisely, given a parameter $\xi(t)$ for trees $t \in \mathcal{T}$ which satisfy the relation

$$\xi(t) = \eta(t) + \sum_{j=1}^{\deg(t)} \sigma(t_j),$$

where $\deg(t)$ is the arity of the root and t_j are its children. Let $\Xi(z)$, $H(z)$ and $\Sigma(z)$ be the associated cumulative functions of ξ, η and σ. That is, $\Xi(z) = \sum_{t\in\mathcal{T}} \xi(t) z^{|t|}$, $H(z) = \sum_{t\in\mathcal{T}} \eta(t) z^{|t|}$ and $\Sigma(z) = \sum_{t\in\mathcal{T}} \sigma(t) z^{|t|}$.

Lemma VII.1 in [10] has an analogue for trees counted by their leaves, and it is proved in a very similar way.

Lemma 2 (Iteration lemma for trees counted by their leaves) *Let $\mathcal{T}$ be a class of trees satisfying $\mathcal{T} = \mathcal{Z} + \Lambda(\mathcal{T})$. The cumulative generating functions are related by*

$$\Xi(z) = H(z) + \Lambda'(T(z))\,\Sigma(z).$$

In particular, if $\sigma \equiv \xi$, one has $\Xi(z) = \frac{H(z)}{1-\Lambda'(T(z))} = H(z)\cdot T'(z)$.

The last equality is a consequence of $T'(z)(1-\Lambda'(T(z))) = 1$, which is obtained by differentiating $T(z) = z + \Lambda(T(z))$ with respect to z.

We make a remark, that if $\sigma \equiv \xi$, we say the parameter is *recursive*; most basic parameters are recursive, and in what follows we shall use this case only. Note also that when analytic treatment applies, $T(z)$ has a square-root singularity, so that $T'(z)$ has an inverse square-root singularity (by analytic derivation). Therefore, whenever $H(z)$ tends to a positive real when $z \to \rho$ (under some analytic conditions), then transfer yields an asymptotic equivalent of the mean value of the parameter of the form $c \cdot n$. This is for instance the case for the number of nodes of fixed arity and the number of internal nodes.

Number of nodes with exactly κ children We "mark" nodes of arity κ by setting

$$\eta(t) = \begin{cases} 1 & \text{if the root of } t \text{ is of arity } \kappa, \\ 0 & \text{otherwise.} \end{cases}$$

Hence if $\kappa \geq 2$, $H(z) = \sum_{t\in\mathcal{T}} \eta(t) z^{|t|} = \sum_{t_1,\ldots,t_\kappa \in \mathcal{T}} \lambda_\kappa z^{|t_1|+|t_2|+\ldots+|t_\kappa|}$ so that $H(z) = \lambda_\kappa\, T(z)^\kappa$. And(i) if $\kappa = 0$, $H(z) = z$ which is not interesting since it is counting the number of leaves *i.e.* the size of the tree.

(i) This is the only other possibility since there can be no unary nodes in a proper specification.

By Lemma 2, for any $\kappa \geq 2$ one has $\Xi(z) = \lambda_\kappa T(z)^\kappa \cdot T'(z)$. Since the singular expansion of $T(z)$ near ρ is

$$T(z) = \tau - \gamma\sqrt{1 - z/\rho} + o\left(\sqrt{1 - z/\rho}\right), \text{with } \gamma = \sqrt{\frac{2\rho}{\Lambda''(\tau)}} \tag{4}$$

then near ρ, one has $T(z)^\kappa = \tau^\kappa + O\left(\sqrt{1 - z/\rho}\right)$. Using the singular differentiation theorem we have

$$T'(z) = \frac{\gamma}{2\rho\sqrt{1 - z/\rho}} + o\left(\frac{1}{\sqrt{1 - z/\rho}}\right), \text{ so that } \Xi(z) = \frac{\lambda_\kappa \gamma \tau^\kappa}{2\rho\sqrt{1 - z/\rho}} + o\left(\frac{1}{\sqrt{1 - z/\rho}}\right),$$

from which we get the asymptotics of the cumulative generating function

$$[z^n]\Xi(z) \sim \frac{\lambda_\kappa \gamma \tau^\kappa \rho^{-n-1}}{2\sqrt{\pi n}}.$$

The asymptotics of the average value across all trees of size n is reported in Table 1.

Number of internal nodes For this parameter, just take the following definition for η:

$$\eta(t) = \begin{cases} 0 & \text{if } t \text{ is just one leaf,} \\ 1 & \text{otherwise.} \end{cases}$$

One has $H(z) = \sum_{t \in \mathcal{T}} \eta(t) z^{|t|} = T(z) - z$, and therefore (with the γ of Equation (4))

$$\Xi(z) = (T(z) - z)\ T'(z) = \frac{\gamma(\tau - \rho)}{2\rho\sqrt{1 - z/\rho}} + o\left(\frac{1}{\sqrt{1 - z/\rho}}\right) \quad \text{and} \quad [z^n]\Xi(z) \sim \frac{\gamma(\tau - \rho)\rho^{-n-1}}{2\sqrt{\pi n}}.$$

Subtree size sum We are interested in the subtree size sum parameter, defined by $\eta(t) = |t|$, hence $H(z) = zT'(z)$. So that

$$\Xi(z) = zT'(z)^2 = \frac{\gamma^2}{4\rho(1 - z/\rho)} + o\left(\frac{1}{1 - z/\rho}\right) \quad \text{and} \quad [z^n]\Xi(z) \sim \frac{\gamma^2}{4\rho} \cdot \rho^{-n}.$$

It is not an inverse of square-root singularity, and we find an asymptotic equivalent in $n^{\frac{3}{2}}$ for the average value of the subtree size sum (see Table 1), which is typical for path length related parameters.

There are many other tree parameters that we could consider in a similar fashion.

3 Strong Interval Trees

Our interest in trees counted by leaves is spawned by *strong interval trees*. They are in a size preserving bijection with permutations. This particular representation of permutations is a very effective data structure for algorithms in reconstruction of genome evolution scenarios, as we briefly mentioned in Section 1. Our analysis builds subclasses that are in fact each a simple variety of trees, and hence are very well understood, particularly given the generic analysis we have completed in Section 2.

3.1 Definition and examples

A description of the bijective correspondence between strong interval trees (sometimes also called (substitution) decomposition trees) and permutations is given in [6]. Truly, it could be viewed as a tree representation of the block decomposition of permutations described by Albert and Atkinson [1], the modular decomposition of permutation graphs of Bérard *et al.* in [4] and even has origins in the PQ-trees of Booth and Lueker [5]. The bijection is completely constructive, and can be computed in linear time, although this is quite difficult to achieve, see [4]. We do not describe the bijection in this work.

The class is a set of trees where some internal nodes are enriched with a simple permutation. A permutation is said to be *simple* if the only intervals $i, i+1, \ldots, k$ mapped to an interval are the singletons, and $1, 2, \ldots, n$. Because we take the convention that 1 2 and 2 1 are not simple permutations, the shortest ones are of size 4 and are $3\ 1\ 4\ 2$ and $2\ 4\ 1\ 3$. An enumerative study is done by Albert *et al.* [2], and we make use of their asymptotic enumeration formulas. Let s_n be the number of simple permutations of size n. This is sequence A111111 in the On-Line Encyclopedia of Integer Sequences [13]. The sequence is not P-recursive, but it does satisfy a simple functional inversion formula, and we have calculated exact values of for s_n for $n < 800$. Albert *et al.* determined the following bounds:

$$\frac{n!}{e^2}\left(1-\frac{4}{n}\right) \leq s_n \leq \frac{n!}{e^2}\left(1-\frac{4}{n}+\frac{2}{n(n-1)}\right). \tag{5}$$

Here are the first few terms in the generating function for simple permutations:

$$S(z) = 2z^4 + 6z^5 + 46z^6 + 338z^7 + 2926z^8 + 28146z^9 + 298526z^{10} + 3454434z^{11} + \ldots \tag{6}$$

Theorem 3 (Reformulated [1]) *The class of permutations is in a size-preserving bijection with the combinatorial class $\mathcal{P}$ of enriched trees defined by the following relations, where size is given by the number of leaves. The class $\mathcal{Z}$ is an atomic class with a single element of size 1, and the $\mathcal{N}$ classes are all epsilon classes containing a single element of size 0, marking internal nodes:*

$$\begin{aligned}
\mathcal{P} &= \mathcal{Z}_\square + \mathcal{N}_\oplus \cdot \mathrm{Seq}_{\geq 2}\,\mathcal{U}_\oplus + \mathcal{N}_\ominus \cdot \mathrm{Seq}_{\geq 2}\,\mathcal{U}_\ominus + \mathcal{N}_\bullet \cdot S(\mathcal{P}),\\
\mathcal{U}_\oplus &= \mathcal{Z}_\square + \mathcal{N}_\ominus \cdot \mathrm{Seq}_{\geq 2}\,\mathcal{U}_\ominus + \mathcal{N}_\bullet \cdot S(\mathcal{P}),\\
\mathcal{U}_\ominus &= \mathcal{Z}_\square + \mathcal{N}_\oplus \cdot \mathrm{Seq}_{\geq 2}\,\mathcal{U}_\oplus + \mathcal{N}_\bullet \cdot S(\mathcal{P}).
\end{aligned} \tag{7}$$

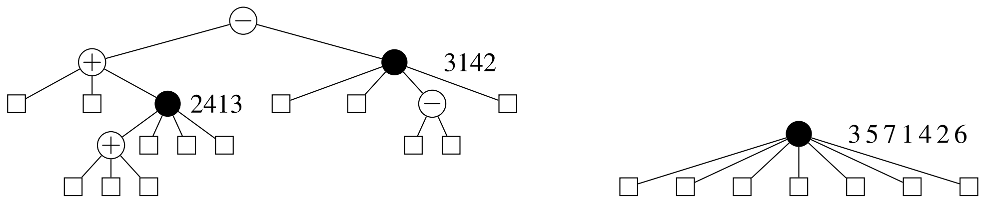
The internal nodes $\mathcal{N}_\bullet$ are called prime nodes *and the internal nodes $\mathcal{N}_\oplus$ and $\mathcal{N}_\ominus$ are called* linear *nodes. The function $S(z)$ is the generating function for simple permutations from Equation* (6)*.*

Figure 1 contains two examples. Figure 1(b) represents a simple permutation. We note that the trees corresponding to simple permutations contain only a single prime node with n children. The root is labeled by the permutation itself.

Notice that $\mathcal{U}_\oplus$ and $\mathcal{U}_\ominus$ define combinatorial classes which are in size-preserving bijection. In the following, in order to deal with one class instead of two, we replace them by the equivalent class $\mathcal{U} = \mathcal{Z}_\square + \mathcal{N}_\circ \cdot \mathrm{Seq}_{\geq 2}\,\mathcal{U} + \mathcal{N}_\bullet \cdot S(\mathcal{P})$. Doing so, we change the labels of the linear nodes having a linear parent (replacing them by $\circ$). This does not affect the enumeration of the class. Indeed, these labels are determined since a linear node and its linear parent have different labels.

Corollary 4 *The following combinatorial equivalences are true:*

$$\mathcal{P} \equiv \mathrm{Seq}_{\geq 1}\,\mathcal{U} \qquad \textit{and} \qquad \mathcal{U} \equiv \mathcal{Z} + \mathrm{Seq}_{\geq 2}\,\mathcal{U} + S(\mathrm{Seq}_{\geq 1}\,\mathcal{U}). \tag{8}$$



(a) $\sigma_1 = 6\ 7\ 9\ 10\ 11\ 13\ 8\ 12\ \ 3\ 1\ 5\ 4\ 2$ (b) $\sigma_2 = 3\ 5\ 7\ 1\ 4\ 2\ 6$

Fig. 1: Two permutations and their associated strong interval trees

Consequently, $\mathcal{U}$ *is in bijection with a class of* Λ*-trees for* $\Lambda(x) = \frac{x^2}{1-x} + \sum_{j\geq 4} s_j \left(\frac{x}{1-x}\right)^j$, *where* s_j *is the number of simple permutations of size* j.

Proof: This equivalence is derived from Equation (7), the fact that $\mathcal{U} \equiv \mathcal{U}_\oplus \equiv \mathcal{U}_\ominus$, and the intermediary equivalence $\mathcal{P} \equiv \mathcal{U} + \mathrm{Seq}_{\geq 2}\,\mathcal{U}$. □

Now, neither $\mathcal{P}$ nor $\mathcal{U}$ are simple varieties of trees because $S(z)$, and hence $\Lambda(x)$, are not analytic at the origin. In this case, we can, of course, use the bijection to permutations to have access to enumeration and random generation tools. However, we propose a different strategy: generate a sequence of analytic Λ_k such that as formal power series, $\lim_{k\to\infty} \Lambda_k = \Lambda$, and consider the set of Λ_k-trees. Can we describe conditions so that the limit of the asymptotics of the subclasses tends to the asymptotics of the whole class? To which extent are the parameter formulas valid under the limit? The example we have in hand is a particularly instructive one, since the limit is known by other means, and allows us to test the limits of analytic inversion.

3.2 A filtration for permutations

Next we describe the central filtration on the class of trees $\mathcal{P}$. The limit of the filtration is the entire class, and each subclass is a simple variety of trees that is very straightforward to analyze. We define the class $\mathcal{P}^{(k)}$ as follows, where $S^{\leq k}(z) = \sum_{j=4}^{k} s_j z^j$:

$$\mathcal{P}^{(k)} = \mathcal{Z} + 2\,\mathrm{Seq}_{\geq 2}\,\mathcal{U}^{(k)} + S^{\leq k}(\mathcal{P}^{(k)}) \qquad \text{and} \qquad \mathcal{U}^{(k)} = \mathcal{Z} + \mathrm{Seq}_{\geq 2}\,\mathcal{U}^{(k)} + S^{\leq k}(\mathcal{P}^{(k)}). \tag{9}$$

That is, we restrict the degree of the prime nodes. The containment $\mathcal{P}^{(k)} \subset \mathcal{P}^{(k+1)}$ is straightforward, and since $\mathcal{P}^{(k)}_n = \mathcal{P}_n$ when $k \geq n$, we can derive the limit of combinatorial classes $\lim_{k\to\infty} \mathcal{P}^{(k)} = \mathcal{P}$.

Furthermore, by the same manipulations as for the full class, we derive:

$$\mathcal{P}^{(k)} \equiv \mathrm{Seq}_{\geq 1}\,\mathcal{U}^{(k)} \qquad \text{and} \qquad \mathcal{U}^{(k)} \equiv \mathcal{Z} + \mathrm{Seq}_{\geq 2}\,\mathcal{U}^{(k)} + S^{\leq k}(\mathrm{Seq}_{\geq 1}\,\mathcal{U}^{(k)}). \tag{10}$$

Remark that $\mathcal{U}^{(k)}$ is isomorphic to a Λ_k-tree with $\Lambda_k(x) = \frac{x^2}{1-x} + \sum_{j=4}^{k} s_j \left(\frac{x}{1-x}\right)^j$. This class is certainly algebraic. It is easy to generate many terms in the enumerative sequence using this algebraic equation. We call the class denoted by $\mathcal{P}^{(k)}$, as *prime-degree restricted* strong interval trees.

More generally, one goal of this work is to illustrate a strategy for the analysis of classes of trees $\mathcal{C}$ that fail to be a simple variety of trees because the series governing the number of children available is not analytic. In such cases, one may look for a parameter such that each subclass of trees $\mathcal{C}^{(k)}$, for which that parameter take value at most k, is algebraic. We can then study the classes $\mathcal{C}^{(k)}$ at fixed k, and hopefully develop techniques to obtain information on $\mathcal{C}$ by letting k go to infinity. We study an example of such a class in the present work, and illustrate some of the challenges of sending the limits of both the parameter value k and the size n to infinity at the same time.

4 Enumerating Prime-Degree Restricted Strong Interval Trees

The enumerative analysis of Section 2 applies directly to these families of trees. Ideally, we would like to preserve k as much as possible in the formulas.

4.1 Asymptotic enumeration

The equations (10) allow us to directly apply Theorem 1 to determine asymptotic formulas for the coefficients of the generating functions.

Theorem 5 *For fixed k, the number of prime-degree restricted strong interval trees of size n, denoted $P_n^{(k)}$ grows asymptotically like*

$$P_n^{(k)} \sim \gamma_k \rho_k^{-n} n^{-3/2} \quad \textit{where} \quad \gamma_k = \sqrt{\frac{\rho_k}{2\pi\Lambda_k''(\tau_k)}} \quad \textit{as } n \to \infty. \tag{11}$$

Here, $\Lambda_k(x) = \frac{x^2}{1-x} + \sum_{j=4}^{k} s_j (\frac{x}{1-x})^j$, τ_k satisfies $1 - \Lambda_k'(\tau_k) = 0$ and $\rho_k = \tau_k - \Lambda_k(\tau_k)$.

Proof: First, we note that since $\sum_{j=4}^{k} s_j (\frac{x}{1-x})^j$ is a polynomial in $\frac{x}{1-x}$, $\Lambda_k(x)$ is certainly analytic at 0. Hence, the enumerative formulas of the first section apply, yielding the asymptotic estimate $U_n^{(k)} \sim \gamma_k \rho_k^{-n} n^{-3/2}$ where $\gamma_k = \sqrt{\frac{\rho_k}{2\pi\Lambda''(\tau_k)}}$.

Next, we note that by the second relation in Equation (10), $P^{(k)}(z) = \frac{U^{(k)}(z)}{1-U^{(k)}(z)}$. This is a subcritical composition, since the value of $U^{(k)}(z)$ at dominant singularity ρ_k is τ_k, which is less than 1 by Theorem 1. Consequently, $P_n^{(k)} \sim \frac{U_n^{(k)}}{1-U_n^{(k)}}$ for large n, hence the approximation stated holds. □

Table 2 contains numeric approximations for τ_k and ρ_k in the range $k = 4 \ldots 13$. Using these estimates gives good asymptotic approximations and the enumerative formulas given in Equation (11) converge quickly for fixed k. Next we apply some refined analysis to bound the asymptotic estimate of Equation (11) – see Equation (16) below.

4.2 Bounding the asymptotic estimate of $P_n^{(k)}$

We can produce an asymptotic estimate for $P_n^{(k)}$ in terms of k from Equation (11) by bounding ρ_k and $\Lambda_k''(\tau_k)$. The first ingredient is a more explicit bound for s_n, the number of simple permutations.

Lemma 6 *For every $n \geq 4$, $s_n \leq \sqrt{2\pi}\, n^{n+1/2}\, e^{-n-2}$.*

Proof: This inequality is a consequence of applying the Stirling bound to the bounds of Equation (5). In particular, we use $n! \leq \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}\, e^{\frac{1}{12n}}$ and the inequality $(1 - \frac{4}{n} + \frac{2}{n(n-1)}) e^{\frac{1}{12n}} \leq 1$ for $n \geq 4$, which can be proved by simple computations. □

k	τ_k	ρ_k	k	τ_k	ρ_k
4	0.2258458016	0.1454726242	9	0.1463252500	0.1102193554
5	0.2043553556	0.1364583031	10	0.1375961304	0.1057725121
6	0.1841224072	0.1277948168	11	0.1300393555	0.1017629085
7	0.1689470150	0.1210046262	12	0.1234001218	0.09810173382
8	0.1565912704	0.1152312243	13	0.1174959122	0.09472586497

Tab. 2: Computed values for ρ_k and τ_k for small values of k. For these values, the bounds of Subsection 4.2 on ρ_k and τ_k are not tight. However, we do note that in the limit, the sequences (τ_k), and (ρ_k) tend to 0, which is consistent with the fact that the ordinary generating function for permutations has zero radius of convergence.

From this estimate, the derivations of the bounds on τ_k and ρ_k are straighforward, but technical. Working with the value $\tilde{\tau}_k = \frac{\tau_k}{1-\tau_k}$ simplifies the expressions. Much of the bounds are then consequences of the inequalities $0 < \rho_k < \tau_k < \tilde{\tau}_k < 1$.

Proposition 7 (Bounds for $\tilde{\tau}_k$) *For any $\alpha < \frac{e-2}{e-1}$, there exists $k(\alpha)$ such that for $k > k(\alpha)$*

$$\left(\frac{\alpha}{ks_k}\right)^{\frac{1}{k-1}} < \tilde{\tau}_k < \left(\frac{1}{ks_k}\right)^{\frac{1}{k-1}}. \tag{12}$$

Consequently,

$$\frac{e}{k}\left(\frac{\alpha e^3}{\sqrt{2\pi}\, k^{5/2}}\right)^{\frac{1}{k-1}} < \tilde{\tau}_k < \frac{e}{k}\left(\frac{e^3}{\sqrt{2\pi}\, k^{3/2}(k-4)}\right)^{\frac{1}{k-1}} < \frac{e}{k}. \tag{13}$$

Computational evidence suggests that $k(\alpha) = 4$, for all α near $\frac{e-2}{e-1}$.

Proof: (sketch) The starting point is the equation $1 = \Lambda_k'(x)$, under the change of variables $y = \frac{x}{1-x} \iff x = \frac{y}{1+y}$. We first remark that the equation $1 = \Lambda_k'(\frac{y}{1+y})$ can be rewritten as

$$1 = (1+y)^2 - 1 + (1+y)^2 \sum_{j=4}^{k} j s_j y^{j-1} \quad \text{which implies} \quad \frac{2-(1+y)^2}{(1+y)^2} = \sum_{j=4}^{k} j s_j y^{j-1}. \tag{14}$$

The next step towards proving the stated inequalities is the fact that for $0 < y < 1$, $1-5y \leq \frac{2-(1+y)^2}{(1+y)^2} \leq 1$. Indeed, Equation (14) is satisfied at $y = \tilde{\tau}_k$, and consequently these inequalities yield an upper and a lower bound for $\sum_{j=4}^{k} j s_j \tilde{\tau}_k^{j-1}$.

The announced upper bound on $\tilde{\tau}_k$ is easily derived from $k s_k \tilde{\tau}_k^{k-1} \leq \sum_{j=4}^{k} j s_j \tilde{\tau}_k^{j-1} \leq 1$.

The lower bound is derived from the upper bound via the inequality $1 - 5\tilde{\tau}_k - \sum_{j=4}^{k-1} j s_j \tilde{\tau}_k^{j-1} \leq k s_k \tilde{\tau}_k^{k-1}$. For this purpose, we also need an upper bound on $\sum_{j=4}^{k-1} j s_j \tilde{\tau}_k^{j-1}$. It is obtained splitting the sum into two parts, which can be bounded separately. More precisely, setting $\lambda_k = \lfloor k^{1/3} \rfloor$, we can show that

$$\sum_{j=4}^{k-\lambda_k-1} j s_j \tilde{\tau}_k^{j-1} = O\left(\frac{1}{k^3}\right) \qquad \text{and that} \qquad \sum_{k-\lambda_k}^{k-1} j s_j \tilde{\tau}_k^{j-1} = \frac{1}{e-1}\left(1 + o(1)\right).$$

Full details are available in the long version [7] of this abstract. □

Theorem 8 (Bounds for ρ_k) *For any $\alpha < \frac{e-2}{e-1}$, there exist $\beta(\alpha)$ and $k(\alpha)$ such that for any $k \geq k(\alpha)$,*

$$\frac{e}{k}\left(\frac{e^3\alpha}{\sqrt{2\pi}\,k^{5/2}}\right)^{\frac{1}{k-1}}\left(1-\frac{\beta(\alpha)}{k}\right) < \rho_k < \frac{e}{k}\left(\frac{e^3}{\sqrt{2\pi}\,k^{3/2}(k-4)}\right)^{\frac{1}{k-1}}.$$

Consequently, $\rho_k = \frac{e}{k}\left(1-\frac{5}{2}\frac{\log k}{k}+\Theta(\frac{1}{k})\right)$.

Proof: The upper bound is immediate from the bound $\rho_k < \tilde{\tau}_k$ and Proposition 7.
The lower bound is derived by showing that $\rho_k = \tau_k - \Lambda_k(\tau_k) = \tilde{\tau}_k\left(1-\frac{2\tilde{\tau}_k}{1+\tilde{\tau}_k}-\sum_{j=4}^k s_j\tilde{\tau}_k^{j-1}\right) = \tilde{\tau}_k(1+\Theta(\frac{1}{k}))$. In much the same fashion as the previous proposition, we leverage upper bounds on $\tilde{\tau}_k$ to build a lower bound. In this case, we use $\frac{2\tilde{\tau}_k}{1+\tilde{\tau}_k} \leq 2\tilde{\tau}_k \leq 2\frac{e}{k}$, and the summation can be bounded by splitting the sum at the same place:

$$\sum_{j=4}^{k} s_j\,\tilde{\tau}_k^{j-1} = \sum_{j=4}^{k-\lambda_k-1} s_j\,\tilde{\tau}_k^{j-1} + \sum_{j=k-\lambda_k}^{k-1} s_j\,\tilde{\tau}_k^{j-1} + s_k\,\tilde{\tau}_k^{k-1}.$$

Even though it is not the same summation, we nonetheless re-use the same bounding process on the partial summations to recover

$$\sum_{j=4}^{k-\lambda_k-1} s_j\,\tilde{\tau}_k^{j-1} = O\left(\frac{1}{k^3}\right) \quad\text{and}\quad \sum_{j=k-\lambda_k}^{k-1} s_j\,\tilde{\tau}_k^{j-1} = \frac{1}{k-\lambda_k}\sum_{j=k-\lambda_k}^{k-1} j s_j\,\tilde{\tau}_k^{j-1} = \Theta\left(\frac{1}{k}\right). \tag{15}$$

Finally, since $k\,s_k\,\tilde{\tau}_k^{k-1} \leq 1$, we have that $\frac{2\tilde{\tau}_k}{1+\tilde{\tau}_k}+\sum_{j=4}^k s_j\tilde{\tau}_k^{j-1} = \Theta\left(\frac{1}{k}\right)$, from which it follows that $\rho_k = \tilde{\tau}_k(1+\Theta(\frac{1}{k}))$. The remaining expressions arise from substituting the lower bounds for $\tilde{\tau}_k$, bounds for s_k, followed by some basic manipulations. $\square$

It was known in [8] that $\rho_k = \frac{e}{k}(1+o(1))$, but we are able to produce a more precise estimate. We require this precision when we consider the limit as $k \to \infty$.

From the series expansion of $\Lambda''(x)$, we have $\Lambda''(\tau_k) \geq 2+6\tilde{\tau}_k$. We could expand this expression further, and use lower bounds on $\tilde{\tau}_k$, but it turns out that for our purposes, the bound $\Lambda''(\tau_k) \geq 2$ is sufficient.

Upper bound for the asymptotic estimate of $P_n^{(k)}$ Finally, we have all of the elements to determine an asymptotic estimate of $P_n^{(k)}$. We substitute the upper and lower bounds for ρ_k, and the bound $\Lambda''(\tau_k) \geq 2$ to obtain:

$$\gamma_k\rho_k^{-n}n^{-3/2} \leq \sqrt{\frac{e}{4k\pi}}\left(\frac{k}{e}\right)^n\left(1+\frac{5}{2}\frac{\log k}{k}+\Theta\left(\frac{1}{k}\right)\right)^n n^{-3/2}. \tag{16}$$

In the limit, Stirling's approximation Our analysis of $\mathcal{P}$ brings together two classic asymptotic facts. The asymptotic growth of a simple variety of trees $\mathcal{T}$ is always of the form $T_n \sim \gamma\rho^{-n}n^{-3/2}$ for some real valued ρ and γ but the classic Stirling's approximation of $n!$ gives $P_n \sim \left(\frac{n}{e}\right)^n\sqrt{2\pi n}$. Subtle analysis is required to reconcile these two estimates.

The trees for all permutations of size n have prime nodes of arity at most n. Thus, if $k \geq n$, $\mathcal{P}_n^{(k)}$ contains all of them, and hence $P_n^{(k)} = n!$ for $k \geq n$. Now, consider Equation (16) with $k = n$. The upper bound is a constant times Stirling's formula[(ii)]. However, when we consider $P_n^{(2n)}$, which is also $n!$, the upper bound gains an unwanted factor of 2^n. This does not contradict the correctness of our asymptotic form, for fixed k, and it rather emphasizes that it is an open problem to develop asymptotic formulas when k is a function of n, and they go to infinity together. This will require a return to the analytic inversion and transfer theorems to study how the error terms depend on k[(iii)].

4.3 Parameter analysis

From Equation (5), simple permutations make up about 1/9 of all permutations, and consequently the average case analysis of parameters is dominated by their very flat shape. However, the prime-degree restricted trees are much more rich and parameter analysis follows from Section 2.

We remark that the perfect sorting scenarios for σ are directly related to the number of internal nodes, and in particular the distribution among prime and linear nodes. The average subtree size is related to the average reversal size. These two parameters give important insight into the average case analysis of perfect sorting by reversals. A more elaborate discussion on the links between these parameters and algorithm analysis is presented in [6].

4.4 Random generation

Since our initial interest is the shape of the trees, and not the particulars of the internal nodes, we have produced a Boltzmann generator which generates trees of size approximately 10000 for k up to 800 without generating the simple permutation labels. Figure 2 illustrates a randomly generated tree from $\mathcal{P}^{(7)}$ with approximately 1000 leaves. Remark that the structure is dominated by prime nodes of arity 7.

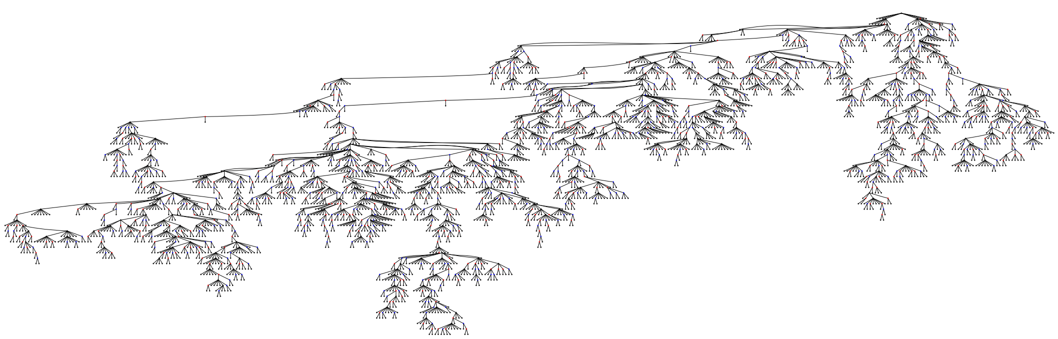


Fig. 2: A tree from $\mathcal{P}^{(7)}$ generated uniformly at random

5 Conclusion

On the biological side, our long term goal is to understand random permutations in order to identify the very specific traits which arise in permutations which encode mammalian genome comparisons. Chauve,

(ii) This constant is $\sqrt{\frac{e}{8\pi^2}}$ obtained replacing k by n in Equation (16).

(iii) The difficulty here lies in Λ being not analytic. Notice however that the same filtration by truncations at order k may also be defined when Λ is analytic: in this case, it is not difficult to prove that we obtain the correct asymptotic formula when taking the limit as k tends to infinity, *i.e.* that limits in n and k commute.

McCloskey and Mishna [9] have taken some preliminary steps in this direction.

On the analytical side, we would like to describe the parameters as functions of k. This will require a very delicate treatment of the bounds, and a much stronger understanding of how to take the limit as $k \to \infty$. This is a much larger undertaking, as essentially we are no longer guided by the inversion theorems.

Finally, one can ask other permutations properties with respect to this filtration. In particular, the model we investigate has a strong connection with the pattern avoiding permutation classes that contain a finite number of simple permutations [1].

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Gelfand Models for Diagram Algebras: extended abstract

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Abstract. A Gelfand model for a semisimple algebra A over $\mathbb{C}$ is a complex linear representation that contains each irreducible representation of A with multiplicity exactly one. We give a method of constructing these models that works uniformly for a large class of combinatorial diagram algebras including: the partition, Brauer, rook monoid, rook-Brauer, Temperley-Lieb, Motzkin, and planar rook monoid algebras. In each case, the model representation is given by diagrams acting via "signed conjugation" on the linear span of their vertically symmetric diagrams. This representation is a generalization of the Saxl model for the symmetric group, and, in fact, our method is to use the Jones basic construction to lift the Saxl model from the symmetric group to each diagram algebra. In the case of the planar diagram algebras, our construction exactly produces the irreducible representations of the algebra.

Résumé. Un modèle de Gelfand pour une algèbre semi-simple A sur $\mathbb{C}$ est une représentation linéaire complexe qui contient chaque représentation irréductible de A avec multiplicité exactement un. Nous fournissons une méthode de construction explicite de ces modèles qui fonctionne de manière uniforme pour une grande classe d'algèbres de schéma combinatoire, y compris: la partition, Brauer, rook-monoid, rook-Brauer, Temperley-Lieb, Motzkin, et algèbres planaires rook monoid. En chaque cas, la représentation du modèle est donnée par les diagrammes agissant par "conjugaison signé" sur l'espace engendré par les diagrammes verticalement symétriques. Cette représentation est une généralisation du modèle Saxl pour le groupe symétrique, et, en fait, notre méthode est d'utiliser le "Jones basic construction" pour étendre le modèle Saxl du groupe symétrique à chaque algèbre diagramme. Dans le cas des algèbres de diagrammes planaires, notre construction produit exactement les représentations irréductibles de l'algèbre.

Keywords: Gelfand model; multiplicity-free representation; symmetric group; partition algebra; Brauer algebra; Temperley-Lieb algebra; Motzkin algebra; rook-monoid

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1 Introduction

A famous consequence of Robinson-Schensted-Knuth (RSK) insertion is that the set of standard Young tableaux with k boxes is in bijection with the set of involutions in the symmetric group S_k (the permutations $\sigma \in \mathsf{S}_k$ with $\sigma^2 = 1$). Furthermore, these standard Young tableux index the bases for the irreducible $\mathbb{C}\mathsf{S}_k$ modules, so it follows that the sum of the degrees (dimensions) of the irreducible S_k modules equals the number of involutions in S_k. This suggests the possibility of a representation of the symmetric group

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on the linear span of its involutions which decomposes into irreducible S_k-modules such that the multiplicity of each irreducible is exactly 1. Indeed, Saxl [22] and Kljačko [13] have constructed such a module. In this representation, the symmetric group acts on its involutions by a twisted, or signed, conjugation (see Section 3). A combinatorial construction of this module was studied recently by Adin, Postnikov, and Roichman [1] and extended to the rook monoid and related semigroups in [14]. A representation for which each irreducible appears with multiplicity one is called a *Gelfand model* (or, simply, a *model*), because of the work in [3] on models for complex Lie groups.

In [9] the RSK algorithm is extended to work for a large class of well-known, combinatorial diagram algebras including the partition, Brauer, rook monoid, rook-Brauer, Temperley-Lieb, Motzkin, and planar rook monoid algebras. A consequence [9, (5.5)] of this algorithm is that the sum of the degrees of the irreducible representations of each of these algebras equals the number of horizontally symmetric basis diagrams in the algebra. This suggests the existence of a model representation of each of these algebras on the span of its symmetric diagrams, and the main result of this paper is to produce a such a model.

Let A_k denote one of the following unital associative $\mathbb{C}$-algebras: the partition, Brauer, rook monoid, rook-Brauer, Temperley-Lieb, Motzkin, or planar rook monoid algebra. Then A_k has a basis of diagrams and a multiplication given by diagram concatenation. The algebra A_k depends on a parameter $x \in \mathbb{C}$ and is semisimple for all but a finite number of choices of x. When A_k is semisimple, its irreducible modules are indexed by a set Λ_{A_k}, and for $\lambda \in \Lambda_{\mathsf{A}_k}$, we let A_k^λ denote the irreducible A_k-module labeled by λ. We construct, in a uniform way, an A_k-module $\mathsf{M}_{\mathsf{A}_k}$ which decomposes into irreducibles as $\mathsf{M}_{\mathsf{A}_k} \cong \bigoplus_{\lambda \in \Lambda_{\mathsf{A}_k}} \mathsf{A}_k^\lambda$, where the multiplicity of each irreducible module is exactly one.

Our model representation is constructed as follows. For a basis diagram d, we let d^T be its reflection across its horizontal axis and say that a diagram t is symmetric if $t^T = t$. A basis diagram d acts on a symmetric diagram t by "signed conjugation": $d \cdot t = \mathsf{sign}(d,t)\, dtd^T$, where $\mathsf{sign}(d,t)$ is the sign on the permutation of the fixed blocks of t induced by conjugation by d (see Section 4 for details). In each example, our basis diagrams are assigned a rank, which is the number of blocks in the diagram that propagate from the top row to the bottom row. We let $\mathsf{M}^r_{\mathsf{A}_k}$ be the linear span of the symmetric diagrams of rank r and our model is the direct sum $\mathsf{M}_{\mathsf{A}_k} = \oplus_{r=0}^k \mathsf{M}^r_{\mathsf{A}_k}$.

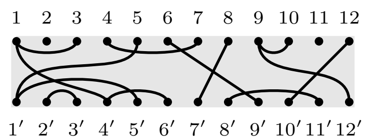
The diagram algebras in this paper naturally form a tower $\mathsf{A}_0 \subseteq \mathsf{A}_1 \subseteq \cdots \subseteq \mathsf{A}_k$, and we are able to use the structure of the Jones basic construction of this tower to derive our model. Each algebra contains a basic construction ideal $\mathsf{J}_{k-1} \subseteq \mathsf{A}_k$ such that $\mathsf{A}_k \cong \mathsf{J}_{k-1} \oplus \mathsf{C}_k$, where $\mathsf{C}_k \cong \mathbb{C}\mathsf{S}_k$ for nonplanar diagram algebras and $\mathsf{C}_k \cong \mathbb{C}\mathbf{1}_k$ for planar diagram algebras. The ideal J_{k-1} is in Schur-Weyl duality with one of A_{k-1} or A_{k-2} (depending on the specific diagram algebra). In this setup, we are able to take a model for each $\mathsf{C}_r, 0 \leq r \leq k$, and lift them to a module for A_k.

For the planar diagram algebras – the Temperley-Lieb, Motzkin, and planar rook monoid algebras – the algebra $\mathsf{C} \cong \mathbb{C}\mathbf{1}_k$ is trivial and the model is trivial. It follows that $\mathsf{M}^r_{\mathsf{A}_k}$ is irreducible and that signed conjugation produces a complete set of irreducible modules for the planar algebras. For the nonplanar diagram algebras, the algebra is $\mathsf{C} \cong \mathbb{C}\mathsf{S}_k$, and we use the Saxl model for S_r. In this case $\mathsf{M}^r_{\mathsf{A}_k}$ is further graded as $\mathsf{M}^r_{\mathsf{A}_k} = \oplus_f \mathsf{M}^{r,f}_{\mathsf{A}_k}$, where $\mathsf{M}^{r,f}_{\mathsf{A}_k}$ is the linear span of symmetric diagrams of rank r having f "fixed blocks" and $\mathsf{M}^{r,f}_{\mathsf{A}_k}$ decomposes into irreducibles labeled by partitions $\lambda \vdash r$ having f odd parts.

Besides being natural constructions, these model representations are useful in several ways. (1) In a model representation, isotypic components are irreducible components, so projection operators map directly onto irreducible modules without being mixed up among multiple isomorphic copies of the same module. (2) A key feature of our model is that we give the explicit action of each basis element of A_k on the basis of $\mathsf{M}_{\mathsf{A}_k}^{r,f}$. For small values of k, and for all values of k in the planar case, these representations are irreducible or have few irreducible components. Thus, in practice, the model provides a natural and easy way to compute the explicit action of basis diagrams on irreducible representations. (3) Gelfand models are useful in the study of Markov chains on related combinatorial objects; see, for example, Chapter 3F of [5] and the references therein, as well as [6], [20].

2 The Partition Algebra and its Diagram Subalgebras

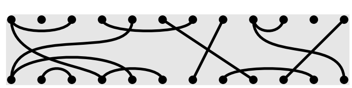
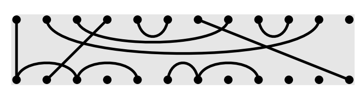
For $k \in \mathbb{Z}_{>0}$, let $\mathcal{P}_k$ denote the set of set partitions of $\{1, 2, \ldots, k, 1', 2', \ldots, k'\}$. We represent a set partition $d \in \mathcal{P}_k$ by a diagram with k vertices in the top row, labeled $1, \ldots, k$, and k vertices in the bottom row, labeled $1', \ldots, k'$. We then assign edges in this diagram so that its connected components equal the underlying set partition d. For example, the following is a diagram $d \in \mathcal{P}_{12}$,



$$= \left\{ \begin{array}{l} \{1,3,4',6'\}, \{2\}, \{4,7\}, \{5,1',5'\}, \{6,9'\}, \{8,7'\}, \\ \{9,10,12'\}, \{11\}, \{12,10'\}, \{2',3'\}, \{8',11'\} \end{array} \right\}.$$

We refer to the parts of a set partition as *blocks*, so that the above diagram has 11 blocks. The diagram of d is not unique, since it only depends on the underlying connected components.

Multiply two set partition diagrams $d_1, d_2 \in \mathcal{P}_k$ as follows. Place d_1 above d_2 and identify each vertex j' in the bottom row of d_1 with the corresponding vertex j in the top row of d_2. Remove any connected components that live entirely in the middle row and let $d_1 \circ d_2 \in \mathcal{P}_k$ be the resulting diagram. For example, if

$d_1 =$  and $d_2 =$ 

then

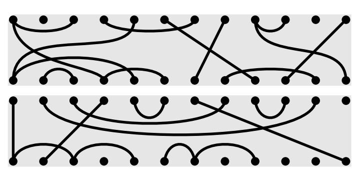
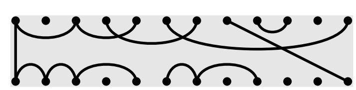
$d_1 \circ d_2 =$  $=$  .

Diagram multiplication is associative and makes $\mathsf{P}_k(x)$ a monoid with identity $\mathbf{1}_k =$  .

Now let $x \in \mathbb{C}$, define $\mathsf{P}_0(x) = \mathbb{C}$, and for $k \geq 1$, let $\mathsf{P}_k(x)$ be the $\mathbb{C}$-vector space with basis $\mathcal{P}_k$. If $d_1, d_2 \in \mathcal{P}_k$, let $\kappa(d_1, d_2)$ denote the number of connected components that are removed from the middle row in computing $d_1 \circ d_2$, and define

$$d_1 d_2 = x^{\kappa(d_1,d_2)}\, d_1 \circ d_2. \tag{1}$$

In the multiplication example of the previous section $\kappa(d_1, d_2) = 1$ and $d_1 d_2 = x(d_1 \circ d_2)$. This product makes $\mathsf{P}_k(x)$ an associative algebra with identity $\mathbf{1}_k$.

We say that a block B in a set partition diagram $d \in \mathcal{P}_k$ is a *propagating* block if B contains vertices from both the top and bottom row of d; that is, both $B \cap \{1, 2, \ldots, k\}$ and $B \cap \{1', 2', \ldots, k'\}$ are nonempty. The *rank* of $d \in \mathcal{P}_k$ (also called the *propagating number*) is

$$\mathsf{rank}(d) = \big(\text{ the number of propagating blocks in } d\ \big). \tag{2}$$

For each $k \in \mathbb{Z}_{>0}$, the following are subalgebras of the partition algebra $\mathsf{P}_k(x)$:

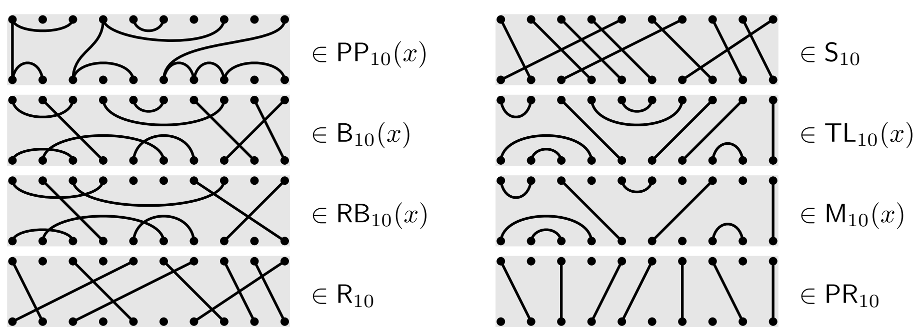
$$\begin{aligned}
\mathbb{C}\mathsf{S}_k &= \mathbb{C}\text{-span}\{\, d \in \mathcal{P}_k \mid \mathsf{rank}(d) = k\},\\
\mathsf{B}_k(x) &= \mathbb{C}\text{-span}\{\, d \in \mathcal{P}_k \mid \text{all blocks of } d \text{ have size } 2\},\\
\mathsf{R}_k &= \mathbb{C}\text{-span}\left\{ d \in \mathcal{P}_k \;\middle|\; \begin{array}{l}\text{all blocks of } d \text{ have at most one vertex in } \{1, \ldots k\}\\ \text{and at most one vertex in } \{1', \ldots k'\}\end{array} \right\},\\
\mathsf{RB}_k(x) &= \mathbb{C}\text{-span}\{\, d \in \mathcal{P}_k \mid \text{all blocks of } d \text{ have size 1 or } 2\}.
\end{aligned}$$

Here, $\mathbb{C}\mathsf{S}_k$ is the group algebra of the symmetric group, $\mathsf{B}_k(x)$ is the Brauer algebra, R_k is the rook monoid algebra [23], and $\mathsf{RB}_k(x)$ is the rook-Brauer algebra [4], [17].

A set partition is *planar* if it can be represented as a diagram without edge crossings inside of the rectangle formed by its vertices. The planar partition algebra [12] is $\mathsf{PP}_k(x) = \mathbb{C}\text{-span}\{\, d \in \mathcal{P}_k \mid d \text{ is planar }\}$. The following are the planar subalgebras of $\mathsf{P}_k(x)$:

$$\begin{aligned}
\mathbb{C}\{\mathbf{1}_k\} &= \mathbb{C}\mathsf{S}_k \cap \mathsf{PP}_k(x), & \mathsf{TL}_k(x) &= \mathsf{B}_k(x) \cap \mathsf{PP}_k(x),\\
\mathsf{PR}_k &= \mathsf{R}_k \cap \mathsf{PP}_k(x), & \mathsf{M}_k(x) &= \mathsf{RB}_k(x) \cap \mathsf{PP}_k(x).
\end{aligned}$$

Here, $\mathsf{TL}_k(x)$ is the Temperley-Lieb algebra, PR_k is the planar rook monoid algebra [7], and $\mathsf{M}_k(x)$ is the Motzkin algebra [2]. The parameter x does not arise when multiplying symmetric group diagrams (as there are never middle blocks to be removed). The parameter is set to be $x = 1$ for the rook monoid algebra and the planar rook monoid algebra. Here are examples from each of these subalgebras:



3 Saxl's Model Representation of the Symmetric Group

An involution $t \in \mathsf{S}_k$ is a permutation such that $t^2 = 1$. In disjoint cycle notation, involutions consist of 2-cycles and fixed points. Let I_k be the set of involutions in S_k and let I_k^f be the involutions in S_k which fix precisely f points. For a fixed involution $t \in \mathsf{I}_k^f$, let $\mathsf{C}(t) \subseteq \mathsf{S}_n$ be the centralizer of t in S_k. If $w \in \mathsf{C}(t)$, then $wtw^{-1} = t$, so w fixes t but possibly permutes the fixed points of t. Let π_f be the linear character of

$\mathsf{C}(t)$ such that $\pi_f(w)$ is the sign of the permutation of w on the fixed points of t. Saxl [22] (see also [13] or [11]) proves the following decomposition of the induced character

$$\varphi^f_{\mathsf{S}_k} := \mathbf{Ind}^{\mathsf{S}_n}_{\mathsf{C}(t)}(\pi_f) = \sum_{\substack{\lambda \vdash k \\ \mathsf{odd}(\lambda)=f}} \chi^\lambda_{\mathsf{S}_k}, \qquad \text{and thus} \qquad \varphi_{\mathsf{S}_k} := \sum_{\ell=0}^{\lfloor k/2 \rfloor} \varphi^{k-2\ell}_{\mathsf{S}_k} = \sum_{\lambda \vdash k} \chi^\lambda_{\mathsf{S}_k}, \tag{3}$$

where $\mathsf{odd}(\lambda)$ is the number of odd parts of the partition λ. This result generalizes the classic result (see [24, Theorem IV]) for fixed-point-free permutations, i.e., the case where $f = 0$. In this case, there are no fixed points and π_0 is the trivial character of $\mathsf{C}(t)$.

We can then explicitly construct the corresponding induced model. If $w \in \mathsf{S}_k$ and $t \in \mathsf{I}_{n,f}$ then $wtw^{-1} \in \mathsf{I}^f_{\mathsf{S}_k}$ is an involution with the same number f of fixed points as t. However, the relative position of the fixed points are permuted in the map $t \mapsto wtw^{-1}$. Define $\mathsf{sign}(w,t)$ to be the sign of the permutation induced on the fixed points of t under conjugation. That is,

$$\mathsf{sign}(w,t) = (-1)^{|\{\, 1 \le i < j \le k \,|\, t(i)=i,\, t(j)=j, \text{ and } w(i)>w(j) \,\}|}. \tag{4}$$

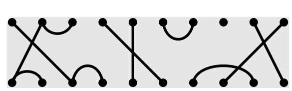
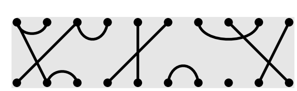
Now, define an action of $w \in \mathsf{S}_k$ on $t \in \mathsf{I}^f_{\mathsf{S}_k}$ by $w \cdot t = \mathsf{sign}(w,t)wtw^{-1}$, which we refer to as *signed conjugation*. Define $\mathsf{M}^f_{\mathsf{S}_k} = \mathbb{C}\text{-span}\{\, t \mid t \in \mathsf{I}^f_{\mathsf{S}_k} \}$, and let S_k act on $\mathsf{M}^f_{\mathsf{S}_k}$ by extending the action linearly. We then prove that $\mathsf{M}^f_{\mathsf{S}_k} \cong \mathbf{Ind}^{\mathsf{S}_k}_{\mathsf{C}(t)}(\mathsf{M}_t)$, and it follows from (3) that

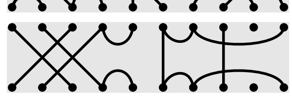
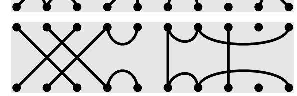
$$\mathsf{M}_{\mathsf{S}_k} = \bigoplus_f \mathsf{M}^f_{\mathsf{S}_k} \cong \bigoplus_{\lambda \vdash n} \mathsf{S}^\lambda_k. \tag{5}$$

Adin, Postnikov, and Roichman [1] study a slightly different combinatorial model for S_k. In this work, the sign is computed as $\overline{\mathsf{sign}}(w,t) = (-1)^{|\{\, 1 \le i < j \le k \,|\, t(i)=j,\, t(j)=i, \text{ and } w(i)>w(j) \,\}|}$. If we let $\overline{\mathsf{M}}^f_k$ denote the corresponding S_k module, then we are able to prove that $\mathsf{M}^f_{\mathsf{S}_k} \cong \overline{\mathsf{M}}^f_{\mathsf{S}_k} \otimes \mathsf{S}^{(1^k)}_k$, where $\mathsf{S}^{(1^k)}_k$ is the sign representation of S_k.

4 Gelfand Models for Diagram Algebras

Let A_k be any one of the diagrams described in Section 2 with the parameter $x \in \mathbb{C}$ chosen such that A_k is semisimple. Let $\mathcal{A}_k$ be the basis of diagrams which span A_k. For $d \in \mathcal{A}_k$, let $d^T \in \mathcal{A}_k$ be the diagram obtained by reflecting d over its horizontal axis. Note that the map $d \to d^T$ corresponds to exchanging $i \leftrightarrow i'$ for all i. For example,

$d_1 =$  $\Rightarrow$ $d_1^T =$  ,

$d_2 =$  $\Rightarrow$ $d_2^T =$  .

We say that a diagram d is *symmetric* if $d^T = d$, so that d_2 is symmetric and d_1 is not. If we let $(i')' = i$ and let $B' = \{\, b' \,|\, b \in B \,\}$ for a block B of a partition diagram d, then d is symmetric if it satisfies: $B \in d$ if and only if $B' \in d$. If d is a partition diagram, then we say that a block $B \in d$ is a *fixed block* if $B' = B$. In our above examples, d_1 has one fixed block, $\{5, 5'\}$, and d_2 has two fixed blocks, $\{8, 8'\}$ and $\{6, 7, 10, 6', 7', 10'\}$. Note that for $a, b \in \mathcal{A}_k$, $(ab)^T = b^T a^T$, and observe that $(dtd^T)^T = (d^T)^T t^T d^T = dtd^T$, so t is symmetric if and only if dtd^T is symmetric. We say that dtd^T is the *conjugate* of t by d.

Remark 6 *The symmetric diagrams in this paper are the same as the type-B set partitions in [18] Sequence A002872 and they are closely related to the type-B set partitions used in [21].*

Remark 7 *If we restrict our diagrams to S_k, then d^T equals d^{-1}, diagram conjugation corresponds to usual group conjugation, symmetric diagrams are involutions, and fixed blocks are fixed points.*

For any of our diagram algebras A_k, we let

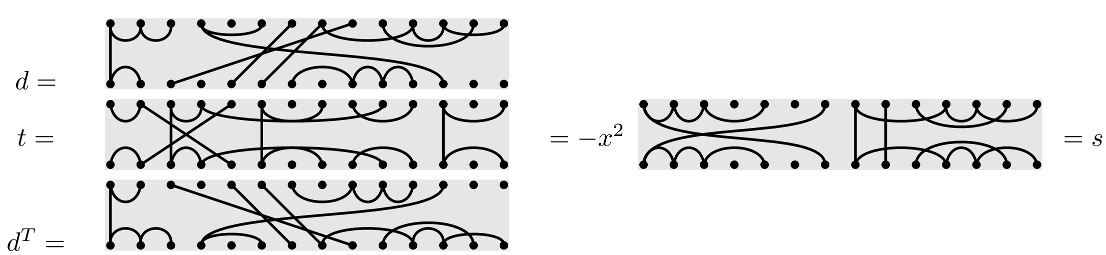
$$\begin{aligned} \mathsf{I}^{r,f}_{\mathsf{A}_k} &= \{\, d \in \mathcal{A}_k \mid d \text{ is symmetric}, \mathsf{rank}(d) = r, \text{ and } d \text{ has } f \text{ fixed blocks} \,\}, \\ \mathsf{I}^{r}_{\mathsf{A}_k} &= \{\, d \in \mathcal{A}_k \mid d \text{ is symmetric}, \mathsf{rank}(d) = r \,\}, \\ \mathsf{I}_{\mathsf{A}_k} &= \{\, d \in \mathcal{A}_k \mid d \text{ is symmetric} \,\}, \end{aligned} \tag{8}$$

If $d \in \mathcal{A}_k$ and $t \in \mathsf{I}^{r,f}_{\mathsf{A}_k}$, then there are two possibilities for the map $t \mapsto d \circ t \circ d^T$. Either $\mathsf{rank}(d \circ t \circ d^T) < \mathsf{rank}(t)$ or $\mathsf{rank}(d \circ t \circ d^T) = \mathsf{rank}(t)$. In the later case, the fixed blocks of t have been permuted, and we let $\mathsf{sign}(d,t)$ be the sign of the permutation of the fixed blocks of t. and for $d \in \mathcal{A}_k$ and $t \in \mathsf{I}^{r,f}_{\mathsf{A}_k}$, we define

$$d \cdot t = \begin{cases} x^{\kappa(d,t)} \mathsf{sign}(d,t)\, d \circ t \circ d^T, & \text{if } \mathsf{rank}(d \circ t \circ d^T) = \mathsf{rank}(t) \\ 0, & \text{if } \mathsf{rank}(d \circ t \circ d^T) < \mathsf{rank}(t) \end{cases} \tag{9}$$

where $\kappa(d,t)$ is the number of blocks removed from the middle row in creating $d \circ t$ as described in (1).

Example 10 (Signed Conjugation) *In the following example, there are two blocks removed in $d \circ t$ yielding x^2. Furthermore, the three fixed blocks of t are permuted as $(B_1, B_2, B_3) \mapsto (B_3, B_2, B_1)$. Hence,* $\mathsf{sign}(d,t) = -1$.



For $0 \le f \le r \le k$, define $\mathsf{M}^{r,f}_{\mathsf{A}_k} = \mathbb{C}\text{-span}\{\, d \mid d \in \mathsf{I}^{r,f}_{\mathsf{A}_k} \,\}$, where $\mathsf{M}^{r,f}_{\mathsf{A}_k} = 0$ if $\mathsf{I}^{r,f}_{\mathsf{A}_k} = \emptyset$, and let

$$\begin{aligned} \mathsf{M}^{r}_{\mathsf{A}_k} &= \mathbb{C}\text{-span}\{\, d \mid d \in \mathsf{I}^{r}_{\mathsf{A}_k} \,\}, \\ &= \bigoplus_{f=0}^{r} \mathsf{M}^{r,f}_{\mathsf{A}_k}, \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathsf{M}_{\mathsf{A}_k} &= \mathbb{C}\text{-span}\{\, d \mid d \in \mathsf{I}_{\mathsf{A}_k} \,\}, \\ &= \bigoplus_{r=0}^{k} \mathsf{M}^{r}_{\mathsf{A}_k} = \bigoplus_{r=0}^{k} \bigoplus_{f=0}^{r} \mathsf{M}^{r,f}_{\mathsf{A}_k}. \end{aligned} \tag{11}$$

Then we prove the following:

Proposition 12 *The action defined in* (9) *makes* $\mathsf{M}^{r,f}_{\mathsf{A}_k}$ *an* A_k*-module.*

The main theorem of this paper is the following.

Theorem 13 *For each* $0 \le f \le r \le k$ *chosen such that* $\mathsf{M}^{r,f}_{\mathsf{A}_k} \neq 0$*, we have*

$$\mathsf{M}^{r,f}_{\mathsf{A}_k} \cong \bigoplus_{\lambda \in \Lambda^f_{\mathsf{C}_r}} \mathsf{M}^{\lambda}_{\mathsf{A}_k} \qquad \text{and thus} \qquad \mathsf{M}_{\mathsf{A}_k} \cong \bigoplus_{\lambda \in \Lambda_{\mathsf{A}_k}} \mathsf{M}^{\lambda}_{\mathsf{A}_k}.$$

Our method of proof of this theorem is to use the Jones basic construction. We have a natural tower of algebras, $\mathsf{A}_0 \subseteq \mathsf{A}_1 \subseteq \mathsf{A}_2 \subseteq \cdots$. where A_{k-1} is embedded as subalgebra of A_k by placing an identity edge to the right of any diagram in A_{k-1}. Let $\mathsf{J}_{k-1} \subseteq \mathsf{A}_k$ be the ideal spanned by the diagrams of A_k having rank $k-1$ *or less*. Then,

$$\mathsf{A}_k \cong \mathsf{J}_{k-1} \oplus \mathsf{C}_k, \tag{14}$$

where C_k is the span of the diagrams of rank exactly equal to k. For us,

$$\begin{array}{ll} \mathsf{C}_k \cong \mathbb{C}\mathsf{S}_k & \text{when } \mathsf{A}_k \text{ is one of the nonplanar algebras } \mathsf{P}_k(x), \mathsf{B}_k(x), \mathsf{RB}_k(x) \text{ or } \mathsf{R}_k, \\ \mathsf{C}_k \cong \mathbb{C}\mathbf{1}_k & \text{when } \mathsf{A}_k \text{ is one of the planar algebras } \mathsf{TL}_k(x), \mathsf{M}_k(x), \text{ or } \mathsf{PR}_k, \end{array} \tag{15}$$

We then are able to lift model representations from $\mathsf{C}_r, 0 \le r \le k$, to a model for A_k.

5 Gelfand Models for Diagram Algebras

We now illustrate some of the combinatorial details that come from applying our model construction to the various diagram algebras.

5.1 The partition algebra $\mathsf{P}_k(x)$

The partition algebra $\mathsf{P}_k(x)$ has dimension equal to the Bell number $B(2k)$ and is semisimple for $x \in \mathbb{C}$ such that $x \notin \{0, 1, \ldots, 2k-1\}$ (see [16] or [10]). When semisimple, its irreducible representations are indexed by partitions in the set $\Lambda_{\mathsf{P}_k} = \{\ \lambda \vdash r \mid 0 \le r \le k\ \}$. Let P^{λ}_k denote the irreducible module indexed by $\lambda \in \Lambda_{\mathsf{P}_k}$.

For each $0 \le \ell \le \lfloor r/2 \rfloor$ there exist symmetric diagrams in $\mathsf{I}^{r,f}_{\mathsf{P}_k}$ of rank r with $f = r - 2\ell$ fixed blocks and ℓ blocks which are transposed (i.e., propagating, nonidentity blocks). The model representation satisfies

$$\mathsf{M}^{r,f}_{\mathsf{P}_k} = \sum_{\substack{\lambda \vdash k \\ \mathsf{odd}(\lambda) = f}} \mathsf{P}_k{}^{\lambda} \qquad \text{and} \qquad \mathsf{M}_{\mathsf{P}_k} = \sum_{r=0}^{k} \sum_{\ell=0}^{\lfloor r/2 \rfloor} \mathsf{M}^{r,r-2\ell}_{\mathsf{P}_k} = \sum_{\lambda \in \Lambda_{\mathsf{P}_k}} \mathsf{P}_k{}^{\lambda}. \tag{16}$$

We show that

$$\dim \mathsf{M}^{r,r-2\ell}_{\mathsf{P}_k} = \left|\mathsf{I}^{r,r-2\ell}_{\mathsf{P}_k}\right| = \sum_{b=r}^{k} S(k,b) \binom{b}{r} \binom{r}{2\ell} (2\ell-1)!! \tag{17}$$

where $S(k,b)$ is a Stirling number of the second kind. If we let $\mathsf{p}_k = |\mathsf{I}_{\mathsf{P}_k}| = \sum_{r=0}^{k} \sum_{\ell=0}^{\lfloor r/2 \rfloor} |\mathsf{I}^{r,r-2\ell}_{\mathsf{P}_k}| = \dim \mathsf{M}_{\mathsf{P}_k}$ denote the total number of symmetric diagrams in $\mathsf{P}_k(x)$, then p_k is the sum of the degrees of the irreducible $\mathsf{P}_k(x)$-modules (which can be found in [15], [10]). The first few values are

k	0	1	2	3	4	5	6	7	8	9	10
$\mathsf{p}_k = \dim \mathsf{M}_{\mathsf{P}_k}$	1	2	7	31	164	999	6841	51790	428131	3827967	36738144

(18)

The sequence p_k is [18] Sequence A002872, which equals the number of type-B set partitions (see Remark 6) and has exponential generating function $e^{(e^{2x}-3)/2+e^x} = \sum_{k=0}^{\infty} \mathsf{p}_k \frac{x^k}{k!}$.

5.2 The Brauer algebra $\mathsf{B}_k(x)$

The Brauer algebra has dimension $\dim \mathsf{B}_k(x) = (2k-1)!!$ and is semisimple for $x \in \mathbb{C}$ chosen to avoid $\{ x \in \mathbb{Z} \,|\, 4-2k \le x \le k-2\}$. When $\mathsf{B}_k(x)$ is semisimple, its irreducible modules are indexed by partitions in the set $\Lambda_{\mathsf{B}_k} = \{ \lambda \vdash (k-2r) \mid 0 \le r \le \lfloor k/2 \rfloor \}$. Let B_k^λ denote the irreducible $\mathsf{B}_k(x)$ module for $\lambda \in \Lambda_{\mathsf{B}_k}$.

For each $0 \le c \le \lfloor k/2 \rfloor$ and each $0 \le \ell \le \lfloor (k-2c)/2 \rfloor$ there exist symmetric diagrams in $\mathsf{I}_{\mathsf{B}_k}^{k-2c,k-2c-2\ell}$ of rank $r = k-2c$ with $f = k-2c-2\ell$ fixed blocks. The $\mathsf{B}_k(x)$ model satisfies

$$\mathsf{M}_{\mathsf{B}_k}^{r,f} \cong \bigoplus_{\substack{\lambda \vdash r \\ \mathrm{odd}(\lambda)=f}} \mathsf{B}_k^\lambda \qquad \text{and} \qquad \mathsf{M}_{\mathsf{B}_k} \cong \bigoplus_{c=0}^{\lfloor k/2 \rfloor} \bigoplus_{\ell=0}^{\lfloor (k-2c)/2 \rfloor} \mathsf{M}_{\mathsf{B}_k}^{k-2c,k-2c-2\ell} \cong \bigoplus_{\lambda \in \Lambda_{\mathsf{B}_k}} \mathsf{B}_k^\lambda. \tag{19}$$

We show that

$$\dim \mathsf{M}_{\mathsf{B}_k}^{r,r-2\ell} = \left| \mathsf{I}_{\mathsf{B}_k}^{r,r-2\ell} \right| = \binom{k}{r} (k-r-1)!! \binom{r}{2\ell} (2\ell-1)!!. \tag{20}$$

If we let $\mathsf{b}_k = |\mathsf{I}_{\mathsf{B}_k}| = \sum_{c=0}^{\lfloor k/2 \rfloor} \sum_{\ell=0}^{\lfloor (k-2c)/2 \rfloor} |\mathsf{I}_{\mathsf{B}_k}^{k-2c,k-2c-2\ell}| = \dim \mathsf{M}_{\mathsf{B}_k}$ denote the total number of symmetric diagrams in $\mathsf{B}_k(x)$, then b_k is the sum of the degrees of the irreducible $\mathsf{B}_k(x)$-modules (which can be found in [19]). The first few values of these dimensions are

k	0	1	2	3	4	5	6	7	8	9	10
$\mathsf{b}_k = \dim \mathsf{M}_{\mathsf{B}_k}$	1	1	3	7	25	81	331	1303	5937	26785	133651

(21)

The sequence b_k is [18] Sequence A047974 and has exponential generating function $e^{x^2+x} = \sum_{k=0}^{\infty} \mathsf{b}_k \frac{x^k}{k!}$.

5.3 The rook monoid algebra R_k

The rook monoid algebra R_k has dimension $\dim \mathsf{R}_k = \sum_{\ell=0}^{k} \binom{k}{\ell}^2 \ell!$ (see [15], [8], [14]) and is semisimple with irreducible modules labeled by $\Lambda_{\mathsf{R}_k} = \{ \lambda \vdash r \mid 0 \le r \le \lfloor k \rfloor \}$. Let R_k^λ denote the irreducible module labeled by $\lambda \in \Lambda_{\mathsf{R}_k}$.

For each $0 \le r \le k$ and each $0 \le \ell \le \lfloor r/2 \rfloor$ there exist symmetric rook monoid diagrams of rank r and $f = r-2\ell$ fixed blocks. The R_k model satisfies

$$\mathsf{M}_{\mathsf{R}_k}^{r,f} \cong \bigoplus_{\substack{\lambda \vdash r \\ \mathrm{odd}(\lambda)=f}} \mathsf{R}_k^\lambda \qquad \text{and} \qquad \mathsf{M}_{\mathsf{R}_k} \cong \bigoplus_{r=0}^{k} \bigoplus_{\ell=0}^{\lfloor r/2 \rfloor} \mathsf{M}_{\mathsf{R}_k}^{r,r-2\ell} \cong \bigoplus_{\lambda \in \Lambda_{\mathsf{R}_k}} \mathsf{R}_k^\lambda. \tag{22}$$

We show that

$$\dim \mathsf{M}_{\mathsf{R}_k}^{r,r-2\ell} = \left| \mathsf{I}_{\mathsf{R}_k}^{r,r-2\ell} \right| = \binom{k}{r} \binom{r}{2\ell} (2\ell-1)!!, \tag{23}$$

If we let $\mathsf{r}_k = |\mathsf{I}_{\mathsf{R}_k}| = \sum_{r=0}^{k}\sum_{\ell=0}^{\lfloor r/2\rfloor} |\mathsf{I}_{\mathsf{R}_k}^{r,r-2\ell}| = \dim \mathsf{M}_{\mathsf{R}_k}$ denote the total number of symmetric diagrams in R_k, then r_k is sum of the degrees of the irreducible R_k-modules (which can be found in [15], [8]). The first few values of these dimensions are

k	0	1	2	3	4	5	6	7	8	9	10
$\dim \mathsf{M}_{\mathsf{R}_k}$	1	2	5	14	43	142	499	1850	7193	29186	123109

(24)

The sequence r_k gives the number of "self-inverse partial permutations" and is [18] Sequence A005425. Furthermore, r_k is related to the number of involutions s_k in the symmetric group by the binomial transform $\mathsf{r}_k = \sum_{i=0}^{k}\binom{k}{i}\mathsf{s}_i$ and thus has exponential generating function $e^{x^2/2+2x} = \sum_{k=0}^{\infty} \mathsf{r}_k \frac{x^k}{k!}$.

5.4 The rook-Brauer algebra $\mathsf{RB}_k(x)$

The rook-Brauer algebra $\mathsf{RB}_k(x)$ (see [4] or [17]) has dimension $\sum_{\ell=0}^{k}\binom{2k}{2\ell}(2\ell-1)!!$ and is semisimple for all but finitely many $x \in \mathbb{C}$. When semisimple, its irreducible representations are indexed by partitions in the set $\Lambda_{\mathsf{RB}_k} = \{\, \lambda \vdash r \mid 0 \le r \le \lfloor k \rfloor \,\}$. Let RB_k^λ denote the irreducible module indexed by $\lambda \in \Lambda_{\mathsf{RB}_k}$.

For each $0 \le r \le k$ and each $0 \le \ell \le \lfloor r/2 \rfloor$ there exist symmetric rook monoid diagrams of rank r and $f = r - 2\ell$ fixed blocks. The $\mathsf{RB}_k(x)$ models satisfy

$$\mathsf{M}_{\mathsf{RB}_k}^{r,f} \cong \bigoplus_{\substack{\lambda\vdash r\\ \mathsf{odd}(\lambda)=f}} \mathsf{RB}_k^\lambda \qquad \text{and} \qquad \mathsf{M}_{\mathsf{RB}_k} \cong \bigoplus_{r=0}^{k}\bigoplus_{\ell=0}^{\lfloor r/2\rfloor} \mathsf{M}_{\mathsf{RB}_k}^{r,r-2\ell} \cong \bigoplus_{\lambda\in\Lambda_{\mathsf{RB}_k}} \mathsf{RB}_k^\lambda. \tag{25}$$

We show that

$$\dim \mathsf{M}_{\mathsf{RB}_k}^{r,r-2\ell} = \left|\mathsf{I}_{\mathsf{RB}_k}^{r,r-2\ell}\right| = \sum_{c=0}^{\lfloor (k-r)/2\rfloor} \binom{k}{r}\binom{k-r}{2c}(2c-1)!!\binom{r}{2\ell}(2\ell-1)!!. \tag{26}$$

If we let $\mathsf{rb}_k = |\mathsf{I}_{\mathsf{RB}_k}| = \sum_{r=0}^{k}\sum_{\ell=0}^{\lfloor r/2\rfloor} |\mathsf{I}_{\mathsf{RB}_k}^{r,r-2\ell}| = \dim \mathsf{M}_{\mathsf{RB}_k}$ denote the total number of symmetric diagrams in $\mathsf{RB}_k(x)$, then rb_k is the sum of the degrees of the irreducible $\mathsf{RB}_k(x)$-modules (these dimensions can be found in [4] or [17]). The first few values of these dimensions are

k	0	1	2	3	4	5	6	7	8	9	10
$\mathsf{rb}_k = \dim \mathsf{M}_{\mathsf{RB}_k}$	1	2	6	20	76	312	1384	6512	32400	168992	921184

(27)

The sequence rb_k is [18] Sequence A000898 and it is related to the number of symmetric diagrams b_k in the Brauer algebra (21) by the binomial transform $\mathsf{rb}_k = \sum_{i=0}^{k}\binom{k}{i}\mathsf{b}_i$ and thus has exponential generating function $e^{x^2+2x} = \sum_{k=0}^{\infty}\mathsf{rb}_k\frac{x^k}{k!}$.

5.5 The Temperley-Lieb algebra $\mathsf{TL}_k(x)$

The Temperley-Lieb algebra $\mathsf{TL}_k(x)$ has dimension equal to the Catalan number $C_k = \frac{1}{k+1}\binom{2k}{k}$ and is semisimple for $x \in \mathbb{C}$ chosen such that x is not the root of the Chebyshev polynomial $U_k(x/2)$ (see [25] or [2]). When semisimple, its irreducible modules are indexed by the following set of integers $\Lambda_{\mathsf{TL}_k} = \{\, k-2\ell \mid 0 \le \ell \le \lfloor k/2 \rfloor \,\}$. We let $\mathsf{TL}_k^{(k-2\ell)}$ denote the irreducible module labeled by $(k-2\ell) \in \Lambda_{\mathsf{TL}_k}$

For each $0 \leq \ell \leq \lfloor k/2 \rfloor$, there exist symmetric Temperley-Lieb diagrams of rank $r = k - 2\ell$ and $f = k - 2\ell$ fixed points. The $\mathsf{TL}_k(x)$ model satisfies

$$\mathsf{M}_{\mathsf{TL}_k}^{(k-2\ell)} \cong \mathsf{TL}_k^{(k-2\ell)} \qquad \text{and} \qquad \mathsf{M}_{\mathsf{TL}_k} \cong \bigoplus_{\ell=0}^{\lfloor k/2 \rfloor} \mathsf{M}_{\mathsf{TL}_k}^{(k-2\ell)} \cong \bigoplus_{(k-2\ell) \in \Lambda_{\mathsf{TL}_k}} \mathsf{TL}_k^{(k-2\ell)}. \tag{28}$$

The number of symmetric Temperley-Lieb diagrams of rank r with $r = f$ fixed points is given by

$$\dim \mathsf{M}_{\mathsf{TL}_k}^{r,f} = \left|\mathsf{I}_{\mathsf{TL}_k}^{k-2\ell}\right| = \left\{ \begin{array}{c} k \\ \ell \end{array} \right\} := \binom{k}{\ell} - \binom{k}{\ell - 1} \tag{29}$$

If we let $\mathsf{tl}_k = |\mathsf{I}_{\mathsf{TL}_k}| = \sum_{\ell=0}^{\lfloor k/2 \rfloor} |\mathsf{I}_{\mathsf{TL}_k}^{k-2\ell}| = \dim \mathsf{M}_{\mathsf{TL}_k}$ denote the total number of symmetric diagrams in $\mathsf{TL}_k(x)$, then tl_k is the sum of the degrees of the irreducible $\mathsf{TL}_k(x)$-modules. We give a bijection between the symmetric Temperley-Lieb diagrams $\mathsf{I}_{\mathsf{TL}_k}$ and subsets of $\{1, 2, \ldots, k\}$ of size $\lfloor k/2 \rfloor$ and thus $\mathsf{tl}_k = \binom{k}{\lfloor k/2 \rfloor}$ (the kth central binomial coefficient), which is [18] Sequence A000984.

5.6 *The Motzkin algebra* $\mathsf{M}_k(x)$

The Motzkin algebra $\mathsf{M}_k(x)$ has dimension equal to the Motzkin number M_{2k} (see [2]) and is semisimple for $x \in \mathbb{C}$ chosen such that x is not the root of the Chebyshev polynomial $U_k((x-1)/2)$. When semisimple, its the irreducible modules are indexed by $\Lambda_{\mathsf{M}_k} = \{0, 1, \ldots, k\}$. We let $\mathsf{M}_k^{(r)}$ denote the irreducible module labeled by $r \in \Lambda_{\mathsf{M}_k}$.

For each $0 \leq r \leq k$ there exist symmetric Motzkin diagrams having rank r and $f = r$ fixed blocks. The $\mathsf{M}_k(x)$ models satisfy

$$\mathsf{M}_{\mathsf{M}_k}^r \cong \mathsf{M}_k^{(r)} \qquad \text{and} \qquad \mathsf{M}_{\mathsf{M}_k} \cong \bigoplus_{r=0}^{k} \mathsf{M}_{\mathsf{M}_k}^r \cong \bigoplus_{r \in \Lambda_{\mathsf{M}_k}} \mathsf{M}_k^{(r)}. \tag{30}$$

We show that

$$\dim \mathsf{M}_{\mathsf{M}_k}^r = \left|\mathsf{I}_{\mathsf{M}_k}^r\right| = \sum_{c=0}^{\lfloor (k-r)/2 \rfloor} \binom{k}{r+2c} \left\{ \begin{array}{c} r+2c \\ c \end{array} \right\}. \tag{31}$$

If we let $\mathsf{m}_k = |\mathsf{I}_{\mathsf{M}_k}| = \sum_{r=0}^{k} |\mathsf{I}_{\mathsf{M}_k}^r| = \dim \mathsf{M}_{\mathsf{M}_k}$ denote the total number of symmetric diagrams in $\mathsf{M}_k(x)$, then m_k is the degree of φ_{M_k} and is the sum of the degrees of the irreducible $\mathsf{M}_k(x)$-modules. The first few values of these dimensions are

k	0	1	2	3	4	5	6	7	8	9	10
$\mathsf{m}_k = \dim \mathsf{M}_{\mathsf{M}_k}$	1	2	5	13	35	96	267	750	2123	6046	17303

(32)

The sequence m_k is [18] Sequence A005773 and it is related to the number of symmetric diagrams tl_k in the Temperley-Lieb algebra by the binomial transform $\mathsf{m}_k = \sum_{i=0}^{k} \binom{k}{i} \mathsf{tl}_i$ and thus has exponential generating function $e^x(I_0(2x) + I_1(2x)) = \sum_{k=0}^{\infty} \mathsf{m}_k \frac{x^k}{k!}$.

5.7 The planar rook monoid algebra PR_k

The planar rook monoid algebra PR_k has dimension $\binom{2k}{k}$ and is semisimple with irreducible modules labeled by $\Lambda_{\mathsf{PR}_k} = \{0, 1, \ldots, k\}$. We let $\mathsf{PR}_k^{(r)}$ denote the irreducible PR_k-module labeled by $r \in \Lambda_{\mathsf{PR}_k}$

For each $0 \leq r \leq k$ there exist $\binom{k}{r}$ symmetric planar rook monoid diagrams having rank r and $f = r$ fixed blocks. The PR_k model satisfies

$$\mathsf{M}^r_{\mathsf{PR}_k} \cong \mathsf{PR}_k^{(r)} \qquad \text{and} \qquad \mathsf{M}_{\mathsf{PR}_k} \cong \bigoplus_{r=0}^{k} \mathsf{M}^r_{\mathsf{PR}_k} \cong \bigoplus_{r \in \Lambda_{\mathsf{PR}_k}} \mathsf{PR}_k^{(r)}. \tag{33}$$

The irreducible modules $\mathsf{PR}_k^{(r)}$ are constructed in [7] on a basis of r-subsets of $\{1, 2, \ldots, k\}$. The action of PR_k on subsets is exactly the same as our conjugation action on symmetric diagrams. If we let $\mathsf{pr}_k = |\mathsf{I}_{PR_k}| = \sum_{r=0}^{k} |\mathsf{I}^r_{PR_k}| = \dim \mathsf{M}_{\mathsf{PR}_k}$ denote the total number of symmetric diagrams in PR_k, then pr_k is the number of subsets of $\{1, 2, \ldots, k\}$, so $\mathsf{pr}_k = \dim \mathsf{M}_{\mathsf{PR}_k} = 2^k$.

Acknowledgements

We thank Arun Ram for suggesting that we look for a model representation of these diagram algebras after seeing the dimension result in [9]. We also thank Michael Reeks and Michael Decker, whose Macalester College honors projects examined the model characters of symmetric groups and partition algebras.

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Moving robots efficiently using the combinatorics of CAT(0) cubical complexes

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Abstract. Given a reconfigurable system X, such as a robot moving on a grid or a set of particles traversing a graph without colliding, the possible positions of X naturally form a cubical complex $\mathcal{S}(X)$. When $\mathcal{S}(X)$ is a CAT(0) space, we can explicitly construct the shortest path between any two points, for any of the four most natural metrics: distance, time, number of moves, and number of steps of simultaneous moves.

CAT(0) cubical complexes are in correspondence with posets with inconsistent pairs (PIPs), so we can prove that a state complex $\mathcal{S}(X)$ is CAT(0) by identifying the corresponding PIP. We illustrate this very general strategy with one known and one new example: Abrams and Ghrist's "positive robotic arm" on a square grid, and the robotic arm in a strip. We then use the PIP as a combinatorial "remote control" to move these robots efficiently from one position to another.

Résumé. Etant donné un système X, qui est reconfigurable, par example un robot se déplaçant sur une grille ou bien un ensemble de particules qui traverse un graphe sans collision, toutes les positions possibles de X forment de façon naturel un c complexe cubique $\mathcal{S}(X)$. Dans le cas ou $\mathcal{S}(X)$ est un espace CAT(0), nous pouvons explicitement construire le chemin le plus court entre deux points quelconques, pour une des quatre mesures les plus naturels: la distance euclidienne, le temps, le nombre de coups, et le nombre d'étapes de mouvements simultanés.

CAT (0) complexes cubiques sont en correspondance avec les ensembles partiellement ordonnés posets des paires incompatibles (PPI), et donc nous pouvons demontrer qu'un état complexe $\mathcal{S}(X)$ est CAT (0), en identifiant le PPI correspondant. Nous illustrons cette stratégie très générale avec un example bien connu et un exemple nouveau: L'example de Abrams et Ghrist du "bras robotique positif" sur une grille carrée, et le bras robotique dans une bande. Ensuite nous utilisons le PPI come une "télécommande" combinatorique pour efficacement déplacer ces robots d'une position à une autre.

Keywords: cubical complexes, combinatorial optimization, posets, reconfigurable systems, state complexes

1 Introduction

There are numerous contexts in mathematics, robotics, and other fields where a discrete system changes according to local, reversible moves. For example, one might consider a robotic arm moving around a

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grid, a number of particles moving around a graph, or a phylogenetic tree undergoing local mutations. Abrams, Ghrist, and Peterson [1, 8] introduced the formalism of *reconfigurable systems* to model a very wide variety of such contexts.

Perhaps the most natural and important question that arises is the *motion-planning* or *shape-planning* question: how does one efficiently get a reconfigurable system X from one position to another one? Abrams, Ghrist, and Peterson observed that the *transition graph* $G(X)$ is the 1-skeleton of the *state complex* $\mathcal{S}(X)$: a cubical complex whose vertices are the states of X, whose edges correspond to allowable moves, and whose cubes correspond to collections of moves which can be performed simultaneously. In fact, $\mathcal{S}(X)$ can be regarded as the space of all possible positions of X, including the positions in between states.

The geometry and topology of the state complex $\mathcal{S}(X)$ can help us solve the motion-planning problem for the system X. More concretely, $\mathcal{S}(X)$ is locally non-positively curved for *any* configuration system. [1, 8] Furthermore, the state complex of *some* reconfigurable systems is globally non-positively curved, or *CAT(0)*. This stronger property implies that for any two points p and q there is a unique shortest path between them. Ardila, Owen, and Sullivant [3] gave an explicit algorithm to find this path.

It is therefore extremely useful to find out when a state complex $\mathcal{S}(X)$ is CAT(0). The first groundbreaking result in this direction is due to Gromov [9], who gave a topological-combinatorial criterion for this geometric property. Roller [13] and Sageev [14], and Ardila, Owen, and Sullivant [3] then gave two completely combinatorial descriptions of CAT(0) cubical complexes. The second descripion is a bijection between rooted CAT(0) cube complexes and *posets with inconsistent pairs* (PIPs).

In this paper, we put into practice the paradigm introduced in [3] to prove that a given cubical complex X is CAT(0). The idea is simple: we identify a PIP whose corresponding (rooted) CAT(0) cubical complex is X. In principle, this method is completely general, though its implementation in a particular situation is not trivial. We illustrate this with one known and one new example of robotic arms. We close by showing how to find the shortest path between states in a CAT(0) state complex $\mathcal{S}(X)$ under four natural metrics.

2 Preliminaries

2.1 Reconfigurable systems and cubical complexes

We now sketch the basic definitions for reconfigurable systems due to Abrams, Ghrist, and Peterson and illustrate them with an example. We refer the reader to [1] and [8] for the details. Let $\mathcal{G} = (V, E)$ be a graph and $\mathcal{A}$ be a set of labels. A *state* u is a labeling of the vertices of $\mathcal{G}$ by elements of $\mathcal{A}$. Roughly speaking, a *reconfigurable system* is given by a collection of states, together with a given set of local moves called *generators* that one can perform to get from one state to another. Given a state s and a set of moves M which can be applied to s, we say that the moves in M *commute* if they can be applied simultaneously to s; that is, they are "physically independent". In this paper we will study two robotic arms moving inside a grid. Here G will represent the grid, and a labelling of G with 0s and 1s will indicate the position of the robot.

Example 2.1 (Metamorphic robots in a hexagonal lattice [7, 8]) *Consider a robot made up of identical hexagonal unit cells in the hexagonal lattice, which has the ability to pivot cells on the boundary whenever they are unobstructed. Figure 1a. shows one move, and b.-e. shows two commutative moves.*

A *cubical complex* X is a polyhedral complex obtained by gluing cubes of various dimensions, in such a way that the intersection of any two cubes is a face of both. Such a space X has a natural piecewise

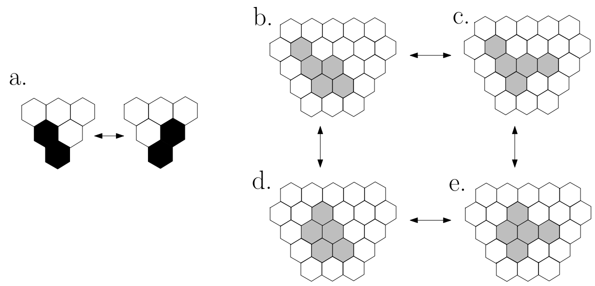

Fig. 1: a. A generator for a metamorphic robot in the hexagonal lattice. b-e. Four possible states

Euclidean metric. Any reconfigurable system gives rise to a cubical complex:

Definition 2.2 *The* state complex $\mathcal{S}(\mathcal{R})$ *of a reconfigurable system $\mathcal{R}$ is a cubical complex whose vertices correspond to the states of $\mathcal{R}$. We draw an edge between two states if they differ by an application of a single move. The k-cubes correspond to k-tuples of commutative moves.*

Figure 2 shows the state complex of a robot of 5 cells which moves following the rules of Fig. 1, and is constrained to stay inside a tunnel of width 3.

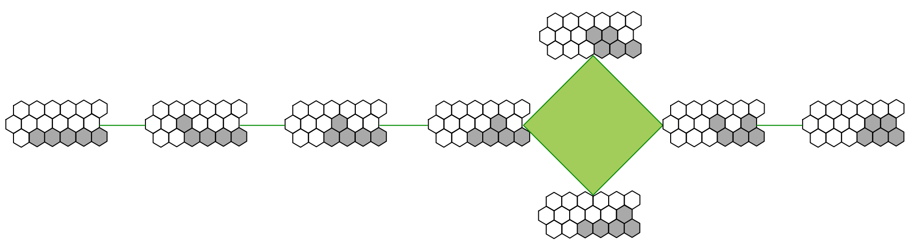

Fig. 2: The state complex of a hexagonal metamorphic robot in a tunnel.

Given a reconfigurable system $\mathcal{R}$ and a state u, there is a natural partial order on the states of $\mathcal{R}$:

Definition 2.3 *Let $\mathcal{R}$ be a reconfigurable system and let u be any "home" state. Define the* poset of states *$\mathcal{R}_u$ to be the set of states ordered by declaring that $p \leq q$ if there is a shortest edge-path from the home state u to q going through p.*

2.2 Combinatorial geometry of CAT(0) cubical complexes

We now define CAT(0) spaces, the spaces of global non-positive curvature that we are interested in. For more information, see [5, 6]. Let X be a geodesic metric space– that is, a metric space where any two points x and y are the endpoints of a curve of length $d(x, y)$. Consider a triangle T in X of side lengths a, b, c, and build a comparison triangle T' with the same lengths in the Euclidean plane. Consider a chord of length d in T which connects two points on the boundary of T; there is a corresponding comparison chord in T', say of length d'. If $d \leq d'$ for any chord in T, we say that T is a *thin triangle* in X.

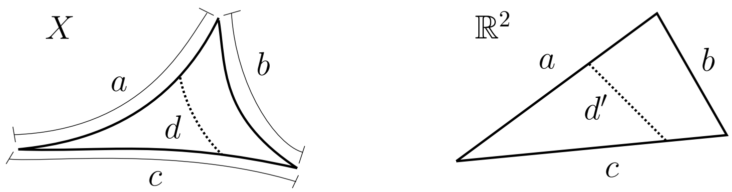

Fig. 3: A chord in a triangle in X, and the corresponding chord in the comparison triangle in the plane. The triangle in X is *thin* if $d \leq d'$ for all such chords.

Definition 2.4 *A* CAT(0) space *is a metric space having a unique geodesic between any two points, such that every triangle is thin.*

A related concept is that of a *locally CAT(0)* or *non-positively curved* metric space X. This is a space where all sufficiently small triangles are thin.

Testing whether a general metric space is CAT(0) is quite subtle. However, Gromov [9] proved that this is easier if the space is a cubical complex. He showed that a cubical complex is CAT(0) if and only if it is simply connected and the link of any vertex is a flag simplicial complex.

Ardila, Owen, and Sullivant [3] gave a purely combinatorial description of CAT(0) cube complexes, which we now describe. If X is a CAT(0) cubical complex and v is any vertex of X, we call (X, v) a *rooted CAT(0) cubical complex*. The right side of Figure 4 shows an example.

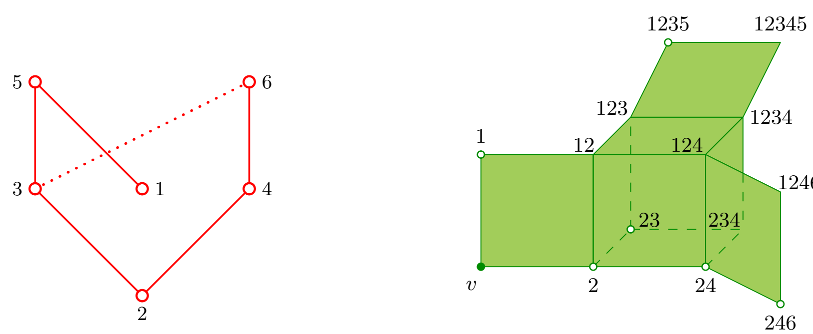

Fig. 4: A poset with inconsistent pairs and the corresponding rooted CAT(0) cubical complex.

Recall that a poset P is *locally finite* if every interval $[i, j] = \{k \in P : i \leq k \leq j\}$ is finite, and it has *finite width* if every antichain (set of pairwise incomparable elements) is finite.

Definition 2.5 *A* poset with inconsistent pairs (PIP) *is a locally finite poset P of finite width, together with a collection of* inconsistent pairs $\{p, q\}$*, such that no two comparable elements are inconsistent, and if p and q are inconsistent and $p' \geq p$ and $q' \geq q$, then p' and q' are inconsistent.*

The *Hasse diagram* of a poset with inconsistent pairs (PIP) is obtained by drawing the poset and connecting each minimal inconsistent pair with a dotted line. An inconsistent pair $\{p, q\}$ is *minimal* if there is no other inconsistent pair $\{p', q'\}$ with $p' \leq p$ and $q' \leq q$. For example, see the left side of Figure 4.

Recall that $I \subseteq P$ is an *order ideal* if $a \leq b$ and $b \in I$ imply $a \in I$. A *consistent order ideal* is one which contains no inconsistent pairs.

Definition 2.6 *If P is a poset with inconsistent pairs, we construct the* cube complex of P*, which we denote $X(P)$. The vertices of $X(P)$ are identified with the consistent order ideals of P. There will be a cube $C(I, M)$ for each pair (I, M) of a consistent order ideal I and a subset $M \subseteq I_{max}$, where I_{max} is the set of maximal elements of I. This cube has dimension $|M|$, and its vertices are obtained by removing from I the $2^{|M|}$ possible subsets of M. The cubes are naturally glued along their faces according to their labels.*

Figure 4 shows a PIP and the corresponding complex. For example, the compatible order ideal $I = \{1, 2, 3, 4\}$ and the subset $M = \{1, 4\} \subseteq I_{max}$ give rise to the square with vertices $1234, 123, 234, 23$.

Theorem 2.7 (Ardila, Owen, Sullivant) *[3] The map $P \mapsto X(P)$ is a bijection between posets with inconsistent pairs and rooted CAT(0) cube complexes.*

2.3 Reconfigurable systems and CAT(0) cubical complexes

The influential paper of Billera, Holmes, and Vogtmann [4] was one of the first to highlight the relevance of the CAT(0) property in applications. Most relevantly to this paper, the space T_n of phylogenetic trees was shown in [4] to be a CAT(0) cubical complex. This led to important consequences, such as the existence of geodesics and of "average trees" in T_n. Furthermore, after numerous partial results by many authors, Owen and Provan [11] recently gave the first polynomial time algorithm to compute geodesics in T_n. The work of Billera, Holmes, and Vogtmann was generalized in the following two directions:

Theorem 2.8 (Ardila-Owen-Sullivant) *[3] There is an algorithm to compute the geodesic between any two points in a CAT(0) cubical complex.*

Theorem 2.9 (Abrams-Ghrist, Ghrist-Peterson) *[1, 8] The state complex of a reconfigurable system is a* ***locally*** *CAT(0) cubical complex; that is, all small enough triangles are thin.*

When the state complex of a reconfigurable system is **globally** CAT(0), we can use the algorithm in Theorem 2.8 to navigate it. That will allow us to get our system from one position to another one in the optimal way. This highlights the importance of the following question:

Question 2.10 *Is the state complex of a given reconfigurable system a CAT(0) space?*

Theorem 2.7 offers a new technique to provide an affirmative answer to Question 2.10: Rooted CAT(0) cubical complexes are in bijection with PIPs; so to prove that a cubical complex is CAT(0), we "simply" have to choose a root for it, and find the corresponding PIP! In principle, this technique works for **any** reconfigurable system whose state complex X is CAT(0). In practice, it is not always easy to identify the corresponding PIP. However, we hope to convince the reader that this can be done in many interesting special cases. We will do it for one old and one new example. We introduce the two relevant robots in Section 3, and provide combinatorial proofs that their state complexes are CAT(0) in Sections 4 and 5.

3 The robotic arms

3.1 The positive robotic arm in a quadrant

The following reconfigurable system, which we call QR_n, was first introduced in [1] and shown to be CAT(0) using Gromov's topological/combinatorial criterion. Consider a robotic arm consisting of n links of unit length, attached sequentially. The robot lives inside an $n \times n$ grid, and its base is affixed to the lower left corner of the grid. Figure 5.a shows a position of the arm.

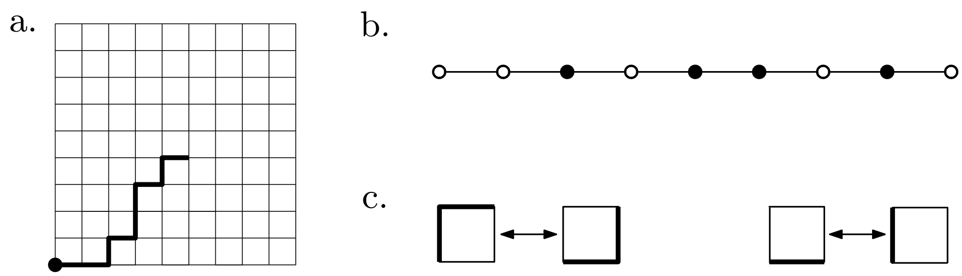

Fig. 5: a. The robotic arm in position 3568 for $n = 9$ b. the corresponding particles on a line (to be introduced later), and c. the local movements of QR_n.

The robot is free to move using the two local moves illustrated in Figure 5.c. They are: *NE-switching corners* (two consecutive links facing north and east can be switched to face east and north, and vice versa), and *NE-flipping the end* (if the last link of the robot is facing east, it can be switched to face north, and vice versa). It is clear that QR_n has 2^n possible positions, corresponding to the paths of length n which start at the southwest corner and always step east or north. We call these simply *NE-paths*.

Notation 3.1 *We will label each state of the robot using the set of its vertical steps: if a position of the robot has k links facing north at positions $a_1, \ldots, a_k$ (counting from the base), then we label it $\{a_1, \ldots, a_k\}$ or simply $a_1 \ldots a_k$.*

Notice that two states of different lengths can have the same label. We assume implicitly that the length of the robot is specified ahead of time.

3.2 The robotic arm in a strip

Now consider a robotic arm SR_n which also consists of n links of unit length, attached sequentially. The robot lives inside a $1 \times n$ grid, and its base is still affixed to the lower left corner of the grid, but the links do not necessarily have to face north and east. Figure 6 shows a position of the arm, as well as the legal moves: *switching corners* and *flipping the end*.

Again, we label a state using its vertical steps shown in Figure 6. One easily checks that the number of states of SR_n is the Fibonacci number F_{n+2}. For this reason, we call a state of SR_n an F-path.

3.2.1 The systems QR_n and SR_n as hopping particles.

Consider a board consisting of n slots on a line, and a system of indistinguishable particles hopping around the board. Any particle can hop to the slot immediately to its left or right whenever that slot is empty.

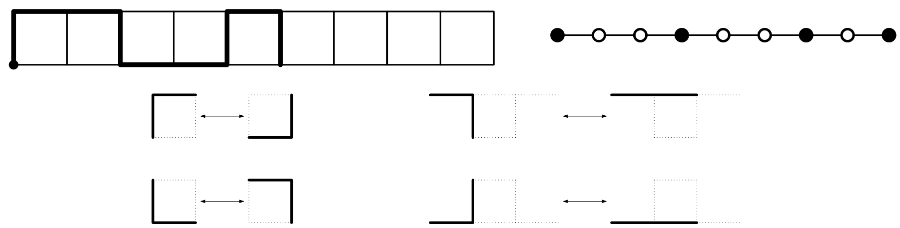

Fig. 6: The robotic arm in position 1479 for $n = 9$, the corresponding particles on a line, and the legal moves.

Particles may enter and leave the board via the rightmost slot. The following proposition is illustrated in Figure 5; for details, see [2].

Proposition 3.2 *The system QR_n is equivalent to the system of hopping particles on a board of length n.*

Now consider a similar board of n slots on a line, with indistinguishable *repellent particles* hopping around the board. The repellent particles must stay at distance at least 2 from each other.

Proposition 3.3 *The system SR_n is equivalent to the system of hopping repellent particles on a board of length n.*

4 The state complex of QR_n is CAT(0)

We now provide combinatorial proofs that the state complexes of the robots QR_n and SR_n are CAT(0). In view of Theorem 2.7, our strategy is as follows. We root the complex $\mathcal{S}(QR_n)$ at a natural vertex v. If $\mathcal{S}(QR_n)$ really is CAT(0), then Theorem 2.7 puts it in correspondence with a PIP (poset with inconsistent pairs) QP_n. We identify the candidate PIP QP_n, and prove that, under the bijection of Theorem 2.7, the PIP QP_n is mapped to the (rooted) state complex of QR_n. Therefore this complex must be CAT(0).

Definition 4.1 *Define the PIP QP_n to be the set of lattice points inside the triangle $y \geq 0, y \leq x$, and $x \leq n-1$, with componentwise order (so $(x,y) \leq (x',y')$ if $x \leq x'$ and $y \leq y'$) and no inconsistent pairs.*

The poset QP_n has the triangular shape shown in Figure 7 for $n = 6$.

Proposition 4.2 *There is a bijection between the states of the robot QR_n and the order ideals of QP_n.*

Recall Definition 2.3. We get the following by Birkhoff's theorem:

Corollary 4.3 *If we declare the "home" state of QR_n to be the fully horizontal state, then the poset of states of QR_n is a distributive lattice.*

Let the *word of a subset* $A = \{a_1 < a_2 < \cdots < a_k\} \subseteq [n]$ be the length n word $w(A) = (a_1, a_2, \ldots a_k, (n+1), (n+1), \ldots, (n+1))$.

Proposition 4.4 *The lattice of states of QR_n is isomorphic to the poset on the subsets of $[n]$, where $A \leq B$ if $w(A) \geq w(B)$ coordinatewise.*

Having established these results about the 1-skeleton of the state complex, we now extend them to the higher-dimensional cubes.

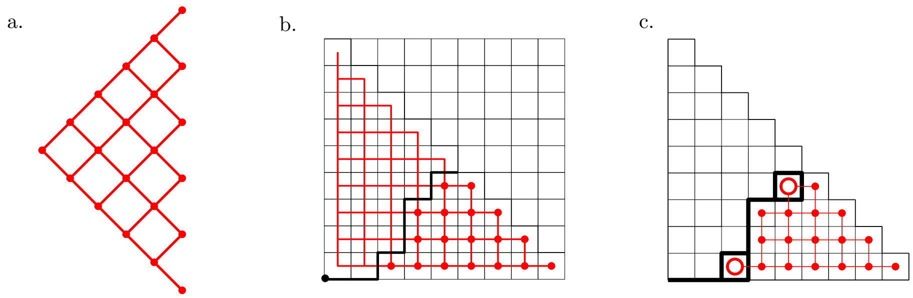

Fig. 7: a. The poset QP_6. b. A state of QR_n corresponds to an order ideal in QP_n c. The bijection between partial NE-paths and pairs (order ideal, maximal elements) of QP_n

Definition 4.5 *A* partial NE-path *is a path consisting of consecutive links which may be north edges, east edges, or unit squares, such that each unit square is attached to the rest of the path by its southwest and northeast corners. The* length *of a partial NE-path is $e + 2f$, where e is the number of edges and f is the number of squares. The partial NE-paths form a poset by containment, whose minimal elements are the NE-paths.*

To illustrate this definition, Figure 7.c shows a partial NE-path which contains the NE-path in b. Recall that $X(QP_n)$ is the rooted cube complex corresponding to the PIP QP_n under the bijection of Theorem 2.7. We use the notation of Definition 2.6.

Lemma 4.6 *The partial NE-paths of length n are in order-preserving bijection with the cubes of $X(QP_n)$.*

Lemma 4.7 *The partial NE-paths of length n are in order-preserving bijection with the cubes of the state complex $\mathcal{S}(QR_n)$.*

Proof: A k-cube C of $\mathcal{S}(QR_n)$ is given by a state u and k commutative moves $\varphi_1, \ldots, \varphi_k$ that can be applied to u. The state u is given by an NE-path, and each one of the k moves $m_1, \ldots, m_k$ corresponds to a corner of the NE-path that could be switched. The two positions of this corner before and after the move m_i form a square. Since the moves are commutative, two of these squares cannot share an edge. Adding these k squares to the NE-path u gives rise to a partial NE-path corresponding to the k-cube C.

Conversely, consider a partial NE-path with k squares. There are 2^k NE-paths contained in it, obtained by "resolving" each square into an NE or an EN corner. The resulting 2^k NE-paths form a cube of $\mathcal{S}(QR_n)$. This bijection is clearly order-preserving. □

Theorem 4.8 *The state complex of the robotic arm in an $n \times n$ grid is a CAT(0) cubical complex.*

Proof: This is an immediate consequence of Lemmas 4.6 and 4.7 and Theorem 2.7. □

As a corollary of our combinatorial description of the state complex of QR_n, we get:

Corollary 4.9 *If $q_{n,d}$ is the number of d-cubes in the state complex of the robot in a quadrant QR_n,*

$$\sum_{n,d \geq 0} q_{n,d}\, x^n y^d = \frac{1 + xy}{1 - 2x - x^2 y}.$$

5 The state complex of SR_n is CAT(0)

Now we carry out the same approach for the robotic arm in a strip SR_n.

Definition 5.1 *Define the PIP SP_n to be the set of lattice points inside the triangle $y \geq 0, y \leq 2x$, and $x \leq n-1$, with componentwise order (so $(x, y) \leq (x', y')$ if $x \leq x'$ and $y \leq y'$) and no inconsistent pairs.*

Proposition 5.2 *There is a bijection between the states of the robot SR_n and the order ideals of SP_n.*

This is proved similarly to Proposition 4.2. Details are given in [2]. If we declare the "home" state of SR_n to be the fully horizontal state h, then the poset of states of SR_n is a distributive lattice. We have,

Proposition 5.3 *The lattice of states of SR_n is isomorphic to the poset on the spread out subsets of $[n]$, where $A \leq B$ if $w(A) \geq w(B)$ coordinatewise.*

Proof: This is clear from the repellent hopping particles model for SR_n of Section 3.2.1. □

Definition 5.4 *A* partial F-path *is a partial NE-path such that the link following any vertical edge or square must be a horizontal edge.*

Recall that $X(SP_n)$ is the rooted cube complex corresponding to the PIP SP_n under the bijection of Theorem 2.7. We then have the following results. The proofs are essentially the same as those of Lemmas 4.6 and 4.7, Theorem 4.8.

Lemma 5.5 *The partial F-paths of length n are in bijection with the cubes of the state complex of $X(SP_n)$.*

Lemma 5.6 *The partial F-paths of length n are in bijection with the cubes of the state complex of SR_n.*

Theorem 5.7 *The state complex of the robotic arm in an $n \times n$ grid is a CAT(0) cubical complex.*

As a corollary of our combinatorial description of the state complex of SR_n, we get:

Corollary 5.8 *If $s_{n,d}$ is the number of d-cubes in the state complex of the robot in a strip SR_n,*

$$\sum_{n,d \geq 0} s_{n,d}\, x^n y^d = \frac{1 + x + xy + x^2 y}{1 - x - x^2 - x^3 y},$$

6 Finding the optimal path between two states

Consider a robot, or some other reconfigurable system $\mathcal{R}$, whose state complex $\mathcal{S}(\mathcal{R})$ is CAT(0). As in the two examples above, there may be a natural choice of a "home state" u, such that the PIP P_u corresponding to the rooted complex $(\mathcal{S}(\mathcal{R}), u)$ has a particularly simple description. Now suppose that we want to take the robot from state a to state b in an optimal way. Equivalently, we wish to get from vertex a to vertex b of the state complex $\mathcal{S}(\mathcal{R})$.

6.1 Rerooting the complex

To find the optimal path from a to b, the first step will be to reroot the complex at a, and find the PIP P_a corresponding to the rooted CAT(0) cubical complex $(\mathcal{S}(\mathcal{R}), a)$. Fortunately, this is very easy to do.

Notation 6.1 *If p and q are an inconsistent pair in a PIP, write $p \nleftrightarrow q$.*

Proposition 6.2 *Let u and a be vertices of the CAT(0) cube complex X and let P_u and P_a be the PIPs corresponding to the rooted complexes (X, u) and (X, a) respectively. Let I be the consistent order ideal of P_u corresponding to a, and let $J = P_u - I$. The PIP P_a has an element p' corresponding to each element $p \in P_u$, and it can be described in terms of P_u as follows:*

- *If $j_1 < j_2$ in P_u, then $j_1' < j_2'$ in P_a.*
- *If $i_1 < i_2$ in P_u then $i_1' < i_2'$ in P_a.*
- *If $i < j$ in P_u then $i' \nleftrightarrow j'$ in P_a.*
- *If $j_1 \nleftrightarrow j_2$ in P_u, then $j_1' \nleftrightarrow j_2'$ in P_a.*
- *If $i \nleftrightarrow j$ in P_u then $i' < j'$ in P_a.*

Here the is and the js represent arbitrary elements of I and J, respectively.[(i)]

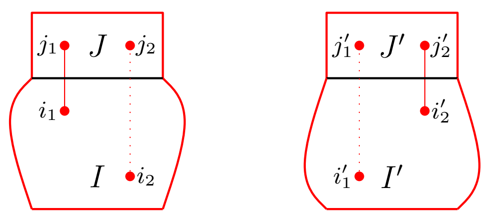

Fig. 8: The PIPs P_u and P_a before and after rerooting the CAT(0) cube complex.

Corollary 6.3 *The Hasse diagram of P_a is obtained from that of P_u by turning I upside down, and converting all solid edges from I to J into dotted edges, and vice versa.*

Note that even if P_u has no inconsistent pairs, the PIP P_a probably will have inconsistent pairs. Now that we have rerooted the complex, our goal is to get from the root a to the vertex b optimally. There are at least four notions of "optimality": we may wish to minimize Euclidean distance, number of moves, simultaneous moves, or time. We can solve these four problems.

(i) Notice that we never have $i > j$ or $i_1 \nleftrightarrow i_2$ in P_u.

6.2 Minimizing the Euclidean distance

Suppose we want to find the shortest path from a to b in the Euclidean metric of the cubical complex $\mathcal{S}(\mathcal{R})$. This can be accomplished using Ardila, Owen, and Sullivant's algorithm [3] to compute the shortest path from a to b. As explained there, a prerequisite for this is to write down the PIP P_a, which we have done in Proposition 6.2. This metric is very useful in some applications, particularly when navigating the space of phylogenetic trees [4, 11]. However, this metric does not seem natural for the robotic applications we have in mind here. It is probably more natural to consider the following three variants.

6.3 Minimizing the number of moves

Suppose we are only allowed to perform one move at a time. Geometrically, we are looking for a shortest edge-path from a to b. Let B be the consistent order ideal of P_a corresponding to vertex b in the rooted complex $(\mathcal{S}(\mathcal{R}), a)$. We can regard B as a subposet of P_a. The following description makes it clear how to construct the minimal shortest paths.

Proposition 6.4 *The shortest edge-paths from a to b are in one-to-one correspondence with the linear extensions of the poset B. Their length is $|B|$.*

6.4 Minimizing the sequence of simultaneous moves

Now suppose that we can move the robot in steps, where at each step we can perform several moves at a time with no penalty. Geometrically, we are looking for a shortest cube path from a to b, where at each step we cross a cube from the current vertex to the one across the diagonal. Again, let B be the consistent order ideal of P_a corresponding to b. Let the *depth* $d(B)$ of B be the size of the longest chain(s) in B.

Definition 6.5 *Let the* normal cube path *from a to b be the cube path given by the sequence of order ideals $\mathbf{M} : \emptyset = M_0 \subset M_1 \subset \cdots \subset M_{d(B)} = B$, where each ideal is obtained from the previous one by adding to it all the minimal elements that have not yet been added. In other words, $M_{k+1} := M_k \cup (B - M_k)_{min}$.*

The previous definition is due to Niblo and Reeves [10] in a different language; the correspondence with PIPs makes these paths more explicit. It also allows us to give a simple proof of the following result from Reeves's Ph.D. thesis [12] in [2]:

Proposition 6.6 *The shortest cube paths from a to b have size $d(B)$. In particular, the normal cube path from a to b is minimal.*

6.5 Minimizing time

Perhaps the most realistic model is to allow ourselves to move the robot continuously in time, where we can perform several moves simultaneously, as long as these moves are physically independent. We can even perform only part of a move, and perform the rest of the move later. Each move still takes one unit of time, and there is no time penalty for multitasking.

Geometrically, we are endowing each cube with the ℓ_∞ metric: For $\mathbf{x}, \mathbf{y}$ in a unit d-cube, we let $||\mathbf{x} - \mathbf{y}|| := \max(x_1 - y_1, \ldots, x_d - y_d)$. Now we are looking for a shortest path from a to b with respect to this ℓ_∞ metric. The following result, stated without proof in [1], shows that the added flexibility of performing partial moves does not actually help us move our robots more quickly.

Proposition 6.7 *The fastest paths from a to b take $d(B)$ units of time. In particular, the normal cube path from a to b is a fastest path.*

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The Robinson–Schensted Correspondence and A_2-webs

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Abstract. The A_2-spider category encodes the representation theory of the sl_3 quantum group. Kuperberg (1996) introduced a combinatorial version of this category, wherein morphisms are represented by planar graphs called *webs* and the subset of *reduced webs* forms bases for morphism spaces. A great deal of recent interest has focused on the combinatorics of invariant webs for tensors powers of V^+, the standard representation of the quantum group. In particular, the invariant webs for the $3n$th tensor power of V^+ correspond bijectively to $[n, n, n]$ standard Young tableaux. Kuperberg originally defined this map in terms of a graphical algorithm, and subsequent papers of Khovanov–Kuperberg (1999) and Tymoczko (2012) introduce algorithms for computing the inverse. The main result of this paper is a redefinition of Kuperberg's map through the representation theory of the symmetric group. In the classical limit, the space of invariant webs carries a symmetric group action. We use this structure in conjunction with Vogan's generalized tau-invariant and Kazhdan–Lusztig theory to show that Kuperberg's map is a direct analogue of the Robinson–Schensted correspondence.

Résumé. La catégorie d'araignée A_2 encode la théorie des représentations du groupe quantique $U_q(sl_3)$. Kuperberg (1996) a introduit une version combinatoire de cette catégorie, dans laquelle les morphismes sont reprśentés par des graphes planaires appelés *toiles* dont le sous-ensemble de *toiles réduites* constitue des bases pour les espaces de morphismes. Beaucoup d'interêt a été fixé recemment sur le combinatoire des toiles invariantes pour les puissances tensorielles de V^+, la représentation standarde de $U_q(sl_3)$. En particulier, les toiles invariantes pour $(V^+)^{3n}$ sont en corresponance bijective au tableaux de Young standards $[n, n, n]$. L'application original de Kuperberg avait une définition en termes d'un algorithme graphique, puis des articles de Khovanov–Kuperberg (1999) et Tymoczko (2012) présentent des algorithmes pour la computation d l'inverse. Le résultat principal de cette article est une redéfinition de l'application de Kuperberg à travers la théorie de représentations du groupe symétrique. Dans la limite classique, l'éspace des toiles invariantes porte une action de S_{3n}. On emploie cette structure de concert avec l'invariant tau généralisé de Vogan et la théorie Kazhdan–Lusztig pour montrer que l'application de Kuperberg est un analogue direct de la correspondance Robinson–Schensted.

Keywords: Robinson–Schensted, Web basis, Kazhdan–Lusztig theory, Young tableau

1 Introduction

The A_2-spider is a category encoding the representation theory of $\mathcal{U}_q(\mathfrak{sl}_3)$, the quantum enveloping algebra of the $\mathfrak{sl}_3$ Lie algebra. The objects in the category are tensor products of V^+ and V^-, the standard and dual representations of the quantum group, while the morphisms are intertwining maps. Kuperberg (1996) defined a diagramatic construction of the A_2-spider in which morphism are represented by planar graphs called *webs*, with the subset of *reduced webs* forming bases for each morphism space.

In order to prove that reduced webs span each morphism space, Kuperberg studied quantum invariants, which may be viewed as morphisms from the trivial representation to other representations. Classical results require that the invariant space for a tensor product of copies of V^+ and V^- has dimension equal to the number of *dominant lattice paths* satisfying certain conditions; Kuperberg developed an explicit graphical algorithm carrying reduced webs to dominant lattice paths and showed that it was bijective. In a subsequent paper Khovanov and Kuperberg (1999) introduced a method to compute the inverse via a recursive growth algorithm. Recent combinatorial interest in Kuperberg's map has focused on the case of invariants for tensor powers of V^+ Petersen et al. (2009); Tymoczko (2012). For $(V^+)^{\otimes 3n}$, the map may be interpreted as a bijection between webs on $3n$ source vertices and standard Young tableaux on the shape $[n, n, n]$.

In this paper, we reinterpret Kuperberg's bijection in terms of the representation theory of the symmetric group. A tensor power of a quantum group representation carries a Hecke algebra action which in the classical limit reduces to a symmetric group action by permutation of tensor factors. The subspace of invariants forms a subrepresentation.

Vogan introduced the generalized τ-invariant to study infinite dimensional representations of semisimple Lie algebras Vogan (1979). Generalized τ-invariants are closely related to the combinatorics of standard Young tableaux (Section 2). The generalized τ-invariant gives a nonalgorithmic way of defining the Robinson–Schensted correspondence between symmetric group elements and same shape pairs of standard Young tableaux (Section 3). In Section 4, we discuss an *in situ* version of the Robinson–Schensted algorithm for parameterizing Kazhdan–Lusztig left cell basis elements by Young tableaux in terms of the symmetric group action on the left cell representation. In Sections 6 and 7, we discuss the symmetric group action on webs and Kuperberg's bijection between reduced webs and standard tableaux. Our main result is in Section 8, where we show that Kuperberg's map can be defined in terms of the Robinson–Schensted algorithm for Kazhdan–Lusztig left cells.

2 Generalized τ-invariants for Tableaux

In this and subsequent sections, we will rely extensively on standard results from the combinatorics of tableaux and the symmetric group. Björner and Brenti (2005) provide an excellent exposition of this material.

Recall that a *Young diagram* is a collection of finitely many boxes arranged in left justified rows so that no row has more boxes than the rows above it. Young diagrams with n boxes correspond naturally to partitions of n by treating the length of each row as an element of a partition. A *standard Young tableau* on a Young diagram with n boxes is a labeling of the boxes with the numbers $1, 2, \ldots, n$ in such a way that the label in each box is less than the labels in the boxes immediately below and immediately to the right, with each label appearing exactly once.

We indicate the set of all standard Young tableaux on n boxes by $\mathscr{T}_n$. Let $s_i \in S_n$ be the simple transposition that exchanges i and $i+1$. Given $Y \in \mathscr{T}_n$, $\tau(Y)$ is a subset of the simple transpositions in S_n, where $s_i \in \tau(Y)$ when $i+1$ is below the row of i in Y. We refer to $\tau(Y)$ as the *τ-invariant* of Y. (Most sources call $\tau(Y)$ the descent set of Y, but we choose our terminology to be consistent with Vogan (1979).) If s_i, s_j are adjacent simple transpositions in S_n, then $D^{\mathrm{YT}}_{i,j}$ is defined as the set of all $Y \in \mathscr{T}_n$

such that $s_i \in \tau(Y)$ and $s_j \notin \tau(Y)$. Let $s_i \cdot Y$ be the (not necessarily standard) tableau obtained from Y by exchanging i and $i+1$.

Lemma 1 *Let s_i, s_j be adjacent simple transpositions in S_n. Given $Y \in D^{\mathrm{YT}}_{i,j}$, exactly one of $s_i \cdot Y$ and $s_j \cdot Y$, denoted $f^{\mathrm{YT}}_{i,j}(Y)$, is a standard tableau in $D^{\mathrm{YT}}_{j,i}$; $f^{\mathrm{YT}}_{i,j} : D^{\mathrm{YT}}_{i,j} \to D^{\mathrm{YT}}_{j,i}$ is a bijection with inverse $f^{\mathrm{YT}}_{j,i} : D^{\mathrm{YT}}_{j,i} \to D^{\mathrm{YT}}_{i,j}$.*

Definition 1 *Let Y and Y' be elements of $\mathscr{T}_n$. If $\tau(Y) = \tau(Y')$, then Y and Y' are* equivalent to order 0*, denoted $Y \underset{0}{\approx} Y'$. We say that $Y \underset{n}{\approx} Y'$ (Y and Y' are equivalent to order n) if $Y \underset{n-1}{\approx} Y'$ and $f^{\mathrm{YT}}_{i,j}(Y) \underset{n-1}{\approx} f^{\mathrm{YT}}_{i,j}(Y')$ whenever Y and Y' are in $D^{\mathrm{YT}}_{i,j}$. If $Y \underset{n}{\approx} Y'$ for all nonnegative integers n, then Y and Y' have the same* generalized τ-invariant. *($\tau_g(Y) = \tau_g(Y')$).*

Theorem 1 (Vogan) *If $Y, Y' \in \mathscr{T}_n$ and $\tau_g(Y) = \tau_g(Y')$, then $Y = Y'$.*

In other words, a standard Young tableau on n boxes is completely determined by its generalized τ-invariant.

3 Generalized τ-invariants and the Robinson–Schensted Correspondence

We can define τ-invariants and generalized τ-invariants for elements of the symmetric group. As we will see in a moment, these constructions are closely related to the previous definitions for Young tableaux.

Take as a generating set for S_n the simple transpositions $s_1, s_2, \ldots$ Given $x, y \in S_n$, let $x \leq y$ if some minimal length expression for x in terms of generators is a subword of some minimal expression for y. This defines the *Bruhat order* on S_n. We define the τ-invariant for $x \in S_n$ by letting $\tau(x)$ be the set of simple transpositions such that $s_i \cdot x < x$.

The set $\tau(x)$ is closely related to the one line notation for x. Recall that the one line notation for $x \in S_n$ is a permutation of the integers $1, 2, \ldots n$; if x has one line notation $x_1 x_2 x_3 \cdots$ then x sends 1 to x_1, 2 to x_2, etc. It is a standard fact from the combinatorics of the symmetric group that $s_i \in \tau(x)$ if and only if i and $i+1$ appear out of order in the one line notation for x.

If s_i and s_j are adjacent simple transpositions, let $D^{S_n}_{i,j}$ be the set of all $x \in S_n$ such that $s_i \in \tau(x)$, $s_j \notin \tau(x)$. One can prove the next lemma by thinking about the one line notation for x.

Lemma 2 *If $x \in D^{S_n}_{i,j}$, then exactly one of $s_i \cdot x$, $s_j \cdot x$ is an element of $D^{S_n}_{j,i}$. Denote this element of $D^{S_n}_{j,i}$ by $f_{i,j}(x)$. This defines a bijection $f^{S_n}_{i,j} : D^{S_n}_{i,j} \to D^{S_n}_{j,i}$ with inverse $f^{S_n}_{j,i} : D^{S_n}_{j,i} \to D^{S_n}_{i,j}$.*

This allows us to define a generalized τ-invariant for symmetric group elements by making the appropriate substitutions in Definition 1, i.e.,

Definition 2 *Let x and y be elements of S_n. If $\tau(x) = \tau(y)$, then x and y are* equivalent to order 0*, denoted $x \underset{0}{\approx} y$. We say that $x \underset{n}{\approx} y$ (x and y are equivalent to order n) if $x \underset{n-1}{\approx} y$ and $f^{S_n}_{i,j}(x) \underset{n-1}{\approx} f^{S_n}_{i,j}(y)$ whenever x and y are in $D^{\mathrm{YT}}_{i,j}$. If $x \underset{n}{\approx} y$ for all nonnegative integers n, then x and y have the same* generalized τ-invariant. *($\tau_g(x) = \tau_g(y)$).*

Recall that the *Robinson-Schensted correspondence* gives a bijection between elements of S_n and the set of same shape ordered pairs of standard Young tableaux with n boxes. (We do not explain the algorithm here, but descriptions are available from many sources.) Given $w \in S_n$, let $P(w)$ and $Q(w)$ denote respectively the left and right tableaux in the pair corresponding to w. One obtains the following by thinking carefully about the Robinson–Schensted algorithm in terms of one line notation.

Lemma 3 *Given $x \in S_n$, $s_i \in \tau(x)$ if and only $s_i \in \tau(P(x))$.*

One can use *dual Knuth relations* to prove the next lemma: (See (Björner and Brenti, 2005, Section 6.4)

Lemma 4 *Given $x \in D_{i,j}^{S_n}$, $P(f_{i,j}^{S_n}(x)) = f_{i,j}^{\mathrm{YT}}(P(x))$ and $Q(f_{i,j}^{S_n}(x)) = Q(x)$.*

Combining the lemmas with Theorem 1 yields:

Theorem 2 *Given $x, y \in S_n$, $\tau_g(x) = \tau_g(y)$ if and only if $\tau_g(P(x)) = \tau_g(P(y))$.*

The definition of the generalized τ-invariant we use here differs from the one given in Vogan (1979) and discussed in Kazhdan and Lusztig (1979); Vogan defines a *right* generalized τ-invariant by using the right action of the symmetric group on itself. Our version is the equivalent obtained by using the *left* action. Vogan's right generalized τ-invariant is given in terms of our left version by $\tau_g(x^{-1})$. Left and right generalized τ-invariants give a nonalgorithmic means of defining the Robinson–Schensted correspondence by comparing generalized τ-invariants of tableaux and permutations in the natural way:

Theorem 3 (The Robinson–Schensted Correspondence) *Given $x \in S_n$, $P(x)$ is the unique element of $\mathscr{T}_n$ such that $\tau_g(P(x)) = \tau_g(x)$, while $Q(x)$ is the unique tableau in $\mathscr{T}_n$ such that $\tau_g(Q(x)) = \tau_g(x^{-1})$.*

4 Generalized τ-invariants and Kazhdan–Lusztig Theory

We now recall some facts from Kazhdan and Lusztig (1979).

Definition 3 *Let A be the ring $\mathbb{Z}[v^{1/2}, v^{-1/2}]$. The* Hecke Algebra *$\mathscr{H}_n$ of the symmetric group is the associative A-algebra with generators $T_{s_1}, T_{s_2}, \ldots, T_{s_{n-1}}$ and relations*

$$T_{s_i}T_{s_j} = T_{s_j}T_{s_i} \text{ if } |i-j| > 1 \tag{1}$$

$$T_{s_i}T_{s_j}T_{s_i} = T_{s_j}T_{s_i}T_{s_j} \text{ if } |i-j| = 1 \tag{2}$$

$$(T_{s_i} + 1)(T_{s_i} - v) = 0. \tag{3}$$

In addition, given $w \in S_n$, let

$$T_w = T_{s_{i_1}}T_{s_{i_2}} \ldots, T_{s_{i_k}}$$

where

$$s_{i_1}s_{i_2} \ldots, s_{i_k}$$

is some reduced expression for w in terms of the simple transpositions $\{s_i\}$.

Note that $\mathscr{H}_n$ reduces to the group algebra of S_n over $\mathbb{Z}$ when v is set to 1.

The Hecke algebra has a remarkable basis defined in Kazhdan and Lusztig (1979), whose basis elements are also parameterized by elements of the symmetric group. We denote the basis element parameterized by $w \in S_n$ as C_w. The *Kazhdan–Lusztig basis* is encoded in the *Kazhdan–Lusztig graph*, whose vertices are labeled with elements of S_n, where the edge between the vertices labeled with x and y has multiplicity indicated by $\mu(x,y)$. The left action of a generator T_{s_i} on $\mathscr{H}_n$ in terms of the basis $\{C_w\}_{w\in S_n}$ is given as follows:

$$T_{s_i}C_w = \begin{cases} -C_w & \text{if } s_i \in \tau(w), \\ vC_w + v^{1/2} \displaystyle\sum_{\substack{y\in S_n \\ s_i\in\tau(y)}} \mu(w,y)C_y & \text{otherwise.} \end{cases} \tag{4}$$

Treating it as a left $\mathscr{H}_n$-module, we wish to decompose $\mathscr{H}_n$ into irreducible subquotients. For this, we need the notion of a *left cell*.

Definition 4 *Define a binary relation $\preceq_L$ on S_n by letting $x \preceq_L x$ and $x \preceq_L y$ whenever $\mu(x,y) \neq 0$ and $\tau(x) \not\subset \tau(y)$. (This is equivalent to requiring that C_x appears as a summand in $T_{s_i}C_y$ for some i.) Extend $\preceq_L$ to a preorder by imposing transitivity. We refer to $\preceq_L$ as the* left preorder *on S_n.*

Definition 5 *Define an equivalence relation $\sim_L$ on S_n by letting $x \sim_L y$ if $x \preceq_L y$ and $y \preceq_L x$. The equivalence classes under $\sim_L$ are called* left cells*; $\preceq_L$ descends to a partial order on left cells.*

Left cells lead us naturally to the definition of *left cell modules* and *left cell representations*. Let $\mathrm{Cell}(S_n)$ denote the set of left cells in S_n, ordered by $\preceq_L$. Given $\mathcal{C} \in \mathrm{Cell}(S_n)$, define

$$\operatorname{span}_A \mathcal{C} = \operatorname{span}_A\{C_w | w \in \mathcal{C}\}.$$

Definition 6 *Given $\mathcal{C} \in \mathrm{Cell}(S_n)$, define the* left cell module *for $\mathcal{C}$ by*

$$\mathrm{KL}_{\mathcal{C}} = \left(\bigoplus_{\substack{\mathcal{C}_i\in\mathrm{Cell}(S_n) \\ \mathcal{C}_i\preceq_L\mathcal{C}}} \operatorname{span}_A \mathcal{C}_i \right) \Bigg/ \left(\bigoplus_{\substack{\mathcal{C}_i\in\mathrm{Cell}(S_n) \\ \mathcal{C}_i\prec_L\mathcal{C}}} \operatorname{span}_A \mathcal{C}_i \right).$$

If we set $v = 1$ and extend scalars to $\mathbb{C}$, then $\mathrm{KL}_{\mathcal{C}}$ is referred to as the left cell representation *corresponding to the cell $\mathcal{C}$.*

For a left cell representation $\mathrm{KL}_{\mathcal{C}}$, Equation 4 becomes

$$T_{s_i}C_w = \begin{cases} -C_w & \text{if } s_i \in \tau(w), \\ vC_w + v^{1/2} \displaystyle\sum_{\substack{y\in \mathcal{C} \\ s_i\in\tau(y)}} \mu(w,y)C_y & \text{otherwise.} \end{cases} \tag{5}$$

We refer to a Kazhdan-Lusztig graph restricted to a left cell as a *left cell graph*. Let $\widehat{S}_n$ indicate the set of irreducible representations of S_n over the complex numbers. The following parameterization is a standard result from algebraic combinatorics:

Theorem 4 *The elements of $\widehat{S}_n$ are parameterized by partitions of n. In particular, let $\mathbf{p} = [p_1, p_2, \ldots]$ be a partition of n and $\mathbf{t} = [t_1, t_2, \ldots]$ its transpose partition. Then, $\pi_{\mathbf{p}}$ is the unique element of $\widehat{S}_n$ whose restriction to $\Pi_{p_i\in\mathbf{p}}S_{p_i}$ contains a copy of the trivial representation and whose restriction to $\Pi_{t_i\in\mathbf{t}}S_{t_i}$ contains a copy of the sign representation.*

Note that we will often refer to the element of $\widehat{S}_n$ corresponding to some Young diagram, obtained by taking the diagram to its corresponding partition. This parameterization of $\widehat{S}_n$ is closely related to the decomposition of S_n into left cells. As in Section 3, let $P(w)$ and $Q(w)$ denote respectively the left and right tableaux in the pair corresponding to $w \in S_n$.

Theorem 5 *For each left cell $\mathcal{C}$ of S_n, there exists some standard Young tableau $Q_{\mathcal{C}}$ with n boxes such that*

$$\mathcal{C} = \{w \in S_n | Q(w) = Q_{\mathcal{C}}\}.$$

Furthermore, $\mathrm{KL}_{\mathcal{C}}$ is isomorphic as a representation over $\mathbb{C}$ to the irreducible representation parameterized by the shape of $Q_{\mathcal{C}}$.

With this description of left cells in hand, suppose that Q and Q' are n-box standard Young tableaux with the same shape and let $\mathcal{C}_Q$ and $\mathcal{C}_{Q'}$ be the cells obtained by fixing these as right tableaux. We define a bijection $\phi_{Q,Q'} : \mathcal{C}_Q \to \mathcal{C}_{Q'}$ by

$$(P, Q) \mapsto (P, Q').$$

Theorem 6 *The map $\phi_{Q,Q'}$ is an isomorphism of graphs and preserves τ-invariant data.*

Thus, we can refer to the Kazhdan–Lusztig left cell basis for some element of $\widehat{S}_n$ without specifying a particular left cell; elements of the basis are naturally parameterized by their left tableaux.

5 An *in situ* Robinson–Schensted Algorithm for Left Cells

Suppose that we are given some element of $\widehat{S}_n$ with its associated Kazhdan–Lusztig left cell basis but without the tableaux attached to basis elements. Is it possible to compute these tableaux in terms of the structure of the (based) representation? In the remainder of this section, we will develop an appropriate definition of the generalized τ-invariant that allows us to do this. We will subsequently use this structure to redefine Kuperberg's map as an analogue of the Robinson–Schensted correspondence.

Observe that if s_i is not in $\tau(x)$, then $C_{s_i x}$ appears with multiplicity one in $T_{s_i} C_x$ (Kazhdan and Lusztig, 1979, p. 171, equation 2.3.a). Since $s_i \notin \tau(x)$ or $s_i \notin \tau(s_i x)$, we have the following lemma:

Lemma 5 *Given any $y, s_i \in S_n$, y and $s_i y$ are connected by an edge of multiplicity one.*

Such an edge is often referred to as a *Bruhat edge*.

Lemma 6 *Let $x \in S_n$ such that $x \in D_{i,j}^{S_n}$. Then, x is connected to $y = f_{i,j}^{S_n}(x)$ by an edge of multiplicity 1; furthermore, if $z \neq y$ is any other element of $D_{j,i}^{S_n}$, then $\mu(x, z) = 0$.*

Now, let $\mathcal{C} \subset S_n$ be some left cell. Then, $\mathrm{KL}_{\mathcal{C}}$ is the associated left cell representation, with basis $B(\mathrm{KL}_{\mathcal{C}}) = \{C_w | w \in \mathcal{C}\}$. We wish to define a τ-invariant for each basis element by using the S_n-action on $\mathrm{KL}_{\mathcal{C}}$ (Equation 5).

Definition 7 *Given some basis element $C \in B(\mathrm{KL}_{\mathcal{C}})$, let $\tau(C) = \{s_i \in S_n | T_{s_i} \cdot C = -C\}$. Given adjacent simple transpositions s_i, s_j, let $D_{i,j}^{\mathrm{KL}_{\mathcal{C}}}$ be the set of basis elements C in $\mathrm{KL}_{\mathcal{C}}$ such that $s_i \in \tau(C)$, $s_j \notin \tau(C)$.*

Observation 1 *Notice that $\tau(C_w) = \tau(w)$ as desired.*

Lemma 6 gives rise to the following definition:

Definition 8 *Given $C \in D_{i,j}^{\mathrm{KL}_{\mathcal{C}}}$, let $f_{i,j}^{\mathrm{KL}_{\mathcal{C}}}(C)$ be the unique basis element $C' \in D_{j,i}^{\mathrm{KL}_{\mathcal{C}}}$ which appears as a summand of $T_{s_j} \cdot C$. Note that $f_{i,j}^{\mathrm{KL}_{\mathcal{C}}}(C_w) = C_{f_{i,j}^{S_n}(w)}$.*

Now, we immediately have a definition of generalized τ-invariant for each basis element in $\mathrm{KL}_{\mathcal{C}}$ simply by substituting $f_{i,j}^{\mathrm{KL}_{\mathcal{C}}}$ in Definition 1. In fact we can compare generalized τ-invariant of basis elements and permutations or tableaux by using the appropriate $f_{i,j}$ maps on each side of the equations in Definition 1.

Theorem 7 (Robinson–Schensted for left cells) *Let C_w be a basis element in $\mathrm{KL}_{\mathcal{C}}$. Then, $P(w)$, the left Robinson–Schensted tableau for w, is the unique standard tableau on n boxes such that $\tau_g(P(W)) = \tau_g(C_w)$.*

Thus, suppose that $\mathbf{p}$ is some partition of n. (This choice equivalent to the choice of a Young diagram with n boxes.) Let $\mathrm{KL}_{\mathbf{p}}$ be the left cell for this shape with permutation and tableau labeling stripped away. (Recall that $\mathrm{KL}_{\mathbf{p}}$ is uniquely determined as a based representation due to Theorem 6 regardless of the particular left cell from which it originally derived.) Then, given some left cell basis element $C \in \mathrm{KL}_{\mathbf{p}}$, there is a unique standard tableau Y_C such that $\tau_g(C) = \tau_g(Y_C)$. Furthermore, this is the same tableau that we would have obtained by taking the left tableau for the permutation label of C before it was removed. The parameterization of left cell basis elements by standard tableaux on the shape $\mathbf{p}$ is in this sense canonical.

6 The Symmetric Group Action on sl_3-Webs

The sl_3 spider, introduced by Kuperberg (1996) and subsequently studied by many others (Khovanov and Kuperberg (1999); Kim (2003); Morrison (2007); Murakami et al. (1998)) is a diagrammatic, braided monoidal category encoding the representation theory of $U_q(sl_3)$. The objects in this category are tensor products of V^+ and V^-, the three-dimensional representations of $\mathcal{U}_q(\mathfrak{sl}_3)$, but these are encoded as finite strings in the alphabet $\{+,-\}$, including the empty string. The morphisms are intertwining maps, which are represented by $\mathbb{Z}[q,q^{-1}]$-linear combinations of certain graphs called webs which we will describe in a moment. (See Figure 1 for an example of a web.) Webs are oriented trivalent graphs drawn in a

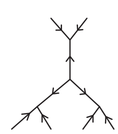
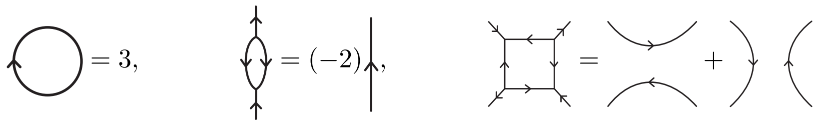


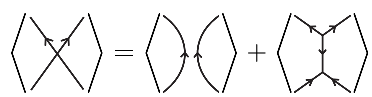
Fig. 1: A web in $\mathrm{Hom}(+\,+\,+\,+\,,\,-\,-)$

square region with boundary points lying on the top and bottom of that region. Edges incident on the boundary points have orientations compatible with the source and target words; edges pointing upward and downward are labeled by $+$ and $-$ respectively. We read webs from bottom to top. All trivalent vertices are either sources or sinks. Webs are also subject to the relations in Equation (6) below, which are often referred to as the circle, bigon, and square relations. (Reduced webs are those with no circles, squares or bigons. For simplicity, we've given these relations in the classical limit.)



(6)

As mentioned in the introduction, there is a Hecke algebra action on invariant webs for tensor powers of V^+. We will work in the classical limit, where this becomes a symmetric group action. In this case, a crossing in the symmetric group reduces to the morphism given in Equation (7). In other words, to act on some invariant web by s_i, we attach the diagram on the right to the i and $i+1$ vertices of the web. This gives a sum of two webs which may or may not be reduced. Reducing summands may give a sum of more than two reduced webs. Petersen et al. (2009) prove that this action is, up to isomorphism, the irreducible S_{3n}-representation corresponding to the partition $[n, n, n]$. (See Theorem 4 above.) For an example of computing the action of a permutation on a web, see Figure 7 on Page 901. Because this computation involves two crossings, each of which can be "smoothed" in two ways as per Figure 2, we initially obtain four web terms. Subsequently, we reduce out squares and bigons and simplify.



(7)

Fig. 2: The symmetric group crossing morphism in the classical sl_3-spider

From now on, we will sometimes omit orientations in our graphs. Orientations are uniquely determined by the fact that edges point away from the bottom boundary.

7 Kuperberg's Bijection

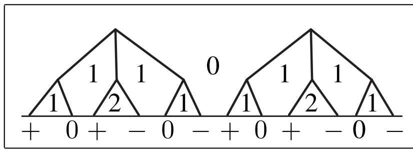
Kuperberg introduced a bijection between webs and dominant lattice paths in the weight lattice of $\mathfrak{sl}_3$. For webs with $3n$ source vertices, this may be interpreted as a bijection between standard tableaux of shape $[n, n, n]$ and reduced webs (Petersen et al. (2009)).

The map sends each web to a *Yamanouchi word* which is then used to build a standard tableau. Given a tableau T of shape $[n, n, n]$, the Yamanouchi word $y_T = y_1 y_2 \cdots y_{3n-1} y_{3n}$ is a string of symbols in the alphabet $\{+, 0, -\}$ where

$$y_i = \begin{cases} + & \text{if } i \text{ is in the top row of } T, \\ 0 & \text{if } i \text{ is in the middle row of } T, \text{ and} \\ - & \text{if } i \text{ is in the bottom row of } T. \end{cases}$$

As an example, the Yamanouchi word for the tableau T in Figure 4 is $y_T = +0+-0-$. Yamanouchi words corresponding to standard fillings of shape $[n, n, n]$ are completely characterized by two properties: They have n of each symbol, and at any point in the word, the number of $+$'s is greater than or equal to the number of 0's which is greater than or equal to the number of $-$'s. These words are called balanced. The algorithm to build Yamanouchi words from webs is as follows:

Start with a reduced web W on $3n$ source vertices aligned along a horizontal line with W drawn in the upper half plane. The horizontal line containing the vertices and edges in the web divides the upper half plane into faces, with one infinite face; label the infinite face 0. Label each additional face with the minimum number of edges that a path must cross to reach this face from the infinite face. Under each base vertex, write $+$, 0 or $-$ to indicate that the labels on the faces directly above the vertex increase, stay the same or decrease as we read from left to right. The string under the horizontal line is the Yamanouchi word of a standard $[n, n, n]$ tableau T. Complete the algorithm by writing down T. We demonstrate this computation in Figure 3.



1	3	7	9
2	5	8	11
4	6	10	12

Fig. 3: A web and its Yamanouchi word obtained from depth labels, with the corresponding standard tableau.

Khovanov and Kuperberg (1999) introduced a method for computing the inverse map, but Tymoczko (2012) recently developed a much simpler approach in terms of *m-diagrams*. Given a tableau T, construct an m-diagram m_T as follows:

Draw a horizontal line with $3n$ equally spaced dots labeled from left to right with the numbers $1, \ldots, 3n$. This line forms the lower boundary for the diagram, and all arcs will lie above it. Starting with the smallest number j on the second row, draw a semi-circular arc connecting j to its nearest unoccupied neighbor i to the left that appears in the first row. The arcs (i, j) are the left arcs in the m-diagram. Starting with the smallest number k on the bottom row, draw a semi-circular arc connecting k to its nearest neighbor j to the left that appears in the second row and does not already have an arc coming to it from the left. The arcs (j, k) are the right arcs of the m-diagram. The collection of left arcs is nonintersecting as is the collection of right arcs, but left arcs can intersect right arcs. Figure 4 shows an example of an m-diagram.

1	3
2	5
4	6

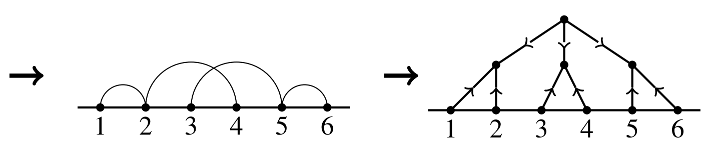


Fig. 4: The m-diagram and web corresponding to a tableau.

From an m-diagram m_T for T, the following straightforward process transforms m_T into an irreducible web W_T. Figure 4 shows a web corresponding to an m-diagram.

At each boundary vertex where two semi-circular arcs meet, replace the portion of the diagram in a small neighborhood of the vertex with a ‘Y‘ shape as shown in Figure 5. Orient all arcs away from the boundary so that the branching point of each ‘Y’ becomes a source. Finally replace any 4-valent intersection point of a left arc and a right arc with a pair of trivalent vertices as shown in Figure 6. There is a unique way to do this preserving orientation of incoming arcs.

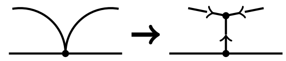


Fig. 5: Modifying the middle vertex of an m.

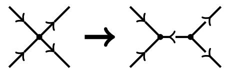


Fig. 6: Replacing a 4-valent vertex with trivalent vertices.

8 Kuperberg's Bijection as Robinson–Schensted Analogue

Recall that $\tau(T)$ is the set of all simple transpositions s_i for which i is in a row above $i+1$ in the tableau T. Our first observation is that $\tau(T)$ is completely determined by looking at W_T.

Lemma 7 *Given a standard tableau T and its associated web W_T, the set $\tau(T)$ consists of all transpositions s_i for which boundary vertices i and $i+1$ are directly connected to the same internal vertex in W_T. Furthermore, $s_i \in \tau(T)$ if and only if $s_i \cdot W_T = -W_T$.*

Thus, we can define $\tau(W)$ for any web W, and this notion agrees with our existing definitions for tableaux and Kazhdan–Lusztig left cell basis elements. In addition, we can define the set $D^{\mathrm{web}}_{i,j}$ as reduced webs whose τ-invariants contain s_i but not s_j.

Lemma 8 *Let $T \in D^{\mathrm{YT}}_{i,j}$. Then $s_j \cdot W_T = W_T + W_{f^{\mathrm{YT}}_{i,j}(T)} + O$ where O is a $\mathbb{Z}$-linear combination of reduced webs, none of which lies in $D^{\mathrm{web}}_{j,i}$.*

Using Lemma 8, we get a definition for $f^{\mathrm{web}}_{i,j}$, which is essentially the same as Definition 8 from the Kazhdan–Lusztig left cell setting:

Definition 9 *Given a reduced web $W \in D^{\mathrm{web}}_{i,j}$, let $f^{\mathrm{web}}_{i,j}(W)$ be the unique reduced web in $D^{\mathrm{web}}_{j,i}$ which appears as a summand in $s_j \cdot W$. This defines a bijection $f^{\mathrm{web}}_{i,j} : D^{\mathrm{web}}_{i,j} \to D^{\mathrm{web}}_{j,i}$ with inverse $f^{\mathrm{web}}_{j,i} : D^{\mathrm{web}}_{j,i} \to D^{\mathrm{web}}_{i,j}$.*

Of course, $f^{\mathrm{web}}_{i,j}(W_T) = W_{f^{\mathrm{YT}}_{i,j}(T)}$. We then obtain generalized τ-invariants for reduced webs by replacing the $f^{\mathrm{YT}}_{i,j}$ maps in Definition 1 with $f^{\mathrm{web}}_{i,j}$. Notice that the definition of the generalized τ-invariant for a reduced web is essentially identical to the *in situ* version for Kazhdan–Lusztig left cell basis elements in terms of the S_n-action. Combining Lemmas 7 and 8 with Theorem 1 brings us to the main theorem of the paper:

Theorem 8 (Robinson–Schensted for Webs) *Kuperberg's bijection carries a reduced web W on $3n$ source vertices to the unique $[n, n, n]$ standard tableau T satisfying $\tau_g(T) = \tau_g(W)$.*

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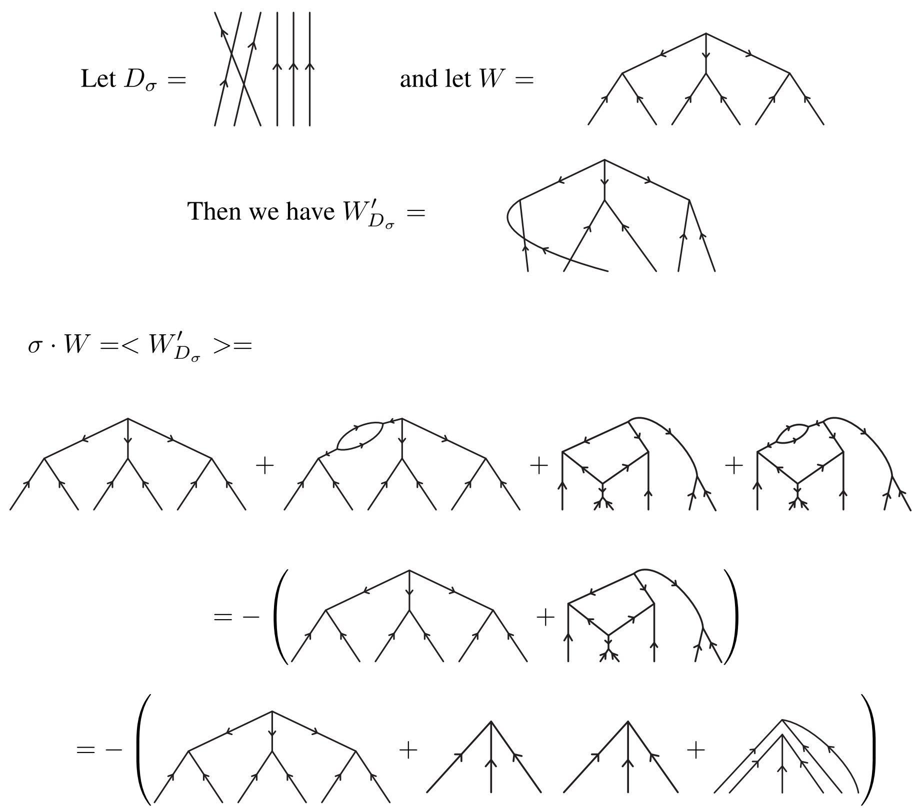

Fig. 7: Computing the action of a permutation on a web.

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Periodic Patterns of Signed Shifts

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Abstract. The periodic patterns of a map are the permutations realized by the relative order of the points in its periodic orbits. We give a combinatorial description of the periodic patterns of an arbitrary signed shift, in terms of the structure of the descent set of a certain transformation of the pattern. Signed shifts are an important family of one-dimensional dynamical systems. For particular types of signed shifts, namely shift maps, reverse shift maps, and the tent map, we give exact enumeration formulas for their periodic patterns. As a byproduct of our work, we recover some results of Gessel and Reutenauer and obtain new results on the enumeration of pattern-avoiding cycles.

Résumé. Les motifs périodiques d'une fonction sont les permutations réalisées par l'ordre relatif des points dans ses orbites périodiques. Nous donnons une description combinatoire des motifs périodiques d'un *shift* signé arbitraire, en termes de la structure de l'ensemble des descentes d'une certaine transformation du motif. Les *shifts* signés sont une familie importante de systèmes dynamiques unidimensionnels. Pour des types particuliers de *shifts* signés, comme les fonctions de *shift*, les fonctions de *shift* inversées, et la fonction de tente, nous donnons des formules exactes pour l'énumération de leurs motifs périodiques. Comme sous-produit de notre travail, nous retrouvons des résultats de Gessel et Reutenauer et obtenons de nouveaux résultats sur l'énumération de cycles qui évitent certain motifs.

Keywords: periodic pattern; signed shift; cyclic permutation; descent; pattern avoidance.

1 Introduction

1.1 Background and motivation

Permutations realized by the orbits of a map on a one-dimensional interval have received a significant amount of attention in the last five years [2]. These are the permutations given by the relative order of the elements of the sequence obtained by successively iterating the map, starting from any point in the interval. One the one hand, understanding these permutations provides a powerful tool to distinguish random from deterministic time series, based on the remarkable fact [6] that every piecewise monotone map has forbidden patterns, i.e., permutations that are not realized by any orbit. Permutation-based tests for this purpose have been developed in [4]. On the other hand, the set of permutations realized by a map (also called allowed patterns) has a rich combinatorial structure. The answer to certain enumerative questions, often involving pattern-avoiding permutations, provides information about the associated dynamical systems. For example, determining the asymptotic growth of the number of allowed patterns of a map reveals its so-called topological entropy, an important measure of the complexity of the system.

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The dynamical systems most commonly studied from the perspective of forbidden patterns are shifts, and more generally signed shifts [1], a large family of maps that includes the tent map (which is equivalent to the logistic map). As we will see, signed shifts have a simple discrete structure which makes them amenable to a combinatorial approach, yet they include many important chaotic dynamical systems.

Permutations realized by shifts were first considered in [3], and later characterized and enumerated in [8]. More recently, permutations realized by the more general β-shifts have been studied in [9]. For the logistic map, some properties of their set of forbidden patterns were given in [10].

If instead of considering an arbitrary initial point in the domain of the map we restrict our attention to periodic points, the permutations realized by the relative order of the entries in the corresponding orbits (up until the first repetition) are called *periodic patterns*. In the case of continuous maps, Sharkovskii's theorem [13] gives a beautiful characterization of the possible periods of these orbits. More refined results that consider which periodic patterns are forced by others are known for continuous maps [7, 12]. However, little is known when the maps are not continuous, as is the case for shifts and, more generally, signed shifts.

The subject of study of this paper are periodic patterns of signed shifts. Our main result is a characterization of the periodic patterns of an (almost) arbitrary signed shift, given in Theorem 2.1. For some particular cases of signed shifts we obtain exact enumeration formulas: the number of periodic patterns of the tent map is given in Theorem 3.2, and the number of periodic patterns of the (unsigned) shift map is given in Theorem 3.5. For the reverse shift, which is not covered in our main theorem, the number of periodic patterns is studied in Sections 3.3 and 3.4.

An interesting consequence of our study of periodic patterns is that we obtain new results regarding the enumeration of cyclic permutations that avoid certain patterns. These are described in Section 4.

1.2 Periodic patterns

Given a linearly ordered set X and a map $f : X \to X$, consider the sequence $\{f^i(x)\}_{i\geq 0}$ obtained by iterating the function starting at a point $x \in X$. If there are no repetitions among the first n elements of this sequence, called the *orbit* of x, then we define the *pattern* of length n of f at x to be

$$\mathrm{Pat}(x, f, n) = \mathrm{st}(x, f(x), f^2(x), \ldots, f^{n-1}(x)),$$

where st is the operation that outputs the permutation of $[n] = \{1, 2, \ldots, n\}$ whose entries are in the same relative order as n entries in the input. For example, $\mathrm{st}(3.3, 3.7, 9, 6, 0.2) = 23541$. If $f^i(x) = f^j(x)$ for some $0 \leq i < j < n$, then $\mathrm{Pat}(x, f, n)$ is not defined. The set of *allowed patterns* of f is $\mathcal{A}(f) = \{\mathrm{Pat}(x, f, n) : n \geq 0, x \in X\}$.

We say that $x \in X$ is an *n-periodic point* of f if $f^n(x) = x$ but $f^i(x) \neq x$ for $1 \leq i < n$. In this case, the permutation $\mathrm{Pat}(x, f, n)$ is denoted $\mathrm{PP}(x, f)$, and called the *periodic pattern* of f at x. Note that if x is an n-periodic point, then $\mathrm{Pat}(x, f, i)$ is not defined for $i > n$. Let $\mathcal{P}(f) = \{\mathrm{PP}(x, f) : x \in X\}$ be the set of *periodic patterns* of f, and let $\mathcal{P}_n(f) = \mathcal{P}(f) \cap \mathcal{S}_n$. For a permutation $\pi = \pi_1\pi_2 \ldots \pi_n \in \mathcal{S}_n$, let $[\pi] = \{\pi_i\pi_{i+1} \ldots \pi_n\pi_1 \ldots \pi_{i-1} : 1 \leq i \leq n\}$ the set of cyclic rotations of π, which we call the equivalence class of π. It is clear that if $\pi \in \mathcal{P}(f)$, then $[\pi] \subset \mathcal{P}(f)$. Indeed, if π is the periodic pattern at a point x, then the other permutations in $[\pi]$ are realized at the other points in the periodic orbit of x. Let $\overline{\mathcal{P}}_n(f) = \{[\pi] : \pi \in \mathcal{P}_n(f)\}$ denote the set of equivalence classes of periodic patterns of f of length n, and let $p_n(f) = |\overline{\mathcal{P}}_n(f)| = |\mathcal{P}_n(f)|/n$.

Given linearly ordered sets X and Y, two maps $f : X \to X$ and $g : Y \to Y$ are said to be *order-isomorphic* if there is an order-preserving bijection $\phi : X \to Y$ such that $\phi \circ f = g \circ \phi$. In this case, $\mathrm{Pat}(x, f, n) = \mathrm{Pat}(\phi(x), g, n)$ for every $x \in X$ and $n \geq 1$. In particular, $\mathcal{A}(f) = \mathcal{A}(g)$ and $\mathcal{P}(f) = \mathcal{P}(g)$.

1.3 Signed shifts

Let $k \geq 2$ be fixed, and let $\mathcal{W}_k$ be the set of infinite words $s = s_1 s_2 \dots$ over the alphabet $\{0, 1, \dots, k-1\}$. Let $<_{\mathrm{lex}}$ denote the lexicographic order on these words. We use the notation $s_{[i,\infty)} = s_i s_{i+1} \dots$, and $\bar{s_i} = k - 1 - s_i$. If q is a finite word, q^m denotes concatenation of q with itself m times, and q^∞ is an infinite periodic word.

Fix $\sigma = \sigma_0 \sigma_1 \dots \sigma_{k-1} \in \{+, -\}^k$. Let $T_\sigma^+ = \{t : \sigma_t = +\}$ and $T_\sigma^- = \{t : \sigma_t = -\}$, and note that these sets form a partition of $\{0, 1, \dots, k-1\}$. We give two definitions of the signed shift with signature σ, and show that they are order-isomorphic to each other.

The first definition, which we denote by Σ'_σ, is the map $\Sigma'_\sigma : (\mathcal{W}_k, <_{\mathrm{lex}}) \to (\mathcal{W}_k, <_{\mathrm{lex}})$ defined by

$$\Sigma'_\sigma(s_1 s_2 s_3 s_4 \dots) = \begin{cases} s_2 s_3 s_4 \dots & \text{if } s_1 \in T_\sigma^+, \\ \bar{s_2} \bar{s_3} \bar{s_4} \dots & \text{if } s_1 \in T_\sigma^-. \end{cases}$$

It is shown in [1] that Σ'_σ is order-isomorphic to the piecewise linear function $M_\sigma : [0, 1] \to [0, 1]$ defined for $x \in [\frac{t}{k}, \frac{t+1}{k})$, for each $0 \leq t \leq k - 1$, as

$$M_\sigma(x) = \begin{cases} kx - t & \text{if } t \in T_\sigma^+, \\ t + 1 - kx & \text{if } t \in T_\sigma^-. \end{cases}$$

As a consequence, the allowed patterns and the periodic patterns of Σ'_σ are the same as those of M_σ, respectively. A few examples of the function M_σ are pictured in Figure 1.

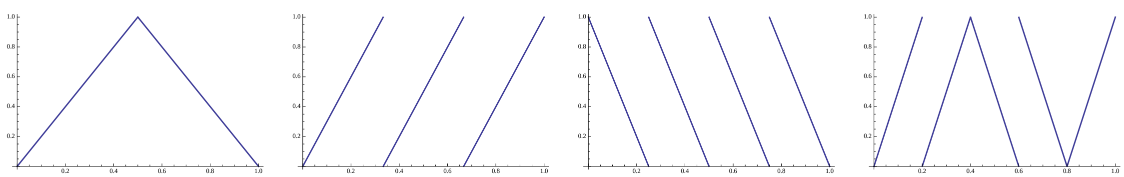

Fig. 1: The graphs of M_σ for $\sigma = +-$, $\sigma = +++$, $\sigma = ----$ and $\sigma = ++--+$, respectively.

The second definition of the signed shift will be more convenient when studying its periodic patterns. Let $\prec_\sigma$ be the linear order on $\mathcal{W}_k$ defined by $s = s_1 s_2 s_3 \dots \prec_\sigma t_1 t_2 t_3 \dots = t$ if either $s_1 < t_1$, $s_1 = t_1 \in T_\sigma^+$ and $s_2 s_3 \dots \prec_\sigma t_2 t_3 \dots$, or $s_1 = t_1 \in T_\sigma^-$ and $t_2 t_3 \dots \prec_\sigma s_2 s_3 \dots$. Equivalently, $s \prec_\sigma t$ if, letting $j \geq 1$ be the smallest such that $s_j \neq t_j$, either $c := |\{1 \leq i < j : s_i \in T_\sigma^-\}|$ is even and $s_j < t_j$, or c is odd and $s_j > t_j$. The signed shift is the map $\Sigma_\sigma : (\mathcal{W}_k, \prec_\sigma) \to (\mathcal{W}_k, \prec_\sigma)$ defined simply by $\Sigma_\sigma(s_1 s_2 s_3 s_4 \dots) = s_2 s_3 s_4 \dots$.

To show that the two definitions of the signed shift as Σ_σ and Σ'_σ are order-isomorphic, consider the order-preserving bijection $\psi_\sigma : (\mathcal{W}_k, \prec_\sigma) \to (\mathcal{W}_k, <_{\mathrm{lex}})$ that maps a word $s = s_1 s_2 s_3 \dots$ to the word $a = a_1 a_2 a_3 \dots$ where

$$a_i = \begin{cases} s_i & \text{if } |\{j < i : s_j \in T_\sigma^-\}| \text{ is even}, \\ \bar{s_i} & \text{if } |\{j < i : s_j \in T_\sigma^-\}| \text{ is odd}. \end{cases}$$

It is easy to check that $\psi_\sigma \circ \Sigma_\sigma = \Sigma'_\sigma \circ \psi_\sigma$, and so $\mathcal{P}(\Sigma_\sigma) = \mathcal{P}(\Sigma'_\sigma)$. From now on we use the second definition Σ_σ only. The n-periodic points of Σ_σ are the words of the form $s = (s_1 s_2 s_3 \dots s_n)^\infty$ where $s_1 s_2 \dots s_n$ is a *primitive* word (sometimes called *aperiodic* word), that is, not a concatenation of copies of a strictly shorter word. Counting these words up to cyclic rotation, we obtain the following result, where μ denotes the Möbius function.

Lemma 1.1 *If $\sigma \in \{+,-\}^k$, where $k \geq 2$, the number of periodic orbits of size n of Σ_σ is*

$$L_k(n) = \frac{1}{n} \sum_{d|n} \mu(d) k^{\frac{n}{d}}.$$

For example, if $\sigma = +--$, then $s = (00110221)^\infty$ is an 8-periodic point of Σ_σ, and $\mathrm{PP}(s, \Sigma_\sigma) = 12453786$, so $12453786 \in \mathcal{P}(\Sigma_{+--})$. One of our main goals is to characterize the sets $\mathcal{P}(\Sigma_\sigma)$.

If $\sigma = +^k$, then $\prec_\sigma$ is the lexicographic order $<_{\mathrm{lex}}$, and Σ_σ is called the *k-shift*. When $\sigma = -^k$, the map Σ_σ is called the *reverse k-shift*. When $\sigma = +-$, the map Σ_σ is the well-known *tent map*.

1.4 Pattern avoidance

Let $\mathcal{S}_n$ denote the set of permutations of $[n]$, and let $\mathcal{S} = \bigcup_{n\geq 0} \mathcal{S}_n$. We write permutations in one line notation as $\pi = \pi_1 \pi_2 \dots \pi_n \in \mathcal{S}_n$. We say that $\tau \in \mathcal{S}_n$ *contains* $\rho \in \mathcal{S}_m$ if there exist indices $i_1 < i_2 < \cdots < i_m$ such that $\mathrm{st}(\tau_{i_1} \tau_{i_2} \dots \tau_{i_m}) = \rho_1 \rho_2 \dots \rho_m$. Otherwise, we say that τ *avoids* ρ. If $\mathscr{A}$ is a set of permutations, we denote by $\mathscr{A}(\rho)$ the set of permutations in $\mathscr{A}$ avoiding ρ, and we define $\mathscr{A}(\rho^{(1)}, \rho^{(2)}, \dots)$ analogously as the set of permutations avoiding all the patterns $\rho^{(1)}, \rho^{(2)}, \dots$. We say that $\mathscr{A}$ is a *(permutation) class* if it is closed under pattern containment, that is, if $\tau \in \mathscr{A}$ and τ contains ρ, then $\rho \in \mathscr{A}$. Sets of the form $\mathcal{S}(\rho^{(1)}, \rho^{(2)}, \dots)$ are permutation classes.

Given classes $\mathscr{A}_0, \mathscr{A}_1, \dots, \mathscr{A}_{k-1}$, their juxtaposition, denoted $[\mathscr{A}_0\, \mathscr{A}_1\, \dots\, \mathscr{A}_{k-1}]$, is the set of permutations that can be expressed as concatenations $\alpha_0 \alpha_1 \dots \alpha_{k-1}$ where $\mathrm{st}(\alpha_t) \in \mathscr{A}_t$ for all $0 \leq t < k$. For example, $[\mathcal{S}(21)\, \mathcal{S}(12)]$ is the set of *unimodal* permutations, i.e., those $\pi \in \mathcal{S}_n$ satisfying $\pi_1 < \pi_2 < \cdots < \pi_j > \pi_{j+1} > \cdots > \pi_n$ for some $1 \leq j \leq n$. The juxtaposition of permutation classes is again a class, and as such, it can be characterized in terms of pattern avoidance. For example, $[\mathcal{S}(21)\, \mathcal{S}(12)] = \mathcal{S}(213, 312)$. Atkinson [5] showed that if $\mathscr{A}_t$ can be characterized by avoidance of a finite set of patterns for each t, then the same is true for $[\mathscr{A}_0\, \mathscr{A}_1\, \dots\, \mathscr{A}_{k-1}]$.

Let $\sigma = \sigma_0 \sigma_1 \dots \sigma_{k-1} \in \{+,-\}^k$ as before. We let $\mathcal{S}^\sigma = [\mathscr{A}_0\ \mathscr{A}_1\ \dots\ \mathscr{A}_{k-1}]$ where, for $0 \leq t < k$,

$$\mathscr{A}_t = \begin{cases} \mathcal{S}(21) & \text{if } \sigma_t = +, \\ \mathcal{S}(12) & \text{if } \sigma_t = -. \end{cases}$$

Let $\mathcal{S}^\sigma_n = \mathcal{S}^\sigma \cap \mathcal{S}_n$. Figure 2 shows two permutations in $\mathcal{S}^{+--}$. Note that since the empty permutation belongs to $\mathcal{S}(21)$ and to $\mathcal{S}(12)$, it is trivial that $\mathcal{S}^{+--} \subset \mathcal{S}^{+-+-}$, for example.

We denote by $\mathcal{C}_n$ (respectively, $\mathcal{C}^\sigma$, $\mathcal{C}^\sigma_n$) the set of cyclic permutations in $\mathcal{S}_n$ (respectively, $\mathcal{S}^\sigma$, $\mathcal{S}^\sigma_n$). In Figure 2, the permutation on the right is in $\mathcal{C}^\sigma$. It will be useful to define the map

$$\begin{array}{cccc} \theta: & \mathcal{S}_n & \to & \mathcal{C}_n \\ & \pi & \mapsto & \hat{\pi}, \end{array}$$

where if $\pi = \pi_1 \pi_2 \dots \pi_n$ in one-line notation, then $\hat{\pi} = (\pi_1, \pi_2, \dots, \pi_n)$ in cycle notation, that is, $\hat{\pi}$ is the cyclic permutation that sends π_1 to π_2, π_2 to π_3, and so on. Writing $\hat{\pi} = \hat{\pi}_1 \hat{\pi}_2 \dots \hat{\pi}_n$ in one-line

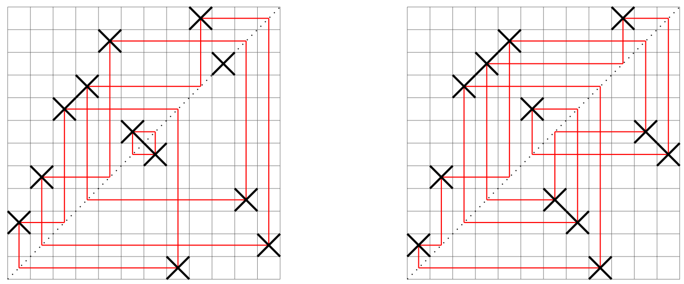

Fig. 2: Two permutations 3 5 8 9 11 7 6 1 12 10 4 2 and 2 5 9 10 11 8 4 3 1 12 7 6 in $\mathcal{S}^{+--}$ and their cycle structure.

notation, we have that $\hat{\pi}_{\pi_i} = \pi_{i+1}$ for $1 \le i \le n$, with the convention that $\pi_{n+1} := \pi_1$. The map θ also plays an important role in [8]. Note that if $\pi \in \mathcal{S}_n$, then $\theta^{-1}(\hat{\pi}) = [\pi]$, the set of cyclic rotations of π.

2 Description of periodic patterns of the signed shift

The main theorem of this paper is the following characterization of the periodic patterns of the signed shift Σ_σ, except in the case of the reverse shift. Throughout the paper we assume that $k \ge 2$.

Theorem 2.1 *Let $\sigma \in \{+,-\}^k$, $\sigma \ne -^k$. Then $\pi \in \mathcal{P}(\Sigma_\sigma)$ if and only if $\hat{\pi} \in \mathcal{C}^\sigma$.*

This theorem, whose proof will require a few lemmas, states that the map θ gives a bijection between $\overline{\mathcal{P}}_n(\Sigma_\sigma)$ and $\mathcal{C}^\sigma_n$. The following lemma describes some conditions satisfied by the periodic patterns of Σ_σ, proving the forward direction of Theorem 2.1.

Lemma 2.2 *Let $\sigma \in \{+,-\}^k$, let $\pi \in \mathcal{P}_n(\Sigma_\sigma)$, and let $s = (s_1 \dots s_n)^\infty \in \mathcal{W}_k$ be such that $\pi = \mathrm{PP}(s, \Sigma_\sigma)$. For $1 \le t \le k$, let $d_t = |\{i \in [n] : s_i < t\}|$, and let $d_0 = 0$. The following statements hold.*

(a) For every i and t, we have $d_t < \pi_i \le d_{t+1}$ if and only if $s_i = t$.

(b) If $d_t < \pi_i < \pi_j \le d_{t+1}$, then $\pi_{i+1} < \pi_{j+1}$ if $t \in T^+_\sigma$, and $\pi_{i+1} > \pi_{j+1}$ if $t \in T^-_\sigma$, where $\pi_{n+1} := \pi_1$.

(c) For $0 \le t < k$, $\hat{\pi}_{d_t+1} \dots \hat{\pi}_{d_{t+1}}$ is increasing if $t \in T^+_\sigma$ and decreasing if $t \in T^-_\sigma$. In particular, $\hat{\pi} \in \mathcal{C}^\sigma$.

Proof: Since $\mathrm{PP}(s, \Sigma_\sigma) = \pi$, it is clear for all $a, b \in [n]$, $\pi_a < \pi_b$ implies $s_a \le s_b$, from where (a) follows. To prove (b), suppose that $d_t < \pi_i < \pi_j \le d_{t+1}$, and so $s_{[i,\infty)} \prec_\sigma s_{[j,\infty)}$. By part (a), we have $s_i = s_j = t$. If $t \in T^+_\sigma$, then $s_{[i+1,\infty)} \prec_\sigma s_{[j+1,\infty)}$, and so $\pi_{i+1} < \pi_{j+1}$. Similarly, if $t \in T^-_\sigma$, then $s_{[j+1,\infty)} \prec_\sigma s_{[i+1,\infty)}$, and so $\pi_{i+1} > \pi_{j+1}$.

Now let $0 \le t < k$, and suppose that the indices j such that $s_j = t$ are $j_1, \dots, j_m$, ordered in such a way that $\pi_{j_1} < \pi_{j_2} < \dots < \pi_{j_m}$, where $m = d_{t+1} - d_t$. Then part (a) implies that $\pi_{j_\ell} = d_{t-1} + \ell$ for $1 \le \ell \le m$, and part (b) implies that $\pi_{j_1+1} < \pi_{j_2+1} < \dots < \pi_{j_m+1}$ if $t \in T^+_\sigma$, and $\pi_{j_1+1} > \pi_{j_2+1} > \dots > \pi_{j_m+1}$ if $t \in T^-_\sigma$. Using that $\pi_{j_\ell+1} = \hat{\pi}_{\pi_{j_\ell}} = \hat{\pi}_{d_t+\ell}$, this is equivalent to $\hat{\pi}_{d_t+1} < \hat{\pi}_{d_t+2} < \dots < \hat{\pi}_{d_t+m}$ if $t \in T^+_\sigma$, and $\hat{\pi}_{d_t+1} > \hat{\pi}_{d_t+2} > \dots > \hat{\pi}_{d_t+m}$ if $t \in T^-_\sigma$. Note that $d_t + m = d_{t+1}$. Since $\hat{\pi}$ is a cyclic permutation, this proves that $\hat{\pi} \in \mathcal{C}^\sigma$. $\square$

The next two lemmas will be used in the proof of the backward direction of Theorem 2.1.

Lemma 2.3 *Let $\sigma \in \{+,-\}^k$, let $\pi \in \mathcal{S}_n$, and suppose that $\hat\pi = \hat\pi_{e_0+1}\dots\hat\pi_{e_1}\hat\pi_{e_1+1}\dots\hat\pi_{e_2}\dots\hat\pi_{e_k}$, where each segment $\hat\pi_{e_t+1}\dots\hat\pi_{e_{t+1}}$ is increasing if $t \in T_\sigma^+$ and decreasing if $t \in T_\sigma^-$ (and so $\hat\pi \in \mathcal{C}^\sigma$). Suppose that $e_t < \pi_i < \pi_j \le e_{t+1}$ for some $1 \le i, j \le n$. Then $\pi_{i+1} < \pi_{j+1}$ if $t \in T_\sigma^+$, and $\pi_{i+1} > \pi_{j+1}$ if $t \in T_\sigma^-$, where $\pi_{n+1} := \pi_1$.*

Proof: Since $e_t < \pi_i < \pi_j \le e_{t+1}$, both $\hat\pi_{\pi_i}$ and $\hat\pi_{\pi_j}$ lie in the segment $\hat\pi_{e_t+1}\dots\hat\pi_{e_{t+1}}$. If $t \in T_\sigma^+$, this segment is increasing, so $\pi_{i+1} = \hat\pi_{\pi_i} < \hat\pi_{\pi_j} = \pi_{j+1}$. The argument is analogous if $t \in T_\sigma^-$. □

Lemma 2.4 *Let $\sigma \in \{+,-\}^k$ be arbitrary. If $\sigma = -^k$, additionally assume that $n \ne 2 \bmod 4$. Let $\pi \in \mathcal{S}_n$ be such that $\hat\pi \in \mathcal{C}^\sigma$. Then there exist $0 = e_0 \le e_1 \le \dots \le e_k = n$ such that*

(a) each segment $\hat\pi_{e_t+1}\dots\hat\pi_{e_{t+1}}$ is increasing if $t \in T_\sigma^+$ and decreasing if $t \in T_\sigma^-$; and

(b) the word $s_1\dots s_n$, defined by $s_i = t$ whenever $e_t < \pi_i \le e_{t+1}$, is primitive, and $s = (s_1\dots s_n)^\infty$ satisfies $\mathrm{PP}(s, \Sigma_\sigma) = \pi$.

Furthermore, if $\sigma = +^k$ or $\sigma = -^k$, then any choice of $0 = e_0 \le e_1 \le \dots \le e_k = n$ satisfying (a) also satisfies (b).

Proof: Since $\hat\pi \in \mathcal{C}^\sigma$, there is some choice of $0 = e_0 \le e_1 \le \dots \le e_k = n$ such that each segment $\hat\pi_{e_t+1}\dots\hat\pi_{e_{t+1}}$ is increasing if $t \in T_\sigma^+$ and decreasing if $t \in T_\sigma^-$. Pick one such choice and define $s_1\dots s_n$ as above. In this proof we take the indices of π mod n, that is, we define $\pi_{i+jn} = \pi_i$ for $i \in [n]$.

Suppose that $s_1\dots s_n$ is not primitive, so it can be written as q^m for some $m \ge 2$ and some primitive word q with $|q| = r = n/m$. Then, $s_i = s_{i+r}$ for all i. Let $g = |\{i \in [r] : s_i \in T_\sigma^-\}|$. Fix i, and let $t = s_i = s_{i+r}$. Because of the way that $s_1\dots s_n$ is defined, we must have $e_t < \pi_i, \pi_{i+r} \le e_{t+1}$, so we can apply Lemma 2.3 to this pair.

Suppose first that g is even. If $\pi_i < \pi_{i+r}$, then applying Lemma 2.3 r times we get $\pi_{i+r} < \pi_{i+2r}$, since the inequality involving $\pi_{i+\ell}$ and $\pi_{i+r+\ell}$ switches exactly g times as ℓ increases from 0 to r. Starting with $i = 1$ and applying this argument repeatedly, we see that if $\pi_1 < \pi_{1+r}$, then $\pi_1 < \pi_{1+r} < \pi_{1+2r} < \dots < \pi_{1+(m-1)r} < \pi_{1+mr} = \pi_1$, which is a contradiction. A symmetric argument shows that if $\pi_1 > \pi_{1+r}$, then $\pi_1 > \pi_{1+r} > \pi_{1+2r} > \dots > \pi_{1+(m-1)r} > \pi_{1+mr} = \pi_1$.

It remains to consider the case that g is odd. If m is even and $m \ge 4$, then letting $q' = qq$ we have $s_1s_2\dots s_n = (q')^{\frac{m}{2}}$. Letting $r' = |q'| = 2r$ and $g' = |\{i \in [2r] : s_i \in T_\sigma^-\}| = 2g$, the same argument as above using r' and g' yields a contradiction. If m is odd, suppose without loss of generality that $\pi_1 < \pi_{1+r}$. Applying Lemma 2.3 r times to the inequality $\pi_i < \pi_{i+r}$ (respectively $\pi_i > \pi_{i+r}$) yields $\pi_{i+r} > \pi_{i+2r}$ (respectively $\pi_{i+r} < \pi_{i+2r}$) in this case, since the inequality involving $\pi_{i+\ell}$ and $\pi_{i+r+\ell}$ switches an odd number of times. Consider two cases:

- If $\pi_1 < \pi_{1+2r}$, then Lemma 2.3 applied repeatedly in blocks of $2r$ times yields $\pi_1 < \pi_{1+2r} < \pi_{1+4r} < \dots < \pi_{1+(m-1)r}$. Applying now Lemma 2.3 r times starting with $\pi_1 < \pi_{1+(m-1)r}$ gives $\pi_{1+r} > \pi_{1+mr} = \pi_1$, which contradicts the assumption $\pi_1 < \pi_{1+r}$.
- If $\pi_1 > \pi_{1+2r}$, applying Lemma 2.3 r times we get $\pi_{1+r} < \pi_{1+3r}$, and by repeated application of the lemma in blocks of $2r$ times it follows that $\pi_{1+r} < \pi_{1+3r} < \pi_{1+5r} < \dots < \pi_{1+(m-2)r} < \pi_{1+mr} = \pi_1$, contradicting again the assumption $\pi_1 < \pi_{1+r}$.

The only case left is when g is odd and $m = 2$, that is, when $s_1 s_2 \dots s_n = q^2$ and q has an odd number of letters in T_σ^-. Note that this situation does not happen when $\sigma = +^k$ (since in this case $g = 0$) and, although it can happen when $\sigma = -^k$, in this case we would have that $T_\sigma^- = \{0, 1, \dots, k-1\}$, and so $n = 2r = 2g = 2 \bmod 4$, which we are excluding in the statement of the theorem.

Thus, we can assume that there exists some $1 \le \ell < k$ such that $\sigma_{\ell-1}\sigma_\ell$ is either $+-$ or $-+$. We will show that there is a choice of $0 = e'_0 \le e'_1 \le \dots \le e'_k = n$ that satisfies the conditions of the lemma, and the resulting word $s'_1 s'_2 \dots s'_n$ is primitive.

Suppose that $\sigma_{\ell-1}\sigma_\ell = +-$ (the case $\sigma_{\ell-1}\sigma_\ell = -+$ is very similar). Then, $\hat\pi_{e_{\ell-1}+1} < \dots < \hat\pi_{e_\ell}$ and $\hat\pi_{e_\ell+1} > \dots > \hat\pi_{e_{\ell+1}}$. If $\hat\pi_{e_\ell} < \hat\pi_{e_\ell+1}$ (respectively, $\hat\pi_{e_\ell} > \hat\pi_{e_\ell+1}$), let $e'_\ell := e_\ell + 1$ (respectively, $e'_\ell := e_\ell - 1$), and $e'_t := e_t$ for all $t \ne \ell$. Clearly, the values e'_t satisfy part (a) of the lemma. Additionally, the word $s'_1 \dots s'_n$ that they define using part (b) differs from the original word $s_1 s_2 \dots s_n = q^2$ by one entry, making the number of ℓs that appear in $s'_1 \dots s'_n$ be odd instead of even. Thus, $s'_1 \dots s'_n$ can no longer be written as $(q')^2$ for any q', so it is primitive by the above argument.

Finally, we prove that if we let $s = (s_1 \dots s_n)^\infty$, then $\mathrm{PP}(s, \Sigma_\sigma) = \pi$. Let $1 \le i, j \le n$ with $\pi_i < \pi_j$. We need to show that $s_{[i,\infty)} \prec_\sigma s_{[j,\infty)}$. Let $a \ge 0$ be the smallest such that $s_{i+a} \ne s_{j+a}$, and let $h = |\{0 \le \ell \le a-1 : s_\ell \in T_\sigma^-\}|$. If h is even, then Lemma 2.3 applied a times shows that $\pi_{i+a} < \pi_{j+a}$. Since $s_{i+a} \ne s_{j+a}$, we must then have $s_{i+a} < s_{j+a}$, by construction of s. Thus, $s_{[i,\infty)} \prec_\sigma s_{[j,\infty)}$ by definition of $\prec_\sigma$, since the word $s_i s_{i+1} \dots s_{i+a-1} = s_j s_{j+1} \dots s_{j+a-1}$ has an even number of letters in T_σ^-. Similarly, if h is odd, then Lemma 2.3 shows that $\pi_{i+a} > \pi_{j+a}$. Since $s_{i+a} \ne s_{j+a}$, we must have $s_{i+a} > s_{j+a}$, and thus $s_{[i,\infty)} \prec_\sigma s_{[j,\infty)}$ by definition of $\prec_\sigma$. □

We can now combine the above lemmas to prove our main theorem.

Proof of Theorem 2.1: If $\pi \in \mathcal{P}(\Sigma_\sigma)$, then $\hat\pi \in \mathcal{C}^\sigma$ by Lemma 2.2(c). Conversely, $\pi \in \mathcal{S}_n$ is such that $\hat\pi \in \mathcal{C}^\sigma$, then the word s given by Lemma 2.4(b) satisfies $\mathrm{PP}(s, \Sigma_\sigma) = \pi$, and so $\pi \in \mathcal{P}(\Sigma_\sigma)$. □

For $\sigma = -^k$, the same proof yields the following weaker result.

Proposition 2.5 *Let $\sigma = -^k$. If $\pi \in \mathcal{P}_n(\Sigma_\sigma)$, then $\hat\pi \in \mathcal{C}_n^\sigma$. Additionally, the converse holds if $n \ne 2 \bmod 4$.*

Define the reversal of $\sigma = \sigma_0\sigma_1 \dots \sigma_{k-1}$ to be $\sigma^R = \sigma_{k-1} \dots \sigma_1\sigma_0$. If $\pi \in \mathcal{S}_n$, then the complement of π is the permutation π^c where $\pi_i^c = n + 1 - \pi_i$ for $1 \le i \le n$. The following result, whose proof is omitted, relates periodic patterns of Σ_σ and Σ_{σ^R}.

Proposition 2.6 *For every $\sigma \in \{+,-\}^k$, $\pi \in \mathcal{P}(\Sigma_\sigma)$ if and only if $\pi^c \in \mathcal{P}(\Sigma_{\sigma^R})$.*

3 Enumeration for special cases

For particular values of σ, we can give a formula for the number of periodic patterns of Σ_σ. This is the case when $\sigma = +-$, $\sigma = +^k$, and $\sigma = -^k$, for any $k \ge 2$.

3.1 The tent map

We denote the tent map by $\Lambda = \Sigma_{+-}$. The characterization of the periodic patterns of Λ follows from Theorem 2.1.

Corollary 3.1 *$\pi \in \mathcal{P}(\Lambda)$ if and only if $\hat\pi$ is unimodal.*

Next we give an exact formula for the number of periodic patterns of the tent map.

Theorem 3.2

$$p_n(\Lambda) = \frac{1}{2n} \sum_{\substack{d|n \\ d \text{ odd}}} \mu(d) 2^{\frac{n}{d}}.$$

Proof: Let $\mathcal{O}_n$ be the set of binary words $s = (s_1 \dots s_n)^\infty$ where $s_1 \dots s_n$ is primitive and has an odd number of ones. We will show that the map $s \mapsto \mathrm{PP}(s, \Lambda)$ is a bijection between $\mathcal{O}_n$ and $\mathcal{P}_n(\Lambda)$. It is clear that this map is well defined. We will prove that for each $\pi \in \mathcal{P}_n(\Lambda)$ there are either one or two periodic binary words s such that $\mathrm{PP}(s, \Lambda) = \pi$, and that exactly one of them is in $\mathcal{O}_n$.

Fix $\pi \in \mathcal{P}_n(\Lambda)$, and recall from Corollary 3.1 that $\hat{\pi}_1 < \hat{\pi}_2 < \cdots < \hat{\pi}_m > \hat{\pi}_{m+1} > \cdots > \hat{\pi}_n$ for some m. Let $s = (s_1 \dots s_n)^\infty$ be such that $\mathrm{PP}(s, \Lambda) = \pi$, and let $d = |\{1 \le i \le n : s_i = 0\}|$. By Lemma 2.2(a), we have that $s_i = 0$ if and only if $\pi_i \le d$. Suppose now that $s_i = s_j$ and $\pi_i < \pi_j$. If $s_i = s_j = 0$, then $\pi_{i+1} < \pi_{j+1}$ by Lemma 2.2(b), and so $\hat{\pi}_{\pi_i} < \hat{\pi}_{\pi_j}$. Since this holds whenever $1 \le \pi_i < \pi_j \le d$, we see that $\hat{\pi}_1 < \hat{\pi}_2 < \cdots < \hat{\pi}_d$. Similarly, if $s_i = s_j = 1$, then $\pi_{i+1} > \pi_{j+1}$ and so $\hat{\pi}_{\pi_i} > \hat{\pi}_{\pi_j}$. Thus, $\hat{\pi}_{d+1} > \hat{\pi}_{d+2} > \cdots > \hat{\pi}_n$.

It follows that $m = d$ or $m = d + 1$, depending on whether $\hat{\pi}_d > \hat{\pi}_{d+1}$ or $\hat{\pi}_d < \hat{\pi}_{d+1}$. Thus, since π was fixed, there are two choices for d, namely $d = m$ or $d = m - 1$. This corresponds to setting s_i, where i is such that $\pi_i = m$, equal to 1 or to 0, respectively. The above argument shows that the rest of the entries of s are forced by π. For exactly one of these two choices, $s_1 \dots s_n$ will have an odd number of ones.

However, as shown in the proof of Theorem 2.1, it is possible for $s_1 \dots s_n$ constructed as above not to be primitive. This can only happen when n is even and $s_1 \dots s_n = q^2$, in which case $s_1 \dots s_n$ has an even number of ones. Thus, the choice where $s_1 \dots s_n$ has an odd number of ones is primitive, so $s \in \mathcal{O}_n$ and it satisfies $\mathrm{PP}(s, \Lambda) = \pi$.

Using the Möbius inversion formula, it can be shown that the number of primitive binary words of length n with an odd number of ones is $|\mathcal{O}_n| = \sum_d \mu(d) 2^{n/d-1}$, where the sum is over all odd divisors of n. Since $p_n(\Lambda) = |\mathcal{O}_n|/n$, the formula follows. $\square$

3.2 The k-shift

Recall that the k-shift is the map Σ_σ where $\sigma = +^k$. We denote this map by Σ_k for convenience. The allowed patterns of the k-shift were characterized and enumerated by Elizalde [8], building up on work by Amigó *et al.* [3].

In this section we describe and enumerate the periodic patterns of the k-shift. Denote the descent set of $\pi \in \mathcal{S}_n$ by $\mathrm{Des}(\pi) = \{i \in [n-1] : \pi_i > \pi_{i+1}\}$, and by $\mathrm{des}(\pi) = |\,\mathrm{Des}(\pi)|$ the number of descents of π. In the case of the k-shift, Theorem 2.1 states that $\pi \in \mathcal{P}(\Sigma_k)$ if and only if $\hat{\pi}$ is a cyclic permutation that can be written as a concatenation of k increasing sequences. The following corollary follows from this description.

Corollary 3.3 $\pi \in \mathcal{P}(\Sigma_k)$ *if and only if* $\mathrm{des}(\hat{\pi}) \le k - 1$.

An equivalent statement is that θ gives a bijection between $\overline{\mathcal{P}}_n(\Sigma_k)$ and permutations in $\mathcal{C}_n$ with at most $k - 1$ descents. It will be convenient to define, for $1 \le i \le n$,

$$C(n, i) = |\{\tau \in \mathcal{C}_n : \mathrm{des}(\tau) = i - 1\}|.$$

In the rest of this section we assume that $n \geq 2$. We start by giving a formula for the number of periodic patterns of the binary shift. Recall the formula for $L_k(n)$ given in Lemma 1.1.

Theorem 3.4 *For $n \geq 2$, we have $p_n(\Sigma_2) = C(n,2) = L_2(n)$.*

Proof: When $n \geq 2$, there are no permutations in $\mathcal{C}_n$ with no descents, so Corollary 3.3 states that $\pi \in \mathcal{P}(\Sigma_2)$ if and only if $\operatorname{des}(\hat{\pi}) = 1$. It follows that $\overline{\mathcal{P}}_n(\Sigma_2)$ is in bijection with permutations in $\mathcal{C}_n$ with exactly one descent, so $p_n(\Sigma_2) = C(n,2)$.

Next we show that $\overline{\mathcal{P}}_n(\Sigma_2)$ is also in bijection with the set of periodic orbits of size n of Σ_2, and thus $p_n(\Sigma_2) = L_2(n)$ by Lemma 1.1. Clearly, to each n-periodic point $s = (s_1 \dots s_n)^\infty$ one can associate the periodic pattern $\pi = \operatorname{PP}(s,\Sigma_2) \in \mathcal{P}_n(\Sigma_2)$, so that the n points in the orbit of s give rise to the patterns in $[\pi]$. Conversely, for each $\pi \in \mathcal{P}_n(\Sigma_2)$ there is some $s \in \mathcal{W}_2$ such that $\operatorname{PP}(s,\Sigma_2) = \pi$. It remains to show that s is unique. Suppose that $\operatorname{Des}(\hat{\pi}) = \{j\}$ and that $\operatorname{PP}(s,\Sigma_2) = \pi$. Letting d be the number of zeros in $s_1 \dots s_n$, we have by Lemma 2.2(c) that $\hat{\pi}_1 < \dots < \hat{\pi}_d$ and $\hat{\pi}_{d+1} < \dots < \hat{\pi}_n$. Thus, $d = j$, and so the word $s_1 \dots s_n$ is uniquely determined by Lemma 2.2(a). □

Theorem 3.5 *For $k \geq 3$ and $n \geq 2$,*

$$p_n(\Sigma_k) - p_n(\Sigma_{k-1}) = C(n,k) = L_k(n) - \sum_{i=2}^{k-1} \binom{n+k-i}{k-i} C(n,i).$$

Proof: It is clear from Corollary 3.3 that $C(n,k) = \frac{1}{n}|\mathcal{P}_n(\Sigma_k) \setminus \mathcal{P}_n(\Sigma_{k-1})| = p_n(\Sigma_k) - p_n(\Sigma_{k-1})$.

To prove the recursive formula for $C(n,k)$, we count periodic orbits of size n of Σ_k in two ways. On one hand, this number equals $L_k(n)$ by Lemma 1.1. On the other hand, to each such orbit one can associate an equivalence class $[\pi] \in \overline{\mathcal{P}}_n(\Sigma_k)$, consisting of the periodic patterns at the n points of the orbit.

Fix $\pi \in \mathcal{P}_n(\Sigma_k)$. We now count how many words $s \in \mathcal{W}_k$ satisfy $\operatorname{PP}(s,\Sigma_k) = \pi$ (equivalently, how many periodic orbits are associated with $[\pi]$). By Lemma 2.4, for each choice of $0 = e_0 \leq e_1 \leq \dots \leq e_k = n$ such that $\hat{\pi}_{e_t+1} \dots \hat{\pi}_{e_{t+1}}$ is increasing for all $0 \leq t < k$, the word s defined in part (b) of the lemma satisfies $\operatorname{PP}(s,\Sigma_k) = \pi$. Conversely, if $s = (s_1 \dots s_n)^\infty \in \mathcal{W}_k$ is such that $\operatorname{PP}(s,\Sigma_k) = \pi$, then, by Lemma 2.2(c), each block $\hat{\pi}_{d_t+1} \dots \hat{\pi}_{d_{t+1}}$ is increasing, with d_t defined as in the lemma, for $0 \leq t < k$. Thus, finding all the words $s \in \mathcal{W}_k$ such that $\operatorname{PP}(s,\Sigma_k) = \pi$ is equivalent to finding all the ways to choose $0 = e_0 \leq e_1 \leq \dots \leq e_k = n$ such that $\hat{\pi}_{e_t+1} \dots \hat{\pi}_{e_{t+1}}$ is increasing for all $0 \leq t < k$. If $\operatorname{des}(\hat{\pi}) = i-1$, it is a simple exercise to show that there are $\binom{n+k-i}{k-i}$ such choices, since $\operatorname{Des}(\hat{\pi})$ has to be a subset of $\{e_1, \dots, e_{k-1}\}$.

By Corollary 3.3, for each $2 \leq i \leq k$, the number of equivalence classes $[\pi] \in \overline{\mathcal{P}}_n(\Sigma_k)$ where $\operatorname{des}(\hat{\pi}) = i-1$ is $C(n,i)$. It follows that

$$L_k(n) = \sum_{i=2}^{k} \binom{n+k-i}{k-i} C(n,i),$$

which is equivalent to the stated formula. □

It follows immediately from Theorem 3.5 that $p_n(\Sigma_k) = \sum_{i=2}^{k} C(n,i)$ for $n \geq 2$.

Let us show an example that illustrates how, in the above proof, the words $s \in \mathcal{W}_k$ with $\mathrm{PP}(s, \Sigma_k) = \pi$ are constructed for given π. Let $k = 5$, and let $\pi = 165398427 \in \mathcal{P}_9(\Sigma_5)$. Then $\hat{\pi} = 679235148$, which has descent set $\mathrm{Des}(\hat{\pi}) = \{3, 6\}$. Choosing $e_1 = 3$, $e_2 = 6$, $e_3 = e_4 = 9$, Lemma 2.4 gives the word $s_1 \dots s_9 = 011022102$. Choosing $e_1 = 2$, $e_2 = 3$, $e_3 = 6$, $e_4 = 7$, we get $s_1 \dots s_9 = 022144203$.

The second equality in Theorem 3.5 also follows from a result of Gessel and Reutenauer [11, Theorem 6.1], which is proved using quasi-symmetric functions.

3.3 The reverse k-shift, when $n \neq 2$ mod 4

The reverse k-shift is the map Σ_σ where $\sigma = -^k$. We denote this map by Σ_k^- in this section. Denote the ascent set of $\pi \in \mathcal{S}_n$ by $\mathrm{Asc}(\pi) = \{i \in [n-1] : \pi_i < \pi_{i+1}\}$, and the number of ascents of π by $\mathrm{asc}(\pi) = |\mathrm{Asc}(\pi)| = n - 1 - \mathrm{des}(\pi)$.

Proposition 2.5 gives a partial characterization of the periodic patterns of Σ_k^-. For patterns of length $n \neq 2 \bmod 4$, it states that $\pi \in \mathcal{P}_n(\Sigma_k^-)$ if and only if $\hat{\pi}$ can be written as a concatenation of k decreasing sequences. The next corollary follows from this description. The case $n = 2 \bmod 4$ will be discussed in Section 3.4.

Corollary 3.6 *Let $\pi \in \mathcal{S}_n$, where $n \neq 2 \bmod 4$. Then $\pi \in \mathcal{P}(\Sigma_k^-)$ if and only if $\mathrm{asc}(\hat{\pi}) \leq k - 1$.*

To enumerate periodic patterns of Σ_k^- of length $n \neq 2 \bmod 4$, we use an argument very similar to the one we used for Σ_k. For $1 \leq i \leq n$, let $C'(n, i) = |\{\tau \in \mathcal{C}_n : \mathrm{asc}(\tau) = i - 1\}|$. By definition, we have $C'(n, i) = C(n, n-i+1)$. The proofs of the following two theorems are similar to those of Theorems 3.4 and 3.5, and thus omitted from this extended abstract.

Theorem 3.7 *For $n \geq 3$ with $n \neq 2 \bmod 4$, we have $p_n(\Sigma_2^-) = C'(n, 2) = L_2(n)$.*

Theorem 3.8 *For $n \geq 3$ with $n \neq 2 \bmod 4$ and $k \geq 3$,*

$$p_n(\Sigma_k) - p_n(\Sigma_{k-1}) = C'(n, k) = L_k(n) - \sum_{i=2}^{k-1} \binom{n+k-i}{k-i} C'(n, i).$$

Combining Theorems 3.4, 3.5, 3.7 and 3.8, we obtain the following.

Corollary 3.9 *For $n \neq 2 \bmod 4$ and $2 \leq k \leq n$, we have $C(n, k) = C'(n, k)$.*

This equality is equivalent to the symmetry $C(n, k) = C(n, n-1-k)$, which is not obvious from the recursive formula in Theorem 3.5. Corollary 3.9 also follows from a more general result of Gessel and Reutenauer [11, Theorem 4.1], which states that if $n \neq 2 \bmod 4$, then for any $D \subseteq [n-1]$,

$$|\{\tau \in \mathcal{C}_n : \mathrm{Des}(\tau) = D\}| = |\{\tau \in \mathcal{C}_n : \mathrm{Asc}(\tau) = D\}|. \tag{1}$$

Their proof involves quasi-symmetric functions. Even though describing a direct bijection proving Eq. (1) remains an open problem, our construction can be used to give the following bijection between $\{\tau \in \mathcal{C}_n : \mathrm{Des}(\tau) \subseteq D\}$ and $\{\tau \in \mathcal{C}_n : \mathrm{Asc}(\tau) \subseteq D\}$.

Given $\hat{\pi} \in \mathcal{C}_n$ such that $\mathrm{Des}(\hat{\pi}) \subseteq D = \{d_1, d_2, \dots, d_{k-1}\}$, let $\pi \in \mathcal{S}_n$ be such that $\theta(\pi) = \hat{\pi}$ and $\pi_1 = 1$. Let $s = (s_1 \dots s_n)^\infty \in \mathcal{W}_k$ be defined by $s_i = t$ if $d_t < \pi_i \leq d_{t+1}$, for $1 \leq i \leq n$, where we let $d_0 = 0$ and $d_k = n$. Let $\pi' = \mathrm{PP}(s, \Sigma_k^-)$. Then $\hat{\pi'} = \theta(\pi') \in \mathcal{C}_n$, $\mathrm{Asc}(\hat{\pi'}) \subseteq D$, and the map $\hat{\pi} \mapsto \hat{\pi'}$ gives the desired bijection.

Another consequence of Theorem 3.8 is that $p_n(\Sigma_k^-) = \sum_{i=2}^k C'(n, k) = p_n(\Sigma_k)$ when $n \neq 2 \bmod 4$.

3.4 The reverse k-shift, when $n = 2 \bmod 4$

When $n = 2$ mod 4, the results in Section 3.3 no longer hold. Corollary 3.6 fails in that there are certain permutations $\pi \in \mathcal{S}_n$ with $\mathrm{asc}(\hat{\pi}) \leq k-1$ that are not periodic patterns for Σ_k^-. For the binary case, the number of periodic patterns of Σ_2^- is given next.

Theorem 3.10 *For $n \geq 3$ with $n = 2 \bmod 4$,*

$$p_n(\Sigma_2^-) = L_2(n) = C'(n,2) - C'(n/2,2).$$

The proof of this theorem, which we omit due to lack of space, is based in the following idea. By Proposition 2.5, the map θ gives an injection from $\overline{\mathcal{P}}_n(\Sigma_2^-)$ to the set $\{\tau \in \mathcal{C}_n : \mathrm{asc}(\tau) = 1\}$. To show that the number of cycles with one ascent that are not in the image of this map is precisely $C'(n/2,2)$, we give a bijection with primitive binary necklaces of length r, constructed by analyzing how Lemma 2.4 fails when $n = 2 \bmod 4$. From Theorems 3.4, 3.7 and 3.10, we get the following.

Corollary 3.11

$$C'(n,2) = \begin{cases} C(n,2) + C(n/2,2) & n = 2 \bmod 4 \\ C(n,2) & n \neq 2 \bmod 4 \end{cases}$$

For $k \geq 3$, the number of periodic patterns is given in the next theorem, whose proof is omitted.

Theorem 3.12 *For $n \geq 3$ with $n = 2 \bmod 4$ and $k \geq 3$,*

$$p_n(\Sigma_k^-) = \sum_{i=2}^{k} C'(n,i) - C'(n/2,k).$$

Since $n/2$ is odd in this case, $C'(n/2,k)$ is easily computed by the recurrence in Theorem 3.8. To compute $C'(n,i)$, one can use the following recurrence.

Theorem 3.13 *For $n \geq 3$ with $n = 2 \bmod 4$ and $k \geq 3$,*

$$C'(n,k) = L_k(n) - \sum_{i=2}^{k-1} \left[\binom{n+k-i}{k-i} C'(n,i) - \binom{n/2+k-i}{k-i} C'(n/2,i) \right] + C'(n/2,k).$$

4 Pattern-avoiding cyclic permutations

The enumeration of pattern-avoiding cycles is a wide-open problem, part of its difficulty stemming from the fact that it combines two different ways to look at permutations: in terms of their cycle structure and in terms of their one-line notation. The question of finding a formula for $|\mathcal{C}_n(\sigma)|$ where σ is a pattern of length 3 was proposed by Richard Stanley and is still open. However, using Theorem 2.1, the formulas that we have found for the number of periodic patterns of the tent map, the k-shift and the reverse k-shift translate into the following related results.

Theorem 4.1 *For $n \geq 2$,*

$$|\mathcal{C}_n(213,312)| = \frac{1}{2n} \sum_{\substack{d|n \\ d \text{ odd}}} \mu(d) 2^{n/d}, \qquad |\mathcal{C}_n(321,2143,3142)| = \frac{1}{n} \sum_{d|n} \mu(d) 2^{n/d},$$

$$|\mathcal{C}_n(123, 2413, 3412)| = \begin{cases} \frac{1}{n}\sum_{d|n}\mu(d)2^{n/d} & \text{if } n \neq 2 \bmod 4, \\ \frac{1}{n}\sum_{d|n}\mu(d)2^{n/d} + \frac{2}{n}\sum_{d|\frac{n}{2}}\mu(d)2^{n/2d} & \text{if } n = 2 \bmod 4. \end{cases}$$

Proof: The formula for $|\mathcal{C}_n(213, 312)|$ is a consequence of Theorem 3.2 and Corollary 3.1, together with the fact that a permutation is unimodal if and only if it avoids 213 and 312. The second formula follows from Theorem 3.4, using that the set of permutations with at most one descent is $\mathcal{S}^{++} = \mathcal{S}(321, 2143, 3142)$ (see [5]). Finally, the third formula is a consequence of Corollary 3.11 and Theorem 3.4, noting that the class of permutations with at most one ascent is $\mathcal{S}(123, 2413, 3412)$. □

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Divisors on graphs, Connected flags, and Syzygies

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Abstract. We study the binomial and monomial ideals arising from linear equivalence of divisors on graphs from the point of view of Gröbner theory. We give an explicit description of a minimal Gröbner basis for each higher syzygy module. In each case the given minimal Gröbner basis is also a minimal generating set. The Betti numbers of I_G and its initial ideal (with respect to a natural term order) coincide and they correspond to the number of "connected flags" in G. Moreover, the Betti numbers are independent of the characteristic of the base field.

Résumé. Nous étudions les idéaux monômiaux et binomiaux résultant de l'équivalence linéaire de diviseurs sur les graphes du point de vue de la théorie de Gröbner. Nous donnons une description explicite d'une base de Gröbner minimale pour chaque module engendré par une syzygie d'ordre supérieur. Dans chaque cas, cette base de Gröbner minimale est aussi une ensemble generateur minimal. Les nombres de Betti de I_G et son idéal initial coïncident et correspondent au nombre de $\ll$ drapeaux connexes $\gg$ de G. En particulier, les nombres de Betti sont indépendants de la caractéristique du corps de référence.

Keywords: Graph, divisors, chip-firing, Gröbner bases, Betti numbers, connected flags.

1 Introduction

The theory of divisors on finite graphs can be viewed as a discrete version of the analogous theory on Riemann surfaces. This notion arises in different fields of research including the study of "abelian sandpiles" ([Dha90, Gab93]), the study of component groups of Néron models of Jacobians of algebraic curves ([Ray70, Lor89]), and the theory of chip-firing games on graphs ([Big97]). Riemann-Roch theory for finite graphs (and generalizations to tropical curves) is developed in this setting ([BN07, GK08, MZ08]).

We are interested in the linear equivalence of divisors on graphs from the point of view of commutative algebra. Associated to every graph G there is a canonical binomial ideal I_G which encodes the linear equivalences of divisors on G. Let R denote the polynomial ring with one variable associated to each vertex. For any two effective divisors $D_1 \sim D_2$ there is a binomial $\mathbf{x}^{D_1} - \mathbf{x}^{D_2}$. The ideal $I_G \subset R$ is generated by all such binomials. Two effective divisors are linearly equivalent if and only if their associated monomials are equal in R/I_G. This ideal is already implicitly defined in Dhar's seminal statistical physics paper [Dha90] ; R/I_G is the "operator algebra" defined there. To our knowledge, this ideal (more precisely, an

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affine piece of it) was first introduced in [CRS02] to address computational questions in chip-firing dynamics using Gröbner basis. From a purely computational point of view there are now much more efficient methods available (see, e.g., [BS13] and references therein). However this ideal seems to encode a lot of interesting information about G and its linear systems. Some of the algebraic properties of I_G (and its generalization for directed graphs) are studied in [PPW11]. In [MS13a], Manjunath and Sturmfels relate Riemann-Roch theory for finite graphs to Alexander duality in commutative algebra using this ideal.

In this paper we study the syzygies and free resolutions of the ideals I_G and $\mathrm{in}(I_G)$ from the point of view of Gröbner theory. Here $\mathrm{in}(I_G)$ denotes the initial ideal with respect to a natural term order which is defined after distinguishing a vertex q (see Definition 2.1). When G is a complete graph, the syzygies and Betti numbers of the ideal $\mathrm{in}(I_G)$ are studied by Postnikov and Shapiro in [PS04]. Again for complete graphs, Manjunath and Sturmfels in [MS13a] study the ideal I_G and show that the Betti numbers coincide with the Betti numbers of $\mathrm{in}(I_G)$. Finding minimal free resolutions for a general graph G was stated as an open problem in both [PS04] and [MS13a] (also in [PPW11], where a conjecture is formulated). It was not even known whether the Betti numbers for a general graph depend on the characteristic of the base field or not.

We construct free resolutions for both $\mathrm{in}(I_G)$ and I_G for a general graph G. Indeed we describe, combinatorially, the minimal Gröbner bases for all higher syzygy modules of I_G and $\mathrm{in}(I_G)$. In each case the minimal Gröbner basis is also a minimal generating set and the given resolution is minimal. In particular the Betti numbers of $\mathrm{in}(I_G)$ and I_G coincide. This gives a positive answer to [CHT06, Question 1.1] for ideal I_G. For a complete graph the minimal free resolution for $\mathrm{in}(I_G)$ is nicely structured by a Scarf complex. The resolution for I_G when G a tree is given by a Koszul complex since I_G is a complete intersection. A more conceptual and geometric proof for a general graph G will be given in [MS13b].

The description of the generating sets and the Betti numbers is in terms of the "connected flags" of G. Fix a vertex $q \in V(G)$ and an integer k. A *connected k-flag* of G (based at q) is a strictly increasing sequence $U_1 \subsetneq U_2 \subsetneq \cdots \subsetneq U_k = V(G)$ such that $q \in U_1$ and all induced subgraphs on vertex sets U_i and $U_{i+1} \backslash U_i$ are connected. Associated to any connected k-flag one can assign a "partial orientation" on G (Definition 3.3). Two connected k-flags are considered equivalent if the associated partially oriented graphs coincide. The Betti numbers correspond to the numbers of the connected flags up to this equivalence. We give a bijective map between the connected flags of G and the minimal Gröbner bases for higher syzygy modules of I_G and $\mathrm{in}(I_G)$. For a complete graph all flags are connected and all distinct flags are inequivalent. So in this case the Betti numbers are simply the face numbers of the order complex of the poset of those subsets of $V(G)$ that contain q (ordered by inclusion). These numbers can be described using classical Stirling numbers (see Example 4.6). Hence our results directly generalize the analogous results in [PS04] and [MS13a]. Analogous results with different methods were obtained independently in [MSW12] and in [DS12].

The paper is structured as follows. In § 2 we fix our notation and provide the necessary background from the theory of divisors on graphs. We also define the ideal I_G and the natural $\mathrm{Pic}(G)$-grading and a term order $<$ on the polynomial ring relevant to our setting. In § 2.2 we quickly recall some basic notions from commutative algebra. Our main goal is to fix our notation for Schreyer's algorithm for computing higher syzygies, which is slightly different from what appears in the existing literature but is more convenient for our application. In § 3 we define connected flags and their equivalence relation. In § 4 we study the free resolution and higher syzygies of our ideals from the point of view of Gröbner theory, and as a corollary we give our description of the graded Betti numbers. In § 5 we describe some connections with the theory of reduced divisors. We refer to [MS12] for proofs and more details.

2 Definitions and background

2.1 Graphs and divisors

Throughout this paper, a *graph* means a finite, connected, unweighted multigraph with no loops. As usual, the set of vertices and edges of a graph G are denoted by $V(G)$ and $E(G)$. We set $n = |V(G)|$. A *pointed graph* (G, q) is a graph together with a choice of a distinguished vertex $q \in V(G)$.

Let $\mathrm{Div}(G)$ be the free abelian group generated by $V(G)$. An element of $\mathrm{Div}(G)$ is written as $\sum_{v \in V(G)} a_v(v)$ and is called a *divisor* on G. The coefficient a_v in D is also denoted by $D(v)$. A divisor D is called *effective* if $D(v) \geq 0$ for all $v \in V(G)$. The set of effective divisors is denoted by $\mathrm{Div}_+(G)$. We write $D \leq E$ if $E - D \in \mathrm{Div}_+(G)$. For $D \in \mathrm{Div}(G)$, let $\deg(D) = \sum_{v \in V(G)} D(v)$. For $D_1, D_2 \in \mathrm{Div}(G)$, the divisor $E = \max(D_1, D_2)$ is defined by $E(v) = \max(D_1(v), D_2(v))$ for $v \in V(G)$.

We denote by $\mathcal{M}(G)$ the group of integer-valued functions on the vertices. For $A \subseteq V(G)$, $\chi_A \in \mathcal{M}(G)$ denotes the $\{0, 1\}$-valued characteristic function of A. The *Laplacian operator* $\Delta : \mathcal{M}(G) \to \mathrm{Div}(G)$ is defined by

$$\Delta(f) = \sum_{v \in V(G)} \sum_{\{v,w\} \in E(G)} (f(v) - f(w))(v) \ .$$

The group of *principal divisors* is defined as the image of the Laplacian operator and is denoted by $\mathrm{Prin}(G)$. It is easy to check that $\mathrm{Prin}(G) \subseteq \mathrm{Div}^0(G)$ where $\mathrm{Div}^0(G)$ denotes the set consisting of divisors of degree zero. The quotient $\mathrm{Pic}^0(G) = \mathrm{Div}^0(G)/\mathrm{Prin}(G)$ is a finite group whose cardinality is the number of spanning trees of G (see, e.g., [BS13] and references therein). The full *Picard group* of G is defined as

$$\mathrm{Pic}(G) = \mathrm{Div}(G)/\mathrm{Prin}(G)$$

which is isomorphic to $\mathbb{Z} \oplus \mathrm{Pic}^0(G)$. Since principal divisors have degree zero, the map $\deg : \mathrm{Div}(G) \to \mathbb{Z}$ descends to a well-defined map $\deg : \mathrm{Pic}(G) \to \mathbb{Z}$. Two divisors D_1 and D_2 are called *linearly equivalent* if they become equal in $\mathrm{Pic}(G)$. In this case we write $D_1 \sim D_2$. The *linear system* $|D|$ of D is defined as the set of effective divisors that are linearly equivalent to D.

To an ordered pair of *disjoint* subsets $A, B \subseteq V(G)$ we assign an effective divisor

$$D(A, B) = \sum_{v \in A} |\{w \in B : \{v, w\} \in E(G)\}|(v) \ .$$

In other words, the support of $D(A, B)$ is a subset of A and for $v \in A$ the coefficient of (v) in $D(A, B)$ is the number of edges between v and B.

Let K be a field and let $R = K[\mathbf{x}]$ be the polynomial ring in the n variables $\{x_v : v \in V(G)\}$. Any effective divisor D gives rise to a monomial

$$\mathbf{x}^D := \prod_{v \in V(G)} x_v^{D(v)} \ .$$

Associated to every graph G there is a canonical ideal which encodes the linear equivalences of divisors on G. Our main object study is the ideal

$$I_G := \langle \mathbf{x}^{D_1} - \mathbf{x}^{D_2} : D_1 \sim D_2 \text{ both effective divisors} \rangle$$

which was introduced in [CRS02].

Once we fix a vertex q, there is a natural monomial order that gives rise to a particularly nice Gröbner basis for I_G. This term order was first introduced in [CRS02]. Fix a pointed graph (G, q). Consider a total ordering of the set of variables $\{x_v : v \in V(G)\}$ compatible with the distances of vertices from q in G :

$$\operatorname{dist}(w, q) < \operatorname{dist}(v, q) \implies x_w < x_v \, . \tag{1}$$

Here, the distance between two vertices in a graph is the number of edges in a shortest path connecting them. The above ordering can be thought of an ordering on vertices induced by running the breadth-first search algorithm starting at the root vertex q.

Definition 2.1 *We denote by $<$ the degree reverse lexicographic ordering on $R = K[\mathbf{x}]$ induced by the total ordering on the variables given in* (1)*.*

Throughout this paper $\operatorname{in}(I_G)$ denotes the initial ideal of I_G with respect to this term order. Note that $\operatorname{in}(I_G)$ is denoted by M_G in [PS04].

2.2 Syzygies and Betti numbers

In this subsection we quickly recall some basic notions from commutative algebra in order to fix our notation. We refer to standard books (e.g. [Eis95, GP08]) for more details.

Let K be any field and let $R = K[\mathbf{x}]$ be the polynomial ring in n variables graded by an abelian group A. The degree map will be denoted by $\deg$. Let M be a graded submodule of a free module and fix a *module ordering* $<_0$ extending the monomial ordering $<$ on R. Assume that the finite totally ordered set $(\mathbf{G}, \prec)$ forms a Gröbner basis for $(M, <_0)$ consisting of homogeneous elements. Let F_0 be the free module generated by $\mathbf{G}$. For $g \in \mathbf{G}$ we let the formal symbol $[g]$ denote the corresponding generator for F_0 ; each element of F_0 can be written as a sum of these formal symbols with coefficients in R. There is a natural surjective homomorphism

$$\varphi_0 : F_0 \longrightarrow M$$

sending $[g]$ to g for each $g \in \mathbf{G}$. Moreover, we enforce this homomorphism to be graded (or homogeneous of degree 0) by defining $\deg([g]) := \deg(g)$ for all $g \in \mathbf{G}$.

By definition the syzygy module of M with respect to $\mathbf{G}$, denoted by $\operatorname{syz}(\mathbf{G})$, is the kernel of this map. Let $\operatorname{syz}_0(\mathbf{G}) := M$ and $\operatorname{syz}_1(\mathbf{G}) := \operatorname{syz}(\mathbf{G})$. For $i > 1$ the higher syzygy modules are defined as $\operatorname{syz}_i(\mathbf{G}) := \operatorname{syz}(\operatorname{syz}_{i-1}(\mathbf{G}))$.

We now discuss a method to compute a Gröbner basis for $\operatorname{syz}(\mathbf{G})$. One can "pull back" the module ordering $<_0$ along φ_0 to get a compatible module ordering $<_1$ on F_0 ; for $f, h \in \mathbf{G}$ define

$$\mathbf{x}^\beta[h] <_1 \mathbf{x}^\alpha[f] \Leftrightarrow \begin{cases} \mathrm{LM}(\mathbf{x}^\beta h) <_0 \mathrm{LM}(\mathbf{x}^\alpha f) \\ \text{or} \\ \mathrm{LM}(\mathbf{x}^\beta h) = \mathrm{LM}(\mathbf{x}^\alpha f) \quad \wedge \quad f \prec h. \end{cases} \tag{2}$$

To simplify the notation we assume the leading coefficients of all elements of $\mathbf{G}$ are 1. For a pair of elements $f \prec h$ of $\mathbf{G}$ assume

$$\mathrm{LM}(f) = \mathbf{x}^{\alpha(f)}[e] \quad \text{and} \quad \mathrm{LM}(h) = \mathbf{x}^{\alpha(h)}[e]$$

for some $e \in E$. Since $\mathbf{G}$ is a Gröbner basis, setting $\gamma(f,h) := \max(\alpha(f), \alpha(h))$, we have the "standard representation" :

$$\mathrm{spoly}(f,h) = \mathbf{x}^{\gamma(f,h)-\alpha(f)} f - \mathbf{x}^{\gamma(f,h)-\alpha(h)} h = \sum_{g \in \mathbf{G}} a_g^{(f,h)} g \tag{3}$$

for some polynomials $a_g^{(f,h)} \in R$. We set

$$s(f,h) = \mathbf{x}^{\gamma(f,h)-\alpha(f)}[f] - \mathbf{x}^{\gamma(f,h)-\alpha(h)}[h] - \sum_{g \in \mathbf{G}} a_g^{(f,h)}[g] \in F_0 \ . \tag{4}$$

Theorem 2.2 (Schreyer [Sch80], [Eis95]) *The set*

$$\mathcal{S}(\mathbf{G}) = \{s(f,h) : \ f,h \in \mathbf{G} \ , \ f \prec h \ , \ \mathrm{LM}(f) = \mathbf{x}^{\alpha(f)}[e] \ , \ \mathrm{LM}(h) = \mathbf{x}^{\alpha(h)}[e] \ \textit{for some } e \in E\}$$

forms a homogeneous Gröbner basis for $(\mathrm{syz}(\mathbf{G}), <_1)$.

To read the Betti numbers for M one needs to find a *minimal generating set* for the syzygy modules. In general the set $\mathcal{S}(\mathbf{G})$ is far from being even a *minimal Gröbner basis*. However there exist some criterions to find a subset $\mathcal{S}_{\min}(\mathbf{G})$ of $\mathcal{S}(\mathbf{G})$ which forms a minimal Gröbner basis for $(\mathrm{syz}(\mathbf{G}), <_1)$; see, e.g., [MS12, Lemma 3.4]. Moreover, Theorem 2.2 gives rise to Algorithm 1 for computing free resolutions. The following result gives a general sufficient criterion for an ideal to have the same Betti numbers as its initial ideal.

Theorem 2.3 *If the constructed resolution by Schreyer's algorithm is a* minimal *graded free resolution then* $\beta_{i,\mathsf{j}}(M) = \beta_{i,\mathsf{j}}(\mathrm{in}(M))$ *for all* $i \geq 0$ *and* $\mathsf{j} \in \mathsf{A}$.

3 Connected flags on graphs

3.1 Connected flags, partial orientations, and divisors

From now on we fix a pointed graph (G,q) and we let $n = |V(G)|$. Consider the poset

$$\mathfrak{C}(G,q) := \{U \subseteq V(G) : \ q \in U\}$$

ordered by inclusion. The following special chains of this poset arise naturally in our setting.

Definition 3.1 *Fix an integer* $1 \leq k \leq n$. *A* connected k-flag *of* (G,q) *is a (strictly increasing) sequence* $\mathcal{U}$ *of subsets of* $V(G)$

$$U_1 \subsetneq U_2 \subsetneq \cdots \subsetneq U_k = V(G)$$

such that $q \in U_1$ *and, for all* $1 \leq i \leq k-1$, *both* $G[U_i]$ *and* $G[U_{i+1} \backslash U_i]$ *are connected.*

The set of all connected k-flags of (G,q) will be denoted by $\mathfrak{F}_k(G,q)$.

Input:
Graded polynomial ring $R = K[\mathbf{x}]$,
Monomial ordering $<$ on R ,
Graded submodule M of the free R-module F_{-1} generated by formal symbols $\{[e]\}_{e \in E}$,
Module ordering $<_0$ on F_{-1} extending the monomial ordering $<$,
Finite set $\mathbf{G}$ forming a homogeneous Gröbner basis for $(M, <_0)$.
Output:
A free resolution : $0 \to \cdots \to F_i \xrightarrow{\varphi_i} F_{i-1} \to \cdots \to F_0 \xrightarrow{\varphi_0} M \to 0$.
Initialization :
$\mathbf{G}_0 := \mathbf{G}$;
F_0 := free R-module generated by formal symbols $\{[g]\}_{g \in \mathbf{G}_0}$; **Output** F_0 ;
$\varphi_0 : F_0 \to M \subseteq F_{-1}$ defined by $[g] \mapsto g$ for each $g \in \mathbf{G}_0$; **Output** φ_0 ;
$i = 0$;
while $F_i \neq 0$ **do**
  $\prec_i$: arbitrary total ordering on $\mathbf{G}_i$;
  $<_{i+1}$: module ordering on F_i obtained from $<_i$ on F_{i-1} (as in (2)) ;
  $\mathbf{G}_{i+1} := \mathcal{S}_{\min}(\mathbf{G}_i) \subset F_i$, a minimal Gröbner basis of $(\mathrm{syz}_{i+1}(\mathbf{G}), <_{i+1})$ (as in Theorem 2.2) ;
  F_{i+1} := free R-module generated by formal symbols $\{[u]\}_{u \in \mathbf{G}_{i+1}}$; **Output** F_{i+1} ;
  $\varphi_{i+1} : F_{i+1} \to F_i$ defined by $[u] \mapsto u$ for each $u \in \mathbf{G}_{i+1}$; **Output** φ_{i+1} ; $i \leftarrow i+1$;
end

Algorithm 1: Algorithm for computing a free resolution of M (Schreyer's algorithm)

Remark 3.2 For a complete graph, $\mathfrak{F}_k(G,q)$ is simply the order complex of $\mathfrak{C}(G,q)$, but in general $\mathfrak{F}_k(G,q)$ is not a simplicial complex.

Definition 3.3 *Given $\mathcal{U} \in \mathfrak{F}_k(G,q)$ we define :*
(a) a "partial orientation" of G by orienting edges from U_i to $U_{i+1} \backslash U_i$ *(for all $1 \leq i \leq k-1$) and leaving all other edges unoriented. We denote the resulting partially oriented graph by $G(\mathcal{U})$.*
(b) an effective divisor $D(\mathcal{U}) \in \mathrm{Div}(G)$ given by $D(\mathcal{U}) := \sum_{i=1}^{k-1} D(U_{i+1} \backslash U_i, U_i)$.

Remark 3.4 It is easy to check that $D(\mathcal{U}) = \sum_{v \in V(G)} (\mathrm{indeg}_{G(\mathcal{U})}(v))(v)$, where $\mathrm{indeg}_{G(\mathcal{U})}(v)$ denotes the number of oriented edges directed *to* v in $G(\mathcal{U})$.

3.2 Total ordering on $\mathfrak{F}_k(G,q)$

We endow each $\mathfrak{F}_k(G,q)$ with a *total orderings* $\prec_k$ for all $1 \leq k \leq n$. These total orderings are compatible with each other for different values of $1 \leq k \leq n$.

Let $\preceq$ denote the ordering on $\mathfrak{C}^{\mathrm{op}}(G,q)$ given by reverse inclusion :

$$U \preceq V \iff U \supseteq V \ .$$

Definition 3.5 *We fix, once and for all, a* total ordering *extending $\preceq$. By a slight abuse of notation, $\preceq$ will be used to denote this total ordering extension. In particular $\prec$ will denote the associated strict total order.*

We consider one of the natural "lexicographic extensions" of $\prec$ to the set of connected k-flags.

Definition 3.6 *For $\mathcal{U} \neq \mathcal{V}$ in $\mathfrak{F}_k(G,q)$ written as*

$$\mathcal{U}: \; U_1 \subsetneq U_2 \subsetneq \cdots \subsetneq U_k = V(G)$$

$$\mathcal{V}: \; V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_k = V(G)$$

we say $\mathcal{U} \prec_k \mathcal{V}$ if for the maximum $1 \leq \ell \leq k-1$ with $U_\ell \neq V_\ell$ we have $\mathcal{U}_\ell \prec \mathcal{V}_\ell$.
As usual, we write $\mathcal{U} \preceq_k \mathcal{V}$ if and only if $\mathcal{U} \prec_k \mathcal{V}$ or $\mathcal{U} = \mathcal{V}$.

Lemma 3.7 *$(\mathfrak{F}_k(G,q), \preceq_k)$ is a totally ordered set.*

It is easy to find two different connected k-flags having identical associated partially oriented graphs. This motivates the following definition.

Definition 3.8 *Two k-flags $\mathcal{U}, \mathcal{V} \in \mathfrak{F}_k(G,q)$ are called equivalent if the associated partially oriented graphs $G(\mathcal{U})$ and $G(\mathcal{V})$ coincide.*

Notation 1 *The set of all equivalence classes in $\mathfrak{F}_k(G,q)$ will be denoted by $\mathfrak{C}_k(G,q)$. The set $\mathfrak{S}_k(G,q)$ denotes the collection of minimal representatives of the classes in $\mathfrak{C}_k(G,q)$ with respect to $\prec_k$.*

Given an element in $\mathfrak{S}_k(G,q)$ there is a canonical way to obtain two related elements in $\mathfrak{S}_{k-1}(G,q)$.

Definition 3.9 *Given $\mathcal{U} \in \mathfrak{F}_k(G,q)$, the elements $\mathcal{U}^{(1)}, \mathcal{U}^{(2)} \in \mathfrak{F}_{k-1}(G,q)$ are obtained from $\mathcal{U}$ by removing the 1^{st} and 2^{nd} elements in the following appropriate sense. Let*

$$\mathcal{U}: \; U_1 \subsetneq U_2 \subsetneq \cdots \subsetneq V(G)\,.$$

(a) $\mathcal{U}^{(1)}$ will denote

$$U_2 \subsetneq U_3 \subsetneq U_4 \subsetneq \cdots \subsetneq V(G)\,.$$

(b) $\mathcal{U}^{(2)}$ will denote

$$\begin{cases} U_1 \subsetneq U_3 \subsetneq U_4 \subsetneq \cdots \subsetneq V(G), & \textit{if } G[U_3 \backslash U_1] \textit{ is connected ;} \\ \textit{or} & \\ (U_1 \cup (U_3 \backslash U_2)) \subsetneq U_3 \subsetneq U_4 \subsetneq \cdots \subsetneq V(G), & \textit{if } G[U_3 \backslash U_1] \textit{ is not connected.} \end{cases}$$

Remark 3.10 [MS12, Section 6.1] Assume that $\mathcal{U} \in \mathfrak{S}_k(G,q)$. Let $G_{/\mathcal{U}}$ be the graph obtained from G by contracting the unoriented edges of $G(\mathcal{U})$ and let $\phi : G \to G_{/\mathcal{U}}$ be the contraction map. More precisely, $G_{/\mathcal{U}}$ is the graph on the vertices $u_1, \ldots, u_k$ corresponding to the collection $(U_i \backslash U_{i-1})_{i=1}^k$, i.e. $u_i = \phi(U_i \backslash U_{i-1})$. For any edge between $U_i \backslash U_{i-1}$ and $U_j \backslash U_{j-1}$ there is an edge between u_i and u_j. The contraction map $\phi : G \to G_{/\mathcal{U}}$ induces the maps

(i) $\phi_* : \mathrm{Div}(G) \to \mathrm{Div}(G_{/\mathcal{U}})$ with $\phi_*(\sum_{v \in V(G)} a_v(v)) = \sum_{v \in V(G)} a_v(\phi(v))$. In particular, a total ordering $u_1, \ldots, u_k$ of $V(G_{/\mathcal{U}})$ gives a total ordering on the collection of subsets $(U_i \backslash U_{i-1})_{i=1}^k$ of $V(G)$. By Definition 3.3 we get a divisor D' on $G_{/\mathcal{U}}$ and a divisor D on G, and $\phi_*(D) = D'$. In other words, such a divisor D' has a canonical section.

(ii) $\phi^* : \mathfrak{S}_s(G_{/\mathcal{U}}, u_1) \to \mathfrak{S}_s(G,q)$ with $\phi^*(\mathcal{V}') = \mathcal{V}$ where $V_j = \bigcup_{u_i \in V'_j} (U_i \backslash U_{i-1})$.

4 Minimal free resolution and Betti numbers for I_G and $\mathrm{in}(I_G)$

Let K be a field and let $R = K[\mathbf{x}]$ be the polynomial ring in n variables $\{x_v : v \in V(G)\}$. Recall that $K[\mathbf{x}]$ has a natural A-grading, where $\mathsf{A} = \mathbb{Z}$ or $A = \mathrm{Pic}(G)$ and I_G is also A-graded. Let the monomial ordering $<$ on R be as in Definition 2.1.

The following theorem gives a generalization of [CRS02, Theorem 14]. Indeed [CRS02, Theorem 14] can be rephrased as providing a bijection between $\mathfrak{S}_2(G,q)$ and $\mathbf{G}(G,q)$.

Theorem 4.1 *Fix a pointed graph (G,q) and let $\mathsf{A} = \mathbb{Z}$ or $\mathsf{A} = \mathrm{Pic}(G)$. For each $k \geq 0$ there exists a natural injection*

$$\psi_k : \mathfrak{S}_{k+2}(G,q) \hookrightarrow \mathrm{syz}_k(\mathbf{G}(G,q))$$

such that

(i) For some module ordering $<_k$, the set $\mathbf{G}_k(G,q) := \mathrm{Image}(\psi_k)$ forms a minimal A-homogeneous Gröbner basis of $(\mathrm{syz}_k(\mathbf{G}(G,q)), <_k)$,

(ii) For $\mathcal{U} \in \mathfrak{S}_{k+2}(G,q)$ of the form $U_1 \subsetneq U_2 \subsetneq \cdots \subsetneq V(G)$ we have

$$\mathrm{LM}(\psi_k(\mathcal{U})) = \mathbf{x}^{D(U_2 \backslash U_1, U_1)}[\psi_{k-1}(\mathcal{U}^{(1)})] \,, \tag{5}$$

(iii) The set $\psi_k(\mathfrak{S}_{k+2}(G,q))$ minimally generates $\mathrm{syz}_k(\mathbf{G}(G,q))$.

Sketch of proof : Here we list the key steps of the proof. For a complete proof we refer to [MS12]. For consistency in the notation we define $\mathrm{syz}_{-1}(\mathbf{G}(G,q)) = \{0\}$ and the map

$$\psi_{-1} : \mathfrak{S}_1(G,q) \hookrightarrow \{0\}$$

sends the canonical connected 1-flag $V(G)$ to 0. The proof is by induction on $k \geq 0$.

Base case. For $k = 0$ the result is proved in [CRS02, Theorem 14]. Here $\mathbf{G}_0(G,q) = \mathbf{G}(G,q)$ and $<_0$ is $<$, and

$$\begin{gathered}\psi_0 : \mathfrak{S}_2(G,q) \hookrightarrow \mathrm{syz}_0(\mathbf{G}(G,q)) = I_G \\ (U_1 \subsetneq U_2) \mapsto (\mathbf{x}^{D(U_2 \backslash U_1, U_1)} - \mathbf{x}^{D(U_1, U_2 \backslash U_1)})[0]\end{gathered}$$

and $\mathrm{LM}(\psi_k(\mathcal{U})) = \mathbf{x}^{D(U_2 \backslash U_1, U_1)}[0]$.

Induction hypothesis. Now let $k > 0$ and assume that there exists a bijection

$$\psi_{k-1} : \mathfrak{S}_{k+1}(G,q) \to \mathbf{G}_{k-1}(G,q) \subseteq \mathrm{syz}_{k-1}(\mathbf{G}(G,q))$$

such that $\mathbf{G}_{k-1}(G,q)$ forms a minimal homogeneous Gröbner basis of $\mathrm{syz}_{k-1}(\mathbf{G}(G,q))$ with respect to $<_{k-1}$), and (5) for the leading monomials holds.

Via the bijection ψ_{k-1}, the set $\mathbf{G}_{k-1}(G,q)$ inherits a total ordering $\prec'_{k-1}$ from the total ordering $\prec_{k+1}$ on $\mathfrak{S}_{k+1}(G,q)$, i.e.

$$f \prec'_{k-1} h \quad \text{in} \quad \mathbf{G}_{k-1}(G,q) \quad \Leftrightarrow \quad \psi_{k-1}^{-1}(f) \prec_{k+1} \psi_{k-1}^{-1}(h) \quad \text{in} \quad \mathfrak{S}_{k+1}(G,q).$$

Inductive step. Given $\mathcal{U} \in \mathfrak{S}_{k+2}(G,q)$ let $\mathcal{U}^{(1)}$ and $\mathcal{U}^{(2)}$ be as defined in Definition 3.9. We define

$$\psi_k : \mathfrak{S}_{k+2}(G,q) \to \mathrm{syz}_k(\mathbf{G}(G,q)) \tag{6}$$

$$\mathcal{U} \mapsto s(\psi_{k-1}(\mathcal{U}^{(1)}), \psi_{k-1}(\mathcal{U}^{(2)}))\ .$$

In the following $\mathcal{U}, \mathcal{V} \in \mathfrak{S}_{k+2}(G, q)$ are of the form

$$U_1 \subsetneq U_2 \subsetneq \cdots \subsetneq V(G)$$

$$V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V(G)\ .$$

The result is follows from a series of claims.

Claim 1. ψ_k is a well-defined and $\mathbf{G}_k(G, q) := \mathrm{Image}(\psi_k)$ consists of homogeneous elements.

Since $\psi_{k-1}(\mathcal{U}^{(1)})$ and $\psi_{k-1}(\mathcal{U}^{(2)})$ are homogeneous by induction hypothesis, it follows that $s(\psi_{k-1}(\mathcal{U}^{(1)}), \psi_{k-1}(\mathcal{U}^{(2)}))$ is also homogeneous.

Claim 2. $\mathrm{LM}(\psi_k(\mathcal{U})) = \mathbf{x}^{D(U_2\backslash U_1, U_1)}[\psi_{k-1}(\mathcal{U}^{(1)})]\ .$

It suffices to show that $D(U_2\backslash U_1, U_1) = \max(\alpha, \beta) - \alpha$ where

$$\mathrm{LM}(\psi_{k-1}(\mathcal{U}^{(1)})) = \mathbf{x}^{\alpha}[\psi_{k-2}(\mathcal{U}^{(1,1)})] \quad , \quad \mathrm{LM}(\psi_{k-1}(\mathcal{U}^{(2)})) = \mathbf{x}^{\beta}[\psi_{k-2}(\mathcal{U}^{(2,1)})]\ .$$

Claim 3. ψ_k is injective.

If $\mathcal{U}, \mathcal{V} \in \mathfrak{S}_{k+2}(G, q)$ be such that $\psi_k(\mathcal{U}) = \psi_k(\mathcal{V})$ then their leading terms are equal :

$$\mathbf{x}^{D(U_2\backslash U_1, U_1)}[\psi_{k-1}(\mathcal{U}^{(1)})] = \mathbf{x}^{D(V_2\backslash V_1, V_1)}[\psi_{k-1}(\mathcal{V}^{(1)})]\ .$$

Therefore $\psi_{k-1}(\mathcal{U}^{(1)}) = \psi_{k-1}(\mathcal{V}^{(1)})$ and $D(U_2\backslash U_1, U_1) = D(V_2\backslash V_1, V_1)$. By induction hypothesis ψ_{k-1} is injective which implies $\mathcal{U}^{(1)} = \mathcal{V}^{(1)}$ and $D(U_2\backslash U_1, U_1) = D(V_2\backslash V_1, V_1)$. Therefore $U_1 = V_1$ and $\mathcal{U} = \mathcal{V}$.

The following claim (proved in [MS12]) will finish the inductive step.

Claim 4. $\mathrm{Image}(\psi_k)$ forms a minimal homogeneous Gröbner basis of $\mathrm{syz}_k(\mathbf{G}(G, q))$ with respect to $<_k$ obtained from $<_{k-1}$ according to (2).

These all together show that $\mathrm{Image}(\psi_k) \subseteq \mathcal{S}(\mathbf{G}_{k-1}(G, q))$. In order to show the reverse inclusion by Theorem 2.2 it remains to show that

(I) $0 \notin \mathrm{Image}(\psi_k)$.

(II) For any element $s(f, h) \in \mathcal{S}(\mathbf{G}_{k-1}(G, q))$ there exists an element $g \in \mathrm{Image}(\psi_k)$ such that $\mathrm{LM}(g) \mid \mathrm{LM}(s(f, h))$.

(III) For any two elements $g, g' \in \mathrm{Image}(\psi_k)$, if $\mathrm{LM}(g) \mid \mathrm{LM}(g')$ then $g = g'$.

Claim 5. For $\mathcal{U} \in \mathfrak{S}_{k+2}(G, q)$ we have $\psi_k(\mathcal{U}) = \sum_{\mathcal{W} \in \mathfrak{S}_{k+1}(G,q)} c(\mathcal{U}, \mathcal{W})\mathbf{x}^{\theta(\mathcal{U},\mathcal{W})}[\psi_{k-1}(\mathcal{W})]$ where $c(\mathcal{U}, \mathcal{W}) \in \{-1, 0, 1\}$ and $\theta(\mathcal{U}, \mathcal{W}) = D(U_i\backslash U_{i-1}, U_j\backslash U_{j-1})$ if $\mathcal{W}$ differs from $\mathcal{U}$ by merging $U_i\backslash U_{i-1}$ and $U_j\backslash U_{j-1}$ for some i, j.

Note that this proves (III) which is equivalent to the minimality of the resolution. $\square$

From Theorem 4.1 we obtain the following important corollaries.

Corollary 4.2 *The Betti numbers of the ideals I_G and $\mathrm{in}(I_G)$ are independent of the characteristic of the base field K.*

Corollary 4.3 *For all $i \geq 0$, $\beta_i(R/I_G) = \beta_i(R/\mathrm{in}(I_G)) = |\mathfrak{S}_{i+1}(G,q)| = |\mathfrak{E}_{i+1}(G,q)|$.*

Let $\mathsf{A} = \mathbb{Z}$ or $\mathsf{A} = \mathrm{Pic}(G)$. Recall that I_G and $\mathrm{in}(I_G)$ are graded (or homogeneous) with respect to the $\mathbb{Z}$ and $\mathrm{Pic}(G)$ gradings. One can also read the graded Betti numbers from Theorem 4.1.

Corollary 4.4 *For all $i \geq 0$ and $\mathsf{j} \in \mathsf{A}$ we have $\beta_{i,\mathsf{j}} = |\mathfrak{S}_{i+1,\mathsf{j}}(G,q)|$ where $\mathfrak{S}_{k,\mathsf{j}}(G,q) = \{\mathcal{U} \in \mathfrak{S}_k(G,q) :\ \deg_{\mathsf{A}}(\mathbf{x}^{D(\mathcal{U})}) = \mathsf{j}\}$.*

We conclude this section with some examples.

Example 4.5 It follows from above descriptions that $\beta_{n-1}(R/I_G) = \beta_{n-1,m}(R/I_G)$ is equal to the number of acyclic orientations of G with unique source.

Example 4.6 Let G be the complete graph K_n on n vertices. Then $\beta_{k-1}(R/I_G) = |\mathfrak{S}_k(G,q)| = (k-1)!\,S(n,k)$ where $S(n,k)$ denotes the Stirling number of the second kind (i.e. the number of ways to partition a set of n elements into k nonempty subsets).

Example 4.7 Let G be a tree on n vertices. Then $\beta_{k-1}(R/I_G) = |\mathfrak{S}_k(G,q)| = \binom{n-1}{k-1}$.

Example 4.8 For the cycle C_n on n vertices and $k \geq 2$ we have $\beta_{k-1}(R/I_{C_n}) = |\mathfrak{S}_k(C_n,q)| = (k-1) \times \binom{n}{k}$.

5 Relation to maximal reduced divisors

Recall the definition of reduced divisors.

Definition 5.1 *Let (Γ, v_0) be a pointed graph. A divisor $D \in \mathrm{Div}(\Gamma)$ is called v_0-*reduced *if it satisfies the following two conditions :*
(i) $D(v) \geq 0$ for all $v \in V(\Gamma) \backslash \{v_0\}$.
(ii) For every non-empty subset $A \subseteq V(\Gamma) \backslash \{v_0\}$, there exists a vertex $v \in A$ such that $D(v) < \mathrm{outdeg}_A(v)$.

These divisors arise precisely from the normal forms with respect to the Gröbner basis given in Theorem 4.1. There is a well-known algorithm due to Dhar for checking whether a given divisor is reduced (see, e.g., [BS13] and references therein).

Lemma 5.2 *For $\mathcal{U} \in \mathfrak{S}_k(G,q)$, $\phi_*(D(\mathcal{U})) = E + \mathbf{1}$, where E is a maximal $(\phi(U_1))$-reduced divisor and $\mathbf{1}$ is the all-one divisor.*

Since different acyclic orientations with unique source at q' give rise to inequivalent q'-reduced divisors we deduce that if $\mathcal{U}, \mathcal{V} \in \mathfrak{S}_k(G,q)$ and the graphs $G_{/\mathcal{U}}$ and $G_{/\mathcal{V}}$ coincide, then $\phi_*(D(\mathcal{U})) - \mathbf{1}$ and $\phi_*(D(\mathcal{V})) - \mathbf{1}$ are two inequivalent maximal reduced divisors. These observations lead to the following formula for Betti numbers which was conjectured in [PPW11] for I_G :

$$\begin{aligned}\beta_i &= \sum_{G_{/\mathcal{U}}} |\{D :\ D \text{ is a maximal } v_0\text{-reduced divisors on } G_{/\mathcal{U}}\}| \\ &= \sum_{G_{/\mathcal{U}}} |\{\text{acyclic orientations of } G_{/\mathcal{U}} \text{ with unique source at } v_0\}|\end{aligned}$$

where the sum is over all *distinct* contracted graphs $G_{/\mathcal{U}}$ as $\mathcal{U}$ varies in $\mathfrak{S}_{i+1}(G,q)$, and v_0 is an arbitrary vertex of $G_{/\mathcal{U}}$.

Here is another connection with reduced divisors. Hochster's formula for computing the Betti numbers topologically (see, e.g., [MS05, Theorem 9.2]), when applied to I_G and the "nice" grading by $\mathrm{Pic}(G)$, says that for each $\mathrm{j} \in \mathrm{Pic}(G)$ the graded Betti number $\beta_{i,\mathrm{j}}(R/I_G)$ is the dimension of the i^{th} reduced homology of the simplicial complex $\Delta_{\mathrm{j}} = \{\mathrm{supp}(E) : 0 \le E \le D' \in |\,\mathrm{j}\,|\}$ where $|\,\mathrm{j}\,|$ denotes the linear system of $\mathrm{j} \in \mathrm{Pic}(G)$.

Remark 5.3

(i) For $\mathrm{j} \in \mathrm{Pic}(G)$, we have $\beta_{n-1,\mathrm{j}}(R/I_G) = 1$ if and only if $\mathrm{j} \sim E + \mathbf{1}$ where E is a maximal q-reduced divisor.

(ii) One can use Corollary 4.3 to read all dimensions of the reduced homologies of Δ_{j}. Although we now know all the Betti numbers, giving an explicit bijection between connected flags and the bases of the reduced homologies of Δ_{j} is an intriguing problem. In a recent work, Horia Mania in [Man12] studies the number of connected components of Δ_{j}.

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Counting Strings over $\mathbb{Z}2^d$ with Given Elementary Symmetric Function Evaluations

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Abstract. Let α be a string over $\mathbb{Z}_q$, where $q = 2^d$. The j-th elementary symmetric function evaluated at α is denoted $e_j(\alpha)$. We study the cardinalities $S_q(m; \tau_1, \tau_2, \ldots, \tau_t)$ of the set of length m strings for which $e_i(\alpha) = \tau_i$. The *profile* $\mathbf{k}(\alpha) = \langle k_1, k_2, \ldots, k_{q-1}\rangle$ of a string α is the sequence of frequencies with which each letter occurs. The profile of α determines $e_j(\alpha)$, and hence S_q. Let $h_n : \mathbb{Z}^{(q-1)}_{2^{n+d-1}} \mapsto \mathbb{Z}_{2^d}[z] \bmod z^{2^n}$ be the map that takes $\mathbf{k}(\alpha) \bmod 2^{n+d-1}$ to the polynomial $1 + e_1(\alpha)z + e_2(\alpha)z^2 + \cdots + e_{2^n-1}(\alpha)z^{2^n-1}$. We show that h_n is a group homomorphism and establish necessary conditions for membership in the kernel for fixed d. The kernel is determined for $d = 2, 3$. The range of h_n is described for $d = 2$. These results are used to efficiently compute $S_4(m; \tau_1, \tau_2, \ldots, \tau_t)$ for $d = 2$ and the number of complete factorizations of certain polynomials.

Résumé. Soit α un mot sur $\mathbb{Z}_q$, où $q = 2^d$. La $j-$ième fonction symmétrique élémentaire évaluée à α est dénotée $e_j(\alpha)$. Nous étudions les cardinalités $S_q(m; \tau_1, \tau_2, \ldots, \tau_t)$ de l'ensemble des mots de longueur m pour lesquels $e_i(\alpha) = \tau_i$. Le *profil* $\mathbf{k}(\alpha) = \langle k_1, k_2, \ldots, k_{q-1}\rangle$ d'un mot α est la suite de fréquences d'apparition de chaque lettre. Le profil de α détermine $e_j(\alpha)$ et donc S_q. Soit $h_n : \mathbb{Z}^{(q-1)}_{2^{n+d-1}} \mapsto \mathbb{Z}_{2^d}[z] \bmod z^{2^n}$ la fonction qui associe à $\mathbf{k}(\alpha) \bmod 2^{n+d-1}$ le polynôme $1 + e_1(\alpha)z + e_2(\alpha)z^2 + \cdots + e_{2^n-1}(\alpha)z^{2^n-1}$. Nous démontrons que h_n est un homomorphisme de groupe et nous établissons des conditions nécessaires à l'appartenance au noyau pour un d fixé. Le noyau est déterminé pour $d = 2, 3$. L'image de h_n est décrite pour $d = 2$. Ces résultats sont utilisés pour calculer de manière efficace $S_4(m; \tau_1, \tau_2, \ldots, \tau_t)$ pour $d = 2$ ainsi que le nombre de factorisations complètes de certains polynômes.

Keywords: elementary symmetric function, monomial factorization, integers mod 2^d, group homomorphism, kernel.

1 Introduction and motivation

Before getting too deeply into the abstract and technical details let us illustrate the types of computations that we will be able to easily carry out after proving our results. Let $[\![P]\!]$ be 1 or 0 depending of whether the proposition P is true or false, respectively. Consider the problem below.

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EXAMPLE 1 *How many strings are there of length* 100 *over the alphabet* $0, 1, 2, 3$ *that satisfy the following six conditions, with arithmetic done mod 4? Conditions: (a) The sum of the characters is 0 and, (b) the sum of the products of all pairs of characters is 3 and, (c) the sum of the products of all 4-tuples of characters is 3, (d) the sum of the products of all 8-tuples of characters is 2, (e) the sum of the products of all 16-tuples of characters is 3, (f) the sum of the products of all 32-tuples of characters is 3. The answer is approximately* 2.33×10^{58}*, or exactly*

$$23283888738988446954113680611180557044216386182393836339200, \tag{1}$$

which is the value of the sum

$$\sum_{k_0+k_1+k_2+k_3=100} \binom{100}{k_0, k_1, k_2, k_3} [\![k_1, k_3 \textit{ even}, k_2 \textit{ odd}, k_1 + k_3 \equiv 54 \bmod 64]\!].$$

That is, the answer is the sum of 667 multinomial coefficients. Furthermore, the sum above applies for strings of length m*; one need only replace the* 100 *by* m*.*

There are several natural questions that should occur to the reader at this point. Firstly, why are the "tuples" involved all powers of two? The reason is that, for example, the sum of products of all 3-tuples is determined already by the value of the sum of products of 1-tuples and 2-tuples. Secondly, why do the mysterious parity and modular conditions arise; in particular why is it some condition mod 64 and not just mod 4? We will answer all these questions in due course, generalizing from arithmetic done mod 4 to arithmetic done mod 2^d.

EXAMPLE 2 *In this example all computations are done mod 8. The following equation illustrates the non-unique factorization of a polynomial into monomials.*

$$(1 + z)^3(1 + 5z)^5 = (1 + 3z)^9(1 + 7z)^1 \tag{2}$$

Given a polynomial factored into monomials, we do not know a nice or efficient way to express the number of its other such factorizations, but we can count them mod z^{2^n} *(simply meaning that we ignore all terms involving* z^{2^n} *for* $k \geq 2^n$*). For example,*

$$(1 + z)^6(1 + 2z)^1(1 + 4z)^1(1 + 6z)^3 = (1 + 3z)^{20}(1 + 5z)^{14}(1 + 7z)^4 \bmod z^8 \tag{3}$$

and we will show that the total number of possible distinct right hand sides in (3) is 2^{22} *if the exponents on the monomials* $(1 + jz)^k$ *(*$j = 1, 2, \ldots, 7$*) are restricted so that* $0 \leq k < 32$*; here 32 is the minimum value required to ensure "periodicity."*

One aim of this paper is to explain this example and to generalize it to other powers of 2. We hope that these examples entice the reader to keep reading.

The theory of symmetric functions has long been a basic tool of combinatorial enumeration. In some combinatorial settings it is useful to enumerate the number of variable substitutions to symmetric functions so that the functions achieve given values. Stanley discusses some of these issues in Section 7.8 of [6]. Our initial interest in the elementary symmetric functions stems from the counting of degree n monic irreducible polynomials over finite fields with prescribed coefficients for x^{n-1} and x^{n-2}. If such a polynomial is factored in a splitting field, these coefficients can be interpreted as the first and second elementary symmetric functions evaluated at the (circular) string of coefficients occurring in the factorization.

If a string α has its alphabet in a finite commutative ring R, we can evaluate the j-th elementary symmetric function e_j at α. This evaluation depends on the profile $\mathbf{k} = \langle k_1, k_2, \ldots\rangle$ of α where k_i is the frequency with which the ring element x_i occurs in α. The relationship between strings, polynomials, an elementary symmetric functions is contained in the map $E_{\mathbf{k}}(z) := \prod(1+jz)^{k_j}$ since $e_j(\alpha) = [z^j]E_{\mathbf{k}}(z)$. This relationship can be refined to give a sequence of mappings $h_n : \mathbb{Z}_m^{(|R|-1)} \to G$, where G is an appropriate multiplicative subgroup of $\mathbb{Z}_\ell[[z]]$ where m and ℓ depend on n.

In [3] we studied the the case $R = \mathbb{Z}_p$, where p is prime. These results were then used in [4] in order to enumerate certain circular strings. Here we choose the substitutions to come from the ring of integers mod 2^d. A fundamental difference between the case considered in [3] is that in the $\mathbb{Z}_p$ case the h_n are one-to-one, whereas in the $\mathbb{Z}_{2^d}$ case, they are not. However, there is an underlying group homomorphism and a periodic repetition which will allow us to provide much structural information and a complete characterization for specific small values of d. As a byproduct, we are able to enumerate the number of non-unique factorizations of certain types of polynomials in $\mathbb{Z}_{2^d}[z]$.

A primary aim in this extended abstract is to state/prove some basic facts about h_n, particularly about its kernel; most proofs have been omitted, although a few proof sketches are given. In doing so we will make use of some binomial coefficient congruences and manipulations of formal power series. Interestingly, it will prove useful to allow the profiles contain negative integers and to use the infinite version of the homomorphism which we call h_∞. In the final part of the paper we apply the necessary conditions established earlier to determine the kernel for $d = 2$ and $d = 3$ and give the range for $d = 2$. In principle, the same approach would work for higher values of d, but the computations required become prohibitive.

2 Notation and Preliminaries

In this section we carefully define the problem and introduce some of the basic tools. All computations are done mod q. We set $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$ to denote the ring of integers mod q.

2.1 Strings

Consider a string $\alpha = a_1a_2\cdots a_m$ where each $a_i \in \mathbb{Z}_q$. The j-th elementary symmetric function evaluated at α, denoted $e_j(\alpha)$, is the sum

$$e_j(\alpha) := \sum_{1\le i_1<i_2<\cdots<i_j\le m} a_{i_1}a_{i_2}\cdots a_{i_j} \pmod q.$$

Clearly, $(-1)^j e_j(\alpha)$ is the coefficient of z^{n-j} in the polynomial $(z-a_1)(z-a_2)\cdots(z-a_m)$.

By $S_q(m;\tau_1,\tau_2,\ldots,\tau_t)$ we denote the number of strings α over $\mathbb{Z}_q$ of length m for which $e_i(\alpha) = \tau_i$ for $i = 1,2,\ldots,t$. Obviously if $t = 0$, then $S_q(m) = q^m$. It is also true that $S_q(m;s) = q^{m-1}$ for any $s \in \mathbb{Z}_q$, since $e_1(\alpha x)$ takes on distinct values for each $x \in \mathbb{Z}_q$. The numbers $S_q(m;\tau_1,\tau_2,\ldots,\tau_t)$ satisfy the following recurrence relation. If $n = 1$, then $S_q(m;\tau_1,\tau_2,\ldots,\tau_t) = [\![\tau_2 = \cdots = \tau_t = 0]\!]$, and for $m > 0$,

$$S_q(m;\tau_1,\tau_2,\ldots,\tau_t) = \sum_{x\in\mathbb{Z}_q} S_q(m-1;\rho_1,\rho_2,\ldots,\rho_t), \tag{4}$$

where $\rho_0 = 1$, and $\rho_i = \tau_i - \rho_{i-1}x$ for $i = 1,2,\ldots,t$.

Recurrence relation (4) implies that the power series $\sum_{m\geq 0} S_q(m;\tau_1,\tau_2,\ldots,\tau_t)z^m$ is rational. We can evaluate $S_q(m;\tau_1,\tau_2,\ldots,\tau_t)$ by creating a table of size mq^t consisting of S_q for all strings of length at most m and over the first t elementary symmetric functions. Each table entry requires $\Theta(qt)$ ring operations and $\Theta(q)$ arithmetic operations, for a total of $\Theta(mtq^{t+1})$ ring operations and $\Theta(mq^{t+1})$ arithmetic operations. An aim of this paper is to reduce the number of ring and arithmetic operations required to evaluate S_q.

2.2 Profiles

Suppose that the string α has k_x occurrences of the symbol x for $x \in \mathbb{Z}_q$. We refer to the $(q-1)$-tuple of natural numbers $\mathbf{k} = \langle k_1, k_2, \ldots, k_{q-1}\rangle$ as the *profile* of the string. The elementary symmetric function $e_j()$ depends only on the profile. Note that k_0 is omitted since it does not affect $e_j()$. It will prove useful to have profiles consisting of integers, positive or negative; and to have profiles consisting of integers mod a natural number. Which case is in effect will usually be obvious from context. From now on, a bold letter will only denote a profile. We add profiles componentwise and define $x\mathbf{k} = \langle xk_1, xk_2, \ldots, xk_{q-1}\rangle$.

For $\mathbf{k} = \langle k_1, k_2, \ldots, k_{q-1}\rangle \in \mathbb{Z}^{q-1}$, define in $\mathbb{Z}_q[[z]]$ the formal power series

$$E_{\mathbf{k}}(z) := \prod_{j=1}^{q-1}(1+jz)^{k_j} \tag{5}$$

We make no assumption here that the k_i are positive.

Observe that $e_j(\alpha) = [z^j]E_{\mathbf{k}}(z)$, where the notation $[z^j]A(z)$ means the coefficient of z^j in the generating function $A(z)$. Clearly,

$$E_{\mathbf{a}+\mathbf{b}}(z) = E_{\mathbf{a}}(z)E_{\mathbf{b}}(z) \tag{6}$$

We also denote the $e_j(\alpha)$ by $e_j(\mathbf{k})$ or $e_j(\langle k_1, k_2, \ldots, k_{q-1}\rangle)$ when we wish to emphasize the role of profiles.

The evaluation of S_q in terms of profiles is given by

$$S_q(m;\tau_1,\tau_2,\ldots,\tau_t) = \sum_{\substack{k_0+k_1+\cdots+k_{q-1}=m \\ \mathbf{k}:=\langle k_1,\ldots,k_{q-1}\rangle}} \binom{m}{k_0, k_1, \ldots, k_{q-1}} \prod_{i=1}^{t} [\![e_i(\mathbf{k}) = \tau_i]\!]. \tag{7}$$

In order to evaluate (7) efficiently we need to be able to determine efficiently those profiles $\mathbf{k}$ for which $e_i(\mathbf{k}) = \tau_i$ for $i = 1, 2, \ldots, t$. We do this by recasting the conditional as

$$\prod_{i=1}^{t} [\![e_i(\mathbf{k}) = \tau_i]\!] = [\![E_{\mathbf{k}}(z) \bmod z^{t+1} = \sum_{i=0}^{t} \tau_i z^i]\!],$$

where τ_0 is defined to be 1.

This approach to the problem was established in [3] in the case where $p = q$ is prime. There it is proven that there is a bijection between the set of all polynomials $\sum_{0\leq i<p} \tau_i z^i$ in $\mathbb{Z}_p[z]$ and $E_{\mathbf{k}}(z) \bmod z^p$ where $\mathbf{k} \in \mathbb{Z}_p^{(p-1)}$. This bijection is then extended to a bijection between polynomials

$$\sum_{j=0}^{m-1}\sum_{i=0}^{p-1} \tau_{ip^j} z^{ip^j} \text{ where } \tau_{ip^j} \in \mathbb{Z}_p$$

and $E_{\mathbf{k}}(z) \bmod z^{p^m}$ where $\mathbf{k} \in \mathbb{Z}_{p^m}^{(p-1)}$. For $q = 2^d$ the situation is considerably more complicated. Our first goal is to determine the algebraic structure of those $\mathbf{k}$ for which $E_{\mathbf{k}}(z) = 1$.

3 General Results

3.1 Periodicity and group structure

Our initial aim is to establish the periodic nature of the profiles $\mathbf{k}$ when used to determine $E_{\mathbf{k}}(z)$. In this section all computation is done mod 2^d unless noted otherwise.

THEOREM 1 *If $0 \leq s \leq d-1$, then as polynomials in two variables y and z,*

$$(1 + (y + 2^{d-s})z)^{2^s} = (1 + yz)^{2^s} \bmod 2^d.$$

LEMMA 1 *With arithmetic mod 2^d and $0 < t \leq d$, where b, t, d, m are integers,*

$$(1 + 2^t bz)^m = (1 + 2^t bz)^{m \bmod 2^{d-t}}.$$

THEOREM 2 *With arithmetic mod 2^d, for any $n \geq 1$, we have $E_{2^{d+n-1}\mathbf{k}}(z) = E_{2^{d-1}\mathbf{k}}(z^{2^n})$.*

Proof: Our proof is by induction on n; details omitted. □

COROLLARY 1 (PERIODICITY) *In $\mathbb{Z}_{2^d}[[z]] \bmod z^{2^n}$,*

$$E_{\mathbf{a}+2^{d+n-1}\mathbf{b}}(z) = E_{\mathbf{a}}(z).$$

Proof: Follows from (6) and Theorem 2. □

This last corollary implies that if we are only considering $e_j()$ with $j < 2^n$, then we need only consider values of the profile taken mod 2^{d+n-1}.

THEOREM 3 *The set $M_n = \{E_{\mathbf{a}}(z) \bmod z^{2^n} \mid \mathbf{a} \in \mathbb{Z}_{2^{d+n-1}}^{(2^d-1)}\}$ is a multiplicative group in $\mathbb{Z}_{2^d}[[z]] \bmod z^{2^n}$, where the multiplication operation is polynomial multiplication mod z^{2^n}.*

For each $n \in \mathbb{Z}^+$, define the map $h_n : \mathbb{Z}_{2^{d+n-1}}^{(2^d-1)} \mapsto M_n$ that takes $\mathbf{a}$ to $E_{\mathbf{a}}(z) \pmod{z^{2^n}}$. We also define the set $M_\infty = \{E_{\mathbf{a}}(z) \subseteq \mathbb{Z}_{2^d}[[z]] \mid \mathbf{a} \in \mathbb{Z}^{(2^d-1)}\}$ and the map $h_\infty : \mathbb{Z}^{(2^d-1)} \mapsto M_\infty$ that takes $\mathbf{a}$ to $E_{\mathbf{a}}(z)$ (no mod-ing by z^{2^n}). Clearly M_∞ is also a group, where the operation is multiplication of power series in $\mathbb{Z}_{2^d}[[z]]$.

THEOREM 4 *For each $n > 0$, the map h_n is a group homomorphism. The map h_∞ is also a group homomorphism.*

The fact that h_n is a homomorphism can be used to garner information about certain polynomials. In general the *kernel*, Ker h of a homomorphism h is the set of elements in the domain that are mapped to the identity element in the range. In our case Ker $h_n = \{\mathbf{a} \in \mathbb{Z}_{2^{d+n-1}}^{(2^d-1)} \mid 1 = E_{\mathbf{a}}(z)\}$. Since there are $2^{(2^d-1)(d+n-1)}$ elements in the domain of h_n, the number of distinct polynomials of the form $E_{\mathbf{a}}(z)$ in the range of h_n is

$$\frac{2^{(2^d-1)(d+n-1)}}{|\text{Ker } h_n|}. \tag{8}$$

Note also that $|\text{Ker } h_n|$ is the number of distinct complete factorizations of any polynomial $E_{\mathbf{k}}(z)$ in $\mathbb{Z}_{2^d}[[z]] \bmod z^{2^n}$. The value of $|\text{Ker } h_n|$ is computed for $d = 2, 3$ later in the paper.

Since Ker h_∞ is closed under component-wise addition and scalar multiplication by integers, Ker h_∞ is a $\mathbb{Z}$-module. Similarly, Ker h_n is a $\mathbb{Z}_{2^{d+n-1}}$-module. We will show below that Ker h_∞ has a basis but Ker h_n does not, and determine the rank of Ker h_∞ for $d = 2, 3$ in later sections of the paper.

THEOREM 5 *A profile* $\mathbf{k} \in$ *Ker* h_∞ *if and only if* $\mathbf{k} \bmod 2^{d+n-1} \in$ *Ker* h_n *for all* $n \geq 0$.

For example, with $d = 2$, the identity $1 = (1+z)^{-2}(1+3z)^2$ holds and thus $\langle -2, 0, 2\rangle \in$ Ker h_∞. Hence, with $n = 3$ we have $\langle 14, 0, 2\rangle \in$ Ker h_3 and so $1 = (1+z)^{14}(1+3z)^2 \bmod z^8$.

We will need a variant of Theorem 5 which says that if $\mathbf{k}$ is in the kernel of h_n and n is large enough, then $\mathbf{k}$, appropriately normalized, is also in the kernel of h_∞. Before stating that result we need to define some notation and prove a small technical lemma. Let $\mathbf{u}_j$ denote the unit profile whose i-th entry is equal to $[\![i = j]\!]$. For $0 \leq s \leq d-1$ and $x, y \in \mathbb{Z}_{2^d}$ we define the profile

$$\mathbf{u}(s; x, y) := 2^s \mathbf{u}_x - 2^s \mathbf{u}_{x+y2^{d-s}},$$

and the set of profiles

$$U_s := \{\mathbf{u}(s; x, y) \mid x, y \in \mathbb{Z}_{2^d}\}.$$

By Theorem 1 $U_s \subseteq$ Ker h_∞ for each s. For example, with $d = 3$ we have $\mathbf{u}(2; 1, 3) = \langle 4, 0, 0, 0, 0, 0, -4\rangle \in$ Ker h_∞ since $(1+z)^4 = (1+3z)^4 = (1+5z)^4 = (1+7z)^4$ by Theorem 1.

LEMMA 2 *For all* $n, d \geq 1$, *if* $2^{n-1} \leq k < 2^{d+n-1}$, *then* $\binom{k}{2^{n-1}} \not\equiv 0 \bmod 2^d$.

THEOREM 6 *There is a smallest value* $N(d)$, *dependent only on* d, *with the following property: If* $n \geq N(d)$ *and* $\mathbf{k} \in$ *Ker* h_n, *then there is a* $\mathbf{k}' \equiv \mathbf{k} \bmod 2^{d+n-1}$ *such that* $\mathbf{k}' \in$ *Ker* h_∞.

Proof: (sketch) Assume that $1 = E_{\mathbf{k}}(z) \bmod z^{2^n}$ for some n. The main idea of the proof is to apply an "exponent reduction" of the k_i with $i > 1$ using the sets U_s for $s = d-1, d-2, \ldots, 2, 1$ for the odd i and Lemma 1 for the even i. At the end of the reduction process, we can express $\mathbf{k} = \mathbf{a} + \mathbf{v}$ where $\mathbf{v} \in$ Ker h_∞ is a linear combination of the $\mathbf{u}(s; x, y)$ and the $\mathbf{u}_i$. In addition $\sum_{i=2}^{2^d-1} a_i \leq D_d$, where

$$D_d := (2^d + 2^{d-1} - 2d - 1) + (d-1) = 2^d + 2^{d-1} - d - 2.$$

We can thus write

$$(1+z)^{a_1 \bmod 2^{d+n-1}} = P(z) + O(z^{2^n}) \tag{9}$$

where $P(z)$ is a polynomial of degree at most D_d.

Below is a table of the values of D_d. Note that $1 + \lceil \lg(D_d+1) \rceil = d + 2$ for $d \geq 4$.

d	2	3	4	5	6	7	8	9	10	11	12
D_d	2	7	18	41	88	183	374	757	1524	3059	6130
$1+\lceil \lg(D_d{+}1)\rceil$	3	4	6	7	8	9	10	11	12	13	14

We now want to show that $(a_1 \bmod 2^{d+n-1}) \le 2^n$ for large enough n. It then follows that $(1+z)^a = P(z)$ where $a = a_1 \bmod 2^{d+n-1}$, which will prove the theorem. Let n be such that $\deg(P(z)) \le D_d < 2^{n-1}$. By Lemma 2, $[z^{2^{n-1}}](1+z)^a \ne 0$ for any a in the range $2^{n-1} \le a < 2^{d+n-1}$. Thus $a < 2^{n-1}$ and so

$$(1+z)^a = P(z) \text{ in } \mathbb{Z}_d[[z]].$$

Taking $n = 1+\lceil \lg(D_d{+}1)\rceil$ the theorem is proven, $1+\lceil \lg(D_d{+}1)\rceil$ is an upper bound on $N(d)$. □

EXAMPLE 3 *We illustrate the proof technique of the preceding theorem. In this example we take* $d=3$ *(so arithmetic is mod 8). Consider the profile* $\mathbf{k} = \langle 63, 67, 3, 1, 61, 5, 65\rangle$. *A Maple calculation reveals that*

$$(1{+}z)^{63}(1{+}2z)^{67}(1{+}3z)^3(1{+}4z)^1(1{+}5z)^{61}(1{+}6z)^5(1{+}7z)^{65} = 1 + O(z^{16})$$

Thus we want $n=4$, *and so* $d+n-1=6$. *The even indexed factors give*

$$(1+2z)^{67}(1+4z)^1(1+6z)^5 = (1+2z)^3(1+4z)^1(1+6z)^1 = 1.$$

We can write the linear combination

$$\begin{aligned} \mathbf{k} &= 16\cdot\langle 4,0,0,0,0,0,-4\rangle + 30\cdot\langle 2,0,0,0,-2,0,0\rangle + \langle 187,3,3,1,1,1,1\rangle \\ &= 16\cdot\mathbf{u}(2;1,3) + 30\cdot\mathbf{u}(1;1,1) + 4\cdot\mathbf{u}_6 + \langle 187,3,3,1,1,1,1\rangle \end{aligned}$$

Thus

$$(1+z)^{-187} = (1+3z)^3(1+5z)^1(1+7z)^1 \bmod z^{16},$$

from which it follows that $(1+z)^{-5}(1+3z)^3(1+5z)(1+7z) = 1$ *and so* $\langle -5,3,3,1,1,1,1\rangle \in$ *Ker* h_∞ *and* $\mathbf{k}' = \langle -129,3,3,1,61,5,65\rangle \in$ *Ker* h_∞, *where* $\mathbf{k}' \equiv \mathbf{k} \bmod 2^{d+n-1}$.

3.2 An even-odd decomposition of the kernel

Define

$$E_n := \{(k_2, k_4, \ldots, k_{2^d-2}) \mid 1 = \prod_{j=1}^{2^d-1} (1+jz)^{k_j [\![j \text{ even}]\!]} \text{ in } \mathbb{Z}_{2^d}[[z]] \bmod z^{2^n}\},$$

$$O_n := \{(k_1, k_3, \ldots, k_{2^d-1}) \mid 1 = \prod_{j=1}^{2^d-1} (1+jz)^{k_j [\![j \text{ odd}]\!]} \text{ in } \mathbb{Z}_{2^d}[[z]] \bmod z^{2^n}\}.$$

The sets E_∞ and O_∞ are defined analogously by removing the $\bmod z^{2^n}$.

THEOREM 7 *The kernels can be decomposed into the following cartesian products*

$$\textit{Ker } h_\infty = E_\infty \times O_\infty, \textit{ and}$$

$$\textit{Ker } h_n = E_n \times O_n, \textit{ if } n \ge N(d),$$

subject to a shuffling of the indices.

Proof: (Sketch.) We first treat h_∞. By Lemma 1 we may assume that all the even indexed profile numbers k_{2i} are non-negative. Re-arranging the equation $E_{\mathbf{k}}(z) = 1$, we have the following equality of *polynomials*

$$\prod_{j=1}^{2^d-1}(1+jz)^{k_j[\![j \text{ even}]\!]}\prod_{j=1}^{2^d-1}(1+jz)^{k_j[\![k_j>0]\!][\![j \text{ odd}]\!]} = \prod_{j=1}^{2^d-1}(1+jz)^{-k_j[\![k_j<0]\!][\![j \text{ odd}]\!]}.$$

The leading coefficient, $\prod_{j \text{ odd}} j^{-k_j}$, of the polynomial on the right must be odd. The leading coefficient of the polynomial on the left will be even unless $1 = \prod_{j=1}^{2^d-1}(1+jz)^{k_j[\![j \text{ even}]\!]}$. Thus $(k_2, k_4, \ldots, k_{2^d-2}) \in E_\infty$ and hence $(k_1, k_3, \ldots, k_{2^d-1}) \in O_\infty$.

If $\mathbf{k} \in$ Ker h_n, then by Theorem 6 there is a $\mathbf{k}' \in$ Ker h_∞ such that $\mathbf{k}' \equiv \mathbf{k} \bmod 2^{d+n-1}$. By our previous discussion $\mathbf{k}' = \mathbf{e}' \times \mathbf{o}'$ where $\mathbf{e}' \in E_\infty$ and $\mathbf{o}' \in O_\infty$. By Theorem 5, it follows that $\mathbf{e} \in E_n$ and $\mathbf{o} \in O_n$, where $\mathbf{e}$ and $\mathbf{o}$ are defined as expected.

The h_n case follows from Theorem 6. □

LEMMA 3 *The following two conditions are necessary for membership in the respective kernels.*

- *If $\mathbf{k} \in$ Ker h_∞, then $\sum_{j=1}^{2^d-1} k_j[\![j \text{ odd}]\!] = 0$. This is an integer sum.*
- *If $\mathbf{k} \in$ Ker h_n and $n \geq N(d)$, then $\sum_{j=1}^{2^d-1} k_j[\![j \text{ odd}]\!] = 0 \bmod 2^{d+n-1}$.*

COROLLARY 2 *The $\mathbb{Z}$-module Ker h_∞ has a basis.*

Proof: (Sketch.) Show that Ker h_∞ is finitely-generated and torsion-free. Any finitely-generated torsion-free module has a basis. □

The rank of Ker h_∞ is at most $2^d - 1$ since it is a sub-module of $\mathbb{Z}^{(2^d-1)}$. After proving the following technical lemma, we will establish a useful necessary condition for membership in Ker h.

LEMMA 4 *For all $j \in \mathbb{Z}_{2^d}$, where $d \geq 4$,*

$$j^{2^{d-2}} \equiv [\![j odd]\!] \pmod{2^d}$$

If $d = 2$ exceptions occur for $j = 2, 3$, since $2^{2^0} \equiv 2$ and $3^{2^0} \equiv 3$ mod 4. If $d = 3$ exceptions occur for $j = 2, 6$, since $2^{2^1} \equiv 6^{2^1} \equiv 4 \bmod 8$.

LEMMA 5 *The logarithmic derivative of $E_{\mathbf{k}}(z)$ can be written as*

$$\frac{d}{dz}\log E_{\mathbf{k}}(z) = \sum_{k=0}^{d-2}(-z)^k \sum_{j=1}^{2^d-1} k_j j^{k+1}[\![j \text{ even}]\!] + \sum_{k\geq 0}(-z)^k \sum_{j=1}^{2^d-1} k_j j^{(k+1) \bmod P}[\![j \text{ odd}]\!],$$

where $P = 2^{d-2}$ if $d \geq 3$ and $P = 2$ if $d = 2$.

Proof: (Sketch.) Expand. The left part of the sum is a polynomial since if j is even and $k+1 \geq d$, then $j^{k+1} = 0 \bmod 2^d$. The right part of the sum has periodic coefficients by Lemma 4. □

LEMMA 6 *The conditions listed below are necessary for a profile* $\mathbf{k}$ *to be in Ker* h_∞ *or in Ker* h_n *if* $n \geq N(d)$.

$$0 = \sum_{j=1}^{2^d-1} k_j j^{k+1} [\![j \textit{ even}]\!] \bmod 2^d, \textit{ for } k = 0, 1, \ldots, d-2 \tag{10}$$

$$0 = \sum_{j=1}^{2^d-1} k_j j^{k+1} [\![j \textit{ odd}]\!] \bmod 2^d \textit{ for } k = 0, 1, \ldots, P-1, \tag{11}$$

where $P = 2^{d-2}$ *if* $d \geq 3$ *and* $P = 2$ *if* $d = 2$.

Proof: Omitted. □

The $k = d-2$ condition in (10) is implied by the $k = d-3$ condition. In a similar vein, when $k = 2^{d-2}-1$ condition (11) becomes $0 = \sum_{j=1}^{2^d-1} k_j [\![j \text{ odd}]\!]$.

To finish this section we will determine the cardinality of Ker h_1. In the case where $n = 1$ the condition $0 = [z]E_{\mathbf{k}}(z) = \sum_j jk_j$ is both necessary and sufficient since $\bmod 2^{d+n-1} = 2^d$. Since we can solve for k_1 for any values of $k_2, k_3, \ldots, k_{2^d-1}$,

$$|\text{Ker } h_1| = 2^{d(2^d-2)}. \tag{12}$$

4 The kernel for small values of d

In this section we determine the kernels of h_∞ and h_n for $d = 2$ and $d = 3$.

4.1 The kernel when $d = 2$

THEOREM 8 *Ker* $h_\infty = \{\mathbf{k} \mid k_1 \equiv k_2 \equiv k_3 \equiv 0 \bmod 2$ *and* $k_1 + k_3 = 0\}$.

COROLLARY 3 *For the* $\mathbb{Z}$*-module Ker* h_∞, $\{\langle -2, 0, 2\rangle, \langle 0, 2, 0\rangle\}$ *is a basis.*

THEOREM 9 *If* $n = 1$, *then*

$$\begin{aligned} \textit{Ker } h_1 \;=\; & \{\langle 0,0,0\rangle, \langle 0,2,0\rangle, \langle 2,0,2\rangle, \langle 2,2,2\rangle, \langle 1,1,3\rangle, \langle 3,1,1\rangle, \langle 1,3,3\rangle, \langle 3,3,1\rangle, \\ & \langle 0,0,2\rangle, \langle 0,2,2\rangle, \langle 2,0,0\rangle, \langle 2,2,0\rangle, \langle 1,0,1\rangle, \langle 3,0,3\rangle, \langle 1,2,1\rangle, \langle 3,2,3\rangle\}. \end{aligned}$$

If $n > 1$, *then*

$$\textit{Ker } h_n = \{\mathbf{a} \mid a_1 = a_2 = a_3 = 0 \bmod 2 \textit{ and } a_1 + a_3 = 0 \bmod 2^{n+1}\}$$

Proof: An exhaustive computation can be used to verify the result for $n = 1$ and $n = 2$. Assume that $n > 2$. The result follows from applying Theorem 6 to the kernel of h_∞ as expressed in Theorem 8. Theorem 6 can be used for any $n \geq N(2) = 3$. □

LEMMA 7

$$|Ker\ h_n| = \begin{cases} 16 & \text{if } n = 1 \\ 2^{2n} & \text{if } n > 1 \end{cases}$$

Proof: The $n = 1$ result is clear from the previous theorem. Use Theorem 9. Mod 2^{n+1} the value of k_3 is determined by the value of k_1. There are 2^n even elements in $\mathbb{Z}_{2^{n+1}}$. Thus there are 2^n choices for k_1 and 2^n choices for k_2, for a total of 2^{2n} choices. □

Since there are 2^{2n} elements in the kernel of h_n, by the properties of homomorphisms, the number of distinct polynomials in the range of h_n is $2^{3n+3}/2^{2n} = 2^{n+3}$ if $n > 1$. Another consequence is that the number of distinct factorizations of $E_{\mathbf{k}}(z) \bmod z^{2^n}$ in $\mathbb{Z}_4[z]$ is 2^{2n} if $n > 1$.

4.2 The kernel when $d = 3$

The necessary conditions from Lemma 6 imply the following for the even indexed profile numbers:

$$k_2 + 2k_4 + 3k_6 \equiv 0 \bmod 4. \tag{13}$$

For the odd indexed profile numbers we have

$$\begin{aligned} k_1 + k_3 + k_5 + k_7 &= 0 \\ k_1 + 3k_3 + 5k_5 + 7k_7 &\equiv 0 \bmod 8 \\ k_1 + k_3 + k_5 + k_7 &\equiv 0 \bmod 8 \end{aligned}$$

These conditions are not sufficient, but the changes required to make them sufficient are small.

THEOREM 10 *The set E_∞, is a $\mathbb{Z}$-module with basis $B = \{(4,0,0), (2,0,2), (3,1,1)\}$.*

Proof: By Lemma 1, we have with arithmetic mod 8, $(1 + 2z)^k = (1 + 2z)^{k \bmod 4}$, $(1 + 6z)^k = (1 + 6z)^{k \bmod 4}$, and $(1 + 4z)^k = (1 + 4z)^{k \bmod 2}$.

The profiles that satisfy the necessary condition (13) can therefore be classified as $(k_2 \bmod 4, k_4 \bmod 2, k_6 \bmod 4)$, where an exhaustive listing gives

$$\{(0,0,0), (2,0,2), (1,1,3), (3,1,1)\} \cup \{(1,0,1), (3,0,3), (0,1,2), (2,1,0)\}.$$

A routine calculation shows that the left set is in the kernel, but the right set is not. To show that B is a basis, we first note that it is linearly independent, since the system of equations (14) has only the solution $n_1 = n_2 = n_3 = 0$.

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \tag{14}$$

To show that B spans E_∞ note that $(0,2,0) = 2 \cdot (3,1,1) - (4,0,0) - (2,0,2)$, $(0,0,4) = 2 \cdot (2,0,2) - (4,0,0)$, and $(1,1,3) = (3,1,1) + (2,0,2) - (4,0,0)$. □

COROLLARY 4 *A profile $\mathbf{k} = \langle k_2, k_4, k_6 \rangle$ is in E_∞ if and only if $k_2 \equiv k_4 \equiv k_6 \bmod 2$ and $k_2 + 2k_4 + 3k_6 \equiv 0 \bmod 4$.*

We now turn our attention to the odd indexed profile numbers.

THEOREM 11 *The set* O_∞ *is a* $\mathbb{Z}$*-module with basis* $B = \{(1, 1, -1, -1), (-1, 1, 1, -1), (4, -4, 0, 0)\}$.

By Theorems 10 and 11, the rank of Ker h_∞ is 6.

LEMMA 8 *The value of* $|E_n| \cdot |O_n|$ *over* $\mathbb{Z}_8$ *is*

$$2^{3n+3} \cdot \begin{cases} 512 & \text{if } n = 1, \\ 1024 & \text{if } n = 2, \\ 2^{3n+3} & \text{if } n \geq 3. \end{cases}$$

Proof: The number of kernel elements in E_n is 2^{3n+3}.

In the case where operations are done mod $2^{d+n-1} = 2^{n+2}$, a certain linear system, used in the proof of the previous theorem, has 8 distinct solutions, namely

$$n_1 = n_2 \in \{0, 2^{n+1}\} \text{ and } n_3 \in \{0, 2^n, 2^{n+1}, 3 \cdot 2^n\}.$$

Note that these solutions are the submodule with basis $\{\langle 2^{n+1}, 2^{n+1}, 0\rangle, \langle 0, 0, 2^n\rangle\}$. The number of kernel elements in O_n is therefore $2^{3(n+2)}/8 = 2^{3n+3}$, since there are three basis elements and any kernel element can be written in exactly 8 distinct ways as linear combination of basis elements, where the coefficients of the combination come from $\mathbb{Z}_{2^{d+n-1}} = \mathbb{Z}_{2^{n+2}}$. □

LEMMA 9 *The value of* $|$*Ker* $h_n|$ *over* $\mathbb{Z}_8$ *is*

$$\begin{cases} 2^{18} = 262144 & \text{if } n = 1, \\ 2^{19} = 524288 & \text{if } n = 2, \\ 2^{22} = 4194304 & \text{if } n = 3, \\ 2^{6n+6} & \text{if } n \geq 4. \end{cases}$$

Proof: The value for $|$Ker $h_1|$ is from (12). The value for $|$Ker $h_2|$ and $|$Ker $h_3|$ is from an exhaustive computer listing [5]. Since $N(3) \leq 4$, the value for $n \geq 4$ follows from Lemma 8. Note that $N(3) = 4$ since $22 \neq 24 = 6 \cdot 3 + 6$. □

4.3 The range of the kernel when $d = 2$

In this subsection all computation is done mod 4. We will show that the indices of certain "critical" elementary symmetric functions determine the remaining elementary symmetric function values. These critical indices occur at the powers of two. We can use this information to get fast algorithms for converting between a profile and elementary symmetric function evaluations.

LEMMA 10 *Let* $k' = 2^n + k$. *Then*

$$[z^{2^{n-1}}]E_{k',x,y}(z) = 2 + [z^{2^{n-1}}]E_{k,x,y}(z).$$

It is easy to see, for example, by an exhaustive listing, that there is a bijection between profiles in $\mathbb{Z}_{16} \times \mathbb{Z}_2^{(2)}$ and triples $(e_1, e_2, e_4) \in \mathbb{Z}_4^{(3)}$.

EXAMPLE 4 *This is an explanation of Example 2 from the Introduction. What is the profile, if any, that corresponds to the sequence of six elementary symmetric function values* $e_1, e_2, e_4, e_8, e_{16}, e_{32} = 0, 3, 3, 2, 3, 3$*? Consider first* $e_1, e_2, e_4 = 0, 3, 3$ *which corresponds to profile* $6, 1, 0 \bmod 16$*. Here* $e_8(6, 1, 0) = 0$*, so Lemma 10 tells us to add 16 to* k_1 *to get* $e_8(22, 1, 0) = 2$*, while preserving the values of* e_1, e_2, e_4*. In a similar manner, since* $e_{16}(22, 1, 0) = 1$*, we add 32 to* k_1 *to get* $e_{16}(54, 1, 0) = 3$*. Now* $e_{32}(54, 1, 0) = 3$*, so we are done. Any profile that has* k_1 *and* k_3 *even,* k_2 *odd, and* $k_1 + k_3 \equiv 54$ *mod 64 has the required trace values. Furthermore, these determine all traces* e_j *where* $j = 1, 2, \ldots, 63$ *as per the theorem stated below.*

We can extrapolate this example to an algorithm whose running time is $O(n)$. The running time of this algorithm is clearly $O(n)$ so long as the values of $e_{2^j}(\mathbf{k})$ can be computed in constant time. We show how to do this, essentially by a table lookup, in the full paper.

THEOREM 12 *The values of* e_{2^i} *for* $i = 0, 1, \ldots, n-1$ *determine the values of* e_j *for* $j = 1, 2, \ldots, 2^n - 1$.

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Patterns in matchings and rook placements

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Abstract. Extending the notion of pattern avoidance in permutations, we study matchings and set partitions whose arc diagram representation avoids a given configuration of three arcs. These configurations, which generalize 3-crossings and 3-nestings, have an interpretation, in the case of matchings, in terms of patterns in full rook placements on Ferrers boards. We enumerate 312-avoiding matchings and partitions, obtaining algebraic generating functions, unlike in the 321-avoiding (i.e., 3-noncrossing) case. Our approach also provides a more direct proof of a formula of Bóna for the number of 1342-avoiding permutations. Additionally, we give a bijection proving the shape-Wilf-equivalence of the patterns 321 and 213 which simplifies existing proofs by Backelin–West–Xin and Jelínek.

Résumé. Étendant la notion de motifs exclus dans des permutations, nous étudions des appariements et partitions dont le diagramme d'arc évite une configuration donnée de trois arcs. Ces configurations, qui généralisent les 3-croissements et les 3-emboîtements, ont une interprétation, dans le cas d'appariements, en termes de motifs dans des placements pleins de tours sur des tables de Ferrers. Nous énumérons les appariements et les partitions qui évitent 312, obtenant des séries génératrices algébriques, contrairement au cas du motif 321. Notre approche fournit aussi une démonstration plus directe d'une formule de Bóna pour le nombre de permutations qui évitent 1342. En plus, nous donnons une preuve bijective de l'équivalence au sens de la forme et de Wilf des motifs 321 et 213 qui simplifie les preuves de Backelin–West–Xin et Jelínek.

Keywords: matching, set partition, bijection, pattern avoidance, shape-Wilf-equivalence, rook placement, Dyck path.

1 Introduction

Pattern avoidance in matchings is a natural extension of pattern avoidance in permutations. Indeed, a permutation of $[n] = \{1, 2, \ldots, n\}$ can be thought of as matching of $[2n]$ where each element of $[n]$ is paired up with an element of $[2n] \setminus [n]$. The natural translation of the definition of patterns in permutations to this type of matchings extends to all perfect matchings, and more generally, to set partitions —which, when all the blocks have size 2, are just perfect matchings. We will use the term matching to refer to a perfect matching, when it creates no confusion. On the other hand, the well-studied notions of k-crossings and k-nestings in matchings and set partitions, in our language, are simply occurrences of the patterns $k \ldots 21$ and $12 \ldots k$, respectively. Additionally, by viewing matchings as certain fillings of Ferrers boards, patterns in matchings relate to patterns in Ferrers boards, and thus to the concept of shape-Wilf-equivalence of permutations.

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Motivated by these connections and by the recent work on crossings, nestings, permutation patterns, and shape-Wilf-equivalence, we study matchings and partitions that avoid patterns of length 3. We consolidate and simplify recent work on the classification of these patterns, and we obtain new results on the enumeration of matchings and partitions that avoid some of these patterns.

1.1 Background

We represent a matching of $[2n]$ as an arc diagram as follows: place $2n$ equally spaced points on a horizontal line, numbered from left to right, and draw an arc between the two vertices of each of the n pairs. The picture on the left of Fig. 2 corresponds to the matching $(1,6),(2,12),(3,4),(5,7),(8,10),(9,11)$. If $i<j<k<\ell$, two arcs (i,k), (j,ℓ) form a crossing, and two arcs (i,ℓ), (j,k) form a nesting. Similarly, a partition of $[n]$ is represented by drawing, for each block $\{i_1,i_2,\ldots,i_a\}$ of size a with $i_1<i_2<\cdots<i_a$, $a-1$ arcs $(i_1,i_2),(i_2,i_3),\ldots,(i_{a-1},i_a)$. A crossing in the partition is then a pair of arcs (i,k), (j,ℓ), and a nesting is a pair of arcs (i,ℓ), (j,k), where $i<j<k<\ell$.

Crossings and nestings in matchings and partitions have been studied for decades. It is well known that the number of perfect matchings on $[2n]$ with no crossings (or with no nestings) is the n-th Catalan number C_n, which also equals the number of partitions of $[n]$ of with no crossings, and the number of those with no nestings. More generally, attention has focused on the study of k-crossings (k-nestings), which are sets of k pairwise crossing (respectively, nesting) arcs. For set partitions, the above definition, which we use throughout the paper, is the same given by Chen, Deng, Du, Stanley and Yan [7] and Krattenthaler [16]. However, we point out that different definitions of pattern avoidance for partitions have been introduced by Klazar [14] and Sagan [17].

The number of 3-nonnesting matchings of $[2n]$ (viewed as fixed-point-free involutions with no decreasing sequence of length 6) was found by Gouyou-Beauchamps [12], who recursively constructed a bijection onto pairs of noncrossing Dyck paths with $2n$ steps, counted by $C_nC_{n+2}-C_{n+1}^2$. More recently, Chen et al. [7] showed that the number of k-noncrossing matchings (i.e., containing no k-crossing) of $[2n]$ equals the number of k-nonnesting (i.e., containing no k-nesting) ones, for all k, and that the analogous results for partitions hold as well. Their proof, which uses vacillating tableaux and a variation of Robinson-Schensted insertion and deletion, also provides a bijection between k-noncrossing matchings and certain $(k-1)$-dimensional closed lattice walks, from where a determinant formula for the generating function in terms of hyperbolic Bessel functions follows.

Less is known about the enumeration of k-noncrossing set partitions. Bousquet-Mélou and Xin [5] settled the case $k=3$ using a bijection into lattice paths to derive a functional equation for the generating function, which then is solved by the kernel method. They showed that the generating function for 3-noncrossing set partitions is D-finite, that is, it satisfies a linear differential equation with polynomial coefficients. This is conjectured not to be the case for $k>3$. For k-nonnesting set partitions, additional functional equations for the generating functions have been obtained by Burrill et al. [6] using generating trees for open arc diagrams.

By interpreting matchings and partitions as rook placements on Ferrers boards and using the growth diagram construction of Fomin, Krattenthaler [16] gave a simpler description of the bijections in [7] proving the symmetry of crossing and nesting number on matchings and partitions. He extended the results to fillings of Ferrers boards with nonnegative integers. Other extensions have been given by de Mier [9] to fillings with prescribed row and column sums.

As mentioned before, k-crossings (respectively, k-nestings) in matchings have a simple interpretation as occurrences of the monotone decreasing (respectively, increasing) pattern of length k. In this paper we study and enumerate matchings that avoid other patterns of length 3, and in some cases, we extend our results to the enumeration of pattern-avoiding partitions. The translation of crossings and nestings to the language of permutation patterns becomes natural via a bijection between matchings and certain fillings of Ferrers boards, called full rook placements, described in Section 2.2. For such fillings, the definitions of pattern containment and avoidance in permutations generalize routinely, and they have been widely studied in the literature. In this setting, Stankova and West [19] introduced the concept of shape-Wilf-equivalence, and they showed that the patterns 231 and 312 are shape-Wilf-equivalent. A simpler proof of this fact was later given by Bloom and Saracino [3]. As we will see, if two patterns are shape-Wilf-equivalent, then the number of matchings avoiding one is the same as the number of those avoiding the other, and the same is true for partitions. Backelin, West and Xin [1] showed that $12\dots k$ and $k\dots 21$ are shape-Wilf-equivalent. A more direct proof of their result, which implies again that k-nonnesting and k-noncrossing matchings are equinumerous, was given by Krattenthaler [16]. It also follows from [1] that 123 and 213 are shape-Wilf-equivalent. Thus, there are three shape-Wilf-equivalence classes of patterns of length 3, namely $123 \sim 321 \sim 213$, $231 \sim 312$, and 132.

Jelínek [13] reproved some of these results independently in the context of matchings, by giving bijections between 231-avoiding matchings and 312-avoiding ones, and between 213-avoiding matchings and 123-avoiding (i.e. 3-nonnesting) ones.

Finally, let us mention that Stankova [18] compared, for each one of the three shape-Wilf-equivalence classes of patterns of length 3, the number of full rook placements on any given Ferrers board avoiding each a pattern in the class. She showed that the number of 231-avoiding placements is no larger than the number of 321-avoiding placements (this is also proved in [13]), which is in turn no larger than the number of 132-avoiding ones.

1.2 Structure of the paper

In Section 2 we define patterns in matchings, in set partitions, and in rook placements on Ferrers boards, and we set the notation for the rest of the paper. In Sections 3 and 4 we study two of the three shape-Wilf-equivalence classes of patterns of length 3. In Section 3 we give a new simple bijection between 123-avoiding matchings and 213-avoiding ones, as well as an extension of work of Gouyou-Beauchamps [12] for matchings with fixed points (i.e., not necessarily perfect). In Section 4 we enumerate 231-avoiding (equivalently, 312-avoiding) matchings and partitions, and we show that their generating functions are algebraic, in contrast to the case of 123-avoiding matchings [12] and partitions [5]. We then use our techniques for matchings to obtain a new proof of Bóna's formula enumerating 1342-avoiding permutations [4]. This leaves one pattern of length 3, namely 132, for which we have been unable to find a formula for the number of 132-avoiding matchings or partitions. We argue in [2] that this question is related to the outstanding open problem of enumerating 1324-avoiding permutations [4, 8]. Finally, Section 5 summarizes some results about matchings and partitions that avoid pairs of patterns of length 3. The proofs that are omitted in this extended abstract can be found in [2].

2 Matchings, partitions, and rook placements

2.1 Ferrers boards

A *Ferrers board* is a left-justified array of unit squares so that the number of squares in each row is less than or equal to the number of squares in the row below. To be precise, consider an $n \times n$ array of unit squares in the xy-plane, whose bottom left corner is at the origin $(0,0)$. The vertices of the unit squares are lattice points in $\mathbb{Z}^2$. For any vertex $V=(a,b)$, let $\Gamma(V)$ be the set of unit squares inside the rectangle $[0,a]\times[0,b]$. Then, a subset F of the $n\times n$ array with the property that $\Gamma(V)\subseteq F$ for each vertex in F is a Ferrers board. Equivalently, F is bounded by the coordinate lines and by a lattice path from $(0,n)$ to $(n,0)$ with east steps $(1,0)$ and south steps $(0,-1)$. We call this path the *border* of F, and we denote its vertices by $V_0,\dots,V_{2n}$, where $V_0=(0,n)$, $V_n=(n,0)$ and V_{i+1} is immediately below or to the right of V_i. An example of these definitions appears in [2, Fig. 1].

Definition 1 *A* full rook placement *is a pair (R,F) where F is a Ferrers board and R is a subset of squares of F (marked by placing a rook in each one of them) such that each row and each column of F contains exactly one rook. Let $\mathcal{R}_F$ denote the set of full rook placements on F.*

In this paper, the term *placement* will always refer to a full rook placement. For a Ferrers board F to admit a full rook placement, the number or nonempty rows must equal the number of nonempty columns, and the coordinates (x,y) of the vertices in the border of F must satisfy $x\geq y$. We denote by $\mathcal{F}_n$ the set of Ferrers boards satisfying this condition and having n nonempty rows and columns. The border of $F\in\mathcal{F}_n$, which we denote by D_F, is a lattice path from $(0,n)$ to $(n,0)$ with steps east ($e=(1,0)$) and south ($s=(0,-1)$) that remains above the line $y=n-x$. We denote by $\mathcal{D}_n$ the set of such paths, which we call *Dyck paths* of semilength n (despite being rotated from other standard ways of drawing them). The map $F\mapsto D_F$ is a trivial bijection between $\mathcal{F}_n$ and $\mathcal{D}_n$. A *peak* on a Dyck path is an occurrence of es (as consecutive steps), and a valley is an occurrence of se.

We let

$$\mathcal{R}_n=\bigcup_{F\in\mathcal{F}_n}\mathcal{R}_F$$

be the set of all placements on boards in $\mathcal{F}_n$. Denote by $\mathcal{S}_n$ the set of permutations of $\{1,2,\dots,n\}$. To each full rook placement (R,F) where $F\in\mathcal{F}_n$, one can associate a permutation $\pi_R\in\mathcal{S}_n$ by letting $\pi_R(i)=j$ if R has a rook in column i and row j (our convention is to number the columns of F from left to right and its rows from bottom to top, as in the usual cartesian coordinates). In the case that $F\in\mathcal{F}_n$ is the square Ferrers board, this map is a bijection between $\mathcal{R}_F$ and $\mathcal{S}_n$. More generally, given a vertex V of the border of F, the restriction of the placement R to the rectangle $\Gamma(V)$, which consists of the squares $R\cap\Gamma(V)$, determines a unique permutation in $\mathcal{S}_k$, where $k=|R\cap\Gamma(V)|$. This permutation is obtained by disregarding empty rows and columns, and then applying the above map. Under this correspondence it makes sense to consider concepts such as the longest increasing sequence in $R\cap\Gamma(V)$.

Recall that a permutation $\pi\in\mathcal{S}_n$ *avoids* another permutation $\tau\in\mathcal{S}_k$ (usually called a *pattern*) if there is no subsequence $\pi(i_1)\dots\pi(i_k)$ with $i_1<\dots<i_k$ that is order-isomorphic to $\tau(1)\dots\tau(k)$. The number of τ-avoiding permutations in $\mathcal{S}_n$ is denoted by $\mathcal{S}_n(\tau)$. Viewing permutations as full rook placements on the square Ferrers board, π avoids τ if the placement corresponding to τ cannot be obtained from the placement corresponding to π by removing rows and columns. This definition has been generalized [1] to rook placements as follows.

Definition 2 *A full rook placement (R,F)* avoids *$\tau\in\mathcal{S}_k$ if and only if for every vertex V on the border of F, the permutation given by $R\cap\Gamma(V)$ avoids τ. Let $\mathcal{R}_F(\tau)$ be the set of full rook placements on F that avoid τ. Similarly, let*

$$\mathcal{R}_n(\tau)=\bigcup_{F\in\mathcal{F}_n}\mathcal{R}_F(\tau).$$

Definition 3 *Two patterns σ and τ are said to be* shape-Wilf-equivalent*, denoted $\sigma \sim \tau$, if for any Ferrers board F we have $|\mathcal{R}_F(\sigma)| = |\mathcal{R}_F(\tau)|$.*

Clearly, if two patterns are shape-Wilf-equivalent, then they are also Wilf-equivalent, meaning that they are avoided by the same number of *permutations*. The converse is not true, as shown by the fact that there is one Wilf-equivalence class for patterns of length 3, but three shape-Wilf-equivalence classes: $123 \sim 321 \sim 213$, $231 \sim 312$, and 132.

Regarding shape-Wilf-equivalence of patterns of arbitrary length, two important results are due to Backelin, West and Xin [1]. One states that $12\dots k \sim k\dots 21$ for all k, and the other one is the following.

Proposition 2.1 ([1]) *Let $\sigma, \tau \in \mathcal{S}_k$ and $\rho \in \mathcal{S}_\ell$. If $\sigma \sim \tau$, then $\sigma\rho' \sim \tau\rho'$, where ρ' is obtained from ρ by adding k to each of its entries.*

Denote by $\mathcal{D}_n^2$ the set of pairs (D_0, D_1) of Dyck paths $D_0, D_1 \in \mathcal{D}_n$ such that D_0 never goes above D_1. We say that D_0 and D_1 are *noncrossing*, and we call D_0 the *bottom path* and D_1 the *top path*. For any $F \in \mathcal{F}_n$, we denote by $\mathcal{D}_F^2$ the set of pairs $(D_0, D_F) \in \mathcal{D}_n^2$, that is, those where the top path is the border of F.

2.2 Matchings

Denote by $\mathcal{M}_n$ the set of perfect matchings on $[2n]$. If (i,j) is a matched pair with $i < j$, we call i an *opener* and j a *closer*. The following natural bijection between $\mathcal{M}_n$ and $\mathcal{R}_n$, which we denote κ, has been used in [9, 13]. Given a matching $M \in \mathcal{M}_n$, construct a path from $(0,n)$ to $(n,0)$ by reading the vertices of M in increasing order, and adding an east step for each opener, and a south step for each closer. This path is clearly a Dyck path, so it is the border of a Ferrers board $F \in \mathcal{F}_n$, which we call the *shape* of M. Each column of F is naturally associated to an opener of M (the vertex that produced the east step at the top of the column), and similarly each row is naturally associated to a closer. Now define a full rook placement on F by placing a rook in the column associated to i and the row associated to j for each matched pair (i,j). Two examples of the bijection κ are given in Fig. 2. For fixed $F \in \mathcal{F}_n$, denote by $\mathcal{M}_F = \kappa^{-1}(\mathcal{R}_F)$ the set of matchings of shape F. Note that $\mathcal{M}_n = \bigcup_{F\in\mathcal{F}_n} \mathcal{M}_F$. In light of this bijection, the definition of pattern avoidance in Ferrers boards translates naturally to matchings.

Definition 4 *We say that a matching $M \in \mathcal{M}_n$* avoids *the pattern $\tau \in \mathcal{S}_k$ if the corresponding rook placement $\kappa(M)$ does. Equivalently, M avoids τ if there are no $2k$ vertices $1 \le i_1 < \ldots < i_{2k} \le n$ such that M contains all the pairs $(i_a, i_{2k+1-\tau(a)})$ for $1 \le a \le k$. Let $\mathcal{M}_F(\tau) = \kappa^{-1}(\mathcal{R}_F(\tau))$ be the set of τ-avoiding matchings of shape F, and let $\mathcal{M}_n(\tau) = \bigcup_{F\in\mathcal{F}_n} \mathcal{M}_F(\tau)$.*

This definition extends the notions of k-noncrossing and k-nonnesting matchings studied in [7, 16]. Recall that a matching is k-noncrossing if it contains no k mutually crossing arcs. In our terminology, this is equivalent to avoiding the pattern $k\dots 21$. Similarly, a matchings is k-nonnesting if it contains no k mutually crossing arcs, which is equivalent to avoiding $12\dots k$.

For patterns $\tau \in \mathcal{S}_3$, which are the focus of this paper, we can describe $\mathcal{M}_n(\tau)$ as the set of matchings $M \in \mathcal{M}_n$ containing no three arcs whose endpoints occur in the same order as in the corresponding configuration in Fig. 1.

Since κ is a bijection, it is clear that $|\mathcal{M}_F(\tau)| = |\mathcal{R}_F(\tau)|$ for any τ. Thus, shape-Wilf-equivalence can be interpreted in terms of pattern-avoiding matchings: $\sigma \sim \tau$ if and only if $|\mathcal{M}_F(\sigma)| = |\mathcal{M}_F(\tau)|$ for every Ferrers board F. In particular, if $\sigma \sim \tau$, then $|\mathcal{M}_n(\sigma)| = |\mathcal{M}_n(\tau)|$ for all n. The converse

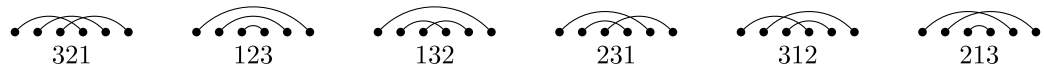

Fig. 1: Forbidden configurations corresponding to $\tau \in \mathcal{S}_3$.

statement is false in general. For example, it is trivial by symmetry that $|\mathcal{M}_n(2341)| = |\mathcal{M}_n(4123)|$ for all n, but the patterns 2341 and 4123 are not shape-Wilf-equivalent, since $|\mathcal{M}_F(2341)| \neq |\mathcal{M}_F(4123)|$ for the Ferrers boards in $\mathcal{F}_6$ consisting of a 6×6 square with two missing boxes.

2.3 Set partitions

Denote by $\mathcal{P}_n$ the set of partitions of $[n]$. For each block $\{i_1, i_2, \dots, i_a\}$ with $i_1 < i_2 < \cdots < i_a$ and $a \geq 2$, we call i_1 an *opener*, i_a a *closer*, and we say that $i_2, \dots, i_{a-1}$ are *transitory* vertices. If $a = 1$, the vertex i_1 is called a singleton. We will use the term partition to refer to a set partition when it creates no confusion. Note that matchings are partitions where all blocks have size 2. The definition of pattern avoidance for matchings extends to partitions as follows.

Definition 5 *We say that a partition $P \in \mathcal{P}_n$* avoids *the pattern $\tau \in \mathcal{S}_k$ if there are no $2k$ vertices $1 \leq i_1 < \ldots < i_{2k} \leq n$ such that P contains all the arcs $(i_a, i_{2k+1-\tau(a)})$ for $1 \leq a \leq k$. Denote by $\mathcal{P}_n(\tau)$ the set of τ-avoiding partitions of $[n]$.*

Note that in the above definition, singleton blocks of P do not contribute to occurrences of any pattern τ.

3 The patterns $123 \sim 321 \sim 213$

The equivalence $123 \sim 321$ was first proved in [1], and later simplified by Chen et al [7] and by Krattenthaler [16]. The equivalence $321 \sim 213$ was proved by Backelin, West and Xin [1], and later by Jelínek [13]. In this section we provide a short bijective proof of the fact that $321 \sim 213$, greatly simplifying the proofs in [1, 13]. For the rest of this section, we fix a Ferrers board $F \in \mathcal{F}_n$, and we let V_i denote the ith vertex on the border of F.

Theorem 3.1 *There are explicit bijections $\Delta_{321} : \mathcal{M}_F(321) \to \mathcal{D}^2_F$ and $\Delta_{213} : \mathcal{M}_F(213) \to \mathcal{D}^2_F$. Therefore, $321 \sim 213$.*

This theorem will follow from Theorems 3.2 and 3.3 below. In a different form, the bijection Δ_{321} was constructed by Chen et al. [7] using vacillating tableaux. Here we provide a short description of this bijection in our language. Recall that matchings can be viewed as full rook placements via the bijection $\kappa : \mathcal{M}_F \to \mathcal{R}_F$ described in Section 2.2.

It will be convenient to identify a Dyck path $D \in \mathcal{D}_n$ with the sequence $d_0 d_1 \dots d_{2n}$ that records the distances from its vertices to the main diagonal $y = n - x$. More precisely, if $V_i = (a, b)$, then $d_i = a + b - n$. We call $d_0 d_1 \dots d_{2n}$ the *height sequence* of D. Fix $h_0 h_1 \dots h_{2n}$ to be the height sequence of D_F.

For $(R, F) \in \mathcal{R}_F$, define the sequence $j_0 \dots j_{2n}$ by letting $j_i = 2\ell_i - h_i$, where ℓ_i is the length of the longest increasing sequence in $R \cap \Gamma(V_i)$. A straightforward argument (see [2]) shows that $j_0 \dots j_{2n}$ is a height sequence for some Dyck path, which we denote by $D_{R,F}$. Additionally, we show that $j_i \leq h_i$ for all i, and so $(D_{R,F}, D_F) \in \mathcal{D}^2_F$. Define the map $\delta_{321} : \mathcal{R}_F(321) \to \mathcal{D}^2_F$ by letting $\delta_{321}(R, F) =$

$(D_{R,F}, D_F)$. Then define $\Delta_{321} = \delta_{321} \circ \kappa$. The proof of following theorem is omitted in this extended abstract, but it may be found in [2].

Theorem 3.2 *The map* $\delta_{321} : \mathcal{R}_F(321) \to \mathcal{D}_F^2$ *is a bijection, and thus so is* $\Delta_{321} : \mathcal{M}_F(321) \to \mathcal{D}_F^2$.

Now we turn to the second part of the proof of Theorem 3.1. Even though a different bijection between $\mathcal{M}_F(213)$ and $\mathcal{D}_F^2$ has already been given by Jelínek in [13], here we present a much simpler bijection Δ_{213} through a short pictorial argument.

As in the case of 321-avoiding matchings, it is convenient to let $\Delta_{213} = \delta_{213} \circ \kappa$, where the map $\delta_{213} : \mathcal{R}_F(213) \to \mathcal{D}_F^2$ is defined as $\delta_{213}(R, F) = (D, D_F)$, with D given by the following construction. As the pattern 213 ends with its largest entry, the fact that (R, F) is 213-avoiding implies that $\pi_R \in \mathcal{S}_n(213)$. Let F_R be the minimal Ferrers board that contains R. In a different language, the bijection between $\mathcal{S}_n(213)$ and $\mathcal{D}_n$ that sends π_R to D_{F_R} appears in [15]. We define the bottom path in $\delta_{213}(R, F)$ to be $D = D_{F_R}$. Note that $F_R \subseteq F$ by definition, so D_{F_R} and D_F are noncrossing Dyck paths. The following theorem is now clear.

Theorem 3.3 *The map* $\delta_{213} : \mathcal{R}_F(213) \to \mathcal{D}_F^2$ *is a bijection, and thus so is* $\Delta_{213} : \mathcal{M}_F(213) \to \mathcal{D}_F^2$.

Examples of the maps δ_{321} and δ_{213}, together with the complete bijection from between $\mathcal{M}_F(321)$ and $\mathcal{M}_F(213)$, is given in Fig. 2.

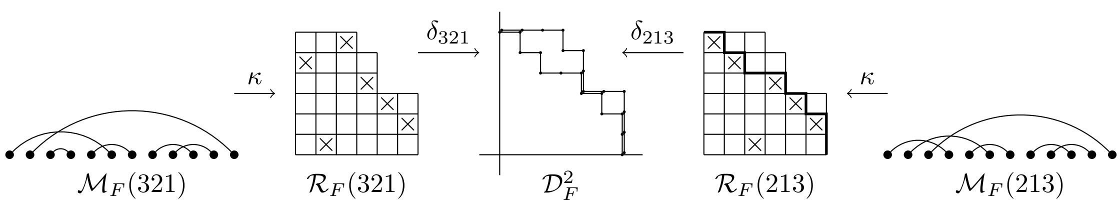

Fig. 2: An example of the bijection between $\mathcal{M}_F(321)$ and $\mathcal{M}_F(213)$. The bold path on the Ferrers board on the right represents the border of F_R.

In the particular case that $F \in \mathcal{F}_n$ is the square board, the composition $\Delta_{213}^{-1} \circ \Delta_{321}$ gives a bijection between $\mathcal{S}_n(321)$ and $\mathcal{S}_n(213)$ which coincides, up to symmetry, with a bijection of Elizalde and Pak [10].

We end this section by mentioning that Δ_{321} and Δ_{213} can be generalized to bijections between pattern-avoiding matchings with fixed points and pairs of noncrossing Dyck paths satisfying a certain condition. These generalizations, which we describe in the full paper [2], extend the results of Gouyou-Beauchamps [12] involving Young tableaux with at most 4 or 5 rows and 54321-avoiding involutions, which in our language become 123-avoiding matchings with fixed points.

4 The patterns $231 \sim 312$

The first proof of the equivalence $231 \sim 312$ was given by Stankova and West [19]. Later, Bloom and Saracino [3] gave a more direct proof. The main ingredient in Bloom and Saracino's construction is a bijection between 231-avoiding full rook placements of a given Ferrers board $F \in \mathcal{F}_n$ and certain labelings of the vertices on the border of F. Recall that the vertices $V_0V_1 \ldots V_{2n}$ are ordered from $(0, n)$ to $(n, 0)$.

We define a *labeled* Dyck path of semilength n to be a pair (D, α) where $D \in \mathcal{D}_n$, and $\alpha = \alpha_0\alpha_1 \dots \alpha_{2n}$ is an integer sequence with the following monotonicity property: if V_{i+1} is to the right of V_i, then $\alpha_i \le \alpha_{i+1} \le \alpha_i + 1$, else $\alpha_i \ge \alpha_{i+1} \ge \alpha_i - 1$. We think of α_i as the label of vertex V_i.

We say that two vertices $V_i = (x_i, y_i)$ and $V_j = (x_j, y_j)$ of D are *aligned* if $x_i - x_j = y_i - y_j$ and the line segment connecting the points V_i and V_j lies strictly below D (except for the endpoints of the segment, which are on D). We say that a labeled Dyck path (D, α) has the *diagonal property* if for any two aligned vertices V_i and V_j with $i < j$, we have $\alpha_i \ge \alpha_j$. We say (D, α) satisfies the *0-condition* if for each i, one has $\alpha_i = 0$ if and only if V_i lies on the diagonal $y = n - x$. For $F \in \mathcal{F}_n$, we denote by $\mathcal{L}_F$ the set of labelings (D_F, α) of the boundary of F that satisfy both the diagonal property and the 0-condition. We also let $\mathcal{L}_n = \bigcup_{F\in\mathcal{F}_n} \mathcal{L}_F$.

Bloom and Saracino's bijection in [3] between placements and labeled Dyck paths is the map $\Pi : \mathcal{R}_F(312) \to \mathcal{L}_F$ that sends $(R, F) \in \mathcal{R}_F(312)$ to the pair (D_F, α) where, for $0 \le i \le 2n$, the label α_i is the length of the longest increasing sequence in $R \cap \Gamma(V_i)$. In a slight abuse of notation, we also denote by Π the bijection induced by Π from $\mathcal{R}_n(312) = \bigcup_{F\in\mathcal{F}_n} \mathcal{R}_F(312)$ to $\mathcal{L}_n = \bigcup_{F\in\mathcal{F}_n} \mathcal{L}_F$.

4.1 312*-avoiding matchings*

In this section we enumerate 312-avoiding matchings, or equivalently, 231-avoiding ones.

Theorem 4.1 *The generating function for* 312*-avoiding matchings is*

$$\sum_{n\ge 0} |\mathcal{M}_n(312)| z^n = \frac{54z}{1 + 36z - (1 - 12z)^{3/2}}. \tag{1}$$

The asymptotic behavior of its coefficients is given by

$$|\mathcal{M}_n(312)| \sim \frac{3^3}{2^5\sqrt{\pi n^5}}\, 12^n. \tag{2}$$

Proof: We first translate the problem into an enumeration of labeled Dyck paths. The composition $\Pi \circ \kappa$ is a bijection between $\mathcal{M}_n(312)$ and $\mathcal{L}_n$, so we have $L(z) := \sum_{n\ge0} |\mathcal{M}_n(312)| z^n = \sum_{n\ge0} |\mathcal{L}_n| z^n$.

We will find an expression for $L(z)$ using the recursive structure of Dyck paths: every $D \in \mathcal{D}_n$ with $n \ge 1$ uniquely decomposes as eD_1sD_2 where e is an east step, s is a south step, and D_1 and D_2 are Dyck paths. Even though this decomposition can be extended to deal with labeled Dyck paths by transferring the label on each vertex of D to the corresponding vertex of eD_1s or D_2, the fact that the labels on eD_1s satisfy the 0-condition does not guarantee that the labels on D_1 do, even if their values are decreased by 1.

To deal with this problem, we relax the 0-condition and consider the larger set $\mathcal{K}_n$ consisting of all labeled Dyck paths (D, α) of semilength n that have the diagonal property and satisfy $\alpha_{2n} = 0$. Let $\mathcal{K} = \bigcup_{n\ge0} \mathcal{K}_n$, and denote by $K(u, z) = \sum_{n\ge0} \sum_{(D,\alpha)\in\mathcal{K}_n} u^{\alpha_0} z^n$ the generating function for such paths according to the value of the first label.

To obtain an equation for $K(u, z)$, first consider the following operation: given $(A, \alpha) \in \mathcal{K}_i$, $(B, \beta) \in \mathcal{K}_j$, let $(A, \alpha) \oplus (B, \beta) \in \mathcal{K}_{i+j}$ be the concatenation of Dyck paths AB with labels $(\alpha_0 + \beta_0)(\alpha_1 + \beta_0) \dots (\alpha_{2i} + \beta_0)\beta_1 \dots \beta_{2j}$. In other words, the labels along A are increased by β_0, and the labels along B do not change. Every nonempty $(D, \gamma) \in \mathcal{K}$ can be decomposed uniquely as $(D, \gamma) = (eD_1s, \alpha) \oplus (D_2, \beta)$ where $(eD_1s, \alpha), (D_2, \beta) \in \mathcal{K}$. Whereas (D_2, β) is an arbitrary element of $\mathcal{K}$, the labeling α of the elevated Dyck path eD_1s can be of four different types, according to whether $\alpha_0 = \alpha_1$ and whether

$\alpha_{2i-1} = \alpha_{2i}$, where i is the semilength of eD_1s. Analyzing these four possibilities, the decomposition translates into the functional equation

$$K(u,z) = 1 + zK(u,z)\left(2K(u,z) + uK(u,z) + \frac{K(u,z) - K(0,z)}{u}\right).$$

We solve this equation using the quadratic method, due to Tutte, as described in [11, p. 515]. Doing so yields an expression for $K(0,z)$ (see [2] for details).

Finally, to find $L(z)$, observe that for any $(D,\alpha) \in \mathcal{L}_n$, the path D can be decomposed uniquely as $D = eA_1seA_2s\dots$, where each A_j is a Dyck path, and if we let $\alpha^{(j)}$ be its sequence of labels decreased by one, then $(A_j, \alpha^{(j)})$ is an arbitrary element of $\mathcal{K}$ with $\alpha_0^{(j)} = 0$. It follows that $L(z) = 1/(1 - zK(0,z))$, which gives Eq. (1).

To find the asymptotic behavior of the coefficients, note that the singularity of $L(z)$ nearest to the origin is a branch point at $z = 1/12$. By [11, Corollary VI.1], its coefficients satisfy Eq. (2). □

It is interesting to observe that the generating function in Theorem 4.1 is algebraic, in contrast with the fact that the generating function for 123-avoiding matchings is D-finite but not algebraic [12, 7]. Compare also the growth rate in Eq. (2) with $|\mathcal{M}_n(123)| = C_{n+2}C_n - C_{n+1}^2 \sim \frac{24}{\pi n^5}\,16^n$.

4.2 312*-avoiding partitions*

A refinement of the methods from Section 4.1 can be used to enumerate 312-avoiding partitions, or equivalently, 231-avoiding ones. For any pattern τ, the set of τ-avoiding set partitions can be generated from the set of all τ-avoiding matchings as follows. Given a matching M, one can first choose, for each closer immediately followed by an opener, either to merge them into one transitory vertex or to leave them as they are; then one can insert singleton vertices in any position. If we let $\mathrm{val}(M)$ denote the number of closers immediately followed by openers in M (we call these *valleys* of M), and $A(v,z) = \sum_{n\geq 0}\sum_{M\in\mathcal{M}_n(\tau)} u^{\mathrm{val}(M)} z^n$ is the generating function for τ-avoiding matchings with respect to the number of valleys, then

$$\sum_{n\geq 0} |\mathcal{P}_n(\tau)| z^n = \frac{1}{1-z}\, A\left(\frac{1}{z}, \frac{z^2}{(1-z)^2}\right). \tag{3}$$

If two patterns satisfy $\sigma \sim \tau$, then $|\mathcal{M}_F(\sigma)| = |\mathcal{M}_F(\tau)|$ for every F, and so the above generating function $A(v,z)$ is the same for σ-avoiding as for τ-avoiding matchings. It follows that $|\mathcal{P}_n(\sigma)| = |\mathcal{P}_n(\tau)|$ for all n. In particular, since $312 \sim 231$, we have $|\mathcal{P}_n(312)| = |\mathcal{P}_n(231)|$.

Theorem 4.2 *The generating function $B(z) = \sum_{n\geq 0} |\mathcal{P}_n(312)| z^n$ for 312-avoiding partitions is a root of the cubic polynomial*

$$\begin{aligned}(z-1)(5z^2-2z+1)^2B^3 &+ (-9z^5+54z^4-85z^3+59z^2-14z+3)B^2\\ &+ (-9z^4+60z^3-64z^2+13z-3)B + (-9z^3+23z^2-4z+1).\end{aligned} \tag{4}$$

The asymptotic behavior of its coefficients is given by

$$|\mathcal{P}_n(312)| \sim \delta n^{-5/2}\,\rho^n, \tag{5}$$

where

$$\rho = \frac{3(9+6\sqrt{3})^{1/3}}{2+2(9+6\sqrt{3})^{1/3}-(9+6\sqrt{3})^{2/3}} \approx 6.97685$$

and $\delta \approx 0.061518$.

Proof sketch: To apply Eq. (3), we need to count 312-avoiding matchings while keeping track of the number of closers immediately followed by an opener. Via the bijection $\Pi \circ \kappa : \mathcal{M}_n(312) \to \mathcal{L}_n$, this is equivalent to counting labeled paths in $\mathcal{L}_n$ with respect to the number of valleys. Proceeding as in the proof of Theorem 4.1, the generating function $K(u,v,z)$ for paths in $\mathcal{K}$ that refines $K(u,z)$ by marking the number of valleys with the variable v satisfies

$$K(u,v,z) = 1 + z(vK(u,v,z) - v + 1)\left(2K(u,v,z) + uK(u,v,z) + \frac{K(u,v,z)-K(0,v,z)}{u}\right).$$

Applying the quadratic method, we obtain an expression for $K(0,v,z)$. Then, letting $L(v,z)$ be the generating function for paths in $\mathcal{L}_n$ where v marks the number of valleys, we have

$$L(v,z) = \frac{1/v}{1-vzK(0,v,z)} - \frac{1}{v} + 1.$$

Using now Eq. (3) to relate $B(z)$ and $L(v,z)$, it follows that $B(z)$ is a root of the polynomial (4).

To describe the asymptotic growth of its coefficients, we use the method described in [11, Section VII.7.1] to compute the singularities of algebraic functions (see [2] for details). □

Again, the generating function in Theorem 4.2 is algebraic, in contrast with the fact that the generating function for 123-avoiding (namely, 3-noncrossing) partitions is D-finite but not algebraic [5]. Compare also Eq. (5) with the growth of the number of 3-noncrossing partitions [5], given by

$$|\mathcal{P}_n(123)| \sim \frac{3^9\, 5\sqrt{3}}{2^5\, \pi} \frac{9^n}{n^7}.$$

4.3 An application to 1342*-avoiding permutations*

The method involving labeled Dyck paths that we have developed to enumerate 312-avoiding matchings can be used to recover the following generating function due to Bóna [4] for the number of 1342-avoiding permutations (which, by symmetry, equals the number of 3124-avoiding ones).

Theorem 4.3 ([4])

$$\sum_{n\geq 0} |\mathcal{S}_n(1342)| z^n = \frac{32z}{1+20z-8z^2-(1-8z)^{3/2}}.$$

Bóna [4] obtained this formula by constructing a bijection between so-called indecomposable 1342-avoiding permutations and certain labeled trees, called $\beta(0,1)$-trees. He then used the fact that the generating function for $\beta(0,1)$-trees had already been found by Tutte [20]. Our approach provides a more direct method to enumerate 1342-avoiding permutations without using $\beta(0,1)$-trees.

Denote by $\mathcal{R}_n^{\times}(312)$ the set of placements $(R,F) \in \mathcal{R}_n(312)$ with the property that F is the smallest Ferrers board that contains R. There is a straightforward bijection $\chi : \mathcal{S}_n(3124) \to \mathcal{R}_n^{\times}(312)$ defined by $\chi(\pi) = (R_\pi, F_\pi)$, where R_π is the placement consisting of the squares $(i, \pi(i))$ for $1 \le i \le n$, and F_π is the smallest board containing R_π. To enumerate $\mathcal{R}_n^\times(312)$, we use the fact (proved in [2]) that the image of the map $\Pi : \mathcal{R}_n(312) \to \mathcal{L}_n$, when restricted to $\mathcal{R}_n^\times(312)$, is the set of labeled Dyck paths $(D, \alpha) \in \mathcal{L}_n$ such that for every peak V_i, the labels around it satisfy $\alpha_{i-1} = \alpha_i + 1 = \alpha_{i+1}$. We denote this set by $\mathcal{L}_n^\times$.

Using the same framework as in the proof of Theorem 4.1, we can obtain the generating function for these paths. Let $\mathcal{K}_n^\times$ be the set of labeled Dyck paths (D, α) that have the diagonal property, satisfy $\alpha_{2n} = 0$, and such that $\alpha_{i-1} = \alpha_i + 1 = \alpha_{i+1}$ if V_i is a peak. Letting $K^\times(u, z) = \sum_{n\ge 0} \sum_{(D,\alpha)\in\mathcal{K}_n^\times} u^{\alpha_0} z^n$ be the generating function for such paths according to the value of the first label, we obtain

$$K^\times(u,z) = 1 + zK^\times(u,z)\left(K^\times(u,z) + (u+1)(K^\times(u,z)-1) + \frac{K^\times(u,z) - K^\times(0,z)}{u}\right).$$

The quadratic method yields a formula for $K^\times(0, z)$, from where

$$\sum_{n\ge 0} |\mathcal{S}_n(3124)| z^n = \sum_{n\ge 0} |\mathcal{L}_n^\times| z^n = \frac{1}{1 - zK^\times(0,z)} = \frac{32z}{1 + 20z - 8z^2 - (1-8z)^{3/2}}.$$

5 Pairs of patterns

Tab. 1 summarizes the results from the full paper [2] on matchings and set partitions that avoid a pair of patterns of length 3. The notions of pattern-avoidance and shape-Wilf-equivalence defined in Section 2 have a straightforward generalization to pairs of patterns. We establish that the 15 pairs of patterns in $\mathcal{S}_3$ are partitioned into 7 shape-Wilf-equivalence classes. Further, we provide enumeration results for matchings and set partitions avoiding a pair of patterns in many cases.

	132	213	231	312	321
123	VI	I	II	III	IV
132		I	I	I	VII
213			I	I	V
231				I	I
312					I

Class	Matchings	Set partitions
I	$\dfrac{4}{3+\sqrt{1-8z}}$	$\dfrac{2-3z+z^2-z\sqrt{1-6z+z^2}}{2(1-3z+3z^2)}$
II, III	Solutions of a cubic	Solutions of a cubic
IV	$\dfrac{1-5z+2z^2}{1-6z+5z^2}$	$\dfrac{1-10z+32z^2-37z^3+12z^4}{(1-z)(1-10z+31z^2-30z^3+z^4)}$
V	Functional equation	Unknown
VI, VII	Unknown	Unknown

Tab. 1: The 7 shape-Wilf-equivalence classes of pairs of patterns, and a summary of our enumeration results.

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Dual Equivalence Graphs Revisited with Applications to LLT and Macdonald Polynomials

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Abstract. In 2007 Sami Assaf introduced dual equivalence graphs as a method for demonstrating that a quasisymmetric function is Schur positive. The method involves the creation of a graph whose vertices are weighted by Ira Gessel's fundamental quasisymmetric functions so that the sum of the weights of a connected component is a single Schur function. In this paper, we improve on Assaf's axiomatization of such graphs, giving locally testable criteria that are more easily verified by computers. We then demonstrate the utility of this result by giving explicit Schur expansions for a family of Lascoux-Leclerc-Thibon polynomials. This family properly contains the previously known case of polynomials indexed by two skew shapes, as was described in a 1995 paper by Christophe Carré and Bernard Leclerc. As an immediate corollary, we gain an explicit Schur expansion for a family of modified Macdonald polynomials in terms of Yamanouchi words. This family includes all polynomials indexed by shapes with less than four cells in the first row and strictly less than three cells in the second row, a slight improvement over the known two column case described in 2005 by James Haglund, Mark Haiman, and Nick Loehr.

Résumé. En 2007, Sami Assaf a introduit la mé thode des graphes d'é quivalence duale pour dé montrer qu'une fonction quasisymmé trique est Schur-positive. La mé thode né cessite la cré ation d'un graphe dont les sommets sont pondé ré s par les fonctions quasisymmé triques fondamentales d'Ira Gessel tel que la somme des poids d'une composante connexe soit une unique fonction de Schur. Dans cet article, nous amé liorons l'axiomatisation d'Assaf pour ces graphes, et nous obtenons des critè res locaux qui sont plus facilement vé rifié s par ordinateur. Puis nous appliquons ces techniques pour pré senter des dé veloppements explicites en fonctions de Schur d'une famille de polynô mes de Lascoux-Leclerc-Thibon. Cette famille contient strictement le cas des polynô mes indexé s par deux formes gauches, qui a é té dé crit dans un article en 1995 de Christophe Carré et Bernard Leclerc. Comme corollaire immé diat, nous obtenons un dé veloppement explicite en fonctions de Schur d'une famille de polynô mes de Macdonald modifié s, exprimé e au moyen de mots de Yamanouchi. Cette famille inclut tous les polynô mes indexé s par des formes de moins de quatre cellules dans la premiè re ligne et strictement moins de trois cellules dans la deuxiè me ligne, ce qui est une lé gè re amé lioration par rapport au cas connu de deux colonnes dé crit en 2005 par James Haglund, Mark Haiman, et Nick Loehr.

Keywords: Dual equivalence graph, LLT polynomial, Macdonald polynomial, Schur expansion, quasisymmetric function

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1 Introduction

Dual equivalence was developed and applied by Mark Haiman in (Haiman, 1992) as an extension of work done by Donald Knuth in (Knuth, 1970). Sami Assaf then introduced the theory of dual equivalence graphs in her Ph.D. dissertation Assaf (2007) and subsequent preprint Assaf (2011). In these papers, she is able to associate a number of symmetric functions to dual equivalence graphs and each component of a dual equivalence graph to a Schur function, thus demonstrating Schur positivity. More recently, variations of dual equivalence graphs are given for k-Schur functions in Assaf and Billey (2012) and for the product of a Schubert polynomial with a Schur polynomial in Assaf et al. (2012).

A key connection between dual equivalence graphs and symmetric functions is the ring of quasisymmetric functions. The quasisymmetric functions were introduced by Ira Gessel (1984) as part of his work on P-partitions. Currently there are a number of functions that are easily expressible in terms of Gessel's *fundamental quasisymmetric functions* that are not easily expressed in terms of Schur functions. For example, such an expansion for plethysms is described in Loehr and Warrington (2012), for Lascoux-Leclerc-Thibon (LLT) polynomials in Haglund et al. (2005b), and for Macdonald polynomials in Haglund et al. (2005a). An expressed goal of developing the theory of dual equivalence graphs is to create a tool for turning such quasisymmetric expansions into explicit Schur expansions.

Previously, dual equivalence graphs were defined by five *dual equivalence axioms* that are locally testable and one that is not. In attempting to apply the theory of dual equivalence graphs, it is often a challenge to demonstrate that this nonlocal axiom is satisfied. The main results of this paper is to give an equivalent definition using only local conditions, as is stated in Theorem 2.9.

The paper concludes by applying the above result to LLT polynomials. LLT polynomials were first introduced in Lascoux et al. (1997) as a q-analogue to products of Schur functions and were later given a description in terms of tuples of skew tableaux in Haglund et al. (2005b). LLT polynomials can, in turn, be used to give an explicit combinatorial description of modified Macdonald polynomials by using the results of Haglund et al. (2005a). First introduced in Macdonald (1988), Macdonald polynomials are often defined as the set of q, t-symmetric functions that satisfy certain orthogonality and triangularity conditions, as is well described in Macdonald (1995). Part of the importance of Macdonald polynomials derives from the fact that they specialize to a wide array of well known functions, including Hall-Littlewood polynomials and Jack polynomials (see Macdonald (1995) for details). In Haiman (2001), Mark Haiman used geometric and representation-theoretic techniques to prove that Macdonald polynomials are Schur positive.

In some cases, nice Schur expansions for LLT and Macdonald polynomials are already known. In particular, the set of LLT polynomials indexed by two skew shapes was described in Carré and Leclerc (1995) and van Leeuwen (2000), and modified Macdonald polynomials indexed by shapes with strictly less than three columns was described in Haglund et al. (2005a) (which in turn drew on the earlier work in Carré and Leclerc (1995), van Leeuwen (2000)). The first combinatorial description of the two column case was given in Fishel (1995), but others were subsequently given in Zabrocki (1998), Lapointe and Morse (2003), and Assaf (2008/09).

This paper is broken into sections as follows. Section 2 is dedicated giving a new axiomatization for dual equivalence graphs in Theorem 2.9. Section 3 applies the results of Section 2 to LLT polynomials and Macdonald polynomials. The graph structure given to LLT polynomials in Assaf (2011) is reviewed. Theorem 3.9 states that the set of LLT polynomials corresponding to said graphs have a Schur expansion indexed by standardized Yamanouchi words. This set strictly contains the set of LLT polynomials indexed by two skew shapes. Corollary 3.10 then gives a Schur expansion for modified Macdonald polynomials indexed by partition shapes with strictly less than four boxes in the first row and strictly less than three boxes in the second row. For a fuller account of all of these results, see Roberts (2013).

2 The Structure of Dual Equivalence Graphs

2.1 Preliminaries

In this section we provide the necessary definitions and results from Assaf (2011). We begin by recalling Mark Haiman's dual to the fundamental Knuth equivalences.

Definition 2.1 Given a permutation in S_n expressed in one-line notation, define an *elementary dual equivalence* as an involution d_i that interchanges the values $i-1, i$, and $i+1$ as

$$\begin{aligned} d_i(\ldots i \ldots i-1 \ldots i+1 \ldots) &= (\ldots i+1 \ldots i-1 \ldots i \ldots), \\ d_i(\ldots i-1 \ldots i+1 \ldots i \ldots) &= (\ldots i \ldots i+1 \ldots i-1 \ldots), \end{aligned} \tag{2.1}$$

and acts as the identity if i occurs between $i-1$ and $i+1$. Two words are $dual\ equivalent$ if one may be transformed into the other by successive elementary dual equivalences.

For example, 21345 is dual equivalent to 41235 because $d_3(d_2(21345)) = d_3(31245) = 41235$.

We may also let d_i act on the entries of a tableau via the row reading word. It is simple to check that the result is again a tableau of the same shape. The transitivity of this action is described in the next theorem.

Theorem 2.2 ((Haiman, 1992, Prop. 2.4)) *Two standard Young tableaux on partition shapes are dual equivalent if and only if they have the same shape.*

The signature of a permutation is a string of 1's and -1's, or +'s and $-$'s for short, where there is a + in the i^{th} position if and only if i comes before $i+1$ in one-line notation. We may then define the signature of a tableau T, denoted $\sigma(T)$, as the signature of the row reading word of T. If we wish to be explicit about this definition of σ, we will refer to the signature function as being *given by inverse descents*.

By definition, d_i is an involution, and so we define a graph on standard Young tableaux by letting each nontrivial orbit of d_i define an edge colored by i. By Theorem 2.2, the graph on SYT(n) with edges labeled by $1 < i < n$ has connected components with vertices in SYT(λ) for each $\lambda \vdash n$. We may further label each vertex with its signature to create a *standard dual equivalence graph* that we will denote $\mathcal{G}_\lambda$. See Figure 1 for examples.

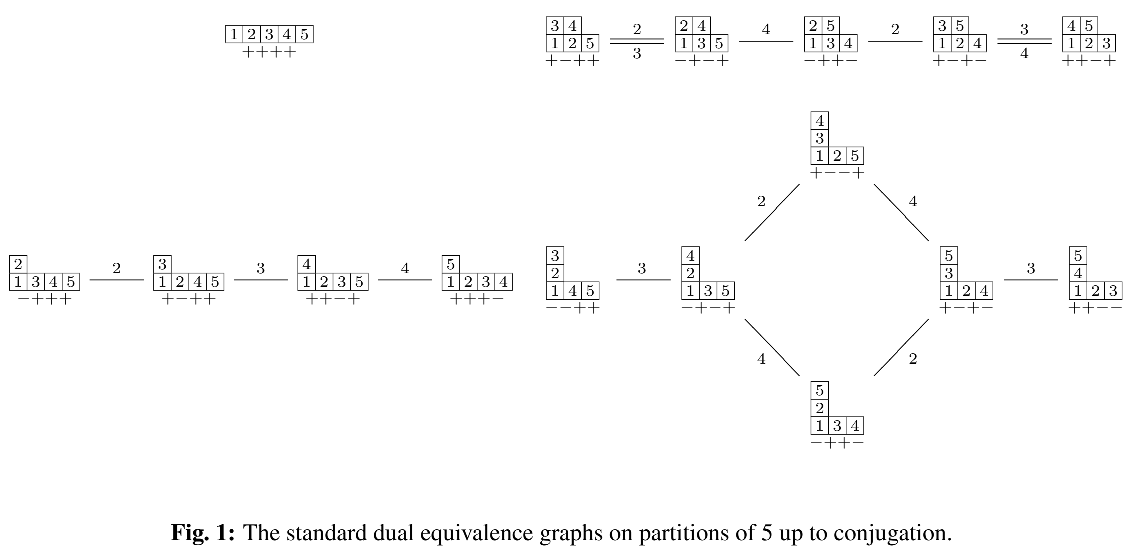

Fig. 1: The standard dual equivalence graphs on partitions of 5 up to conjugation.

Here, $\mathcal{G}_\lambda$ is an example of the following broader class of graphs.

Definition 2.3 An *edge colored graph* consists of the following data:

1. *a finite vertex set* V*,*
2. *a collection* E_i *of unordered pairs of distinct vertices in* V *for each* $i \in \{m+1, \ldots, n-1\}$*, where* m *and* n *are positive integers.*

A *signed colored graph* is an edge colored graph with the following additional data:

3. *a signature function* $\sigma : V \to \{\pm 1\}^{N-1}$*, for some positive integer* $N \geq n$*.*

We denote a signed colored graph by $\mathcal{G} = (V, \sigma, E_{m+1} \cup \cdots \cup E_{n-1})$ or simply $\mathcal{G} = (V, \sigma, E)$. If a signed colored graph has $m = 1$, then it is said to have *type* (n, N) and is termed an (n, N)-signed colored graph.

We may also restrict signed colored graphs. If $\mathcal{G}$ is an (n, N)-signed colored graph, $M \leq N$, and $m \leq n$, then the (m, M)*-restriction* of $\mathcal{G}$ is the result of excluding E_i for $i \geq m$ and projecting each signature onto its first $M - 1$ coordinates.

In order to present structural results about signed colored graphs, we first need to define isomorphisms.

Definition 2.4 A map $\phi : \mathcal{G} \to \mathcal{H}$ between edge colored graphs $\mathcal{G} = (V, E_{m+1} \cup \ldots \cup E_{n-1})$ and $\mathcal{H} = (V', E'_{m+1} \cup \ldots \cup E'_{n-1})$ is called a $morphism$ if it preserves i-edges. That is, $\{v, w\} \in E_i$ implies $\{\phi(v), \phi(w)\} \in E'_i$ for all $v, w \in V$ and all $m < i < n$.

A map $\phi : \mathcal{G} \to \mathcal{H}$ between signed colored graphs $\mathcal{G} = (V, \sigma, E_{m+1} \cup \ldots \cup E_{n-1})$ and $\mathcal{H} = (V', \sigma', E'_{m+1} \cup \ldots \cup E'_{n-1})$ is called a *morphism* if it is a morphism of edge colored graphs that also preserves signatures. That is, $\sigma'(\phi(v)) = \sigma(v)$.

In both cases, a morphism is an $isomorphism$ if it admits an inverse morphism.

Notice that in a standard dual equivalence graph, a vertex v is included in an i-edge if and only if $\sigma(v)_{i-1} = -\sigma(v)_i$, motivating the following definition.

Definition 2.5 Let $\mathcal{G} = (V, \sigma, E)$ be a signed colored graph. We say that $w \in V$ *admits an* i*-neighbor* if $\sigma(w)_{i-1} = -\sigma(w)_i$.

We are now ready to present an axiomatization of the structure inherent in a standard dual equivalence graph.

Definition 2.6 A signed colored graph $\mathcal{G} = (V, \sigma, E_{m+1} \cup \ldots \cup E_{n-1})$ is a *dual equivalence graph* if the following hold:

(ax1): For $m < i < n$, each E_i is a complete matching on the vertices of V that admit an i-neighbor.

(ax2): If $\{v, w\} \in E_i$, then $\sigma(v)_i = -\sigma(w)_i, \sigma(v)_{i-1} = -\sigma(w)_{i-1}$, and $\sigma(v)_h = \sigma(w)_h$ for all $h < i-2$ and all $h > i+1$.

(ax3): For $\{v, w\} \in E_i$, if σ_{i-2} is defined, then v or w (or both) admits an $(i-1)$-neighbor, and if σ_{i+1} is defined, then v or w (or both) admits an $(i+1)$-neighbor.

(ax4): For all $m+1 < i < n$, any component of the edge colored graph $(V, E_{i-2} \cup E_{i-1} \cup E_i)$ is isomorphic to a component of the restriction of some $\mathcal{G}_\lambda = (V', \sigma', E')$ to $(V', E'_{i-2} \cup E'_{i-1} \cup E'_i)$, where E_{i-2} is omitted if $i = m+2$ (see Figure 2).

(ax5): For all $1 < i, j < n$ such that $|i - j| > 2$, if $\{v, w\} \in E_i$ and $\{w, x\} \in E_j$, then there exists $y \in V$ such that $\{v, y\} \in E_j$ and $\{x, y\} \in E_i$.

(ax6): For all $m < i < n$, any two vertices of a connected component of $(V, \sigma, E_{m+1} \cup \cdots \cup E_i)$ may be connected by some path crossing at most one E_i edge.

A dual equivalence graph that is also an (n, N)-signed colored graph is said to have *type* (n, N) and is termed an (n, N)-*dual equivalence graph.*

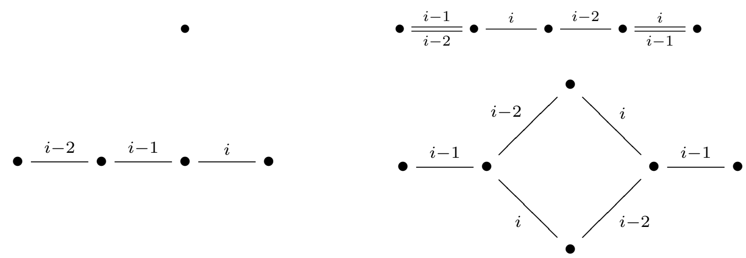

Fig. 2: Allowable $E_{i-2} \cup E_{i-1} \cup E_i$ components of Axiom 4.

The next theorem links the definitions of dual equivalence graphs and standard dual equivalence graphs.

Theorem 2.7 ((Assaf, 2011, Theorems 3.5 and 3.9)) *A connected component of an (n, n)-signed colored graph is a dual equivalence graph if and only if it is isomorphic to a unique $\mathcal{G}_\lambda$.*

2.2 Local Conditions for Axiom 6

Axiom 6 is the only dual equivalence axiom that cannot be tested locally, and in practice it is often a barrier to applying the theory of dual equivalence graphs. In this section, we show how to replace Axiom 6 with a locally testable axiom.

Definition 2.8 A signed colored graph $\mathcal{G} = (V, \sigma, E_{m+1} \cup \ldots \cup E_{n-1})$ is said to obey Axiom 4^+ if for all $m + 1 < i < n$, any component of the edge colored graph $(V, E_{i-3} \cup E_{i-2} \cup E_{i-1} \cup E_i)$ is isomorphic to a component of the restriction of some $\mathcal{G}_\lambda = (V', \sigma', E')$ to $(V', E'_{i-3} \cup E'_{i-2} \cup E'_{i-1} \cup E'_i)$, where E_{i-3}, E_{i-2}, or E_{i-1} is omitted if $i \leq m + 3, i \leq m + 2$, or $i = m + 1$, respectively.

Notice that Axiom 4^+ is just an extension of Axiom 4 to components with 4 consecutive edge colors. The next theorem states that this extension to Axiom 4^+ allows for the omission of Axiom 6 in the dual equivalence axioms.

Theorem 2.9 *A signed colored graph satisfies Axioms 1, 2, 3, 4^+, and 5 if and only if it is a dual equivalence graph.*

Proof Sketch: The crux of the proof is to show that Axioms 1, 2, 3, 4^+, and 5 imply Axiom 6. We provide a sketch of this argument here, proceeding by induction on n. For $n \leq 6$ the result is largely a consequence of the definition of Axiom 4^+. We may then assume the result when $n - 1 \leq 6$ and prove that an arbitrary (n, n)-signed colored graph $\mathcal{G}$ satisfying Axioms 1, 2, 3, 4^+, and 5 must also satisfy Axiom 6.

It follows from the proof of Theorem 2.7 that $\mathcal{G}$ admits a morphism onto some standard dual equivalence graph $\mathcal{G}_\lambda$ and that each $(n-1, n-1)$-component of $\mathcal{G}$ is connected in $\mathcal{G}$ to exactly one $(n-1, n-1)$-components of each possible isomorphism type. We may then show that every $(n-1, n-1)$-component of $\mathcal{G}$ has a unique isomorphism type, establishing Axiom 6.

To prove this last point, we consider two cases. In the first case, we assume that λ is not a staircase. In the second case, we assume that λ has at least four Northeast corners. In both cases, induction is used to provide the required connectivity result. Finally, the only cases that are staircases but have less than four Northeast corners have at most six cells, and so are covered in our base case. $\square$

Remark 2.10 We may readily classify the set of edge colored graphs described in Definition 2.8, i.e., the set of edge colored graphs that arise as components of the restriction of some $\mathcal{G}_\lambda = (V', \sigma', E')$ to $(V', E'_{i-3} \cup E'_{i-2} \cup E'_{i-1} \cup E'_i)$. Each such edge colored graph is the result of restricting some $\mathcal{G}_\lambda = (V, \sigma, E)$ to (V, E) and adding h to each edge label, where $\lambda \vdash 6$ and h is some nonnegative integer.

Let $\mathcal{F}$ be the set of edge colored graphs with edge sets $E_{i-3} \cup \ldots \cup E_i$ that satisfy Axioms 4 and 5 but not Axiom 6. A canonical graph in $\mathcal{F}$ is presented in Figure 3. The following corollary reformulates Theorem 2.9 in terms of $\mathcal{F}$.

Corollary 2.11 *Let $\mathcal{G} = (V, \sigma, E_{m+1} \cup \ldots \cup E_{n-1})$ be a signed colored graph satisfying Axioms 1, 2, 3, 4, and 5. Then $\mathcal{G}$ is a dual equivalence graph if and only if for all $m + 4 < i < n$, the restriction of $\mathcal{G}$ to the edge colored graph $(V, E_{i-4} \cup \ldots \cup E_i)$ has no components isomorphic to an element of $\mathcal{F}$.*

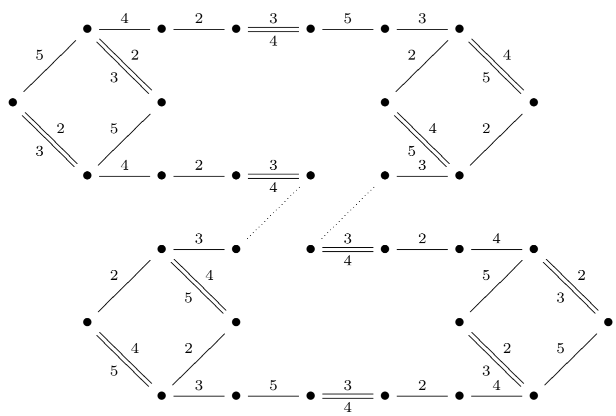

Fig. 3: A generic graph in $\mathcal{F}$ with edge labels in $\{2, 3, 4, 5\}$.

Remark 2.12 For any edge colored graph in $\mathcal{F}$, every vertex shares an edge with at least two other vertices. We may then give yet another characterization of dual equivalence graphs. Let $\mathcal{G} = (V, \sigma, E_{m+1} \cup \ldots \cup E_{n-1})$ be a signed colored graph obeying Axioms 1, 2, 3, 4, and 5. Choose $\mathcal{C}$ to be any component of the restriction of $\mathcal{G}$ to the edge colored graph $(V, E_{i-3} \cup E_{i-2} \cup E_{i-1} \cup E_i)$ such that $m + 3 < i < n$ and the vertices of $\mathcal{C}$ all have at least two adjacent vertices in $\mathcal{C}$. Then $\mathcal{G}$ is a dual equivalence graph if and only if $\mathcal{C}$ is not in $\mathcal{F}$ for any choice of $\mathcal{C}$. This characterization of dual equivalence graphs is used in the computer verification of Theorem 3.7.

3 LLT and Macdonald Polynomials

In this section, we demonstrate the utility of the results in Section 2 by applying them to LLT Polynomials.

3.1 Symmetric Functions

We begin by recalling the definitions of the necessary symmetric functions. Crucial to our definitions will be the definition of the fundamental quasisymmetric function.

Definition 3.1 Given any signature $\sigma \in \{\pm 1\}^{n-1}$, define the *fundamental quasisymmetric function* $F_\sigma(X) \in \mathbb{Z}[x_1, x_2, \ldots]$ by

$$F_\sigma(X) := \sum_{\substack{i_1 \leq \ldots \leq i_n \\ i_j = i_{j+1} \Rightarrow \sigma_j = +1}} x_{i_1} \cdots x_{i_n}.$$

We take the unorthodox approach of using a result by Gessel (1984) to give our next definition.

Definition 3.2 Gessel (1984) Given any skew shape λ/ρ, define

$$s_{\lambda/\rho}(X) := \sum_{T \in \mathrm{SYT}(\lambda/\rho)} F_{\sigma(T)}(X), \tag{3.1}$$

where $s_{\lambda/\rho}$ is termed a Schur function if λ/ρ is a straight shape and a skew Schur function in general.

Definition 3.2 and Theorem 2.2 determine the connection between Schur functions and dual equivalence graphs as highlighted in (Assaf, 2011, Cor. 3.10). Given any standard dual equivalence graph $\mathcal{G}_\lambda = (V, \sigma, E)$,

$$\sum_{v \in V} F_{\sigma(v)} = s_\lambda. \tag{3.2}$$

Given a a diagram of shape λ/ρ in French notation, the *content* of cell x, denoted $c(x)$, is $j - i$, where j is the column of x and i is the row of x in Cartesian coordinates. Here, the lower left corner of λ is assumed to be at the origin. Given a k-tuple of skew shapes $\boldsymbol{\nu} = (\nu^{(0)}, \ldots, \nu^{(k-1)})$, we write $|\boldsymbol{\nu}| = n$ if $\sum_{i=0}^{k-1} |\nu^{(i)}| = n$. A standard filling $\mathbf{T} = (T^{(0)}, \ldots, T^{(k-1)})$ of $\boldsymbol{\nu}$ is a bijective filling of the diagram of $\boldsymbol{\nu}$ with entries in $[n]$ such that each $T^{(i)}$ is strictly increasing up columns and across rows from left to right. Denote the set of standard fillings of $\boldsymbol{\nu}$ as SYT($\boldsymbol{\nu}$). Define the *shifted content* of cell x in $\nu^{(i)}$ as,

$$\tilde{c}(x) = k \cdot c(x) + i, \tag{3.3}$$

where $c(x)$ is the content of x in $\nu^{(i)}$. The *shifted content word* of $\mathbf{T}$ is defined as the word retrieved from reading off the values in the cells from lowest shifted content to highest, reading northeast along diagonals of constant shifted content. We may then define $\sigma(\mathbf{T})$ as the signature of the shifted content word of $\mathbf{T}$.

Letting $\mathbf{T}(x)$ denote the entry in cell x, the set of *k-inversions of* $\mathbf{T}$ is

$$\mathrm{Inv}_k(\mathbf{T}) := \{(x, y) \mid k > \tilde{c}(y) - \tilde{c}(x) > 0 \text{ and } \mathbf{T}(x) > \mathbf{T}(y)\}. \tag{3.4}$$

The *k-inversion number of* $\mathbf{T}$ is defined as

$$\mathrm{inv}_k(\mathbf{T}) := |\mathrm{Inv}_k(\mathbf{T})|. \tag{3.5}$$

Now define the set of LLT polynomials by

$$\widetilde{G}_{\boldsymbol{\nu}}(X; q) := \sum_{\mathbf{T} \in \mathrm{SYT}(\boldsymbol{\nu})} q^{\mathrm{inv}_k(\mathbf{T})} F_{\sigma(\mathbf{T})}(X). \tag{3.6}$$

We now move on to the definition of the modified Macdonald polynomials $\widetilde{H}_{\mu/\rho}(X; q, t)$. We will use (Haglund et al., 2005a, Theorem 2.2) to give a strictly combinatorial definition, using the statistics "inv", "maj", and "a" from this paper without defining them here. We will, however, need one new definition.

Given any skew shape μ/ρ with each cell represented by a pair (i, j) in Cartesian coordinates, let TR(μ/ρ) be the set of tuples of ribbons $\boldsymbol{\nu} = (\nu^{(0)}, \ldots, \nu^{(k-1)})$, such that $\nu^{(i)}$ has a cell with content j if

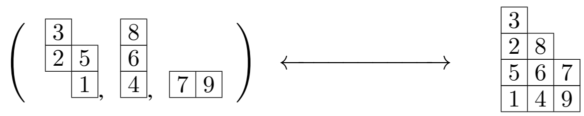

Fig. 4: An example of the bijection between standard fillings of shapes in TR(μ/ρ) and bijective fillings of μ/ρ. At left, the cells labeled 1, 4, and 9 have content 0 in their respective ribbons.

and only if $(i, -j)$ is a cell in μ/ρ. There is then a bijection between standard fillings of shapes in TR(μ/ρ) and bijective fillings of μ/ρ given by turning each ribbon into a column of μ/ρ as demonstrated in Figure 4.

We are now able to define the modified Macdonald polynomials and show their relationship with LLT polynomials.

$$\widetilde{H}_{\mu/\rho}(X;q,t) := \sum_{\substack{\boldsymbol{\nu}\in \mathrm{TR}(\mu/\rho)\\ \mathbf{T}\in \mathrm{SYT}(\boldsymbol{\nu})}} q^{\mathrm{inv}(\mathbf{T})} t^{\mathrm{maj}(\mathbf{T})} F_{\sigma(\mathbf{T})} = \sum_{\boldsymbol{\nu}\in \mathrm{TR}(\mu/\rho)} q^{-a(\boldsymbol{\nu})} t^{\mathrm{maj}(\boldsymbol{\nu})} \widetilde{G}_{\boldsymbol{\nu}}(X;q) \tag{3.7}$$

By using this definition, results about LLT polynomials can be easily translated into results about Macdonald polynomials.

3.2 LLT graphs

We follow Assaf (2011) in defining an involution that will provide the edge sets of a signed colored graph. In this section, $\boldsymbol{\nu}$ will always denote a k-tuple of skew shapes whose sizes sum to $|\boldsymbol{\nu}| = n$. Also, w will always denote a permutation in S_n.

Let the involution $\tilde{d}_i : S_n \to S_n$ act by permuting the entries $i-1, i$, and $i+1$ as defined by,

$$\begin{aligned} \tilde{d}_i(\ldots i \ldots i-1 \ldots i+1 \ldots) &= (\ldots i-1 \ldots i+1 \ldots i \ldots), \\ \tilde{d}_i(\ldots i \ldots i+1 \ldots i-1 \ldots) &= (\ldots i+1 \ldots i-1 \ldots i \ldots), \end{aligned} \tag{3.8}$$

and by acting as the identity if i occurs between $i-1$ and $i+1$. For instance, $\tilde{d}_3 \circ \tilde{d}_2(4123) = \tilde{d}_3(4123) = 3142$.

To decide when to apply d_i and when to use $\tilde{d}_i$, we appeal to the shifted content. Numbering the cells of a fixed $\boldsymbol{\nu}$ from 1 to n in shifted content reading order, let $\tilde{c}_i$ be the shifted content of the i^{th} cell. Define the weakly increasing word $\tau = \tau_1\tau_2\ldots\tau_n$ by

$$\tau_i = \max\{j \in [n] : \tilde{c}_j - \tilde{c}_i \leq k\}. \tag{3.9}$$

See Figure 5 for an example. To emphasize the relationship between τ and $\boldsymbol{\nu}$, we will sometimes write $\tau = \tau(\boldsymbol{\nu})$. Notice that there are finitely many possible τ of any fixed length n. Specifically, τ will always satisfy $\tau_n = n$ and $i \leq \tau_i \leq \tau_{i+1}$ for all $i < n$. Next, let $m(i)$ be the index of the the value in $\{i-1, i, i+1\}$ that occurs first in w, and let $M(i)$ be the index of the the value in $\{i-1, i, i+1\}$ that occurs last in w. We now define the desired involution,

$$D_i^{(\tau)}(w) := \begin{cases} d_i(w) & \tau_{m(i)} < M(i) \\ \tilde{d}_i(w) & \tau_{m(i)} \geq M(i). \end{cases} \tag{3.10}$$

As an example, we may take $\tau = 456667899$ and $w = 534826179$, as in Figure 5. Then $D_3^{(\tau)}(w) = \tilde{d}_i(w) = 542836179$ and $D_5^{(\tau)}(w) = d_i(w) = 634825179$.

Fig. 5: On the left, the shifted contents of a pair of skew diagrams with $\tau = 456667899$. On the right, a standard filling of the same tuple with shifted content word 534826179.

Direct inspection shows that if $\tau = \tau(\boldsymbol{\nu})$, then $D_i^{(\tau)}$ takes shifted content words of standard fillings of $\boldsymbol{\nu}$ to shifted content words of other standard fillings of $\boldsymbol{\nu}$. Thus, $D_i^{(\tau)}$ has a well defined action on SYT($\boldsymbol{\nu}$) inherited from the action of $D_i^{(\tau)}$ on shifted content words. We may then define the following.

Definition 3.3 Given some tuple of skew shapes $\boldsymbol{\nu}$, the *LLT graph* $\mathcal{L}_{\boldsymbol{\nu}} = (V, \sigma, E)$ is defined to be the (n, n)-signed colored graph with the following data:

1. $V = \{w \in S_n : w$ is the shifted content word of some $\mathbf{T} \in$ SYT($\boldsymbol{\nu}$)$\}$,
2. The signature function σ is given by the inverse descents of $w \in V$,
3. The edge sets E_i are defined by the nontrivial orbits of $D_i^{(\tau)}$ for all $1 < i < |\boldsymbol{\nu}|$, where $\tau = \tau(\boldsymbol{\nu})$.

Example 3.4 Consider $\boldsymbol{\nu} = ((2), (2), (1), (1))$. A portion of the LLT graph $\mathcal{L}_{\boldsymbol{\nu}}$ is presented in Figure 6. Here, $\mathcal{L}_{\boldsymbol{\nu}}$ is a subgraph of $\mathcal{G}_6^{(\tau)}$ with $\tau = 566666$. In the figure, the edge $\{312654, 412653\}$ is defined by the action of d_3 and d_4, while all other edges are defined by the action of $\tilde{d}_i$ for $1 < i < 6$.

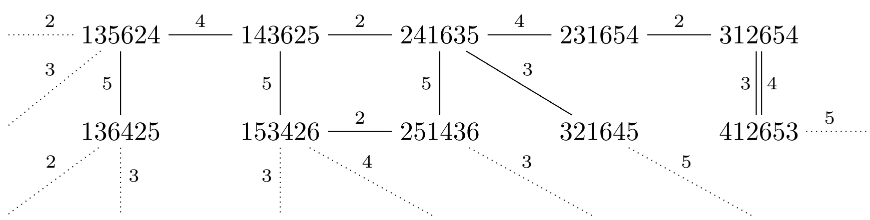

Fig. 6: A portion of $\mathcal{L}_{\boldsymbol{\nu}}$ with signatures omitted. Here $\boldsymbol{\nu} = ((2), (2), (1), (1))$.

While LLT graphs do not necessarily satisfy Axiom 4 or Axiom 6, they do satisfy a subset of the dual equivalence axioms. This is made precise in the following proposition.

Proposition 3.5 (Assaf (2011) Prop. 4.6) *Any LLT graph $\mathcal{L}_{\boldsymbol{\nu}}$ obeys Axioms 1, 2, 3, and 5. Furthermore, the* inv *statistic is constant on each connected component of $\mathcal{L}_{\boldsymbol{\nu}}$.*

To state the main theorem of this section, we will also need the following definition.

Definition 3.6 Given a k-tuple of skew shapes $\boldsymbol{\nu}$, let $S(\boldsymbol{\nu})$ be the set of distinct shifted contents of the cells in $\boldsymbol{\nu}$. Define the *diameter* of $\boldsymbol{\nu}$, denoted diam($\boldsymbol{\nu}$), as

$$\text{diam}(\boldsymbol{\nu}) := \max\{|R| : R \subset S(\boldsymbol{\nu}) \text{ and } |x - y| \leq k \text{ for all } x, y \in R\}.$$

See Figure 7 for an example.

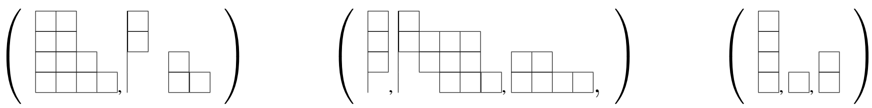

Fig. 7: The first two tuples of skew shapes have diameter 3. The third tuple of skew shapes has diameter 4.

Theorem 3.7 *The LLT graph $\mathcal{L}_{\boldsymbol{\nu}}$ is a dual equivalence graph if and only if* $\mathrm{diam}(\boldsymbol{\nu}) \leq 3$.

The proof of this theorem is primarily an application of Proposition 3.5, followed by a computer verification using Corollary 2.11. This computer verification can be found at

$<$http://www.math.washington.edu/$\sim$austinis/Proof_LLTandDEG.sws$>$.

Finally, we give a lemma that allows us to find a representative vertex in each component of an LLT graph. It is crucial to results in the following section.

Lemma 3.8 *Let $\boldsymbol{\nu}$ be a tuple of skew shapes such that $|\boldsymbol{\nu}| = n$ and* $\mathrm{diam}(\boldsymbol{\nu}) \leq 3$. *Let $\mathcal{C}$ be any connected component of $\mathcal{L}_{\boldsymbol{\nu}}$, and let $\phi : \mathcal{C} \to \mathcal{G}_\lambda$ be an isomorphism, where $\lambda \vdash n$. Then there is exactly one standardized Yamanouchi word w in $\mathcal{C}$, and $P(w) = \phi(w) = U_\lambda$.*

3.3 The Schur Expansion when $\mathrm{diam}(\nu) \leq 3$

The goal of this section is to apply Theorem 3.7 to get specific Schur expansions for a family of LLT polynomials and a family of Macdonald polynomials.

Theorem 3.9 *Let $\boldsymbol{\nu}$ be any tuple of skew shapes with* $\mathrm{diam}(\boldsymbol{\nu}) \leq 3$. *Further, let $\mathbf{T}(\lambda)$ be the set of standard fillings in* $\mathrm{SYT}(\boldsymbol{\nu})$ *whose shifted content words are in* $\mathrm{SYam}(\lambda)$. *Then*

$$\widetilde{G}_{\boldsymbol{\nu}}(X;q) = \sum_{\lambda \vdash |\boldsymbol{\nu}|} \sum_{\mathbf{T} \in \mathbf{T}(\lambda)} q^{\mathrm{inv}(\mathbf{T})} s_\lambda.$$

In particular, the set of $\boldsymbol{\nu}$ such that $\mathrm{diam}(\boldsymbol{\nu}) \leq 3$ properly contains the set of $\boldsymbol{\nu}$ that are 2-tuples.

Proof Sketch: We begin by reducing Theorem 3.9 to a statement about signed colored graphs. Let $V_1, V_2, \ldots, V_m$ be the vertex sets of the connected components of $\mathcal{L}_{\boldsymbol{\nu}} = (V, \sigma, E)$. Then

$$\widetilde{G}_{\boldsymbol{\nu}}(X;q) = \sum_{\mathbf{T} \in \mathrm{SYT}(\boldsymbol{\nu})} q^{\mathrm{inv}_k(\mathbf{T})} F_{\sigma(\mathbf{T})}(X) = \sum_{v \in V} q^{\mathrm{inv}_{\boldsymbol{\nu}}(v)} F_{\sigma(v)}(X) = \sum_{j=1}^{m} \sum_{v \in V_j} q^{\mathrm{inv}_{\boldsymbol{\nu}}(v)} F_{\sigma(v)}(X), \quad (3.11)$$

We may then apply Theorem 3.7, (3.2), and Lemma 3.8 to express $\widetilde{G}_{\boldsymbol{\nu}}(X;q)$ in terms of standardized Yamanouchi words as given in the statement of the theorem. $\square$

The next corollary follows immediately by applying Theorem 3.9 to the definition of modified Macdonald polynomials in (3.7). We also use the easily verified fact that tuples of ribbons in $\mathrm{TR}(\mu/\rho)$ have diameter less than or equal to three if and only if μ/ρ does not contain (3,3) or (4) as a subdiagram.

Corollary 3.10 *Let μ/ρ be a skew shape not containing $(3,3)$ or (4) as a subdiagram, and let $T(\lambda)$ be the set of bijective fillings of μ/ρ whose row reading words are in* $\mathrm{SYam}(\lambda)$. *Then*

$$\widetilde{H}_{\mu/\rho}(X;q,t) = \sum_{\lambda \vdash |\mu/\rho|} \sum_{T \in T(\lambda)} q^{\mathrm{inv}(T)} t^{\mathrm{maj}(T)} s_\lambda.$$

In particular, Corollary 3.10 applies to all $\tilde{H}_\mu(X;q,t)$ with $\mu_1 \le 3$ where $\mu_2 \le 2$.

Remark 3.11 The conditions on $\boldsymbol{\nu}$ and μ/ρ in Theorem 3.9 and Corollary 3.10, respectively, are sharp in the following sense. Let $\lambda = (2,2)$ and $\boldsymbol{\nu} = ((1),(1),(1,1))$ or $((1),(1),(1),(1))$. In particular, $\mathrm{diam}(\boldsymbol{\nu}) = 4$. Then

$$\widetilde{G}_{\boldsymbol{\nu}}(X;q)\big|_{q^4 s_\lambda} = 1 \quad \text{and} \quad \sum_{T \in T(\lambda)} q^{\mathrm{inv}(T)} s_\lambda\big|_{q^4 s_\lambda} = 2. \tag{3.12}$$

If $\lambda = (2,2)$ and $\mu/\rho = (4)$, then

$$\widetilde{H}_{\mu/\rho}(X;q,t)\big|_{q^4 s_\lambda} = 1 \quad \text{and} \quad \sum_{T \in T(\lambda)} q^{\mathrm{inv}(T)} t^{\mathrm{maj}(T)} s_\lambda\big|_{q^4 s_\lambda} = 2. \tag{3.13}$$

If $\lambda = (2,2,2)$ and $\mu/\rho = (3,3)$, then

$$\widetilde{H}_{\mu/\rho}(X;q,t)\big|_{q^3 t^3 s_\lambda} = 1 \quad \text{and} \quad \sum_{T \in T(\lambda)} q^{\mathrm{inv}(T)} t^{\mathrm{maj}(T)} s_\lambda\big|_{q^3 t^3 s_\lambda} = 2. \tag{3.14}$$

By using a symmetry of Macdonald polynomial, we also have the following immediate corollary.

Corollary 3.12 *Let μ/ρ be a skew shape not containing $(2,2,2)$ or $(1,1,1,1)$ as a subdiagram, and let $\widetilde{T}(\lambda)$ be the set of bijective fillings of $\tilde{\mu}/\tilde{\rho}$ whose row reading words are in* $\mathrm{SYam}(\lambda)$. *Then*

$$\widetilde{H}_{\mu/\rho}(X;q,t) = \sum_{\lambda \vdash |\mu/\rho|} \sum_{T \in \widetilde{T}(\lambda)} q^{\mathrm{maj}(T)} t^{\mathrm{inv}(T)} s_\lambda.$$

In particular, Corollary 3.10 applies to all $\tilde{H}_\mu(X;q,t)$ where μ has at most three rows and $\mu_3 \le 1$.

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Rational Catalan Combinatorics: The Associahedron

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Abstract. Each positive rational number $x > 0$ can be written **uniquely** as $x = a/(b-a)$ for coprime positive integers $0 < a < b$. We will identify x with the pair (a, b). In this extended abstract we use *rational Dyck paths* to define for each positive rational $x > 0$ a simplicial complex $\mathsf{Ass}(x) = \mathsf{Ass}(a, b)$ called the *rational associahedron*. It is a pure simplicial complex of dimension $a - 2$, and its maximal faces are counted by the *rational Catalan number*

$$\mathsf{Cat}(x) = \mathsf{Cat}(a, b) := \frac{(a + b - 1)!}{a!\, b!}.$$

The cases $(a, b) = (n, n + 1)$ and $(a, b) = (n, kn + 1)$ recover the classical associahedron and its Fuss-Catalan generalization studied by Athanasiadis-Tzanaki and Fomin-Reading. We prove that $\mathsf{Ass}(a, b)$ is shellable and give nice product formulas for its h-vector (the *rational Narayana numbers*) and f-vector (the *rational Kirkman numbers*). We define $\mathsf{Ass}(a, b)$.

Résumé. Tout nombre rationnel positif $x > 0$ peut être exprimé de façon **unique** par $x = a/(b - a)$ avec $0 < a < b$ deux entiers positifs premiers entre eux. Nous identifierons x avec la paire (a, b). Dans cet article, nous utilisons les *chemins de Dyck rationnels* pour définir pour tout rationnel positif $x > 0$ un complexe simplicial $\mathsf{Ass}(x) = \mathsf{Ass}(a, b)$ que nous appelons l'*associahedron rationnel*. Il s'agit d'un complexe simplicial pur de dimension $a - 2$, et ses faces maximales sont comptées par le *nombre rationnel de Catalan*

$$\mathsf{Cat}(x) = \mathsf{Cat}(a, b) := \frac{(a + b - 1)!}{a!\, b!}.$$

Les cas $(a, b) = (n, n+1)$ et $(a, b) = (n, kn+1)$ permettent de retrouver l'associhedron classique et sa généralisation Fuss-Catalan, étudiée par Athanasiadis-Tzanaki et Fomin-Reading. Nous démontrons que $\mathsf{Ass}(a, b)$ est shellable et nous donnons des formules de produits simples pour son h-vecteur (les *nombres rationnels de Narayana*) et son f-vecteur (les *nombres rationnels de Kirkman*).

Keywords: associahedron, Dyck path, f-vector, h-vector, shelling, noncrossing partition

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1 Motivation

This extended abstract is one of a pair of papers (see also [ALW]) that initiate the research program of *rational Catalan combinatorics*. The motivation for this program is both combinatorial and representation-theoretic.

The *classical Catalan numbers*[(i)]

$$\mathsf{Cat}(n, n+1) = \frac{1}{n+1}\binom{2n}{n}$$

are among the most important sequences in combinatorics. As of this writing, they are known to count at least 201 distinct families of combinatorial objects [Stan]. For our current purpose, the following three are the most important:

1. Dyck paths from $(0, 0)$ to (n, n),
2. Triangulations of a convex $(n + 2)$-gon, and
3. Noncrossing partitions of a cycle $(1, 2, \ldots, n)$.

There are two observations that have spurred recent progress in this field. The first is that Catalan objects are revealed to be type A phenomena (corresponding to the symmetric group) when properly interpreted in the context of reflection groups. The second is that many definitions of Catalan objects can be further generalized to accommodate an additional parameter, so that the resulting objects are counted by Fuss-Catalan numbers (see [Arm, Chapter 5]).

Both of these generalizations can be motivated from Garsia's and Haiman's [GH] observation that the Catalan numbers play a deep role in representation theory. The symmetric group $\mathfrak{S}_n$ acts on the polynomial ring $DS_n := \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ by permuting variables diagonally. That is, for $w \in \mathfrak{S}_n$ we define $w.x_i = x_{w(i)}$ and $w.y_i = y_{w(i)}$. Weyl [W] proved that the subring of diagonal invariants is generated by the polarized power sums $p_{r,s} = \sum_i x_i^r y_i^s$ for $r + s \geq 0$ with $1 \leq r + s \leq n$. The quotient ring of diagonal coinvariants $DR_n := DS_n/(p_{r,s})$ inherits the structure of an $\mathfrak{S}_n$-module which is bigraded by x-degree and y-degree. Garsia and Haiman conjectured that $\dim DR_n = (n + 1)^{n-1}$ (a number famous from Cayley's formula [Cay]) and that the dimension of the sign-isotypic component is the Catalan number $\mathsf{Cat}(n, n+1)$. These conjectures turned out to be difficult to resolve, and were proved about ten years later by Haiman using the geometry of Hilbert schemes.

An excellent introduction to this subject is Haiman's paper [Hai1], in which he laid the foundation for generalizing the theory of diagonal coinvariants to other reflection groups. Let W be a Weyl group, so that W acts irreducibly on $\mathbb{R}^\ell$ by reflections and stabilizes a full-rank lattice $\mathbb{Z}^\ell \approx Q \subseteq \mathbb{R}^\ell$, called the *root lattice*. The group also comes equipped with special integers $d_1 \leq \cdots \leq d_\ell$ called *degrees*, of which the largest $h := d_\ell$ is called the *Coxeter number*. Haiman showed that number of orbits of W acting on the finite torus $Q/(h + 1)Q$ is equal to

$$\mathsf{Cat}(W) := \prod_i \frac{h + d_i}{d_i},$$

which we now refer to as the *Catalan number* of W.

From this modern perspective, our three examples above become:

(i) This notation will we justified shortly.

1. W-orbits of the finite torus $Q/(h+1)Q$ [Shi1, Hai1, Ath1, CP],

2. Clusters in Fomin and Zelevinsky's finite type cluster algebras [FZ], and

3. Elements beneath a Coxeter element c in the absolute order on W [Rei, Arm].

More generally, given any positive integer p **coprime** to the Coxeter number h, Haiman showed that the number of orbits of W acting on the finite torus Q/pQ is equal to

$$\mathsf{Cat}(W,p) := \prod_i \frac{p+d_i-1}{d_i}, \tag{1}$$

which we now refer to as a *rational Catalan number*.

The cases $p = mh+1$ have been extensively studied as the Fuss-Catalan analogues, which further generalize our initial three examples to:

1. Dominant regions in the m-Shi arrangement [Ath2, FV],

2. Clusters in the generalized cluster complex [FR], and

3. m-multichains in the noncrossing partition lattice. [Edel, Arm].

The broad purpose of *rational Catalan combinatorics* is to complete the generalization from $p = +1 \bmod h$ to all parameters p. That is, we wish to define and study Catalan objects such as parking functions, Dyck paths, triangulations, and noncrossing partitions for each pair (W,p), where W is a finite reflection group and p is a positive integer coprime to the Coxeter number h. We may think of this as a two-dimensional problem with a "type axis" W and a "parameter axis" p. The level set $p = h+1$ is understood fairly well, and the Fuss-Catalan cases $p = +1 \bmod h$ are discussed in Chapter 5 of Armstrong [Arm]. However, it is surprising that the type A level set (i.e. $W = \mathfrak{S}_n$) is an open problem. This could have been pursued fifty years ago, but no one has done so in a systematic way.

Thus, we propose to begin the study of *rational Catalan combinatorics* with the study of *classical* rational Catalan combinatorics corresponding to a pair $(\mathfrak{S}_a, b)$ with b coprime to a. In this case, we have the *classical rational Catalan number*

$$\mathsf{Cat}(\mathfrak{S}_a, b) = \frac{1}{a+b}\binom{a+b}{a,b} = \frac{(a+b-1)!}{a!\,b!}. \tag{2}$$

Note the surprising symmetry between a and b; i.e. that $\mathsf{Cat}(\mathfrak{S}_a, b) = \mathsf{Cat}(\mathfrak{S}_b, a)$. This will show up as a conjectural Alexander duality in our study of rational associahedra.

First we will set down notation for the rational Catalan numbers $\mathsf{Cat}(\mathfrak{S}_a, b)$ in Section 2. Then in Section 3 we will define the *rational Dyck paths* which are the heart of the theory. In Section 4 we will use the Dyck paths to define and study *rational associahedra*. (In the full version of this paper we will also study the closely related *rational noncrossing partitions*.) In a separate paper [ALW] the Dyck paths will be used to define and study *rational parking functions* and q,t-statistics on these. The project of generalizing these constructions to reflection groups beyond $\mathfrak{S}_n$ is left for the future.

2 Rational Catalan Numbers

Given a rational number $x \in \mathbb{Q}$ outside the range $[-1, 0]$, there is a unique way to write $x = a/(b-a)$ where $a \neq b$ are coprime positive integers. We consider this a canonical form, and we will identify $x \in \mathbb{Q}$ with the ordered pair $(a, b) \in \mathbb{N}^2$ when convenient.

Inspired by the formulas (1) and (2) above, we define the *rational Catalan number*:

$$\mathsf{Cat}(x) = \mathsf{Cat}(a, b) := \frac{1}{a+b}\binom{a+b}{a, b} = \frac{(a+b-1)!}{a!\, b!}.$$

The most important feature of the rational Catalan numbers is that they are backwards-compatible:

$$\mathsf{Cat}(n) = \mathsf{Cat}(n/1) = \mathsf{Cat}(n, n+1) = \frac{1}{2n+1}\binom{2n+1}{n, n+1} = \frac{1}{n+1}\binom{2n}{n}.$$

But note also that $\mathsf{Cat}(a, b)$ is symmetric in a and b. This, together with the fact that $a/(b-a) = x$ if and only if $b/(a-b) = -x-1$, gives us

$$\mathsf{Cat}(x) = \mathsf{Cat}(a, b) = \mathsf{Cat}(b, a) = \mathsf{Cat}(-x-1).$$

That is, the function $\mathsf{Cat} : \mathbb{Q} \setminus [-1, 0] \to \mathbb{N}$ is symmetric about $x = -1/2$. Now observe that $-\frac{1}{x-1} - 1 = \frac{x}{1-x}$, and hence $\mathsf{Cat}(1/(x-1)) = \mathsf{Cat}(x/(1-x))$. We call this value the *derived Catalan number*:

$$\mathsf{Cat}'(x) := \mathsf{Cat}(1/(x-1)) = \mathsf{Cat}(x/(1-x)).$$

Furthermore, note that $\frac{1}{(1/x)-1} = \frac{x}{1-x}$, hence

$$\mathsf{Cat}'(x) = \mathsf{Cat}'(1/x). \tag{3}$$

We call this equation *rational duality* and it will play an important role in our study of rational associahedra. Equation (3) can also be used to extend the domain of Cat' from $\mathbb{Q} \setminus [-1, 0]$ to $\mathbb{Q} \setminus \{0\}$, but we don't know if this holds combinatorial significance. In terms of a and b we can write

$$\mathsf{Cat}'(x) = \mathsf{Cat}'(a, b) = \begin{cases} \binom{b}{a}/b & \text{if } a < b, \\ \binom{a}{b}/a & \text{if } b < a. \end{cases}$$

The "derivation" of Catalan numbers can be viewed as a "categorification" of the Euclidean algorithm. For example, consider $x = 5/3$ (that is, $a = 5$ and $b = 8$). The continued fraction expansion of x is

$$\frac{5}{3} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}$$

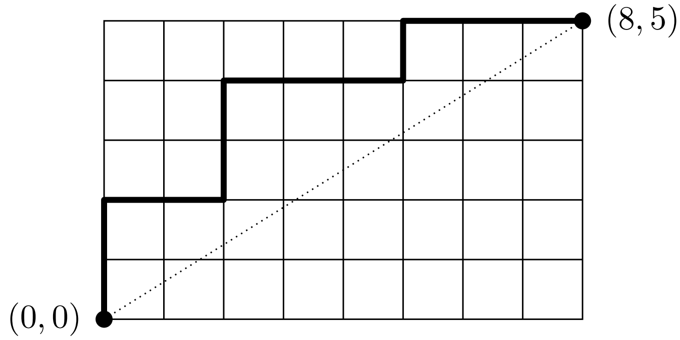

Fig. 1: This is a $(5,8)$-Dyck path.

with "convergents" (that is, successive truncations) $\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}$. Thus we have

$$\begin{aligned}
\mathsf{Cat}(5/3) &= 99, \\
\mathsf{Cat}'(5/3) &= \mathsf{Cat}(3/2) = 7, \\
\mathsf{Cat}''(5/3) &= \mathsf{Cat}'(3/2) = \mathsf{Cat}(2) = 2, \\
\mathsf{Cat}'''(5/3) &= \mathsf{Cat}''(3/2) = \mathsf{Cat}'(2) = \mathsf{Cat}(1) = 1.
\end{aligned}$$

The process stabilizes because $\mathsf{Cat}'(1) = 1$.

3 Rational Dyck Paths

At the heart of our constructions lies a family of lattice paths called rational Dyck paths. A *rational Dyck path* is a path from $(0,0)$ to (b,a) in the integer lattice $\mathbb{Z}^2$ using steps of the form $(1,0)$ and $(0,1)$ and staying above the diagonal $y = \frac{a}{b}x$. (Because a and b are coprime, it will never **touch** the diagonal.) More specifically, we call this an x-Dyck path or an (a,b)-Dyck path. For example, Figure 1 displays a $(5,8)$-Dyck path. When a and b are clear from context, we will sometimes refer to (a,b)-Dyck paths as simply *Dyck paths*.

Note that the final step of an $(n,n+1)$-Dyck path must travel from (n,n) to $(n,n+1)$. Upon removing this step we obtain a path from $(0,0)$ to (n,n) that stays *weakly* above the line of slope 1; that is, we obtain a classical Dyck path. The following result generalizes the fact that there are $\mathsf{Cat}(n,n+1)$ classical Dyck paths, can be proven using the Cycle Lemma of Dvorestky and Motzkin [DM], and is perhaps best attributed to 'folklore'.

Theorem 1 *For $a \neq b$ coprime positive integers, the number of (a,b)-Dyck paths is the Catalan number* $\mathsf{Cat}(a,b) = \frac{1}{a+b}\binom{a+b}{a,b}$.

The following refinement from [ALW] can also be proven using the Cycle Lemma.

Theorem 2 *The number of (a,b)-Dyck paths with i nontrivial vertical runs is the* Narayana number

$$\mathsf{Nar}(a,b;i) := \frac{1}{a}\binom{a}{i}\binom{b-1}{i-1},$$

and the number of (a, b)-Dyck paths with r_j vertical runs of length j is the Kreweras number

$$\mathsf{Krew}(a, b; \mathbf{r}) := \frac{1}{b}\binom{b}{r_0, r_1, \ldots, r_a} = \frac{(b-1)!}{r_0! r_1! \cdots r_a!}.$$

Equivalently, the first formula counts the (a, b)-Dyck paths with $i - 1$ valleys. We include trivial vertical runs of length 0 in the second formula for aesthetic reasons. For example, the path in Figure 1 has 3 nontrivial vertical runs (i.e. 2 valleys) and $\mathbf{r} = (5, 1, 2, 0, 0, 0)$. The rational Narayana numbers will appear below as the h-vector of the rational associahedron.

4 Rational Associahedra

4.1 Simplicial Complexes

We recall a collection of definitions related to simplicial complexes. A *simplicial complex* Δ on a finite ground set E is a collection of subsets of E such that if $S \in \Delta$ and $T \subseteq S$, then $T \in \Delta$. The elements of Δ are called *faces*, the maximal elements of Δ are called *facets*, and Δ is called *pure* if all of its facets have the same cardinality. The *dimension* of a face $S \in \Delta$ is $\dim(S) := |S| - 1$ and the *dimension* of Δ is the maximum dimension of a face in Δ. Observe that the 'empty face' $\emptyset$ has dimension -1.

If Δ is a d-dimensional simplicial complex, the f*-vector* of Δ is the integer sequence $f(\Delta) = (f_{-1}, f_0, \ldots, f_d)$, where $f_{-1} = 1$ and f_i is the number of i-dimensional faces in Δ for $0 \leq i \leq d$. The *reduced Euler characteristic* $\chi(\Delta)$ is given by $\chi(\Delta) := \sum_{i=-1}^{d} (-1)^i f_i$. The h*-vector* of Δ is the sequence $h(\Delta) = (h_{-1}, h_0, \ldots, h_d)$ defined by the following polynomial equation in t: $\sum_{i=-1}^{d} f_i (t-1)^{d-i} = \sum_{k=-1}^{d} h_k t^k$. The sequences $f(\Delta)$ and $h(\Delta)$ determine one another completely for any simplicial complex Δ.

Shellability is a key property possessed by some pure simplicial complexes which determines the homotopy type and h-vector of the complex. Let Δ be a pure d-dimensional simplicial complex. A total order $F_1 \prec \cdots \prec F_r$ on the facets $F_1, \ldots, F_r$ of Δ is called a *shelling order* if for $2 \leq k \leq r$, the subcomplex of the simplex F_k defined by $C_k := (\bigcup_{i=1}^{k-1} F_i) \cap F_k$ is a pure $(d-1)$-dimensional simplicial complex. The complex Δ is called *shellable* if there exists a shelling order on its facets; it can be shown that any pure d-dimensional shellable simplicial complex is homotopy equivalent to a wedge of spheres, all of dimension d.

The number of the spheres in the homotopy type of a shellable complex can be read off from the shelling order. More precisely, let Δ be a pure d-dimensional simplicial complex which is shellable and let $F_1 \prec \cdots \prec F_r$ be a shelling order on its facets. For $1 \leq k \leq r$, there exists a unique minimal face M_k of the simplex F_k which is not contained in the union $\bigcup_{i=1}^{k-1} F_i$ of the facets which appear earlier in the shelling order. The multiset of dimensions $\{\dim(M_1), \dim(M_2), \ldots, \dim(M_k)\}$ of these minimal added faces is independent of the shelling order. In fact, we have that i^{th} entry h_i of the h-vector $h(\Delta)$ equals the number of minimal faces M_k with $\dim(M_k) = i$. Moreover, the complex Δ is homotopy equivalent to a wedge of h_d copies of the d-dimensional sphere. For future use, we recall the well-known fact that adding a unique minimal face at each stage characterizes shelling orders.

Lemma 3 *Let Δ be a pure simplicial complex and let $F_1 \prec \cdots \prec F_r$ be a total order on the facets of Δ. The order $\prec$ is a shelling order if and only if for $1 \leq k \leq r$ there exists a unique minimal face M_k of the simplex F_k which is not contained in $\bigcup_{i=1}^{k-1} F_i$.*

4.2 Construction, Basic Facts, and Conjectures

For $n \geq 3$, let $\mathbb{P}_n$ denote the regular n-gon. Recall that the (dual of the) classical associahedron $\mathsf{Ass}(n, n+1)$consists of all (noncrossing) collections of diagonals of $\mathbb{P}_{n+2}$—the *dissections* of $\mathbb{P}_{n+2}$—ordered by inclusion. The diagonals of $\mathbb{P}_{n+2}$ are therefore the vertices of $\mathsf{Ass}(n, n+1)$ and the facets of $\mathsf{Ass}(n, n+1)$ are labeled by the maximal dissections—the triangulations of $\mathbb{P}_{n+2}$. Associahedra were introduced by Stasheff [St] in the context of nonassociative products arising in algebraic topology. Since its introduction, the associahedron has become one of the most well-studied complexes in geometric combinatorics, with connections to the permutohedron and exchange graphs of cluster algebras.

The classical associahedron has a Fuss analog defined as follows. Let $m \geq 1$ be a Fuss parameter. The Fuss associahedron $\mathsf{Ass}(n, mn+1)$ has as its facets the collection of all dissections of $\mathbb{P}_{mn+2}$ into $(m+2)$-gons. Fuss associahedra arise in the study of the generalized cluster complexes of Fomin and Reading.

We define our further generalization $\mathsf{Ass}(a, b)$ of the classical associahedron as follows. The vertices of $\mathsf{Ass}(a, b)$ will correspond to certain diagonals in $\mathbb{P}_{b+1}$ and the faces will correspond to certain dissections of $\mathbb{P}_{b+1}$. Label the vertices of $\mathbb{P}_{b+1}$ clockwise with $1, 2, \dots, b+1$.

Given any Dyck path D and any lattice point $P = (i, j)$ which is the bottom of a north step in D, we associate a diagonal $e(P)$ in $\mathbb{P}_{b+2}$ as follows. Consider the line ℓ with equation $(y - j) = \frac{a}{b}(x - i)$. This line intersects the path D in the lattice point P and in at least one other point to the right of P. Let Q be the leftmost such point to the right of P and let (r, s) be the coordinates of Q. By coprimality and the fact that $b > a$, we have that $i + 1 < s < b$ and s is not an integer. Let $e(P)$ be the diagonal $(i, \lceil s \rceil)$ in $\mathbb{P}_{b+1}$, where $\lceil s \rceil$ is the smallest integer $\geq s$. Define a subset $F(D)$ of diagonals of $\mathbb{P}_{b+1}$ by

$$F(D) := \{e(P) \,:\, P \text{ is the bottom of a north step in } D\}. \tag{4}$$

The right of Figure 2 shows the collection $F(D)$ of diagonals corresponding to the given Dyck path D on $\mathbb{P}_9$. It is topologically clear that the collection $F(D)$ of diagonals in $\mathbb{P}_{b+1}$ is noncrossing for any Dyck path D. The sets $F(D)$ form the facets of our simplicial complex.

Definition 4 *For $a < b$, the simplicial complex $\mathsf{Ass}(a, b)$ has as its ground set the collection of diagonals of $\mathbb{P}_{b+1}$ and facets $\{F(D) \,:\, D$ is an (a, b)-Dyck path $\}$.*

The following basic facts about $\mathsf{Ass}(a, b)$ can be proven directly from its definition.

Proposition 5 *1. The simplicial complex $\mathsf{Ass}(a, b)$ is pure and has dimension $a - 2$.*

2. The number of facets in $\mathsf{Ass}(a, b)$ is $\mathsf{Cat}(a, b)$.

3. Define a subset $S(a, b)$ of $[b-1]$ by $S(a, b) = \{\lfloor \frac{ib}{a} \rfloor \,:\, 1 \leq i < a\}$, where $\lfloor s \rfloor$ is the greatest integer $\leq s$. A diagonal of $\mathbb{P}_{b+1}$ which separates i vertices from $b - i - 1$ vertices appears as a vertex in the complex $\mathsf{Ass}(a, b)$ if and only if $i \in S(a, b)$.

We will call a diagonal e of $\mathbb{P}_{b+1}$ which satisfies the hypothesis in Part 3 of Proposition 5 (a, b)-*admissible*. The vertex set of $\mathsf{Ass}(a, b)$ consists precisely of the (a, b)-admissible diagonals in $\mathbb{P}_{b+1}$.

Proof: Part 1 follows from the fact that an (a, b)-Dyck path contains a north steps. For Part 2, observe that if D and D' are distinct Dyck paths, the multisets of x-coordinates of the bottoms of the north steps of D and D' are distinct. In particular, this means that $F(D)$ and $F(D')$ are distinct sets of diagonals in

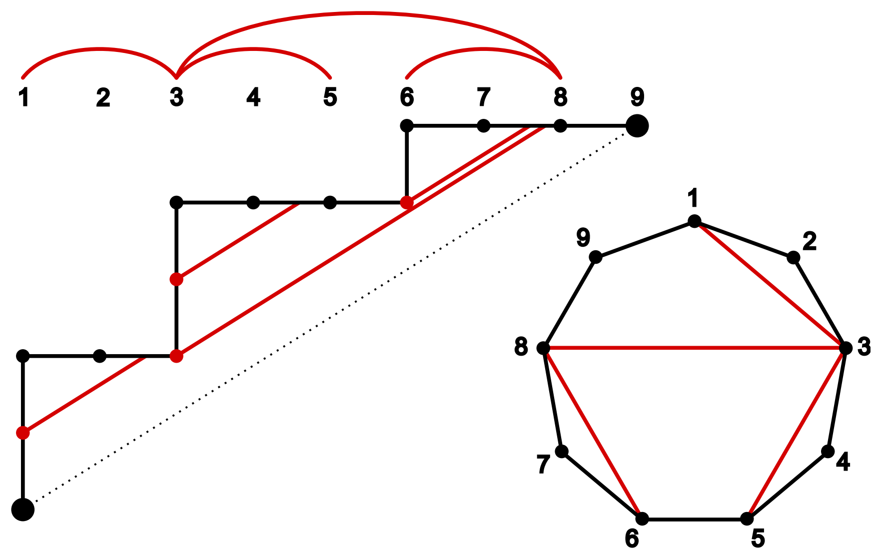

Fig. 2: A (5,8)-Dyck path and the corresponding dissection of $\mathbb{P}_9$.

$\mathbb{P}_{b+1}$. Part 2 follows from the fact that there are $\mathsf{Cat}(a, b)$ Dyck paths. Part 3 is a geometric observation about lines of slope $\frac{a}{b}$. $\square$

In the classical case $b = a + 1$ and the Fuss case $b = ka + 1$, the faces of the associahedron $\mathsf{Ass}(a, b)$ are given by collections of (a, b)-admissible diagonals in $\mathbb{P}_{b+1}$ which are mutually noncrossing. Given this characterization, it is clear that the associahedron carries an action of the cyclic group $\mathbb{Z}_{b+1}$ given by rotation in these cases. Neither of these statements remains true at the rational level of generality. Indeed, when $(a, b) = (3, 5)$, the diagonals $(1, 5)$ and $(3, 5)$ of $\mathbb{P}_6$ are $3, 5$-admissible and mutually noncrossing. However, the set $\{(1, 5), (3, 5)\}$ is not a face of $\mathsf{Ass}(3, 5)$. It can also be checked that $\mathsf{Ass}(3, 5)$ is not closed under rotation of $\mathbb{P}_6$.

In spite of the last paragraph, we conjecture that $\mathsf{Ass}(a, b)$ carries a rotation action 'up to homotopy'. More precisely, let $\mathsf{Ass}'(a, b)$ denote the simplicial complex whose faces are collections of mutually noncrossing (a, b)-admissible diagonals in $\mathbb{P}_{b+1}$. It is clear that $\mathsf{Ass}'(a, b)$ carries a rotation action and that $\mathsf{Ass}(a, b)$ is a subcomplex of $\mathsf{Ass}'(a, b)$.

Before stating our conjecture, we recall what it means for a complex to collapse onto a subcomplex; this is a combinatorial deformation retraction. Let Δ be a simplicial complex, $F \in \Delta$ be a facet, and suppose $F' \subset F$ satisfies $|F'| = |F| - 1$. If F' is not contained in any facet of Δ besides F, we can perform an *elementary collapse* by replacing Δ with $\Delta - \{F, F'\}$. A simplicial complex is said to *collapse* onto a subcomplex if the subcomplex can be obtained by a sequence of elementary collapses.

Conjecture 6 *The complexes* $\mathsf{Ass}(a, b)$ *and* $\mathsf{Ass}'(a, b)$ *are homotopy equivalent. In fact, the complex* $\mathsf{Ass}'(a, b)$ *collapses onto the subcomplex* $\mathsf{Ass}(a, b)$.

Figure 3 displays $\mathsf{Ass}(2, 5)$ (shown in blue) and $\mathsf{Ass}(3, 5)$ (shown in red) as subcomplexes of the sphere $\mathsf{Ass}(4, 5)$. The complex $\mathsf{Ass}'(2, 5)$ coincides with $\mathsf{Ass}(2, 5)$ and the complex $\mathsf{Ass}'(3, 5)$ is obtained from

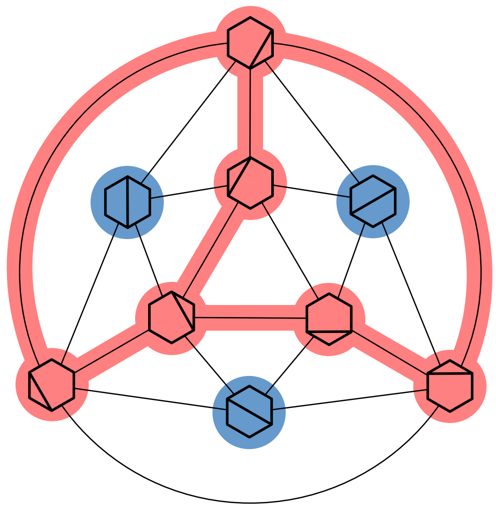

Fig. 3: $\mathsf{Ass}(2,5)$ and $\mathsf{Ass}(3,5)$ are Alexander dual within $\mathsf{Ass}(4,5)$.

the complex $\mathsf{Ass}(3,5)$ by adding the middle and exterior triangles to the red complex. Observe that $\mathsf{Ass}(3,5)$ can be obtained by performing two elementary collapses on $\mathsf{Ass}'(3,5)$.

Conjecture 6 would also have implications regarding Alexander duality. Recall that two topological subspaces X and Y of a fixed sphere S are said to be *Alexander dual* to one another if Y is homotopy equivalent to the complement of X in S. With $b > 1$ fixed, we have that a and b are coprime for $1 \leq a < b$ if and only if $b - a$ and b are coprime. Both of the complexes $\mathsf{Ass}(a,b)$ and $\mathsf{Ass}(a-b,b)$ sit within the classical associahedron $\mathsf{Ass}(b-1,b)$. The proof of Conjecture 6 would imply that $\mathsf{Ass}(a,b)$ and $\mathsf{Ass}(a-b,b)$ are Alexander dual.

Proposition 7 *Let $a < b$ be coprime for $b > 1$. The subcomplexes* $\mathsf{Ass}(a,b)$ *and* $\mathsf{Ass}(b-a,b)$ *are Alexander dual within the sphere* $\mathsf{Ass}(b-1,b)$*. If Conjecture 6 is true, then the subcomplexes* $\mathsf{Ass}(a,b)$ *and* $\mathsf{Ass}(b-a,b)$ *are also Alexander dual within* $\mathsf{Ass}(b-1,b)$.

Proof: It is routine to check that any diagonal of $\mathbb{P}_{b+1}$ is either (a,b)-admissible or $((b-a),b)$-admissible, but not both. This means that the vertex sets of $\mathsf{Ass}'(a,b)$ and $\mathsf{Ass}'(b-a,b)$ partition the vertex set of the simplicial sphere $\mathsf{Ass}'(b-1,b)$. By definition, the faces of $\mathsf{Ass}'(a,b)$ and $\mathsf{Ass}'(b-a,b)$ are precisely the faces of $\mathsf{Ass}(b-1,b)$ whose vertex sets are contained in $\mathsf{Ass}'(a,b)$ and $\mathsf{Ass}'(b-a,b)$, respectively. It follows that the complement of $\mathsf{Ass}'(a,b)$ inside $\mathsf{Ass}(b-1,b)$ deformation retracts onto $\mathsf{Ass}'(b-a,b)$. This proves the first statement. The second statement is clear. $\square$

4.3 Shellability and f- and h-vectors

We will prove that the simplicial complex $\mathsf{Ass}(a,b)$ is shellable by giving an explicit shelling order on its facets. This shelling order will be induced by lexicographic order on the partitions whose Ferrers diagrams lie to the northwest of (a,b)-Dyck paths.

Theorem 8 *The simplicial complex* $\mathsf{Ass}(a,b)$ *is shellable, hence homotopy equivalent to a wedge of spheres. Moreover, there is a total order* $D_1 \prec D_2 \prec \cdots \prec D_{\mathsf{Cat}(a,b)}$ *on the set of* (a,b)*-Dyck paths which induces a shelling order on the facets of* $\mathsf{Ass}(a,b)$ *such that the dimension of the minimal face added upon addition of the facet* $F(D_i)$ *equals the number of nonempty vertical runs in* D_i*, less one.*

Proof: (Sketch.) We will find it convenient to identify the facets of $\mathsf{Ass}(a,b)$ with both Dyck paths and partitions. We define a family of concepts which will be used for this proof only.

A *partition* λ is a weakly decreasing sequence $(\lambda_1 \geq \cdots \geq \lambda_k > 0)$ of positive integers. The *Ferrers diagram* associated with λ consists of λ_i left justified boxes in row i (we are using English notation). We will use the *lexicographic* total order $\prec$ on partitions defined by $\lambda \prec \mu$ if there exists an $i \geq 1$ such that $\lambda_j = \mu_j$ for $1 \leq j < i$ and $\lambda_i < \mu_i$. (We append an infinite string of zeros to the ends of λ and μ, if necessary, for these relations to make sense.)

Given any (a,b)-Dyck path D, let $\lambda(D)$ be the partition whose Ferrers diagram consists of the lattice boxes to the northwest of D in the rectangle with corners $(0,0)$ and (b,a). For example, if $(a,b)=(5,8)$ and D is the path in Figure 2, then $\lambda(D)=(5,2,2)$. It is clear that distinct Dyck paths give rise to distinct partitions, so the facets of $\mathsf{Ass}(a,b)$ are labeled by either (a,b)-Dyck paths or by partitions λ which satisfy $\lambda_i \leq \max(\lfloor \frac{(a-i)b}{a} \rfloor, 0)$ for all i.

Let $\lambda^{(1)} \prec \cdots \prec \lambda^{(\mathsf{Cat}(a,b))}$ be the restriction of lexicographic order to set of partitions which satisfy $\lambda_i \leq \max(\lfloor \frac{(a-i)b}{a} \rfloor, 0)$ for all i. In particular, we have that $\lambda^{(1)}$ is the empty partition and $\lambda_i^{(\mathsf{Cat}(a,b))} = \max(\lfloor \frac{(a-i)b}{a} \rfloor, 0)$. The total order $\prec$ induces a total order $D_1 \prec \cdots \prec D_{\mathsf{Cat}(a,b)}$ on (a,b)-Dyck paths and a total order $F(D_1) \prec \cdots \prec F(D_{\mathsf{Cat}(a,b)})$ on the facets of $\mathsf{Ass}(a,b)$.

In the case $(a,b)=(3,5)$, our order on partitions is

$$(0,0) \prec (1,0) \prec (1,1) \prec (2,0) \prec (2,1) \prec (3,0) \prec (3,1).$$

The corresponding order on facets of $\mathsf{Ass}(3,5)$ (written as diagonal sets in $\mathbb{P}_6$) is

$$\{(1,3),(1,5)\} \prec \{(2,4),(1,5)\} \prec \{(2,4),(2,6)\} \prec$$
$$\{(1,3),(3,5)\} \prec \{(2,6),(3,5)\} \prec \{(1,3),(4,6)\} \prec \{(2,4),(4,6)\}.$$

We will prove that $\prec$ is a shelling order on the facets of $\mathsf{Ass}(a,b)$ and that the minimal added faces corresponding to $\prec$ have the required dimensions. In fact, we will be able to describe these minimal added faces explicitly. Given any Dyck path D, recall that the corresponding facet $F(D)$ in $\mathsf{Ass}(a,b)$ is given by $F(D) = \{e(P) \,:\, P \text{ is the bottom of a north step in } D\}$. We define the *valley face* $V(D)$ to be the subset of $F(D)$ given by $V(D) := \{e(P) \,:\, P \text{ is a valley in } D\}$.

Claim: $1 \leq k \leq \mathsf{Cat}(a,b)$, the valley face $V(D_k)$ is the unique minimal face of $F(D_k)$ which is not contained in $\bigcup_{i=1}^{k-1} F(D_i)$.

By Lemma 3 and the discussion preceding it, this claim implies that $\prec$ is a shelling order with the dimension of the minimal added face as described (observe that in any Dyck path, the number of valleys equals the number of vertical runs), completing the proof of the theorem. The proof of this claim is omitted in this extended abstract. □

As a corollary to the above result, we get product formulas for the f- and h-vectors of $\mathsf{Ass}(a,b)$, as well as its reduced Euler characteristic. Define the *rational Kirkman numbers* by

$$\mathsf{Kirk}(a,b;i) := \frac{1}{a}\binom{a}{i}\binom{b+i-1}{i-1}. \tag{5}$$

Corollary 9 *Let* $(f_{-1}, f_0, \ldots, f_{a-2})$ *and* $(h_{-1}, h_0, \ldots, h_{a-2})$ *be the* f*- and* h*-vectors of* $\mathsf{Ass}(a,b)$*. For* $1 \leq i \leq a$ *we have that* $f_{i-1} = \mathsf{Kirk}(a,b;i)$ *and* $h_{i-1} = \mathsf{Nar}(a,b;i)$*. The reduced Euler characteristic of* $\mathsf{Ass}(a,b)$ *is the derived Catalan number* $\mathsf{Cat}'(a,b)$*.*

Proof: By Theorem 8 and Lemma 3, we have that h_{i-1} equals the number of (a,b)-Dyck paths which have exactly i vertical runs. By Theorem 2, this equals the Narayana number $\mathsf{Nar}(a,b;i)$.

To prove the statement about the f-vector, one must check that

$$\sum_{i=-1}^{a-2} \mathsf{Kirk}(a,b;i+1)(t-1)^{a-i-2} = \sum_{k=1}^{a-2} \mathsf{Nar}(a,b;k+1)t^k.$$

The statement about the Euler characteristic reduces to proving that

$$\sum_{i=-1}^{a-2} (-1)^{i+1} \mathsf{Kirk}(a,b;i+1) = \mathsf{Cat}'(a,b).$$

Both of these identities are left to the reader. □

Conjecture 6 and Proposition 7 assert that the symmetry $(a < b) \leftrightarrow (b-a < b)$ on pairs of coprime positive integers shows up in rational associahedra as an instance of Alexander duality. Corollary 9 shows that the categorification $\mathsf{Cat}(x) \mapsto \mathsf{Cat}'(x)$ of the Euclidean algorithm presented in Section 2 sends the number of facets of $\mathsf{Ass}(a,b)$ to the reduced Euler characteristic of $\mathsf{Ass}(a,b)$. This 'categorifies' the number theoretic properties of rational Catalan numbers to the context of associahedra.

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Homomesy in products of two chains

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Abstract. Many cyclic actions τ on a finite set $\mathcal{S}$ of combinatorial objects, along with a natural statistic f on $\mathcal{S}$, exhibit "homomesy": the average of f over each τ-orbit in $\mathcal{S}$ is the same as the average of f over the whole set $\mathcal{S}$. This phenomenon was first noticed by Panyushev in 2007 in the context of antichains in root posets; Armstrong, Stump, and Thomas proved Panyushev's conjecture in 2011. We describe a theoretical framework for results of this kind and discuss old and new results for the actions of promotion and rowmotion on the poset that is the product of two chains.

Résumé. Plusieurs actions cycliques τ sur un ensemble fini $\mathcal{S}$ d'objets combinatoires muni d'une statistique naturelle f sur $\mathcal{S}$ démontrent une "homomesie": la moyenne de f sur une orbite de τ en $\mathcal{S}$ est la même que la moyenne de f sur la totalité de l'ensemble $\mathcal{S}$. Ce phénomène a été d'abord remarqué par Panyushev en 2007 dans le contexte des anti-chaînes dans des posets de racine; Armstrong, Stump, et Thomas ont demontré la conjecture de Panyushev en 2011. On décrit un contexte de travail pour énoncer des résultats de ce genre et on discute de nouveaux et d'anciens résultats pour des actions de "promotion" et "rowmotion" sur le poset qui est le produit de deux chaînes.

Keywords: antichains, combinatorial ergodicity, homomesy, orbit, order ideals, poset, product of chains, promotion, rowmotion, sandpile, toggle group.

1 Introduction

We begin with the definition of our main unifying concept, and supporting nomenclature.

Definition 1. *Given a finite set $\mathcal{S}$ of combinatorial objects, an invertible map τ from $\mathcal{S}$ to itself, and a function (or "statistic") $f : \mathcal{S} \to K$ taking values in some field K of characteristic zero, we say the triple $(\mathcal{S}, \tau, f)$ exhibits* **homomesy** *iff there exists a constant $c \in K$ such that for every τ-orbit $\mathcal{O} \subset \mathcal{S}$*

$$\frac{1}{\#\mathcal{O}} \sum_{x \in \mathcal{O}} f(x) = c. \tag{1}$$

In this situation we say that the function $f : \mathcal{S} \to K$ is **homomesic** *(Greek for "same middle") relative to the action of τ on $\mathcal{S}$. If c is 0, we say that f is* **0-mesic**.

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We also apply the term "homomesic" more broadly to situations in which $\mathcal{S}$ is not a set of combinatorial objects or the statistic f takes values in a vector space over a field of characteristic 0 (as in sections 2.2 and 2.3). Homomesy can be restated equivalently as all orbit-averages being equal to the global average:

$$\frac{1}{\#\mathcal{O}} \sum_{x \in \mathcal{O}} f(x) = \frac{1}{\#\mathcal{S}} \sum_{x \in \mathcal{S}} f(x). \tag{2}$$

This is the form in which Panyushev [Pan08] stated his conjecture.

We have found many instances of (2) where the actions τ and the statistics f are natural ones. Many (but far from all) situations that support examples of homomesy also support examples of the *cyclic sieving phenomenon* of Reiner, Stanton, and White [RSW04]. Examples of homomesy are given starting with Section 2.

At the stated level of generality this notion appears to be new, but specific instances can be found in earlier literature. In particular, in 2007, Panyushev [Pan08] conjectured and in 2011, Amstrong, Stump, and Thomas [AST11] proved that if $\mathcal{S}$ is the set of antichains in the root poset of a finite Weyl group, Φ is the operation variously called the Brouwer-Schrijver map [BS74], the Fon-der-Flaass map [Fon93, CF95], the reverse map [Pan08], Panyushev complementation [AST11], and rowmotion [SW12], and $f(A)$ is the cardinality of the antichain A, then $(\mathcal{S}, \Phi, f)$ satisfies (2).

Our main results for this paper involve studying the action of this rowmotion operator and also the promotion operator on the poset $P = [a] \times [b]$, the product of two chains. (Here and throughout this article we use $[n]$ to denote the set $\{1, 2, \ldots, n\}$ and the associated n-element poset.) We show that the statistic $f := \#A$, the size of the antichain, is homomesic with respect to the promotion operator, and that the statistic $f = \#I(A)$, the size of the corresponding order ideal, is homomesic with respect to both the promotion and rowmotion operators.

Although these results are of intrinsic interest, we think the main contribution of the paper is its identification of homomesy as a phenomenon that (as we expect future articles to show) occurs quite widely. Within any linear space of functions on $\mathcal{S}$, the functions that are homomesic under τ, like the functions that are invariant under τ, form a subspace, and there is a loose sense in which the notions of invariance and homomesy (or, more strictly speaking, 0-mesy) are dual; an extremely clean case of this duality is outlined in subsection 2.2. This extended abstract gives a general overview of the broader picture as well as a few specific examples done in more detail for the operators of promotion and rowmotion on the poset $[a] \times [b]$.

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2 Examples of Homomesy

2.1 Rotation of bit-strings

We begin with a simple concrete example to clarify the definition.

$$\begin{array}{ccccccccccc} \ldots & \xrightarrow{\tau} & 0011 & \xrightarrow{\tau} & 0110 & \xrightarrow{\tau} & 1100 & \xrightarrow{\tau} & 1001 & \xrightarrow{\tau} & \ldots \\ & & f\downarrow & & f\downarrow & & f\downarrow & & f\downarrow & & \\ & & 0 & & 2 & & 4 & & 2 & & \end{array}$$

$$\begin{array}{ccccccc} \ldots & \xrightarrow{\tau} & 1010 & \xrightarrow{\tau} & 0101 & \xrightarrow{\tau} & \ldots \\ & & f\downarrow & & f\downarrow & & \\ & & 3 & & 1 & & \end{array}$$

Fig. 1: The two orbits of the action of the cyclic shift $\tau = C_L$ on binary strings consisting of two 0's and two 1's. The average value of the inversion statistic f is $(0+2+4+2)/4 = 2$ on the orbit of size 4 and $(3+1)/2 = 2$ on the orbit of size 2.

Example 1. Let $\mathcal{S} := \binom{[n]}{k}$, thought of as the set of length n binary strings with k 1's and $n-k$ 0's. Set $f(b) := \mathrm{inv}(b) := \#\{i < j : b_i > b_j\}$ and $\tau := C_L : \mathcal{S} \to \mathcal{S}$ the leftward cyclic shift operator given by $b = b_1 b_2 \cdots b_n \mapsto b_2 b_3 \cdots b_n b_1$. Then over any orbit $\mathcal{O}$ we have

$$\frac{1}{\#\mathcal{O}} \sum_{s \in \mathcal{O}} f(s) = \frac{k(n-k)}{2} = \frac{1}{\#\mathcal{S}} \sum_{s \in \mathcal{S}} f(s).$$

This fact can be proved in isolation, but it also follows from one of our results from Section 3, with $a = k$ and $b = n-k$. More specifically, we have a bijection between order-ideals in the poset $P = [a] \times [b]$ and strings consisting of a -1's and b $+1$'s (which in turn correspond to bit strings, if one replaces the -1's by 0's). Then promotion on $J(P)$ is equivariant with leftward cyclic shift on strings, and the cardinality of an order ideal is equal to the number of inversions in the associated string. Theorem 11 then yields the claimed result on bit-strings.

In the specfic case $n = 4, k = 2$, the six-element set decomposes into two orbits, shown in Figure 1. As frequently happens, not all orbits are the same size; however, one could also view the orbit of size 2 as a "multiset orbit" of size 4, cycling through the same set of elements twice. This perspective, where we view all orbits as multiset orbits of the same size, facilitates the discussion of certain comparisons.

2.2 Linear actions on vector spaces

Let V be a (not necessarily finite-dimensional) vector space over a field K of characteristic zero, and define $f(v) = v$ (that is, our "statistic" is just the identity function). Let $T : V \to V$ be a linear map such that $T^n = I$ (the identity map on V) for some fixed $n \geq 1$ (i.e., $I - T^n$ is the 0-map). Say v is *invariant* under T if $Tv = v$ (i.e., v is in the kernel of $I - T$) and *0-mesic* under T if $(v + Tv + \cdots + T^{n-1}v)/n = 0$ (i.e., v is in the kernel of $I + T + T^2 + \cdots + T^{n-1}$). Every $v \in V$ can be written uniquely as the sum of an invariant vector $\overline{v}$ and a 0-mesic vector $\hat{v}$ (specifically, one can check that $\overline{v} = (v + Tv + \cdots + T^{n-1}v)/n$ and $\hat{v} = v - \overline{v}$ work, and no other solution is possible because that would yield a nonzero vector that is both invariant and 0-mesic, which does not exist). In representation-theoretic terms, we are applying symmetrization to v to extract from it the invariant component $\overline{v}$ associated with the trivial representation of the cyclic group, and the homomesic (0-mesic) component $\hat{v}$ consists of everything else.

This picture relates more directly to our earlier definition if we use the dual space V^* of linear functionals on V as the set of statistics on V. As a concrete example, let $V = \mathbf{R}^n$ and let T be the cyclic shift of coordinates sending $(x_1, x_2, ..., x_n)$ to $(x_n, x_1, ..., x_{n-1})$. The T-invariant functionals form a 1-dimensional subspace of V^* spanned by the functional $(x_1, x_2, ..., x_n) \mapsto x_1 + x_2 + ... + x_n$, while the homomesic functionals form an $(n-1)$-dimensional subspace of V^* spanned by the $n-1$ functionals $(x_1, x_2, ..., x_n) \mapsto x_i - x_{i+1}$ (for $1 \leq i \leq n-1$).

2.3 Sandpile dynamics

Let G be a directed graph with vertex set V. For $v \in V$ let $\mathrm{outdeg}(v)$ be the number of directed edges emanating from v, and for $v, w \in V$ let $\deg(v, w)$ be the number of directed edges from v to w (which we will permit to be larger than 1, even when $v = w$). Define the *combinatorial Laplacian* of G as the matrix Δ (with rows and columns indexed by the vertices of V) whose v, vth entry is $\mathrm{outdeg}(v) - \deg(v, v)$ and whose v, wth entry for $v \neq w$ is $-\deg(v, w)$. Specify a *global sink* t with the property that for all $v \in V$ there is a forward path from v to t, let $V^- = V \setminus \{t\}$, and let Δ' (the reduced Laplacian) be the matrix Δ with the row and column associated with t removed. By the Matrix-Tree theorem, Δ' is nonsingular. A *sandpile configuration* on G (with sink at t) is a function σ from V^- to the nonnegative integers. (For more background on sandpiles, see Holroyd, Levine, Mészáros, Peres, Propp, & Wilson [HLMPPW08].) We say σ is *stable* if $\sigma(v) < \mathrm{outdeg}(v)$ for all $v \in V^-$. For any sandpile-configuration σ, Dhar's least-action principle for sandpile dynamics (see Levine & Propp [LP10]) tells us that the set of nonnegative-integer-valued functions u on V^- such that $\sigma - \Delta' u$ is stable has a minimal element $\phi = \phi(\sigma)$ in the natural (pointwise) ordering; we call ϕ the *firing vector* for σ and we call $\sigma - \Delta'\phi$ the *stabilization* of σ, denoted by σ°. If we choose a *source vertex* $s \in V^-$, then we can define an action on sandpile configurations via $\tau(\sigma) = (\sigma + 1_s)^\circ$, where 1_v denotes the function that takes the value 1 at v and 0 elsewhere. Say that σ is *recurrent* (relative to s) if $\tau^m(\sigma) = \sigma$ for some $m > 0$. (This notion of recurrence is slightly weaker than that of [HLMPPW08]; they are equivalent when every vertex is reachable by a path from s.) Then τ restricts to an invertible map from the set of recurrent sandpile configurations to itself. Let $f(\sigma) = \phi(\sigma + 1_s)$. Since $\tau(\sigma) = \sigma + 1_s - \Delta' f(\sigma)$ we have $\tau(\sigma) - \sigma = 1_s - \Delta' f(\sigma)$; if we average this relation over all σ in a particular τ-orbit, the left side telescopes, giving $0 = 1_s - \Delta'\overline{f}$, where $\overline{f}$ denotes the average of f over the orbit. Hence:

Proposition 2. *For the action of τ on recurrent sandpile configurations, the function $f : \sigma \mapsto \phi(\sigma + 1_s)$ is homomesic, and its orbit-average is the function f^* on V^- such that $\Delta' f^* = 1_s$ (unique because Δ' is nonsingular).*

Example 2. Figure 2 shows an example of the τ-orbits for the case where G is the bidirected cycle graph with vertices 1, 2, 3, and 4, with a directed edge from i to j iff $i - j = \pm 1$ mod 4; here the discrete Laplacian is

$$\Delta = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}.$$

Let the source be $s = 2$ and global sink be $t = 4$. The sandpile configuration σ is represented by the triple $(\sigma(1), \sigma(2), \sigma(3))$. The four recurrent configurations σ are $(1, 0, 1)$, $(1, 1, 1)$, $(0, 1, 1)$, and $(1, 1, 0)$, and the respective firing vectors $f(\sigma)$ are $(0, 0, 0)$, $(1, 2, 1)$, $(0, 1, 1)$, and $(1, 1, 0)$. The average value of the firing vector statistic f is $f^* = (\frac{1}{2}, 1, \frac{1}{2})$ on each orbit. Treating f^* as a column vector and multiplying

$$\begin{array}{ccccccc} \dots & \xrightarrow{\tau} & (1,0,1) & \xrightarrow{\tau} & (1,1,1) & \xrightarrow{\tau} & \dots \\ & & f\downarrow & & f\downarrow & & \\ & & (0,0,0) & & (1,2,1) & & \end{array}$$

$$\begin{array}{ccccccc} \dots & \xrightarrow{\tau} & (0,1,1) & \xrightarrow{\tau} & (1,1,0) & \xrightarrow{\tau} & \dots \\ & & f\downarrow & & f\downarrow & & \\ & & (0,1,1) & & (1,1,0) & & \end{array}$$

Fig. 2: The two orbits in the action of the sandpile map τ on recurrent configurations on the cycle graph of size 4, with source at 2 and sink at 4. There are two orbits, each of size 2, and the average of f along each orbit is $(1/2, 1, 1/2)$.

on the left by Δ' gives the column vector $(0, 1, 0) = 1_s$:

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Similar instances of homomesy were known for a variant of sandpile dynamics called rotor-router dynamics; see Holroyd-Propp [HP10]. It was such instances of homomesy that led the second author to seek instances of the phenomenon in other, better-studied areas of combinatorics.

3 Promotion and rowmotion in products of two chains

For a finite poset P, we let $J(P)$ denote the set of order ideals (or down-sets) of P, $F(P)$ denote the set of (order) filters (or up-sets) of P, and $\mathcal{A}(P)$ be the set of antichains of P. (For standard definitions and notation about posets and ideals, see Stanley [EC1].) There is a bijection $J(P) \leftrightarrow \mathcal{A}(P)$ given by taking the maximal elements of $I \in J(P)$ or conversely by taking the order ideal geneated by an antichain $A \in \mathcal{A}(P)$. Similarly, there is a bijection $F(P) \leftrightarrow \mathcal{A}(P)$. Composing these with the complementation bijection between $J(P)$ and $F(P)$ leads to an interesting map that has been studied in several contexts [BS74, Fon93, CF95, Pan08, AST11, SW12], namely $\Phi_A := \mathcal{A}(P) \to J(P) \to F(P) \to \mathcal{A}(P)$ and the companion map $\Phi_J := J(P) \to F(P) \to \mathcal{A}(P) \to J(P)$, where the subscript indicates whether we consider the map to be operating on antichains or order ideals. We often drop the subscript and just write Φ when context makes clear which is meant. Following Striker and Williams [SW12] we call this map **rowmotion**.

Let $[a] \times [b]$ denote the poset that is a product of chains of lengths a and b. Figure 5 shows an orbit of the action of Φ_J starting from the ideal generated by the antichain $\{(2, 1)\}$. Note that the elements of $[4] \times [2]$ here are represented by the *squares* rather than the points in the picture, with covering relations represented by shared edges. One can also view this as an orbit of Φ_A if one just considers the maximal elements in each shaded order ideal.

This section contains our main specific results, namely that the following triples exhibit homomesy:

$$(J([a] \times [b]), \Phi_J, \#I)\ ; \qquad (\mathcal{A}([a] \times [b]), \Phi_A, \#A)\ ; \text{ and } \qquad (J([a] \times [b]), \partial_J, \#I)\ .$$

Here ∂_J is the promotion operator to be defined in the next subsection, and $\#I$ (resp. $\#A$) denotes the statistic on $J(P)$ (resp. $\mathcal{A}(P)$) that is the cardinality of the order ideal I (resp. the antichain A). All maps operate on the left (e.g., we write $\partial_J I$, not $I\partial_J$).

3.1 Background on the toggle group

Several of our examples arise from the *toggle group* of a finite poset (first explicitly defined in [SW12]; see also [CF95, Sta09, SW12]). We review some basic facts and provide some pointers to relevant literature.

Definition 3. *Given $x \in P$, we define the toggle operation $\sigma_x : J(P) \to J(P)$ ("toggling at x") via*

$$\sigma_x(I) = \begin{cases} I \triangle \{x\} & \text{if } I \triangle \{x\} \in J(P); \\ I & \text{otherwise.} \end{cases}$$

Proposition 4 ([CF95])**.** *(a) For every $x \in P$, σ_x is an involution, i.e., $\sigma_x^2 = 1$.*
(b) For every $x, y \in P$ where neither x covers y nor y covers x, the toggles commute, i.e., $\sigma_x\sigma_y = \sigma_y\sigma_x$.

Proposition 5 ([CF95])**.** *Let $x_1, x_2, \ldots, x_n$ be any linear extension (i.e., any order-preserving listing of the elements) of a poset P with n elements. Then the composite map $\sigma_{x_1}\sigma_{x_2}\cdots\sigma_{x_n}$ coincides with the rowmotion operator Φ_J.*

Corollary 6 ([SW12], Cor. 4.9)**.** *Let P be a graded poset of rank r, and set $T_k := \prod_{|x|=k} \sigma_x$, the product of all the toggles of elements of fixed rank k. (This is well-defined by Proposition 4.) Then the composition $T_1T_2\cdots T_r$ coincides with Φ_J, i.e., rowmotion is the same as toggling by ranks from top to bottom.*

We focus on the case $P = [a] \times [b]$, whose elements we write as (k, ℓ); the Hasse diagram of P consists of the points $(i, j) := (\ell - k, \ell + k - 2)$ $(1 \le k \le a, 1 \le \ell \le b)$ so that the poset-elements $(k, \ell) = (1, 1)$, $(a, 1)$, $(1, b)$, and (a, b) are respectively the bottom, left, right, and top elements of the Hasse diagram.

Definition 7. *In this situation, we call the sets with constant j **ranks** (in accordance with standard poset terminology), sets with constant i **files**, sets with constant $j - i$ **positive fibers**, and sets with constant $j + i$ **negative fibers**. (The words "positive" and "negative" indicate the slopes of the lines on which the fibers lie in the Hasse diagram.) More specifically, the element $(k, \ell) \in [a] \times [b]$ belongs to* rank $k + \ell - 2$*, to* file $\ell - k$*, to* positive fiber k*, and to* negative fiber ℓ.

To each order ideal $I \in J([a] \times [b])$ we associate a lattice path of length $a + b$ joining the points $(-a, a)$ and (b, b) in the plane, where each step is of type $(i, j) \to (i + 1, j + 1)$ or of type $(i, j) \to (i + 1, j - 1)$, as follows. Given $1 \le k \le a$ and $1 \le \ell \le b$, represent $(k, \ell) \in [a] \times [b]$ by the square centered at $(\ell - k, \ell + k - 1)$ with vertices $(\ell - k, \ell + k - 2)$, $(\ell - k, \ell + k)$, $(\ell - k - 1, \ell + k - 1)$, and $(\ell - k + 1, \ell + k - 1)$. Then the squares representing the elements of the order ideal I form a "Russian-style" Young diagram whose upper border is a path joining some point on the line of slope -1 to some point on the line of slope $+1$. Adding extra edges of slope -1 at the left and extra edges of slope $+1$ at the right, we get a path joining $(-a, a)$ to (b, b). See Figure 5 for several examples of this correspondence.

Definition 8. *We can think of this path as the graph of a (real) piecewise-linear function $h_I : [-a, b] \to [0, a + b]$; we call this function (or its restriction to $[-a, b] \cap \mathbf{Z}$) the* **height function** *representation of the ideal I. To this height function we can in turn associate a word consisting of a -1's and b $+1$'s, whose ith term (for $1 \le i \le a + b$) is $h_I(i - a) - h_I(i - a - 1) = \pm 1$; we call this the* **sign-word** *associated with the order ideal I.*

Note that the sign-word simply lists the slopes of the segments making up the path, and that either the sign-word or the height-function encodes all the information required to determine the order ideal.

Proposition 9. *Let $I \in J([a] \times [b])$ correspond with height function $h_I : [-a, b] \to \mathbf{R}$. Then*

$$\sum_{k=-a}^{b} h_I(k) = \frac{a(a+1)}{2} + \frac{b(b+1)}{2} + 2\#I\,.$$

So to prove that the cardinality of I is homomesic, it suffices to prove that the function $h_I(-a) + h_I(-a+1) + \cdots + h_I(b)$ is homomesic (where our combinatorial dynamical system acts on height functions h via its action on order ideals I).

3.2 Promotion in products of two chains

In general a ranked poset P may not have an embedding in $\mathbf{Z} \times \mathbf{Z}$ that allows files to be defined; when they are, however, then all toggles corresponding to elements within the same file commute by Proposition 4, so their product is a well defined operation on $J(P)$. This allows one to define an operation on $J(P)$ by successively toggling all the files from left to right, in analogy to Corollary 6.

Theorem 10 (Striker-Williams [SW12, § 6.1])**.** *Let $x_1, x_2, \ldots, x_n$ be any enumeration of the elements (k, ℓ) of the poset $[a] \times [b]$ arranged in order of increasing $\ell - k$. Then the action on $J(P)$ given by $\partial := \sigma_{x_n} \circ \sigma_{x_{n-1}} \circ \cdots \circ \sigma_{x_1}$ viewed as acting on the paths (or binary strings representing them) is just a leftward cyclic shift.*

Striker and Williams call this well-defined composition ∂ **promotion** (since it is related to Schützenberger's notion of promotion on linear extensions of posets). They show that it is conjugate to rowmotion in the toggle group, obtaining a much simpler bijection to prove Panyushev's conjecture in Type A, and generalizing an equivariant bijection for $[a] \times [b]$ of Stanley [Sta09, remark after Thm 2.5]. This definition and their results apply more generally to the class they define of *rc-posets*, whose elements fit neatly into "rows" and "columns" (which we call here "ranks" and "files"). As with Φ, we can think of ∂ as operating either on $J(P)$ or $\mathcal{A}(P)$, adding subscripts ∂_J or ∂_A if necessary. Since the cyclic left-shift has period $a + b$, so does ∂.

Theorem 11. *The cardinality of order ideals is homomesic under the action of promotion ∂_J.*

Proof: To show that $\#I$ is homomesic, by Proposition 9 it suffices to show that $h_I(k)$ is homomesic for all $-a \le k \le b$. Note that here we are thinking of I as varying over $J(P)$, and h_I as being a *function*-valued function on $J(P)$.

We can write $h_I(k)$ as the telescoping sum $h_I(-a) + (h_I(-a+1) - h_I(-a)) + (h_I(-a+2) - h_I(-a+1)) + \cdots + (h_I(k) - h_I(k-1))$; to show that $h_I(k)$ is homomesic for all k, it will be enough to show that all the increments $h_I(k) - h_I(k-1)$ are homomesic. Note that these increments are precisely the terms of the sign-word of I. Create a square array with $a + b$ rows and $a + b$ columns, where the rows are the sign-words of I and its successive images under the action of ∂; each row is just the cyclic left-shift of the row before. Since each row contains a -1's and b $+1$'s, the same is true of each column. Thus, for all k, the average value of the kth terms of the sign-words of I, ∂I, $\partial^2 I$, ..., $\partial^{a+b-1} I$ is $(b-a)/(b+a)$. This show that the increments are homomesic, as required, which suffices to prove the theorem. □

Our proof actually shows the more refined result that the restricted cardinality functions $\#(I \cap S)$ where S is any file of $[a] \times [b]$ are homomesic with respect to the action of ∂_J.

The next example shows that the cardinality of the antichain A_I associated with the order ideal I is *not* homomesic under the action of promotion ∂.

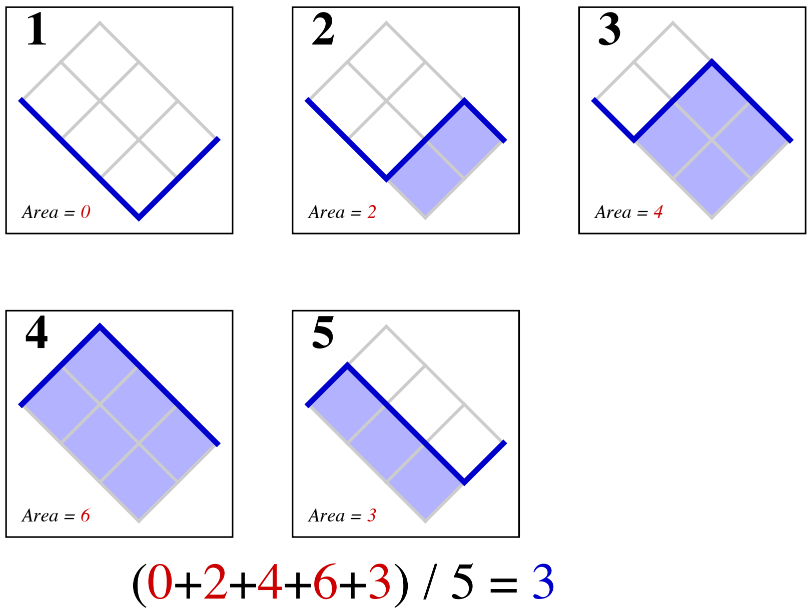

Fig. 3: One promotion orbit in $J([3] \times [2])$

Example 3. Consider the two promotion orbits of ∂_A shown in Figures 3 and 4. Although the statistic $\#I$ is homomesic, giving an average of 3 in both cases, the statistic $\#A$ averages to $\frac{1}{5}(0+1+1+1+1) = \frac{4}{5}$ in the first orbit and to $\frac{1}{5}(1+2+2+1+2) = \frac{8}{5}$ in the second.

3.3 Rowmotion in products of two chains

Unlike promotion, the rowmotion operator turns out to exhibit homomesy with respect to both the statistic that counts the size of an order ideal and the statistic that counts the size of an antichain.

3.3.1 Rowmotion on order ideals in $J([a] \times [b])$

We can describe rowmotion nicely in terms of the sign-word. We define **blocks** within the sign-word as occurrences of the subword $-1, +1$ (that is, a -1 followed immediately by a $+1$). Once we have found all the blocks, we identify all the **gaps** between the blocks, where a gap is bounded by two consecutive blocks, or between the beginning of the word and the first block, or between the last block and the end of the word. (In the case where there are no blocks at all, the entire word is considered a gap.) To apply rowmotion to a sign-word, reverse all the blocks and all the gaps. For example, consider the binary word $-1, +1, +1, -1, -1, -1, +1, +1$. To apply rowmotion to it, we first divide it into blocks and gaps as $-1, +1, |\ +1, -1, -1, |\ -1, +1, |\ +1$, and then reverse each block and gap in place, obtaining $+1, -1, |$ $-1, -1, +1, |\ +1, -1, |\ +1$, or (dropping the dividers) $+1, -1, -1, -1, +1, +1, -1, +1$. Note that the

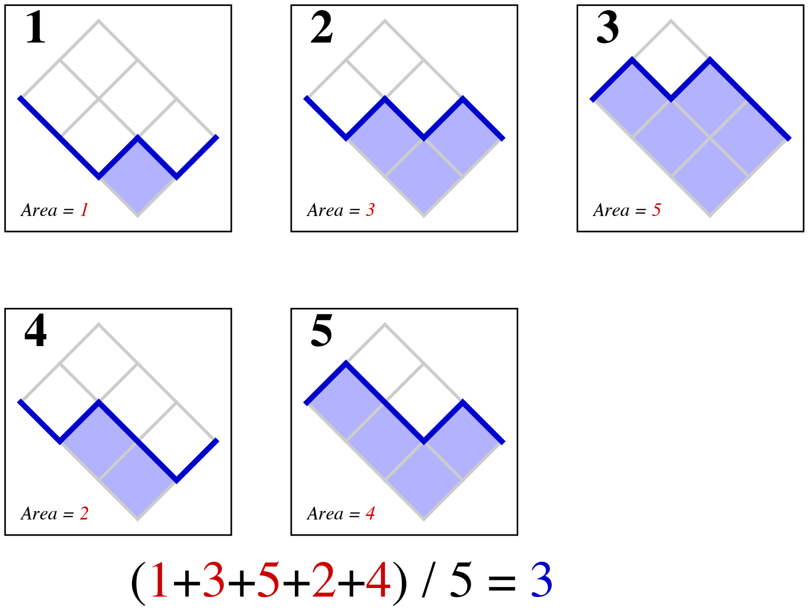

Fig. 4: The other promotion orbit in $J([3] \times [2])$

dividers correspond to the red dots in Figure 5, so one can visualize Φ_J as reversing (180° rotation of) each lattice-path segment that corresponds to a block or a gap in the sign-word. (See animations within talk slides at `http://www.math.uconn.edu/~troby/combErg2012kizugawa.pdf`.)

It turns out that all we really need to know for purposes of proving homomesy is that the sign-word for I has $-1, +1$ in a pair of adjacent positions if and only if the sign-word for ΦI has $+1, -1$ in the same two positions. This can be seen directly for $J([a] \times [b])$ from the description of Φ_J given at the start of Section 3. (See also Figure 5.) This situation occurs if and only if the antichain $A(\Phi(I))$ contains an element in the associated file of $[a] \times [b]$.

Theorem 12. *The cardinality of order ideals is homomesic under the action of rowmotion Φ_J.*

Proof: As in the previous section, to prove that $\#I$ is homomesic under rowmotion, it suffices to prove that all the increments $h_I(k) - h_I(k-1)$ are homomesic. A result of Fon-der-Flaass [Fon93, Theorem 2], states that the size of any Φ-orbit in $[a] \times [b]$ is a divisor of $a + b$, so this is equivalent to showing that for all k, the kth element of the sign-word of $\sum_{m=0}^{a+b-1} \Phi^m I$ under the action of rowmotion is independent of I.

Create a square array with $a + b$ rows and $a + b$ columns, where the rows are the sign-words of I and its successive images under the action of Φ. Consider any two consecutive columns of the array, and the width-2 subarray they form. There are just four possible combinations of values in a row of the subarray:

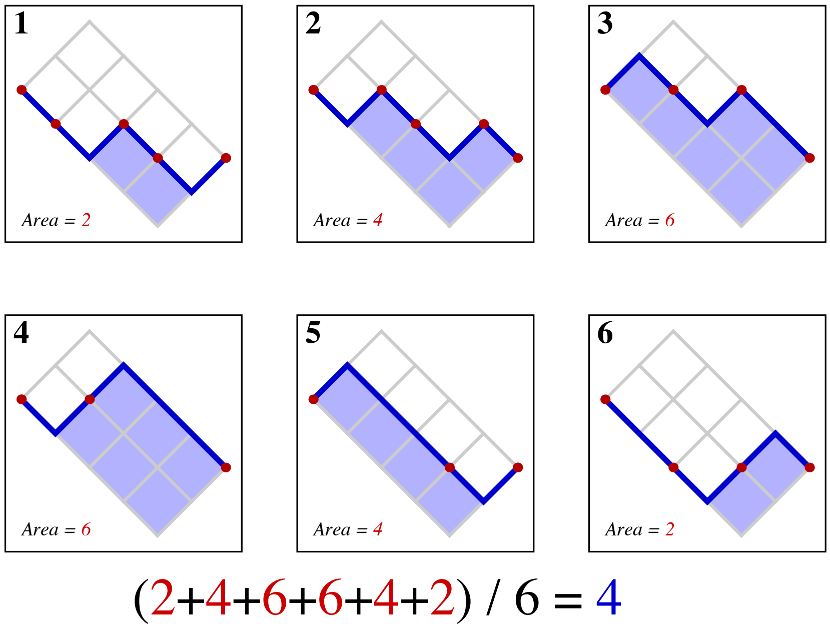

Fig. 5: A rowmotion orbit in $J([4] \times [2])$

$(+1,+1)$, $(+1,-1)$, $(-1,+1)$, and $(-1,-1)$. However, we have just remarked that a row is of type $(-1,+1)$ if and only if the next row is of type $(+1,-1)$ (where we consider the row after the bottom row to be the top row). Hence the number of rows of type $(-1,+1)$ equals the number of rows of type $(+1,-1)$. It follows that any two consecutive column-sums of the full array are equal, since other row types contribute the same value to each column sum. That is, within the original square array, every two consecutive columns have the same column-sum. Hence all columns have the same column-sum. This common value of the column-sum must be $1/(a+b)$ times the grand total of the values of the square array. But since each row contains a -1's and b $+1$'s, each row-sum is $b-a$, so the grand total is $(a+b)(b-a)$, and each column-sum is $b-a$. Since this is independent of which rowmotion orbit we are in, we have proved homomesy for elements of the sign-word of I as I varies over $J([a] \times [b])$, and this gives us the desired result about $\#I$, just as in the proof of Theorem 11. $\square$

3.3.2 Rowmotion on antichains in $\mathcal{A}([a] \times [b])$

In his survey article on promotion and evacuation, Stanley [Sta09, remark after Thm 2.5] gave a concrete equivariant bijection between rowmotion Φ_A acting on antichains in $\mathcal{A}([a] \times [b])$ and cyclic rotation of certain binary strings. Armstrong (private communication) gave a variant description that clarified the correspondence, which he learned from Thomas and which we use in what follows.

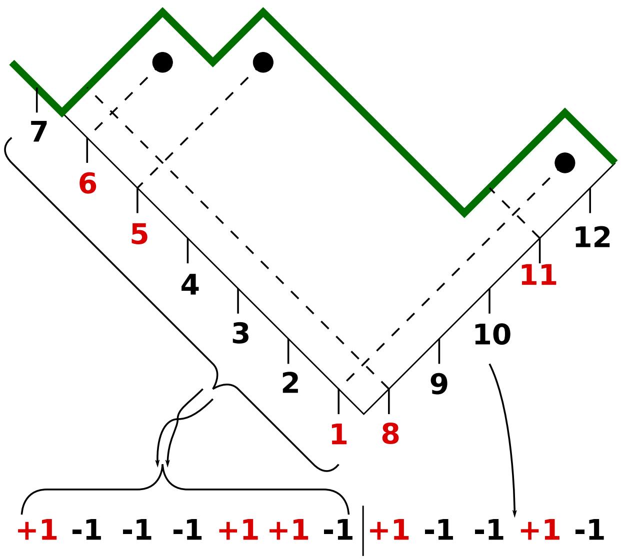

Fig. 6: The Stanley-Thomas word for a 3-element antichain in $\mathcal{A}([7] \times [5])$.

Definition 13. *Fix a, b, and $n = a + b$. Call sets in $[a] \times [b]$ of the form $\{(k, \ell) : \ell \in [b]\}$ (with k fixed)* ***rows*** *and sets of the form $\{(k, \ell) : k \in [a]\}$ (with ℓ fixed)* ***columns****. Define the* ***Stanley-Thomas word*** *$w(A)$ of an antichain A in $[a] \times [b]$ to be $w_1 w_2 \cdots w_{a+b} \in \{-1, +1\}^{a+b}$ with*

$$w_i := \begin{cases} +1, & \textit{if } A \textit{ has an element in } \text{row } i \ (i \in [a]) \textit{ or } A \textit{ has NO element in } \text{column } i \ (a+1 \leq i \leq n); \\ -1 & \textit{otherwise.} \end{cases}$$

Example 4. As illustrated in Figure 6, let $A = \{(1, 5), (5, 3), (6, 2)\}$. By definition, the Stanley-Thomas word $w(A)$ should have $+1$ in entries 1, 5, and 6 (rows where A appears) and in entries 8 and 11 (columns where A does not appear, with indices shifted by $7 = a$). Indeed one sees that $w(A) = +1, -1, -1, -1, +1, +1, -1, | \ +1, -1, -1, +1, -1$ (where the divider between a and $a + 1$ is just for ease of reading). Note that applying rowmotion gives $A' = \Phi(A) = \{(2, 4), (6, 3), (7, 1)\}$ with Stanley-Thomas word $w(A') = -1, +1, -1, -1, -1, +1, +1, | \ -1, +1, -1, -1, +1 = C_R w(A)$, the rightward cyclic shift of $w(A)$.

Proposition 14 (Stanley-Thomas)**.** *The correspondence $A \longleftrightarrow w_A$ is a bijection from $\mathcal{A}([a] \times [b])$ to binary words $w \in \{-1, +1\}^{a+b}$ with exactly a -1's and b $+1$'s. Furthermore, this bijection is equivariant with respect to the actions of rowmotion Φ_A and rightward cyclic shift C_R.*

Note that the classical result that Φ_A^{a+b} is the identity map follows immediately.

Theorem 15. *The cardinality of antichains is homomesic under the action of rowmotion Φ_A.*

Proof: It suffices to prove a more refined claim, namely, that if S is any row or column of $[a] \times [b]$, the cardinality of $A \cap S$ is homomesic under the action of rowmotion on A. By the previous result, rowmotion corresponds to cyclic shift of the Stanley-Thomas word, and the entries in the Stanley-Thomas word tell us which fibers (rows or columns) contain an element of A and which do not. Specifically, for $1 \leq k \leq a$, if S is the kth row, then A intersects S iff the kth symbol of the Stanley-Thomas word is a $+1$. Since the Stanley-Thomas word contains a -1's and b $+1$'s, the multiset orbit of A of size $a+b$ has exactly b elements that are antichains that intersect S. That is, the sum of $\#(A \cap S)$ over the multiset orbit of size $a+b$ is exactly b, for each of the a rows of $[a] \times [b]$. Summing over all the rows, we see that the sum of $\#A$ over the multiset orbit is ab. Hence $\#A$ is homomesic with average $ab/(a+b)$. $\square$

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The height of the Lyndon tree

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Abstract. We consider the set $\mathcal{L}_n$ of n-letters long Lyndon words on the alphabet $\mathcal{A} = \{0, 1\}$. For a random uniform element L_n of the set $\mathcal{L}_n$, the binary tree $\mathfrak{L}(L_n)$ obtained by successive standard factorization of L_n and of the factors produced by these factorization is the *Lyndon tree* of L_n. We prove that the height H_n of $\mathfrak{L}(L_n)$ satisfies

$$\lim_n \frac{H_n}{\ln n} = \Delta,$$

in which the constant Δ is solution of an equation involving large deviation rate functions related to the asymptotics of Eulerian numbers ($\Delta \simeq 5.092\ldots$). The convergence is the convergence in probability of random variables.

Résumé. Pour un mot L_n choisi au hasard uniformément dans l'ensemble des mots de Lyndon de longueur n sur l'alphabet $\{0, 1\}$, on montre que la hauteur H_n de l'arbre de Lyndon associé possède le comportement asymptotique suivant

$$\lim_n \frac{H_n}{\ln n} = \Delta.$$

La constante Δ est définie à l'aide de fonctions de taux liées au comportement asymptotique des nombres eulériens (incidemment, $\Delta \simeq 5.092\ldots$). La convergence est la convergence en probabilité des variables aléatoires.

Keywords: Lyndon word, Lyndon tree, branching random walk, Galton-Watson tree, Yule process, binary search tree

1 Introduction

1.1 Lyndon words and Lyndon trees

We recall some notations of Lothaire (1997) for readability. For an alphabet $\mathcal{A}$, $\mathcal{A}^n$ is the set of n-letters words, and the language, i.e. the set of finite words,

$$\{\emptyset\} \cup \mathcal{A} \cup \mathcal{A}^2 \cup \mathcal{A}^3 \cup \ldots,$$

is denoted by $\mathcal{A}^\star$. The length of a word $w \in \mathcal{A}^\star$ is denoted by $|w|$. A total order, $\prec$, on the alphabet $\mathcal{A}$, induces a corresponding lexicographic order, again denoted by $\prec$, on the language $\mathcal{A}^\star$: the word w_1 is smaller than the word w_2 (for the lexicographic order, $w_1 \prec w_2$) at one of the following conditions: either w_1 is a prefix of w_2, or there exist words p, v_1, v_2 in $\mathcal{A}^\star$ and letters $a_1 \prec a_2$ in $\mathcal{A}$, such that $w_1 = pa_1v_1$ and $w_2 = pa_2v_2$.

The notion of *Lyndon word* has many equivalent definitions, to be found, for instance, in Lothaire (1997). For any factorization $w = uv$ of w, vu is called a rotation of w, and the set $\langle w \rangle$ of rotations of w is called the *necklace* of w. A word w is primitive if $|w| = \#\langle w \rangle$.

Definition 1 *A word w is Lyndon if it is primitive and if it is the smallest element of its necklace.*

Example 1 *The word $w = \mathtt{aabaab}$ is the smallest in its necklace*

$$\langle w \rangle = \{\mathtt{aabaab}, \mathtt{abaaba}, \mathtt{baabaa}\}$$

but is not Lyndon; baac *is not Lyndon, nor* acba *or* cbaa*, but* aacb *is Lyndon.*

Here is a recursive characterization of Lyndon words:

Proposition 1 *One-letter words are Lyndon. A word w with length $n \geq 2$ is a Lyndon word if and only if there exists two Lyndon words u and v such that $w = uv$ and $u \prec v$.*

Among such decompositions of w, the decomposition with the longest second factor (or suffix) v is called the *standard* decomposition.

Example 2 $\mathtt{0011} = (\mathtt{001})(\mathtt{1}) = (\mathtt{0})(\mathtt{011})$ *is a Lyndon word with two such decompositions. The latter is the standard decomposition.*

The set of Lyndon words is denoted by $\mathcal{L}$, and we set $\mathcal{L}_n = \mathcal{L} \cap \mathcal{A}^n$. The Lyndon tree (or standard bracketing tree, cf. Barcelo (1990)) of the Lyndon word w is a binary tree obtained by iteration of the standard decomposition:

Definition 2 (Lyndon tree) *For $w \in \mathcal{L}$, the Lyndon tree $\mathfrak{L}(w)$ of w is a labelled finite binary tree defined as follows:*

- *if $|w| = 1$, $\mathfrak{L}(w)$ has a unique node labelled w, and no edges;*
- *if (u, v) is the standard decomposition of w, then $\mathfrak{L}(w)$ is the binary tree with label w at its root, $\mathfrak{L}(u)$ as its left subtree and $\mathfrak{L}(v)$ as its right subtree.*

Remark 1 *Note that the labels of the leaves of a Lyndon tree are letters. Also, the label of an internal node is the concatenation of the labels of its two children, and, if $|w| = n$, $\mathfrak{L}(w)$ is a binary tree with n leaves, and $n - 1$ internal nodes.*

1.2 Context

The asymptotic behavior of the size of the right and left subtrees of $\mathfrak{L}(L_n)$, for n large, have been studied in Bassino et al. (2005); Chassaing and Zohoorian Azad (2010), for L_n a random element of $\mathcal{L}_n$. The height $h(\mathfrak{L}(L_n))$ of $\mathfrak{L}(L_n)$ is of interest for the analysis of algorithms, cf. Sawada and Ruskey (2003), but it seems to have resisted analysis up to now.

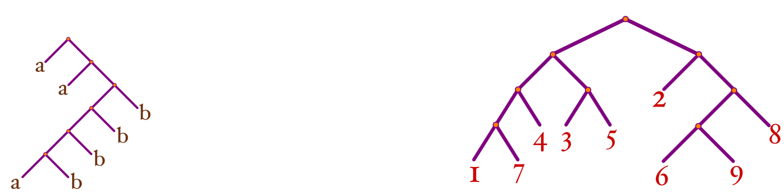

Fig. 1: a. $\mathcal{A} = \{\mathtt{a}, \mathtt{b}\}$ and $\mathfrak{L}(\mathtt{a}^3\mathtt{b}^4)$, b. $\mathcal{A} = \{\mathtt{1}, \mathtt{2}, \ldots, \mathtt{9}\}$ and $\mathfrak{L}(\mathtt{174352698})$.

1.3 Result

For a 2-letter alphabet, say $\mathcal{A} = \{0,1\}$, and for $n \geq 1$, let L_n denote a uniform random word in $\mathcal{L}_n$. Let $(A(n,k))_{n,k}$ denote the Eulerian numbers, i.e. $A(n,k)$ is the number of permutations σ of n symbols having exactly k descents (k places where $\sigma(i) \geq \sigma(i+1)$) . Set

$$\begin{aligned}
\Xi(\theta) &= \lim_n \frac{1}{n} \ln(A(n, \lfloor \theta n \rfloor)/n!), & (1)\\
\Psi(\lambda,\mu,\nu) &= \ln\left(\frac{(1+\mu)^{1+\mu}}{\mu^\mu}\,\frac{(e\lambda\ln 2)^\nu \ln 2}{\nu^\nu 2^\lambda}\right) + \Xi(\lambda-\mu), & (2)\\
\Delta &= \sup_{\lambda,\mu,\nu>0} \frac{1+\nu+\mu}{\lambda} + \frac{\Psi(\lambda,\mu,\nu)}{\lambda\ln 2} = 5.092\ldots & (3)
\end{aligned}$$

See Lemma 1 for an expression of Ξ (Giladi and Keller, 1994, p. 299). We shall prove that:

Theorem 1

$$\frac{h(\mathfrak{L}(L_n))}{\ln n} \xrightarrow{\mathbb{P}} \Delta. \tag{4}$$

The two factors of the standard decomposition of a uniform Lyndon word are not uniform Lyndon words, and they are not even independent, which seems to preclude a recursive approach to the proof of this Theorem. We shall rather use a coupling method: in Section 2 we sketch the main steps of the construction, on the same probability space, of a random Lyndon tree, and of two well-known trees, the binary search tree of a random uniform permutation, and a Yule tree, in such a way that the height of the Lyndon tree is closely related to some statistics of the two other trees. Then Theorem 1 follows from a large deviation result presented in Section 3.

2 Coupling results

2.1 Reduction to a Bernoulli source

The Lyndon tree of a non-Lyndon word u is defined as the Lyndon tree of the unique Lyndon word in the necklace of u, if u is primitive. If u is periodic, we define the Lyndon word of u as the word $0^{|u|-1}1$, and the Lyndon tree of u is defined accordingly. Then the following algorithm:

- let W_∞ be a infinite word of uniformly random characters, obtained through the binary expansion of a number U uniformly distributed on $[0,1]$;

- let W_n be the word W_∞ truncated after n letters, and let L_n be the Lyndon word of W_n.

produces a n-letter long random Lyndon word L_n, that fails to be uniform due to the small probability that W_n is periodic. However, the total variation distance between the probability distribution of L_n and the uniform distribution on $\mathcal{L}_n$ is $\mathcal{O}\left(2^{-n/2}\right)$ (cf. e.g. (Chassaing and Zohoorian Azad, 2010, Lemma 2.1)), thus any property that holds true asymptotically almost surely with respect to either distribution, holds true a.a.s. for both. We shall, from now on, consider that L_n is produced by the previous algorithm.

2.2 Exponential transform

In the first steps of the recursive construction of $\mathfrak{L}(L_n)$, the sizes of the factors of the successive standard decompositions are predicted by the positions of the longest runs of 0's, and the structure of the top levels of $\mathfrak{L}(L_n)$ is given by the lexicographic comparisons between the suffixes of L_n beginning at these longest runs. But when n is large, the number of runs of 0's is typically $n/4$, and several among these runs are tied for the title of the longest run. Actually the lengths of the runs behave pretty much as a sample of $n/4$ i.i.d. geometric random variables with parameter $1/2$, and, according to Brands et al. (1994), for any strictly increasing sequence n_k such that $\lim_k \log_2 n_k - \lfloor \log_2 n_k \rfloor = \alpha \in [0, 1)$, the probability p_{ℓ,n_k} that $m \geq 1$ among the n_k elements of such a sample are tied for the maximum is given, approximately, by

$$p_{m,n_k} \simeq \sum_{j \in \mathbb{Z}} e^{-2^{\alpha+j}} \frac{(2^{\alpha+j-1})^m}{m!}. \tag{5}$$

Thus the number of ties does not converge in distribution, but has a set of limit distributions indexed by $\alpha \in \mathbb{R}/\mathbb{Z}$. Such a complex behavior does not bode well, so we shall rather analyze a transform of this problem, in the form of the Lyndon tree of a word with random length. Consider the finite word W^ℓ formed by a letter 1 followed by the truncation of W_∞ at the position τ_ℓ of the ℓth 0 in the first run of ℓ consecutive 0's of W_∞. Then W^ℓ is primitive, and L^ℓ denotes the Lyndon word of W^ℓ, *i.e.*:

$$W^\ell = 1\,\underbrace{010110\ldots1\overbrace{000000}^{\ell 0s}}_{\text{prefix of } W_\infty} \quad \text{and} \quad L^\ell = \overbrace{000000}^{\ell 0s}1\,\underbrace{010110\ldots1}_{\text{prefix of } W_\infty}.$$

If σ_ℓ is the position of the last 1 before τ_ℓ, L^ℓ is the concatenation of $0^\ell 1$ and of the truncation of the word W_∞ at position σ_ℓ.

Now, there exists a unique longest run of 0's in W^ℓ as well as in L^ℓ, and this run is ℓ letters long, to be compared with the behavior revealed by (5). Moreover, if Z_k denotes the number of runs longer than $\ell - k - 1$, then $Z_0 = 1$ and $(Z_j)_{0 \leq j \leq \ell-1}$ is a Galton-Watson process with offspring distribution $2^{-k}\mathbb{1}_{k\geq 1}$, so that Z_j has a geometric distribution with parameter 2^{-j}, see for instance Devroye (1992). The family tree of this Galton-Watson process gives a lot of information on $\mathfrak{L}(L^\ell)$, leading to the proof of Theorem 2 below. Note that the typical length of L^ℓ grows exponentially fast with ℓ:

$$\mathbb{E}[|L^\ell|] = 2^{\ell+1} - 1. \tag{6}$$

Thus, expectedly, the typical height of the Lyndon tree of L^ℓ grows linearly with ℓ:

Theorem 2 $\frac{h(\mathfrak{L}(L^\ell))}{\ell} \xrightarrow[\ell\to\infty]{\mathbb{P}} \Delta \ln 2.$

Due to space requirements, let us mention briefly how we obtain Theorem 1 using Theorem 2: we choose $\ell(n) = \log_2 n - \varepsilon_n$ in such a way that $\mathbb{P}\left(\tau_{\ell(n)} \geq n\right)$ is small, so that, with a large probability, $W^{\ell(n)}$ is a factor of W_n. Then we compare carefully $\mathfrak{L}(W^{\ell(n)})$ and $\mathfrak{L}(W_n)$. In the rest of the paper, we give more details about the proof of Theorem 2.

2.3 Reduction to a skeleton

In the top levels of the tree $\mathfrak{L}(L^\ell)$, the successive standard decompositions of the Lyndon word L^ℓ, at the smallest suffixes of L^ℓ, split the word L^ℓ at the longest runs of 0's. For ℓ large enough, the longest runs are sparse enough to preserve some degree of independence between the factors. This is not true anymore at the lowest levels of the tree $\mathfrak{L}(L^\ell)$. For this reason, it is easier to split the study of the Lyndon tree in two parts: the first one focuses on the top of the tree, where the runs of $\mathtt{0}$'s are still above a threshold a_ℓ, and the second part studies a forest of shrubs at the bottom of the tree, each of them labeled with a factor of L^ℓ that contains only runs of $\mathtt{0}$'s shorter than the treshold. The top part is a tree itself (a subtree of $\mathfrak{L}(L^\ell)$), and each shrub of the forest at the bottom of $\mathfrak{L}(L^\ell)$ is rooted at (or grafted on) a leaf of the top tree. We follow here the same path as Broutin and Devroye (2008), our *shrubs* playing the same rôle as their *spaghetti-like subtrees*. Let us define by induction the tree above the threshold k, with $k \geq 1$:

Definition 3 (Top tree) *If w denotes a Lyndon word, $\mathfrak{L}^k(w)$ is a finite labelled binary tree defined by:*

- *if w has one factor $\mathtt{0}^k$ or less (thus $\mathtt{0}^{k+1}$ is not a factor of w), $\mathfrak{L}^k(w)$ is a single node, labelled w;*
- *otherwise, let (u, v) be the standard decomposition of w. Then the root of $\mathfrak{L}^k(w)$ has label w, the left subtree of $\mathfrak{L}^k(w)$ is $\mathfrak{L}^k(u)$ and the right subtree is $\mathfrak{L}^k(v)$.*

$\mathfrak{L}^k(w)$ is called the top tree associated to w, with threshold k.

Since we focus on $\mathcal{L}(L^\ell)$, the threshold a_ℓ depends on ℓ. It has to be large enough for the top tree to retain the independence properties between the factors, but small enough that we can handle the shrubs, though they lack these nice independence properties. We set

$$a_\ell = \lfloor \log_2 \ell \rfloor = \Theta\left(\log\log \mathbb{E}\left[|L^\ell|\right]\right) \qquad \text{and} \qquad \mathfrak{T}_\ell = \mathfrak{L}^{a_\ell}(L^\ell). \tag{7}$$

2.4 A binary search tree

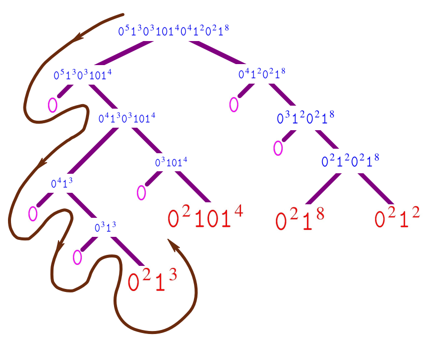

Fig. 2: For $L^5 = \mathtt{0}^5\mathtt{1}^3\mathtt{0}^3\mathtt{1}\mathtt{0}\mathtt{1}^4\mathtt{0}^4\mathtt{1}^8\mathtt{0}^2\mathtt{1}^2$, the tree $\mathfrak{T}_5 = \mathfrak{L}^2(L^5)$ has 6 needles and 4 blades. In brown, its contour traversal.

Let us take a closer look at $\mathfrak{T}_\ell$: observe that if $w \in \mathcal{L}$, then $\mathtt{0}w \in \mathcal{L}$, and the two factors of the standard factorization of $\mathtt{0}w$ are $\mathtt{0}$ and w. Thus either a leaf v of $\mathfrak{T}_\ell$ has label $\mathtt{0}$, and v is called a *needle*, or the label of v is a factor of w beginning with $\mathtt{0}^{a_\ell}\mathtt{1}$, and v is called a *blade*. The number N_ℓ of blades of $\mathfrak{T}_\ell$ has a geometric distribution with parameter $2^{-\ell+a_\ell}$, and the set of blades has a natural order related to the contour traversal (see Figure 2), that allows to identify it to $[\![1, N_\ell]\!]$. The number of needles has a simple expression in terms of a Galton-Watson process with geometric offspring distribution. In the analysis of the shape of $\mathfrak{T}_\ell$, the configuration of the needles is a special concern.

We shall need some notations: in the contour traversal of $\mathfrak{T}_\ell$, there is a sequence of $n_v - a_\ell \geq 0$ needles between a blade v and the previous blade (or between v and the root, if there exists no previous blade). The concatenation, starting at this sequence of needles, included, of the labels of the leaves in the order of the contour traversal of $\mathfrak{T}_\ell$, is a suffix of L^ℓ that can be written $\mathtt{0}^{n_v}\mathtt{1}\, t_v$, the run $\mathtt{0}^{n_v}$ being maximal in the sense that $\mathtt{0}^{n_v+1}\mathtt{1}\, t_v$ is not a suffix of L^ℓ.

The words of the sequence $(t_v)_{1\leq v\leq N_\ell}$ have different lengths, being proper suffixes of each other, so they are all different, and we can give a reformulation of the algorithm that produces $\mathfrak{T}_\ell$, or more generally $\mathfrak{L}^k(w)$, in terms of the family $T_\ell = ((n_v, t_v))_{1\leq v\leq N_\ell}$ of the blades (with labels $\mathtt{0}^k\mathtt{1}\, t_v$), in which only the n_v's and the relative order of the t_v's matter. With this reformulation of the algorithm, a slight perturbation of the t_v's produces a new tree, $\mathfrak{S}_\ell$, that is easier to handle than $\mathfrak{T}_\ell$ due to its property of independence of labels, but that has essentially the same profile as $\mathfrak{T}_\ell$ (i.e. it has the same repartition of blades with respect to the height). Let $\epsilon^{(j)}$ denote the sequence of integers defined, for $j \in I$, by

$$\epsilon_i^{(j)} = \delta_{i,j}.$$

For $\mathcal{R}$ a totally ordered set, $\mathbb{N}_0 \times \mathcal{R}$ inherits a lexicographic order, $\prec$, from $\mathcal{R}$: $(n,t) \prec (m,u)$ if $n > m$ or if $n = m$ and $t < u$. Let $B = (l_i, r_i)_{1\leq i\leq N}$ (resp. $L = (l_i)_{1\leq i\leq N}$, $R = (r_i)_{1\leq i\leq N}$) be a finite sequence of elements of $\mathbb{N}_0 \times \mathcal{R}$ (resp. $\mathbb{N}_0$, $\mathcal{R}$), with no repetitions in the sequence R. Assume that $l_j \geq k$ for each j, and that (l_1, r_1) is the smallest element of B, for $\prec$.

Definition 4 *The Lyndon tree $\mathfrak{L}^k(B)$ is defined by induction by:*

1. *If $N = 1$ and $l_1 = k$, $\mathfrak{L}^k(B)$ has no edge and its unique vertex, with label (k, r_1), is a blade.*
2. *Otherwise, consider the new sequence B' formed from $L - \epsilon^{(1)}$ and R and let i_0 denote the index of the smallest element in B', for $\prec$.*
   (a) *If $i_0 = 1$, then $\mathfrak{L}^k(B)$ is the binary tree with a needle (labelled $\mathtt{0}$) as its left child and $\mathfrak{L}^k(B')$ as its right child.*
   (b) *If $i_0 \geq 2$, then the left subtree of the binary tree $\mathfrak{L}^k(B)$ is $\mathfrak{L}^k((l_i, r_i)_{1\leq i\leq i_0-1})$ while the right subtree of $\mathfrak{L}^k(B)$ is $\mathfrak{L}^k((l_{i+i_0}, r_{i+i_0})_{0\leq i\leq N-i_0})$.*

Remark 2 *Since $\sum(l_i+1)$ is strictly decreasing at each step, and the l_i's are not allowed to drop under k, $\mathfrak{L}^k(B)$ is well-defined as long as the r_i's are distinct. The N blades of $\mathfrak{L}^k(B)$ are labelled $(k, r_i)_{1\leq i\leq N}$, and, during the contour traversal, they appear in this order.*

Remark 3 *For $\mathcal{R} = \{t_v \mid 1 \leq v \leq N_\ell\}$ and $T_\ell = ((n_v, t_v))_{1\leq v\leq N_\ell}$ defined in this section,*

$$\mathfrak{L}^{a_\ell}(T_\ell) = \mathfrak{T}_\ell,$$

or more precisely, the shapes are the same, but the labels are different. When the label of some node of $\mathfrak{L}^{a_\ell}(T_\ell)$ is (k, t_v), the corresponding label of $\mathfrak{T}_\ell$ is the prefix of $\mathtt{0}^k\mathtt{1}t_v$ that stops before the beginning of the next occurrence of $\mathtt{0}^k$.

For the analysis of $\mathfrak{T}_\ell$, the fact that the t_v's are suffixes of t_1, precluding any form of independence, is bothering. In order to fix the problem, in T_ℓ, we replace the sequence $(t_v)_{1\leq v\leq N_\ell}$ with a new sequence $(s_v)_{1\leq v\leq N_\ell}$ of infinite binary words, close to the t_v's but independent, defined as follows: let $(\zeta_i)_{i\in\mathbb{N}}$ be an i.i.d. sequence of uniform infinite words, independent of T_ℓ and let s_v be the concatenation of p_v, the prefix formed by the first a_ℓ letters of t_v, with ζ_v. When t_{N_ℓ} is shorter than a_ℓ letters, it is completed with the appropriate number of $\mathtt{0}$'s, before the concatenation with ζ_{N_ℓ}. This way, we obtain a new sequence $S_\ell = ((n_v, s_v))_{1\leq v\leq N_\ell}$, and we set

$$\mathfrak{S}_\ell \;=\; \mathfrak{L}^{a_\ell}\left(S_\ell\right).$$

Differences between $\mathfrak{T}_\ell$ and $\mathfrak{S}_\ell$ occurs scarcely, only when at least a_ℓ letters are used to distinguish two suffixes, so that $\mathfrak{T}_\ell$ and $\mathfrak{S}_\ell$ are close in some sense, see Proposition 3. The probability distribution of S_ℓ is given by:

Proposition 2 *The sequence of words $\tilde{S}_\ell = (\mathtt{0}^{n_v-a_\ell}\mathtt{1}s_v)_{2\leq v\leq N_\ell}$, followed by the word $\mathtt{0}^{n_1-a_\ell}\mathtt{1}s_1$, is distributed as a sequence of uniform infinite random words observed until the first occurrence of the prefix $\mathtt{0}^{\ell-a_\ell}$, this first occurrence $\mathtt{0}^{n_1-a_\ell}\mathtt{1}s_1$ being eventually truncated of any $\mathtt{0}$ in excess of $\mathtt{0}^{\ell-a_\ell}\mathtt{1}\ldots$, so that $n_1 = \ell$.*

Remark 4 *By properties of repeated coin flipping, equivalently, $n_1 = \ell$ and for $v \geq 2$, $n_v - a_\ell$ is a geometric random variable with parameter $\frac{1}{2}$, conditioned to be smaller than $\ell - a_\ell$, s_v is an infinite uniform random word, for all v n_v and s_v are independent, N_ℓ is geometric with parameter $2^{a_\ell-\ell}$, and the sequence $(n_v, s_v)_{v\in\mathbb{N}}$ is independent of N_ℓ.*

Remark 5 *When ℓ grows, with a large probability the prefix p_v is short compared to t_v and the corresponding, much longer suffix of t_v determines, with p_v, the shape of the corresponding shrub at the bottom of $\mathfrak{L}_\ell$. These suffixes are independent of S_ℓ, and for that matter, independent of $\mathfrak{S}_\ell$, which happens to be crucial in the study of the bottom of the tree.*

For a blade $v \in [\![1, N_\ell]\!]$, let h_v (resp. $\tilde{h}_v$) be its height in $\mathfrak{S}_\ell$ (resp. in $\mathfrak{T}_\ell$). Set

$$d_v = \left|h_v - \tilde{h}_v\right|, \qquad D_\ell = \max_{1\leq v\leq N_\ell} d_v.$$

Then

Proposition 3

$$\lim_\ell \mathbb{P}\left(D_\ell \;\geq\; \frac{\ell}{\sqrt{\ln \ell}}\right) \;=\; 0.$$

Due to space requirements, we omit the proof, that uses branching random walks arguments, as in Biggins (1977). We shall now be interested in the height of leaves of $\mathfrak{S}_\ell$. Due to Proposition 2, conditionally given N_ℓ, the ranks of the terms of the sequence $\tilde{S}_\ell = (\mathtt{0}^{n_v-a_\ell}\mathtt{1}s_v)_{2\leq v\leq N_\ell}$ with respect to the lexicographic order form a uniform permutation on $N_\ell - 1$ symbols. As a consequence, the subtree $\mathfrak{A}_\ell$ of $\mathfrak{S}_\ell$ induced by the root and the blades, once the needles and the first blade erased, forms a *binary search tree*. With the help of a coupling of $\mathfrak{A}_\ell$ with a Yule process, Chauvin et al. (2005) provides a very precise analysis of the depths of the leaves of a binary search tree. Though, the depths of blades in $\mathfrak{S}_\ell$, though they depend on their depths in $\mathfrak{A}_\ell$, are also affected by the positions of the needles, and we need to tweak the arguments of Chauvin et al. (2005) in order to include the needles in their analysis.

2.5 A Yule process

A *Yule process* $\mathcal{Y}$ (Athreya and Ney, 1972, page 109) models a population in which each individual lives forever, and gives birth to new individuals at the times of a Poisson process with rate λ. We assume that the population starts at time 0 with a single individual, called the *ancestor*. One can keep track of the genealogy of the population through the *Yule tree* $\mathfrak{Y}$ (cf. Chauvin et al. (2005)), a family tree of the population: for each individual, a vertical life line is drawn downward, on the right of the life line of its father, and starting at an ordinate given by *minus* the date of birth of this individual ; this vertical life line is connected to the life line of its father by a dotted horizontal line. Let $\mathfrak{Y}_t$ denote the family tree $\mathfrak{Y}$ truncated at time t, i.e. at ordinate $-t$.

The Yule tree has yet another construction, starting from a sample $Y = (Y_n)_{n\geq 1}$ of i.i.d. exponential random variables with rate λ: consider the sequence $Z^{(t)}$ defined by

$$T_t = \inf\{n \geq 1 \,|\, Y_n \geq t\}, \quad Z^{(t)} = (Z_k^{(t)})_{0\leq k\leq T_t-1} = (t, Y_1, Y_2, \ldots, Y_{T_t-1})\,; \tag{8}$$

now, picture each term $Z_k^{(t)}$ of the sequence $Z^{(t)}$ by a vertical line of the corresponding length $Z_k^{(t)}$, drawn at abscissa $T_t - k$, and connect with an horizontal line the top of the kth line, but the first, to the next line on its left whose height exceeds $Z_k^{(t)}$, to obtain a family tree $\mathfrak{E}_t$. Then $\mathfrak{E}_t$ and $\mathfrak{Y}_t$ have the same probability distribution, by the lack of memory of the exponential distribution, and, from now on, we shall retain the notation $\mathfrak{Y}_t$ for both of them.

Let us denote by U_v the real number whose dyadic development is $\mathtt{0}^{n_v-a_\ell}\mathtt{1}s_v$. Then, as a consequence of Proposition 2, $(U_2, U_3, \ldots, U_{N_\ell}, U_1)$ is distributed as a sequence of uniform random variables observed until the first term smaller than $2^{-\ell+a_\ell}$ occurs, this last term U_1 being eventually multiplied by a power of 2, so as to belong to $[2^{-\ell+a_\ell-1}, 2^{-\ell+a_\ell}[$. Equivalently, the sequence $(X_2, X_3, \ldots, X_{N_\ell}, X_1)$ defined by

$$X_i \;=\; -\log_2 U_i$$

is distributed as a sequence of exponential random variables observed until the first term larger than $\ell - a_\ell$ occurs, this last term X_1 being eventually shifted by an integer, so as to belong to $[\ell - a_\ell, \ell - a_\ell + 1[$. Then the construction of Figure 3, based on the sequence $(\ell - a_\ell, X_2, X_3, \ldots, X_{N_\ell})$, gives a Yule family tree $\mathfrak{Y}_{\ell-a_\ell}$ with lifetime $\ell - a_\ell$, and with intensity $\ln 2$. Now, as observed for instance in Chauvin et al. (2005), $\mathfrak{Y}_{\ell-a_\ell}$ seen as a planar tree, with no edge length, or with edge length 1, is the binary search tree $\mathfrak{A}_\ell$: their planar tree structure depends only on the relative order of the words $\mathtt{0}^{n_v-a_\ell}\mathtt{1}s_v$ for $\mathfrak{A}_\ell$, and of the real numbers X_v, through the same algorithm, and the mapping $\mathtt{0}^{n_v-a_\ell}\mathtt{1}s_v \longrightarrow X_v$ is monotone. Thus the depth of the blade v in the binary search tree $\mathfrak{A}_\ell$ induced by $\mathfrak{S}_\ell$ is the depth of the leaf v corresponding to

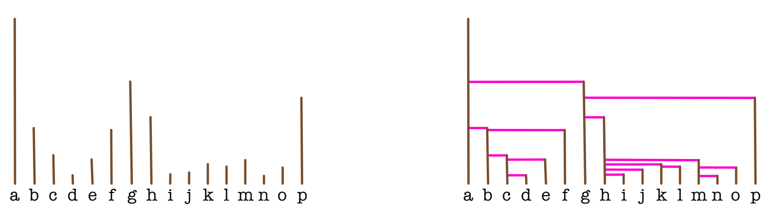

Fig. 3: A sample of exponential random variables, until hitting t, and the related tree $\mathfrak{E}_t$, distributed as $\mathfrak{Y}_t$ (in which the pink horizontal lines are to be seen as vertices rather than as edges).

X_v in $\mathfrak{Y}_{\ell-a_\ell}$. The difference between the depth of the blade v in $\mathfrak{A}_\ell$ and its depth in $\mathfrak{S}_\ell$ is the number of needles on the path to the root, whose expression is given by Proposition 4 below.

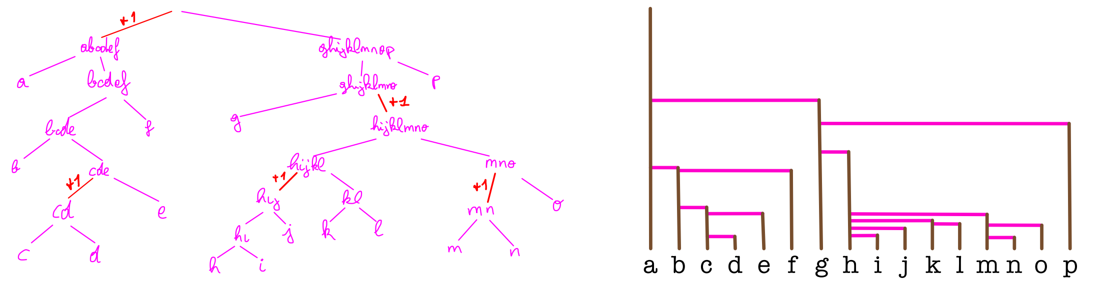

Fig. 4: Trees $\mathfrak{S}_\ell$ and $\mathfrak{Y}_\ell$. The sign $+1$ marks the presence of a needle.

For a given blade v of $\mathfrak{S}_\ell$, consider the marked point process Π_v formed by the vertices of $\mathfrak{Y}_{\ell-a_\ell}$ on the path from v to the root, the leaf v and the root excluded. This path is naturally identified with $[0, \ell - a_\ell]$, the leaf being at 0 and the root at $\ell - a_\ell$, for instance, and the mark being 0 or 1, respectively, according to the side of the branching, say left or right, leading to a decomposition

$$\Pi_v \;=\; \Pi_v^{(0)} \cup \Pi_v^{(1)}.$$

By convention, the mark for both points 0 and $\ell - a_\ell$ is one, and unless mentioned otherwise, they are not included in the point processes. For a point process $\pi = \{\xi_1 < \xi_2 < \ldots < \xi_{k-1} < \xi_k\}$ in some interval $[0, m]$, $G(\pi)$ denotes

$$G(\pi) \;=\; \sum_{s=1}^{k+1} \lfloor \xi_r - \xi_{r-1} \rfloor, \tag{9}$$

in which $\xi_{k+1} = m$ and $\xi_0 = 0$. We have

Proposition 4 *For $1 \le v \le N_\ell$,*

$$0 \le h_v - (\#\Pi_v + 1 + G(\Pi_v^{(1)})) \le 1.$$

Here $\#\Pi_v + 1$ accounts for the depth of v in $\mathfrak{A}_\ell$, $\lfloor \xi_1 \rfloor$ is $n_v - a_\ell$, and one can check that $\lfloor \xi_r - \xi_{r-1} \rfloor = k$ if the corresponding labels satisfy $(n_{w_r} - k, s_{w_r}) \prec (n_{w_{r-1}} - k, s_{w_{r-1}})$ but $(n_{w_r} - k - 1, s_{w_r}) \succ (n_{w_{r-1}} - k, s_{w_{r-1}})$, in which case the corresponding edge in $\mathfrak{A}_\ell$ is obtained by erasing k needles of $\mathfrak{S}_\ell$. The special rôle of $\Pi_v^{(1)}$ in Proposition 4 reflects the asymmetry of the Lyndon tree. Since $G(\Pi_v^{(1)})$ tends to be large when $\#\Pi_v^{(1)}$ is small, and small when $\#\Pi_v^{(1)}$ is large, the difference between the height and the saturation level should be smaller for the Lyndon tree than for the binary search tree.

3 Large deviations and final result

3.1 The many-to-one formula

A leaf v of $\mathfrak{Y}_\ell$ is said to be *of type* (m, n, A) if its left (resp. right) depth in $\mathfrak{Y}_\ell$ is m (resp. n) and if $\Pi_v^{(1)}$ belongs to A. Let $\pi_{\ell,m,n,A}$ denote the average number of leaves of type (m, n, A) in $\mathfrak{Y}_\ell$ and let $\mathbb{U}_{n,\ell}$ denote the uniform probability distribution on the simplex $\{0 < \xi_1 < \xi_2 < \ldots < \xi_n < \ell\}$. Then

Proposition 5

$$\pi_{\ell,m,n,A} \;=\; (\ell \ln 2)^{m+n} 2^{-\ell}\, \mathbb{U}_{n,\ell}(A)/m!n!. \tag{10}$$

Up to a factor 2^ℓ, the right hand of (10) is the probability that two independent Poisson processes with intensity $\ln 2$ on $[0,\ell]$, $\Pi^{(0)}$ (resp. $\Pi^{(1)}$), have m (resp. n) points, and that $\Pi^{(1)}$ belongs to A. This is an instance of the many-to-one formula for branching random walks.

When A_k (resp. A_I) is the set of point processes Π on $[0,\ell]$ such that $G(\Pi) = k$ (resp. such that $G(\Pi) \in I$) , we set

$$\pi_{\ell,m,n,A_k} \;=\; \pi_{\ell,m,n,k}, \qquad \text{resp. } \pi_{\ell,m,n,I}.$$

3.2 Final argument and expression of Δ

For the final argument, it will be convenient to think of $\Psi(\lambda,\mu,\nu)$ as the following limit:

$$\Psi(\lambda,\mu,\nu) \;=\; \lim_n n^{-1} \,\ln\left(\pi_{\lambda_n n,\nu_n n,n,\mu_n n}\right), \tag{11}$$

when the sequence $(\lambda_n,\nu_n,\mu_n) = (\ell/n, m/n, k/n)$ converges to (λ,μ,ν), though we do not really need the existence of this limit for the proof of Theorem 2. We follow Broutin and Devroye (2008): roughly, the average number of blades of type $(\nu,1,\mu)\ell/\lambda$ in $\mathfrak{Y}_\ell$ or in $\mathfrak{S}_\ell$ is approximately $e^{\ell\Psi(\lambda,\mu,\nu)/\lambda}$, and their depth is approximately $\frac{1+\nu+\mu}{\lambda}\,\ell$. On each of these blades we graft shrubs that are Lyndon trees for Lyndon words that contain approximately 2^{a_ℓ} runs of `0`'s, all shorter than a_ℓ, and the same number of runs of `1`'s, whose lengths are independent geometric random variables with parameter $1/2$. We prove that the maximum height of a set of k such shrubs behaves like the maximum of a sample of k independent geometric random variables with parameter $1/2$, i.e. the maximum is essentially $\log_2 k$. As a consequence, the total height of the highest leaf in any shrub that is grafted on a blade of type $(\nu,1,\mu)\ell/\lambda$ in $\mathfrak{S}_\ell$ is approximately

$$\frac{1+\nu+\mu}{\lambda}\,\ell \;+\; \log_2\left(e^{\ell\Psi(\lambda,\mu,\nu)/\lambda}\right) \;=\; \frac{1+\nu+\mu}{\lambda}\,\ell \;+\; \ell\,\frac{\Psi(\lambda,\mu,\nu)}{\lambda\ln 2}.$$

Thus the highest leaf in such a tree is $\Delta\ell$ high, in which

$$\Delta \;=\; \sup_{\lambda,\mu,\nu>0} \left((1+\nu+\mu)\ln 2 \;+\; \Psi(\lambda,\mu,\nu)\right)/\lambda\ln 2. \tag{12}$$

Due to Proposition 3, the same is true when the shrubs are grafted on the blades of $\mathfrak{T}_\ell$, producing $\mathfrak{L}(L^\ell)$.

3.3 Large deviations for G under $\mathbb{U}_{n,\ell}$

Let us compute Ψ. For a sequence b of positive numbers, set

$$|b| \;=\; \sum_{j\in I} b_j, \quad \langle b\rangle \;=\; \sum_{j\in I} j b_j, \quad \text{and} \quad \mathcal{H}(b) \;=\; -\sum_{j\in I} b_j \ln(b_j),$$

and if the entries are integers, set:

$$b! \;=\; \prod b_i! \quad \text{and} \quad \binom{|b|}{b} \;=\; \frac{|b|\,!}{\prod b_i!},$$

whenever they are defined. If $|b| = 1$, $\mathcal{H}(b)$ is the Shannon entropy of b. Under $\mathbb{U}_{n,\ell}$, rather than the ξ_j's, we shall consider the vector $\gamma = (\gamma_j)_{1\leq j\leq n+1}$ of gaps between the order statistics, defined by

$$\gamma_j = \xi_j - \xi_{j-1},\ 1 \leq j \leq n+1,$$

with the usual convention, $\xi_0 = 0$ and $\xi_{n+1} = \ell$. The random vector γ is uniformly distributed on

$$\mathcal{D}_{n,\ell} \ = \ \{\sum_{j=1}^{n+1} x_j = \ell \quad \text{and} \quad \forall j,\ x_j \geq 0\},$$

and its distribution is denoted $\mathbb{U}_{n,\ell}$ again. Also, set $\rho = (\rho_j)_{1\leq j\leq n+1}$, in which $\rho_j = \lfloor \gamma_j \rfloor$. Then $G = |\rho|$. The probability $\mathbb{U}_{n,\ell}(\rho = r)$ depends only on $|r|$ and is given by:

$$\mathbb{U}_{n,\ell}\,(\rho = r) = n!\,\ell^{-n}\mathbb{P}(\ell - |r| - 1 < \sum_{j=1}^{n} U_j < \ell - |r|), \tag{13}$$

in which the U_i's are a sequence of i.i.d. uniform random variables on $[0, 1]$, $\mathbb{P}\,(\ldots)$ is the volume of the domain $\{\rho = r\}$, and $\ell^n/n!$ is the volume of $\mathcal{D}_{n,\ell}$. Consider the sequence $\beta = (\beta_j)_{j\geq 0}$ defined by

$$\beta_j = \#\,\{1 \leq i \leq n+1 \,|\, \rho_i = j\}\,,\ \text{satisfying}\ |\beta| = n+1,\ \text{and} \quad \langle\beta\rangle = |\rho| = G(\xi),$$

in order to take advantage of the symmetric rôle of the ρ_j's. Then

$$\mathbb{U}_{n,\ell}\,(\beta = b) \ = \ \frac{n!}{\ell^n}\binom{|b|}{b}\,\mathbb{P}(\ell - \langle b\rangle - 1 < \sum_{j=1}^{n} U_j < \ell - \langle b\rangle).$$

For distributions b such that $b_j = c_j n + o(n)$, $|c| = 1$, and $\langle c\rangle = \mu$, we find that

$$\frac{1}{n}\ \ln\left(\frac{n!}{\ell^n}\binom{|b|}{b}\right) \ = \ -1 - \ln\lambda + \mathcal{H}\,(c) + o(1), \tag{14}$$

and we know [(i)] that

$$\lim_n n^{-1}\ln\mathbb{P}(\theta n - 1 < \sum_{j=1}^{n} U_j < \theta n) \ = \ \Xi(\theta). \tag{15}$$

According to formulas (6.12) to (6.16) (Giladi and Keller, 1994, p. 299), Ξ is given by

Lemma 1 *For* $0 < \theta < 1$*,*

$$\Xi(\theta) \ = \ \ln\sinh\alpha - \alpha\coth\alpha + 1 - \ln\alpha,$$

in which α is given implicitly by $-\alpha^{-1} + 1 + \coth\alpha = 2\theta$.

[(i)] Due to an interpretation of $A(n, k)$ in terms of uniform random variables going back to Tanny (1973).

Now, if $|c| = 1$ and $\langle c \rangle = \mu$,

$$\mathcal{H}(c) \;\leq\; (1+\mu)\ln(1+\mu) - \mu\ln\mu, \tag{16}$$

with equality only if

$$c_k = \mu^k (1+\mu)^{-k-1}\, 1\!\!1_{k\geq 0} = d_k^{(\mu)},$$

cf. Lemma 8.3.1 in (Ash, 1965, p. 238). As usual in large deviation theory, only the leading term, provided by (16), contributes, so that, using (13), (14) and (15), we obtain

$$\begin{aligned} n^{-1}\,\ln\left(\mathbb{U}_{n,\ell}\,(G = \mu n)\right) &= n^{-1}\,\ln\left(\mathbb{U}_{n,\ell}\,(\langle\beta\rangle = \mu n)\right) \\ &\simeq n^{-1}\,\ln\left(\mathbb{U}_{n,\ell}\left(\beta = d^{(\mu)} n\right)\right) \\ &= -1 - \ln\lambda + \mathcal{H}\left(d^{(\mu)}\right) \;+\; \Xi(\lambda - \mu). \end{aligned}$$

Together with (10), this leads to the expression of Ψ given in (2).

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A Murnaghan-Nakayama Rule for Generalized Demazure Atoms

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Abstract. We prove an analogue of the Murnaghan-Nakayama rule to express the product of a power symmetric function and a generalized Demazure atom in terms of generalized Demazure atoms.

Résumé. Nous prouvons un analogue de la règle Murnaghan-Nakayama à exprimer le produit d'une fonction de puissance symétrique et un Demazure généralisée atomes en termes de généralisées atomes de Demazure.

Keywords: generalized Demazure atoms, nonsymmetric Macdonald polynomials, permuted basement fillings

1 Introduction

Haglund, Mason, and Remmel [HMR12] introduced a family of polynomials $\widehat{E}_\gamma^\sigma(x_1,\ldots,x_n)$ indexed by weak compositions γ of n and permutations σ in the symmetric group S_n that they called generalized Demazure atoms. The $\widehat{E}_\gamma^\sigma(x_1,\ldots,x_n)$ interpolate between the Schur functions and the Demazure atoms studied by Mason [Mas08]. The main goal of this paper is to develop a generalization of the Murnaghan-Nakayama rule to express the product of a power symmetric function $p_r(x_1,\ldots,x_n)$ and a generalized Demazure atom $\widehat{E}_\gamma^\sigma(x_1,\ldots,x_n)$ as a sum of generalized Demazure atoms $\widehat{E}_\delta^\sigma(x_1,\ldots,x_n)$. That is, we shall give a combinatorial definition of the coefficients $c_{\gamma,\sigma,\delta}^{(r)}$ where

$$p_r(x_1,\ldots,x_n)\widehat{E}_\gamma^\sigma(x_1,\ldots,x_n) = \sum_\delta c_{\gamma,\sigma,\delta}^{(r)}\widehat{E}_\delta^\sigma(x_1,\ldots,x_n). \tag{1}$$

Let $\mathbb{N}$ denote the set of natural numbers, $\{0,1,2,\ldots\}$, and let $\mathbb{P}$ denote the set of positive integers, $\{1,2,\ldots\}$. We say that $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_n)$ is a partition of m into n parts if each $\lambda_i \in \mathbb{N}$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\sum_{i=1}^n \lambda_i = m$. We say that $\gamma = (\gamma_1,\gamma_2,\ldots,\gamma_n)$ is a weak composition of m into n parts if each $\gamma_i \in \mathbb{N}$ and $\sum_{i=1}^n \gamma_i = m$. A composition of m is a weak composition which has no zeros. The *diagram* of γ, $dg(\gamma)$, is the set of m cells arranged in n columns so that there are γ_i cells in the i^{th} column and all the columns are flush with the bottom of the diagram. For example, the diagram of $\gamma = (2,0,1,0,3)$ is pictured in Figure 1. The *augmented diagram* of γ, $\widehat{dg}(\gamma)$, consists of $dg(\gamma)$ plus a row of n cells attached below. This lowest row is called the *basement*. Let $\lambda(\gamma)$ be the rearrangement of the parts of γ into weakly decreasing order. Thus, $\lambda(\gamma)$ produces the partition associated with each weak composition. For example, $\lambda(2,0,1,0,3) = (3,2,1,0,0)$.

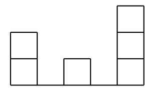

Fig. 1: The diagram of $\gamma = (2, 0, 1, 0, 3)$

Macdonald's well-known symmetric polynomials [Mac79], denoted $P_\lambda(x_1, x_2, \ldots, x_n; q, t)$ for λ a partition with n parts, have certain defining characteristics, including an orthogonality condition. Macdonald [Mac95] showed that many of these same characteristics were shared by a family of nonsymmetric polynomials indexed by weak compositions with n parts, denoted $E_\gamma(x_1, x_2, \ldots, x_n; q, t)$. These polynomials were given a combinatorial interpretation by Haglund, Haiman, and Loehr [HHL08] as the generating functions for fillings of the diagram of γ with positive integer entries satisfying some conditions.

Mason [Mas08], [Mas09] studied a slight variation of the $E_\gamma(x_1, x_2, \ldots, x_n; q, t)$ polynomials, called $\widehat{E}_\gamma(x_1, x_2, \ldots, x_n; q, t)$. The polynomial $\widehat{E}_\gamma$ is obtained from E_γ by reversing the order of the x_i's and sending q and t to their reciprocals. Mason considered the specialization arising from setting $q = t = 0$ in $\widehat{E}_\gamma(x_1, x_2, \ldots, x_n; q, t)$, hereafter referred to as $\widehat{E}_\gamma(x_1, x_2, \ldots, x_n)$. These polynomials arise in [LS90] as "standard bases" and are also called *Demazure atoms*. Using the work of [HHL08], Mason showed that $\widehat{E}_\gamma(x_1, x_2, \ldots, x_n)$ can be interpreted as the sum of the weights of certain fillings of $\hat{dg}(\gamma)$, which she called semi-standard augmented fillings. An important outcome of Mason's work is a generalization of the Robinson-Schensted-Knuth (RSK) insertion algorithm for semi-standard augmented fillings that she used to give combinatorial proofs of many results involving Demazure atoms. For example, this generalization of RSK was used to exhibit a bijection that showed that, for any partition β, the Schur function s_β could be expressed as

$$s_\beta(x_1, \ldots, x_n) = \sum_{\lambda(\gamma)=\beta} \widehat{E}_\gamma(x_1, \ldots, x_n). \tag{2}$$

Mason's work has led to several lines of further research. Using Mason's extension of RSK, Haglund, Luoto, Mason, and van Willigenburg [HLMvW11a] developed the theory for the quasisymmetric Schur functions, a new basis for the ring of quasisymmetric functions. In [HLMvW11b], they developed analogues of the Littlewood-Richardson rule to express the product of a Schur function and a Demazure atom (quasisymmetric Schur function) in terms of Demazure atoms (quasisymmetric Schur functions).

Haglund, Mason, and Remmel [HMR12] further generalized Mason's work by viewing semi-standard augmented fillings as fillings of augmented diagrams with entries in the basement equal to the identity permutation, ϵ_n. They also viewed reverse row-strict tableaux as fillings of augmented diagrams with entries in the basement equal to the reverse of the identity permutation, $\bar{\epsilon}_n$. Their work further generalizes Mason's extension of RSK to apply to fillings of diagrams with arbitrary permutations in the basement cells, called *permuted basement fillings*, or PBFs. The permuted basement fillings with basement σ generate the polynomials called *generalized Demazure atoms* and denoted $\widehat{E}^\sigma_\gamma(x_1, x_2, \ldots, x_n)$. The $\widehat{E}^\sigma_\gamma$'s can be viewed as intermediates between the Schur functions and the Demazure atoms. In fact, Haglund, Mason, and Remmel [HMR12] showed that for any permutation σ,

$$s_\beta(x_1, \ldots, x_n) = \sum_{\lambda(\gamma)=\beta} \widehat{E}^\sigma_\gamma(x_1, \ldots, x_n). \tag{3}$$

They also showed that $\widehat{E}^\sigma_\gamma(x_1, x_2, \ldots, x_n) = 0$ unless $\gamma_i \geq \gamma_j$ whenever $i < j$ and $\sigma_i > \sigma_j$. Note that when $\sigma = \epsilon_n$, we can recover (2) from (3). Also, when $\sigma = \overline{\epsilon}_n$, there is only one nonzero term in (3), so $s_\beta = \widehat{E}_\beta^{\overline{\epsilon}_n}$. We shall see that in the special case where $\sigma = \overline{\epsilon}_n$, our generalized Murnaghan-Nakayama rule reduces to the classical Murnaghan-Nakayama rule.

One of the motivations for studying the $\widehat{E}_\gamma^\sigma(x_1, \ldots, x_n)$ is an unpublished result of Haiman and Haglund which can be described briefly as follows. Let $\widehat{E}_\gamma^\sigma(x_1, \ldots, x_n; q, t)$ denote the polynomial obtained by modifying $\widehat{E}_\gamma(x_1, \ldots, x_n; q, t)$ in the following way. Under the interpretation of [HHL08] which associates $\widehat{E}_\gamma(x_1, \ldots, x_n; q, t)$ with fillings of $\widehat{dg}(\gamma)$, replace the basement permutation ϵ_n with $\sigma = \sigma_1\sigma_2\cdots\sigma_n$, and change nothing else. Then if $i+1$ occurs to the left of i in the basement $\sigma_1\sigma_2\cdots\sigma_n$, we have

$$T_i\widehat{E}_\gamma^\sigma(x_1, \ldots, x_n; q, t) = t^A\widehat{E}_\gamma^{\sigma'}(x_1, \ldots, x_n; q, t). \tag{4}$$

Here A equals one if the height of the column of $\widehat{dg}(\gamma)$ above $i+1$ in the basement is greater than or equal to the height of the column above i in the basement, and equals zero otherwise. Also, σ' is the permutation obtained by interchanging i and $i+1$ in σ. The T_i are generators for the affine Hecke algebra which act on monomials in the $\{x_1, x_2, \ldots, x_n\}$ by

$$T_i x^\lambda = t x^{s_i(\lambda)} + (t-1)\frac{x^\lambda - x^{s_i(\lambda)}}{1 - x^{\alpha_i}},$$

with $x^{\alpha_i} = x_i/x_{i+1}$. See [HHL08] for a more detailed description of the T_i and their relevance to nonsymmetric Macdonald polynomials. The $\widehat{E}_\gamma^\sigma(x_1, \ldots, x_n)$ can be obtained by setting $q = t = 0$ in $\widehat{E}_\gamma^\sigma(x_1, \ldots, x_n; q, t)$, and hence are a natural generalization of the $\widehat{E}_\gamma(x_1, \ldots, x_n)$ to investigate. If we set $q = t = 0$ in the Hecke operator T_i, it reduces to a divided difference operator similar to those appearing in the definition of Schubert polynomials. By (4), $\widehat{E}_\gamma^\sigma(x_1, \ldots, x_n)$ can be expressed, up to a power of t, as a series of the divided difference operators applied to $\widehat{E}_\gamma^{\overline{\epsilon}_n}(x_1, \ldots, x_n)$.

Haglund, Mason, and Remmel [HMR12] proved analogues of the Pieri rules for generalized Demazure atoms. Let $h_r(x_1, \ldots, x_n)$ denote the r^{th} homogeneous symmetric function and $e_r(x_1, \ldots, x_n)$ denote the r^{th} elementary symmetric function. Then Haglund, Mason, and Remmel gave combinatorial interpretations to the coefficients $a_{\gamma,\sigma,\delta}^{(r)}$ and $b_{\gamma,\sigma,\delta}^{(r)}$ where

$$h_r(x_1, \ldots, x_n)\widehat{E}_\gamma^\sigma(x_1, \ldots, x_n) = \sum_\delta a_{\gamma,\sigma,\delta}^{(r)}\widehat{E}_\delta^\sigma(x_1, \ldots, x_n). \tag{5}$$

and

$$e_r(x_1, \ldots, x_n)\widehat{E}_\gamma^\sigma(x_1, \ldots, x_n) = \sum_\delta b_{\gamma,\sigma,\delta}^{(r)}\widehat{E}_\delta^\sigma(x_1, \ldots, x_n). \tag{6}$$

We will indicate how we can derive our combinatorial interpretation of the coefficients $c_{\gamma,\sigma,\delta}^{(r)}$ from these two rules. The same technique can be used to derive an analogue of the Murnaghan-Nakayama rule to express the product of a power symmetric function p_r and a quasisymmetric Schur function in terms of quasisymmetric Schur functions, but we will not pursue this topic in this paper.

The outline of this paper is as follows. In the next section, we will define permuted basement fillings and the generalized RSK insertion algorithm as well as introduce some results from [HMR12]. In section 3, we will state and prove our refinement of the Murnaghan-Nakayama rule.

2 PBFs and the Insertion Procedure

In this section, we shall formally define permuted basement fillings (PBFs) and the generalization of the RSK insertion algorithm due to Haglund, Mason, and Remmel [HMR12]. Let $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ be a weak composition of m into n parts. We will denote the cell in column i and row j of $\widehat{dg}(\gamma)$ by (i, j). The basement is row zero, and row indices increase from bottom to top. The leftmost column is column 1, and column indices increase moving from left to right. This way, it is easy to think of $\widehat{dg}(\gamma)$ as $dg(\gamma)$ augmented by row zero.

A filling F of an augmented diagram is a function $F : \widehat{dg}(\gamma) \to \mathbb{P}$, which can be pictured as an assignment of positive integers to the cells of $\widehat{dg}(\gamma)$. We will use $F(i, j)$ to denote the integer assigned to cell (i, j) by the function F. We will be interested only in fillings where the basement entries are a permutation $\sigma = \sigma_1\sigma_2 \ldots \sigma_n$ of $\{1, 2, \ldots, n\}$ and the column entries weakly decrease reading from bottom to top.

A set of cells $(i, k), (j, k), (i, k-1)$ in $\widehat{dg}(\gamma)$ is a *type A triple* if $i < j$, $k > 0$, and $\gamma_i \geq \gamma_j$. A type A triple is an *inversion triple* in F if $F(j, k) < F(i, k) \leq F(i, k-1)$ or $F(i, k) \leq F(i, k-1) < F(j, k)$. A set of cells $(j, k+1), (i, k), (j, k)$ in $\widehat{dg}(\gamma)$ is a *type B triple* if $i < j$, $k \geq 0$, and $\gamma_i < \gamma_j$. A type B triple is an *inversion triple* in F if $F(i, k) < F(j, k+1) \leq F(j, k)$ or $F(j, k+1) \leq F(j, k) < F(i, k)$.

A filling F is said to satisfy the *B-increasing condition* if, whenever $i < l$ and $\gamma_i < \gamma_l$, it is true that $F(i, j-1) < F(l, j)$ for all $j \geq 1$.

A PBF F^σ of shape γ and basement σ is a filling of $\widehat{dg}(\gamma)$ with positive integer entries such that

1. the basement is filled with $\sigma_1, \sigma_2, ..., \sigma_n$ from left to right,
2. column entries weakly decrease reading from bottom to top,
3. every triple of type A or B is an inversion triple in F^σ, and
4. the B-increasing condition is satisfied.

It was observed in [HMR12] that the B-increasing condition plus the fact that column entries are weakly decreasing automatically implies that all type B triples are inversion triples.

The weight of a PBF F^σ with shape γ and basement σ is defined to be

$$wt(F^\sigma) = \prod_{(i,j) \in dg(\gamma)} x_{F^\sigma(i,j)}. \tag{7}$$

Note that the basement cells do not contribute to the weight of the PBF. The nonsymmetric polynomials $\widehat{E}^\sigma_\gamma(x_1, x_2, \ldots, x_n)$ are then defined by

$$\widehat{E}^\sigma_\gamma(x_1, x_2, \ldots, x_n) = \sum_{F^\sigma} wt(F^\sigma) \tag{8}$$

where the sum is over PBFs F^σ of shape γ and basement σ.

We say that a shape γ is *σ-compatible* if $\gamma_i \geq \gamma_j$ whenever $i < j$ and $\sigma_i > \sigma_j$. In [HMR12], it is shown that $\widehat{E}^\sigma_\gamma(x_1, x_2, \ldots, x_n) = 0$ unless γ is σ-compatible. Thus, there are no PBFs of shape γ and basement σ if γ is not σ-compatible.

When σ is the identity permutation, ϵ_n, a PBF is a semi-standard augmented filling as defined by Mason [Mas08], [Mas09]. When σ is the reverse of the identity permutation, $\overline{\epsilon}_n$, a PBF is strictly decreasing from left to right and weakly decreasing from bottom to top. The only $\overline{\epsilon}_n$-compatible shapes are partition shapes λ, so that PBFs can be described as "reverse row-strict tableaux." It is not hard to see that these reverse row-strict tableaux are in one-to-one correspondence with column-strict tableaux, so that for λ a partition, $\widehat{E}_{\lambda}^{\overline{\epsilon}_n} = s_{\lambda}$.

The *reading order* of the cells of $\widehat{dg}(\gamma)$ is obtained by moving across the rows from left to right, beginning with the highest row. Formally, (a, b) comes before (c, d) in reading order if $b > d$ or $b = d$ and $a < c$. As mentioned above, Mason defined an insertion procedure $k \to F$ analogous to the RSK insertion procedure that inserts a positive integer k into a semi-standard augmented filling to produce another semi-standard augmented filling. Haglund, Mason, and Remmel [HMR12] generalized this procedure to PBFs with arbitrary basements. To define this insertion $k \to F^{\sigma}$, let $\overline{F^{\sigma}}$ be the filling that extends the basement permutation by first adding a j in each cell $(j, 0)$ with $n < j \leq k$ and then adding an extra cell filled with a 0 on top of each column. Let $(x_1, y_1), (x_2, y_2), \ldots$ be the cells of $\overline{F^{\sigma}}$ listed in reading order. To insert k into F^{σ}, go through the cells of $\overline{F^{\sigma}}$ in reading order looking for the first (x_i, y_i) such that $\overline{F^{\sigma}}(x_i, y_i) < k \leq \overline{F^{\sigma}}(x_i, y_i - 1)$. Replace $\overline{F^{\sigma}}(x_i, y_i)$ with k and insert the cell's previous value into the remaining cells in reading order, beginning with (x_{i+1}, y_{i+1}). Continue in this way until some 0 is replaced by a positive integer. Finally, remove any zeros from the tops of the columns. Notice that $k \to F^{\sigma}$ creates a new cell at the top of some column in F^{σ}.

A fundamental result of [HMR12], used to prove many properties about PBFs, is the fact that this insertion procedure $k \to F^{\sigma}$ is well-defined and produces a PBF. Also, in the case that $\sigma = \overline{\epsilon}_n$, this insertion algorithm reduces to a reverse row-strict version of the usual RSK algorithm.

Another important question addressed in [HMR12] is whether this insertion procedure can be reversed. To answer this question, the authors define the term *removable cell*. Let γ and δ be weak compositions with n parts such that $dg(\gamma) \subseteq dg(\delta)$. We use $dg(\delta/\gamma)$ to denote the cells of $dg(\delta)$ which are not in $dg(\gamma)$. Suppose $dg(\delta/\gamma)$ consists of a single cell $c = (x, y)$. Then c is a removable cell from δ if there is no cell to the right of c that is at the top of a column in $dg(\delta)$. That is, there is no j with $x < j \leq n$ and $\delta_j = y$. It is shown that if F^{σ} is a PBF of shape γ and basement σ, and $G^{\sigma} = k \to F^{\sigma}$ is a PBF of shape δ, then the cell c in $dg(\delta/\gamma)$ is a removable cell. This terminology is used because it means that the insertion procedure can be reversed starting with c. That is, begin with the entry, say a, in cell c and read through the cells of G^{σ} in reverse reading order starting with c until an entry, say b, is found which is greater than a and positioned below an number less than or equal to a. Now replace b with a and continue reversing the insertion procedure with b. The entry that emerges from the first cell in reading order is the k which was originally inserted into F^{σ} to produce G^{σ}. So long as cell c is removable, the insertion can be reversed in this way.

A lemma used in [HMR12] to prove the Pieri rules for the products $h_r \widehat{E}_{\gamma}^{\sigma}$ and $e_r \widehat{E}_{\gamma}^{\sigma}$ will also be useful for our version of the Murnaghan-Nakayama rule. It is reproduced below:

Lemma 1 *Suppose that F^{σ} is a PBF, $G^{\sigma} = a \to F^{\sigma}$, and $H^{\sigma} = b \to G^{\sigma}$. Suppose F^{σ} is of shape α, G^{σ} is of shape β, and H^{σ} is of shape γ. Suppose A is the cell in $dg(\beta/\alpha)$ and B is the cell in $dg(\gamma/\beta)$. Then*

a. if $b \leq a$, then B is strictly above A, and

b. if $b > a$, then B is after A in reading order.

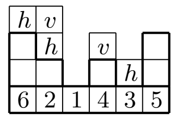

Fig. 2: A satisfactory labeling of $dg((3,3,0,2,1,2)/(2,1,0,1,0,2))$

3 Murnaghan-Nakayama Rule

The power symmetric function $p_r(x_1, x_2, \ldots, x_n)$ is defined as $\sum_{i=1}^{n} x_i^r$. Our goal is to express the product $p_r(x_1, x_2, \ldots, x_n)\widehat{E}_\gamma^\sigma(x_1, x_2, \ldots, x_n)$ as a sum of $\widehat{E}_\delta^\sigma$'s. In particular, we would like to find the coefficients $c_{\gamma,\sigma,\delta}^{(r)}$ in the expansion

$$p_r(x_1, x_2, \ldots, x_n)\widehat{E}_\gamma^\sigma(x_1, x_2, \ldots, x_n) = \sum_\delta c_{\gamma,\sigma,\delta}^{(r)} \widehat{E}_\delta^\sigma(x_1, x_2, \ldots, x_n). \tag{9}$$

Let γ be a weak composition of m with n parts, and let $\sigma \in S_n$. Let $\delta = (\delta_1, \ldots, \delta_n)$ be a weak composition of $m + r$ such that $dg(\gamma) \subseteq dg(\delta)$. Define a *satisfactory labeling of* $dg(\delta/\gamma)$ to be a labeling of the cells in $dg(\delta/\gamma)$ with v's and h's that is consistent with the following four rules:

(a) Assign an h to all but the rightmost cell in each row of $dg(\delta/\gamma)$.

(b) Assign a v to any cell above another cell of $dg(\delta/\gamma)$.

(c) Assign an h to any cell above a cell of $dg(\gamma)$ and having a cell of $dg(\delta/\gamma)$ one row below and anywhere to the left.

(d) Assign an h to the cells of the lowest row of $dg(\delta/\gamma)$.

For an example of a satisfactory labeling, see Figure 2

The following lemma gives a characterization for when $dg(\delta/\gamma)$ has a satisfactory labeling.

Lemma 2 *Let γ and δ be weak compositions such that $dg(\gamma) \subseteq dg(\delta)$. Then $dg(\delta/\gamma)$ has a satisfactory labeling if and only if there is no cell (x, y) in $dg(\delta/\gamma)$ such that $(x + j, y)$ and $(x, y - 1)$ are both cells in $dg(\delta/\gamma)$ for some $j > 0$. In other words, $dg(\delta/\gamma)$ has a satisfactory labeling if and only if it avoids the configuration .*

A satisfactory labeling of $dg(\delta/\gamma)$ is called a *satisfactory k-hook labeling* if it produces $k + 1$ h's and $r - (k + 1)$ v's and meets the following two additional conditions. Let $c_1 = (x_1, y_1), \ldots, c_{k+1} = (x_{k+1}, y_{k+1})$ be the cells labeled h listed in reading order. Let $c_{k+2} = (x_{k+2}, y_{k+2}), \ldots, c_r = (x_r, y_r)$ be the cells labeled v listed in reverse reading order. Let $dg(\delta^{(i)})$ be the diagram of γ plus the cells $c_1, c_2, \ldots, c_i$. A satisfactory k-hook labeling must also satisfy, for $i = 1, 2, \ldots, r$,

1. $\delta^{(i)}$ is a σ-compatible weak composition shape and
2. c_i is a removable cell from $\delta^{(i)}$.

Fig. 3: Unique satisfactory labelings of $dg((5,0,1)/(2,0,1))$ and $dg((3,0,3)/(2,0,1))$

Note that rule (d) in the definition of a satisfactory labeling requires that there is at least one h, so a satisfactory k-hook labeling has $k \geq 0$.

We say that the shape δ/γ is a *γ-transposed k-hook relative to basement σ* if rules (a)-(d) can be used to assign v and h labels to all of the cells of $dg(\delta/\gamma)$ in a unique way and that labeling is a satisfactory k-hook labeling. With these definitions, we can state the main theorem.

Theorem 1 *If $\gamma = (\gamma_1, \ldots, \gamma_n)$ is a weak composition of m and $\sigma \in S_n$, then*

$$p_r(x_1, x_2, \ldots, x_n)\widehat{E}_\gamma^\sigma(x_1, x_2, \ldots, x_n) = \sum_\delta (-1)^k \widehat{E}_\delta^\sigma(x_1, x_2, \ldots, x_n) \tag{10}$$

where the sum is over all weak compositions $\delta = (\delta_1, \ldots, \delta_n)$ of $m + r$ such that $dg(\gamma) \subseteq dg(\delta)$ and δ/γ is a γ-transposed k-hook relative to basement σ.

Theorem 1 says that the nonzero coefficients $c_{\gamma,\sigma,\delta}^{(r)}$ in (9) are ± 1, depending on the number of h's in the unique satisfactory k-hook labeling of $dg(\delta/\gamma)$.

For example, consider the product $p_3(x_1, x_2, x_3)\widehat{E}_{(2,0,1)}^{312}(x_1, x_2, x_3)$. The shapes δ that appear on the right hand side of (10) must be 312-compatible compositions of 6 into 3 parts containing $(2,0,1)$. Since the first and second basement entries are out of order, we require $\delta_1 \geq \delta_2$. Similarly, we require $\delta_1 \geq \delta_3$. Take $dg(2,0,1)$ and consider all the ways to add three cells around the outside of the diagram. Adding three cells to the first column creates the 312-compatible shape $(5,0,1)$. Moreover, this shape has a unique satisfactory labeling as shown in Figure 3, and it is easy to check that this is a satisfactory 0-hook labeling. Thus, $\widehat{E}_{(5,0,1)}^{312}(x_1, x_2, x_3)$ appears on the right hand side of (10) with coefficient $(-1)^0 = 1$. If instead we add two cells to the first column and one cell to another column, the lower of the two cells in the first column will not be assigned a v or h label by rules (a)-(d). Thus, there will be more than one satisfactory labeling of the diagram of such a shape, and it will not contribute to the sum. If one cell is added to the first column and two cells are added elsewhere, note that rules (b) through (d) will not apply to the highest cell in the first column. Therefore, in order to get a unique satisfactory labeling, one where each cell's label is forced by rules (a) through (d), we must make sure that the highest cell in the first column is not the rightmost cell in its row. The only shape that achieves this requirement is $(3,0,3)$. This shape has a unique satisfactory labeling as shown in Figure 3, and this labeling is a satisfactory 1-hook labeling. So $\widehat{E}_{(3,0,3)}^{312}(x_1, x_2, x_3)$ appears on the right hand side of (10) with coefficient $(-1)^1 = -1$. Finally, if we add no cells to the first column, the only way to keep the height of the first column greater than or equal to the heights of the other columns, as required by the basement permutation 312, is to create the shape $(2,2,2)$. In this case, however, $dg((2,2,2)/(2,0,1))$ contains the configuration . Lemma 2 says there is no satisfactory labeling, so this shape will not contribute to the sum. Putting this all together gives

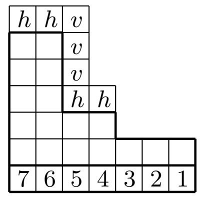

Fig. 4: A rim hook and its satisfactory labeling

$p_3(x_1, x_2, x_3)\widehat{E}^{312}_{(2,0,1)}(x_1, x_2, x_3) = \widehat{E}^{312}_{(5,0,1)}(x_1, x_2, x_3) - \widehat{E}^{312}_{(3,0,3)}(x_1, x_2, x_3)$, which can be verified by direct computation.

Note that in the special case where $\sigma = \bar{\epsilon}_n$ and γ is actually a partition shape, $\widehat{E}^{\sigma}_{\gamma} = s_\gamma$ by (3). Our rule then reduces to the reverse row-strict version of the classical Murnaghan-Nakayama rule, which says

$$p_r(x_1, x_2, \ldots, x_n)s_\gamma(x_1, x_2, \ldots, x_n) = \sum_\lambda (-1)^{w(\lambda/\gamma)-1} s_\lambda(x_1, x_2, \ldots, x_n) \tag{11}$$

Here, the sum is over all partition shapes λ such that $dg(\lambda/\gamma)$ is a *rim hook* of size r, that is a connected skew shape with no 2×2 square. Also, the width of λ/γ, $w(\lambda/\gamma)$, is the number of columns over which the cells of $dg(\lambda/\gamma)$ stretch.

First, if λ is a shape that appears on the right hand side of (11), then it also appears on the right hand side of (10) with the same coefficient. If $dg(\lambda/\gamma)$ is a connected skew shape around the outside of $dg(\gamma)$ avoiding the 2×2 square, then every cell of $dg(\lambda/\gamma)$ either has another cell of $dg(\lambda/\gamma)$ to its right or below it, with the exception of the last cell in reading order. This means that rule (d) labels the last cell in reading order, and rules (a) and (b) label all the other cells of $dg(\lambda/\gamma)$ in a unique way, as in Figure 4. Let $c_1 = (x_1, y_1), \ldots, c_{k+1} = (x_{k+1}, y_{k+1})$ be the cells labeled h listed in reading order. Let $c_{k+2} = (x_{k+2}, y_{k+2}), \ldots, c_r = (x_r, y_r)$ be the cells labeled v listed in reverse reading order. Let $dg(\lambda^{(i)})$ be the diagram of γ plus the cells $c_1, c_2, \ldots, c_i$. Clearly, each $\lambda^{(i)}$ is a partition shape, making it compatible with basement $\bar{\epsilon}_n$. Also, each c_i is on the rightmost column of its height in $\lambda^{(i)}$, making each c_i a removable cell. This means the unique labeling is a satisfactory k-hook labeling, so λ/γ is a γ-transposed k-hook relative to basement $\bar{\epsilon}_n$. Then this s_λ appears as a term in the sum with coefficient $(-1)^k$. This coefficient is the same as that given in the classical Murnaghan-Nakayama rule, because the width of λ/γ is the number of cells labeled h. The labeling assigns an h to $k+1$ cells so that $(-1)^{w(\lambda/\gamma)-1} = (-1)^k$.

Now, if λ is not counted by (11), it is also not counted by (10). If $dg(\lambda/\gamma)$ is not a connected skew shape, then consider the first connected component of $dg(\lambda/\gamma)$ in reading order. The last cell in reading order of this component will not have its label forced by rules (a) through (d). It is neither above nor to the left of another cell. It is not in the lowest row of $dg(\lambda/\gamma)$ because there is another component following this one, and the situation described by rule (c) does not apply to partition shapes. Thus, there is not a unique satisfactory of labeling of $dg(\lambda/\gamma)$, meaning δ/γ is not a γ-transposed k-hook relative to basement $\bar{\epsilon}_n$, so s_λ will not appear as a term in the sum. Similarly, if $dg(\lambda/\gamma)$ contains a 2×2 square, Lemma 2 says that $dg(\lambda/\gamma)$ has no satisfactory labeling. In this case, clearly λ/γ is not a γ-transposed k-hook relative to basement $\bar{\epsilon}_n$. Thus the shapes λ for which λ/γ is a γ-transposed k-hook relative to basement $\bar{\epsilon}_n$ are exactly the shapes λ such that $dg(\lambda/\gamma)$ is a rim hook.

To prove Theorem 1, we start with a well-known result from symmetric function theory, which says that the power symmetric functions can be expressed as an alternating sum of hook Schur functions:

$$p_r(x_1, x_2, \ldots, x_n) = \sum_{k=0}^{r-1} (-1)^k s_{(r-k,1^k)}. \tag{12}$$

Using (12) allows us to write

$$p_r(x_1, x_2, \ldots, x_n)\widehat{E}^{\sigma}_{\gamma}(x_1, x_2, \ldots, x_n) = \sum_{k=0}^{r-1} (-1)^k s_{(r-k,1^k)} \widehat{E}^{\sigma}_{\gamma}(x_1, x_2, \ldots, x_n) \tag{13}$$

thereby reducing the problem to one of multiplying a hook Schur function and a generalized Demazure atom.

If $A_k(\delta/\gamma)$ is the number of satisfactory k-hook labelings of $dg(\delta/\gamma)$, the following lemma answers the question of how to multiply a hook Schur function and a generalized Demazure atom. While it is possible to give a proof of this lemma using the Pieri rules of [HMR12], we have chosen to give a direct proof here.

Lemma 3 *If $\gamma = (\gamma_1, \ldots, \gamma_n)$ is a weak composition of m and $\sigma \in S_n$, then*

$$s_{(r-k,1^k)}(x_1, x_2, \ldots, x_n)\widehat{E}^{\sigma}_{\gamma}(x_1, x_2, \ldots, x_n) = \sum_{\delta} A_k(\delta/\gamma)\widehat{E}^{\sigma}_{\delta}(x_1, x_2, \ldots, x_n) \tag{14}$$

where the sum is over all weak compositions $\delta = (\delta_1, \ldots, \delta_n)$ of $m + r$ such that $dg(\gamma) \subseteq dg(\delta)$ and δ/γ has a satisfactory k-hook labeling.

Proof: The left hand side of (14) can be interpreted as the sum of the weights of all pairs (w, F^σ), where F^σ is a PBF of shape γ with basement σ and $w = w_1w_2 \ldots w_r$ satisfies $1 \le w_1 < \cdots < w_k < w_{k+1} \le n$ and $n \ge w_{k+1} \ge \cdots \ge w_r \ge 1$. This condition on w comes from interpreting $s_{(r-k,1^k)}$ as the sum of the weights of all reverse row-strict tableaux of shape $(r-k, 1^k)$ read by columns. Here, the weight of a pair (w, F^σ) is defined to be $wt(F^\sigma)\prod_{i=1}^{r} x_{w_i}$. The right hand side of (14) can be interpreted as the sum of the weights of all pairs (L, G^σ) where G^σ is a PBF with basement σ and shape $\delta = (\delta_1, \ldots, \delta_n)$ of size $m + r$ such that $dg(\gamma) \subseteq dg(\delta)$ and L is a satisfactory k-hook labeling of $dg(\delta/\gamma)$. Here, the weight of a pair (L, G^σ) is just $wt(G^\sigma)$.

Let Θ be the function which takes such a (w, F^σ) pair to the pair (L, G^σ) where G^σ is the PBF that results from the insertion of w into F^σ and L is the labeling induced by this insertion, defined below. Let δ be the shape of this resulting PBF G^σ. Let $G^\sigma_0 = F^\sigma$ and $G^\sigma_i = w_1 \ldots w_i \to F^\sigma$ for $i = 1, \ldots, r$. Let $\delta^{(i)}$ be the shape of G^σ_i, and let c_i be the cell in $dg(\delta^{(i)}/\delta^{(i-1)})$. To obtain the induced labeling L of $dg(\delta/\gamma)$, simply label cells $c_1, \ldots, c_{k+1}$ with h, and label cells $c_{k+2}, \ldots, c_r$ with v. Note that by Lemma 1, $c_1, \ldots, c_{k+1}$ appear in reading order because $w_1 < \cdots < w_k < w_{k+1}$, and $c_{k+2}, \ldots, c_r$ appear in reverse reading order because $w_{k+2} \ge \cdots \ge w_r$. Also, L produces the correct number of h's and v's to be a satisfactory k-hook labeling. We must check that the two additional conditions are met, as well as that L is a satisfactory labeling.

We know that for each $i = 1, \ldots, r$, $\delta^{(i)}$ is a σ-compatible weak composition of $m + i$ because it arises from insertion. Also, insertion creates only removable cells, so each c_i is a removable cell. This means conditions 1 and 2 in the definition of a satisfactory k-hook labeling hold. It remains to show that the labeling L satisfies the definition of a satisfactory labeling.

First, note that none of $\{c_{k+2}, \ldots, c_r\}$ have another cell in $dg(\delta/\gamma)$ to their right in the same row. Suppose that some c_l with $k+2 \leq l \leq r$ has another cell c_i to its right. Since c_l is removable, it must be the case that $i > l$. This means that $w_i \leq w_l$ so by Lemma 1, c_i should be strictly above c_l. Since the assumption is that c_i and c_l are in the same row, it must not be possible for any of $\{c_{k+2}, \ldots, c_r\}$ to have another cell in $dg(\delta/\gamma)$ to the right. This means every cell with another cell to the right must be among $\{c_1, \ldots, c_{k+1}\}$, and our labeling L assigns them all h. Thus, part (a) of the definition holds.

Next, note that none of $\{c_1, \ldots, c_{k+1}\}$ have another cell in $dg(\delta/\gamma)$ immediately below. Suppose that some c_l with $1 \leq l \leq k+1$ is directly above some other c_i. For $\delta^{(l)}$ to be a weak composition shape, it must be the case that $i \leq l$. This means $w_i < w_l$ so that c_l should come after c_i in reading order by Lemma 1. This contradicts the fact that c_l is above c_i, which means all cells with another cell directly below must be among $\{c_{k+2}, \ldots, c_r\}$ and our labeling assigns them all v, as required by part (b) of the definition.

Now suppose that some c_l with $k+2 \leq l \leq r$ is above a cell of $dg(\gamma)$ and has some other $c_i \in dg(\delta/\gamma)$ one row below and to the left. Then $i > l$ otherwise c_i would not be removable. Then $w_l \geq w_i$ so c_i should be strictly above c_l. Since this is not the case, it must not be possible for one of $\{c_{k+2}, \ldots, c_r\}$ to be above a cell in $dg(\gamma)$ and have some other cell $dg(\delta/\gamma)$ one row below and to the left. Any such cell must therefore be among $\{c_1, \ldots, c_{k+1}\}$, and L assigns them all h. Thus, part (c) of the definition holds.

Finally, we show that none of $\{c_{k+2}, \ldots, c_r\}$ can end up in the lowest row of $dg(\delta/\gamma)$. For any w_l with $k+2 \leq l \leq r$, $w_l \leq w_{k+1}$ so c_l must fall strictly above c_{k+1} in $dg(\delta/\gamma)$. Since each such c_l must fall strictly above another cell in the diagram, none can be in the bottom row. Therefore, all cells in the bottom row of the diagram are among $\{c_1, \ldots, c_{k+1}\}$. The labeling L assigns them all h, thereby satisfying part (d) of the definition.

We have shown that G^σ is a PBF with basement σ and shape $\delta = (\delta_1, \ldots, \delta_n)$ of size $m+r$ such that $dg(\gamma) \subseteq dg(\delta)$ and L is a satisfactory k-hook labeling. Also, the weight of the pair (w, F^σ) is clearly the same as the weight of G^σ. Since the insertion procedure can be reversed, the map Θ is one-to-one.

It remains to show that Θ is surjective, or that for each G^σ with a given satisfactory k-hook labeling, there is a pair (w, F^σ) that maps to it under Θ. Suppose G^σ is a PBF with basement σ and shape $\delta = (\delta_1, \ldots, \delta_n)$ of size $m+r$ such that $dg(\gamma) \subseteq dg(\delta)$ and L is a satisfactory k-hook labeling of $dg(\delta/\gamma)$. Label the cells with v's and h's according to L. Then label the h's in reading order with $c_1, \ldots, c_{k+1}$ and the v's in reverse reading order with $c_{k+2}, \ldots, c_r$. Since c_r is a removable cell, we can reverse the insertion procedure to produce a PBF which we will call F^σ_{r-1} and a letter w_r such that $G^\sigma = w_r \to F^\sigma_{r-1}$. Since each c_i is removable, we can continue to reverse the insertion procedure starting with c_{r-1} next, all the way down to c_1. Each step in this reversal produces a new PBF F^σ_i and a letter w_{i+1} such that $G^\sigma = w_{i+1} \ldots w_r \to F^\sigma_i$. The shape of F^σ_i is δ with the cells $c_{i+1}, \ldots, c_r$ removed. This means F^σ_0 is a PBF of shape γ and basement σ such that $w = w_1 \ldots w_r$ inserted into F^σ_0 produces G^σ and induces the labeling L.

We must show, however, that the resulting word $w = w_1 \ldots w_r$ satisfies $1 \leq w_1 < \cdots < w_k < w_{k+1} \leq n$ and $n \geq w_{k+1} \geq \cdots \geq w_r \geq 1$. It is clear that each w_i satisfies $1 \leq w_i \leq n$, as w_i was an element of a PBF with basement $\sigma \in S_n$. Suppose for a contradiction that some $w_i \geq w_{i+1}$ for $1 \leq i \leq k$. Then Lemma 1 says that c_{i+1} should be strictly above c_i, which contradicts the fact that $\{c_1, \ldots, c_{k+1}\}$ were labeled in reading order. So $1 \leq w_1 < \cdots < w_k < w_{k+1} \leq n$. Now suppose that some $w_i < w_{i+1}$ for $k+1 \leq i \leq r$. Then Lemma 1 says that c_{i+1} should appear after c_i in reading order, which contradicts the fact that $\{c_{k+1}, \ldots, c_r\}$ were labeled in reverse reading order. Thus, it is also true that $n \geq w_{k+1} \geq \cdots \geq w_r \geq 1$. Therefore, Θ is a bijection, which proves (14).

□

Replacing 14 in 13 allows us to write

$$p_r(x_1, x_2, \ldots, x_n)\widehat{E}_\gamma^\sigma(x_1, x_2, \ldots, x_n) = \sum_{k=0}^{r-1}(-1)^k \sum_\delta A_k(\delta/\gamma)\widehat{E}_\delta^\sigma(x_1, x_2, \ldots, x_n) \tag{15}$$

where the sum is over all weak compositions $\delta = (\delta_1, \ldots, \delta_n)$ of $m + r$ such that $dg(\gamma) \subseteq dg(\delta)$ and $dg(\delta/\gamma)$ has a satisfactory k-hook labeling. Next we state a technical lemma for which we will not supply a proof due to lack of space. This lemma will help us define an involution which will allow us to simplify the right hand side of (15).

Lemma 4 *Suppose δ is a shape such that $dg(\delta/\gamma)$ has a satisfactory j-hook labeling for some j and rules (a) through (d) cannot be used to assign labels to all cells of $dg(\delta/\gamma)$. Then every satisfactory labeling of $dg(\delta/\gamma)$ is a satisfactory k-hook labeling for some k.*

Using these lemmas, we can prove Theorem 1.

Proof of Theorem 1: We have from (15) an expression for $p_r\widehat{E}_\gamma^\sigma$ in terms of $\widehat{E}_\delta^\sigma$'s. Suppose δ is a shape such that $\widehat{E}_\delta^\sigma$ appears on the right hand side of (15). By Lemma 3, δ is a shape such that $dg(\delta/\gamma)$ has a satisfactory j-hook labeling for some j. Suppose also that δ/γ is not a γ-transposed k-hook relative to basement σ, so that there is more than one way to label its cells in accordance with rules (a) through (d).

We will associate with each satisfactory labeling L of $dg(\delta/\gamma)$ the sign $(-1)^k$ if L is a satisfactory k-hook labeling. Note that Lemma 4 implies that each satisfactory labeling L is a satisfactory k-hook labeling for some value of k, so this notion of sign is well-defined. Now define an involution I on the set of satisfactory labelings of $dg(\delta/\gamma)$. Take any satisfactory labeling L of $dg(\delta/\gamma)$. By Lemma 4, L is a satisfactory k-hook labeling for some k. To define $I(L)$, take the first cell c in reading order of $dg(\delta/\gamma)$ which was not forced to be a v or an h by rules (a) through (d). If L labeled cell c with an h, change it to a v. Otherwise, change it from a v to an h. This new labeling is $I(L)$. $I(L)$ is still a satisfactory labeling because we have not changed the label of any cell to which rules (a) through (d) apply. The number of h's in $I(L)$ is either one fewer or one more than the number of h's in L. By Lemma 4, $I(L)$ is a satisfactory $k+1$-hook or $k-1$-hook labeling. Furthermore the sign of $I(L)$ is the opposite of the sign of L. I is an involution because I applied to $I(L)$ will change the label of the same cell c.

Applying this involution to the labelings of $dg(\delta/\gamma)$ when δ/γ is not a γ-transposed k-hook relative to basement σ gives a way to pair $\widehat{E}_\delta^\sigma$ terms with opposite signs on the right hand side of (15). Since I has no fixed points, all such $\widehat{E}_\delta^\sigma$ terms will cancel. This means that if δ is a shape for which any cells of $dg(\delta/\gamma)$ are left undetermined by rules (a)-(d), then $\widehat{E}_\delta^\sigma$ will not appear in the expansion of $p_n\widehat{E}_\gamma^\sigma$. The terms that do appear are for those δ's for which rules (a)-(d) assign a v or h label to every cell of $dg(\delta/\gamma)$, which are exactly γ-transposed k-hooks relative to basement σ. □

If $\lambda = (\lambda_1, \ldots, \lambda_s)$ is a partition, this theorem can be used to multiply $p_\lambda\widehat{E}_\gamma^\sigma$ by first writing $p_\lambda = p_{\lambda_1}p_{\lambda_2}\ldots p_{\lambda_s}$ and then repeatedly applying 10.

In conclusion, we note that one can define a quasisymmetric Schur function QS_α and a row-strict quasisymmetric Schur function RS_α for each composition α of n; see [HLMvW11a] and [MR11], respectively. The methods of this paper can easily be modified to prove analogues of the Murnaghan-Nakayama rule for quasisymmetric Schur functions and row-strict quasisymmetric Schur functions. That is, we can give combinatorial interpretations to the coefficients $u^{(r)}_{\alpha,\beta}$ and $v^{(r)}_{\alpha,\beta}$ where

$$\begin{aligned} p_r(x_1,\ldots,x_n)QS_\alpha(x_1,\ldots,x_n) &= \sum_\beta u^{(r)}_{\alpha,\beta} QS_\beta(x_1,\ldots,x_n) \text{ and} \\ p_r(x_1,\ldots,x_n)RS_\alpha(x_1,\ldots,x_n) &= \sum_\beta v^{(r)}_{\alpha,\beta} RS_\beta(x_1,\ldots,x_n). \end{aligned}$$

This work will appear in a subsequent paper.

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Semi-skyline augmented fillings and non-symmetric Cauchy kernels for stair-type shapes

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Abstract. Using an analogue of the Robinson-Schensted-Knuth (RSK) algorithm for semi-skyline augmented fillings, due to Sarah Mason, we exhibit expansions of non-symmetric Cauchy kernels $\prod_{(i,j)\in\eta}(1-x_iy_j)^{-1}$, where the product is over all cell-coordinates (i,j) of the stair-type partition shape η, consisting of the cells in a NW-SE diagonal of a rectangle diagram and below it, containing the biggest stair shape. In the spirit of the classical Cauchy kernel expansion for rectangle shapes, this RSK variation provides an interpretation of the kernel for stair-type shapes as a family of pairs of semi-skyline augmented fillings whose key tableaux, determined by their shapes, lead to expansions as a sum of products of two families of key polynomials, the basis of Demazure characters of type A, and the Demazure atoms. A previous expansion of the Cauchy kernel in type A, for the stair shape was given by Alain Lascoux, based on the structure of double crystal graphs, and by Amy M. Fu and Alain Lascoux, relying on Demazure operators, which was also used to recover expansions for Ferrers shapes.

Résumé. En utilisant an analogue de l'algorithme de Robinson-Schensted-Knuth (RSK) pour remplissages des lignes d'horizon augmentées, proposé par Sarah Mason, nous donnons des développements d'un noyau de Cauchy non symétrique, $\prod_{(i,j)\in\eta}(1-x_iy_j)^{-1}$, dans le cas où les paires (i,j) sont les coordonnées des cellules d'une partition η du type escalier dans un rectangle, contenant la plus grande partition escalier de ce rectangle. Dans l'esprit du développement classique sur les diagrammes rectangulaires, cette variation de RSK fournit une somme des produits de deux familles de polynômes clefs, engendrée par paires de remplissages des lignes d'horizon augmentées dont les formats définissent tableaux clefs, à savoir, la base des caractères de Demazure du type A et les Demazure atomes. Un développement du noyau de Cauchy non symétrique pour le type A, dans le cas de la partition escalier, a été donné par Alain Lascoux en employant la structure des graphes cristallins doublés, et par Amy M. Fu et Alain Lascoux, en se basant aux opérateurs de Demazure, qui a été aussi utilisé pour obtenir des expansions sur diagrammes de Ferrers.

Keywords: Non-symmetric Cauchy kernels, Demazure character, key polynomial, Demazure operator, semi-skyline augmented filling, RSK analogue.

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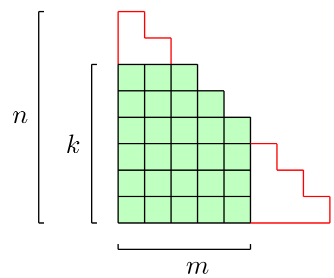
1 Introduction

Given the general Lie algebra $\mathfrak{gl}_n(\mathbb{C})$, and its quantum group $U_q(\mathfrak{gl}_n)$, finite-dimensional representations of $U_q(\mathfrak{gl}_n)$ are also classified by the highest weight. Let λ be a dominant integral weight (that is, a partition) and $V(\lambda)$ the integrable representation with highest weight λ and u_λ the highest weight vector. For a given permutation w in the symmetric group $\mathfrak{S}_n$, minimum for the Bruhat order in the class modulo the stabilizer of λ, the Demazure module is defined to be $V_w(\lambda) := U_q(\mathfrak{g})^{>0}.u_{w\lambda}$, and the Demazure character is the character of $V_w(\lambda)$. Kashiwara (1991) has associated with λ a crystal graph $\mathfrak{B}_\lambda$, which can be realised as a coloured directed graph whose vertices are all semi-standard Young tableaux (SSYTs) of shape λ in the alphabet $[n]$, and the edges are coloured with a colour i, for each pair of crystal operators $f_i,\ e_i$, such that there exists a coloured i-arrow from the vertex P to P' if and only if $f_i(P) = P'$, equivalently, $e_i(P') = P$, for $1 \leq i \leq n-1$. Littelmann (1995) conjectured and Kashiwara (1993) proved that the intersection of a crystal basis of V_λ with $V_w(\lambda)$ is a crystal basis for $V_w(\lambda)$. The resulting subset $\mathfrak{B}_{w\lambda} \subseteq \mathfrak{B}_\lambda$ is called Demazure crystal, and the Demazure character corresponding to λ and w, is the sum of the monomial weights of SSYTs in the Demazure crystal $\mathfrak{B}_{w\lambda}$.

Demazure characters (or key polynomials) are also defined through Demazure operators (or isobaric divided differences). They were introduced by Demazure (1974) for all Weyl groups and were studied combinatorially, in the case of $\mathfrak{S}_n$, by Lascoux and Schützenberger (1990) who produce a crystal structure. The simple transpositions s_i of $\mathfrak{S}_n$ act on vectors $v \in \mathbb{N}^n$ by $s_i v := (v_1, \dots, v_{i+1}, v_i \dots, v_n)$, for $1 \leq i \leq n-1$, and induce an action of $\mathfrak{S}_n$ on $\mathbb{Z}[x_1, \dots, x_n]$ by considering vectors v as exponents of monomials $x^v := x_1^{v_1} x_2^{v_2} \cdots x_n^{v_n}$. Two families of Demazure operators $\pi_i,\ \widehat{\pi}_i$ on $\mathbb{Z}[x_1, \dots, x_n]$ are defined by $\pi_i f = \frac{x_i f - x_{i+1} s_i(f)}{x_i - x_{i+1}}$ and $\widehat{\pi}_i f = \pi_i f - f$, for $1 \leq i \leq n-1$. For the partition λ and $w = s_{i_N} \cdots s_{i_2} s_{i_1}$ a reduced decomposition in $\mathfrak{S}_n$, one defines the type A key polynomials indexed by $w\lambda$, $\kappa_{w\lambda}(x) = \pi_{i_N} \cdots \pi_{i_2} \pi_{i_1} x^\lambda$ and $\widehat{\kappa}_{w\lambda}(x) = \widehat{\pi}_{i_N} \cdots \widehat{\pi}_{i_2} \widehat{\pi}_{i_1} x^\lambda$, the latter consisting of all monomials in $\kappa_{w\lambda}$ which do not appear in $\kappa_{\sigma\lambda}$ for any $\sigma < w$ in the Bruhat order. Thereby key polynomials can be decomposed into non intersecting pieces $\kappa_{w\lambda}(x) = \sum_{\nu \leq w} \widehat{\kappa}_{\nu\lambda}(x)$, where the ordering on permutations is the Bruhat order in $\mathfrak{S}_n$. In Lascoux and Schützenberger (1990) they are called *standard basis* and in Mason (2009) Demazure atoms. The Demazure character corresponding to w and λ can be expressed in terms of the Demazure operator and is equivalent to the key polynomial $\kappa_{w\lambda}$. Lascoux and Schützenberger (1990) have given a combinatorial interpretation for Demazure operators in terms of crystal operators to produce a crystal graph structure. Let P be a SSYT of shape λ and define the set $f_{s_i}(P) := \{f_i^m(P) : m \geq 0\} \setminus \{0\}$. If P is the head of an i-string of the crystal graph $\mathfrak{B}_\lambda$, $\pi_i(x^P)$ is the sum of the monomial weights of all SSYTs in $f_{s_i}(P)$. In particular, when Y is the Yamanouchi tableau of shape λ, the set $f_w(Y) := \{f_{i_N}^{m_N} \dots f_{i_1}^{m_1}(Y) : m_k \geq 0\} \setminus \{0\}$ constitutes the vertices of the Demazure crystal $\mathfrak{B}_{w\lambda}$, and $\kappa_{w\lambda}$ is the sum of all monomial weights over the Demazure crystal. The top of this crystal graph $\widehat{\mathfrak{B}}_{w\lambda} := \mathfrak{B}_{w\lambda} \setminus \bigcup_{\sigma < w} \mathfrak{B}_{\sigma\lambda}$ defines the Demazure atom $\widehat{\kappa}_{w\lambda}(x)$ which is combinatorially characterised by Lascoux and Schützenberger (1990) as the sum of the monomial weights of all SSYTs whose right key is $key(w\lambda)$.

As the sum of the monomial weights over all crystal graph $\mathfrak{B}_\lambda$ gives the Schur polynomial s_λ, each SSYT of shape λ appears in precisely one such polynomial, henceforth, the Demazure atoms form a decomposition of Schur polynomials. Specialising the combinatorial formula for nonsymmetric Macdonald polynomials $E_\gamma(x; q; t)$, given in Haglund et al. (2008), by setting $q = t = 0$, implies that $E_\gamma(x; 0; 0)$ is the sum of the monomial weights of all semi-skyline augmented fillings (SSAF) of shape γ which are fillings of composition diagrams with positive integers, weakly decreasing upwards along columns, and the rows satisfy an inversion condition. These polynomials are also a decomposition of the Schur polynomial s_λ, with $\gamma^+ = \lambda$. Semi-skyline augmented fillings are in bijection with semi-standard Young tableaux of the same content whose right key is the unique key with content the shape of the SSAF, Mason (2006/08). Therefore, Demazure atoms $\widehat{\kappa}_{w\lambda}(x)$ and $E_{w\lambda}(x;0;0)$ are equal, Mason (2009). Semi-skyline augmented fillings also satisfy a variation of the Robinson-Schensted-Knuth algorithm which commutes with the usual RSK and retains its symmetry. We are, therefore, endowed with a machinery to exploit expansions of non-symmetric Cauchy kernels $\prod_{(i,j)\in\eta}(1-x_iy_j)^{-1}$, where the product is over all cell-coordinates (i,j) of the diagram η in the French convention. Our main Theorem 4.2 exhibits a bijection between biwords in lexicographic order, whose biletters are cell-coordinates in a NW-SE diagonal of a rectangle and below it, containing the biggest stair shape, and pairs of SSAFs whose shapes satisfy an inequality in the Bruhat order. This allows to apply this variation of RSK for SSAFs to provide expansions for the green diagram $\eta = (m^{n-m+1}, m-1, \ldots, n-k+1)$, $1 \le m, k \le n$, $n+1 \le m+k$, depicted below. The formulas are explicit in the tableaux generating them.



The paper is organised as follows. In Section 2, we recall the tableau criterion for the Bruhat order in $\mathfrak{S}_n$, and its extension to weak compositions. In Section 3, we review the necessary theory of SSAFs, the variation of Schensted insertion and RSK for SSAFs. In Section 4, we give our main result, Theorem 4.2, and, in the last section, we apply it to the expand the Cauchy kernel for stair-type shapes.

2 Key tableaux a criterion for the Bruhat order in $\mathfrak{S}_n$

Let $\mathbb{N}$ denote the set of non-negative integers. Fix a positive integer n, and define $[n]$ the set $\{1,\ldots,n\}$. A weak composition $\gamma = (\gamma_1 \ldots, \gamma_n)$ is a vector in $\mathbb{N}^n$. If $\gamma_i = \cdots = \gamma_{i+k-1}$, for some $k \ge 1$, then we also write $\gamma = (\gamma_1 \ldots, \gamma_{i-1}, \gamma_i^k, \gamma_{i+k}, \ldots, \gamma_n)$. A partition is a weak composition whose entries are in weakly decreasing order, that is, $\gamma_1 \ge \cdots \ge \gamma_n$. Every composition γ determines a unique partition γ^+ obtained by arranging the entries of γ in weakly decreasing order. A partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ is identified with its Young diagram $dg(\lambda)$ in French convention, an array of left-justified cells with λ_i cells in row i from the bottom, for $1 \le i \le n$. The cells are located in the diagram $dg(\lambda)$ by their row and column indices (i,j), where $1 \le i \le n$ and $1 \le j \le \lambda_i$. A filling of shape λ is a map $T : dg(\lambda) \to [n]$. A semi-standard Young tableau (SSYT) of shape λ is a filling of $dg(\lambda)$ weakly increasing in each row from left to right and strictly increasing up in each column. The content or weight of SSYT T is the weak composition $c(T) = (\alpha_1, \ldots, \alpha_n)$ such that T has α_i cells with entry i. A key is a SSYT such that the set of entries in the $(j+1)^{th}$ column is a subset of the set of entries in the j^{th} column, for all j. There is a bijection in Reiner and Shimozono (1995) between weak compositions in $\mathbb{N}^n$ and keys in the alphabet $[n]$

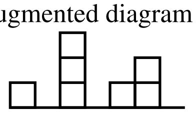
given by $\gamma \to key(\gamma)$, where $key(\gamma)$ is the key such that for all j, the first γ_j columns contain the letter j. Any key tableau is of the form $key(\gamma)$ with γ its content and γ^+ the shape.

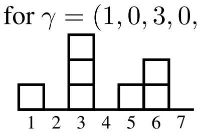
Suppose u and v are two rearrangements of a partition λ. We write $u \leq v$ in the (strong) *Bruhat order* whenever $key(u) \leq key(v)$ for the entrywise comparison. If σ and β are in $\mathfrak{S}_n$, $\sigma \leq \beta$ in the Bruhat order if and only if $\sigma(n, n-1, \ldots, 1) \leq \beta(n, n-1, \ldots, 1)$ as weak compositions.

3 Semi-skyline augmented fillings

3.1 Definitions and properties

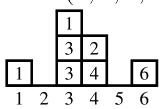
We follow most of the time the conventions and terminology in Haglund et al. (2005, 2008) and Mason (2006/08, 2009). A weak composition $\gamma = (\gamma_1, \ldots, \gamma_n)$ is visualised as a diagram consisting of n columns, with γ_j boxes in column j. Formally, the *column diagram* of γ is the set $dg'(\gamma) = \{(i,j) \in \mathbb{N}^2 : 1 \leq j \leq n, 1 \leq i \leq \gamma_j\}$ where the coordinates are in French convention, the abscissa i indexing the rows, and the ordinate j indexing the columns. (The prime reminds that the components of γ are the columns.) The number of cells in a column is called the height of that column and a cell a in a column diagram is denoted $a = (i,j)$, where i is the row index and j is the column index. The *augmented diagram* of γ, $\widehat{dg}(\gamma) = dg'(\gamma) \cup \{(0,j) : 1 \leq j \leq n\}$, is the column diagram with n extra cells adjoined in row 0. This adjoined row is called the *basement* and it always contains the numbers 1 through n in strictly increasing order. The shape of $\widehat{dg}(\gamma)$ is defined to be γ. For example, the column diagram $dg'(\gamma)$ and the augmented diagram $\widehat{dg}(\gamma)$ for $\gamma = (1,0,3,0,1,2,0)$ are respectively,





An *augmented filling* F of an augmented diagram $\widehat{dg}(\gamma)$ is a map $F : \widehat{dg}(\gamma) \to [n]$, which can be pictured as an assignment of positive integer entries to the non-basement cells of $\widehat{dg}(\gamma)$. Let $F(i)$ denote the entry in the i^{th} cell of the augmented diagram encountered when F is read across rows from left to right, beginning at the highest row and working down to the bottom row. This ordering of the cells is called the *reading order*. A cell $a = (i,j)$ precedes a cell $b = (i',j')$ in the reading order if either $i' < i$ or $i' = i$ and $j' > j$. The reading word of F is obtained by recording the non-basement entries in reading order. The content of an augmented filling F is the weak composition $c(F) = (\alpha_1, \ldots, \alpha_n)$ where α_i is the number of non-basement cells in F with entry i, and n is the number of basement elements. The *standardization* of F is the unique augmented filling that one obtains by sending the i^{th} occurrence of j in the reading order to $i + \sum_{m=1}^{j-1} \alpha_m$. Let $a, b, c \in \widehat{dg}(\gamma)$ three cells situated as follows, $\begin{array}{lc} \boxed{a}\cdots & \boxed{c} \\ \boxed{b} & \end{array}$ where a and c are in the same row, possibly the first row, possibly with cells between them, and the height of the column containing a and b is greater than or equal to the height of the column containing c. Then the triple a, b, c is an *inversion triple of type 1* if and only if after standardization the ordering from smallest to largest of the entries in cells a, b, c induces a counterclockwise orientation. Similarly, consider three cells $a, b, c \in \widehat{dg}(\gamma)$ situated as follows, $\begin{array}{cc} & \boxed{b} \\ \boxed{a}\cdots & \boxed{c} \end{array}$ where a and c are in the same row (possibly the basement) and the column containing b and c has strictly greater height than the column containing a. The triple a, b, c is an *inversion triple of type 2* if and only if after standardization ordering from smallest to largest of the entries in cells $a,\ b,\ c$ induces a clockwise orientation.

Define a *semi-skyline augmented filling* (SSAF) of an augmented diagram $\widehat{dg}(\gamma)$ to be an augmented filling F such that every triple is an inversion triple and columns are weakly decreasing from bottom to top. The shape of the semi-skyline augmented filling is γ and denoted by $sh(F)$. The picture below is an example of a semi-skyline augmented filling with shape $(1, 0, 3, 2, 0, 1)$, reading word 1321346 and content $(2, 1, 2, 1, 0, 1)$.



The entry of a cell in the first row of a SSAF is equal to the basement element where it sits and, thus, in the first row the cell entries increase from left to the right. For any weak composition γ in $\mathbb{N}^n$, there is at least one SSAF with shape γ, by putting γ_i cells with entries i in the top of the basement element i.

In Mason (2006/08) a sequence of lemmas provide several conditions on triples of cells in a SSAF. We recall a property regarding an inversion triple of type 2 which will be used in the proof of our main theorem. Given a cell a in SSAF F define $F(a)$ to be the entry in a.

Remark 3.1 1. *If* $\{a, b, c\}$ *is a type* 2 *inversion triple in* F *then* $F(a) < F(b) \leq F(c)$.

3.2 An analogue of Schensted insertion and RSK for SSAF.

The fundamental operation of the Robinson-Schensted-Knuth (1970) (RSK) algorithm is Schensted insertion which is a procedure for inserting a positive integer k into a SSYT T. Mason (2006/08) defines a similar procedure for inserting a positive integer k into a SSAF F, which is used to describe an analogue of the RSK algorithm. If F is a SSAF of shape γ, we set $F := (F(j))$, where $F(j)$ is the entry in the j^{th} cell in reading order, with the cells in the basement included, and j goes from 1 to $n + \sum_{i=1}^{n} \gamma_i$. If $\hat{j}$ is the cell immediately above j and the cell is empty, set $F(\hat{j}) = 0$. The operation $k \to F$, for $k \leq n$, is defined as follows.

Procedure. The insertion $k \to F$:

1. Set $i := 1$, set $x_1 := k$, set $p_0 = \emptyset$, and set $j := 1$.
2. If $F(j) < x_i$ or $F(\hat{j}) \geq x_i$, then increase j by 1 and repeat this step. Otherwise, set $x_{i+1} := F(\hat{j})$ and set $F(\hat{j}) := x_i$. Set $p_i = (b + 1, a)$, where (b, a) is the j^{th} cell in reading order. (This means that the entry x_i "$bumps$" the entry x_{i+1} from the cell p_i.)
3. If $x_{i+1} \neq 0$ then increase i by 1, increase j by 1, and repeat step 2.
4. Set t_k equal to p_i, which is the termination cell, and terminate the algorithm.

The procedure terminates in finitely many steps and the result is a SSAF. Based on this Schensted insertion analogue, it is given a weight preserving and a shape rearranging bijection Ψ between SSYT and SSAF over the alphabet $[n]$. The bijection Ψ is defined to be the insertion, from right to left, of the column word which consists of the entries of each column, read top to bottom from columns left to rigth, of a SSYT into the empty SSAF with basement $[n]$. The bijection together with the shape of $\Psi(T)$ provides the right key of T, $K_+(T)$, a notion due to Lascoux and Schützenberger (1990).

Theorem 3.1 [Mason (2009)] *Given an arbitrary SSYT* T*, let* γ *be the shape of* $\Psi(T)$. *Then* $K_+(T) = key(\gamma)$.

It should be observed that Willis (2011) gives another way to calculate the right key of a SSYT.

Given the alphabet $[n]$, the RSK algorithm is a bijection between biwords in lexicographic order and pairs of SSYT of the same shape over $[n]$. Equipped with the Schensted insertion anlogue Mason

(2006/08) applies the same procedure to find an analogue Φ of the RSK for SSAF. This bijection has an advantage over the classical RSK because it comes along with the extra pair of right keys.

The two line array $w = \begin{pmatrix} i_1 & i_2 & \cdots & i_l \\ j_1 & j_2 & \cdots & j_l \end{pmatrix}$, $i_r < i_{r+1}$, or $i_r = i_{r+1}$ & $j_r \le j_{r+1}$, $1 \le i, j \le l-1$, with $i_r,\ j_r \in [n]$, is called a biword in lexicographic order over the alphabet $[n]$. The map Φ defines a bijection between the set $\mathbb{A}$ of all biwords w in lexicographic order in the alphabet $[n]$, and pairs of SSAFs whose shapes are rearrangements of the same partition in $\mathbb{N}^n$ and the contents are respectively those of the second and first rows of w. Let $\mathbb{SSAF}$ be the set of all SSAFs with basement $[n]$.

Procedure. The map $\Phi : \mathbb{A} \longrightarrow \mathbb{SSAF} \times \mathbb{SSAF}$. Let $w \in \mathbb{A}$.

1. Set $r := l$, where l is the number of biletters in w. Let $F = \emptyset = G$, where $\emptyset$ is the empty $SSAF$.
2. Set $F := (j_r \to F)$. Let h_r be the height of the column in $(j_r \to F)$ at which the insertion procedure $(j_r \to F)$ terminates.
3. Place i_r on top of the leftmost column of height $h_r - 1$ in G such that doing so preserves the decreasing property of columns from bottom to top. Set G equal to the resulting figure.
4. If $r - 1 \neq 0$, repeat step 2 for $r := r - 1$. Else terminate the algorithm.

Remark 3.2 1. *The entries in the top row of the biword are weakly increasing when read from left to right. Henceforth, if $h_r > 1$, placing i_r on top of the leftmost column of height $h_r - 1$ in G preserves the decreasing property of columns. If $h_r = 1$, the i_r^{th} column of G does not contain an entry from a previous step. It means that number i_r sits on the top of basement i_r.*
2. *Let h be the height of the column in F at which the insertion procedure $(j \to F)$ terminates. Remark 3.1, implies that there is no column of height $h+1$ in F to the right.*

Corollary 3.2 [Mason (2006/08, 2009)] *The RSK algorithm commutes with the above analogue Φ. That is, if (P, Q) is the pair of SSYT produced by RSK algorithm applied to biword w, then $(\Psi(P), \Psi(Q)) = \Phi(w)$, and $K_+(P) = key(sh(\Psi(P)))$, $K_+(Q) = key(sh(\Psi(Q)))$.*

This result is summarised in the following scheme from which, in particular, it is clear the RSK analogue Φ also shares the symmetry of RSK.

$$\begin{tikzcd} & w \arrow[dl, dash, "\mathrm{RSK}"'] \arrow[dr, dash, "\Phi"] & \\ (P,Q) \arrow[rr, dash, "\Psi"'] & & (F,G) \end{tikzcd}$$

$$c(P) = c(Q) = c(F) = c(G),$$
$$sh(F)^+ = sh(G)^+ = sh(P) = sh(Q),$$
$$K_+(P) = key(sh(F)),\ K_+(Q) = key(sh(G)).$$

4 Main Theorem

We give a bijection between biwords, in lexicographic order, whose biletters are cell-coordinates in a NW-SE diagonal of a rectangle diagram, and below it, containing the biggest stair shape, and pairs of SSAFs whose shapes satisfy an inequality in the Bruhat order.

Lemma 4.1 *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ be two weak compositions in $\mathbb{N}^n$, rearrangements of each other, with $key(\beta) \le key(\alpha)$. Given $k \in \{1, \ldots, n\}$, let $k' \in \{1, \ldots, n\}$ be such that $\beta_{k'}$ is the left most entry of β satisfying $\alpha_k = \beta_{k'}$. Then if $\tilde{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k + 1, \ldots, \alpha_n)$ and $\tilde{\beta} = (\beta_1, \beta_2, \ldots, \beta_{k'} + 1, \ldots, \beta_n)$, it holds $key(\tilde{\beta}) \le key(\tilde{\alpha})$.*

Proof: Let $k,\ k' \in \{1, \ldots, n\}$ as in the lemma, and put $\alpha_k = \beta_{k'} = m \geq 1$. (The proof for $m = 0$ is left to the reader. The case of interest for our problem is $m > 0$ which is related with the procedure of map Φ.) This means that k appears exactly in the first m columns of $key(\alpha)$, and k' is the smallest number that does not appear in column $m + 1$ of $key(\beta)$ but appears exactly in the first m columns. Let t be the row index of the cell with entry k' in column m of $key(\beta)$. Every entry less than k' in column m of $key(\beta)$ appears in column $m+1$ as well, and since in a key tableau each column is contained in the previous one, this imply that the first t rows of columns m and $m+1$ of $key(\tilde{\beta})$ are equal. The only difference between $key(\tilde{\beta})$ and $key(\beta)$ is in columns $m + 1$, from row t to the top. Similarly if z is the row index of the cell with entry k in column $m + 1$ of $key(\tilde{\alpha})$, the only difference between $key(\tilde{\alpha})$ and $key(\alpha)$ is in columns $m + 1$ from row z to the top. To obtain column $m + 1$ of $key(\tilde{\beta})$, shift in the column $m + 1$ of $key(\beta)$ all the cells with entries $> k'$ one row up, and add to the position left vacant (of row index t) a new cell with entry k'. The column $m + 1$ of $key(\tilde{\alpha})$ is obtained similarly, by shifting one row up in the column $m + 1$ of $key(\alpha)$ all the cells with entries $> k$ and adding a new cell with entry k in the vacant position. Put $p = \min\{t, z\}$ and $q = \max\{t, z\}$. We divide the columns $m + 1$ in each pair $key(\beta),\ key(\tilde{\beta})$ and $key(\alpha),\ key(\tilde{\alpha})$ into three parts: the first, from row one to row $p - 1$; the second, from row p to row q; and the third, from row $q + 1$ to the top row. The first parts of column $m + 1$ of $key(\tilde{\beta})$ and $key(\beta)$ are the same, equivalently, for $key(\tilde{\alpha})$ and $key(\alpha)$. The third part of column $m+1$ of $key(\tilde{\beta})$ consists of row q plus the third part of $key(\beta)$, equivalently, for $key(\tilde{\alpha})$ and $key(\alpha)$. As columns $m + 1$ of $key(\beta)$ and $key(\alpha)$ are entrywise comparable, the same happens to the third parts of columns $m + 1$ in $key(\tilde{\beta})$ and $key(\tilde{\alpha})$. It remains to analyse the second parts of the pair $key(\tilde{\beta}),\ key(\tilde{\alpha})$ which we split into two cases according to the relative magnitude of p and q.

Case 1. $p = t < q = z$. Let $k' < b_t < \cdots < b_{z-1}$ and $d_t < \cdots < d_{z-1} < k$ be respectively the cell entries of the second parts of columns $m + 1$ in the pair $key(\tilde{\beta}),\ key(\tilde{\alpha})$. By construction $k' < b_t \leq d_t < d_{t+1}$, $b_i < b_{i+1} \leq d_{i+1}$, $t < i < z - 2$, and $b_{z-1} \leq d_{z-1} < k$, and, therefore, the second parts are also comparable.

Case 2. $p = z \leq q = t$. In this case, the assumption on k' implies that the first q rows of columns m and $m + 1$ of $key(\tilde{\beta})$ are equal. On the other hand, since column m of $key(\beta)$ is less or equal than column m of $key(\alpha)$, which is equal to the column m of $key(\tilde{\alpha})$ and in turn is less or equal to column $m + 1$ of $key(\tilde{\alpha})$, forces by transitivity that the second part of column $m + 1$ of $key(\tilde{\beta})$ is less or equal than the corresponding part of $key(\tilde{\alpha})$. $\square$

We illustrate the lemma with $\beta = (3, 2^2, 1, 0^2, 1)$, $\alpha = (2, 0, 3, 0, 1, 2, 1)$, $\tilde{\beta} = (3, 2^3, 0^2, 1)$, and $\tilde{\alpha} = (2, 0, 3, 0, 2^2, 1)$,

$$key(\beta) = \begin{array}{|c|cc}\hline 7 \\ \cline{1-1} 4 \\ \cline{1-2} 3 & \multicolumn{1}{|c|}{3} \\ \cline{1-2} 2 & \multicolumn{1}{|c|}{2} \\ \hline 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{1} \\ \hline \end{array} \leq key(\alpha) = \begin{array}{|c|cc}\hline 7 \\ \cline{1-1} 6 \\ \cline{1-2} 5 & \multicolumn{1}{|c|}{6} \\ \cline{1-2} 3 & \multicolumn{1}{|c|}{3} \\ \hline 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{3} \\ \hline \end{array} \qquad key(\tilde{\beta}) = \begin{array}{|c|cc}\hline 7 \\ \cline{1-2} 4 & \multicolumn{1}{|c|}{4} \\ \cline{1-2} 3 & \multicolumn{1}{|c|}{3} \\ \cline{1-2} 2 & \multicolumn{1}{|c|}{2} \\ \hline 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{1} \\ \hline \end{array} \leq key(\tilde{\alpha}) = \begin{array}{|c|cc}\hline 7 \\ \cline{1-2} 6 & \multicolumn{1}{|c|}{6} \\ \cline{1-2} 5 & \multicolumn{1}{|c|}{5} \\ \cline{1-2} 3 & \multicolumn{1}{|c|}{3} \\ \hline 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{3} \\ \hline \end{array}.$$

Theorem 4.2 *Let w be a biword in lexicographic order in the alphabet $[n]$, and let $\Phi(w) = (F, G)$. For each biletter $\begin{pmatrix} i \\ j \end{pmatrix}$ in w one has $i + j \leq n + 1$ if and only if $key(sh(G)) \leq key(\omega sh(F))$, where ω is the longest permutation of $\mathfrak{S}_n$. Moreover, if the first [respectively the second] row of w is a word in the alphabet $[m]$, with $1 \leq m \leq n$, the shape of G [respectively F] has the last $n - m$ entries equal to zero.*

Proof: *"Only if part"*. We prove by induction on the number of biletters of w. If w is the empty word then F and G are the empty semi-skyline and there is nothing to prove. Let $w' = \begin{pmatrix} i_{p+1} & i_p & \cdots & i_1 \\ j_{p+1} & j_p & \cdots & j_1 \end{pmatrix}$ be a biword in lexicographic order such that $p \geq 0$ and $i_t + j_t \leq n+1$ for all $1 \leq t \leq p+1$, and $w = \begin{pmatrix} i_p \cdots i_1 \\ j_p \cdots j_1 \end{pmatrix}$ such that $\Phi(w) = (F, G)$. Let $F' := (j_{p+1} \to F)$ and h the height of the column in F' at which the insertion procedure terminates. There are two possibilities for h which the third step of the algorithm procedure of Φ requires to consider.

• $h = 1$. It means j_{p+1} is sited on the top of the basement element j_{p+1} in F and therefore i_{p+1} goes to the top of the basement element i_{p+1} in G. Let G' be the semi-skyline obtained after placing i_{p+1} in G. As $i_{p+1} \leq i_t$, for all t, i_{p+1} is the bottom entry of the first column in $key(sh(G'))$ whose remain entries constitute the first column of $key(sh(G)$. Suppose $n+1-j_{p+1}$ is added to the row z of the first column in $key(\omega sh(F))$ by shifting one row up all the entries above it. Let $i_{p+1} < a_1 < \cdots < a_z < a_{z+1} < \cdots < a_l$ and $b_1 < b_2 < \cdots < n+1-j_{p+1} < b_z < \cdots < b_l$ be respectively the cell entries of the first columns in the pair $key(sh(G'))$, $key(\omega sh(F'))$, where $a_1 < \cdots < a_z < \cdots < a_l$ and $b_1 < \cdots < b_z < \cdots < b_l$ are respectively the cell entries of the first columns in the pair $key(sh(G))$, $key(\omega sh(F))$. If $z = 1$, as $i_{p+1} \leq n+1-j_{p+1}$ and $a_i \leq b_i$ for all $1 \leq i \leq l$, then $key(sh(G')) \leq key(\omega sh(F'))$. If $z > 1$, as $i_{p+1} < a_1 \leq b_1 < b_2$, we have $i_{p+1} \leq b_1$ and $a_1 \leq b_2$. Similarly $a_i \leq b_i < b_{i+1}$, and $a_i < b_{i+1}$, for all $2 \leq i \leq z-2$. Moreover $a_{z-1} \leq b_{z-1} < n+1-j_{p+1}$, therefore, $a_{z-1} < n+1-j_{p+1}$. Also $a_i \leq b_i$ for all $z \leq i \leq l$. Hence. $key(sh(G')) \leq key(\omega sh(F'))$.

• $h > 1$. Place i_{p+1} on the top of the leftmost column of height $h-1$. This means by Lemma 4.1 $key(sh(G')) \leq key(\omega sh(F'))$.

"*If part*". We prove the contrapositive statement. If there exists a biletter $\begin{pmatrix} i \\ j \end{pmatrix}$ in w such that $i + j > n+1$, then at least one entry of $key(sh(G))$ is strictly bigger than the corresponding entry of $key(\omega sh(F))$. Let $w = \begin{pmatrix} i_p \cdots i_1 \\ j_p \cdots j_1 \end{pmatrix}$ be a biword in lexicographic order on the alphabet $[n]$, and $\begin{pmatrix} i_t \\ j_t \end{pmatrix}$ the first biletter in w, from right to left, with $i_t + j_t > n+1$. Set $F_0 = G_0 := \emptyset$, and for $d \geq 1$, let (F_d, G_d) be the pair of SSAFs obtained by the procedure of map Φ applied to $\begin{pmatrix} i_d \\ j_d \end{pmatrix}$ and (F_{d-1}, G_{d-1}). First apply the map Φ to the biword $\begin{pmatrix} i_{t-1} \cdots i_1 \\ j_{t-1} \cdots j_1 \end{pmatrix}$ to obtain the pair (F_{t-1}, G_{t-1}) of SSAFs whose right keys satisfy, by the "only if part" of the theorem, $key(sh(G_{t-1})) \leq key(\omega sh(F_{t-1}))$. Now insert j_t to F_{t-1}. As $i_k + j_k \leq n+1$, for $1 \leq k \leq t-1$, $i_k + j_k \leq n+1 < i_t + j_t$ and $i_t \leq i_k,\ 1 \leq k \leq t-1$, then $j_t > j_k,\ 1 \leq k \leq t-1$ and since w is in lexicographic order it implies $i_t < i_{t-1}$. Therefore j_t sits on the top of the basement element j_t in F_{t-1} and i_t sits on the top of the basement element i_t in G_{t-1}. It means that $n+1-j_t$ is added to the first row and first column of $key(\omega sh(F_{t-1}))$ and all entries in this column are shifted one row up. Similarly i_t is added to the first row and first column of $key(sh(G_{t-1}))$ and all the entries in this column are shifted one row up. As $i_t > n+1-j_t$ then the first columns of $key(sh(G_t))$ and $key(\omega sh(F_t))$ respectively, are not entrywise comparable, and we say that we have a "*problem*" in the key-pair $(key(sh(G_t)), key(\omega sh(F_t)))$. From now on "*problem*" means $i_t > n+1-j_t$ in some row of a pair of columns in the key-pair $(key(sh(G_d)), key(\omega sh(F_d)))$, with $d \geq t$. Let $d \geq t$ and denote by J the column with basement j_t in F_d, and by I the column with basement i_t in G_d. Let

$|J|$ and $|I|$ denote respectively the height of J and I, and let r_i and k_i denote the number of columns of height $\geq i \geq 1$, respectively, to the right of J and to the left of I.

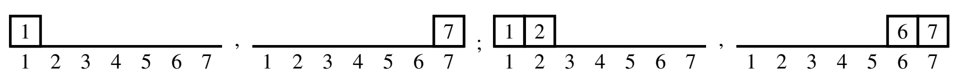
Classification of the "problem": For any $d \geq t$, either there exists $s \geq 1$ such that $|J|, |I| \geq s$, $r_s = k_s > 0$; or $1 \leq |J| \leq |I|$, and there exists $1 \leq f \leq |J|$, such that $k_i > r_i$, for $1 \leq i < f$, and $k_i = r_i = 0$, for $i \geq f$. In the first case, one has a "problem" in the $(r_s+1)^{th}$ rows of the s^{th} columns in the key-pair $(key(sh(G_d)), key(\omega sh(F_d)))$. In the second case, one has a problem in the bottom of the $|J|^{th}$ columns.

The proof of this classification is mainly based on the Remark 3.2 which says that no insertion can terminate, to the left of J, on the top of a column of height $|J|-1$ or $h-1$ such that $r_h > r_{h+1}$, and, on the fact, that an insertion terminating to the right of J or on the top of J will contribute with a cell to the left or to the top of I. Therefore the original "problem" in the key-pair $(key(sh(G_t)), key(\omega sh(F_t)))$ will appear in another row or column in $(key(sh(G_d)), key(\omega sh(F_d)))$ but will never disappear with new insertions. Finally, if the second row of w is over the alphabet $[m]$, there is no cell on the top of the basement of F greater than m. Therefore, the shape of F has the last $n-m$ entries equal to zero. The other case is similar. □

Remark 4.1 *In the previous theorem if the rows of* w *are swapped, one obtains the biword* $\tilde{w}$ *such that* $\Phi(\tilde{w}) = (G, F)$ *with* $key(sh(F) \leq key(\omega sh(G))$. *Moreover, given* $\nu \in \mathbb{N}^n$ *and* $\beta \leq \omega\nu$, *there exists always a pair* (F, G) *of SSAFs with shapes* ν *and* β *respectively.*

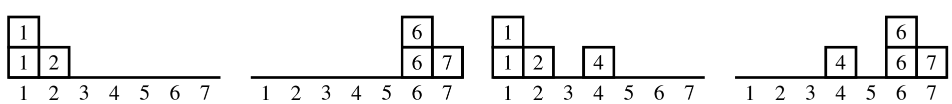
Two examples are now given to illustrate Theorem 4.2.

1. Given $w = \begin{pmatrix} 4 & 6 & 6 & 7 \\ 4 & 1 & 2 & 1 \end{pmatrix}$, $\Phi(w)$ and the key-pair $key(sh(G)) \leq key(\omega sh(F))$ are calculated.



$sh(F_1) = (1, 0^6)$ $sh(G_1) = (0^6, 1)$ $sh(F_2) = (1, 1, 0^5)$ $sh(G_2) = (0^5, 1, 1)$

$key(sh(G_1)) = 7 = key(\omega sh(F_1))$ $key(sh(G_2)) = 67 = key(\omega sh(F_2))$

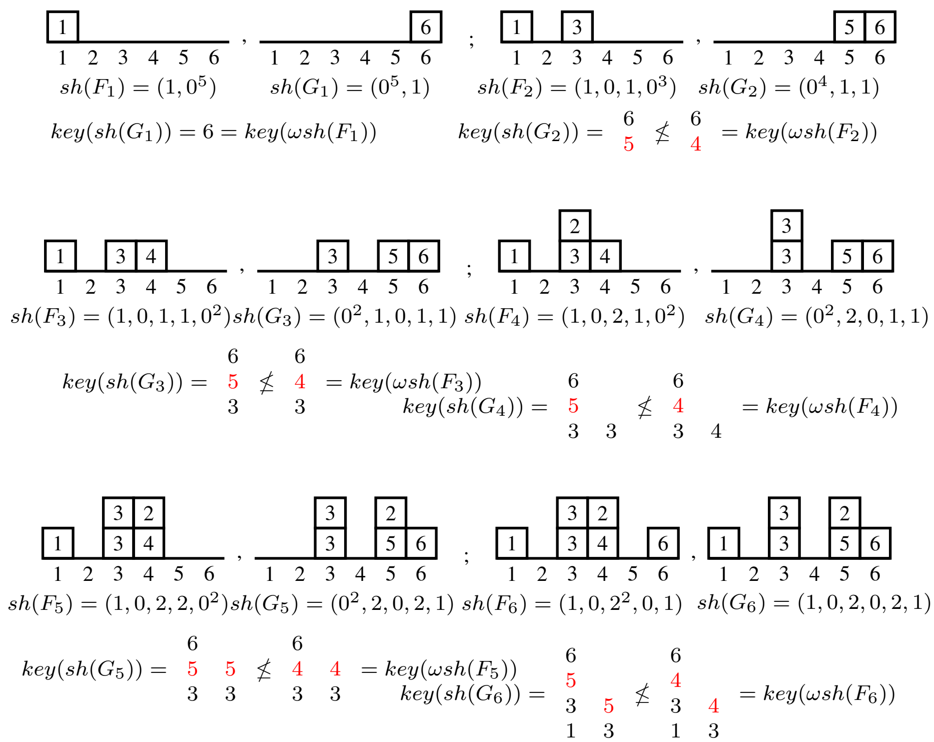


$sh(F_3) = (2, 1, 0^5)$ $sh(G_3) = (0^5, 2, 1)$ $sh(F_4) = (2, 1, 0, 1, 0^3)$ $sh(G_4) = (0^3, 1, 0, 2, 1)$

$key(sh(G_3)) = \begin{array}{cc} 7 & \\ 6 & 6 \end{array} \leq \begin{array}{cc} 7 & \\ 6 & 7 \end{array} = key(\omega sh(F_3))$

$key(sh(G_4)) = \begin{array}{cc} 7 & \\ 6 & \\ 4 & 6 \end{array} \leq \begin{array}{cc} 7 & \\ 6 & \\ 4 & 7 \end{array} = key(\omega sh(F_4))$

2. Let $w = \begin{pmatrix} 1 & 2 & 3 & 3 & 5 & 6 \\ 6 & 3 & 2 & 4 & 3 & 1 \end{pmatrix}$, with $n = 6$, $i_2 = 5 > 6 + 1 - 3$. We calculate $\Phi(w)$ whose key-pair $key(sh(G)), key(\omega sh(F))$ is not entrywise comparable.



5 Expansions of Cauchy kernels in stair-type shapes

The well-known Cauchy identity expresses the product $\prod_{i=1}^{n}\prod_{j=1}^{m}(1-x_iy_j)^{-1}$ as a sum of products of Schur functions in $x=(x_1,x_2,\ldots,x_n)$ and $y=(y_1,y_2,\ldots,y_m)$,

$$\prod_{(i,j)\in(m^n)}(1-x_iy_j)^{-1}=\prod_{i=1}^{n}\prod_{j=1}^{m}(1-x_iy_j)^{-1}=\sum_{\lambda}s_\lambda(x)s_\lambda(y), \tag{1}$$

over all partitions λ of length $\leq \min\{n,m\}$. Using either the RSK correspondence or the Φ correspondence, the Cauchy formula (1) can be interpreted as a bijection between monomials on the left hand side and pairs of SSYTs or SSAFs on the right. Now we replace in the Cauchy kernel the rectangle (m^n) by the stair-type shape $\lambda=(m^{n-m+1},m-1,\ldots,n-k+1)$, with $1\leq m,k\leq n$, and $n+1\leq m+k$. In particular, when $m=n=k$, one has the stair-partition $\lambda=(n,n-1,\ldots,1)$, that is, the cells (i,j) in the NW-SE diagonal of the square diagram (n^n) and below it. Thus $(i,j)\in\lambda$ if and only if $i+j\leq n+1$. Lascoux (2003) has given the following expansion for the non-symmetric Cauchy kernel in the stair shape, using double crystal graphs, and also Fu and Lascoux (2009), based on algebraic properties of Demazure operators,

$$\prod_{i+j\leq n+1}(1-x_iy_j)^{-1}=\sum_{\nu\in\mathbb{N}^n}\widehat{\kappa}_\nu(x)\kappa_{\omega\nu}(y), \tag{2}$$

where κ and $\hat{\kappa}$ are the two families of key polynomials, and ω is the longest permutation of $\mathfrak{S}_n$. Theorem 4.2 allows us to give an expansion of the non-symmetric Cauchy kernel for $\lambda = (m^{n-m+1}, m-1, m-2, \ldots, 1)$, for $1 \le m \le n$, and its conjugate $\overline{\lambda}$, which includes, in particular, the stair case shape (2),

$$\prod_{(i,j)\in\lambda}(1-x_iy_j)^{-1} = \sum_{\substack{\nu\in\mathbb{N}^n\\ \nu=(\nu_1,\ldots,\nu_m,0^{n-m})}} \widehat{\kappa}_\nu(y)\kappa_{\omega\nu}(x), \tag{3}$$

$$\prod_{(i,j)\in\overline{\lambda}}(1-x_iy_j)^{-1} = \sum_{\substack{\nu\in\mathbb{N}^n\\ \nu=(\nu_1,\ldots,\nu_m,0^{n-m})}} \widehat{\kappa}_\nu(x)\kappa_{\omega\nu}(y). \tag{4}$$

Write $\prod_{(i,j)\in\lambda}(1-x_iy_j)^{-1} = \sum_{c\ge 0} x_{i_1}y_{j_1}\cdots x_{i_c}y_{j_c}$, where $(i_l, j_l)\in\lambda,\ i_l+j_l\le n+1,\ 1\le i\le n,$ $1\le j\le m,\ \ 1\le l\le c.$ Each monomial $x_{i_1}y_{j_1}\cdots x_{i_c}y_{j_c}$ is in correspondence with the biword $\binom{i_c\ \cdots\ i_1}{j_c\ \cdots\ j_1}$, whose image by Φ is the pair (F, G) of SSAFs. That is, $x_{i_1}y_{j_1}\cdots x_{i_c}y_{j_c} = y^Fx^G$, where $sh(F)$ has the last $n-m$ entries equal zero, and $sh(G)\le \omega sh(F)$. Therefore,

$$\prod_{(i,j)\in\lambda}(1-x_iy_j)^{-1} = \sum_{\substack{\nu\in\mathbb{N}^n\\ \nu=(\nu_1,\ldots,\nu_m,0^{n-m})}} \sum_{\substack{(F,G)\in SSAF\\ sh(F)=\nu\\ sh(G)\le\omega\nu}} y^Fx^G = \sum_{\substack{\nu\in\mathbb{N}^n\\ \nu=(\nu_1,\ldots,\nu_m,0^{n-m})}} \sum_{\substack{F\in SSAF\\ sh(F)=\nu}} y^F \sum_{\substack{G\in SSAF\\ sh(G)\le\omega\nu}} x^G$$

$$= \sum_{\substack{\nu\in\mathbb{N}^n\\ \nu=(\nu_1,\ldots,\nu_m,0^{n-m})}} \Big(\sum_{\substack{P\in SSYT\\ sh(P)=\nu^+\\ K_+(P)=key(\nu)}} y^P\Big)\Big(\sum_{\substack{Q\in SSYT\\ sh(Q)=\nu^+\\ K_+(Q)=key(\beta)\\ \beta\le\omega\nu}} x^Q\Big) = \sum_{\substack{\nu\in\mathbb{N}^n\\ \nu=(\nu_1,\ldots,\nu_m,0^{n-m})}} \widehat{\kappa}_\nu(y)\kappa_{\omega\nu}(x). \tag{5}$$

The Cauchy kernel expansion (4) for the conjugate shape $\overline{\lambda} = (n, n-1, \ldots, n-m+1)$, with $1\le m\le n$, is a consequence of (3), since $(i,j)\in\overline{\lambda}$ if and only if $(j,i)\in\lambda$, and the symmetry of Φ. When $n = m,\ \lambda = (n, n-1, \ldots, 1) = \overline{\lambda}$, and the symmetry of Φ means the two identities (2) and (3) are equivalent. Finally, as a refinement of (5), we obtain the expansion for the shape $\lambda = (m^{n-m+1}, m-1, \ldots, n-k+1)$, where $1\le m\le k\le n$, and $n+1\le m+k$,

$$\begin{aligned}\prod_{(i,j)\in\lambda}(1-x_iy_j)^{-1} &= \sum_{\substack{\nu\in\mathbb{N}^n\\ \nu=(\nu_1,\ldots,\nu_m,0^{n-m})}} \sum_{sh(F)=\nu} y^F \sum_{\substack{\beta\in\mathbb{N}^n\\ \beta=(\beta_1,\ldots,\beta_k,0^{n-k})\\ \beta\le\omega\nu}} \sum_{sh(G)=\beta} x^G\\ &= \sum_{\substack{\nu\in\mathbb{N}^n\\ \nu=(\nu_1,\ldots,\nu_m,0^{n-m})}} \widehat{\kappa}_\nu(y)\pi^{-1}_{>k}\kappa_{\omega\nu}(x),\end{aligned} \tag{6}$$

where $\pi^{-1}_{>k}\kappa_{\omega\nu}$ is the polynomial weight of the crystal subgraph defined by the colours $1, \ldots, k-1$, in the Demazure crystal graph $\mathfrak{B}_{\omega\nu}$. It means we are considering all the tableaux in the $\mathfrak{B}_{\omega\nu}$ with entries less or equal than k, and so all the tableaux in $\mathfrak{B}_{\omega\nu}$ with right key such that the entries are less or equal than k. It is equivalent to all SSAFs with content in $\mathbb{N}^k$, and shape rearrangement of $\omega\nu$ with zeros in the $n-k$

last entries. For $\overline{\lambda} = (m^{n-m+1}, m-1, \ldots, n-k+1)$, where $1 \leq k \leq m \leq n$, and $n+1 \leq m+k$, one has from (6),

$$\prod_{(i,j)\in\overline{\lambda}} (1-x_iy_j)^{-1} = \prod_{(j,i)\in\lambda} (1-x_iy_j)^{-1} = \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu=(\nu_1,\ldots,\nu_k,0^{n-k})}} \widehat{\kappa}_\nu(x)\pi^{-1}_{>m}\kappa_{\omega\nu}(y),$$

where $\pi^{-1}_{>m}\kappa_{\omega\nu}(y)$ is defined similarly as above, swapping k with m. All these identities are equivalent to those obtained by Lascoux (2003) regarding the shapes discussed here.

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A Divided Difference Operator for the Highest root Hessenberg variety

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Abstract. We construct a divided difference operator using GKM theory. This generalizes the classical divided difference operator for the cohomology of the complete flag variety. This construction proves a special case of a recent conjecture of Shareshian and Wachs. Our methods are entirely combinatorial and algebraic, and rely heavily on the combinatorics of root systems and Bruhat order.

Résumé. Nous construisons un opérateur de différence divisée par la théorie GKM. Cette construction généralise l'opérateur de différence divisée pour la cohomologie de la variété de drapeaux. Cette construction s'avère un cas particulier d'une conjecture récente de Shareshian et Wachs. Nos méthodes sont entièrement combinatoire et algébrique, dèpendent en grande partie de combinatoire des systèmes de racines et de l'ordre de Bruhat.

Keywords: Weyl groups, Bruhat order, root systems, Schubert calculus.

1 Intoduction

This article is an extended abstract of the article [Tef] of the same title. Most of the details of the proofs are omitted.

A classical problem of Schubert calculus is to define explicit classes $\mathcal{S}^{[w]}$ to represent Schubert varieties in cohomology rings of a partial flag variety. For geometric reasons these classes form an additive basis for the cohomology. In equivariant cohomology this problem reduces to finding the polynomials $\mathcal{S}^{[w]}([v])$ which are nonzero only if $[v] \geq [w]$ in Bruhat order. For more general spaces the uniqueness or even existence of generalized Schubert classes named *flow-up classes* is not guaranteed. When they exist it is natural to ask for some combinatorial formula defining the polynomials. This is the type of question we adress here.

A motivating example for our work is the complete flag variety G/B. By a combinatorial construction called *GKM theory* (named after Goresky, Kottwitz and MacPherson) the equivariant cohomology is computed directly from the *Bruhat graph* Γ_W of the Weyl group W (for definitions see Section 2) [GKM98, Tym08]. The Schubert classes classes are constructed by divided difference operators

$$\partial_i : \mathcal{S}^w(u) \longmapsto \frac{\mathcal{S}^w(u) - s_i\mathcal{S}^w(s_i u)}{\alpha_i}.$$

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These operators were first introduced by Berstein, Gelfand, and Gelfand; and Demazure for ordinary cohomology, and Konstant and Kumar generalized them to equivariant cohomology [BGG73, Dem73, KK86]. More recently, employing GKM theory Tymoczko uses a left action of W and defines new divided difference operators [Tym08]. Flow-up classes for G/B are unique, so this construction agrees with the earlier work.

A benefit of divided difference operators is that they are recursive maps. This means if $\mathcal{S}^w$ is known and $s_iw < w$, then $\mathcal{S}^{s_iw} := \partial_i\mathcal{S}^w$. Billey uses this recursion of the Konstant and Kumar operators to define a closed combinatorial formula for the polynomial $\mathcal{S}^w(v)$ [Bil99]. Billey's formula is a positive formula involving the reduced expressions of w obtained as a subexpression of a fixed reduced expression for v [Bil99, Theorem 3].

In this paper *GKM rings* (a combinatorial analog of equivariant cohomology) are defined for certain subgraphs of the Bruhat graph. As with the Bruhat graph these rings construct the equivariant cohomology of algebraic varieties called the *regular semisimple Hessenberg varieties*. Two important examples of regular semisimple Hessenberg varieties are the complete flag variety G/B and the toric variety associated to the Coxeter complex [DMPS92].

Hessenberg varieties were first arose in numerical analysis in the context of calculating the Hessenberg form of a matrix, and have received recent attention in the work of Tymoczko generalizing Springer theory to nilpotent Hessenberg varieties [Spr76, Tym07]. The cohomology ring of regular semisimple Hessenberg varieties carry a representation of W, of which little is known. In fact, it remains an open question when $W \cong \mathfrak{S}_n$ the symmetric group. In this case your author has provided an irreducible decomposition of this representation for a large family called *parabolic Hessenberg varieties* [Tef11].

In another direction, the representation for $\mathfrak{S}_n$ has appeared in a recent conjecture of Shareshian and Wachs in their work on *chromatic quasisymmetric functions* [SW11, Conjecture 5.3]. They conjecture that the under the Frobenius isomorphism between the representation ring of $\mathfrak{S}_n$ and the ring of symmetric functions that the image of the ordinary cohomology ring is the chromatic symmetric function they study.

Our main result (Theorem 3.3) generalizes the divided difference operator for G/B to what we call the *highest root Hessenberg variety*. This result is a model first step toward defining bases which would allow us to investigate the representation on the cohomology (ordinary and equivariant). With this basis in hand we end this paper by announcing that for the highest root Hessenberg variety the Shareshian and Wachs conjecture is true (Theorem 3.12).

Our problem originates in algebraic geometry, but our methods are combinatorial and algebraic, a primary advantage of GKM theory. We will see the construction of divided difference operators and the flow-up classes relies heavily on Bruhat order and root systems. In this abstract to emphasize the combinatorial nature of this construction we have left out the formal definitions of Hessenberg varieties and GKM theory. The curious reader is directed to [DMPS92, Tef11, Tym07] for Hessenberg varieties and to [GKM98, GT09, GZ03, Tym08] for GKM theory.

2 Hessenberg graphs

We begin with the definition of a *Weyl group* W [Hum90]. Let V be a k-dimensional real vector space with a symmetric positive definite bilinear form $(\ , \ )$. A *reflection* in V is a linear map which negates a non-zero vector $\alpha \in V$ and fixes point-wise the hyperplane orthogonal to α. A formula for the reflection through α is $s_\alpha(v) = v - 2(\alpha, v)(\alpha, \alpha)^{-1}\alpha$.

A ***(crystalographic) root system*** in V is a finite set of vectors Φ (called *roots*) which satisfy the following axioms

(1.) $\mathbb{R}\alpha \cap \Phi = \pm\alpha$ for all $\alpha \in \Phi$;

(2.) $s_\alpha \Phi = \Phi$ for all $\alpha \in \Phi$;

(3.) $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$.

The integer $c_{\alpha\beta} := \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ is called a *Cartan integer*. A *base* $\Delta \subset \Phi$ is a basis of V such that for each $\alpha \in \Phi$ the coefficients of the expansion $\alpha = \sum_\Delta c_i \alpha_i$ are either all non-negative or all non-positive.

With a fixed base Δ the *positve roots* Φ^+ are those with all non-negative coefficients and respectively call $\Phi^- = -\Phi^+$ the negative roots. There is a partial order $(\prec)$ on Φ where $\alpha \prec \beta$ means $\beta - \alpha$ is a sum of positive roots. We say $\mathcal{I} \subset \Phi$ is an ***ideal*** if whenever $\beta \in \mathcal{I}$ and $\beta \in \Phi$ with $\beta \prec \alpha$, then $\alpha \in \mathcal{I}$.

The Weyl group W is the group generated by the *simple reflections* $s_i := s_{\alpha_i}$ for $\alpha_i \in \Delta$. For $w \in W$ the *length* $\ell(w)$ is the length of a reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_j}$. Finally, the ***Bruhat graph*** Γ_W has vertices W and edges $u \longrightarrow w$ if $w = s_\alpha u$ for $\alpha \in \Phi^+$ and $w^{-1}\alpha \in \Phi^-$ (or equivalently $\ell(w) > \ell(u)$), and the ***Bruhat order*** $<$ is the transitive closure of the edge relations.

Example 2.1 (The type A_n root system.) *Consider $\mathbb{R}^{n+1}$ with dot product defined on the standard coordinate basis $\mathtt{t_i}$ for $i = 1, \cdots, n+1$. Let V be the span of the roots $\Phi = \{\mathtt{t_i} - \mathtt{t_j} : i \neq j\}$. The simple roots are the $\mathtt{t_i} - \mathtt{t_{i+1}}$ and the positive roots are the $\mathtt{t_i} - \mathtt{t_j}$ for $i < j$. The reflection in $\mathtt{t_i} - \mathtt{t_j}$, denoted $s_{(ij)}$, interchanges $\mathtt{t_i}$ and $\mathtt{t_j}$ and fixes the other $\mathtt{t_k}$. Hence, mapping this reflection to the transposition (ij) defines an isomporism of the Weyl group with $\mathfrak{S}_{n+1}$.*

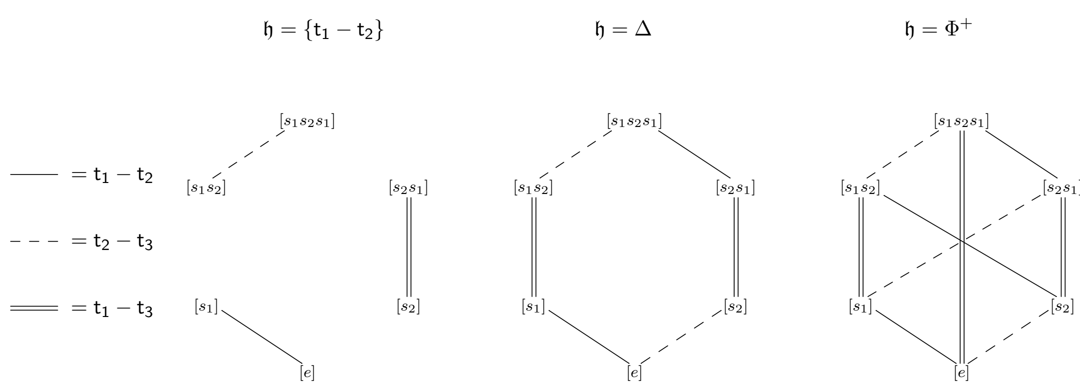

Fig. 1: Hessenberg graphs in type A_2

Definition 2.2 *Let (V, Φ, Δ, W) be as defined above. A* **Hessenberg set** $\mathfrak{h}$ *is the complement of an ideal of $\mathcal{I}_\mathfrak{h} \subset \Phi^+$. The* **Hessenberg graph** $\Gamma_\mathfrak{h}$ *has vertices W and edges $u \longrightarrow w$ if $w = s_\alpha u$ for $\alpha \in \Phi^+$ and $w^{-1}\alpha \in -\mathfrak{h}$. The* **GKM ring of** $\mathfrak{h}$ *is the subring of Maps$(W,\ \mathbb{R}[\alpha_i, \cdots, \alpha_k])$ defined from $\Gamma_\mathfrak{h}$ by*

$$H_T^*(\mathfrak{h}) = \left\{ \mathcal{P} : W \longrightarrow \mathbb{R}[\alpha_1, \cdots, \alpha_k] : \begin{array}{c} \text{for each edge } w \longrightarrow s_\alpha w \\ \mathcal{P}(w) - \mathcal{P}(s_\alpha w) \in \langle \alpha \rangle \end{array} \right\}.$$

The relations $\mathcal{P}(w) - \mathcal{P}(s_\alpha w) \in \langle \alpha \rangle$ are the *GKM conditions*. The GKM ring is a graded ring; we say $\mathcal{P} \in H_T^k(\mathfrak{h})$ if each non-zero polynomial $\mathcal{P}(w)$ is homogeneous of degree k. Elements of $H_T^*(\mathfrak{h})$ are represented by labeling the vertices of $\Gamma_\mathfrak{h}$ by polynomials (cf Figure 2).

The GKM rings carry an action of W obtained by first extending the action of W on Φ to the polynomial ring $\mathbb{R}[\Delta] := \mathbb{R}[\alpha_1, \cdots, \alpha_{k]}$ from which we obtain an action on Maps$(W,\ \mathbb{R}[\Delta])$ by the rule

$$(w \cdot \mathcal{P})(u) = w\mathcal{P}(w^{-1}u) \tag{1}$$

where on the right w is the acting on the *polynomial* $\mathcal{P}(w^{-1}u) \in \mathbb{R}[\Delta]$.

Proposition 2.3 *The GKM ring $H_T^*(\mathfrak{h})$ is W-stable with respect to the action defined in Equation 1.*

Proof: Let $\mathcal{P} \in H_T^*(\mathfrak{h})$ and $w \in W$. We must check the GKM conditions, *i.e.* for every edge $u \longrightarrow s_\alpha u$ is $(w\cdot\mathcal{P})(u) - (w\cdot\mathcal{P})(s_\alpha u) \in \langle\alpha\rangle$. The undirected edge $u \longleftrightarrow s_\alpha u$ is in $\Gamma_\mathfrak{h}$ if and only if the undirected edge $w^{-1}u \longleftrightarrow s_{w^{-1}\alpha}w^{-1}u\,(= w^{-1}s_\alpha u)$ is too. The GKM conditions ignore the edge orientation, so $\mathcal{P}(w^{-1}u) - \mathcal{P}(w^{-1}s_\alpha u) \in \langle w^{-1}\alpha \rangle$ is equivalent to $w\mathcal{P}(w^{-1}u) - w\mathcal{P}(w^{-1}s_\alpha u) \in \langle\alpha\rangle$. The last expression is $(w\cdot\mathcal{P})(u) - (w\cdot\mathcal{P})(v)$ proving the claim. □

This action is easily describe on the graph when $w = s_\alpha$ a reflection; the action of s_α interchanges polynomials across edges corresponding to s_α (some may have been deleted) and permutes the roots.

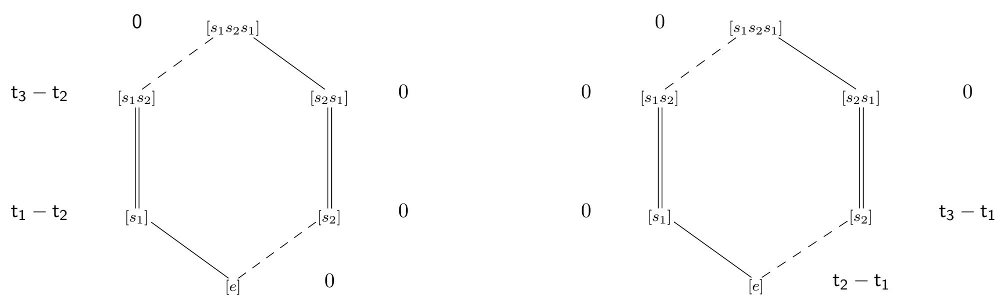

Fig. 2: A class and its image under $s_1\cdot$

In order to study this representation we need to construct a basis of $H_T^*(\mathfrak{h})$. For the GKM ring $H_T^*(\Phi^+)$, this basis consists of Schubert classes $\mathcal{S}^w$ [Tym08]. These are homogenous classes of degree $\ell(w)$ and the polynomial $\mathcal{S}^w(v)$ is nonzero only if $v > w$ in Bruhat order, *i.e.* there is exists a path $w \longrightarrow \cdots \longrightarrow v$ in the Bruhat graph.

These notions are generalized as follows. Fix $\mathfrak{h}$ a Hessenberg set. The ***flow-up of*** $x \in W$ are all the vertices y such that there is a path $x \longrightarrow \cdots \longrightarrow y$ in $\Gamma_\mathfrak{h}$. If y is in the flow up we denote this by $x <_\mathfrak{h} y$, and $\ell_\mathfrak{h}(x) = k$ if there are k edges ending at x.

Definition 2.4 $\mathcal{P}^x \in H_T^{\ell_{\mathfrak{h}}(x)}(\mathfrak{h})$ *is a **flow-up class** at* $x \in W$ *if*

(1) $\mathcal{P}^x(x) = \prod_{s_\alpha x \longrightarrow x} \alpha$, *where the product is over the edges ending at* x*; and*

(2) if $\mathcal{P}^x(y) \neq 0$, *then* $y \geq_{\mathfrak{h}} x$.

These classes have been studied previously by Guillemin and Zara for a general construction of GKM rings [GZ03]. If for every $w \in W$ flow-up classes exist (which is not always true) the family forms a basis of $H_T^*(\mathfrak{h})$ as a free-$\mathbb{R}[\alpha_1, \cdots, \alpha_k]$ module [GZ03]. Fortunately, for $H_T^*(\mathfrak{h})$ flow-up classes always exist.

Theorem 2.5 *Let* $\mathfrak{h}$ *be a Hessenberg set, then the GKM ring* $H_T^*(\mathfrak{h})$ *has a basis of flow-up classes.*

Proof: This follows because the GKM rings $H_T^*(\mathfrak{h})$ are the equivariant cohomology of the regular semisimple Hessenberg variety [Tef11], and [DMPS92, Theorem 8] proves for each i that $\text{rank}_{\mathbb{R}[\Delta]} H_T^i(\mathfrak{h})$ satisfy [GZ03, Theorem 2.1]. □

A drawback of this Theorem (besides its intentionally opaque nature) is that it only guarantees the existence of a flow-up basis. We are still left with the problem of constructing the basis elements. The construction of flow-up classes for GKM rings is important for several reasons. First, an open problem of Schubert calculus is to determine the coefficients c_{uv}^w defined in the expansion of the product of Schubert classes $\mathcal{S}^u\mathcal{S}^v = \sum c_{uv}^w \mathcal{S}^w$, so constructing generalized Schubert classes presents a new context to study this problem. Second, flow-up classes form a basis of the representation of W and without knowing a basis it will be essentially impossible to study the representation.

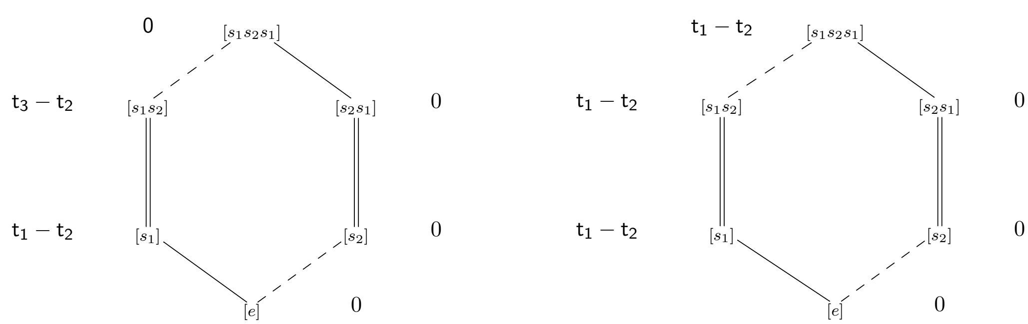

Fig. 3: Non-unique flow-classes

There do exist algorithms for the polynomials $\mathcal{P}^x(y)$ in general GKM rings [GZ03, GT09]. We adopt an alternative approach which emulates the construction of Schubert classes. We use the representation of W on $H_T^*(\mathfrak{h})$ (defined in Equation (1)) to recursively build a new flow-up class. This allows us to define a divided difference operator which as in the classical case recursive defines the flow-up class, *i.e.* if $\mathcal{P}^w$ is know and $s_i w < w$, then $\partial_i^\gamma \mathcal{P}^w = \mathcal{P}^{s_i w}$. A fundamental difficulty for us is that for a fixed $w \in W$ a flow-up class at w is not unique (cf Figure 3), a property enjoyed by Schubert classes.

2.1 $\mathfrak{h}$-inversions

The *inversions of* w *i.e.* $N_w := \left\{\alpha \in \Phi^+ \mid w^{-1}\alpha \in \Phi^-\right\}$ describe the edges ending at w in the Bruhat graph. This motivates the following

Definition 2.6 *Let $\mathfrak{h}$ be a Hessenberg set. For $w \in W$ the set $N_w^{\mathfrak{h}} := \left\{\alpha \in \Phi^+ \mid w^{-1}\alpha \in -\mathfrak{h}\right\}$ is called the **$\mathfrak{h}$-inversions** of w.*

The roots in $N_w^{\mathfrak{h}}$ describe the edges ending at w in $\Gamma_{\mathfrak{h}}$, so knowing only $N_w^{\mathfrak{h}}$ for all $w \in W$ alone determines the GKM ring. Therefore, it is important to understand how $\mathfrak{h}$-inversions change as $w \in W$ varies.

Definition 2.7 *Let $w, v \in W$, we say v is a* **cover** *of w if $w \longrightarrow v \in \Gamma_W$ and*

(1) $\ell(v) = \ell(w) + 1$ and

(2) $v = s_\alpha w$

The following Proposition determines how the set N_w and N_v differ when v is a cover of w (cf. [Tym08]).

Proposition 2.8 *Suppose v is a cover of w, then*

$$N_v = \{\alpha\} \cup (s_\alpha N_w \cap \Phi^+) \cup (N_w \cap s_\alpha \Phi^-).$$

This Proposition generalizes to $\mathfrak{h}$-inversions.

Proposition 2.9 *Suppose v is a cover of w. For $\beta \in N_v$ and*

(1) if $\beta \in s_\alpha N_w \cap \Phi^+$ it follows $\beta \in N_v^{\mathfrak{h}}$ if and only if $s_\alpha\beta \in N_w^{\mathfrak{h}}$ or

(2) if $\beta \in N_w \cap s_\alpha \Phi^-$ it follows when $\beta \in N_v^{\mathfrak{h}}$ then $\beta \in N_w^{\mathfrak{h}}$.

Proof: For Part (1) if $s_\alpha\beta \in N_w$ the equivalence follows because $v^{-1}\beta = w^{-1}s_\alpha\beta$.

For Part (2) we show $v^{-1}\beta \prec w^{-1}\beta$ which by definition of $\mathfrak{h}$ implies $w^{-1}\beta \in \mathfrak{h}$ because $v^{-1}\beta \in \mathfrak{h}$. The hypothesis $\beta \in N_w \cap s_\alpha\Phi^-$ implies $s_\alpha\beta = \beta - c_{\alpha\beta}\alpha \in \Phi^-$, so the Cartan integer $c_{\alpha\beta} > 0$. Therefore, since $v = s_\alpha w$ we have $w^{-1}\beta - v^{-1}\beta = c_{\alpha\beta}w^{-1}\alpha$. Since v is a cover of w and $v = s_\alpha w$ it follows $w^{-1}\alpha \in \Phi^+$ which implies $v^{-1}\beta \prec w^{-1}\beta$. $\square$

Corollary 2.10 *Let v be a cover of w. If $\alpha \in \Delta$ and $\alpha \in N_v^{\mathfrak{h}}$, then $N_v^{\mathfrak{h}} = \{\alpha\} \cup s_\alpha N_w^{\mathfrak{h}}$, otherwise if $\alpha \notin N_v^{\mathfrak{h}}$, then $N_v^{\mathfrak{h}} = s_\alpha N_w^{\mathfrak{h}}$.*

Corollary 2.11 *Let v be a cover of w, then $|N_v| - |\Phi^- \setminus \mathfrak{h}| \leq |N_v^{\mathfrak{h}}| \leq |N_w^{\mathfrak{h}}| + 1$*

The next Proposition determines the values of flow-up classes at the covers in the Bruhat order. It is key to constructing a family of flow-up classes later.

Proposition 2.12 *Let $\mathcal{P}$ be any flow-up class at w. Suppose v is a cover of w, then $\mathcal{P}(v)$ can be determined as follows*

(1) if $v^{-1}(\alpha) \notin \mathfrak{h}$ then $\mathcal{P}(v) = 0$ (i.e. the edge $w \longrightarrow v \in \Gamma_W$ is deleted in $\Gamma_{\mathfrak{h}}$); otherwise

(2) if $\alpha \in \Delta \cap N_v^{\mathfrak{h}}$ then $\mathcal{P}(v) = s_\alpha \mathcal{P}(w)$;

(3) if $|N_v^{\mathfrak{h}}| = |N_w^{\mathfrak{h}}| + 1$ then

$$\mathcal{P}(v) = \prod_{\beta \in N_v^{\mathfrak{h}} \setminus \{\alpha\}} \beta; \text{ or}$$

(4) if $|N_v^{\mathfrak{h}}| \leq |N_w^{\mathfrak{h}}|$ then

$$\mathcal{P}(v) = f \prod_{\beta \in N_v^{\mathfrak{h}} \setminus \{\alpha\}} \beta$$

for some $f \in \mathbb{R}[\Delta]$ of degree $|N_w^{\mathfrak{h}}| - |N_v^{\mathfrak{h}} \setminus \{\alpha\}|$ with $f \equiv \prod_{\mu \in (N_w^{\mathfrak{h}} \cap N_v) - N_v^{\mathfrak{h}}} \mu \pmod{\langle \alpha \rangle}$.

Proof: Use Proposition 2.9 and the GKM conditions to determine these values. $\square$

3 Highest root Hessenberg sets

Suppose Φ is an irreducible root system, *i.e.* Φ cannot be expressed as a disjoint union $\Phi = \Psi \cup \Psi'$ both of which are root systems. For Φ irreducible there exists a unique **highest root** $\gamma \in \Phi^+$ such that $\alpha \prec \gamma$ for all $\alpha \in \Phi$ [Hum90, Section 2.9(3)]. If $\mathfrak{h}_\gamma = \Phi^+ \setminus \{\gamma\}$, then $\mathfrak{h}_\gamma$ is a Hessenberg set.

For $w \in W$ let $N_w^\gamma = N_w^{\mathfrak{h}_\gamma}$ and $\ell_\gamma(w) = \ell_{\mathfrak{h}_\gamma}(w)$. We will be working with both the partial order defined by the flow-up $<_\gamma$ and the Bruhat order $<$. Working with the highest root Hessenberg set simplifies much of the variation which occurs between N_w and N_w^γ. For example

Lemma 3.1 *Suppose $v > w \in W$. We have $\ell_\gamma(w) = \ell_\gamma(v)$ if and only if v is a cover of w; $N_w = N_w^\gamma$; and there exists $\beta \in N_v$ such that $v^{-1}\beta = -\gamma$.*

Proof: The converse follows by definition. Therefore, suppose $\ell_\gamma(w) = \ell_\gamma(v)$. Since $|\mathfrak{h}| = |\Phi^+| - 1$ we have inequality $\ell(v) - 1 \leq \ell_\gamma(v) = \ell_\gamma(w) < \ell(v)$, which implies $\ell_\gamma(v) = \ell(v) - 1$. Therefore, there exists a $\beta \in N_v$ such that $v^{-1}\beta = -\gamma$. Further, the equality $\ell_\gamma(v) = \ell_\gamma(w)$ forces equality in $\ell_\gamma(v) = \ell(v) - 1 \geq \ell(w) \geq \ell_\gamma(w)$. Hence, $\ell(v) = \ell(w) + 1$, *i.e.* v is a cover of w and $N_w = N_w^\gamma$. $\square$

This Lemma with Proposition 2.9 identifies an inversion $\beta \in N_w^\gamma \cap N_v$ such that $v^{-1}\beta = -\gamma$. For a fixed β, the v of Lemma 3.1 is unique.

Corollary 3.2 *Suppose $v > w$ and $\beta \in N_w^\gamma \cap N_v$. If $\ell_\gamma(w) = \ell_\gamma(v)$ and $v^{-1}\beta = -\gamma$, then v is unique.*

We are now ready to state the main Theorem of this paper.

Theorem 3.3 *These exist $\mathbb{R}[\Delta]$-module divided difference operators $\partial_i^\gamma : H_T^*(\mathfrak{h}_\gamma) \longrightarrow H_T^*(\mathfrak{h}_\gamma)$ and a family of flow-up classes $\{\mathcal{P}^w\}_{w \in W}$ such that*

$$\partial_i^{\gamma}\mathcal{P}^w = \begin{cases} \mathcal{P}^{s_i w} & \text{if } s_i w < w; \\ 0 & \text{if } s_i w > w. \end{cases}$$

Further, if $w = s_{i_1}\cdots s_{i_n}$ is any reduced expression for $w \in W$, then the operator $\partial_w := \partial_{i_1}\cdots\partial_{i_n}$ is well-defined. In other words, if $w = s_{j_1}\cdots s_{j_n}$ is another reduced expression for w, then

$$\partial_{i_1}\cdots\partial_{i_n} = \partial_{j_1}\cdots\partial_{j_n}.$$

3.1 Proof of Theorem 3.3

In order to prove Theorem 3.3 we give an explicit formula for the divided difference operator. With this we work by induction on the length function $\ell(w)$ to define simultaneously the action of the simple reflection $s_i\cdot$ on the previously defined flow-up classes AND define a new flow-up class satisfying Theorem 3.3.

The base case of our induction is the longest element $w_\circ \in W$ (cf [Hum90, Theorem 1.8]) for which it is straightforward to define a flow-up class. Since $N_{w_\circ} = \Phi^+$ it follows $N_{w_\circ}^{\mathfrak{h}} = \mathfrak{h}$, so $\mathcal{P}^{w_\circ}$ is the class whose value at $w_\circ$ is the product of the roots in $\mathfrak{h}$ and 0 otherwise. Proceeding by induction, suppose for all $w \in W$ with $\ell(w) \geq k$ that flow-up classes satisfying Theorem 3.3 have been defined.

First a bit of notation, we say $s_\alpha w \lessdot w$ if $s_\alpha w < w$ in Bruhat order and the edge $s_\alpha w \longrightarrow w$ has been deleted in $\Gamma_{\mathfrak{h}}$, or in other words $w^{-1}\alpha = -\gamma$.

Definition 3.4 (Formula for ∂_i^{γ}) *Let $w \in W$ with $\ell(w) = k$ and suppose $\{\mathcal{P}^u\}_{\ell(u)\geq k}$ are flow-up classes in $H_T^*(\mathfrak{h}_\gamma)$. For each $s_i \in \Delta$ define the i^{th} divided difference operator by*

$$\partial_i^{\gamma}\mathcal{P}^w = \begin{cases} s_i\cdot\mathcal{P}^w & \text{if } s_i w \lessdot w; \\ \dfrac{\mathcal{P}^w - s_i\cdot\mathcal{P}^w + c_{\alpha\alpha_i}\left(\mathcal{P}^v - \mathcal{P}^{s_i v}\right)}{\alpha_i} & \text{if } s_i w < w; \\ 0 & \text{if } s_i w > w, \end{cases} \tag{2}$$

where $c_{\alpha\alpha_i}$ is the Cartan integer of $s_\alpha(\alpha_i)$, and when $v \in W$ exists it is the unique cover of w such that $\ell_\gamma(w) = \ell_\gamma(v)$ and $v^{-1}\alpha_i = -\gamma$.

Example 3.5 *For the type A_2 root system the highest root set is $\mathfrak{h} = \Delta$. The family of flow-up classes constructed by Definition 3.4 is described in Table 3.5. The reader is encouraged to replicate this data, for guidance Γ_Δ is given in Figure 1.*

$\mathcal{P}^v(w)$	$w=e$	s_1	s_2	s_1s_2	s_2s_1	$s_1s_2s_1$
$\mathcal{P}^e$	1	1	1	1	1	1
$\mathcal{P}^{s_1}$	0	$\mathsf{t}_1-\mathsf{t}_2$	0	$\mathsf{t}_3-\mathsf{t}_2$	0	0
$\mathcal{P}^{s_2}$	0	0	$\mathsf{t}_2-\mathsf{t}_3$	0	$\mathsf{t}_2-\mathsf{t}_1$	0
$\mathcal{P}^{s_1s_2}$	0	0	0	$\mathsf{t}_1-\mathsf{t}_3$	0	$\mathsf{t}_1-\mathsf{t}_2$
$\mathcal{P}^{s_2s_1}$	0	0	0	0	$\mathsf{t}_1-\mathsf{t}_3$	$\mathsf{t}_2-\mathsf{t}_3$
$\mathcal{P}^{s_1s_2s_1}$	0	0	0	0	0	$(\mathsf{t}_1-\mathsf{t}_2)(\mathsf{t}_2-\mathsf{t}_3)$

Tab. 1: The family of flow-up classes for $\mathfrak{h} = \Delta$ in type A_2.

It is not obvious that ∂_i^γ is a $\mathbb{R}[\Delta]$-module homorphism, this is a consequence of the next theorem. We prove this in [Tef], its proof requires a careful case-by-case analysis.

Theorem 3.6 *Suppose Theorem 3.3 has uniquely determined $\mathcal{P}^w$ for $w \in W$ with $\ell(w) \geq k$, then*

$$s_i \cdot \mathcal{P}^w = \begin{cases} \mathcal{P} & \text{if } s_i w \lessdot w \text{ or } s_i w \gtrdot w \\ \mathcal{P}^w & \text{if } s_i w > w \\ \mathcal{P}^w - \alpha_i \mathcal{P} + c_{\alpha\alpha_i}(\mathcal{P}^v - \mathcal{P}^{s_i v}) & \text{if } s_i w < w \end{cases}$$

where $\mathcal{P}$ is a flow-up class at $s_i w$ and when $v \in W$ exists it is the unique cover of w such that $\ell_\gamma(w) = \ell_\gamma(v)$ and $v^{-1}\alpha_i = -\gamma$.

This provides the inductive step to Theorem 3.3. The consequence is that the class $\mathcal{P}$ is a new flow-up class at $s_i w$ where $\ell(s_i w) = k - 1$. Repeating this process for all w' with $\ell(w') = k - 1$ proves the induction. In fact this process uniquely defines a flow-up class at $s_i w$. Before proving the uniqueness we show ∂_i^γ is a module map. Since, $\mathbb{R}[\Delta]$ is a UFD over $\mathbb{R}[\alpha_1, \cdots, \widehat{\alpha}_i, \cdots, \alpha_k]$, where $\widehat{\alpha}_i$ means α_i is removed, dividing by α_i is well-defined. Therefore, in the third case of Theorem 3.6 there exists a well-defined flow-up class such that

$$\partial_i^\gamma \mathcal{P}^w := \frac{\mathcal{P}^w - s_i \cdot \mathcal{P}^w + c_{\alpha\alpha_i}(\mathcal{P}^v - \mathcal{P}^{s_i v})}{\alpha_i} = \mathcal{P}.$$

This proves

Corollary 3.7 *The divided difference operator $\partial_i^\gamma : H_T^*(\mathfrak{h}_\gamma) \longrightarrow H_T^*(\mathfrak{h}_\gamma)$ is a $\mathbb{R}[\Delta]$-module homorphism.*

The next is a technical Lemma we need frequently (cf. [Hum90, Lemma 5.11]). It is important because it says that left multiplication by a simple transposition s_i preserves the flow-up, *i.e.* if v is a cover of w, and $w \longrightarrow s_i w$ if and only if $s_i v$ is a cover of $s_i w$.

Lemma 3.8 (Diamond Lemma) *Let v be a cover of w. Suppose $\ell(s_i w) = \ell(w)+1 = \ell(v)$ and $s_i w \neq v$, then both $s_i v > v$ and $\ell(s_i v) = \ell(s_i w) + 1$. Further, $w \longrightarrow v$ is in $\Gamma_\mathfrak{h}$ if and only if $s_i w \longrightarrow s_i v$ is in $\Gamma_\mathfrak{h}$.*

Next, we prove the flow-up class $\mathcal{P}$ defined in Theorem 3.3 is uniquely determined. This requires a new induction, which again our base case is $w_\circ$ which is uniquely defined. Suppose by induction that if $\ell(v) > k$ that the flow-up classes are uniquely determined, and let $\ell(w) = k$. This next Proposition determines the polynomials at all the covers of w for the flow-up class $\mathcal{P}$ defined in Definition 3.4. We include the proof as an example of how to prove these results.

Proposition 3.9 *Suppose $\mathcal{P}$ is a flow-up classes at $w \in W$ defined by Definition 3.4, i.e. $\mathcal{P} = \partial_i^\gamma \mathcal{P}^{s_i w}$ for $\ell(s_i w) = \ell(w) + 1$. Whenever v covers w, then*

$$\mathcal{P}(v) = \begin{cases} s_\alpha \mu \displaystyle\prod_{\beta \in N_v^\gamma \setminus \{\alpha\}} \beta & \text{if } \alpha \in N_v^\gamma, \\ 0 & \text{if } \alpha \notin N_v^\gamma \end{cases} \tag{3}$$

where $\mu \in (N_w^\gamma \cap N_v) \setminus N_v^\gamma$ or $\mu = 1$ otherwise.

Proof: When μ exists it is the root associated to the edge $s_\mu v \longrightarrow v$ missing in $\Gamma_{\mathfrak{h}_\gamma}$. Define the polynomial $q = s_\alpha \mu \prod_{\beta \in N_v^\gamma \setminus \{\alpha\}} \beta$. When $\alpha \notin N_v^\gamma$ or $\ell_\gamma(v) = \ell_\gamma(w) + 1$ (when μ does not exist) this is proved in Proposition 2.12(1)-(3).

Therefore, we may assume $\ell_\gamma(v) = \ell_\gamma(w)$, $\alpha \in N_v^\gamma$, and $\mu \in (N_w^\gamma \cap N_v) \setminus N_v^\gamma$ exists. We work by induction, for $w = w_\circ$ there is nothing to prove. Suppose by induction for $w' \in W$ with $\ell(w') > k$ the result is true, and let $w \in W$ with $\ell(w) = k$. In this case, there exists a simple reflection s_i so that $\ell(s_i w) = \ell(w)+1$. By Lemma 3.8, $s_i v > v$, and $s_i v = s_{s_i\alpha} w$, therefore $\mathcal{P}^{s_i w}(s_i v)$ satisfies the inductive hypothesis.

If $s_i w \gtrdot w$, then by Equation (2) $\mathcal{P} := s_i \cdot \mathcal{P}^{s_i w}$. It follows from Corollary 2.10 that $N_{s_i v}^\gamma = s_i N_v^\gamma$, so deduce $\mathcal{P}^{s_i w}(s_i v) = s_i q$. This shows $\mathcal{P}(v) = s_i \mathcal{P}^{s_i w}(s_i v) = q$ as desired.

If $s_i w > w$, Equation (2) gives $\alpha_i \mathcal{P} = \mathcal{P}^{s_i w} - s_i \cdot \mathcal{P}^{s_i w} + c_{\beta\alpha_i}(\mathcal{P}^{v'} - \mathcal{P}^{s_i v'})$ where v' may or may not exist. Evaluating both sides of this expression at v we claim

$$\alpha_i \mathcal{P}(v) = -s_i \mathcal{P}^{s_i w}(s_i v).$$

To prove this first note $\mathcal{P}^{s_i w}(v) = \mathcal{P}^{v'}(v) = 0$ since v is not in the flow-up. Next, when $\mathcal{P}^{s_i v'}(v) \neq 0$ since $\ell(s_i v') = \ell(v)$ it must be that $s_i v' = v$. This leads to a contradiction. The hypothesis on v' is that $s_i v' \lessdot v'$, but $s_i v' = v$ and $\ell_\gamma(v) = \ell(v) - 1$. This means at vertex v in $\Gamma_{\mathfrak{h}_\gamma}$ there are two edges deleted from the $\Gamma_{\mathfrak{h}_\gamma}$, *i.e.* v^{-1} maps two roots to $-\gamma$, a contradiction since v is invertible.

Therefore, $\alpha_i \mathcal{P}(v) = -s_i \mathcal{P}^{s_i w}(s_i v)$, and the inductive hypothesis shows $\mathcal{P}^{s_i w}(s_i v)$ is the product $s_{s_i\alpha} s_i \mu = s_i s_\alpha \mu$ times the product of the roots in $N_{s_i v}^\gamma = \{\alpha_i\} \cup s_i N_v^\gamma$ except $s_i \alpha$. Equivalently $\mathcal{P}(v)$ is the product of $s_\alpha \mu$ and the roots in N_v^γ except α, which is q as desired. □

This will prove no matter how you arrive at w the class $\mathcal{P}$ is uniquely determined.

Corollary 3.10 *The flow-up classes defined by Definition 3.4 are unique, i.e. if $sv = w = tu$ where s, t are simple reflections, then $\partial_s \mathcal{P}^{sv} = \mathcal{P} = \partial_t \mathcal{P}^{tu}$.*

Proof: Let $\partial_s \mathcal{P}^{sv} = \mathcal{P}$ and $\partial_t \mathcal{P}^{su} = \mathcal{P}'$. We want to show $\mathcal{P} = \mathcal{P}'$. Since $\mathcal{P}'$ is non-zero only on $x >_\gamma w$ and homogeneous of degree $\ell_\gamma(w)$ we have a $\mathbb{R}[\Delta]$-linear combination

$$\mathcal{P}' = \sum_{\substack{x >_\gamma w \\ \ell_\gamma(x) \leq \ell_\gamma(w)}} f_x \mathcal{P}^x + f_w \mathcal{P}.$$

Evaluating both sides of this expression at w we have $\mathcal{P}'(w) = f_w \mathcal{P}(w)$, but $\mathcal{P}'(w) = \mathcal{P}(w)$ which determines that $f_w = 1$. Next, evaluation at any $x >_\gamma w$ in the summation gives

$$\mathcal{P}'(x) = f_x \mathcal{P}^x(x) + \mathcal{P}(x).$$

Since $\mathcal{P}^x(x) \neq 0$ and $\mathcal{P}'(x) = \mathcal{P}(x)$ by Proposition 3.9 we conclude all the $f_x = 0$. Therefore $\mathcal{P}' = \mathcal{P}$. □

As a consequence we can define a unique class $\mathcal{P}^{s_i w} := \partial_i^\gamma \mathcal{P}^w$, and by induction this proves the first half of Theorem 3.3; that is there exists a family of flow-up classes $\{\mathcal{P}^w\}_W$ and divided difference operators ∂_i^γ. Next, we prove the second half, that is if $w = s_{i_1} \cdots s_{i_n}$ is a reduced expression, then $\partial_w = \partial_{i_1} \cdots \partial_{i_n}$ is independent of the reduced expression.

Theorem 3.11 *If $w \in W$ and $w = s_{i_1} \cdots s_{i_n}$ a reduced expression, then $\partial_w := \partial_{i_1} \cdots \partial_{i_n}$ is independent of the reduced expression, that is if $w = s_{j_1} \cdots s_{j_n}$, then*

$$\partial_{i_1} \cdots \partial_{i_n} = \partial_{j_i} \cdots \partial_{\jmath_n}.$$

Proof Sketch: Since any two expressions for $w \in W$ can obtained by a sequence of braid relations [Hum90, Theorem 1.9] it suffices to check if the Theorem is true for the braid relations. Therefore, suppose that $v = stst \cdots = tsts \cdots$ and let u and u' be suffixes of v, *i.e.* $su = v = tu'$ such that $\ell(u) = \ell(v) - 1 = \ell(u')$. Then, ∂_u and $\partial_{u'}$ are well-defined since they have unique expressions in terms of the simple reflections. To show ∂_v is well-defined it suffices to show $\partial_s \partial_u = \partial_t \partial_{u'}$ by acting on the basis $\{\mathcal{P}^w\}_{w \in W}$.

Now, we need only check the $x \in W$ such that $\ell(vx) = \ell(x) - \ell(v)$ or else by induction with Definiton 3.4

$$\partial_s \partial_u \mathcal{P}^x = 0 = \partial_t \partial_{u'} \mathcal{P}^x.$$

In this case, we have $\ell(ux) = \ell(x) - \ell(u)$ and $\partial_u \mathcal{P}^x = \mathcal{P}^{ux}$ (respectively for u'). The product vx is well-defined, so conclude $sux = vx < ux$ if and only if $tu'x = vx < u'x$. An application of Corollary 3.10 proves $\partial_s \partial_u \mathcal{P}^x = \partial_s \mathcal{P}^{ux} = \mathcal{P}^{vx} = \partial_t \mathcal{P}^{u'x} = \partial_t \partial_{u'} \mathcal{P}^x$. $\square$

3.2 Future work

This work provides a model construction of divded difference operators and flow-up classes for all the GKM rings $H_T^*(\mathfrak{h})$. A difficulty which needs to be overcome before we can obtain the equivalent of Theorem 3.6 we need a better understanding of flow-up classes then Proposition 2.12 provides. Namely, here we take advantage that covers of w essential determine $\mathcal{P}^w$. In general, we will need to understand flow-up classes further up the flow of w then just at the covers.

An advantage of this approach is that it does determine the representation on $H_T^*(\mathfrak{h})$ when Φ is simply-laced, i.e. all the Cartan integers $c_{\alpha\beta} = \pm 1$. This next result will appear in [Tef].

Theorem 3.12 *Suppose Φ is simply-laced. Let $m_V = \frac{|W|}{|\Phi|}$ and $m_{\mathbb{R}} = |W| - |\Delta| m_V$, then as a W-module*

$$H_T^*(\mathfrak{h}) = (V^{\oplus m_V} \bigoplus \mathbb{R}^{\oplus m_{\mathbb{R}}}) \bigotimes_{\mathbb{R}} \mathbb{R}[\Delta],$$

where V is the reflection representation (cf. Section 2) , $\mathbb{R}$ is the trivial representation and $\mathbb{R}[\Delta]$ is the polynomial representation of W.

In the case where Φ is the type A root system we have

Theorem 3.13 *If Φ is the type A_{n-1} root system, then as a $\mathfrak{S}_n$-module*

$$H_T^*(\mathfrak{h}) = (V^{\oplus (n-2)!} \bigoplus \mathbb{R}^{\oplus (n-1)!(n-1)}) \bigotimes_{\mathbb{R}} \mathbb{R}[\Delta].$$

Furthermore, this proves the Shareshian-Wachs conjecture [SW11, Conjecture 5.3].

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A Parking Function Setting for Nabla Images of Schur Functions

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Abstract. In this article, we show how the compositional refinement of the "Shuffle Conjecture" due to Jim Haglund, Jennifer Morse, and Mike Zabrocki can be used to express the image of a Schur function under the Bergeron-Garsia Nabla operator as a weighted sum of a suitable collection of "Parking Functions." The validity of these expressions is, of course, going to be conjectural until the compositional refinement of the Shuffle Conjecture is established.

Résumé. Dans cet article, nous montrons comment le raffinement compositionel de la "Conjecture Shuffle" due à Jim Haglund, Jennifer Morse et Mike Zabrocki peut être utilisé pour exprimer l'image d'une fonction de Schur sous l'opérateur Nabla de Bergeron-Garsia comme une somme pondérée d'un ensemble convenable de "fonctions parking." La validité de ces expressions, bien sûr, va être conjecturale jusqu'à ce que le raffinement de la composition de la "Conjecture Shuffle" est établie.

Keywords: Parking Function, Nabla, Hall-Littlewood operators

1 Introduction

Parking Functions in the $n \times n$ lattice square are represented in the computer by two line arrays

$$PF \quad = \quad \begin{bmatrix} v_1 & v_2 & \cdots & v_n \\ u_1 & u_2 & \cdots & u_n \end{bmatrix}$$

with $u_1, u_2, \ldots, u_n$ integers satisfying

$$u_1 = 0 \quad \text{and} \quad 0 \le u_i \le u_{i-1} + 1$$

and $V = (v_1, v_2, \ldots, v_n)$ a permutation in the symmetric group S_n satisfying

$$u_i = u_{i-1} + 1 \implies v_i > v_{i-1}.$$

We will denote by $\sigma(PF)$ the permutation obtained by successive right to left readings of the components of the vector $V = (v_1, v_2, \ldots, v_n)$ according to decreasing values of $u_1, u_2, \ldots, u_n$. We will call $\sigma(PF)$ the "diagonal word" of PF. We will also let $ides(PF)$ denote the descent set of the inverse of $\sigma(PF)$.

† work accomplished with NSF support

This given, each Parking Function is assigned the "weight"

$$w(PF) \quad = \quad t^{area(PF)} q^{dinv(PF)} Q_{ides(PF)}[X]$$

where

$$area(PF) \quad = \quad \sum_{i=1}^{n} u_i, \tag{1}$$

$$dinv(PF) = \sum_{1 \le i < j \le n} \chi(u_i = u_j \,\&\, v_i < v_j) + \sum_{1 \le i < j \le n} \chi(u_i = u_j + 1 \,\&\, v_i > v_j),$$

and, for a subset $S \subset \{1, 2, \cdots, , n-1\}$, $Q_S[X]$ denotes Gessel's fundamental quasi-symmetric function.

In the figure below we have a Parking Function as we usually conveniently depict it. The vector $U = (u_1, u_2, \ldots, u_n)$ is on its left and the vector $V = (v_1, v_2, \ldots, v_n)$ is on its right. The shaded cells give the "main diagonal" (or 0-diagonal) of PF. The numbers in the lattice cells are the "cars". The path along whose vertical steps we have set the cars is the supporting "Dyck path" of PF. The components of $U = (u_1, u_2, \ldots, u_n)$ give the orders of the diagonals containing the cars. Note that reading the cars by diagonals from right to left starting with the highest diagonal gives

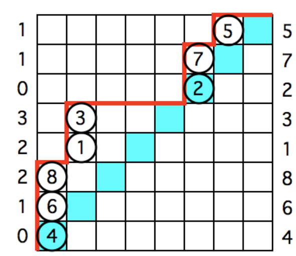
$$\sigma(PF) \quad = \quad 3\,1\,8\,5\,7\,6\,2\,4.$$

Thus

$$ides(PF) = \{2, 4, 6, 7\}. \tag{2}$$

It is easily seen that the sum in (1) gives the total number of cells between the supporting Dyck path and the main diagonal. Note that two cars in the same diagonal with the car on the left smaller than the car on the right will contribute a unit to $dinv(PF)$ called a "primary dinv". Likewise, a car on the left that is bigger than a car on the right with the latter in the adjacent lower diagonal contributes a unit to $dinv(PF)$ called a "secondary dinv".

$$PF = \begin{bmatrix} 4 & 6 & 8 & 1 & 3 & 2 & 7 & 5 \\ 0 & 1 & 2 & 2 & 3 & 0 & 1 & 1 \end{bmatrix} \quad \Longleftrightarrow \tag{3}$$



Thus for the Parking Function in (3) we have

$$area(PF) = 10, \quad dinv(PF) = 4,$$

which together with (2) give

$$w(PF) \; = \; t^{10} q^4 Q_{\{2,4,6,7\}}[X].$$

In Haglund et al. (2012) Haglund, Morse and Zabrocki introduce an additional statistic, the "diagonal composition" of a Parking function, which we denote by "$p(PF)$." This is the composition whose parts determine the position of the zeros in the vector $U = (u_1, u_2, \ldots, u_n)$, or equivalently give the lengths of the segments between successive diagonal touches of its Dyck path. For the present example we have

$$p(PF) = (5, 3).$$

Denoting by $\mathcal{PF}_n$ the collection of Parking Functions in the $n \times n$ lattice square, one of the compositional refinements of the Shuffle conjecture due to Haglund-Morse-Zabrocki in Haglund et al. (2012) states that for any composition $p = (p_1, p_2, \ldots, p_k)$ of n we have

$$\nabla \mathbf{C}_{p_1} \mathbf{C}_{p_2} \cdots \mathbf{C}_{p_k} 1 \quad = \sum_{\substack{PF \in \mathcal{PF} \\ p(PF)=(p_1,p_2,\ldots,p_k)}} t^{area(PF)} q^{dinv(PF)} Q_{ides(PF)}[X] \tag{4}$$

where " ∇ " is the Bergeron-Garsia operator introduced in [1] and, for each integer a, $\mathbf{C}_a$ is the operator plethystically defined by setting for any symmetric function $P[X]$

$$\mathbf{C}_a P[X] \quad = \quad \left(\tfrac{-1}{q}\right)^{a-1} \textstyle\sum_{k \geq 0} P\big[X - \tfrac{1-1/q}{z}\big] z^k h_k[X] \Big|_{z^a}.$$

Using the device θ_i which acts on the operator $\mathbf{C}_p = \mathbf{C}_{p_1} \mathbf{C}_{p_2} \cdots \mathbf{C}_{p_k}$ according to the formula

$$\theta_i \mathbf{C}_p \;=\; \mathbf{C}_{p-e_i}$$

where e_i is the coordinate vector with 1 in the i^{th} position, we will show that

Theorem 1 *For any composition $(p_1, p_2, \ldots, p_k)$ we have*

$$s_{p_1,p_2,\ldots,p_k}[X] \;=\; (-q)^{p_1+\cdots+p_k-k} \prod_{1 \leq i < j \leq n} \big(1 - \theta_j / q\theta_i\big)\, \boldsymbol{C}_p\, \boldsymbol{1} \tag{5}$$

where "$s_{p_1,p_2,\ldots,p_k}[X]$" denotes the Schur function indexed by the composition $(p_1, p_2, \ldots, p_k)$.

To get across the significance of this identity it is best to have a close look at a few special cases. To begin, for $k = 2$ with $a \geq b \geq 1$, $p_1 = a$ and $p_2 = b$, (5) becomes

$$s_{a,b}[X] \;=\; (-q)^{a+b-2}(1 - \theta_2/q\theta_1)\mathbf{C}_{[a,b]}\mathbf{1} \;=\; (-q)^{a+b-2}(\mathbf{C}_{[a,b]}\mathbf{1} - \mathbf{C}_{[a+1,b-1]}\mathbf{1}/q). \tag{6}$$

Similarly, for $k = 3$ with $a \geq b \geq c \geq 1$, $p_1 = a$, $p_2 = b$ and $p_3 = c$, we get

$$\begin{aligned} s_{a,b,c}[X] \;&=\; (-q)^{a+b+c-3}(1 - \theta_2/q\theta_1)(1 - \theta_3/q\theta_1)(1 - \theta_3/q\theta_2)\mathbf{C}_a\mathbf{C}_b\mathbf{C}_c\mathbf{1} \;= \\ &=\; (-q)^{a+b+c-3}\Big(\mathbf{C}_{[a,b,c]}\mathbf{1} \;-\; \mathbf{C}_{[a,b+1,c-1]}\mathbf{1}/q \;-\; \mathbf{C}_{[a+1,b,c-1]}\mathbf{1}/q \;-\; \mathbf{C}_{[a+1,b-1,c]}\mathbf{1}/q \\ &\qquad\qquad +\; \mathbf{C}_{[a+1,b,c-1]}\mathbf{1}/q^2 \;+\; \mathbf{C}_{[a+1,b+1,c-2]}\mathbf{1}/q^2 \\ &\qquad\qquad\quad +\; \mathbf{C}_{[a+2,b-1,c-1]}\mathbf{1}/q^2 \;-\; \mathbf{C}_{[a+2,b,c-2]}\mathbf{1}/q^3\Big). \end{aligned} \tag{7}$$

These identities suggest that it may be possible to obtain a Parking Function interpretation for ∇ of a Schur function via the compositional refinement of the Shuffle conjecture in (4).

For example, from (6) we obtain that

$$\nabla s_{4,3} \,=\, q^4\big(\nabla \mathbf{C}_5\mathbf{C}_2\mathbf{1} \,-\, q\nabla \mathbf{C}_4\mathbf{C}_3\mathbf{1}\big).$$

From the Haglund-Morse-Zabrocki conjectures, it follows that the sum of the weights of the collection $\Pi[5,2]$ of Parking Functions with diagonal composition $[5,2]$ should be given by the polynomial $\nabla\mathbf{C}_5\mathbf{C}_2$ and the sum of the weights of the collection $\Pi[4,3]$ of Parking Functions with diagonal composition $[4,3]$ should be given by $\nabla\mathbf{C}_4\mathbf{C}_3$. This given, to obtain a combinatorial setting for $\nabla s_{4,3}$ it suffices to construct an injection ϕ of $\Pi[4,3]$ into $\Pi[5,2]$ that preserves $area$ and $ides$ but increases $dinv$ by one unit, and then identify the complementary collection $\Pi[5,2]\setminus\phi(\Pi[4,3])$: thereby obtaining the identity

$$\nabla s_{[4,3]} \,=\, q^4 \sum_{PF\in\Pi[5,2]\setminus(\phi\Pi[4,3])} t^{area(PF)} q^{dinv(PF)} Q_{ides(PF)}[X]$$

A look at the identity in (7) suggests that a combinatorial setting for $\nabla s_{a,b,c}$ may be obtained by carrying out an "inclusion-exclusion" process on the collections of Parking Functions with diagonal compositions the indices of the operators occurring in (7).

The task of carrying out the injections yielding such Parking Function settings for the Nabla image of Schur functions is the topic of the author's doctoral thesis which is still in progress. In this article we show how this can be systematically carried out in a variety of examples of Schur functions such as those indexed by two-row or two-column partitions.

We should mention that in Loehr and Warrington (2008) another combinatorial model is conjectured for Nabla Schurs by means of labeled nested Dyck paths. The Loehr-Warrington model stems naturally from the Jacobi-Trudi formula for Schur functions, while the present model stems naturally from Theorem (1) which may be viewed as a q-analogue of Jacobi-Trudi. It would make an interesting combinatorial project to see how their model relates to ours, in particular whether their nested labeled Dyck paths can be naturally unraveled into collections of Parking Functions. Even more importantly, if the latter unraveling is carried out, any progress in the resolution of the Loehr-Warrington conjecture may be conducive to significant progress in the resolution of the Haglund-Morse-Zabrocki conjectures.

This writing is divided into three sections. In the first section we give a proof of Theorem (1), in the second section we give some examples in the two part partition cases, in the third and final section we show how the **B** operators of Haglund et al. (2012) can be used to give a Parking Function setting to the Nabla image of two-column Schur functions.

2 A q-analogue of the Jacobi-Trudi identity

In this section it will be convenient to use plethystic notation in dealing with symmetric function identities. A brief introduction to this notational device can be found in the first section of Garsia et al. (2011). Recall that the "row adder" for Schur functions is the operator

$$\mathbf{S}_a P[X] \,=\, P[X-\tfrac{1}{z}]\Omega[zX]\Big|_{z^a} \tag{8}$$

where

$$\Omega[zX] \,=\, \sum_{m\geq 0} z^m h_m[X]$$

is the generating function of the homogeneous symmetric functions in the alphabet X.

Proposition 1 *For any integral vector $p = (p_1, p_2, \dots, p_k)$ we have*

$$s_{p_1,p_2,\dots,p_k}[X] \;=\; \Omega[Z_k X] \prod_{1 \le i < j \le k} (1 - z_j/z_i) \Big|_{z_1^{p_1}, z_2^{p_2}, \dots, z_k^{p_k}} \tag{9}$$

where $Z_k = z_1 + z_2 + \cdots + z_k$.

Proof:

It is well known (see Macdonald (1995)) that

$$s_{p_1,p_2,\dots,p_k}[X] \;=\; \mathbf{S}_{p_1}\mathbf{S}_{p_2}\cdots\mathbf{S}_{p_k}\,\mathbf{1} \tag{10}$$

From the definition in (8), with $F[X] = \mathbf{1}$, we get,

$$\begin{aligned}\mathbf{S}_{p_1}\mathbf{S}_{p_2}\,\mathbf{1} \;=\; \mathbf{S}_{p_1}\Omega[z_2X]\Big|_{z_2^{p_2}} \;&=\; \Omega\big[z_2(X - \tfrac{1}{z_1})\big]\Omega[z_1X]\Big|_{z_1^{p_1}, z_2^{p_2}} \\ &=\; \Omega[(-z_2/z_1)]\Omega[z_1X + z_2X]\Big|_{z_1^{p_1}, z_2^{p_2}} \;=\; (1 - z_2/z_1)\Omega[z_1X + z_2X]\Big|_{z_1^{p_1}, z_2^{p_2}}\end{aligned}$$

and by iteration we obtain

$$\mathbf{S}_{p_1}\mathbf{S}_{p_2}\cdots\mathbf{S}_{p_k}\,\mathbf{1} \;=\; \Omega[Z_kX] \prod_{1 \le i < j \le k} (1 - z_j/z_i)\Big|_{z_1^{p_1}, z_2^{p_2}, \dots, z_k^{p_k}}.$$

Therefore (9) follows from (10).

□

We are now in a position to give our

Proof of Theorem 1:

Recall that by definition we have set for any symmetric function $F[X]$

$$\mathbf{C}_a F[X] \;=\; (-\tfrac{1}{q})^{a-1} F\big[X - \tfrac{1-1/q}{z}\big]\Omega[zX]\Big|_{z^a}.$$

Then

$$\begin{aligned}(-q)^{p_1+p_2-2}\mathbf{C}_{p_1}\mathbf{C}_{p_2}F[X] \;&=\; (-q)^{p_1-1}\mathbf{C}_{p_1}F\big[X - \tfrac{1-1/q}{z_2}\big]\Omega[z_2X]\Big|_{z_2^{p_2}} \\ &=\; F\big[X - \tfrac{1-1/q}{z_1} - \tfrac{1-1/q}{z_2}\big]\Omega\big[z_2(X - \tfrac{1-1/q}{z_1})\big]\Omega[z_1X]\Big|_{z_1^{p_1}, z_2^{p_2}} \\ &=\; F\big[X - \tfrac{1-1/q}{z_1} - \tfrac{1-1/q}{z_2}\big]\Omega[-z_2\tfrac{1-1/q}{z_1}]\Omega[z_1X + z_2X]\Big|_{z_1^{p_1}, z_2^{p_2}}.\end{aligned}$$

Since $\Omega[-z_2\frac{1-1/q}{z_1}] \;=\; \frac{1-z_2/z_1}{1-z_2/qz_1}$, we finally have that

$$(-q)^{p_1+p_2-2}\mathbf{C}_{p_1}\mathbf{C}_{p_2}F[X] \;=\; F\big[X - \tfrac{1-1/q}{z_1} - \tfrac{1-1/q}{z_2}\big]\Omega[z_1X + z_2X]\,\frac{1-z_2/z_1}{1-z_2/qz_1}\Big|_{z_1^{p_1}, z_2^{p_2}}.$$

By iteration we obtain

$$(-q)^{p_1+\cdots+p_k-k}\mathbf{C}_{p_1}\cdots\mathbf{C}_{p_k}F[X] \;=\; F\big[X-\textstyle\sum_{i=1}^{k}\frac{1-1/q}{z_i}\big]\Omega[Z_kX]\prod_{1\le i<j\le k}\,\frac{1-z_j/z_i}{1-z_j/qz_i}\,\Big|_{z_1^{p_1},\dots,z_k^{p_k}}$$

with

$$Z_k = z_1+z_2+\cdots+z_k.$$

Now note that for $F[X]=1$ and for any vector $a=(a_1,a_2,\dots a_k)$ we get

$$\begin{aligned}(-q)^{p_1-a_1+\cdots+p_k-a_k-k}\,&\mathbf{C}_{p_1-a_1}\cdots\mathbf{C}_{p_k-a_k}\;\mathbf{1}\;=\\ &=\prod_{1\le i<j\le k}\;\frac{1-z_j/z_i}{1-z_j/qz_i}\;\Omega[XZ_k]\;\Big|_{z_1^{p_1-a_1}z_2^{p_2-a_2}\cdots z_k^{p_k-a_k}}\\ &=\prod_{1\le i<j\le k}\;\frac{1-z_j/z_i}{1-z_j/qz_i}\;\Omega[XZ_k]\,z_1^{a_1}z_2^{a_2}\cdots z_k^{a_k}\;\Big|_{z_1^{p_1}z_2^{p_2}\cdots z_k^{p_k}}.\end{aligned}\tag{11}$$

Recalling that the device θ_i acts on the operator $\mathbf{C}_p=\mathbf{C}_{p_1}\mathbf{C}_{p_2}\cdots\mathbf{C}_{p_k}$, according to the formula

$$\theta_i\mathbf{C}_p \;=\; \mathbf{C}_{p-e_i}$$

we can rewrite (11) as

$$\begin{aligned}(-q)^{p_1+\cdots+p_k-k}&(-\theta_1/q)^{a_1}(-\theta_2/q)^{a_2}\cdots(-\theta_k/q)^{a_k}\mathbf{C}_p\;\mathbf{1}\;=\\ &=\prod_{1\le i<j\le k}\;\frac{1-z_j/z_i}{1-z_j/qz_i}\;\Omega[XZ_k]z_1^{a_1}z_2^{a_2}\cdots z_k^{a_k}\;\Big|_{z_1^{p_1}z_2^{p_2}\cdots z_k^{p_k}}.\end{aligned}$$

Thus

$$(-q)^{p_1+\cdots+p_k-k}\prod_{1\le i<j\le n}\big(1-\theta_j/q\theta_i\big)\mathbf{C}_p\;\mathbf{1}\;=\;\prod_{1\le i<j\le n}\;\big(1-z_j/z_i\big)\;\Omega[XZ_k]\;\Big|_{z_1^{p_1}z_2^{p_2}\cdots z_k^{p_k}}.$$

This identity combined with (9) gives

$$(-q)^{p_1+\cdots+p_k-k}\prod_{1\le i<j\le n}\big(1-\theta_j/q\theta_i\big)\mathbf{C}_p\;\mathbf{1}\;=\;s_{p_1,2,\dots,p_n}[X]$$

as desired. □

3 The two row Schur function case

To illustrate the combinatorial reasoning that is needed to give a Parking Function setting to the Nabla image of a two row Schur function we will carry out in full detail the case of $\nabla(s_{[n-3,3]})$ which is the simplest non trivial case.

Let $n > 5$. Let $\mathcal{NS}_3$ denote the collections of Parking Functions with diagonal composition $[n-2, 2]$ whose Dyck path terminates according to one of the following three patterns and the cars adjacent to the north steps are required to satisfy the inequalities indicated by the arrows.

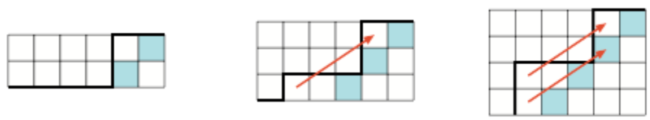


Fig. 1: In the second pattern $v_{n-2} < v_n$ and in the third pattern $v_{n-2} < v_n$ and $v_{n-3} < v_{n-1}$.

Theorem 2 *Assume that the compositional refinement of the Shuffle conjecture in (4) holds. Then,*

$$\nabla(-1)^n s_{[n-3,3]} = q^{n-3} \sum_{PF \in \mathcal{NS}_3} t^{area(PF)} q^{dinv(PF)} Q_{ides(PF)}[X]$$

Proof:

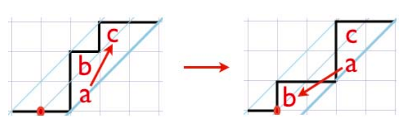
We start by constructing an injection ϕ_3 from the collection $\Pi[n-3,3]$ of Parking Functions with diagonal composition $[n-3,3]$ to $\Pi[n-2,2]$, those Parking Functions with diagonal composition $[n-2,2]$. Furthermore, this injection will preserve the $area$ and $ides$ of the Parking Functions while increasing the $dinv$ by exactly 1.

Let PF be a Parking Function with diagonal composition $[n-3,3]$. There are two possible shapes for the rightmost three columns of the Dyck path of PF.

Fig. 2: The rightmost possible columns.

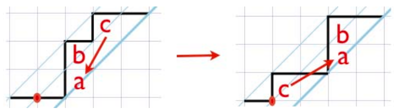
In either shape, since $n > 5$, PF does not hit the diagonal twice in a row. Hence the two steps preceding the last three columns must both be going east.

Between these two shapes, we will have five cases for defining $\phi_3(PF)$. For the first, suppose that the last four columns of PF are as in the left side of the figure below. Suppose also that $a < c$. Then replacing the last four columns of PF with the right side of this figure gives a legal Parking Function. Let this be denoted by PF'



Notice that $area(PF) = area(PF')$. Furthermore, these two Parking Functions have the same diagonal word and hence $ides(PF) = ides(PF')$. Notice also that all pairs contributing to the $dinv$ (primary or secondary) are unchanged except that the pair (b, a) now also contributes to the secondary $dinv$. Hence $dinv(PF') = dinv(PF) + 1$. Therefore we will set $\phi_3(PF) = PF'$.

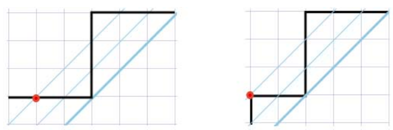
For the second case, suppose that the last four columns of PF are as in the left side of the figure below with $a > c$. Again, let PF' be the Parking Function obtained by replacing the last four columns of PF with the columns of the right side of the figure below.



We have that $c < a < b$, so PF' is a valid Parking Function. The $area$ is unchanged. However, the diagonal word has changed. In particular, b and c have switched places and we need to show that $ides(PF)$ has not been changed. To see this recall that the descent of the inverse of a permutation is the set of all j such that $j + 1$ occurs before j in the permutation. This given, the interchange of the order of c and b alters $ides(PF)$ only if c and b are consecutive but this is excluded by the inequalities $c < a < b$.

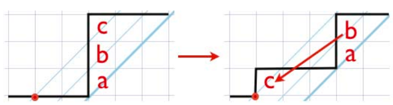
It remains to show that $dinv(PF') = dinv(PF) + 1$. But this is true since the pair (c, b) contributes to $dinv(PF')$, though it did not contribute to $dinv(PF)$, and the pair (c, a) is not contributing to the secondary $dinv$ since $c < a$. Hence we can again let $\phi_3(PF) = PF'$.

We have exhausted the cases corresponding to the right side of **Fig. 2**. Therefore we will move on to the left side. As we noted before, the two steps preceeding the last three columns must both be east steps. However the step preceeding that could be either north or east.



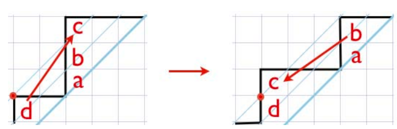
Note that on the right side of the figure above, the step directly before the ones shown must be an east step since $n > 5$ and another north step would result in hitting the diagonal.

For the third case, suppose the last three columns of PF are as on the left. Then we can construct PF' in the usual way corresponding to the diagram below.



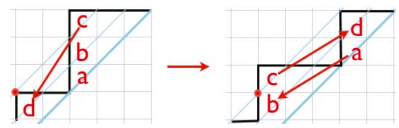
Note $area(PF) = area(PF')$. Also the diagonal word is unchanged for $ides(PF) = ides(PF')$. Furthermore, the only change to the $dinv$ is that (c, b) contributes to the secondary $dinv$ in PF'. Hence $dinv(PF') = dinv(PF) + 1$. Therefore we again set $\phi_3(PF) = PF'$.

For the fourth case, suppose that the last three columns of PF are as on the left side of the figure below. Suppose also that $c > d$. Let PF' be the Parking Function obtained by replacing the last five columns with those on the right side of the figure below.



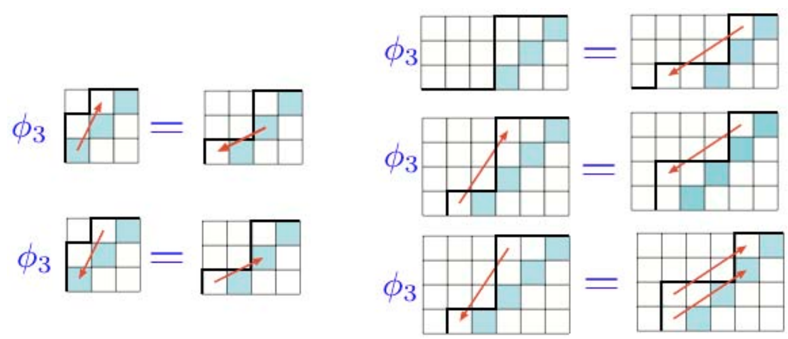
We have $area(PF) = area(PF')$, and $ides(PF) = ides(PF')$ since the diagonal word is unchanged. As in the last case, the only change to the $dinv$ being that (c, b) contributes to $dinv(PF')$ but not to $dinv(PF)$. Hence $\phi_3(PF) = PF'$.

Now for the fifth and final case, suppose that PF is as in the left side of the figure below and $c < d$. Then $a < b < c < d$, so replacing the last 5 columns with those of the right side of the figure gives a Parking Function PF'



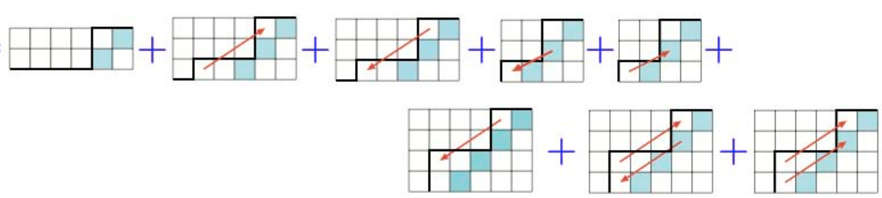
Clearly $area(PF) = area(PF')$. The diagonal word has changed however. As in the second case, two labels have switched places in the diagonal word, namely b and d. Again, these two labels are not consecutive ($b < c < d$) so, arguing in just the same way as in the second case, we see that $ides(PF) = ides(PF')$. It remains to consider the change to the $dinv$. Since no labels have been moved to a different diagonal, the only changes to $dinv$ must occur due to pairs of the labels a, b, c, d. In PF, the pair (b, d) does not contribute to the primary $dinv$ while the pair (a, d) contributes 1 to the secondary $dinv$. In PF', the pairs (b, d) and (b, a) contribute to the primary and secondary $dinv$, respectively, but the pair (a, d) no longer contributes. Hence $dinv(PF') = dinv(PF) + 1$ as desired. Therefore we let $\phi_3(PF) = PF'$.

The following display summarizes the action of ϕ_3 in each of the five cases



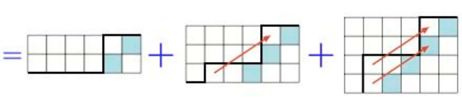
We can thus see that ϕ_3 is an injection $\Pi[n-3, 3]$ into $\Pi[n-2, 2]$ since the images are disjoint sub collections of $\Pi[n-2, 2]$. Moreover note that $\Pi[n-2, 2]$ may be partitioned as follows

$\Pi[n-2, 2]$ =



We can thus deduce that

$\Pi[n-2, 2] \setminus \phi_3\Pi[n-3, 3]$ =



Now by (6), and the Haglund-Morse-Zabrocki conjectures we finally obtain that

$$\begin{aligned}(-1)^n\nabla(s_{[n-3,3]}) &= q^{n-3}\Big(\sum_{PF\in\Pi[n-2,2]} w(PF) - q\sum_{PF\in\Pi[n-3,3]} w(PF)\Big)\\ &= q^{n-3}\Big(\sum_{PF\in\Pi[n-2,2]} w(PF) - \sum_{PF\in\Pi[n-3,3]} w(\phi_3(PF))\Big)\\ &= q^{n-3}\sum_{PF\in\mathcal{NS}_3} t^{area(PF)}q^{dinv(PF)}Q_{ides(PF)}[X].\end{aligned}$$

□

4 The Two Column Schur function case

In the Haglund-Morse-Zabrocki paper Haglund et al. (2012) another Hall-Littlewood type operator "$\mathbf{B}_b$" is introduced whose action on a symmetric function $F[X]$ is defined by setting

$$\mathbf{B}_b = \omega\widetilde{\mathbf{B}}_b\omega \tag{12}$$

with

$$\widetilde{\mathbf{B}}_b F[X] = F\big[X - \tfrac{1-q}{z}\big]\Omega[zX]\Big|_{z^b} \tag{13}$$

The significance of these operators in the present context stems from the following identity

Proposition 2 *For any integral pair of integers $a > b \geq 1$ we have*

$$s_{2^b,1^{a-b}}[X] = \boldsymbol{B}_a\boldsymbol{B}_b\,\boldsymbol{1} - q\,\boldsymbol{B}_{a+1}\boldsymbol{B}_{b-1}\,\boldsymbol{1} \tag{14}$$

Proof: Note that from (13) for $F = \mathbf{1}$ we get

$$\widetilde{\mathbf{B}}_b\,\mathbf{1} = h_b[X]$$

Thus, again from (13) it follows that

$$\widetilde{\mathbf{B}}_a\widetilde{\mathbf{B}}_b\,\mathbf{1} = h_b\big[X - \tfrac{1-q}{z}\big]\Omega[zX]\Big|_{z^a} = \sum_{r=0}^{b} h_{b-r}[X]h_r[q-1]h_{r+a}[X]$$

Likewise we get

$$\widetilde{\mathbf{B}}_{a+1}\widetilde{\mathbf{B}}_{b-1}\,\mathbf{1} = \sum_{r=0}^{b-1} h_{b-1-r}[X]h_r[q-1]h_{r+1+a}[X] = \sum_{r=1}^{b} h_{b-r}[X]h_{r-1}[q-1]h_{r+a}[X]$$

Now it can easily be shown that we have $h_r[q-1] = q^r - q^{r-1}$ if $r > 0$ and $h_r[q-1] = 1$ if $r = 0$. Thus using these identities we may write

$$\widetilde{\mathbf{B}}_a\widetilde{\mathbf{B}}_b\,\mathbf{1} = h_b[X]h_a[X] + \sum_{r=1}^{b} h_{b-r}[X](q^r - q^{r-1})h_{r+a}[X]$$

and

$$q\widetilde{\mathbf{B}}_{a+1}\widetilde{\mathbf{B}}_{b-1}\,\mathbf{1} = qh_{b-1}[X]h_{1+a}[X] + \sum_{r=2}^{b} h_{b-r}[X](q^r - q^{r-1})h_{r+a}[X]$$

By subtraction we get

$$\begin{aligned}\widetilde{\mathbf{B}}_a\widetilde{\mathbf{B}}_b\,\mathbf{1} \;-\; q\,\widetilde{\mathbf{B}}_{a+1}\widetilde{\mathbf{B}}_{b-1}\,\mathbf{1} \;&=\; h_b[X]h_a[X] \;+\; h_{b-1}[X](q-1)h_{1+a}[X] \;-\; qh_{b-1}[X]h_{1+a}[X]\\ &=\; h_b[X]h_a[X] \;-\; h_{b-1}[X]h_{1+a}[X] \;=\; s_{[a,b]}[X]\end{aligned}$$

and since $\omega\,\mathbf{1} \;=\; \mathbf{1}$ from (12) we derive that

$$\mathbf{B}_a\mathbf{B}_b\,\mathbf{1} \;-\; q\,\mathbf{B}_{a+1}\mathbf{B}_{b-1}\,\mathbf{1} \;=\; \omega s_{[a,b]}[X] \;=\; s_{2^b,1^{a-b}}[X]$$

as desired. □

In Haglund et al. (2012) it is shown s that the $\mathbf{B}_b$ and $\mathbf{C}_a$ operators satisfy the commutativity relation

$$\mathbf{B}_b\mathbf{C}_a \;=\; q\,\mathbf{C}_a\mathbf{B}_b \tag{15}$$

and it is also shown that

$$\mathbf{B}_b\,\mathbf{1} \;=\; \sum_{k=1}^{b}\;\sum_{(p_1,p_2,\ldots,p_k)\models b} \mathbf{C}_{p_1}\mathbf{C}_{p_2}\cdots\mathbf{C}_{p_k}\,\mathbf{1} \tag{16}$$

By combining (15) with (16) and (14) we can then easily derive that

$$s_{2^b,1^{a-b}}[X] \;=\; \sum_{\alpha\models a}\sum_{\beta\models b} q^{l(\beta)}\mathbf{C}_\beta\mathbf{C}_\alpha\;\mathbf{1} \;-\; q\sum_{\gamma\models a+1}\;\sum_{\delta\models b-1} q^{l(\delta)}\mathbf{C}_\delta\mathbf{C}_\gamma\;\mathbf{1}$$

with $l(\beta)$ and $l(\delta)$ denoting the lengths of the compositions β and δ respectively.

The number of summands on the right side of this identity can be further reduced and better organized to facilitate the combinatorial steps needed to obtain the desired Parking function setting for $\nabla s_{2^b,1^{a-b}}[X]$.

To begin by breaking up the sums according as the sizes of the last part of β is 1 and first part of γ are equal to 1 or not gives

$$\begin{aligned}s_{2^b,1^{a-b}}[X] \;=\; &\sum_{\alpha\models a}\sum_{\tilde\beta\models b-1} q^{l(\tilde\beta)+1}\mathbf{C}_{\tilde\beta}\mathbf{C}_1\mathbf{C}_\alpha\;\mathbf{1} + \sum_{\alpha\models a}\;\sum_{\beta\models b\,;l\,\beta_{l(\beta)}>1} q^{l(\beta)}\mathbf{C}_\beta\mathbf{C}_\alpha\;\mathbf{1}\\ &- \sum_{\tilde\gamma\models a}\sum_{\delta\models b-1} q^{l(\delta)+1}\mathbf{C}_\delta\mathbf{C}_1\mathbf{C}_{\tilde\gamma}\;\mathbf{1} \;-\; q\sum_{\gamma\models a+1\,;\,\gamma_1>1}\;\sum_{\delta\models b-1} q^{l(\delta)}\mathbf{C}_\delta\mathbf{C}_\gamma\;\mathbf{1}\end{aligned}$$

Canceling the common terms we obtain

$$s_{2^b,1^{a-b}}[X] \;=\; \sum_{\alpha\models a}\;\sum_{\substack{\beta\models b\\ \beta_{l(\beta)}>1}} q^{l(\beta)}\mathbf{C}_\beta\mathbf{C}_\alpha\;\mathbf{1} \;-\; q\sum_{\substack{\gamma\models a+1\\ \gamma_1>1}}\;\sum_{\delta\models b-1} q^{l(\delta)}\mathbf{C}_\delta\mathbf{C}_\gamma\;\mathbf{1}$$

Now splitting once more the sums according to the sizes of the first part of α and the last part of β which we will denote u and v respectively, setting $\gamma = u+1, \tilde\gamma$ and $\delta = \tilde\delta, v-1$ we get

$$s_{2^b,1^{a-b}}[X] \;=\; \sum_{\substack{2\le u\le a\\ 1\le v\le b}} \Big(\sum_{\tilde\alpha\models a-u}\;\sum_{\tilde\beta\models b-v} q^{l(\tilde\beta)+1}\mathbf{C}_{\tilde\beta}\mathbf{C}_v\mathbf{C}_u\mathbf{C}_{\tilde\alpha}\;\mathbf{1} - q\sum_{\tilde\gamma\models a-u}\;\sum_{\tilde\delta\models b-v} q^{l(\tilde\delta)+1}\mathbf{C}_{\tilde\delta}\mathbf{C}_{v-1}\mathbf{C}_{u+1}\mathbf{C}_{\tilde\gamma}\;\mathbf{1}\Big)$$

This can be rewritten in the form

$$s_{2^b,1^{a-b}}[X] \;=\; \sum_{\substack{2\le u\le a\\ 1\le v\le b}} \sum_{\tilde\alpha\models a-u} \sum_{\tilde\beta\models b-v} q^{l(\tilde\beta)+1}\mathbf{C}_{\tilde\beta}\big(\mathbf{C}_v\mathbf{C}_u \;-\; q\,\mathbf{C}_{v-1}\mathbf{C}_{u+1}\big)\mathbf{C}_{\tilde\alpha}\;\mathbf{1}$$

and the Haglund-Morse-Zabrocki conjectures give

$$\nabla s_{2^b,1^{a-b}}[X] \;=\; \sum_{\substack{2\le u\le a\\ 1\le v\le b}} \sum_{\tilde\alpha\models a-u} \sum_{\tilde\beta\models b-v} q^{l(\tilde\beta)+1}\big(W[\tilde\beta,v,u,\tilde\alpha] \;-\; q\,W[\tilde\beta,v,u,\tilde\alpha]\big)$$

where for convenience we have let $W[\tilde\beta,v,u,\tilde\alpha]$ and $W[\tilde\beta,v,u,\tilde\alpha]$ denote the sum of the weight of the collections $\Pi[\tilde\beta,v,u,\tilde\alpha]$ and $\Pi[\tilde\beta,v,u,\tilde\alpha]$.

This identity shows that to obtain a Parking Function setting for $\nabla s_{2^b,1^{a-b}}[X]$ we need to construct an injection of $\Pi[\tilde\beta,v,u,\tilde\alpha]$ into $\Pi[\tilde\beta,v,u,\tilde\alpha]$ which preserves area and ides and increases dinv by 1. Now it is not difficult to see that to construct this injection it suffices to be able to carry it out for $\Pi[v,u,]$ into $\Pi[v,u]$ and then appropriately transfer the resulting injection to the pairs $\Pi[\tilde\beta,v,u,\tilde\alpha]$ and $\Pi[\tilde\beta,v,u,\tilde\alpha]$. and the desired properties will be automatically satisfied as long as all moved cars remain in their diagonal as we have illustrated in the example worked out in section **3**.

The general case can be obtained as an inclusion-exclusion of Parking Functions based on the identity

$$\omega s_{p_1,p_2,\ldots,p_k} \;=\; \prod_{1\le i\le j\le n}(1-q\,\theta_j/\theta_i)\mathbf{B}_{p_1}\mathbf{B}_{p_2}\ldots\mathbf{B}_{p_k}\mathbf{1}$$

which can be established in a manner analogous to our proof of (5). The realization of this plan is part of the author's ongoing thesis research.

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q-Rook placements and Jordan forms of upper-triangular nilpotent matrices

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Abstract. The set of n by n upper-triangular nilpotent matrices with entries in a finite field $\mathbf{F}_q$ has Jordan canonical forms indexed by partitions $\lambda \vdash n$. We study a connection between these matrices and non-attacking q-rook placements, which leads to a combinatorial formula for the number $F_\lambda(q)$ of matrices of fixed Jordan type as a weighted sum over rook placements.

Résumé. L'ensemble des matrices triangulaires supérieures nilpotentes d'ordre n sur un corps fini $\mathbb{F}_q$ a des formes canoniques de Jordan indexées par les partitions $\lambda \vdash n$. Nous étudions une connexion entre ces matrices et les placements de tours, et nous présentons une formule combinatoire pour le nombre $F_\lambda(q)$ des matrices comme une somme sur les placements de tours.

Keywords: q-rook placements, Jordan canonical form, nilpotent matrices, set partitions

1 Introduction

In the beautiful paper *Variations on the Triangular Theme*, Kirillov (1995) studied various structures on the set of triangular matrices. Denote by $G_n(\mathbb{F}_q)$, the group of n by n upper-triangular matrices over the field $\mathbb{F}_q$ having q elements, and let $\mathfrak{g}_n(\mathbb{F}_q) = \mathrm{Lie}(G_n(\mathbb{F}_q))$ denote the corresponding Lie algebra of n by n upper-triangular nilpotent matrices over $\mathbb{F}_q$. It is known, for example, that the conjugacy classes of $G_n(\mathbb{F}_q)$ are in bijection with the adjoint orbits in $\mathfrak{g}_n(\mathbb{F}_q)$. To study the adjoint orbits we consider the Jordan canonical form. Each matrix $X \in \mathfrak{g}_n(\mathbb{F}_q)$ is similar to a block diagonal matrix consisting of elementary Jordan blocks with eigenvalue zero:

$$J_i = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{i \times i} .$$

If the Jordan canonical form of X has block sizes $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$, then X is said to have *Jordan type* λ, where λ is a partition of n. The Jordan type of X depends only on its adjoint orbit, so the similarity classes of nilpotent matrices are indexed by the partitions of n.

Let $\mathfrak{g}_{n,\lambda}(\mathbb{F}_q) \subseteq \mathfrak{g}_n(\mathbb{F}_q)$ be the set of matrices of fixed Jordan type λ. It was shown by Springer that $\mathfrak{g}_{n,\lambda}(\mathbb{F}_q)$ an algebraic manifold with f^λ irreducible components, each of which has the same dimension $\binom{n}{2} - n_\lambda$. Here, f^λ is the number of standard Young tableaux of shape λ, and n_λ is given in Equation 9. Let

$$F_\lambda(q) = |\mathfrak{g}_{n,\lambda}(\mathbb{F}_q)| \tag{1}$$

be the number of matrices of Jordan type λ. We note that $\sum_{\lambda \vdash n} F_\lambda(q) = |\mathfrak{g}_n(\mathbb{F}_q)| = q^{\binom{n}{2}}$. The cases $F_{(1^n)}(q) = 1$ and $F_{(n)}(q) = (q-1)^{n-1} q^{\binom{n-1}{2}}$ are readily computed, since the matrix of Jordan type (1^n) is the matrix of rank zero, and the matrices of Jordan type (n) are the matrices of rank $n-1$.

In **Section 2**, we present a simple recurrence equation for $F_\lambda(q)$ (see Proposition 2). As a consequence of the recurrence equation, it follows that $F_\lambda(q)$ is a polynomial in q with nonnegative integer coefficients, $\deg F_\lambda(q) = \binom{n}{2} - n_\lambda$, and the coefficient of the highest degree term in $F_\lambda(q)$ is f^λ.

A connection with q-Rook placements

In their study of a formula of Frobenius, Garsia and Remmel (1986) introduced the *q-rook polynomial*

$$R_k(q, B) = \sum_{c \in \mathcal{C}(B,k)} q^{\mathrm{inv}(c)}, \tag{2}$$

which is a sum over the set $\mathcal{C}(B, k)$ of non-attacking placements of k rooks on the Ferrers board B, and $\mathrm{inv}(c)$, defined in Equation 11, is the number of inversions of c. In the case when $B = \delta_n$ is the staircase-shaped board, Garsia and Remmel showed that $R_k(q, \delta_n) = S_{n,n-k}(q)$ is a *q-Stirling number of the second kind*. These numbers are defined by the recurrence

$$S_{n,k}(q) = q^{k-1} S_{n-1,k-1}(q) + [k]_q S_{n-1,k}(q) \quad \text{for} \quad 0 \le k \le n, \tag{3}$$

with initial conditions $S_{0,0}(q) = 1$, and $S_{n,k}(q) = 0$ for $k < 0$ or $k > n$.

It was shown by Solomon (1990) that non-attacking placements of k rooks on rectangular $m \times n$ boards are naturally associated to m by n matrices with rank k over $\mathbb{F}_q$. By identifying a Ferrers board B inside an n by n grid with the entries of an n by n matrix, Haglund (1998) generalized Solomon's result to the case of non-attacking placements of k rooks on Ferrers boards, and obtained a formula for the number of n by n matrices with rank k whose support is contained in the Ferrers board region. As a special case of Haglund's formula, the number of nilpotent matrices of rank k is

$$P_k(q) = (q-1)^k q^{\binom{n}{2} - k} R_k(q^{-1}, \delta_n). \tag{4}$$

Now, a matrix in $\mathfrak{g}_{n,\lambda}(\mathbb{F}_q)$ has rank $n - \ell(\lambda)$, where $\ell(\lambda)$ is the number of parts of λ, so the number of matrices in $\mathfrak{g}_n(\mathbb{F}_q)$ with rank k is

$$P_k(q) = \sum_{\lambda \vdash n:\ \ell(\lambda) = n-k} F_\lambda(q). \tag{5}$$

Given Equations 4 and 5, it is natural to ask whether it is possible to partition the placements $\mathcal{C}(\delta_n, k)$ into disjoint subsets so that the sum over each subset of placements gives $F_\lambda(q)$. The goal of this paper is to study the connection between upper-triangular nilpotent matrices over $\mathbf{F}_q$ and non-attacking q-rook placements on the staircase-shaped board δ_n.

Section 3 forms the heart of this paper. Inspired by Equation 7 for $F_\lambda(q)$, we define a graph $\mathcal{Z}$ closely related to Young's lattice. The main result is Theorem 9, which states that there is a weight-preserving bijection between rook placements and paths in $\mathcal{Z}$. As a result, we obtain a formula for $F_\lambda(q)$ as a sum over certain weighted q-rook placements (see Corollary 10), which can be viewed as a generalization of Haglund's formula in Equation 4.

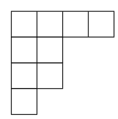
There is a well-known bijection between rook placements on the staircase-shaped board δ_n with k rooks, and set partitions of $\{1, \ldots, n\}$ with $n-k$ blocks. In **Section 4**, we describe how paths in $\mathcal{Z}$ gives a new bijection between these sets, and how this gives a definition of a lattice of compositions which appears to be new. Finally, in **Section 5**, we mention some further problems to pursue. In this article, the proofs are either omitted or briefly sketched. Full details can be found in the preprint Yip (2013).

2 The recurrence equation for $F_\lambda(q)$

We define a *partition* λ of a nonnegative integer n, denoted by $\lambda \vdash n$, is a non-increasing sequence of nonnegative integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ with $|\lambda| = \sum_{i=1}^n \lambda_i = n$. If λ has k positive parts, write $\ell(\lambda) = k$. Represent a partition λ by its Ferrers diagram in the English notation, which is an array of λ_i boxes in row i, with the boxes justified upwards and to the left. Let λ'_j denote the size of the jth column of λ.

Example 1 *The partition*

$\lambda = (4, 2, 2, 1) \vdash 9$ *has diagram*



and columns $\lambda'_1 = 4, \lambda'_2 = 3, \lambda'_3 = 1, \lambda'_4 = 1.$

Young's lattice $\mathcal{Y}$ is the lattice of partitions ordered by the inclusion of their Ferrers diagrams. In particular, write $\mu \prec \lambda$ if $\mu \subseteq \lambda$ and $|\lambda| = |\mu| + 1$. In other words, μ is covered by λ in $\mathcal{Y}$ if the Ferrers diagram of λ can be obtained by adding a box to the Ferrers diagram of μ. If this box is added in the ith row and jth column of the diagram, assign a weight $c_{\mu\lambda}(q)$ to the edge between μ and λ, where

$$c_{\mu\lambda}(q) = \begin{cases} q^{|\mu| - \mu'_j}, & \text{if } j = 1, \\ q^{|\mu| - \mu'_{j-1}} \left(q^{\mu'_{j-1} - \mu'_j} - 1 \right), & \text{if } j \geq 2. \end{cases} \tag{6}$$

See Figure 1 for an illustration.

The following recurrence formula for $F_\lambda(q)$ can be found in Borodin (1995), where he considers the matrices as particles of a certain mass. An elementary proof of a different flavour is outlined below.

Proposition 2 *Let $\lambda \vdash n$. The number of n by n upper-triangular nilpotent matrices over $\mathbb{F}_q$ of Jordan type λ is*

$$F_\lambda(q) = \sum_{\mu \prec \lambda} c_{\mu\lambda}(q) F_\mu(q), \qquad \text{with} \qquad F_\emptyset(q) = 1.$$

Proof: Proceed by induction on n. Given $\lambda \vdash n$, first notice that any matrix in $\mathfrak{g}_{n,\lambda}(\mathbb{F}_q)$ has a leading principal submatrix of type μ where $\mu \prec \lambda$. Furthermore, let J_μ denote the Jordan matrix which is the

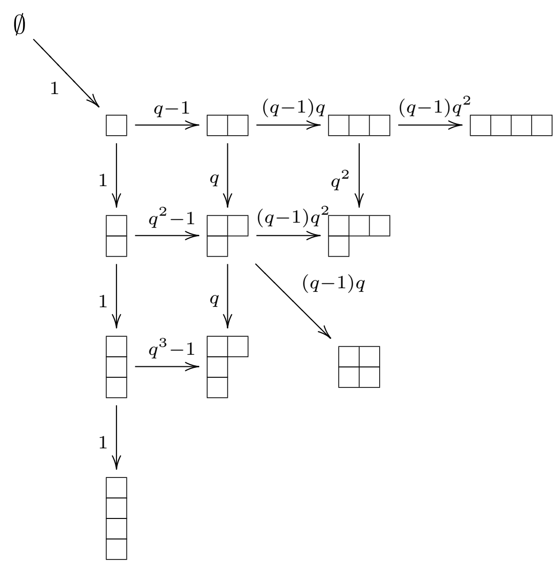


Fig. 1: Young's lattice with edge weights $c_{\mu\lambda}(q)$, up to $n = 4$.

direct sum of elementary nilpotent Jordan blocks of sizes $\mu_1, \dots, \mu_k$. There are $c_{\mu\lambda}(q)$ matrices of Jordan type λ having J_μ as its leading principal submatrix, and by similarity, this result continues to hold if the matrix J_μ is replaced by any matrix Y of Jordan type μ. Summing over all $\mu \prec \lambda$ gives the desired formula. □

The formula for $F_\lambda(q)$ in Proposition 2 can be rephrased as a sum over the set $\mathcal{P}(\lambda)$ of paths in the Young lattice $\mathcal{Y}$ from the empty partition $\emptyset$ to λ. Suppose

$$\varpi : \emptyset \xrightarrow{\epsilon_1} \lambda^{(1)} \xrightarrow{\epsilon_2} \lambda^{(2)} \xrightarrow{\epsilon_3} \cdots \xrightarrow{\epsilon_n} \lambda^{(n)} = \lambda$$

is a path in $\mathcal{Y}$, where the weight of the path

$$w(\varpi) = \prod_{r=1}^{n} c_{\lambda^{(r-1)}\lambda^{(r)}}(q) \tag{7}$$

is the product of the weights on it edges. Then Proposition 2 is equivalent to the statement

$$F_\lambda(q) = \sum_{\varpi \in \mathcal{P}(\lambda)} w(\varpi). \tag{8}$$

Example 3 *There are two partitions of* 4 *with* 2 *parts, namely* $(3,1)$ *and* $(2,2)$*. There are three paths from* $\emptyset$ *to* $(3,1)$*, giving*

$$\begin{aligned} F_{(3,1)}(q) &= (q-1)\cdot(q-1)q\cdot q^2 + (q-1)\cdot q\cdot(q-1)q^2 + \cdot(q^2-1)\cdot(q-1)q^2 \\ &= (q-1)^2\left(3q^3+q^2\right), \end{aligned}$$

and there are two paths from $\emptyset$ to $(2,2)$, giving

$$\begin{aligned} F_{(2,2)}(q) &= (q-1)\cdot q\cdot(q-1)q + (q^2-1)\cdot(q-1)q \\ &= (q-1)^2(2q^2+q). \end{aligned}$$

Two observations about $F_\lambda(q)$ now follow readily from Proposition 2. For $\lambda \vdash n$, let

$$n_\lambda = \sum_{i\geq 1}(i-1)\lambda_i. \tag{9}$$

Suppose $\varpi :\ \emptyset \longrightarrow \lambda^{(1)} \longrightarrow \lambda^{(2)} \longrightarrow \cdots \longrightarrow \lambda^{(n)} = \lambda$ is a path in $\mathcal{Y}$ such that $\lambda^{(r)}$ is obtained by adding a box to $\lambda^{(r-1)}$ in row i and column j. Then $\deg c_{\lambda^{(r-1)}\lambda^{(r)}}(q) = r - i$, and therefore,

$$\deg w(\varpi) = \sum_{r=1}^{n} \deg c_{\lambda^{(r-1)}\lambda^{(r)}}(q) = \sum_{r=1}^{n} r - \sum_{i\geq 1} i\lambda_i = \binom{n}{2} - n_\lambda.$$

In particular, each polynomial $w(\varpi)$ arising from a path $\varpi \in \mathcal{P}(\lambda)$ has the same degree, so

$$\deg F_\lambda(q) = \binom{n}{2} - n_\lambda. \tag{10}$$

Moreover, each $w(\varpi)$ is monic, so the coefficient of the highest degree term in $F_\lambda(q)$ is the number of paths in $\mathcal{Y}$ from $\emptyset$ to λ, which is the number f^λ of standard Young tableaux of shape λ.

Second, the edge weight $c_{\lambda^{(r-1)}\lambda^{(r)}}(q)$ corresponding to the rth step in the path ϖ contributes a factor of $q-1$ to $w(\varpi)$ if and only if the rth box added along the path is in column $j \geq 2$. Therefore, the multiplicity of $q-1$ in each $w(\pi)$ is $n - \lambda_1' = n - \ell(\lambda)$, and so, the multiplicity of $q-1$ in $F_\lambda(q)$ is $n - \ell(\lambda)$.

3 Jordan canonical forms and q-rook polynomials

A *board* B is a subset of an n by n grid of squares. We follow Haglund (1998), and index the squares following the convention for the entries of a matrix. A *Ferrers board* is a board B where if a square $s \in B$, then all squares lying north and/or east of s is also in B. Let δ_n denote the staircase-shaped board with n columns of sizes $0, 1, \ldots, n-1$. Let $\operatorname{area}(B)$ be the number of squares in B, so $\operatorname{area}(\delta_n) = \binom{n}{2}$ in particular.

A placement of k rooks on a board B is *non-attacking* if there is at most one rook in each row and each column of B. Let $\mathcal{C}(B,k)$ be the set of non-attacking placements of k rooks on B. For a placement $\gamma \in \mathcal{C}(B,k)$, let $\operatorname{ne}(\gamma)$ be the number of squares in B lying directly north or directly east of a rook. Also define the *inversion* of the placement to be the number

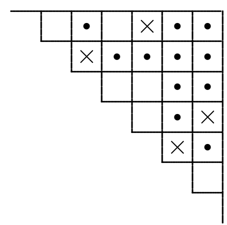
$$\operatorname{inv}(\gamma) = \operatorname{area}(B) - k - \operatorname{ne}(\gamma). \tag{11}$$

See Example 4 for an illustration. As noted in Garsia and Remmel (1986), the statistic $\operatorname{inv}(\gamma)$ is a generalization of the number of inversions of a permutation, since permutations can be identified with non-attacking placements of rooks on a square-shaped board. In terms of the rook placement, $\operatorname{inv}(\gamma)$ is the number of squares left blank.

Define the *weight* of a rook placement $\gamma \in \mathcal{C}(B,k)$ by

$$w(\gamma) = (q-1)^k q^{\mathrm{ne}(\gamma)}. \tag{12}$$

Example 4 *We use $\times$ to mark a rook and use $\bullet$ to mark squares lying directly north or directly east of a rook. (These squares shall be referred to as the north-east squares of the placement.) The following illustration is a non-attacking placement of four rooks on the staircase-shaped board δ_7.*



This rook placement has $\mathrm{ne}(\gamma) = 11$, $\mathrm{inv}(\gamma) = 6$, *and weight* $w(\gamma) = (q-1)^4 q^{11}$.

For $k \geq 0$, the *q-rook polynomial* of a Ferrers board B is defined in (Garsia and Remmel, 1986, I.4) by

$$R_k(q,B) = \sum_{c \in \mathcal{C}(B,k)} q^{\mathrm{inv}(c)}. \tag{13}$$

The following result is due to (Haglund, 1998, Theorem 1).

Proposition 5 *If B is a Ferrers board, then the number $P_{B,k}(q)$ of n by n matrices of rank k with support contained in B is*

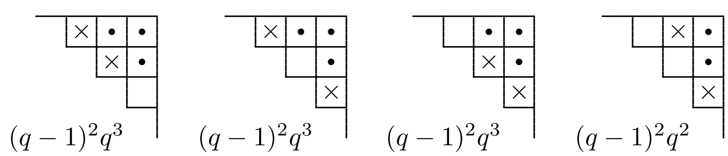
$$P_{B,k}(q) = (q-1)^k q^{\mathrm{area}(B)-k} R_k(q^{-1}, B).$$

□

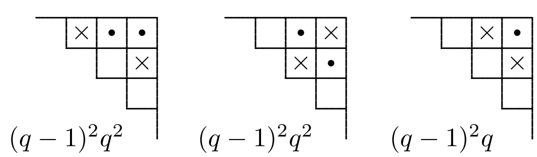
Looking ahead, it will be convenient to consider Theorem 5 in the following equivalent form:

$$P_{B,k}(q) = \sum_{\gamma \in \mathcal{C}(B,k)} (q-1)^k q^{\mathrm{ne}(\gamma)} = \sum_{\gamma \in \mathcal{C}(B,k)} w(\gamma). \tag{14}$$

Example 6 *There are seven non-attacking placements on δ_4 with two rooks:*



(15)



(16)

This gives $P_2(q) = (q-1)^2(3q^3 + 3q^2 + q)$.

Recall that if $\lambda \vdash n$ is obtained from μ by adding a box in row i and column j, then the edge in $\mathcal{Y}$ from μ to λ has weight

$$c_{\mu\lambda}(q) = \begin{cases} q^{|\mu|-\mu'_j}, & \text{if } j = 1, \\ (q-1)q^{|\mu|-\mu'_j-1} + \cdots + (q-1)q^{|\mu|-\mu'_{j-1}}, & \text{if } j \geq 2. \end{cases} \tag{17}$$

Thus each edge weight is a sum of terms of the form $(q-1)^d q^e$ where $d = 1 - \delta_{j1}$ (δ_{j1} denotes the Kronecker delta function), and $e \in \mathbb{Z}_{\geq 0}$. This observation inspires the following definitions.

Based on the Young lattice $\mathcal{Y}$, we construct a graph $\mathcal{Z}$ (see Figure 2). The vertices of $\mathcal{Z}$ are partitions. If there is an edge from μ to λ in $\mathcal{Y}$ of weight $q^{|\mu|-\mu'_{j-1}}(q^{\mu'_{j-1}-\mu'_j} - 1)$, then in $\mathcal{Z}$ the edge is replaced by $\mu'_{j-1} - \mu'_j$ edges with weights

$$(q-1)q^{|\mu|-\mu'_j-1}, \ldots, \ (q-1)q^{|\mu|-\mu'_{j-1}}.$$

A *primitive path* π

$$\emptyset \xrightarrow{\epsilon_1} \pi^{(1)} \xrightarrow{\epsilon_2} \pi^{(2)} \xrightarrow{\epsilon_3} \cdots \xrightarrow{\epsilon_n} \pi^{(n)}$$

is a path in the graph $\mathcal{Z}$. The weight $w(\pi)$ of a primitive path is the product of its edge weights.

Let $\mathcal{PP}(\lambda)$ denote the set of primitive paths from $\emptyset$ to λ. Then each path in Young's lattice corresponds to a set of primitive paths, and

$$F_\lambda(q) = \sum_{\pi \in \mathcal{PP}(\lambda)} w(\pi). \tag{18}$$

Remark 7 *Let $\mu \vdash n-1$ where $\ell(\mu) = \ell$. Let $\pi' : \emptyset \longrightarrow \lambda^{(1)} \longrightarrow \lambda^{(2)} \longrightarrow \cdots \longrightarrow \lambda^{(n-1)} = \mu$ be a primitive path. If λ is obtained by adding a box to the first column of μ, then there is a unique way to extend the primitive path π' by one edge, and by Equation 17, that edge has weight*

$$q^{|\mu|-\ell}.$$

Furthermore, consider all possible λ which can be obtained by adding a box to μ in a column $j \geq 2$. Then by Equation 17, there are

$$\sum_{j=2}^{\ell(\mu')+1} \mu'_{j-1} - \mu'_j = \ell$$

ways to extend the primitive path π' by one edge.

In summary, the out-degree of μ in $\mathcal{Z}$ is $\ell + 1$. Moreover, the weights

$$q^{|\mu|-\ell}, \ (q-1)q^{|\mu|-\ell}, \ (q-1)q^{|\mu|-\ell+1}, \ \ldots, \ (q-1)q^{|\mu|-1},$$

for the edges have unique degrees $|\mu| - \ell \leq d \leq |\mu|$. This observation is crucial for the proof of the next lemma.

Lemma 8 *Let $n \geq 1$ and $0 \leq k \leq n-1$. Suppose $\gamma \in \mathcal{C}(\delta_n, k)$ is a rook placement with columns $\gamma^{(1)}, \ldots, \gamma^{(n)}$. Then the sequence of weights $w(\gamma^{(1)}), \ldots, w(\gamma^{(n)})$ determines a unique primitive path $\pi : \emptyset \xrightarrow{\epsilon_1} \pi^{(1)} \xrightarrow{\epsilon_2} \pi^{(2)} \xrightarrow{\epsilon_3} \cdots \xrightarrow{\epsilon_n} \pi^{(n)}$ such that $\epsilon_r = w(\gamma^{(r)})$. Moreover, $\ell(\pi^{(n)}) = n - k$.*

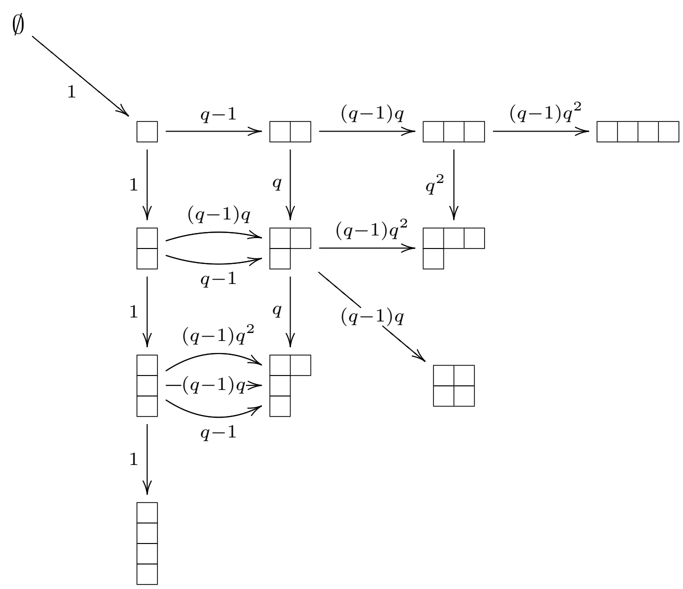


Fig. 2: The graph $\mathcal{Z}$, up to $n = 4$.

Proof: Proceed by induction on $n+k$. Suppose $\gamma \in \mathcal{C}(\delta_n, k)$ and let γ' be the placement consisting of the first $n-1$ columns of γ. By induction, the sequence $\epsilon_1 = w(\gamma^{(1)}), \ldots, \epsilon_{n-1} = w(\gamma^{(n-1)})$ determines a unique primitive path π' : $\emptyset \xrightarrow{\epsilon_1} \pi^{(1)} \xrightarrow{\epsilon_2} \cdots \xrightarrow{\epsilon_{n-1}} \pi^{(n-1)}$. There are two cases to consider; γ' has either k or $k-1$ rooks.

If γ' has k rooks, then the column $\gamma^{(n)}$ then has k north-east squares only and weight $q^k = q^{|\mu|-\ell}$. By the previous remark, there is a unique way to extend the primitive path corresponding to γ' by adding a box to the first column of $\pi^{(n-1)}$, and that edge has weight $q^{|\mu|-\ell}$.

In the second case, if γ' has $k-1$ rooks, then there are $n-1-k$ available boxes in column $\gamma^{(n)}$ to place a rook. The placement of the rook uniquely determines the degree of the weight $w(\gamma^{(n)})$ of the column, which ranges from $k+1, \ldots, n-1$. From the remark, there are precisely $\ell = n-1-k$ edges emanating from μ in $\mathcal{Z}$ with weights having degrees $|\mu| - \ell + 1, \ldots, |\mu|$. □

Let $\mathcal{PP}(n, n-k) = \{\pi \in \mathcal{PP}(\lambda) \mid \lambda \vdash n \text{ and } \ell(\lambda) = n-k\}$ be the set of primitive paths in $\mathcal{Z}$ from $\emptyset$ to a partition with $n-k$ parts. Define a map $\Theta : \mathcal{C}(\delta_n, k) \to \mathcal{PP}(n, n-k)$ as follows. Suppose a rook placement γ has columns $\gamma^{(1)}, \ldots, \gamma^{(n)}$. Let $\Theta(\gamma)$ be the primitive path

$$\emptyset \xrightarrow{\epsilon_1} \pi^{(1)} \xrightarrow{\epsilon_2} \pi^{(2)} \xrightarrow{\epsilon_3} \cdots \xrightarrow{\epsilon_n} \pi^{(n)} \qquad \text{with} \qquad \epsilon_r = w(\gamma^{(r)}).$$

Theorem 9 *The map $\Theta : \mathcal{C}(\delta_n, k) \to \mathcal{PP}(n, n-k)$ is a weight-preserving bijection.* □

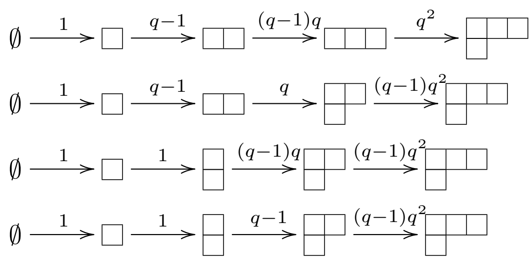
It follows from Theorem 9 that we may associate a partition type to each rook placement on δ_n. The *partition type* of a rook placement γ is the partition at the endpoint of the primitive path $\Theta(\gamma)$. Let $\mathcal{C}(\lambda)$ be the set of rook placements of partition type λ. As a Corollary, we obtain a formula for $F_\lambda(q)$ as a sum over rook placements, and the equation can be viewed as a generalization of Haglund's result in Equation 4.

Corollary 10 *Let $\lambda \vdash n$ be a partition with $\ell(\lambda) = n - k$ parts. Then*

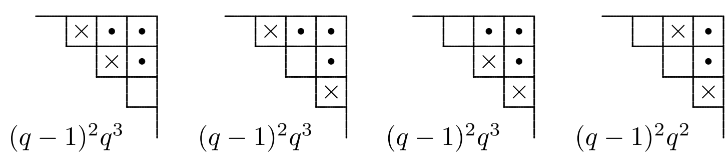
$$F_\lambda(q) = \sum_{\gamma \in \mathcal{C}(\lambda)} (q-1)^k q^{\mathrm{ne}(\gamma)}.$$

Proof: The result follows from Equation 18 and the bijection Θ in Theorem 9. $\square$

Example 11 *There are four primitive paths from $\emptyset$ to $\lambda = (3,1)$.*



Respectively, they correspond to the following rook placements, each having partition type $(3,1)$.



Therefore,

$$F_{(3,1)}(q) = (q-1)^2(3q^3 + q^2).$$

4 A refinement to compositions

A *set partition* is a set $\sigma = \{B_1, \ldots, B_k\}$ of nonempty disjoint subsets of $\mathcal{N}_n$ such that $\bigcup_{i=1}^k B_i = \mathcal{N}_n$. The B_is are the *blocks* of σ. Let $\Pi(n,k)$ be the set of set partitions of $\mathcal{N}_n$ with k blocks. We adopt the convention of listing the blocks in order so that $\min B_i < \min B_j$ if $i < j$. This allows us to represent a set partition with a diagram similar to that of a Young tableau; the ith row of the diagram consists of the elements in block B_i listed in increasing order. A *composition* α of a nonnegative integer n is a sequence of positive integers $(\alpha_1, \ldots, \alpha_k)$ such that $|\alpha| = \sum_{i=1}^k \alpha_i = n$. If α has k positive parts, write $\ell(\alpha) = k$. A set partition $\sigma = \{B_1, \ldots, B_k\}$ has *composition type* α if $\alpha = (|B_1|, \ldots, |B_k|)$.

The number of set partitions of $\mathcal{N}_n$ with k blocks is the Stirling number $S_{n,n-k}(1)$ (see Equation 3). In addition, $S_{n,n-k}(1)$ is also the number of placements of k rooks on the staircase board δ_n. This follows from the following well-known bijection (see Stanley (1999)); given a placement $\gamma \in \mathcal{C}(\delta_n, k)$, construct a set partition of $\mathcal{N}_n$ where the integers i and j are in the same block if and only if there is a rook in square $(i,j) \in \gamma$.

We shall give another bijection $\Psi : \mathcal{C}(\delta_n,k) \to \Pi(n,n-k)$ that arises from the primitive paths in the graph $\mathcal{Z}$. Let $\gamma \in \mathcal{C}(\delta_n,k)$ be a rook placement. Construct a diagram for a set partition using the following procedure (also see Example 14).

- Let $\lambda^{(1)}$ be the diagram with a single box labelled 1 placed in the first row and the first column.
- For $k \geq 2$, if the weight of the kth column in γ has degree d, then place the box labelled k in the $(k-d)$th row of $\lambda^{(k-1)}$, and rearrange the rows of the diagram into a partition shape $\lambda^{(k)}$, so that the rows of the same length have first column entries in increasing order.

Note that if $\pi = \Theta(\gamma)$ is the primitive path in $\mathcal{Z}$ which corresponds to the placement γ, then $\pi^{(k)} = \lambda^{(k)}$. Let $\lambda^{(n)}$ be the partition shape of the diagram after the nth box has been placed. Let $\operatorname{order}\left(\lambda^{(n)}\right)$ be the diagram of the set partition resulting from ordering the rows of $\lambda^{(n)}$ so that the first column entries are increasing. Define $\Psi(\gamma) = \operatorname{order}\left(\lambda^{(n)}\right)$.

Proposition 12 $\Psi : \mathcal{C}(\delta_n,k) \to \Pi(n,n-k)$ *is a bijection.* □

The *composition type* of a rook placement $\gamma \in \mathcal{C}(\delta_n,k)$ is the composition type of $\Psi(\gamma)$. Let $\mathcal{C}(\alpha)$ be the set of rook placements with composition type α. For a composition α of n with $\ell(\alpha) = n-k$ parts, define

$$F_\alpha(q) = \sum_{c \in \mathcal{C}(\alpha)} (q-1)^k q^{\operatorname{ne}(c)}. \tag{19}$$

Let $\operatorname{rearr}(\alpha)$ be the partition resulting from the rearrangement of the parts of the composition α so that they are nondecreasing.

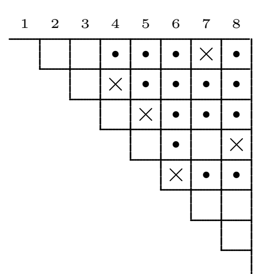
Corollary 13 *Let* $\lambda \vdash n$. *Then*

$$F_\lambda(q) = \sum_{\lambda = \operatorname{rearr}(\alpha)} F_\alpha(q).$$

□

We extend the definition of n_λ to compositions and let $n_\alpha = \sum_{i\geq 1}(i-1)\alpha_i$. Then $\deg F_\alpha(q) = \binom{n}{2} - n_\alpha$, the multiplicity of the factor $q-1$ in $F_\alpha(q)$ is $n = \ell(\alpha)$, and the coefficient of the highest degree term is the number of set partitions whose diagrams are increasing along rows and columns.

Example 14 *Consider the rook placement* $\gamma \in \mathcal{C}(\delta_8, 5)$:



The set partition diagrams $\lambda^{(1)}, \ldots, \lambda^{(n)}$ correspond to the following primitive path in $\mathcal{Z}$.

$$\begin{array}{|c|}\hline 1\\\hline\end{array} \xrightarrow{\epsilon_2} \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \xrightarrow{\epsilon_3} \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array} \xrightarrow{\epsilon_4} \begin{array}{|c|c|}\hline 2 & 4\\\hline \multicolumn{1}{|c|}{1}\\\cline{1-1} \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array} \xrightarrow{\epsilon_5} \begin{array}{|c|c|}\hline 1 & 5\\\hline 2 & 4\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array} \xrightarrow{\epsilon_6} \begin{array}{|c|c|c|}\hline 1 & 5 & 6\\\hline 2 & \multicolumn{1}{c|}{4}\\\cline{1-2} \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array} \xrightarrow{\epsilon_7} \begin{array}{|c|c|c|}\hline 1 & 5 & 6\\\hline 2 & \multicolumn{1}{c|}{4}\\\cline{1-2} 3 & \multicolumn{1}{c|}{7}\\\cline{1-2}\end{array} \xrightarrow{\epsilon_8} \begin{array}{|c|c|c|}\hline 1 & 5 & 6\\\hline 3 & 7 & 8\\\hline 2 & \multicolumn{1}{c|}{4}\\\cline{1-2}\end{array}$$

The edge weights of this primitive path are the same as the column weights in the rook placement. Ordering the rows of the diagram of the endpoint of the primitive path so that the entries in the first column are increasing gives the set partition

$$\Psi(\gamma) = \text{order } \cdot \begin{array}{|c|c|c|}\hline 1 & 5 & 6\\\hline 3 & 7 & 8\\\hline 2 & \multicolumn{1}{c|}{4}\\\cline{1-2}\end{array} = \begin{array}{|c|c|c|}\hline 1 & 5 & 6\\\hline 2 & \multicolumn{1}{c|}{4}\\\hline 3 & 7 & 8\\\hline\end{array}$$

Therefore, the rook placement γ has partition type $\lambda = (3, 3, 2)$ and composition type $\alpha = (3, 2, 3)$.

Remark 15 *We can define a lattice $\mathcal{X}$ of compositions by requiring that each path from $\emptyset$ to α encodes a rook configuration of composition type α. The paper of Björner and Stanley (2005) considers two different lattices of compositions, but $\mathcal{X}$ is different from the two presented in their paper. It may be interesting to investigate the combinatorial properties of $\mathcal{X}$, particularly as paths in $\mathcal{X}$ are equivalent to set partitions, and they are known to play a crucial role in the supercharacter theory of unipotent upper-triangular matrices (see Thiem (2010) for example).*

5 Closing remarks

5.1 Inverse Kostka-polynomials

Let $P_\lambda(x; t)$ denote the Hall-Littlewood function indexed by the partition λ, and let $m_\mu(x)$ denote the monomial symmetric function indexed by μ. See (Macdonald, 1995, Ch. III) for definitions. For $\lambda, \mu \vdash n$, the transition coefficients $L_{\lambda,\mu}(t)$ are defined by

$$P_\lambda(x; t) = \sum_\mu L_{\lambda,\mu}(t) m_\mu(x). \tag{20}$$

The recurrence formula for $F_\lambda(q)$ in Proposition 2 is essentially the same as the one for $L_{\lambda,1^n}(t)$ (Macdonald, 1995, Equation 5.9'), so that $L_{\lambda,1^n}(t) = t^{\binom{n}{2} - n_\lambda} F_\lambda(t^{-1})$. It would be interesting to see if other entries in the transition matrix can be obtained as sums over rook placements on boards of another shape.

5.2 Matrices satisfying $X^2 = 0$

Kirillov and Melnikov (1995) considered the number $A_n(q)$ of n by n upper-triangular matrices over $\mathbb{F}_q$ satisfying $X^2 = 0$. In their first characterization of these polynomials, they considered the number $A_n^r(q)$ of matrices of a given rank r, so that $A_n(q) = \sum_{r \geq 0} A_n^r(q)$, and observed that $A_n^r(q)$ satisfies the recurrence

$$A_n^r(q) = q^r A_{n-1}^r(q) + \left(q^{n-r} - q^r\right) A_n^r(q), \qquad A_n^0(q) = 1.$$

We may think of $A_n(q)$ as the sum of $F_\lambda(q)$ over $\lambda \vdash n$ with at most two columns, so Proposition 2 is, in a sense, a generalization of this recurrence.

It was also conjectured by Kirillov and Melnikov that the same sequence of polynomials arise in a number of different ways. Ekhad and Zeilberger (1996) proved that one of the conjectured alternate definitions of $A_n(q)$, namely

$$C_n(q) = \sum_s c_{n+1,s} q^{\frac{n^2}{4} + \frac{1-s^2}{12}},$$

is a sum over all $s \in [-n-1, n+1]$ which satisfy $s \equiv n+1 \bmod 2$ and $s \equiv (-1)^n \bmod 3$, and $c_{n+1,s}$ are entries in the signed Catalan triangle, is indeed the same as $A_n(q)$. It would be interesting to see what other combinatorics may arise from considering the sum of $F_\lambda(q)$ over $\lambda \vdash n$ with at most k columns for a fixed k.

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On the poset of weighted partitions

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Abstract. In this extended abstract we consider the poset of weighted partitions Π_n^w, introduced by Dotsenko and Khoroshkin in their study of a certain pair of dual operads. The maximal intervals of Π_n^w provide a generalization of the lattice Π_n of partitions, which we show possesses many of the well-known properties of Π_n. In particular, we prove these intervals are EL-shellable, we compute the Möbius invariant in terms of rooted trees, we find combinatorial bases for homology and cohomology, and we give an explicit sign twisted $\mathfrak{S}_n$-module isomorphism from cohomology to the multilinear component of the free Lie algebra with two compatible brackets. We also show that the characteristic polynomial of Π_n^w has a nice factorization analogous to that of Π_n.

Résumé. Dans ce résumé étendu, nous considèrons l'ensemble ordonné des partitions pondérées Π_n^w, introduit par Dotsenko et Khoroshkin dans leur étude d'une certaine paire d'opérades duales. Les intervalles maximaux de Π_n^w généralisent le treillis Π_n des partitions et, comme nous le montrons, possèdent beaucoup de propriétés classiques de Π_n. En particulier, nous prouvons que ces intervalles sont "EL-shellable", nous exprimons leur invariant de Möbius en fonction d'arbres enracinés, nous trouvons des bases combinatoires pour l'homologie et la cohomologie et nous donnes un morphisme explicite de $\mathfrak{S}_n$-modules gauches entre la cohomologie et la composante multilinéaire de l'algèbre de Lie libre avec deux crochets de Lie compatibles. Nous montrons aussi que le polynôme caractéristique de Π_n^w admet une factorisation sympathique, analogue à celle de Π_n.

Keywords: poset topology, partitions, free Lie algebra, rooted trees

1 Introduction

We recall some combinatorial, topological and representation theoretic properties of the lattice Π_n of partitions of the set $[n] := \{1, 2, \ldots, n\}$ ordered by refinement. The Möbius invariant of Π_n is given by $\mu(\Pi_n) = (-1)^{n-1}(n-1)!$ and the characteristic polynomial by $\chi_{\Pi_n}(x) = (x-1)(x-2)\ldots(x-n+1)$ (see [18, Example 3.10.4]). The order complex $\Delta(P)$ of a poset P is the simplicial complex whose faces are the chains of P; and the proper part $\bar{P}$ of a bounded poset P is the poset obtained by removing the minimum element $\hat{0}$ and the maximum element $\hat{1}$. It was proved by Björner [2], using an edge labeling of Stanley [16], that Π_n is EL-shellable; consequently the order complex $\Delta(\overline{\Pi}_n)$ has the homotopy type of a wedge of $(n-1)!$ spheres of dimension $n-3$. The falling chains of the EL-labeling provide a basis for cohomology of $\Delta(\overline{\Pi}_n)$ called the Lyndon basis. See [21] for a discussion of this basis and other nice bases for the homology and cohomology of the partition lattice.

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The symmetric group $\mathfrak{S}_n$ acts naturally on Π_n and this induces isomorphic representations of $\mathfrak{S}_n$ on the unique nonvanishing reduced simplicial homology $\tilde{H}_{n-3}(\overline{\Pi}_n)$ of the order complex $\Delta(\overline{\Pi}_n)$ and on the unique nonvanishing simplicial cohomology $\tilde{H}^{n-3}(\overline{\Pi}_n)$. (Throughout this paper homology and cohomology are taken over $\mathbb{C}$.) Joyal [12] observed that a formula of Stanley [17] for the character of this representation is a sign twisted version of an earlier formula of Brandt [5] for the character of the representation of $\mathfrak{S}_n$ on the multilinear component $Lie(n)$ of the free Lie algebra on n generators. Hence the following $\mathfrak{S}_n$-module isomorphism holds,

$$\tilde{H}_{n-3}(\overline{\Pi}_n) \simeq_{\mathfrak{S}_n} \mathcal{L}ie(n) \otimes \mathrm{sgn}_n, \tag{1.1}$$

where sgn_n is the sign representation of $\mathfrak{S}_n$. Joyal [12] gave a species theoretic proof of the isomorphism. The first purely combinatorial proof was obtained by Barcelo [1] who gave a bijection between known bases for the two $\mathfrak{S}_n$-modules (Björner's NBC basis for $\tilde{H}_{n-3}(\bar{\Pi}_n)$ and the Lyndon basis for $\mathcal{L}ie(n)$). Later Wachs [21] gave a more general combinatorial proof by providing a natural bijection between generating sets of $\tilde{H}^{n-3}(\overline{\Pi}_n)$ and $\mathcal{L}ie(n)$, which revealed the strong connection between the two $\mathfrak{S}_n$-modules.

In this paper we explore analogous properties for a weighted version of Π_n, introduced by Dotsenko and Khoroshkin [6] in their study of Koszulness of certain quadratic binary operads. A weighted partition of $[n]$ is a set $\{B_1^{v_1}, B_2^{v_2}, ..., B_t^{v_t}\}$ where $\{B_1, B_2, ..., B_t\}$ is a partition of $[n]$ and $v_i \in \{0, 1, 2, ..., |B_i| - 1\}$ for all i. The *poset of weighted partitions* Π_n^w is the set of weighted partitions of $[n]$ with covering relation given by $\{A_1^{w_1}, A_2^{w_2}, ..., A_s^{w_s}\} \lessdot \{B_1^{v_1}, B_2^{v_2}, ..., B_t^{v_t}\}$ if the following conditions hold

- $\{A_1, A_2, \ldots, A_s\} \lessdot \{B_1, B_2, \ldots, B_t\}$ in Π_n
- if $B_k = A_i \cup A_j$, where $i \neq j$, then $v_k - (w_i + w_j) \in \{0, 1\}$.

In Figure 1 below the set brackets and commas have been omitted.

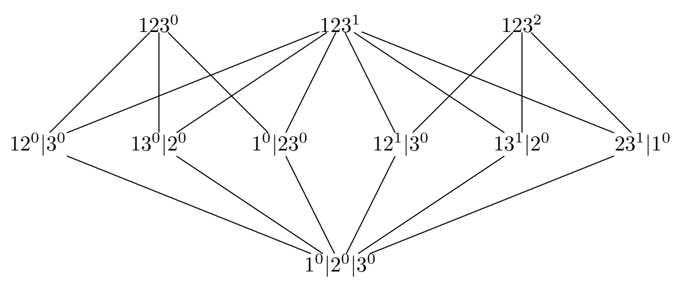


Fig. 1: Weighted partition poset for $n = 3$

We remark that Π_n^w resembles, but is not the same as, a Rees product poset introduced by Björner and Welker [4] and studied in [15].

The poset Π_n^w has a minimum element $\hat{0} = 1^0|2^0|\ldots|n^0$ and n maximal elements $[n]^0,\ [n]^1, \ldots, [n]^{n-1}$. Note that for all i, the maximal intervals $[\hat{0}, [n]^i]$ and $[\hat{0}, [n]^{n-1-i}]$ are isomorphic to each other, and the two maximal intervals $[\hat{0}, [n]^0]$ and $[\hat{0}, [n]^{n-1}]$ are isomorphic to Π_n.

In this paper[(i)] we prove that the augmented weighted partition poset $\widehat{\Pi_n^w} := \Pi_n^w \cup \{\hat{1}\}$ is EL-shellable by providing an interesting weighted analog of the Björner-Stanley EL-labeling of Π_n. In fact our labeling restricts to the Björner-Stanley EL-labeling on the intervals $[\hat{0}, [n]^0]$ and $[\hat{0}, [n]^{n-1}]$. A consequence of shellability is that $\widehat{\Pi_n^w}$ is Cohen-Macaulay, which implies a result of Dotsenko and Khoroshkin [7], obtained through operad theory, that all maximal intervals $[\hat{0}, [n]^i]$ of Π_n^w are Cohen-Macaulay. (Two prior attempts [6, 19] to establish Cohen-Macaulayness of $[\hat{0}, [n]^i]$ are discussed in Remark 2.3.) The falling chains of our EL-labeling also provide a generalization of the Lyndon basis for cohomology of Π_n.

It follows from an operad theoretic result of Vallette [20] and the Cohen-Macaulayness of each maximal interval $[\hat{0}, [n]^i]$ that the following $\mathfrak{S}_n$-module isomorphism holds:

$$\bigoplus_{i=0}^{n-1} \tilde{H}_{n-3}((\hat{0}, [n]^i)) \simeq_{\mathfrak{S}_n} \mathcal{L}ie_2(n) \otimes \mathrm{sgn}_n, \tag{1.2}$$

where $\mathcal{L}ie_2(n)$ is the representation of $\mathfrak{S}_n$ on the multilinear component of the free Lie algebra on n generators with two compatible brackets (defined in Section 3.2) and $\tilde{H}_{n-3}((\hat{0}, [n]^i))$ is the reduced simplicial homology of the order complex of the open interval $(\hat{0}, [n]^i)$. A graded version of this isomorphism, which can also be proved using Vallette's technique, implies (1.1). In Section 3.3 we reveal the connection between these graded modules by providing an explicit bijection between generating sets of $\bigoplus_{i=0}^{n-1} \tilde{H}^{n-3}((\hat{0}, [n]^i))$ and $\mathcal{L}ie_2(n)$, which generalizes the bijection that Wachs [21] used to prove (1.1).

In [13] Liu proves a conjecture of Feigin that $\dim \mathcal{L}ie_2(n) = n^{n-1}$ by constructing a combinatorial basis for $\mathcal{L}ie_2(n)$ indexed by rooted trees. (It is well-known that n^{n-1} is the number of rooted trees on node set $[n]$.) An operad theoretic proof of Feigin's conjecture was obtained by Dotsenko and Khoroshkin [6], but with a gap pointed out in [19] and corrected in [7].

Liu and Dotsenko/Khoroshkin obtain a more general graded version of Feigin's conjecture. In this paper we take a different path to proving the graded version. In Section 2.2 we compute the Möbius invariant of the maximal intervals of Π_n^w by exploiting the recursive nature of Π_n^w and applying the compositional formula. From our computation and the fact that $\widehat{\Pi_n^w}$ is Cohen-Macaulay we conclude that

$$\sum_{i=0}^{n-1} \dim \tilde{H}_{n-3}((\hat{0}, [n]^i)) t^i = \prod_{i=1}^{n-1} ((n-i) + it). \tag{1.3}$$

The Liu and Dotsenko/Khoroshkin result is a consequence of this and (1.2). Since, as was proved by Drake [8], the right hand side of (1.3) is the generating function for rooted trees on $[n]$ with i descents, it follows that $\dim \tilde{H}_{n-3}((\hat{0}, [n]^i))$ is equal to the number of rooted trees on $[n]$ with i descents. In Section 4 we construct a nice combinatorial basis for $\tilde{H}_{n-3}((\hat{0}, [n]^i))$ indexed by such rooted trees, which generalizes Björner's basis for $\tilde{H}_{n-3}(\bar{\Pi}_n)$. A generalization of the comb basis for cohomology of Π_n and the generalization of the Lyndon basis for cohomology of Π_n given by the falling chains of the EL-labeling of $[\hat{0}, [n]^i]$ are also presented in Section 4.

In the final section of this paper, we show that the characteristic polynomial of Π_n^w equals $(x-n)^{n-1}$ and present some consequences. Finally we mention some generalizations of what is presented here.

(i) i.e., the full version of this paper [11]

2 The topology of the poset of weighted partitions

2.1 EL-Shellability

We assume familiarity with basic terminology and results in poset topology; see [22].

For each $i \in [n]$, let $P_i := \{(i,j)^u | i < j \leq n+1,\ u \in \{0,1\}\}$. We partially order P_i by letting $(i,j)^u \leq (i,k)^v$ if $j \leq k$ and $u \leq v$. Note P_i is isomorphic to the direct product of the chain $i+1 < i+2 < \cdots < n+1$ and the chain $0 < 1$. Now define Λ_n to be the ordinal sum $\Lambda_n := P_1 \oplus P_2 \oplus \cdots \oplus P_n$.

The Hasse diagram of the poset Λ_n when $n = 3$ is given in Figure 2(b).

Theorem 2.1 *The labeling* $\lambda : \mathcal{E}(\widehat{\Pi_n^w}) \to \Lambda_n$ *defined by:*

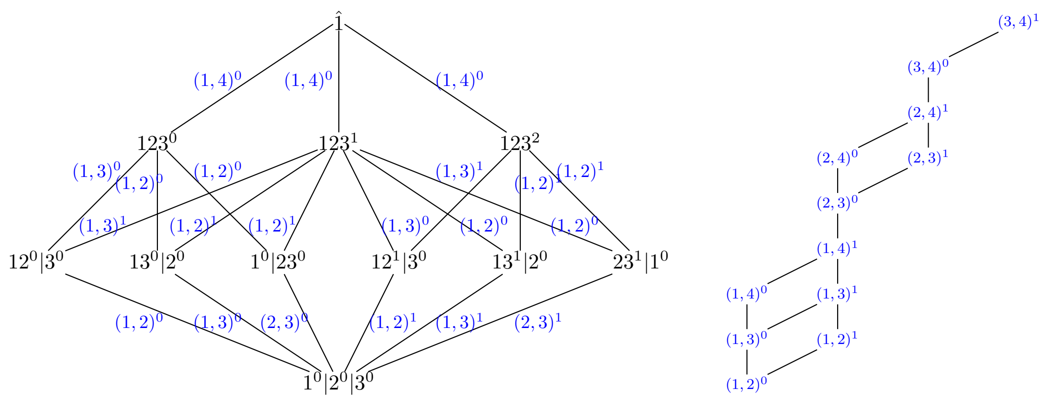
$$\lambda(A_1^{w_1}|A_2^{w_2}|\cdots|A_l^{w_l} \lessdot A_1^{w_1}|A_2^{w_2}|\cdots|(A_i \cup A_j)^{w_i+w_j+u}|\cdots|A_l^{w_l}) := (\min A_i, \min A_j)^u\,,$$

and

$$\tilde{\lambda}([n]^r \lessdot \hat{1}) = (1, n+1)^0,$$

where $\min A_i < \min A_j$*, is an EL-labeling of* Π_n^w.

The proof is given in the full version of this paper. In Figure 2(a) we give an example.



(a) Labeling λ (b) Λ_3

Fig. 2: EL-labeling of the poset $\widehat{\Pi_3^w}$

In [7] Dotsenko and Khoroshkin use operad theory to prove that all intervals of Π_n^w are Cohen-Macaulay. We have the following extension of their result.

Corollary 2.2 *The poset* $\widehat{\Pi_n^w}$ *is Cohen-Macaulay.*

Remark 2.3 In a prior attempt to establish Cohen-Macaulayness of each maximal interval $[\hat{0}, [n]^i]$ of Π_n^w, it is argued in [6] that the intervals are totally semimodular (and hence CL-shellable). In [19] it is noted that this is not the case and a proposed recursive atom ordering of each maximal interval $[\hat{0}, [n]^i]$ is given in order to establish CL-shellability. We note here that one of the requisite conditions in the definition of recursive atom ordering fails to hold when $n = 4$ and $i = 2$. Indeed, assuming (without loss of generality) that the first two atoms in the atom ordering of $[\hat{0}, [4]^2]$ given in [19] are 12^0 and 12^1 (the singleton blocks have been omitted), it is not difficult to see that the proposed atom ordering of the interval $[12^1, [4]^2]$ given in [19] will fail to satisfy the condition that the covers of 12^0 come first.

2.2 Möbius Invariant

Since each maximal interval of Π_n^w is Cohen-Macaulay, we can determine the dimension of the top (unique nonvanishing) (co)homology of the interval by the use of the Möbius invariant. We use the recursive definition of the Möbius function and the compositional formula to derive the following result.

Proposition 2.4 *For all $n \geq 1$,*

$$\sum_{i=0}^{n-1} \mu_{\Pi_n^w}(\hat{0}, [n]^i) t^i = (-1)^{n-1} \prod_{i=1}^{n-1} ((n-i) + it). \tag{2.1}$$

Consequently,

$$\sum_{i=0}^{n-1} \mu_{\Pi_n^w}(\hat{0}, [n]^i) = (-1)^{n-1} n^{n-1}.$$

Let T be a rooted tree on node set $[n]$. A *descent* of T is a node that is smaller than its parent. We denote by $\mathcal{T}_{n,i}$ the set of rooted trees on node set $[n]$ with exactly i descents. In [8] Drake proves that

$$\sum_{i=0}^{n-1} |\mathcal{T}_{n,i}| t^i = \prod_{i=1}^{n-1} ((n-i) + it). \tag{2.2}$$

The following result is a consequence of this and Proposition 2.4.

Corollary 2.5 *For all $n \geq 1$ and $i \in \{0, 1, \ldots, n-1\}$,*

$$\mu_{\Pi_n^w}(\hat{0}, [n]^i) = (-1)^{n-1} |\mathcal{T}_{n,i}|.$$

Recall that if P is an EL-shellable poset of length ℓ then $\Delta(\bar{P})$ has the homotopy type of a wedge of $|\mu(P)|$ spheres of dimension $\ell - 2$. The following result is therefore a consequence of Theorem 2.1 and Corollary 2.5.

Theorem 2.6 *For all $n \geq 1$ and $i \in \{0, 1, \ldots, n-1\}$, the simplicial complex $\Delta((\hat{0}, [n]^i))$ has the homotopy type of a wedge of $|\mathcal{T}_{n,i}|$ spheres of dimension $n - 3$.*

Corollary 2.7 ([6, 7]) *For all $n \geq 1$ and $i \in \{0, 1, \ldots, n-1\}$,*

$$\dim \tilde{H}^{n-3}((\hat{0}, [n]^i)) = |\mathcal{T}_{n,i}| \qquad \textit{and} \qquad \dim \bigoplus_{i=0}^{n-1} \tilde{H}^{n-3}((\hat{0}, [n]^i)) = n^{n-1}.$$

In the full version of the paper we also obtain the following result by using the same methods.

Theorem 2.8 *For all $n \geq 1$, the simplicial complex $\Delta(\Pi_n^w \setminus \{\hat{0}\})$ has the homotopy type of a wedge of $(n-1)^{n-1}$ spheres of dimension $n - 2$.*

3 Connection with the doubly bracketed free Lie algebra

3.1 The doubly bracketed free Lie algebra

Recall that a *Lie bracket* on a vector space V is a bilinear binary product $[\cdot,\cdot] : V \times V \to V$ such that for all $x, y, z \in V$,

$$[x,y] = -[y,x] \qquad \text{(Antisymmetry)} \tag{3.1}$$

$$[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0 \qquad \text{(Jacobi Identity)} \tag{3.2}$$

The *free Lie algebra on* $[n]$ is the complex vector space generated by the elements of $[n]$ and all the possible bracketings involving these elements subject only to the relations (3.1) and (3.2). Let $\mathcal{L}ie(n)$ denote the *multilinear* component of the free Lie algebra on $[n]$, ie., the subspace generated by bracketings that contain each element of $[n]$ exactly once. For example $[[2,3],1]$ is an element of $\mathcal{L}ie(3)$, while $[[2,3],2]$ is not.

Now let V be a vector space equipped with two Lie brackets $[\cdot,\cdot]$ and $\langle\cdot,\cdot\rangle$. The brackets are said to be *compatible* if any linear combination of them is a Lie bracket. As pointed out in [6, 13], compatibility is equivalent to the condition that for all $x, y, z \in V$

$$[x,\langle y,z\rangle] + [z,\langle x,y\rangle] + [y,\langle z,x\rangle] + \langle x,[y,z]\rangle + \langle z,[x,y]\rangle + \langle y,[z,x]\rangle = 0 \quad \text{(Mixed Jacobi)} \tag{3.3}$$

Let $\mathcal{L}ie_2(n)$ be the multilinear component of the free Lie algebra on $[n]$ with two compatible brackets $[\cdot,\cdot]$ and $\langle\cdot,\cdot\rangle$, that is, the multilinear component of the vector space generated by (mixed) bracketings of elements of $[n]$ subject only to the five relations given by (3.1) and (3.2), for each bracket, and (3.3). We will call the bracketed words that generate $\mathcal{L}ie_2(n)$ *bracketed permutations*.

It will be convenient to refer to the bracket $[\cdot,\cdot]$ as the *blue* bracket and the bracket $\langle\cdot,\cdot\rangle$ as the *red* bracket. For each i, let $\mathcal{L}ie_2(n,i)$ be the subspace of $\mathcal{L}ie_2(n)$ generated by bracketed permutations with exactly i red brackets and $n-1-i$ blue brackets.

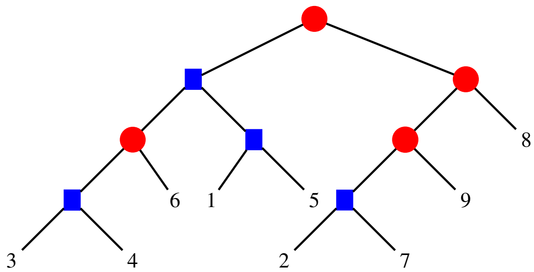
A permutation $\tau \in \mathfrak{S}_n$ acts on the bracketed permutations by replacing each letter i by $\tau(i)$. For example $(1,2)\ \langle[\langle 3,5\rangle,[2,4]],1\rangle = \langle[\langle 3,5\rangle,[1,4]],2\rangle$. Since this action respects the five relations, it induces a representation of $\mathfrak{S}_n$ on $\mathcal{L}ie_2(n)$. Since this action also preserves the number of blue and red brackets, we have the following decomposition into $\mathfrak{S}_n$-submodules: $\mathcal{L}ie_2(n) = \oplus_{i=0}^{n-1} \mathcal{L}ie_2(n,i)$. Note that $\mathcal{L}ie_2(n,i) \simeq_{\mathfrak{S}_n} \mathcal{L}ie_2(n,n-1-i)$ for all i, and that $\mathcal{L}ie_2(n,0) \simeq_{\mathfrak{S}_n} \mathcal{L}ie_2(n,n-1) \simeq_{\mathfrak{S}_n} \mathcal{L}ie(n)$.

A *bicolored binary tree* is a complete planar binary tree (i.e., every internal node has a left child and a right child) for which each internal node has been colored blue or red. For a bicolored binary tree T with n leaves and $\sigma \in \mathfrak{S}_n$, define the *labeled bicolored binary tree* (T,σ) to be the tree T whose jth leaf from left to right has been labeled $\sigma(j)$. We denote by $\mathcal{BT}_n$ the set of labeled bicolored binary trees with n leaves and by $\mathcal{BT}_{n,i}$ the set of labeled bicolored binary trees with n leaves and i red internal nodes. It will also be convenient to allow the leaf labeling σ to be a bijection from $[n]$ to any subset of $\mathbb{Z}^+$ of size n.

We can represent the bracketed permutations that generate $\mathcal{L}ie_2(n)$ with labeled bicolored binary trees. More precisely, let (T_1,σ_1) and (T_2,σ_2) be the respective left and right labeled subtrees of the root r of $(T,\sigma) \in \mathcal{BT}_n$. Then define recursively

$$[T,\sigma] = \begin{cases} [[T_1,\sigma_1],[T_2,\sigma_2]] & \text{if } r \text{ is blue and } n>1 \\ \langle[T_1,\sigma_1],[T_2,\sigma_2]\rangle & \text{if } r \text{ is red and } n>1 \\ \sigma(1) & \text{if } n=1 \end{cases} \tag{3.4}$$

Clearly $(T,\sigma) \in \mathcal{BT}_{n,i}$ if and only if $[T,\sigma]$ is a bracketed permutation of $\mathcal{L}ie_2(n,i)$. See Figure 3.



$$\langle [\langle [3,4],6\rangle ,[1,5]],\langle \langle [2,7],9\rangle ,8\rangle \rangle$$

Fig. 3: Example of a tree $(T,[346152798]) \in \mathcal{BT}_{9,4}$ and $[T,[346152798]] \in \mathcal{L}ie_2(9,4)$

3.2 A generating set for $\tilde{H}^{n-3}((\hat{0},[n]^i))$

The top dimensional cohomology of a pure poset P, say of length ℓ, has a particularly simple description. Let $\mathcal{M}(P)$ denote the set of maximal chains of P and let $\mathcal{M}'(P)$ denote the set of chains of length $\ell-1$. We view the coboundary map δ as a map from the chain space of P to itself, which takes chains of length d to chains of length $d+1$ for all d. Since the image of δ on the top chain space (i.e. the space spanned by $\mathcal{M}(P)$) is 0, the kernel is the entire top chain space. Hence top cohomology is the quotient of the space spanned by $\mathcal{M}(P)$ by the image of the space spanned by $\mathcal{M}'(P)$. The image of $\mathcal{M}'(P)$ is what we call the coboundary relations. We thus have the following presentation of the top cohomology

$$\tilde{H}^{\ell}(P) = \langle \mathcal{M}(P) |\text{ coboundary relations}\rangle.$$

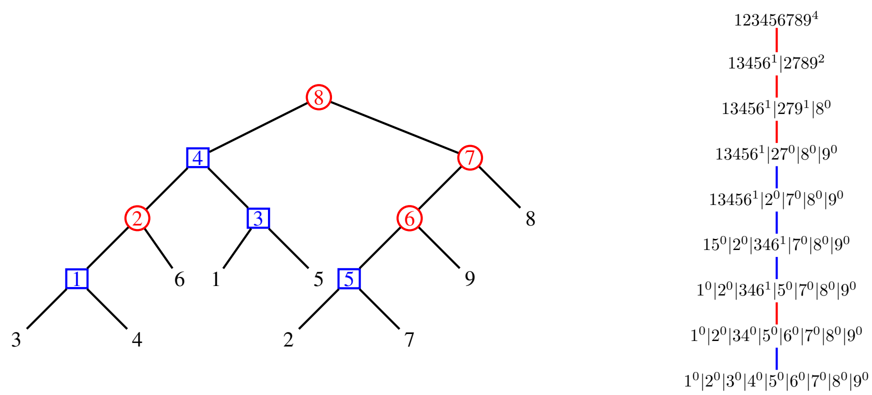
Recall that the *postorder listing* of the internal nodes of a binary tree T is defined recursively as follows: first list the internal nodes of the left subtree in postorder, then list the internal nodes of the right subtree in postorder, and finally list the root.

The postorder listing of the internal nodes of the binary tree of Figure 3 is illustrated in Figure 4(a) by labeling the internal nodes.

Definition 3.1 *For $(T,\sigma) \in \mathcal{BT}_n$, let x_k be the kth internal node in the postorder listing of the internal nodes of T. The chain $\mathrm{c}(T,\sigma) \in \mathcal{M}(\Pi_n^w)$ is the one whose rank k weighted partition π_k is obtained from the rank $k-1$ weighted partition π_{k-1} by merging the blocks L_k and R_k, where $k \geq 1$, L_k is the set of leaf labels in the left subtree of the node x_k, and R_k is the set of leaf labels in the right subtree of the node x_k. The weight attached to the new block $L_k \cup R_k$ is the sum of the weights of the blocks L_k and R_k in π_{k-1} plus u, where $u=0$ if x_k is colored blue and $u=1$ if x_k is colored red. See Figure 4.*

Not all maximal chains in $\mathcal{M}(\Pi_n^w)$ are of the form $\mathrm{c}(T,\sigma)$. It can be shown, however, that any maximal chain $c \in \mathcal{M}(\Pi_n^w)$ is cohomology equivalent to a chain of the form $\mathrm{c}(T,\sigma)$; more precisely, in cohomology $\bar{c} = \pm\bar{\mathrm{c}}(T,\sigma)$, where $\bar{c}$ is the chain obtained from c by removing the top and bottom elements.

Let $I(\Upsilon)$ denote the number of internal nodes of a labeled bicolored binary tree Υ. Let $\Upsilon_1 \overset{\mathrm{col}}{\wedge} \Upsilon_2$ denote a labeled bicolored binary tree whose left subtree is Υ_1, right subtree is Υ_2 and root is colored col, where $\mathrm{col} \in \{\text{blue}, \text{ red}\}$. If Υ is a labeled bicolored binary tree then $\alpha(\Upsilon)\beta$ denotes a labeled bicolored binary tree with Υ as a subtree. The following result generalizes [21, Theorem 5.3].



(a) $(T, \sigma) \in \mathcal{BT}_9$ (b) $\mathrm{c}(T, \sigma)$

Fig. 4: Example of postorder labeling of the binary tree T of Figure 3 and the chain $\mathrm{c}(T, \sigma)$

Theorem 3.2 *The set $\{\bar{\mathrm{c}}(T,\sigma) | (T,\sigma) \in \mathcal{BT}_{n,i}\}$ is a generating set for $\tilde{H}^{n-3}((\hat{0}, [n]^i))$, subject only to the relations*

- $\bar{\mathrm{c}}(\alpha(\Upsilon_1 \overset{\mathrm{col}}{\wedge} \Upsilon_2)\beta) = (-1)^{I(\Upsilon_1)I(\Upsilon_2)}\bar{\mathrm{c}}(\alpha(\Upsilon_2 \overset{\mathrm{col}}{\wedge} \Upsilon_1)\beta)$ *where* $\mathrm{col} \in \{blue, red\}$.
- $\bar{\mathrm{c}}(\alpha(\Upsilon_1 \overset{\mathrm{col}}{\wedge} (\Upsilon_2 \overset{\mathrm{col}}{\wedge} \Upsilon_3))\beta) + (-1)^{I(\Upsilon_3)}\bar{\mathrm{c}}(\alpha((\Upsilon_1 \overset{\mathrm{col}}{\wedge} \Upsilon_2) \overset{\mathrm{col}}{\wedge} \Upsilon_3)\beta)$
  $+ (-1)^{I(\Upsilon_1)I(\Upsilon_2)}\bar{\mathrm{c}}(\alpha(\Upsilon_2 \overset{\mathrm{col}}{\wedge} (\Upsilon_1 \overset{\mathrm{col}}{\wedge} \Upsilon_3))\beta) = 0,$ *where* $\mathrm{col} \in \{blue, red\}$.
- $\bar{\mathrm{c}}(\alpha(\Upsilon_1 \overset{\mathrm{red}}{\wedge} (\Upsilon_2 \overset{\mathrm{blue}}{\wedge} \Upsilon_3))\beta) + (-1)^{I(\Upsilon_3)}\bar{\mathrm{c}}(\alpha((\Upsilon_1 \overset{\mathrm{blue}}{\wedge} \Upsilon_2) \overset{\mathrm{red}}{\wedge} \Upsilon_3)\beta)$
  $+ (-1)^{I(\Upsilon_1)I(\Upsilon_2)}\bar{\mathrm{c}}(\alpha(\Upsilon_2 \overset{\mathrm{red}}{\wedge} (\Upsilon_1 \overset{\mathrm{blue}}{\wedge} \Upsilon_3))\beta) + \bar{\mathrm{c}}(\alpha(\Upsilon_1 \overset{\mathrm{blue}}{\wedge} (\Upsilon_2 \overset{\mathrm{red}}{\wedge} \Upsilon_3))\beta)$
  $+ (-1)^{I(\Upsilon_3)}\bar{\mathrm{c}}(\alpha((\Upsilon_1 \overset{\mathrm{red}}{\wedge} \Upsilon_2) \overset{\mathrm{blue}}{\wedge} \Upsilon_3)\beta) + (-1)^{I(\Upsilon_1)I(\Upsilon_2)}\bar{\mathrm{c}}(\alpha(\Upsilon_2 \overset{\mathrm{blue}}{\wedge} (\Upsilon_1 \overset{\mathrm{red}}{\wedge} \Upsilon_3))\beta) = 0$

The proof is given in the full version of this paper. The bicolored comb basis of Theorem 4.1 and the formula for the Möbius function given in Proposition 2.4 play a key role in showing that these relations generate all the relations.

3.3 The isomorphism

Define the sign of a bicolored binary tree T recursively by $\mathrm{sgn}(T) = 1$ if $I(T) = 0$ and $\mathrm{sgn}(T_1 \overset{\mathrm{col}}{\wedge} T_2) = (-1)^{I(T_2)}\,\mathrm{sgn}(T_1)\,\mathrm{sgn}(T_2)$ otherwise.

Theorem 3.3 *For each $i \in \{0, 1, \ldots, n-1\}$, there is an $\mathfrak{S}_n$-module isomorphism $\phi : \mathcal{L}ie_2(n,i) \to \tilde{H}^{n-3}((\hat{0}, [n]^i)) \otimes \mathrm{sgn}_n$ determined by*

$$\phi([T,\sigma]) = \mathrm{sgn}(\sigma)\,\mathrm{sgn}(T)\bar{\mathrm{c}}(T,\sigma),$$

for all $(T,\sigma) \in \mathcal{BT}_{n,i}$.

The map ϕ maps generators to generators and clearly respects the $\mathfrak{S}_n$ action. In the full version of the paper we prove that the map ϕ is a well-defined $\mathfrak{S}_n$-module isomorphism by showing that the relations for $\mathcal{L}ie_2(n)$ given in (3.1), (3.2), (3.3) map to the relations in Theorem 3.2.

4 Combinatorial bases

4.1 The bicolored comb basis and Lyndon basis for cohomology

In this section we present generalizations of both the comb basis and the Lyndon basis for cohomology of Π_n (see [21]). Define a *normalized tree* to be a labeled bicolored binary tree in which the leftmost leaf of each subtree has the smallest label in the subtree. A *bicolored comb* is a normalized tree in which each internal node with an internal right child is colored red and the right child is colored blue. Note that if a bicolored comb is monochromatic then the right child of every internal node is a leaf. Hence the monochromatic ones are the usual combs in the sense of [21].

Given a normalized tree, we can extend the leaf labeling to the internal nodes by labeling each internal node with the smallest leaf label in its right subtree. A *bicolored Lyndon tree* is a normalized tree in which each internal node that has an internal left child with a smaller label is colored blue and the left child is colored red. It is easy to see that the monochromatic ones are the classical Lyndon trees.

Let $\mathsf{Comb}^2_{n,i} \subseteq \mathcal{BT}_{n,i}$ be the set of all bicolored combs with label set $[n]$ and i red internal nodes and let $\mathsf{Lyn}^2_{n,i} \subseteq \mathcal{BT}_{n,i}$ be the set of all bicolored Lyndon trees with label set $[n]$ and i red internal nodes.

Theorem 4.1 *The sets $\mathcal{C}_{n,i} := \{\bar{c}(T,\sigma) | (T,\sigma) \in \mathsf{Comb}^2_{n,i}\}$ and $\mathcal{L}_{n,i} := \{\bar{c}(T,\sigma) | (T,\sigma) \in \mathsf{Lyn}^2_{n,i}\}$ are bases for $\tilde{H}^{n-3}((\hat{0},[n]^i))$.*

The proof for $\mathcal{C}_{n,i}$ is obtained by "straightening" via the relations in Theorem 3.2 and counting bicolored combs. The proof for $\mathcal{L}_{n,i}$ follows from the observation that $\mathcal{L}_{n,i}$ is the falling chain basis coming from the EL-labeling of $[\hat{0},[n]^i]$ given in Section 2.1.

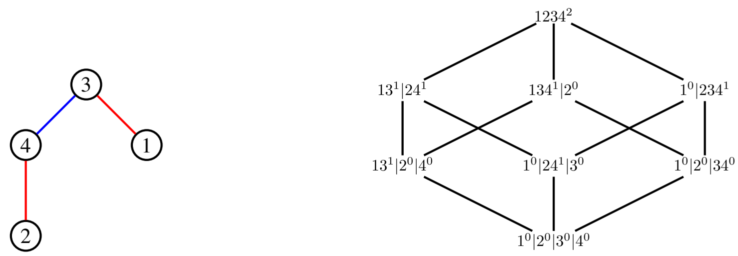
It is a consequence of Theorem 3.3 then that the sets $\{[T,\sigma] | (T,\sigma) \in \mathsf{Comb}^2_{n,i}\}$ and $\{[T,\sigma] | (T,\sigma) \in \mathsf{Lyn}^2_{n,i}\}$ are bases for $\mathcal{L}ie_2(n,i)$. The bicolored comb basis for $\mathcal{L}ie_2(n,i)$ was first obtained in [6]. The bicolored Lyndon basis for $\mathcal{L}ie_2(n,i)$ is not the same as a bicolored Lyndon basis obtained in [13].

It follows from Theorem 4.1 that $|\mathsf{Comb}^2_{n,i}| = |\mathsf{Lyn}^2_{n,i}|$. In [9] an explicit color preserving bijection between the bicolored combs and bicolored Lyndon trees is given. These trees also provide a combinatorial interpretation of γ-positivity of the tree analog of the Eulerian polynomials, $\sum_{T\in\mathcal{T}_n} t^{\operatorname{des}(T)}$ (see [9]).

4.2 The tree basis for homology

We now present a generalization of an NBC basis of Björner for the homology of Π_n (see [3, Proposition 2.2]). Let T be a rooted tree on node set $[n]$. For each subset A of the edge set $E(T)$ of T, let T_A be the subgraph of T with node set $[n]$ and edge set A. Clearly T_A is a forest on $[n]$ consisting of rooted trees $T_1, T_2, \ldots, T_k$.

Consider the weighted partition $\pi(T_A) := \{N(T_i)^{\operatorname{des}(T_i)} \mid i \in [k]\}$, where $N(T_i)$ is the node set of the tree T_i and $\operatorname{des}(T_i)$ is the number of descents of T_i. We define Π_T to be the induced subposet of Π_n^w on the set $\{\pi(T_A) \mid A \subseteq E(T)\}$. The poset Π_T is clearly isomorphic to the boolean algebra $\mathcal{B}_{n-1}$. Hence $\Delta(\overline{\Pi_T})$ is the barycentric subdivision of the boundary of the $(n-2)$-simplex. We let ρ_T denote a fundamental cycle of the spherical complex $\Delta(\overline{\Pi_T})$. See Figure 5 for an example of Π_T.



(a) T (b) Π_T

Fig. 5: Example of a tree T with two descents (in red) and the corresponding poset Π_T

Theorem 4.2 *Let $\mathcal{T}_{n,i}$ be the set of rooted trees on node set $[n]$ with i descents. Then $\{\rho_T \mid T \in \mathcal{T}_{n,i}\}$ is a basis for $\tilde{H}_{n-3}((\hat{0}, [n]^i))$.*

To prove this theorem we only need to show that the set $\{\rho_T \mid T \in \mathcal{T}_{n,i}\}$ is linearly independent since we know from Corollary 2.7 that $\dim \tilde{H}_{n-3}((\hat{0}, [n]^i)) = |\mathcal{T}_{n,i}|$. We prove linear independence (in the full version of the paper) by showing that the the fundamental cycles in $\{\rho_T \mid T \in \mathcal{T}_{n,i}\}$ have a triangular incidence relationship with a basis of maximal chains corresponding (under Theorem 3.3) to a bicolored version of the Lyndon basis for $\mathcal{L}ie_2(n, i)$ proposed by Liu [13].

5 Other combinatorial and algebraic properties

5.1 The characteristic polynomial and rank generating polynomial

Recall that the characterstic polynomial of Π_n factors nicely. In the full version of this paper we prove that the same is true for Π_n^w.

Theorem 5.1 *For all $n \geq 1$, the characteristic polynomial of Π_n^w is given by*

$$\chi_{\Pi_n^w}(x) := \sum_{\alpha \in \Pi_n^w} \mu_{\Pi_n^w}(\hat{0}, \alpha) x^{n-1-\rho(\alpha)} = (x-n)^{n-1},$$

where $\rho(\alpha)$ is the rank of α, and the rank generating function is given by

$$\mathcal{F}(\Pi_n^w, x) := \sum_{\alpha \in \Pi_n^w} x^{\rho(\alpha)} = \sum_{k=0}^{n-1} \binom{n}{k} (n-k)^k x^k.$$

Recall that for a poset P of length r the *Whitney number of the first kind* $w_k(P)$ is the coefficient of x^{r-k} in the characteristic polynomial $\chi_P(x)$ and the *Whitney number of the second kind* $W_k(P)$ is the coefficient of x^k in the rank generating function $\mathcal{F}(P, x)$; see [18]. It follows from Theorem 5.1 that

$$w_k(\Pi_n^w) = (-1)^k \binom{n-1}{k} n^k \quad \text{and} \quad W_k(\Pi_n^w) = \binom{n}{k} (n-k)^k.$$

A pure bounded poset P, with rank function ρ, is said to be *uniform* if there is a family of posets $\{P_i\}$ such that for all $x \in P$ with $\rho(\hat{1}_P) - \rho(x) = i$, the intervals $[x, \hat{1}]$ and P_i are isomorphic. It is easy to see that Π_n and $\widehat{\Pi}_n^w$ are uniform.

Proposition 5.2 ([18, Exercise 3.130(a)]) *Let P be a uniform poset of length r. Then the matrices $[w_{i-j}(P_i)]_{0 \le i,j \le r}$ and $[W_{i-j}(P_i)]_{0 \le i,j \le r}$ are inverses of each other.*

From this and the uniformity of $\widehat{\Pi_n^w}$, we have the following corollary of Theorem 5.1.

Corollary 5.3 *The matrices $[(-1)^{i-j}\binom{i-1}{j-1} i^{i-j}]_{1 \le i,j \le n}$ and $[\binom{i}{j} j^{i-j}]_{1 \le i,j \le n}$ are inverses of each other.*

This result is not new and an equivalent dual version was already obtained by Sagan in [14], also by using essentially Proposition 5.2, but with a completely different poset. So we can consider this to be a new proof of that result.

5.2 Further work

In [10] one of the authors discusses some of the results presented here in greater generality. The author considers partitions with blocks weighted by k-tuples of nonnegative integers that sum to one less the cardinality of the block, and shows that the resulting poset of weighted partitions is EL-shellable. Then he finds the dimensions of the top cohomology of the maximal intervals and gives generalizations of the Lyndon basis and comb basis for these modules. The analogue of the explicit sign twisted isomorphism from the direct sum of the cohomology of the maximal intervals to the multilinear component of the free Lie algebra then can be constructed, this time with k linearly compatible brackets.

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The Gaussian free field and strict plane partitions

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Abstract. We study height fluctuations around the limit shape of a measure on strict plane partitions. It was shown in our earlier work that this measure is a Pfaffian process. We show that the height fluctuations converge to a pullback of the Green's function for the Laplace operator with Dirichlet boundary conditions on the first quadrant. We use a Pfaffian formula for higher moments to show that the height fluctuations are governed by the Gaussian free field. The results follow from the correlation kernel asymptotics which is obtained by the steepest descent method.

Résumé. Nous étudions les fluctuations de la hauteur autour de la forme limite d'une mesure sur les partitions planes strictes. Nous avons déjà montré que cette mesure est un processus Pfaffien. Nous montrons que les fluctuations convergent vers un "pullback" de la fonction de Green pour l'opérateur de Laplace avec des conditions de bord de Dirichlet sur le premier quadrant. Nous utilisons une formule Pfaffienne pour les moments d'ordre supérieur pour montrer que les fluctuations sont gouvernées par le champ libre gaussien. Ces résultats découlent de l'asymptotique du noyau de corrélation qui est obtenue par la méthode du col.

Keywords: Gaussian free field, Pfaffian process, limit shape, height function fluctuations, plane partitions, correlation function asymptotics

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1 Introduction

In this paper we study fluctuations of the height function around the limit shape of a measure on strict plane partitions that was introduced in our earlier paper [V]. In that earlier paper, we found the correlation function for this measure and showed that it is given in terms of a Pfaffian. We used the correlation function to obtain the limit shape of large strict plane partitions. In this paper, we study fluctuations around the limit shape. We show that the fluctuations are governed by the Gaussian free field. We do this by finding the leading term in the asymptotic expansion of the correlation kernel using the steepest descent method.

Kenyon was the first to show that the Gaussian free field describes height fluctuations of two random surface models, see [K1, K2]. After that there have been other similar results [BF, D, Ku, P, RV]. All these processes are determinantal. Here we present a Pfaffian process whose fluctuations are given by the Gaussian free field.

We start by explaining our results in more detail.

1.1 Measure on strict plane partitions

A plane partition is a filling of a Young diagram with positive integers in such a way that each row and each column is a non-increasing sequence of integers, see the left part of Figure 1. A plane partition π can be represented as a 3-dimensional diagram (object), by stacking $\pi_{i,j}$ cubes above the (i,j)-box of its Young diagram, see the right part of Figure 1. The weight of a plane partition π, denoted with $|\pi|$, is the sum of

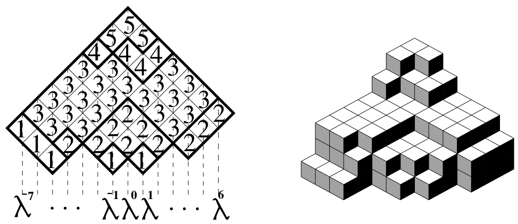

Fig. 1: A plane partition and its 3-dimensional diagram

all entries, or alternatively, the volume of the 3-dimensional diagram. A connected component of a plane partition is a set of rook-wise connected boxes filled with a same integer. In the 3-dimensional diagram connected components are white terraces seen from the above. The number of connected components of a plane partition π is denoted with $k(\pi)$. For the plane partition shown in Figure 1 the number of connected components is 10.

Diagonals of a plane partition are all (ordinary) partitions. A (diagonally) strict plane partition is a partition whose all diagonals are strict partitions. A strict plane partition π can be represented with a 2-dimensional diagram, which is a subset of

$$\mathfrak{X} = \{(t,x) \in \mathbb{Z} \times \mathbb{Z} \mid x > 0\}$$

containing points $(j-i, \pi_{i,j})$, where (i,j) is a box in the Young diagram of π and $\pi_{i,j}$ is its filling number, see Figure 2.

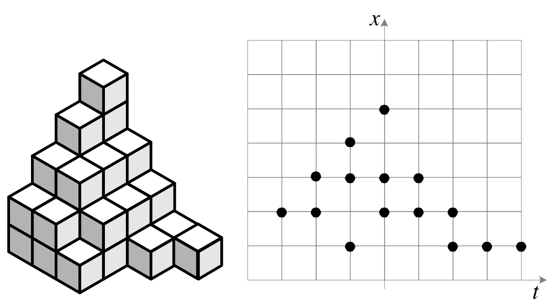

Fig. 2: A strict plane partition and its 2-dimensional diagram

A probability measure $\mathfrak{M}_q$ on the set of strict plane partitions is defined by

$$\mathfrak{M}_q(\pi) \propto 2^{k(\pi)} q^{|\pi|}.$$

The normalization constant is given by the shifted MacMahon formula, see [V]:

$$\sum_{\substack{\pi \text{ is a strict} \\ \text{plane partition}}} 2^{k(\pi)} q^{|\pi|} = \prod_{n=1}^{\infty} \left(\frac{1+q^n}{1-q^n} \right)^n .$$

The limit shape of large strict plane partitions distributed according to $\mathfrak{M}_q$ is shown in Figure 3, see [V]. It is parameterized on the domain $\mathcal{D}$ representing a half of the amoeba of the polynomial $P(z,w) = -1 + z + w + zw$, see (1) for the definition.

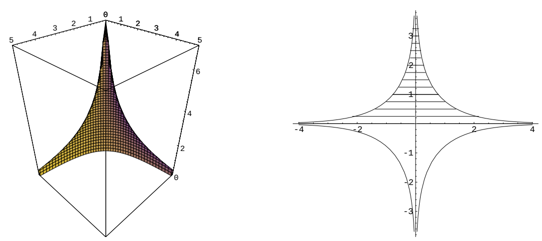

Fig. 3: The limit shape and the half-amoeba (the domain of the limit shape parametrization)

The limit shape for $\tau \geq 0$ (symmetrically for $\tau < 0$) is given by

$$\begin{aligned} x(\tau,\chi) &= R(\tau,\chi), \\ y(\tau,\chi) &= R(\tau,\chi) + \tau, \\ z(\tau,\chi) &= \chi, \end{aligned}$$

for $(\tau, \chi) \in \mathcal{D}$, where

$$\mathcal{D} = \{(\xi, \omega) = (\log|z|, \log|w|) \in \mathbb{R}^2,\ \omega \geq 0 \mid (z,w) \in (\mathbb{C}\backslash\{0\})^2, -1 + z + w + zw = 0\}. \quad (1)$$

and $R(\tau, \chi)$ is the Ronkin function of $-1 + z + w + zw$ at $(\tau/2, -\chi/2)$, i.e.

$$R(\tau,\chi) = \frac{1}{2\pi^2} \int_0^{2\pi} \int_0^{2\pi} \log|1 - e^{\tau/2+iu} - e^{-\chi/2+iv} - e^{\tau/2+iu-\chi+iv}| du dv.$$

The limit shape was obtained using the correlation function that can be written as a Pfaffian of a certain skew-symmetric matrix, more precisely, we exploit the fact that the process is Pfaffian, see below for a definition.

1.2 Determinantal and Pfaffian processes

Let $\mathcal{X}$ be a discrete space. A random point process P on $\mathcal{X}$ is a probability measure on the set $2^{\mathcal{X}}$ of all subsets of $\mathcal{X}$. For $x_1, x_2, \ldots, x_n \in \mathcal{X}$, the correlation function ρ is

$$\rho(x_1, x_2, \ldots, x_n) = \Pr(X \in 2^{\mathcal{X}} | \{x_1, x_2, \ldots, x_n\} \subset X).$$

P is called a determinantal process if there exists a function $K(x, y)$ such that for any finite set $\{x_1, x_2, \ldots, x_n\} \subset \mathcal{X}$ one has

$$\rho(x_1, x_2, \ldots, x_n) = \det(K(x_i, x_j))_{i,j=1}^n.$$

K is called the correlation kernel of the determinantal point process P.

P is called a Pfaffian process if there exists a 2×2 matrix valued kernel such that $K(x, y) = -K^T(y, x)$ and such that for any finite set $(x_1, x_2, \ldots, x_n) \subset \mathcal{X}$ one has

$$\rho(x_1, x_2, \ldots, x_n) = \mathrm{Pf}(K(x_i, x_j))_{i,j=1}^n.$$

The explicit formula for the correlation kernel of $\mathfrak{M}_q$ is given in Section 2 and was derived in our earlier paper. In that paper, we started asymptotic analysis of the correlation kernel and computed the limit shape as a result. Here, we continue with the asymptotic analysis and obtain that the height fluctuations around the limit shape are given by a pullback of the Gaussian free field on the first quadrant. We define the Gaussian free field below.

1.3 The Gaussian free field

Continuous Gaussian free field is a random field (collection of random variables) over a domain $D \in \mathbb{R}^d$. The collection consists of zero mean Gaussians with the covariance

$$E[GFF(z_1)GFF(z_2)] = G(z_1, z_2),$$

where $G(z_1, z_2)$ is the Green's function for the Laplacian on D (i.e. $\Delta G(z_1, z_2) = \delta_{z_1}(z_2)$) satisfying the Dirichlet boundary conditions (i.e. G vanishes on ∂D).

In the case $d = 1$ the Gaussian free field is either the Brownian motion or Brownian bridge. In higher dimensions we need to define it as a random generalized function. Let $C_0^1(D)$ be the set of test functions on D, i.e. smooth compactly supported functions on D, and let $H(D)$ be its Hilbert space closure under the Dirichlet inner product

$$\langle \varphi_1, \varphi_2 \rangle = \int_D \nabla\varphi_1(z) \cdot \nabla\varphi_2(z) |dz|^2 = \langle \varphi_1, \varphi_2 \rangle = \int_D \int_D \varphi_1(z_1)\varphi_2(z_2) G(z_1, z_2) |dz_1|^2 |dz_2|^2.$$

The Gaussian free field GFF is the formal sum $\sum_i \alpha_i f_i$, where $\{f_i\}$ is an orthonormal basis for $H(D)$ and α_i are i.i.d. standard Gaussians.

The higher moments of GFF (in the generalized function sense) are

$$E\left[GFF(z_1)\cdots GFF(z_n)\right] = \begin{cases} \sum_\sigma \prod_{i=1}^{n/2} G(z_{\sigma(2i-1)}, z_{\sigma(2i)}) & n \text{ is even} \\ 0 & n \text{ is odd} \end{cases},$$

where the sum is taken over all fixed point free involutions σ on $\{1, 2, \ldots, n\}$.

Any process whose higher moments are given by the formula above is the Gaussian free field. In this paper we use this property to show that height fluctuations are governed by the Gaussian free field on the first quadrant $Q \subset \mathbb{R}^2$. The Green's function for the Laplace operator with the Dirichlet boundary conditions on Q is given by

$$G(z_1, z_2) = \frac{1}{2\pi} \log \left| \frac{|z_1 - z_2|\,|z_1 + z_2|}{|\bar{z}_1 - z_2|\,|\bar{z}_1 + z_2|} \right|. \tag{2}$$

1.4 Results

The Gaussian free field is a random object that has been associated with many determinantal processes: uniform planar domino tilings [K1], a honeycomb dimer model [K2], a KPZ class model [BF], interlacing particles [D, Ku], uniformly random lozenge tilings of polygons [P], and a random matrix model [RV]. For all these models it has been shown that certain fluctuations behave like the Gaussian free field.

In this paper, we show that this holds for our model too. As far as we know, this is the first time where the Gaussian free field is associated with a Pfaffian process.

For a plane partition π and $(t,x) \in \mathfrak{X}$, we define the height function $h_\pi(t,x)$ by

$$h_\pi(t,x) = \#\{(t,y) \in \pi | y \geq x\}.$$

In the 3-dimensional diagram of π, the height function represents the distance between the top surface and the right wall for $t \geq 0$ and the left wall for $t < 0$, see Figure 2.

For $(t,x) \in \mathfrak{X}$, we define the height fluctuation by

$$H(t,x) = \sqrt{\pi}\left[h(t,x) - E(h(t,x))\right].$$

The main result of this paper is that there is a map between the domain $\mathfrak{X}$, representing strict plane partitions, and the quadrant Q under which the pushforward of H converges to the Gaussian free field on Q.

In this paper we only give an outline of the proofs. The full version of the paper with details will be published somewhere else. The paper is organized as follows. In Section 2 we recall the correlation function of $\mathfrak{M}_q$ which was derived in [V]. In Section 3 we derive the main terms in the asymptotic expansion of the correlation kernel. In Section 4 we use the asymptotic formula to obtain that the covariance of the height fluctuation is given by the Green's function (2). In Section 5 we show that the fluctuations are given by the pullback of the Gaussian free field on the first quadrant. We finish with some possible directions in which this work can be continued. This is given in Section 6.

2 Background

The correlation kernel for the measure $\mathfrak{M}_q$ was derived in [V].

Let

$$J_q(t,z) = \begin{cases} \dfrac{(q^{1/2}z^{-1};q)_\infty(-q^{t+1/2}z;q)_\infty}{(-q^{1/2}z^{-1};q)_\infty(q^{t+1/2}z;q)_\infty} & t \geq 0, \\[2ex] \dfrac{(-q^{1/2}z;q)_\infty(q^{-t+1/2}z^{-1};q)_\infty}{(q^{1/2}z;q)_\infty(-q^{-t+1/2}z^{-1};q)_\infty} & t < 0, \end{cases} \tag{3}$$

where

$$(z;q)_\infty = \prod_{n=0}^{\infty}(1-q^n z)$$

is the quantum dilogarithm function.

Theorem 2.1 *[V] Let $X = \{(t_i, x_i) : i = 1, \ldots, n\} \subset \mathfrak{X}$. The correlation function has the form*

$$\rho(X) = \mathrm{Pf}(M_X)$$

where M_X is a skew-symmetric $2n \times 2n$ matrix

$$M_X(i,j) = \begin{cases} K_{++}^{i,j} = K_{x_i,x_j}(t_i,t_j) & 1 \le i < j \le n, \\ K_{+-}^{i,j'} = (-1)^{x_{j'}} K_{x_i,-x_j'}(t_i,t_{j'}) & 1 \le i \le n < j \le 2n, \\ K_{--}^{i',j'} = (-1)^{x_{i'}+x_{j'}} K_{-x_{i'},-x_{j'}}(t_{i'},t_{j'}) & n < i < j \le 2n, \end{cases}$$

where $i' = 2n - i + 1$ and $K_{x,y}(t_i,t_j)$ is the coefficient of $z^x w^y$ in the formal power series expansion of

$$\frac{z-w}{2(z+w)} J_q(z,t_i) J_q(w,t_j)$$

in the region $|z| > |w|$ if $t_i \ge t_j$ and $|z| < |w|$ if $t_i < t_j$.

By the definition,

$$\begin{aligned} K_{++}^{i,j} &= \frac{1}{(2\pi i)^2} \iint\limits_{\substack{|z|=1\pm\epsilon \\ |w|=1\mp\epsilon}} \frac{z-w}{2(z+w)} J_q(t_i,z) J_q(t_j,w) \frac{1}{z^{x_i+1} w^{x_j+1}} dz dw, \\ K_{+-}^{i,j} &= \frac{(-1)^{x_j}}{(2\pi i)^2} \iint\limits_{\substack{|z|=1\pm\epsilon \\ |w|=1\mp\epsilon}} \frac{z-w}{2(z+w)} J_q(t_i,z) J_q(t_j,w) \frac{1}{z^{x_i+1} w^{-x_j+1}} dz dw, \\ K_{--}^{i,j} &= \frac{(-1)^{x_i+x_j}}{(2\pi i)^2} \iint\limits_{\substack{|z|=1\pm\epsilon \\ |w|=1\mp\epsilon}} \frac{z-w}{2(z+w)} J_q(t_i,z) J_q(t_j,w) \frac{1}{z^{-x_i+1} w^{-x_j+1}} dz dw, \end{aligned}$$

where we take the upper signs if $t_i \ge t_j$ and the lower signs otherwise.

Observe that $J_q(-z) = 1/J_q(z)$. If we make the change of variables $w \mapsto -w$ in the second kernel and $z \mapsto -z$ and $w \mapsto -w$ in the third one we obtain

$$\begin{aligned} K_{++}^{i,j} &= \frac{1}{(2\pi i)^2} \iint\limits_{\substack{|z|=1\pm\epsilon \\ |w|=1\mp\epsilon}} \frac{z-w}{2(z+w)} J_q(t_i,z) J_q(t_j,w) \frac{1}{z^{x_i+1} w^{x_j+1}} dz dw, \\ K_{+-}^{i,j} &= \frac{1}{(2\pi i)^2} \iint\limits_{\substack{|z|=1\pm\epsilon \\ |w|=1\mp\epsilon}} \frac{z+w}{2(z-w)} \frac{J_q(t_i,z)}{J_q(t_j,w)} \frac{1}{z^{x_i+1} w^{-x_j+1}} dz dw, \\ K_{--}^{i,j} &= \frac{1}{(2\pi i)^2} \iint\limits_{\substack{|z|=1\pm\epsilon \\ |w|=1\mp\epsilon}} \frac{z-w}{2(z+w)} \frac{1}{J_q(t_i,z) J_q(t_j,w)} \frac{1}{z^{-x_i+1} w^{-x_j+1}} dz dw. \end{aligned}$$

3 Asymptotics of the correlation kernel

Let (τ_i, χ_i) and (τ_j, χ_j) be two different fixed points in $\mathcal{D}$ and $r = -\log q$. We are interested in finding the first terms in the asymptotic expansion of the kernels $K^{i,j}_{++}$, $K^{i,j}_{+-}$ and $K^{i,j}_{--}$ where $rt_i \to \tau_i$, $rt_j \to \tau_j$, $rx_i \to \chi_i$ and $rx_j \to \chi_j$ when $r \to +0$. We do this using the steepest descent method.

We start with a simpler problem first. Let $rt \to \tau$ and $rx \to \chi$ when $r \to +0$ for $(\tau, \chi) \in \mathcal{D}$. We want to find the main term in the asymptotic expansion of

$$I = \frac{1}{2\pi i} \int\limits_{|z|=e^{\tau/2}} \frac{1}{z} J_q(t,z) \left(\frac{z}{e^{\tau/2}}\right)^{-x} dz. \tag{4}$$

This integral contains terms of the form

$$\exp\left[\log(z;q)_\infty\right],$$

see (3), so we need the asymptotic behavior of $\log(z;q)_\infty$. We have that

$$\log(z;q)_\infty = -\frac{1}{r}\mathrm{dilog}(z) - \frac{1}{2}\log(1-z) + O(r), \quad r \to +0,$$

where

$$\mathrm{dilog}(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}$$

is the dilogarithm function which is an analytic function everywhere in $\mathbb{C}^2 \backslash (1, +\infty)$.

Then the exponentially large term in (4) that determines the asymptotics is

$$\exp\left[\log[J_q(t,z)(z/e^{\tau/2})^{-x}]\right] = \exp\left[\frac{1}{r}S(z,\tau,\chi)\right]\left[G(z,\tau) + O(r)\right], \quad r \to +0,$$

where

$$S(z,\tau,\chi) = -\mathrm{dilog}(-e^{-\tau}z) - \mathrm{dilog}(z^{-1}) + \mathrm{dilog}(e^{-\tau}z) + \mathrm{dilog}(-z^{-1}) - \chi(\log z - \tau/2)$$

and

$$G(z,\tau) = \sqrt{\frac{(1+e^{-\tau}z)(z-1)}{(1-e^{-\tau}z)(z+1)}},$$

with branch cuts $(-\infty, -e^{\tau})$, $(-1,0)$, $(0,1)$ and (e^{τ}, ∞) .

So, we need to find the main term in the asymptotic expansion of

$$\frac{1}{2\pi i} \int\limits_{|z|=e^{\tau/2}} \frac{1}{z} G(z,\tau) \exp\left[\frac{1}{r}S(z,\tau,\chi)\right] dz.$$

We deform the contour $|z| = e^{\tau/2}$ to a new contour γ such that $\mathrm{Re}S(z,\tau,\chi) < 0$ except at the two critical points that lie on $|z| = e^{\tau/2}$. In addition, we choose γ that passes through two line segments in the direction of the steepest descent of ReS, which we denote with $\gamma_{\rm loc}$ and $-\overline{\gamma}_{\rm loc}$. The main asymptotic term will come from the portion of the integral obtained by the integration over these line segments.

The two critical points are $z_c = z(\tau,\chi)$ and $\overline{z}_c = \overline{z(\tau,\chi)}$, where

$$z(\tau,\chi) = \exp\left[\tau/2 + i \arccos \frac{(e^\tau+1)(e^\chi-1)}{2e^{\tau/2}(e^\chi+1)}\right]. \tag{5}$$

The two line segments in the direction of the steepest descent of ReS, $\gamma_{\rm loc}$ and $-\overline{\gamma}_{\rm loc}$, are given by

$$\gamma_{\rm loc} = z_{\rm c} + \hat\theta_{z_{\rm c}} x, \quad \hat\theta_{z_{\rm c}} = e^{i\gamma_{z_{\rm c}}}, x\in[-\delta,\delta], \delta>0,$$

where

$$\gamma_{z_c} = \frac{\pi}{2} - \frac{\arg S''(z_c,\tau,\chi)}{2} = \frac{\pi}{4} + \arg(z_c).$$

Let γ_1 be the portion of γ without $\gamma_{\rm loc}$ and $-\overline{\gamma}_{\rm loc}$. Let $M>0$ be such that $\mathrm{Re}(S(z,\tau,\chi)) \le -M$ on γ_1. Then the portion of the integral coming from the integration over γ_1 is

$$I_1 = \frac{1}{2\pi i}\int_{\gamma_1} \frac{1}{z} G(z,\tau) \exp\left[\frac{1}{r} S(z,\tau,\chi)\right] = O\left(\exp\left[-M/r\right]\right).$$

Now, for

$$\begin{aligned} I_{\rm loc} &= \frac{1}{2\pi i}\int_{\gamma_{\rm loc}} \frac{1}{z} G(z,\tau) \exp\left[\frac{1}{r} S(z,\tau,\chi)\right] \\ &= \frac{1}{2\pi i}\int_{-\delta}^{\delta} \frac{1}{z_{\rm c}} \left(G(z_c,\tau)+O(x)\right) \exp\left[\frac{1}{r} S(z_c,\tau,\chi) - \frac{1}{2r}x^2|S''(z_c,\tau,\chi)| + \frac{1}{r}O(x^3)\right]\hat\theta_{z_{\rm c}} dx \end{aligned}$$

we get (using Gaussian integral)

$$I_{\rm loc} = \frac{1}{2\pi i}\exp\left[\frac{1}{r}S(z_c,\tau,\chi)\right]\frac{\hat\theta_{z_{\rm c}}}{z_{\rm c}} G(z_c,\tau)\sqrt{\frac{2\pi r}{|S''(z_c,\tau,\chi)|}}\left(1+O(\sqrt{r})\right).$$

Similarly, the integral over $-\overline{\gamma}_{\rm loc}$ is

$$\frac{1}{2\pi i}\exp\left[\frac{1}{r}S(\overline{z}_c,\tau,\chi)\right]\frac{\hat\theta_{\overline{z}_{\rm c}}}{\overline{z}_{\rm c}} G(\overline{z}_c,\tau)\sqrt{\frac{2\pi r}{|S''(\overline{z}_c,\tau,\chi)|}}\left(1+O(\sqrt{r})\right).$$

Finally, combining

$$I = \mathrm{Re}\left(\frac{1}{\pi i}\exp\left[\frac{1}{r}\mathrm{Im}S(z_c,\tau,\chi)\right]\frac{\hat\theta_{z_{\rm c}}}{z_{\rm c}} G(z_c,\tau)\sqrt{\frac{2\pi r}{|S''(z_c,\tau,\chi)|}}\right)\left(1+O(\sqrt{r})\right).$$

3.1 Correlation kernel asymptotics

Now, we are ready to find the main terms in the asymptotic expansions of the kernels $K^{i,j}_{++}$, $K^{i,j}_{+-}$ and $K^{i,j}_{--}$. For convenience, we take out exponential terms in the expressions for the kernels (they cancel out in the Pfaffian formula for the correlation function). For example,

$$K^{i,j}_{++} = e^{(-\tau_i x_i - \tau_j x_j)/2} \cdot I^{i,j}_{++},$$

where

$$I^{i,j}_{++} = \frac{1}{(2\pi i)^2} \iint \frac{z-w}{2zw(z+w)} J_q(t_i, z) J_q(t_j, w) \left(\frac{z}{e^{\tau_i/2}}\right)^{-x_i} \left(\frac{w}{e^{\tau_j/2}}\right)^{-x_j} dz dw,$$

with integration contours $|z| = e^{\tau_i/2}$ and $|w| = e^{\tau_j/2}$.

The main term in the asymptotics of $I^{i,j}_{++}$ will be the same as the main term in the asymptotics of

$$\frac{1}{(2\pi i)^2} \iint \frac{z-w}{2zw(z+w)} G(z, \tau_i) G(w, \tau_j) \exp\left[\frac{1}{r}\left[S(z, \tau_i, \chi_i) + S(w, \tau_j, \chi_j)\right]\right] dz dw,$$

with integration contours $|z| = e^{\tau_i/2}$ and $|w| = e^{\tau_j/2}$.

Using the same steepest descent analysis as for I we get the following:

Theorem 3.1 *When $r \to 0+$*

$$\begin{aligned}
I^{ij}_{++} &\approx \frac{r}{4\pi} \overline{\sum} \frac{z_c - w_c}{z_c + w_c} \frac{1}{z_c w_c} \frac{1}{\sqrt{S''(z_c) S''(w_c)}} F(z_c) F(w_c) \\
I^{ij}_{+-} &\approx \frac{ri}{4\pi} \overline{\sum} \frac{z_c + w_c}{z_c - w_c} \frac{1}{z_c w_c} \frac{1}{\sqrt{S''(z_c) S''(w_c)}} \frac{F(z_c)}{F(w_c)} (-1)^{\mathbb{1}(w_c)} \\
I^{ij}_{--} &\approx \frac{r}{4\pi} \overline{\sum} \frac{z_c - w_c}{z_c + w_c} \frac{1}{z_c w_c} \frac{1}{\sqrt{S''(z_c) S''(w_c)}} \frac{1}{F(z_c) F(w_c)} (-1)^{\mathbb{1}(z_c) + \mathbb{1}(w_c)},
\end{aligned}$$

where

$$\overline{\sum} f(z, w) = f(z, w) + f(\overline{z}, w) + f(z, \overline{w}) + f(\overline{z}, \overline{w}),$$

$$F(z) = G(z, \tau) \exp\left[\frac{1}{r} S(z, \tau, \chi)\right], \quad \mathbb{1}(z) = \begin{cases} 0 & \mathrm{Re} z > 0 \\ 1 & \mathrm{Re} z < 0. \end{cases}$$

$$z_c = z(\tau_i, \chi_i),\ w_c = z(\tau_j, \chi_j).$$

4 Covariance

Recall, the height function $h_\pi(t, x)$ is defined by $h_\pi(t, x) = \#\{(t, y) \in \pi | y \geq x\}$ and fluctuations by $H(t, x) = \sqrt{\pi}\,[h(t, x) - E(h(t, x))]$. We want to show

Theorem 4.1

$$\lim_{r \to 0+} E\left(H(t_1, x_1) H(t_2, x_2)\right) = G(z_1, z_2)$$

when $r t_i \to \tau_i$ and $r x_i \to \chi_i$ for $(\tau_i, \chi_i) \in \mathcal{D}$, where $z_i = z(\tau_i, \chi_i)$, see (5). G is the Green's function given by (2).

We start with

$$
\begin{aligned}
& E(h(t_1,x_1)h(t_2,x_2)) - E(h(t_1,x_1))E(h(t_2,x_2)) \\
= & \sum_{y_1\geq x_1}\sum_{y_2\geq x_2}\rho_2\left((t_1,y_1),(t_2,y_2)\right) - \sum_{y_1\geq x_1}\rho_1\left(t_1,y_1\right)\sum_{y_2\geq x_2}\rho_1\left(t_2,y_2\right) \\
= & \sum_{y_1\geq x_1}\sum_{y_2\geq x_2}\left(\mathrm{Pf}\begin{bmatrix} 0 & K_{++}^{y_1,y_2} & K_{+-}^{y_1,y_2} & K_{+-}^{y_1,y_1} \\ & 0 & K_{+-}^{y_2,y_2} & K_{+-}^{y_2,y_1} \\ & & 0 & K_{--}^{y_2,y_1} \\ & & & 0 \end{bmatrix} - \mathrm{Pf}\begin{bmatrix} 0 & K_{+-}^{y_1,y_1} \\ & 0 \end{bmatrix}\mathrm{Pf}\begin{bmatrix} 0 & K_{+-}^{y_2,y_2} \\ & 0 \end{bmatrix}\right) \\
= & \sum_{y_1\geq x_1}\sum_{y_2\geq x_2} K_{++}^{y_1,y_2}K_{--}^{y_2,y_1} - K_{+-}^{y_1,y_2}K_{+-}^{y_2,y_1}
\end{aligned}
$$

Since, $K_{++}^{1,2}K_{--}^{2,1} - K_{+-}^{1,2}K_{+-}^{2,1} = I_{++}^{1,2}I_{--}^{2,1} - I_{+-}^{1,2}I_{+-}^{2,1}$ we start with

$$\lim_{r\to+0}\sum_{\substack{ry_1\geq\chi_1\\ y_1\in\mathbb{N}}}\sum_{\substack{ry_2\geq\chi_2\\ y_2\in\mathbb{N}}} I_{++}((t_1,y_1),(t_2,y_2))I_{--}((t_1,y_1),(t_2,y_2)). \tag{6}$$

By Theorem 3.1 we have that this is equal to

$$
\begin{aligned}
& \lim_{r\to+0}\sum_{\substack{ry_1\geq\chi_1\\ y_1\in\mathbb{N}}}\sum_{\substack{ry_2\geq\chi_2\\ y_2\in\mathbb{N}}}\frac{r}{4\pi}\overline{\sum}\frac{z_c-w_c}{z_c+w_c}\frac{1}{z_cw_c}\frac{1}{\sqrt{S''(z_c)S''(w_c)}}F(z_c)F(w_c) \\
& \cdot\frac{r}{4\pi}\overline{\sum}\frac{z_c-w_c}{z_c+w_c}\frac{1}{z_cw_c}\frac{1}{\sqrt{S''(z_c)S''(w_c)}}\frac{1}{F(z_c)F(w_c)}(-1)^{\mathbb{1}(z_c)\mathbb{1}(w_c)},
\end{aligned}
$$

where $z_c = z(\tau_1,\chi_1)$ and $w_c = z(\tau_2,\chi_2)$, see (5). When we expand the product, there are 16 terms in the above expression and in the limit only 4 terms survive. We get that the above limit is equal to

$$\lim_{r\to+0}\sum_{\substack{ry_1\geq\chi_1\\ y_1\in\mathbb{N}}}\sum_{\substack{ry_2\geq\chi_2\\ y_2\in\mathbb{N}}}\frac{r^2}{(4\pi)^2}\overline{\sum}\left(\frac{z_c-w_c}{z_c+w_c}\right)^2\frac{1}{z_c^2w_c^2}\frac{1}{S''(z_c)S''(w_c)}(-1)^{\mathbb{1}(z_c)\mathbb{1}(w_c)},$$

The above sum can be written as an integral

$$\int_{\chi_1}^{b(\chi_1)}\int_{\chi_2}^{b(\chi_2)}\frac{1}{4\pi^2}\overline{\sum}\left(\frac{z_c-w_c}{z_c+w_c}\right)^2\frac{1}{(z_cw_c)^2}\frac{1}{S''(z_c)S''(w_c)}(-1)^{\mathbb{1}(z_c)\mathbb{1}(w_c)}d\chi_1 d\chi_2,$$

where b gives the boundary of $\mathcal{D}$. Because $S''(z) = \dfrac{1}{z}\dfrac{\partial z}{\partial\chi}$, this integral is equal to

$$
\begin{aligned}
& \frac{1}{4\pi^2}\int_{\substack{|z|=e^{\tau/2}\\ z_c\to\overline{z}_c}}\int_{\substack{|w|=e^{\tau/2}\\ w_c\to\overline{w}_c}}\frac{1}{zw}\left(\frac{z-w}{z+w}\right)^2 dzdw \\
= & \frac{1}{4\pi^2}\left[\log\frac{z_c}{\overline{z}_c}\log\frac{w_c}{\overline{w}_c} + 2\log\left|\frac{z_c+w_c}{\overline{z}_c+w_c}\right|\right].
\end{aligned}
$$

Similarly,

$$\lim_{r\to+0} \sum_{\substack{ry_1\geq\chi_1\\ y_1\in\mathbb{N}}} \sum_{\substack{ry_2\geq\chi_2\\ y_2\in\mathbb{N}}} I_{+-}((t_1,y_1),(t_2,y_2))I_{+-}((t_1,y_1),(t_2,y_2))$$

$$= \frac{1}{4\pi^2}\left[\log\frac{z_c}{\overline{z}_c}\log\frac{w_c}{\overline{w}_c} - 2\log\left|\frac{z_c-w_c}{\overline{z}_c-w_c}\right|\right]. \tag{7}$$

When we combine both limits (6) and (7) we get

$$\frac{1}{2\pi^2}\log\frac{|z_c+w_c||z_c-w_c|}{|\overline{z}_c+w_c||\overline{z}_c-w_c|} = \frac{1}{\pi}G(z_c,w_c).$$

5 Higher moments. The Gaussian free field

We now show that the height fluctuations are given by the pullback of the Gaussian free field on the first quadrant Q by $z = z(\tau,\chi)$ (defined by (5)):

Theorem 5.1 *Let $rt_i \to \chi_i$ and $rx_i \to \tau_i$ when $r \to 0+$ then*

$$\lim_{r\to 0+} E[H(t_1,x_1)\cdots H(t_n,x_n)] = \begin{cases} \sum_\sigma \prod_{i=1}^{n/2} G(z_{\sigma(2i-1)}, z_{\sigma(2i)}) & n \textit{ is even} \\ 0 & n \textit{ is odd} \end{cases},$$

where the sum is taken over all fixed point free involutions σ on $\{1,2,\ldots,n\}$ and G is given by (2).

In order to prove the theorem we first derive a Pfaffian formula for the higher moments.

Lemma 5.2 *Let $X = \{(t_i,x_i) : i = 1,\ldots,n\} \subset \mathfrak{X}$.*

$$E[H(t_1,x_1)\cdots H(t_n,x_n)] = (\sqrt{\pi})^n \mathrm{Pf}[M_X^0]$$

where M_X^0 is the same as M_X in Theorem 2.1 except it has zeros on the minor diagonal too (they both have zeros on the main diagonal).

The formula is derived in a similar way as in the case $n = 2$ which was done in Section 4.

It remains to show that higher moments are given by the sum over fixed point free involutions of the products of the Green's function. This is done using the following lemma.

Lemma 5.3 *Let $M = (K_{i,j})_{i,j=1\ldots n}$*

$$K_{i,j} = \begin{pmatrix} \dfrac{z_i - z_j}{z_i + z_j} & \dfrac{z_i + z_j}{z_i - z_j} \\[2ex] \dfrac{z_i + z_j}{z_i - z_j} & \dfrac{z_i - z_j}{z_i + z_j} \end{pmatrix}$$

Then

$$Pf[M] = \begin{cases} \sum_\sigma \prod_{i=1}^{n/2} G_{\sigma(2i-1),\sigma(2i)} & n \textit{ is even} \\ 0 & n \textit{ is odd} \end{cases},$$

where $G_{i,j} = -\dfrac{8z_iz_j(z_i^2+z_j^2)}{(z_i^2-z_j^2)^2}$.

6 Concluding Remarks

We have found that the Gaussian free field arises from $\mathfrak{M}_q$ which is a Pfaffian process. Previously, this was found only for determinantal processes. We could generalize this result to a class of Pfaffian processes whose kernel possesses certain properties. The properties arise from technical requirements. It would be interesting to construct physical models with these kernels.

We would like to draw attention to Lemma 5.3. As was said before, several determinantal processes were studied that gave rise to the Gaussian free field. The proofs require a small lemma similar to Lemma 5.3. It is interesting that although the models are very different, and that kernels have different asymptotics they all need the same small lemma to transform the determinantal formula for higher moments into the sum of products over different fixed point free involutions. We plan to study other Pfaffian processes in the future and see if they give rise to the Gaussian free field. It would be interesting to see if we would need Lemma 5.3 for the proof. If yes, can we understand why this is the case?

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Antipode and Convolution Powers of the Identity in Graded Connected Hopf Algebras

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Abstract. We study convolution powers id^{*n} of the identity of graded connected Hopf algebras H. (The antipode corresponds to $n = -1$.) The chief result is a complete description of the characteristic polynomial—both eigenvalues and multiplicity—for the action of the operator id^{*n} on each homogeneous component H_m. The multiplicities are independent of n. This follows from considering the action of the (higher) Eulerian idempotents on a certain Lie algebra $\mathfrak{g}$ associated to H. In case H is cofree, we give an alternative (explicit and combinatorial) description in terms of palindromic words in free generators of $\mathfrak{g}$. We obtain identities involving partitions and compositions by specializing H to some familiar combinatorial Hopf algebras.

Résumé. Nous étudions les puissances de convolution id^{*n} de l'identité d'une algèbre de Hopf graduée et connexe H quelconque. (L'antipode correspond à $n = -1$.) Le résultat principal est une description complète du polynôme caractéristique (des valeurs propres et de leurs multiplicités) de l'opérateur id^{*n} agissant sur chaque composante homogène H_m. Les multiplicités sont indépendants de n. Ceci résulte de l'examen de l'action des idempotents eulériens (supérieures) sur une algèbre de Lie $\mathfrak{g}$ associé à H. Dans le cas où H est colibre, nous donnons une description alternative (explicite et combinatoire) en termes de mots palindromes dans les générateurs libres de $\mathfrak{g}$. Nous obtenons des identités impliquant des partitions et compositions en choisissant comme H certaines algèbres de Hopf combinatoires connues.

Keywords: Hopf power, antipode, Eulerian idempotent, graded connected Hopf algebra, Schur indicator.

Dedicated to the memory of Jean-Louis Loday.

1 Introduction

As the practice of algebraic combinatorics often involves breaking and joining like combinatorial structures (planar trees, permutations, set partitions, etc.), it is right to say that bialgebras are ubiquitous in the theory. This was the argument put forth by G.C. Rota and others, and increasingly, researchers are taking it to heart. On the other hand, the defining property of "Hopf algebra"—the existence of the *antipode*—is

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less often explicitly considered. To be sure, there is a general result stating that the bialgebras built within algebraic combinatorics are *automatically* Hopf algebras (see Section 2).

The *antipode problem* (Aguiar and Mahajan, 2013, Section 5.4) asks for explicit knowledge of the antipode. This can be a source of interesting combinatorial results. Consider Lam et al. (2011), where the antipode plays a crucial role in proving a skew-Littlewood Richardson rule conjectured in Assaf and McNamara (2011). Here is a small illustration of the utility of the antipode (an application we obtain in Section 6.1). If $p_k(n)$ denotes the number of partitions of length k of a positive integer n, and $c(n)$ denotes the number of self-conjugate partitions of n, then

$$(-1)^n\, c(n) = \sum_{k=1}^{n} (-1)^k\, p_k(n).$$

If a Hopf algebra H is commutative or cocommutative, then it is well-known that its antipode $S\colon H \to H$ is an involution: $S^2 = \mathrm{id}$. In particular, its eigenvalues are ± 1. We prove in Corollary 5 that in case H is graded connected, the eigenvalues of the antipode are always ± 1, regardless of (co)commutativity, even if S may have infinite order on any homogeneous component. This is a consequence of our main result (Theorem 4), which provides a complete description of the characteristic polynomial for the convolution power id^{*n} acting on each homogeneous component of H. (The antipode satisfies $S = \mathrm{id}^{*(-1)}$.)

This note is organized as follows. In Section 2, we introduce the Hopf and Lie preliminaries needed to state and prove Theorem 4, which is carried out in Section 3. In Section 4, we give two refinements of our main result in the presence of additional (co)freeness assumptions. Section 5 applies the preceding to higher Schur indicators, and Section 6 provides illustrations of the results and derives some applications.

2 Hopf and Lie preliminaries

Throughout, we assume $\Bbbk$ is a field of characteristic zero. A Hopf algebra is a vector space H over $\Bbbk$ with a host of maps—product ($\mu\colon H \otimes H \to H$), unit ($\iota\colon \Bbbk \to H$), coproduct ($\Delta\colon H \to H \otimes H$), counit ($\varepsilon\colon H \to \Bbbk$), and antipode ($S\colon H \to H$)—satisfying various compatibility axioms, *e.g.*, Δ and ε are algebra maps. The convolution product of two linear maps $P, Q\colon H \to H$ is defined by $P * Q := \mu \circ (P \otimes Q) \circ \Delta$. This is an associative product, making $\mathrm{End}(H)$ into a $\Bbbk$-algebra, with unit element $\iota\varepsilon$. The antipode is the convolution-inverse of the identity map id; that is, $S * \mathrm{id} = \iota\varepsilon = \mathrm{id} * S$.

2.1 Coradical filtration and primitive elements

Let $H_{(0)}$ denote the *coradical* of a Hopf algebra H. This is the sum of the simple subcoalgebras of H. Given any two subspaces U, V of H, define their wedge by

$$U \wedge V := \Delta^{-1}(U \otimes H + H \otimes V). \tag{1}$$

Putting $H_{(n)} = H_{(0)} \wedge H_{(n-1)}$ for all $n \geq 1$ affords H with the *coradical filtration:*

$$H_{(0)} \subseteq H_{(1)} \subseteq \cdots \subseteq H_{(n)} \subseteq \cdots H \quad \text{and} \quad H = \bigcup_{n\geq 0} H_{(n)}. \tag{2}$$

The unit element (as well as any other *group-like* element) of H belongs to $H_{(0)}$. The Hopf algebra H is *connected* if $H_{(0)}$ is spanned by the unit element. In this case, $H_{(1)} = H_{(0)} \oplus \mathcal{P}(H)$, where

$$\mathcal{P}(H) = \{x \in H \mid \Delta(x) = 1 \otimes x + x \otimes 1\}$$

is the space of *primitive elements* of H. It is a Lie subalgebra of H under the commutator bracket.

If $\mathfrak{g}$ is a Lie algebra, its universal enveloping algebra $U(\mathfrak{g})$ is a Hopf algebra for which the space of primitive elements is $\mathfrak{g}$. If H is connected and cocommutative, the Cartier–Milnor–Moore (CMM) theorem states that H is isomorphic as Hopf algebra to $U(\mathcal{P}(H))$. The Poincaré–Birkhoff–Witt (PBW) identifies the vector space $U(\mathfrak{g})$ with $S(\mathfrak{g})$, the symmetric algebra on $\mathfrak{g}$.

We associate a commutative Hopf algebra to any connected Hopf algebra. This will enable us to use the above Lie machinery in a wider class of algebras. Given a Hopf algebra with coradical filtration $H = \bigcup_{n\geq 0} H_{(n)}$, let $\operatorname{gr} H$ denote the associated graded space

$$\operatorname{gr} H = H_{(0)} \oplus \big(H_{(1)}/H_{(0)}\big) \oplus \big(H_{(2)}/H_{(1)}\big) \oplus \big(H_{(3)}/H_{(2)}\big) \oplus \cdots . \tag{3}$$

It is a graded Hopf algebra for which the component of degree n is $H_{(n)}\big/H_{(n-1)}$ (Montgomery, 1993, Ch. 5). If H is connected, then $\operatorname{gr} H$ is commutative by a result of Foissy (Aguiar and Sottile, 2005b, Proposition 1.6 and Remark 1.7).

The Hopf algebra H is *graded* if there is given a vector space decomposition $H = \bigoplus_{m\geq 0} H_m$ such that $\mu(H_p \otimes H_q) \subseteq H_{p+q}$, $\Delta(H_m) \subseteq \bigoplus_{p+q=m} H_p \otimes H_q$, $S(H_m) \subseteq H_m$, $1 \in H_0$, and $\epsilon(H_m) = 0$ for all $m > 0$. In this situation, $H_{(0)} \subseteq H_0$. It follows that if $\dim H_0 = 1$, then H is connected. In this case we say that H is *graded connected* and we have that $H_m \subseteq H_{(m)}$ for all m.

If H is graded, then so is each subspace $H_{(n)}$ with $(H_{(n)})_m = H_{(n)} \cap H_m$. Hence, $\operatorname{gr} H$ inherits a second grading for which $(\operatorname{gr} H)_m$ is the direct sum of the spaces $(H_{(n)})_m\big/(H_{(n-1)})_m$.

2.2 Antipode and Eulerian idempotents

Let H be a connected Hopf algebra. We introduce some notation useful for discussing convolution powers. Put $\Delta^{(0)} = \mathrm{id}$, $\Delta^{(1)} = \Delta$, and $\Delta^{(n)} = (\Delta \otimes \mathrm{id}^{\otimes(n-1)}) \circ \Delta^{(n-1)}$ for all $n \geq 2$. So the superscript is one less than the number of tensor factors in the codomain. Similarly, $\mu^{(n)}$ denotes the map that multiplies $n+1$ elements of H, with $\mu^{(0)} = \mathrm{id}$. Convolution powers of any $P \in \operatorname{End}(H)$ can be written as follows:

$$P^{*0} = \iota\varepsilon \quad \text{and} \quad P^{*n} = \mu^{(n-1)} \circ P^{\otimes n} \circ \Delta^{(n-1)} \ \text{(for } n \geq 1\text{)}.$$

Proposition 1 *Any connected bialgebra is a Hopf algebra with antipode*

$$S = \sum_{k\geq 0} \big(\iota\varepsilon - \mathrm{id}\big)^{*k}. \tag{4}$$

This basic result can be traced back to Sweedler (Sweedler, 1969, Lemma 9.2.3) and Takeuchi (Takeuchi, 1971, Lemma 14); see also Montgomery (Montgomery, 1993, Lem. 5.2.10). It follows by expanding $x^{-1} = \frac{1}{1-(1-x)} = \sum_k (1-x)^k$ in the convolution algebra, with $x = \mathrm{id}$ and $1 = \iota\epsilon$. Connectedness guarantees that the sum in (4) is finite when evaluated on any $h \in H$. More precisely, if $h \in H_{(m)}$, then $(\mathrm{id} - \iota\varepsilon)^{*k}(h) = 0$ for all $k > m$. In particular, this holds if H is graded connected and $h \in H_m$.

We will also need the series expansions of $\log(\mathrm{id})$ in the convolution algebra:

$$\log(\mathrm{id}) = -\sum_{k\geq 1} \frac{1}{k} (\iota\varepsilon - \mathrm{id})^{*k}. \tag{5}$$

Definition 2 To any connected Hopf algebra H are associated *(higher) Eulerian idempotents* $\mathsf{e}^{(k)}$ for $k \geq 0$, given by

$$\mathsf{e}^{(0)} = \iota\varepsilon\,, \quad \mathsf{e}^{(1)} = \log(\mathrm{id})\,, \quad \mathsf{e}^{(k)} = \frac{1}{k!}\big(\mathsf{e}^{(1)}\big)^{*k} \ \ (\text{for } k > 1). \tag{6}$$

The “first” Eulerian idempotent is $\mathsf{e}^{(1)}$. In case H is commutative and cocommutative, the $\mathsf{e}^{(k)}$ form a *complete orthogonal system* of idempotent operators on H. That is,

$$\mathrm{id} = \sum_{k \geq 0} \mathsf{e}^{(k)}, \quad \mathsf{e}^{(k)} \circ \mathsf{e}^{(k)} = \mathsf{e}^{(k)}, \quad \text{and} \quad \mathsf{e}^{(j)} \circ \mathsf{e}^{(k)} = 0 \ \ (\text{for } j \neq k). \tag{7}$$

In addition, if H is cocommutative, $\mathsf{e}^{(k)}$ projects onto the subspace spanned by k-fold products of primitive elements of H (Section 2.1). In particular, $\mathsf{e}^{(1)}$ projects onto $\mathcal{P}(H)$. For proofs of these results, see (Loday, 1992, Ch. 4). It follows from (6) and the identity $x^{*n} = \exp(n \log(x))$ that

$$\mathrm{id}^{*n} = \sum_{k \geq 0} n^k \, \mathsf{e}^{(k)} \ \ (\text{for all } n \in \mathbb{Z}). \tag{8}$$

Some instances of these operators in the recent literature include Aguiar and Mahajan (2013), Diaconis et al. (2012), Novelli et al. (2011), and Patras and Schocker (2006). For references to earlier work, see (Aguiar and Mahajan, 2013, §14).

3 Characteristic polynomials for convolution powers

We need two standard results from linear algebra.

Lemma 3 *Fix finite-dimensional spaces $U \subseteq V$, and suppose U is Θ-invariant for some $\Theta \in \mathrm{End}(V)$.*

(i) If $\bar{\Theta}$ denotes the element of $\mathrm{End}(V/U)$ induced by Θ, and Θ_U denotes the restriction of Θ to U, then the characteristic polynomials of these three maps satisfy $\chi_\Theta(x) = \chi_{\Theta_U}(x)\,\chi_{\bar{\Theta}}(x)$.

(ii) The characteristic polynomials of Θ and of the dual map $\Theta^* \in \mathrm{End}(V^*)$ are equal. □

We are now ready to prove our main result. From now on we assume that H is a graded connected Hopf algebra for which the homogeneous components H_m are finite-dimensional. We consider the associated graded Hopf algebra $\mathrm{gr}\, H$ and its graded dual $\widetilde{H} = (\mathrm{gr}\, H)^*$. Here $\mathrm{gr}\, H$ is endowed with the grading inherited from that of H (as discussed at the end of Section 2.1), and the dual is with respect to this grading: $\widetilde{H}_m = \big((\mathrm{gr}\, H)_m\big)^*$.

Theorem 4 *For every $n \in \mathbb{Z}$ and $m \in \mathbb{N}$, the characteristic polynomial of $\mathrm{id}^{*n}\big|_{H_m}$ takes the form*

$$\chi\big(\mathrm{id}^{*n}\big|_{H_m}\big) = \prod_{k=0}^{m} (x - n^k)^{\mathsf{eul}(k,m)} \tag{9}$$

for some nonnegative integers $\mathsf{eul}(k,m)$, independent of n. More precisely, we have

$$\mathsf{eul}(k,m) = \dim \mathsf{e}^{(k)}\big(\widetilde{H}_m\big).$$

Moreover, these integers depend only on the graded vector space underlying $\mathcal{P}(\widetilde{H})$.

Corollary 5 *The eigenvalues of the antipode for any graded connected Hopf algebra are* ± 1.

This holds since $S = \mathrm{id}^{*(-1)}$.

Remarks: 1. The previous result fails for general Hopf algebras. Let ω be a primitive cube root of unity and consider Taft's Hopf algebra $T_3(\omega)$ Taft (1971), with generators $\{g, x\}$, and relations $\{g^3 = 1, x^3 = 0, gx = \omega\, xg\}$. The coproduct and antipode are determined by $\Delta(g) = g \otimes g$, $S(g) = g^{-1}$, $\Delta(x) = 1 \otimes x + x \otimes g$, and $S(x) = -xg^{-1}$. Here $x^2 + \omega\, x^2 g$ is an eigenvector of S with eigenvalue ω.

2. Corollary 5 implies that the antipode of a graded connected Hopf algebra is diagonalizable if and only if it is an involution.

3. The antipode of a graded connected Hopf algebra need not be an involution (hence diagonalizable). Take for example the Malvenuto–Reutenauer Hopf algebra, (Aguiar and Sottile, 2005a, Remark 5.6).

Proof of Theorem 4: Since id^{*n} preserves both the grading and the coradical filtration of H, it preserves the filtration

$$(H_{(0)})_m \subseteq (H_{(1)})_m \subseteq \cdots \subseteq (H_{(m)})_m = H_m$$

for each m. By repeated application of Lemma 3(i) we deduce that

$$\chi\big(\mathrm{id}^{*n}\big|_{H_m}\big) = \chi\big(\mathrm{id}^{*n}\big|_{(\mathrm{gr}\, H)_m}\big).$$

The map $\Theta \mapsto \Theta^*$ is an isomorphism of convolution algebras $\mathrm{End}(H) \cong \mathrm{End}(H^*)$ (where duals and endomorphisms are in the graded sense). Together with Lemma 3(ii) this implies that

$$\chi\big(\mathrm{id}^{*n}\big|_{(\mathrm{gr}\, H)_m}\big) = \chi\big(\mathrm{id}^{*n}\big|_{\widetilde{H}_m}\big).$$

Thus, we may work with the cocommutative graded connected Hopf algebra $\widetilde{H}$ instead of H.

In this setting the Eulerian idempotents are available, and from (8) we have that

$$\chi\big(\mathrm{id}^{*n}\big|_{\widetilde{H}_m}\big) = \sum_{k\geq 0} n^k \chi\big(\mathsf{e}^{(k)}\,\big|_{\widetilde{H}_m}\big).$$

It thus suffices to calculate the characteristic polynomial of the $\mathsf{e}^{(k)}$.

Let $\mathfrak{g} = \mathcal{P}(\widetilde{H})$. By CMM, $\widetilde{H} \simeq U(\mathfrak{g})$, and by PBW, $\mathrm{gr}\, U(\mathfrak{g}) \cong S(\mathfrak{g})$. The former is the associated graded Hopf algebra with respect to the coradical filtration of $U(\mathfrak{g})$.

If f and g are filtration-preserving maps, then $\mathrm{gr}(f * g) = (\mathrm{gr}\, f) * (\mathrm{gr}\, g)$. Together with $\mathrm{gr}\,\mathrm{id} = \mathrm{id}$, this implies that $\mathrm{gr}\,\mathsf{e}^{(k)} = \mathsf{e}^{(k)}$, or more precisely, that the following diagram commutes.

$$\begin{tikzcd}
\mathrm{gr}\, U(\mathfrak{g}) \arrow[r, "\mathrm{gr}\,\mathsf{e}^{(k)}"] & \mathrm{gr}\, U(\mathfrak{g}) \\
S(\mathfrak{g}) \arrow[u, "\cong"] \arrow[r, "\mathsf{e}^{(k)}"'] & S(\mathfrak{g}) \arrow[u, "\cong"']
\end{tikzcd}$$

Since by Lemma 3(i) characteristic polynomials (of filtration-preserving maps) are also invariant under gr, we are reduced to computing the characteristic polynomial of $\mathsf{e}^{(k)}$ acting on $S(\mathfrak{g})$.

The action of $\mathsf{e}^{(k)}$ on $S(\mathfrak{g})$ is just projection onto $\mathfrak{g}^k$, the subspace spanned by k-fold products of elements of $\mathfrak{g}$. This follows from the easily verified fact that, for $x_i \in \mathfrak{g}$,

$$\mathrm{id}^{*n}(x_1 \cdots x_k) = n^k x_1 \cdots x_k.$$

It follows that

$$\chi\big(\mathsf{e}^{(k)}\,\big|_{S(\mathfrak{g})_m}\big) = (x - n^k)^{\mathsf{eul}(k,m)},$$

where

$$\mathsf{eul}(k,m) = \dim \mathsf{e}^{(k)}\big(S(\mathfrak{g})_m\big) = \dim\,(\mathfrak{g}^k)_m.$$

and this completes the proof. □

Remark: Since $\mathfrak{g}^k = S^k(\mathfrak{g})$ (the k-th symmetric power of $\mathfrak{g}$), one can be more explicit about the integers $\mathsf{eul}(k,m)$. Let $g_m = \dim \mathfrak{g}_m$ be the dimension of the homogeneous component of degree m of $\mathfrak{g}$. Given a partition λ of the form $\lambda = 1^{k_1}2^{k_2}\cdots r^{k_r}$, put

$$\binom{\mathfrak{g}}{\lambda} := \binom{g_1 + k_1 - 1}{k_1} \cdots \binom{g_r + k_r - 1}{k_r}.$$

If $|\lambda|$ and $\ell\,(\lambda)$ denote the size and number of parts of λ, respectively, then we have

$$\mathsf{eul}(k,m) = \sum_{\substack{|\lambda|=m\\ \ell(\lambda)=k}} \binom{\mathfrak{g}}{\lambda}. \tag{10}$$

We record an easy corollary to the proof of Theorem 4.

Corollary 6 *If H is graded, connected, then*

$$\mathrm{trace}\big(\mathrm{id}^{*n}\big|_{H_m}\big) = \sum_{k\geq 0} n^k\,\mathsf{eul}(k,m) \tag{11}$$

for all $n \in \mathbb{Z}$ and $m \in \mathbb{N}$. In particular,

$$\mathrm{trace}\big(S\big|_{H_m}\big) = \sum_{k\geq 0} (-1)^k\,\mathsf{eul}(k,m). \tag{12}$$

4 The trace of the antipode and palindromic words

In this section we assume that the graded connected Hopf algebra H is *cofree*. Let $V = \mathcal{P}(H)$. The first observation is that the integers $\mathsf{eul}(k,m)$ in Theorem 4 depend only on the dimensions of the homogeneous components of V. This holds since the dimensions of the homogeneous components of H determine and are determined either by those of V or those of $\mathfrak{g}$. As a result, these two are related by

$$1 - \sum_{n\geq 1} v_n x^n = \prod_{i\geq 1} \big(1 - x^i\big)^{g_i},$$

where $v_n := \dim V_n$ for each positive integer n. This is Witt's formula (Reutenauer, 1993, Cor. 4.14).

Let $\mathsf{pal}(k, m)$ be the number of palindromic words of length k and weight m in an alphabet with v_n letters of weight n. (A word's *length* is its number of letters; its *weight* is the sum of its letters' weights.)

Let $\mathsf{epal}(m)$ and $\mathsf{opal}(m)$ denote the number of palindromes of weight m with even and odd length, respectively. (A palindrome is a word that equals its reversal.) Let h_m be the dimension of H_m, and put $\mathsf{npal}(m) = h_m - \mathsf{epal}(m) - \mathsf{opal}(m)$.

In case H is graded connected and cofree, we have an alternative description of the characteristic polynomial for $S = \mathrm{id}^{*(-1)}$ acting on H_m.

Theorem 7 *In the above situation,*

$$\chi\big(S\big|_{H_m}\big) = (x+1)^{\mathsf{opal}(m)}(x-1)^{\mathsf{epal}(m)}(x^2-1)^{\mathsf{npal}(m)/2} \,. \tag{13}$$

In particular, the trace of the antipode is given by the formula

$$\mathrm{trace}\big(S\big|_{H_m}\big) = \sum_{k=0}^{m}(-1)^k \, \mathsf{pal}(k, m) \,. \tag{14}$$

Proof: By arguments similar to those in used in the proof of Theorem 4, one may take H to be the shuffle algebra $T(V)$, with its canonical Hopf structure. The antipode then acts on a word w in a basis for V by reversing letters: $S(w_1w_2\cdots w_r) = (-1)^r w_r \cdots w_2 w_1$ and (13) follows. Finally, note that

$$\mathsf{epal}(m) - \mathsf{opal}(m) = \sum_{k=0}^{m}(-1)^k \, \mathsf{pal}(k, m)$$

to deduce (14) and finish the proof. □

We deduce from (9) and (14) that

$$\sum_{k=0}^{m}(-1)^k \, \mathsf{pal}(k, m) \;=\; \sum_{k=0}^{m}(-1)^k \, \mathsf{eul}(k, m) \,, \tag{15}$$

though these two triangles of integers are generally different.

Example 8 Consider the Malvenuto–Reutenauer Hopf algebra. The alphabet is the set of permutations without global descents. See (Aguiar and Sottile, 2005a, Cor. 6.3) and sequence A003319 in Sloane (OEIS). Looking at the degree three component $\mathfrak{S}Sym_3$, we have

length (k)	1	2	3
permutations	$123, 132, 213$	$231, 312$	321
descent words	$123, 132, 213$	$12\|1, 1\|12$	$1\|1\|1$
$\mathsf{pal}(k, 3)$	3	0	1

(Beneath each permutation, we have recorded its expression in terms of letters in the alphabet. On the last line, we count only those words that are palindromic.) The integers $\mathsf{eul}(k, 3)$ are computed from (10), where $\mathfrak{g}$ is the free Lie algebra on V and $v(x) = x + x^2 + 3x^3 + 13x^4 + 71x^5 + \cdots$. From Witt's formula, we have $g(x) = x + x^2 + 4x^3 + 17x^4 + 92x^5 + 572x^6 + \cdots$ See A112354 in Sloane (OEIS). So we get $\mathsf{eul}(k, 3) = 4, 1, 1$ as $k = 1, 2, 3$.

If we move to the degree four component of $\mathfrak{S}Sym$, one checks that there are 13 permutations without any global descents, and one palindromic permutation with each of 1, 2, and 3 global descents:

$$3412 \equiv 12|12, \qquad 4231 \equiv 1|12|1, \qquad \text{and} \qquad 4321 \equiv 1|1|1|1.$$

Once again, the integers $\mathsf{eul}(k, m)$ are quite different:

$\mathsf{pal}(k, 4)$	13	1	1	1
$\mathsf{eul}(k, 4)$	17	5	1	1

One checks that (15) holds for $m = 3$ and $m = 4$.

5 Schur indicators

A theme occurring in the recent Hopf algebra literature involves a generalization of the *Frobenius–Schur indicator function* of a finite group. If $\rho\colon G \to \mathrm{End}(V)$ is a complex representation of G, then the (second) indicator is

$$\nu_2(G, \rho) = \frac{1}{|G|} \sum_{g \in G} \operatorname{trace} \rho(g^2).$$

The only values this invariant can take are $0, 1, -1$, and this occurs precisely when V is a complex, real or pseudo-real representation, respectively. In Linchenko and Montgomery (2000), a reformulation of the definition was given in terms of convolution powers of the integral[(i)] in $\mathbb{C}G$. This extended the notion of (higher) Schur-indicators to all finite-dimensional Hopf algebras, and has since become a valuable tool for the study of these algebras Kashina et al. (2002); Ng and Schauenburg (2008); Shimizu (2012). In case ρ is the regular representation (and H is semisimple), it is shown in Kashina et al. (2006) that the higher Schur indicators can be reformulated further, removing all mention of the integral:

$$\nu_n(H) = \operatorname{trace}(S \circ \mathrm{id}^{*n}) \quad \text{for} \quad n \geq 0.$$

See also Kashina et al. (2012). These invariants are not well-understood at present. Indeed, the possible eigenvalues of id^{*n} are not even known, much less their multiplicities. Our results lead to the following formula for ν_n in case H is graded, connected (instead of finite-dimensional).

Corollary 9 *If H is a graded connected Hopf algebra, then*

$$\nu_n(H_m) = \sum_{k \geq 0} (-n)^k \, \mathsf{eul}(k, m),$$

where $\mathsf{eul}(k, m)$ is as in Theorem 4.

Proof: As in the proof of Theorem 4, we may assume that H is commutative. Then S is an algebra map, and we have $S \circ \mathrm{id}^{*n} = S \circ \mu^{(n)} \circ \Delta^{(n)} = \mu^{(n)} \circ S^{\otimes n} \circ \Delta^{(n)} = S^{*n}$. Finally, observe that $S^{*n} = \mathrm{id}^{*(-n)}$ and apply Theorem 4. $\square$

(i) A construct present for finite-dimensional Hopf algebras that is unavailable for general graded connected Hopf algebras.

6 Examples and applications

6.1 Symmetric functions

Take $H = \mathit{Sym}$, the Hopf algebra of symmetric functions. On the Schur function basis, the antipode acts by $S(s_\lambda) = (-1)^{|\lambda|} s_{\lambda'}$, where λ' is the partition conjugate to λ. Therefore,

$$\mathrm{trace}\big(S\big|_{H_m}\big) = (-1)^m c(m),$$

where $c(m)$ is the number of self-conjugate partitions of m.

We turn to Corollary 6. For this Hopf algebra, $g_i = 1$ for all $i \geq 1$. Hence $\binom{\mathfrak{g}}{\lambda} = 1$ for all λ, and $\mathsf{eul}(k, m) = p_k(m)$, the number of partitions of m into k parts. From (12) we deduce

$$(-1)^m c(m) = \sum_{k=0}^{m} (-1)^k p_k(m), \tag{16}$$

the identity announced in the introduction. (Note that $p_0(m) = 0$ for $m > 0$.)

We point out that it is possible to obtain this result by considering the power sum basis of *Sym*. Since $S(p_\lambda) = (-1)^{\ell(\lambda)} p_\lambda$, we have

$$\mathrm{trace}(\,S|_{H_m}) = \#\big\{\text{partitions of } m \text{ of even length}\big\} - \#\big\{\text{partitions of } m \text{ of odd length}\big\}.$$

Equating to the former expression for the trace gives (16).

We further illustrate Corollary 6 by deriving certain identities involving the Littlewood–Richardson coefficients $c^\lambda_{\mu,\nu}$. Recall that the latter are the structure constants for the product and coproduct on the Schur basis of *Sym*:

$$s_\mu \cdot s_\nu = \sum_\lambda c^\lambda_{\mu,\nu}\, s_\lambda \qquad \text{and} \qquad \Delta(s_\lambda) = \sum_{\mu,\nu} c^\lambda_{\mu,\nu}\, s_\mu \otimes s_\nu .$$

Formula (11) (with $n = \pm 2$) yields the following identities, for all $m \geq 1$:

$$\sum_{\lambda,\mu,\nu \vdash m} (c^\lambda_{\mu,\nu})^2 \;=\; \sum_{k=1}^{m} 2^k\, p_k(m) \qquad \text{and} \qquad \sum_{\lambda,\mu,\nu \vdash m} c^\lambda_{\mu,\nu} c^\lambda_{\mu',\nu'} \;=\; \sum_{k=1}^{m} (-1)^{m-k}\, 2^k\, p_k(m).$$

Note, incidentally, that the fact that the antipode preserves (co)products says that $c^\lambda_{\mu,\nu} = c^{\lambda'}_{\mu',\nu'}$.

6.2 Schur P-functions

Let Γ denote the subalgebra of *Sym* generated by the Schur P-functions, P_λ. See (Macdonald, 1995, III.8) for definitions, as well as the results used below. A partition is *strict* if its parts are all distinct. A basis for Γ_m consists of those P_λ with λ a strict partition of m. Let $d(m)$ denote the number of such partitions. For λ strict, $S(P_\lambda) = (-1)^{|\lambda|} P_\lambda$. Therefore,

$$\mathrm{trace}(\,S|_{\Gamma_m}) = (-1)^m d(m).$$

It is well-known that $d(m)$ is also the number of *odd* partitions of m (partitions into odd parts). In fact, Γ is the $\mathbb{Q}$-subalgebra of *Sym* generated by the odd power sums p_{2i+1}, $i \geq 0$. It also follows from this that $\binom{\mathfrak{g}}{\lambda} = 1$ when λ is odd and $\binom{\mathfrak{g}}{\lambda} = 0$ otherwise. Therefore, $\mathsf{eul}(k, m)$ is the number of odd partitions of m of length k. In an odd partition, the parities of m and k are the same. Thus, identity (12) simply counts odd partitions according to their length.

6.3 Quasisymmetric functions

Let us turn to the Hopf algebra $H = QSym$ of quasisymmetric functions, and consider the two standard homogeneous bases for H_m, the *fundamental* F_α and *monomial* M_α quasisymmetric functions, with α a composition of m. The antipode has the following descriptions:

$$S(F_\alpha) = (-1)^m F_{\widetilde{\alpha}'} \quad \text{and} \quad S(M_\alpha) = (-1)^{\ell(\alpha)} \sum_{\beta \leq \alpha} M_{\widetilde{\beta}},$$

where $\widetilde{\gamma}$ is the reversal of the word γ and γ' is the transpose (when drawn as a ribbon-shaped skew-diagram). Note that $\alpha = \widetilde{\alpha}'$ if and only if α is symmetric with respect to reflection across the anti-diagonal (when drawn as a ribbon). There are precisely $2^{(m-1)/2}$ of these when m is odd, and zero when m is even. Calculating the trace on the fundamental basis we thus obtain

$$\operatorname{trace}(\,S|_{H_m}) = \begin{cases} -2^{(m-1)/2} & \text{if } m \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

The compositions α that contribute to the trace on the monomial basis satisfy $\widetilde{\alpha} \leq \alpha$. Since reversal is an order-preserving involution, this happens if and only if $\widetilde{\alpha} = \alpha$, that is if and only if α is palindromic. Let $\mathsf{pal}(m)$ denote the number of palindromic compositions of m. In m is even, exactly half of the palindromes of length m have odd length; if m is odd, all of them do. We conclude that

$$\operatorname{trace}(\,S|_{H_m}) = \begin{cases} -\,\mathsf{pal}(m) & \text{if } m \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

Comparing (17) and (18) we deduce that, for all odd m,

$$\mathsf{pal}(m) = 2^{(m-1)/2}.$$

This simple fact can also be deduced by establishing the recursion $\mathsf{pal}(m) = 2\,\mathsf{pal}(m-1)$ for m even and $\mathsf{pal}(m) = \mathsf{pal}(m-1)$ for m odd.

QSym is cofree, so Theorem 7 applies. We have that

$$\mathsf{pal}(k, m) = \begin{cases} \dbinom{\lceil m/2 \rceil - 1}{\lceil k/2 \rceil - 1}, & \text{if } m \text{ is even, or if } m \text{ is odd and } k \text{ is odd,} \\ \qquad 0, & \text{if } m \text{ is odd and } k \text{ is even.} \end{cases}$$

Formula (14) boils down in this case to the basic formula $2^h = \sum_{j=0}^{h} \binom{h}{j}$.

6.4 Peak quasisymmetric functions

Let H denote the *peak* Hopf algebra. It is a subalgebra of *QSym*. As *QSym*, it is cofree, and a basis for H_m is indexed by compositions α of m into odd parts. The number of odd compositions of m is the Fibonacci number f_m (with $f_1 = f_2 = 1$).

In Stembridge (1997) and Billera et al. (2003), an analog of the fundamental basis of *QSym* is developed, θ_α. The antipode is $S(\theta_\alpha) = (-1)^m \theta_{\widetilde{\alpha}}$. It follows that

$$\mathrm{trace}(\, S|_{\Gamma_m}) = \begin{cases} f_{m/2}, & \text{if } m \text{ is even,} \\ -f_{\lceil m/2 \rceil + 1}, & \text{if } m \text{ is odd,} \end{cases}$$

(as palindromic odd compositions of m come from odd compositions of $m/2$). A little more work shows that

$$\mathsf{pal}(k,m) = \begin{cases} \binom{(m+k)/4-1}{(m-k)/4}, & \text{if } m \text{ is even and } 4 \mid (m-k), \\ \binom{\lfloor (m+k-1)/4 \rfloor}{\lfloor (m-k+1)/4 \rfloor}, & \text{if } m \text{ and } k \text{ are odd,} \\ 0, & \text{otherwise.} \end{cases}$$

Formula (14) yields the following basic identities:

$$f_h = \sum_{j=0}^{\lfloor h/2 \rfloor} \binom{h-j-1}{j} \ \text{(for } h \geq 1\text{)} \quad \text{and} \quad f_h = \sum_{j=0}^{h-2} \binom{\lfloor (h+j)/2 \rfloor}{\lfloor (h-j)/2 \rfloor} \ \text{(for } h \geq 2\text{).}$$

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Coefficients of algebraic functions: formulae and asymptotics

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accepted We dedicate this article to the memory of Philippe Flajolet.

Abstract. This paper studies the coefficients of algebraic functions. First, we recall the too-little-known fact that these coefficients f_n have a closed form. Then, we study their asymptotics, known to be of the type $f_n \sim CA^n n^\alpha$. When the function is a power series associated to a context-free grammar, we solve a folklore conjecture: the appearing critical exponents α can not be $1/3$ or $-5/2$, they in fact belong to a subset of dyadic numbers. We extend what Philippe Flajolet called the Drmota–Lalley–Woods theorem (which is assuring $\alpha = -3/2$ as soon as a "dependency graph" associated to the algebraic system defining the function is strongly connected): We fully characterize the possible critical exponents in the non-strongly connected case. As a corollary, it shows that certain lattice paths and planar maps can not be generated by a context-free grammar (i.e., their generating function is not $\mathbb{N}$-algebraic). We end by discussing some extensions of this work (limit laws, systems involving non-polynomial entire functions, algorithmic aspects).

Résumé. Cet article a pour héros les coefficients des fonctions algébriques. Après avoir rappelé le fait trop peu connu que ces coefficients f_n admettent toujours une forme close, nous étudions leur asymptotique $f_n \sim CA^n n^\alpha$. Lorsque la fonction algébrique est la série génératrice d'une grammaire non-contextuelle, nous résolvons une vieille conjecture du folklore : les exposants critiques α ne peuvent pas être $1/3$ ou $-5/2$ et sont en fait restreints à un sous-ensemble des nombres dyadiques. Nous étendons ce que Philippe Flajolet appelait le théorème de Drmota–Lalley–Woods (qui affirme que $\alpha = -3/2$ dès lors qu'un "graphe de dépendance" associé au système algébrique est fortement connexe) : nous caractérisons complètement les exposants critiques dans le cas non fortement connexe. Un corolaire immédiat est que certaines marches et cartes planaires ne peuvent pas être engendrées par une grammaire non-contextuelle non ambigüe (*i. e.*, leur série génératrice n'est pas $\mathbb{N}$-algébrique). Nous terminons par la discussion de diverses extensions de nos résultats (lois limites, systèmes d'équations de degré infini, aspects algorithmiques).

Keywords: analytic combinatorics, generating function, algebraic function, singularity analysis, context-free grammars, critical exponent, non-strongly connected positive systems, Gaussian limit laws, $\mathbb{N}$-algebraic function

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1 Introduction

The theory of context-free grammars and its relationship with combinatorics was initiated by the article of Noam Chomsky and Marcel-Paul Schützenberger in 1963 [CS63], where it is shown that the generating function of the number of words generated by a non ambiguous context-free grammar is algebraic. Since then, there has been much use of algebraic functions in combinatorics, see e.g. [Sta99, BM06, FS09].

Quite often, they come from a tree-like structure (dissections of polygons: a result going back to Euler in 1751, one of the founding problems of analytic combinatorics!), or from a grammar description (polyominoes [DV84], lattice paths [Duc00]), or from the "diagonal" of rational functions [BH12], or as solution of functional equations (solvable by the kernel method and its variants, e.g. for avoiding-pattern permutations [Knu98]). Their asymptotics is crucial for establishing (inherent) ambiguity of context-free languages [Fla87], for the analysis of lattice paths [BF02], walks with an infinite set of jumps [Ban02] (which are thus not coded by a grammar on a finite alphabet), or planar maps [BFSS01].

Plan of this article:

- In Section 2, we give a few definitions, mostly illustrating the link between context-free grammars, solutions of positive algebraic systems and $\mathbb{N}$-algebraic functions.
- In Section 3, we survey some closure properties of algebraic functions and give a closed form for their coefficients.
- In Section 4, we state and sketch a proof of our main theorem on the possible asymptotics of algebraic functions (associated to a context-free grammar with positive weights).
- We end with a conclusion pinpointing some extensions (limit laws, algorithmic considerations, extension to infinite systems, or systems involving entire functions).

2 Definitions: $\mathbb{N}$-algebraic functions, context-free grammars and pushdown automata

For the notions of automata, pushdown automata, context-free grammars, we refer to the survey [PS09]. Another excellent compendium on the subject is the handbook of formal languages [RS97] and the Lothaire trilogy. We now consider $\mathbb{S}$-algebraic functions, that is, a function $y_1(z)$ that is solution of a system[(i)]:

$$\begin{cases} y_1 = P_1(z, y_1, \ldots, y_d) \\ \vdots \\ y_d = P_d(z, y_1, \ldots, y_d) \end{cases} \tag{1}$$

where each polynomial P_i has coefficients in any set $\mathbb{S}$ (in this article, we consider $\mathbb{S} = \mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}_+$, or $\mathbb{R}_+$). We restrict (with only minor loss of generality) to systems satisfying: $P_i(0, \ldots, 0) = 0$, each P_i is involving at least one y_j, the coefficient of y_j in $P_i(0, \ldots, 0, y_j, 0 \ldots, 0)$ is 0, and there is at least one $P_i(z, 0, \ldots, 0)$ which is not 0. Such systems are called "well defined" (or "proper" or "well founded" or "well posed", see [BD13]), and correspond to proper context-free grammars for which one has no "infinite

[(i)] In this article, we will often summarize the system (1) via the convenient short notation $\mathbf{y} = \mathbf{P}(z, \mathbf{y})$, where bold fonts are used for vectors.

chain rules". On the set of power series, $d(F(z), G(z)) := 2^{-\mathrm{val(F(z)-G(z))}}$ is an ultrametric distance, this distance extends to vectors of functions, and allows to apply the Banach fixed-point theorem: it implies existence and uniqueness of a solution of the system as a d-tuple of power series $(y_1, \dots, y_d)$ (and they are analytic functions in 0, as we already know that they are algebraic by nature). A common mistake is to forget that there exist situations for which the system (1) can admit several solutions as power series for y_1 (*nota bene:* there is no contradiction with our previous claim, which is considering *tuples*). By elimination theory (resultant or Gröbner bases), $\mathbb{S}$-algebraic functions are algebraic functions.

We give now few trivial/folklore results: $\mathbb{N}$-algebraic functions correspond to generating functions of context-free grammars (this is often called the Chomsky–Schützenberger theorem), or, equivalently, push-down automata (via e.g. a Greibach normal form). $\mathbb{Z}$-algebraic functions have no natural simple combinatorial structures associated to them, but they are the difference of two $\mathbb{N}$-algebraic functions (as can be seen by introducing new unknowns splitting in two the previous ones, and writing the system involving positive coefficients on one side, and negative coefficients on the other side). They also play an important rôle as any algebraic generating with integer coefficients can be considered as a $\mathbb{Z}$-algebraic function. An $\mathbb{N}$-rational function is a function solution of a system (1) where each polynomial P_i has coefficients in $\mathbb{N}$ and only linear terms in y_j. Such functions correspond to generating functions of regular expressions or, equivalently, automata (a result essentially due to Kleene), and formula for their coefficients and their asymptotics are well-known, so we restrict from now on our attention to $\mathbb{N}$-algebraic functions which are not rational.

3 Closed form for coefficient of algebraic function

A first natural question is how can we compute the n-th coefficient f_n of an algebraic power series? An old theorem due to Abel states that algebraic functions are D-finite functions. A function $F(z)$ is D-finite if it satisfies a differential equation with coefficients which are polynomials in z; equivalently, its coefficients f_n satisfy a linear recurrence with coefficients which are polynomials in n. They are numerous algorithms to deal with this important class of functions, which includes a lot of special functions from physics, number theory and also combinatorics [Sta99]. The linear recurrence satisfied by f_n allows to compute in linear time all the coefficients $f_0, \dots, f_n$.

A less known fact is that these coefficients admit a closed form expression as a finite linear combination of weighted multinomial numbers. More precisely, one has the following theorem:

Theorem 1 (The Flajolet–Soria formula for coefficients of algebraic function) *Let $P(z, y)$ be a bivariate polynomial such that $P(0, 0) = 0$, $P_y(0, 0) = 0$ and $P(z, 0) \neq 0$. Consider the algebraic function implicitly defined*[(ii)] *by $f(z) = P(z, f(z))$ and $f(0) = 0$. Then, the Taylor coefficients of $f(z)$ are given by the following finite sum*

$$f_n = \sum_{m \geq 1} \frac{1}{m} [z^n y^{m-1}] P^m(z, y). \tag{2}$$

(ii) If an algebraic function $f(z)$ with $f(0) = 0$ satisfies $Q(z, f(z)) = 0$, where $Q(z, y)$ is a polynomial with $Q(0, 0) = 0$ and $Q_y(0, 0) \neq 0$ then $f(z)$ satisfies the equation $f(z) = P(z, f(z))$, where $P(z, y) = y - Q(y, z)/Q_y(0, 0)$ satisfies $P(0, 0) = 0$ and $P_y(0, 0) = 0$.

Accordingly, applying the multinomial theorem on $P(z,y)=\sum_{i=1}^{d} a_i z^{b_i} y^{c_i}$ leads to

$$f_n=\sum_{m\geq 1}\frac{1}{m}\sum_{\substack{m_1+\dots+m_d=m\\ b_1m_1+\dots+b_dm_d=n\\ c_1m_1+\dots+c_dm_d=m-1}}\binom{m}{m_1,\dots,m_d}{a_1}^{m_1}\dots {a_d}^{m_d}\,. \tag{3}$$

Proof: Consider $y=P(z,y)$ as the perturbation at $u=1$ of the equation $y=uP(z,y)$. Then, applying the Lagrange inversion formula (considering u as the main variable, and z as a fixed parameter) leads to the theorem. The second formula comes from the definition of the multinomial number, which is is the number of ways to divide m objects into d groups, of cardinality $m_1,\dots,m_d$ (with $m_1+\dots+m_d=m$): $[{u_1}^{m_1}\dots {u_d}^{m_d}](u_1+\dots+u_d)^m=\binom{m}{m_1,\dots,m_k}=\frac{m!}{m_1!\dots m_d!}$. □

This Flajolet–Soria formula was first published in the habilitation thesis of Michèle Soria in 1990 (and was later rediscovered independently by Gessel and Sokal).

As it is an alternating sign nested sum (indeed, as one reduces the set of positive equations describing our $\mathbb{N}$-algebraic function to a single equation, the elimination process will lead to some non positive a_i's), it is not suitable to get general asymptotics from this formula, so we now proceed with another approach, which leads to a nice universal result for the critical exponents, but to the price of a rather technical proof.

4 Asymptotics for coefficients of algebraic function

A Puiseux series $f=f(z)$ is a series of the form $f=\sum_{k=k_0}^{\infty}a_k(z-z_0)^{k/N}$, where k_0 is an integer, N a positive integer. Let $k_c:=\min\{k\in\mathbb{Z}-\{0\}|a_k\neq 0\}$, then $\alpha=k_c/N$ is called the *critical exponent* (loosely speaking, α is the "first non zero exponent" appearing in the series, and if z_0 is not precised, it is by default the radius of convergence of $f(z)$). The theory of Puiseux expansions (or the theory of G-functions) implies that every algebraic function has a Puiseux series expansion and, thus, the critical exponents are rational numbers. The following proposition shows that all rational numbers are reached.

Theorem 2 ($\mathbb{Q}$ is the set of critical exponents) *For every rational number α that is not a positive integer there exists an algebraic power series with positive integer coefficients for which its Puiseux expansion at the radius of convergence has exactly the critical exponent α.*

Proof: First consider $F(z):=\frac{1-(1-a^2z)^{1/a}}{z}$, where a is any positive or negative integer. Accordingly, its coefficients are given by $f_n=\binom{1/a}{n+1}a^{2n+1}(-1)^n$. The proof that the f_n are positive integers was proven in [Lan00], via a link with a variant of Stirling numbers. We give here a simpler proof: first, via the Newton binomial theorem, the algebraic equation for $F(z)$ is $F(z)=1+\sum_{k=2}^{a}\binom{a}{k}a^{k-2}(-1)^k z^{k-1}(F(z))^k$. Then, if one sees this equation as a fixed point equation (as a rewriting rule in the style of context-free grammars), it is obvious that the f_n's belong to $\mathbb{Z}$. But as $f_{n+1}=a\,\frac{(an+(a-1))f_n}{n+2}$, it is clear that the f_n's are finally positive integers. Finally, if b is any positive integer (such that b is not a multiple of a), considering $G(z)=e(zF(z)-1)^b$ (where $e=1$ if $a>b \mod (2a)$ and $e=-1$ elsewhere) leads to a series with integer coefficients (because of the integrality of the coefficients of F), positive coefficients (excepted a few of its first coefficients, for some monomials of degree less than b, as it comes from the Newton binomial expansion). Removing these negative coefficients gives a power series with only positive integer coefficients, with a Puiseux expansion of the form $(1-a^2z)^{b/a}$. □

One may then wonder if there is something stronger. For example, is it the case that for any radius of convergence, any critical exponent is possible? It happens not to be the case, as can be seen via a result of Fatou: a power series with integer coefficients and radius of convergence 1 is either rational or transcendental. However, one has the following neat generic behavior:

Theorem 3 (First main result: dyadic critical exponents for $\mathbb{N}$-algebraic function) *The critical exponent of an $\mathbb{N}$-algebraic (or even $\mathbb{R}_+$-algebraic) function is either 2^{-k} for some $k \geq 1$ or $-m2^{-k}$ for some $m \geq 1$ and some $k \geq 0$.*

If $f_1(z)$ is aperiodic, that is, the radius of convergence ρ is the only singularity on the circle of convergence $|z| = \rho$ then the transfer principle of Flajolet and Odlyzko [FO90] implies that the coefficients $f_n = [z^n]\, f_1(z)$ are asymptotically given by $f_n \sim A\rho^{-n}n^{-1-\alpha}$, when α is the critical exponent. In the general (non-aperiodic) case, we have to distinguish between residue classes but the asymptotics are still of the same form (when we restrict to these residue classes).

Theorem 4 (Second main result: asymptotics for coefficients of $\mathbb{N}$-algebraic function) *Let $f_1(z)$ be the power series expansion of a well defined $\mathbb{N}$-algebraic (or even $\mathbb{R}_+$-algebraic) system $\mathbf{y} = \mathbf{P}(z, \mathbf{y})$. Then there is an integer $M \geq 1$ such that for every $j \in \{0, 1, \ldots, M-1\}$ we either have $f_n = 0$ for all (but finitely many) n with $n \equiv j \bmod M$ or there exist positive numbers A_j, ρ_j and β_j that is either $-1-2^{-k}$ for some $k \geq 1$ or $-1+m2^{-k}$ for some $m \geq 1$ and some $k \geq 0$ such*

$$f_n \sim A_j \rho_j^{-n} n^{\beta_j}, \quad (n \to \infty,\ n \equiv j \bmod M).$$

It is easy to see that all possible exponents actually appear.

Proposition 1 *All the dyadic numbers of Theorem 3 appear as critical exponents of $\mathbb{N}$-algebraic functions.*

Proof: Let us first consider the system of equations $y_1 = z(y_2 + y_1^2), y_2 = z(y_3 + y_2^2), y_3 = z(1 + y_3^2)$ has the following (explicit) solution

$$f_1(z) = \frac{1 - (1-2z)^{1/8}\sqrt{2z\sqrt{2z\sqrt{1+2z} + \sqrt{1-2z}} + (1-2z)^{3/4}}}{2z}$$

$$f_2(z) = \frac{1 - (1-2z)^{1/4}\sqrt{2z\sqrt{1+2z} + \sqrt{1-2z}}}{2z} \quad \text{and} \quad f_3(z) = \frac{1 - \sqrt{1-4z^2}}{2z}\,.$$

Here $f_1(z)$ has dominant singularity $(1-2z)^{1/8}$ and it is clear that this example can be generalized: indeed, consider the system $y_i = z(y_{i+1} + y_i^2)$ for $i = 1, \ldots, k-1$, and $y_k = z(1 + y_k^2)$, it leads to behavior $(1-2z)^{2^{-k}}$ for each $k \geq 1$. Now, taking the system of equations $y = z(y_0^m + y), y_0 = z(1 + 2y_0 y_1)$ leads to a behavior $(1-2z)^{-m2^{-k}}$ for each $m \geq 1$ and $k \geq 0$. See also [TB12] for another explicit combinatorial structure (a family of colored tree related to a critical composition) exhibiting all these critical exponents. □

On the other hand, we can use the result of Theorems 3 and 4 to identify classes that cannot be *counted* with the help of an $\mathbb{N}$- (or $\mathbb{R}_+$-)algebraic system.

Proposition 2 *Planar maps and several families of lattice paths (like Gessel walks) are not $\mathbb{N}$-algebraic (i.e., they can not be generated by an unambiguous context-free grammar).*

Proof: This comes as a nice consequence of our Theorem 3: all the families of planar maps of [BFSS01] cannot be generated by an unambiguous context-free grammar, because of their critical exponent $3/2$. Also, the tables [BK09] of lattice paths in the quarter plane and their asymptotics (where some of the connection constants are guessed, but all the critical exponents are proved, and this is enough for our point) allow to prove that many sets of jumps are giving a non algebraic generating function, as they lead to a critical exponent which is a non-negative integer or involving 1/3. One very neat example are Gessel walks (their algebraicity were a nice surprise [BK10]), where the hypergeometric formula for their coefficients leads to an asymptotic in $4^n/n^{2/3}$ not compatible with $\mathbb{N}$-algebraicity. (iii) □

The critical exponents $-1/4, -3/4, -5/4$ which appear for walks on the slit plane [BMS02] and other lattice paths questions [BK10] are compatible with $\mathbb{N}$-algebraicity, but these lattice paths are in fact not $\mathbb{N}$-algebraic (it is possible to use Ogden's pumping lemma, to prove that these walks can be not generated by a context-free grammar). To get a constructive method to decide $\mathbb{N}$-algebraicity (input: a polynomial equation, output: a context-free specification, whenever it exists) is a challenging task.

We now dedicate the two next subsections to the proof of Theorem 3. The proof of Theorem 4 is a considerable extension, where (at least) all singularities on the circle of convergence have to identified (see [BD13]). Due to space limitation we do not work out the latter proof in this extended abstract.

4.1 Dependency graph and auxiliary results

A main ingredient of the proof of Theorem 3 is the analysis of the *dependency graph* G of the system $y_j = P_j(z, y_1, \ldots, y_K)$, $1 \le j \le K$. The vertex set is $\{1, \ldots, K\}$ and there is a directed edge from i to j if P_j depends on y_i (see Figure 1). If the dependency graph is strongly connected then we are in very special case of Theorem 3, for which one has one the following two situations (see [Drm97]):

Lemma 1 (rational singular behavior) *Let $\mathbf{y} = \mathbf{A}(z)\mathbf{y} + \mathbf{B}(z)$ a positive and well defined linear system of equations, where the dependency graph is strongly connected. Then the functions $f_j(z)$ have a joint polar singularity ρ or order one as the dominant singularity, that is, the critical exponent is -1:*

$$f_j(z) = \frac{c_j(z)}{1 - z/\rho},$$

where $c_j(z)$ is non-zero and analytic at $z = \rho$.

Lemma 2 (algebraic singular behavior) *Let $\mathbf{y} = \mathbf{P}(z, \mathbf{y})$ a positive and well defined polynomial system of equations that is not affine and where the dependency graph is strongly connected. Then the functions $f_j(z)$ have a joint square-root singularity ρ as the dominant singularity, that is, the critical exponent is $1/2$:*

$$f_j(z) = g_j(z) - h_j(z)\sqrt{1 - \frac{z}{\rho}},$$

where $g_j(z)$ and $h_j(z)$ are non-zero and analytic at $z = \rho$.

In the proof of Theorem 3 we will use in fact extended version of Lemma 1 and 2, where we introduce additional (polynomial) parameters, that is, we consider systems of functional equations of the form

(iii) The fact that critical exponents involving 1/3 were not possible was an informal conjecture in the community for years. We thank Philippe Flajolet, Mireille Bousquet-Mélou and Gilles Schaeffer, who encouraged us to work on this question.

$$\begin{cases} y_1 = P_1(z, y_1, y_2, y_5) \\ y_2 = P_2(z, y_2, y_3, y_5) \\ y_3 = P_3(z, y_3, y_4) \\ y_4 = P_4(z, y_3, y_4) \\ y_5 = P_5(z, y_5, y_6) \\ y_6 = P_6(z, y_5, y_6). \end{cases}$$

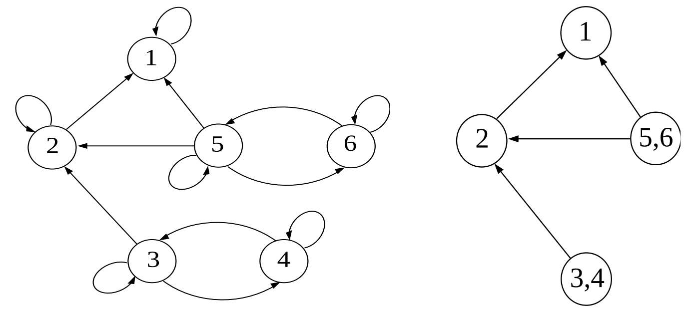

Fig. 1: A positive system, its dependency graph G and its reduced dependency graph $\widetilde{G}$. None of these graphs are here strongly connected: e.g. the state 1 is a sink; it is thus a typical example of system not covered by the Drmota–Lalley–Woods theorem, but covered by our new result implying dyadic critical exponents.

$\mathbf{y} = \mathbf{P}(z, \mathbf{y}, \mathbf{u})$, where $\mathbf{P}$ is now a polynomial in $z, \mathbf{y}, \mathbf{u}$ with non-negative coefficients and where the dependency graph (with respect to $\mathbf{y}$) is strongly connected. We also assume that $\mathbf{u}$ is strictly positive such that the spectral radius of the Jacobian $\mathbf{P}_\mathbf{y}(0, \mathbf{f}(0, \mathbf{u}), \mathbf{u})$ is smaller than 1.[iv] Hence, we can consider the solution that we denote by $\mathbf{f}(z, \mathbf{u})$.

If we are in the affine setting ($\mathbf{y} = \mathbf{A}(z, \mathbf{u})\mathbf{y} + \mathbf{B}(z, \mathbf{u})$) it follows that $\mathbf{y}(z, \mathbf{u})$ has a polar singularity:

$$f_j(z, \mathbf{u}) = \frac{c_j(z, \mathbf{u})}{1 - z/\rho(\mathbf{u})}, \tag{4}$$

where the functions $\rho(\mathbf{u})$ and $c_j(z, \mathbf{u})$ are non-zero and analytic. We have to distinguish two cases. If $\mathbf{A}(z, \mathbf{u}) = \mathbf{A}(z)$ does not depend on $\mathbf{u}$ then $\rho(\mathbf{u}) = \rho$ is constant and the dependence from $\mathbf{u}$ just comes from $\mathbf{B}(z, \mathbf{u})$. Of course, if $\mathbf{A}(z, \mathbf{u})$ depends on $\mathbf{u}$ then $\rho(\mathbf{u})$ is not constant. More precisely it depends exactly on those parameters that appear in $\mathbf{A}(z, \mathbf{u})$.

Similarly in the non-affine setting we obtain representations of the form

$$f_j(z, \mathbf{u}) = g_j(z, \mathbf{u}) - h_j(z, \mathbf{u})\sqrt{1 - \frac{z}{\rho(\mathbf{u})}}, \tag{5}$$

where the functions $\rho(\mathbf{u})$, $g_j(z, \mathbf{u})$, and $h_j(z, \mathbf{u})$ are non-zero and analytic. In this case $\rho(\mathbf{u})$ is always non-constant and depends on all parameters.

We denote by D_0 the set of positive real vectors $\mathbf{u}$, for which $r(\mathbf{P}_\mathbf{y}(0, \mathbf{f}(z, \mathbf{0}), \mathbf{u})) < 1$. It is easy to show that $\rho(\mathbf{u})$ tends to 0 when $\mathbf{u}$ approaches the boundary of D_0.

4.2 Proof of our Theorem 3 on dyadic critical exponents

We fix some notation. Let G denote the dependency graph of the system and $\widetilde{G}$ the reduced dependency graph. Its vertices are the strongly connected components $C_1, \ldots C_L$ of G. We can then *reduce* the dependency graph to its components (see Figure 1).

Let $\mathbf{y}_1, \ldots, \mathbf{y}_L$ denote the system of vectors corresponding to the components $C_1, \ldots C_L$ and let $\mathbf{u}_1, \ldots, \mathbf{u}_L$ denote the input vectors related to these components. In the above example, we have $C_1 =$

[iv] This condition assures that we have a unique analytic solution $z \mapsto \mathbf{f}(z, \mathbf{u})$ locally around $z = 0$.

$\{1\}$, $C_2 = \{2\}$, $C_3 = \{3,4\}$, $C_4 = \{5,6\}$, $\mathbf{y}_1 = y_1$, $\mathbf{y}_2 = y_2$, $\mathbf{y}_3 = (y_3, y_4)$, $\mathbf{y}_4 = (y_5, y_6)$, and $\mathbf{u}_1 = (y_2, y_5)$, $\mathbf{u}_2 = (y_3, y_5)$, $\mathbf{u}_3 = \emptyset$, $\mathbf{u}_4 = \emptyset$.

Finally, for each component C_ℓ we define the set D_ℓ of real vectors $\mathbf{u}_\ell$ for which the spectral radius of the Jacobian of ℓ-th subsystem evaluated at $z = 0$, $\mathbf{y}_\ell = \mathbf{0}$ is smaller than 1.

The first step we for each strongly connected component C_ℓ we solve the corresponding subsystem in the variables z and $\mathbf{u}_\ell$ and obtain solutions $\mathbf{f}(z, \mathbf{u}_\ell)$, $1 \le \ell \le L$. In our example these are the functions $\mathbf{f}_1(z, \mathbf{u}_1) = f_1(z, y_2, y_5)$, $\mathbf{f}_2(z, \mathbf{u}_2) = f_2(z, y_3, y_5)$, $\mathbf{f}_3(z, \mathbf{u}_3) = (f_3(z), f_4(z))$, $\mathbf{f}_4(z, \mathbf{u}_4) = (f_5(z), f_6(z))$.

Since the dependency graph $\widetilde{G}$ is acyclic, there are components $C_{\ell_1}, \ldots, C_{\ell_m}$ with no input, that is, they corresponding functions $\mathbf{y}_{\ell_1}(z), \ldots, \mathbf{y}_{\ell_m}(z)$ can be computed without any further information. By Lemma 1 and 2 they either have a polar singularity or a square-root singularity, that is, they are are precisely of the types that are stated in Theorem 3.

Now we proceed inductively. We consider a strongly connected component C_ℓ with the function $\mathbf{f}_\ell(z, \mathbf{u}_\ell)$ and assume that all the functions $f_j(z)$ that are contained in $\mathbf{u}_\ell$ are already known and and that their leading singularities of the two types stated in Theorem 3: the solutions $f_j(z)$ have positive and finite radii of convergence ρ_j. Furthermore, the singular behavior of $f_j(z)$ around ρ_j is either of algebraic type

$$f_j(z) = f_j(\rho_j) + c_j(1 - z/\rho_j)^{2^{-k_j}} + c_j'(1 - z/\rho_j)^{2 \cdot 2^{-k_j}} + \cdots, \tag{6}$$

where $c_j \ne 0$ and where k_j is a positive integer or of type

$$f_j(z) = \frac{d_j}{(1 - z/\rho_j)^{m_j 2^{-k_j}}} + \frac{d_j'}{(1 - z/\rho_j)^{(m_j - 1)2^{-k_j}}} + \cdots, \tag{7}$$

where $d_j \ne 0$, m_j are positive integers and k_j are non-negative integers.

By the discussion following Lemma 1 and 2 it follows that functions contained in $\mathbf{f}_\ell(z, \mathbf{u}_\ell)$ have either a common polar singularity or a common square-root singularity $\rho(\mathbf{u}_\ell)$.

We distinguish between three cases:

1. $\mathbf{f}_\ell(z, \mathbf{u}_\ell)$ comes from an affine system and is, thus, of the form (4) but the function $\rho(\mathbf{u}_\ell)$ is constant.
2. $\mathbf{f}_\ell(z, \mathbf{u}_\ell)$ comes from an affine system and the function $\rho(\mathbf{u}_\ell)$ is not constant.
3. $\mathbf{f}_\ell(z, \mathbf{u}_\ell)$ comes from an non-affine system and is, thus, of the form (5).

ad 1. The first case is very easy to handle. We just have to observe that $c(z, \mathbf{u}_\ell)$ is a polynomial with non-negative coefficients in $\mathbf{u}_\ell$ and that the class of *admissible functions* (that is, functions, where the critical exponent at the radius of convergence is either 2^{-k} for some $k \ge 1$ or $-m2^{-k}$ for some $m \ge 1$ and some $k \ge 0$) is closed under addition and multiplication. Hence the resulting function $f_\ell(z)$ is of admissible form.

ad 2. In the second, we have to be more careful. Let J_ℓ' denote the set of j for which the function $\rho(\mathbf{u}_\ell)$ really depends on.

First we discuss the denominator. For the sake of simplicity we will work with the difference $\rho(\mathbf{u}_j) - z$. Let ρ' denote the smallest radius of convergence of the functions $f_j(z)$, $j \in J_\ell'$. Then we consider the difference $\delta(z) = \rho((f_j(z))_{j \in J_\ell'}) - z$. We have to consider the following cases for the denominator:

2.1. $\delta(\rho'') = 0$ for some $\rho'' < \rho'$ such that $(f_j(\rho''))_{j\in J'_\ell} \in D_\ell$:
First we note that $\delta(z)$ has at most one positive zero since $\rho((f_j(z))_{j\in J'_\ell})$ is decreasing and z is increasing. Furthermore, the derivative satisfies $\delta'(\rho'') > 0$. Consequently, we have a simple zero ρ'' in the denominator.

2.2. We have $\delta(\rho') = 0$ such that $(f_j(\rho'))_{j\in J'_\ell} \in D_\ell$:
In this case all functions $f_j(z)$, $j \in J'_\ell$, with $\rho_j = \rho'$ have to be of type (6). Consequently $\delta(z)$ behaves like $c(1 - z/\rho')^{2^{-\tilde{k}}} + \ldots$, where $c > 0$ and $\tilde{k}$ is the largest appearing k_j (among those functions $f_j(z)$ with $\rho_j = \rho'$).

2.3. We have $\delta(\rho') > 0$ such that $(f_j(\rho'))_{j\in J'_\ell} \in D_\ell$:
In this case, all functions $f_j(z)$, $j \in J'_\ell$, with $\rho_j = \rho'$ have to be (again) of type (6). Consequently $\delta(z)$ behaves like $c_0 - c_1(1 - z/\rho')^{2^{-\tilde{k}}} + \ldots$, where $c_0 > 0$ and $c_1 > 0$ and $\tilde{k}$ is the largest appearing k_j (among those functions $f_j(z)$ with $\rho_j = \rho'$).

Finally, we have to discuss the numerator. Since the numerator $c(z, \mathbf{u_j})$ is just a polynomial in those u_j for which $j \notin J'_\ell$, we can handle them as in the first case.

Summing up leads to a function $f_j(z)$ that is either of type (6) or type (7).

ad 3. In the last case the function $\mathbf{f}_\ell(z, \mathbf{u}_\ell)$ has an representation of the form (5), where In this case $\rho(\mathbf{u}_\ell)$ depends on all components of $\mathbf{u}_\ell$. As above we will study the behavior of the square-root $\sqrt{\rho(\mathbf{u}_\ell) - z}$ instead of $\sqrt{1 - z/\rho(\mathbf{u}_\ell)}$ since the non-zero factor $\sqrt{\rho(\mathbf{u}_\ell)}$ can be put to $h(z, \mathbf{u}_\ell)$.

Let ρ' denote the smallest radius of convergence of the functions $f_j(z)$ that correspond to $\mathbf{u}_\ell$. Here, we have to consider the following cases:

(3.1) $\delta(\rho'') = 0$ for some $\rho'' < \rho'$ such that $(f_j(\rho'')) \in D_\ell$:
This means that $\rho((f_j(z)) - z$ has a simple zero. By the Weierstrass preparation theorem we can, thus, represent this function as $\rho((f_j(z)) - z = (\rho'' - z)H(z)$, where $H(z)$ is non-zero and analytic at ρ''. Consequently, we observe that $f(z)$ has a (simple) square-root singularity.

(3.2) We have $\delta(\rho') = 0$ such that $(f_j(\rho')) \in D_\ell$:
In this case all functions $f_j(z)$ with $\rho_j = \rho'$ have to be of type (6). Hence the square-root of $\delta(z)$ behaves as

$$\sqrt{c(1 - z/\rho')^{2^{-\tilde{k}}} + \ldots} = \sqrt{c}(1 - z/\rho')^{2^{-\tilde{k}-1}} + \ldots,$$

where the corresponding k equals the largest appearing k_j plus 1. Thus, $f(z)$ is of type (6).

(3.3) We have $\delta(\rho') > 0$ such that $(f_j(\rho'))_{j\in J'_\ell} \in D_\ell$:
In this case all functions $f_j(z)$, with $\rho_j = \rho'$ have to be (again) of type (6). Consequently the square-root of $\delta(z)$ behaves like

$$\sqrt{c_0 - c_1(1 - z/\rho')^{2^{-\tilde{k}}} + \ldots} = \sqrt{c_0}\left(1 - \frac{c_1}{2c_0}(1 - z/\rho')^{2^{-\tilde{k}}} + \ldots\right),$$

where $c_0 > 0$ and $c_1 > 0$ and $\tilde{k}$ is the largest appearing k_j (among those functions $f_j(z)$ with $\rho_j = \rho'$). Hence, $f(z)$ is of type (6).

This completes the induction proof of Theorem 3.

5 Conclusion

Now that we have a better picture of the behavior of algebraic coefficients, several extensions are possible and in the full version of this article [BD13], we say more on

- **Algorithmic aspects:** In order to automatize the asymptotics, one has to follow the right branch of the algebraic equations, this is doable by a disjunction of cases following the proof of our main theorem, coupled with an inspection of the associated spectral radii, this leads to a more ”algebraic” approach suitable for computer algebra, shortcutting some numerical methods like e.g. the Flajolet–Salvy ACA (analytic continuation of algebraic) algorithm [FS09]. Giving an algorithm to decide in a constructive way if a function is $\mathbb{N}$-algebraic would be nice. (This is doable for $\mathbb{N}$-rational functions). With respect to the Pisot problem (i.e., deciding if one, or an infinite number of f_n are zeroes), finding the best equivalent for $\mathbb{N}$-algebraic functions of the Skolem–Lerch–Mahler theorem for $\mathbb{N}$-rational functions is also a nice question. The binomial formula of Section 3 leads to many identities, simplifications of the corresponding nested sums are related to fascinating aspects of computer algebra.

- **Extension to entire functions system:** Most parts of the analysis of positive polynomial systems of equations also works for positive entire systems, however, one quickly gets ”any possible asymptotic behavior” as illustrated by the system of equations $y_1 = z(e^{y_2} + y_1), y_2 = z(1+2y_2y_3), y_3 = z(1+y_3^2)$, as it has the following explicit solution $f_1(z) = \frac{z}{1-z}\exp\left(\frac{z}{\sqrt{1-4z^2}}\right)$, which exhibits a non-algebraic behavior. However, adding the constraints $\frac{\partial^2 P_j}{\partial y_j^2} \neq 0$ or if P_j is affine in y_j leads to the same conclusion as Theorem 3, with a smaller set of possible critical exponents (now, all $m_j = 1$).

- **Extension to infinite systems:** If one considers systems having an infinite (but countable) number of unknowns $y_i(z)$, it is proved in [Mor10] that strongly connected systems also lead to a square-root behavior. The fact that the limit law is Gaussian (as soon as a Jacobian operator associated to the system is compact) is proved in [DGM12]. When the conditions of strong connectivity or of compactness are dropped, a huge diversity of behavior appears, but it is however possible to give interesting subclasses having a regular behaviors.

- **Extension to attributed grammars:** Attribute grammars were introduced by Knuth. Many interesting parameters (like internal paths length in trees or area below lattice paths [BG06, Duc99, Ric09]) are well captured by such grammars. They lead to statistics with a mean which is no more linear. For a large class of strongly connected positive systems, it leads to the Airy function, and it is expected that it is also the case for a class of functional equations with non positive coefficients.

- **Extension to limit laws:** Philippe Flajolet called Borges' theorem the principle that motif statistics follow a Gaussian limit law [FS09]. They are however some technical conditions to ensure such a Gaussian behavior, like a strong connectivity of the associated system of equations; indeed, in the non strongly connected case, even very simple motifs in rational languages can then follow any limit law [BBPT12]. For algebraic systems, the strongly connected case leads to a Gaussian distribution, as illustrated by the limit law version of the Drmota–Lalley–Woods theorem [Drm97, BKP09]. In the full version of our article, we give an extension of this result to non strongly connected cases.

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Long Cycle Factorizations : Bijective Computation in the General Case

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Abstract. This paper is devoted to the computation of the number of ordered factorizations of a long cycle in the symmetric group where the number of factors is arbitrary and the cycle structure of the factors is given. Jackson (1988) derived the first closed form expression for the generating series of these numbers using the theory of the irreducible characters of the symmetric group. Thanks to a direct bijection we compute a similar formula and provide the first purely combinatorial evaluation of these generating series.

Résumé. Cet article est dédié au calcul du nombre de factorisations d'un long cycle du groupe symétrique pour lesquels le nombre de facteurs est arbitraire et la structure des cycles des facteurs est donnée. Jackson (1988) a dérivé la première expression compacte pour les séries génératrices de ces nombres en utilisant la théorie des caractères irréductibles du groupe symétrique. Grâce à une bijection directe nous démontrons une formule similaire et donnons ainsi la première évaluation purement combinatoire de ces séries génératrices.

Keywords: Jackson's formula, factorizations, symmetric group, connection coefficients

1 Introduction

For integer n we note S_n the symmetric group on n elements and γ_n the permutation in S_n defined by $\gamma_n = (12\dots n)$. If r is an integer we call *strictly increasing subsequence of* $1\dots r$ any sequence of the form $(i_1, i_2, \dots i_u)$ where $1 \leq i_1 < i_2 < \dots < i_u \leq r$. Given such a subsequence t containing i, we define $succ_t(i)$ the index following i in t. If no such index exists $succ_t(i)$ is the first index of the sequence or i itself if $t = (i)$.

This paper is devoted to the computation of the numbers $k^n_{p_1,p_2,\dots,p_r}$ of factorizations of γ_n as an ordered product of permutations $\alpha_1\alpha_2\dots\alpha_r = \gamma_n$ such that for $1 \leq i \leq r$, α_i belongs to S_n and is composed of exactly p_i disjoint cycles. More precisely, we use a direct bijection to show the following formula:

Theorem 1 (Main result)

$$\frac{1}{(n-1)!^{r-1}} \sum_{p_1,p_2,\dots,p_r} k^n_{p_1,p_2,\dots,p_r} \prod_{1\leq i\leq r} x_i^{p_i} = \sum_{p_1,p_2,\dots,p_r} \sum_{\mathbf{a}} \Delta_r(\mathbf{a}) \binom{n}{\mathbf{a}} \prod_{1\leq i\leq r} \binom{x_i}{p_i} \tag{1}$$

The last sum runs over sequences $\mathbf{a} = (a_t)$ *of* $2^r - 1$ *non-negative integers* a_t *with the index t being any non empty strictly increasing subsequence of integers of* $1\dots r$ *such that*

$$\sum_t a_t = n, \quad \sum_{t; l\notin t} a_t = p_l \ \textit{for}\ \ 2 \leq l \leq r, \quad \sum_{t;1\notin t} a_t = p_1 - 1 \tag{2}$$

Furthermore, the multinomial coefficient is defined by

$$\binom{n}{\mathbf{a}} = \frac{n!}{\prod_t a_t!}$$

Finally, $\Delta_r(\mathbf{a})$ *is the determinant of the* $r \times r$ *matrix with coefficients* $m_{i,j}$, $1 \le i, j \le r$, *where*

$$m_{i,i} = p_i \ (1 \le i \le r),$$
$$\text{for } j \neq i \text{ (modulo } r\text{)}, \ m_{i,j} = -\sum_{t; i \in t, succ_t(j) - j \ge j - i + 1} a_t$$

(the subtractions on indices are modulo r*).*

Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_p) \vdash n$ an integer partition of n with $\ell(\lambda) = p$ parts sorted in decreasing order. We note C_λ the conjugacy class of S_n containing the permutations of cycle type λ and $m_\lambda(\mathbf{x})$ and $p_\lambda(\mathbf{x})$ the monomial and power sum symmetric functions respectively indexed by λ on indeterminate $\mathbf{x}$. Given r integer partitions $\lambda^1, \lambda^2, \dots, \lambda^r$ of n, a more refined problem is to compute the numbers $k^n_{\lambda^1,\lambda^2,\dots,\lambda^r}$ of ordered factorizations $\alpha_1\alpha_2 \dots \alpha_r$ of γ_n such that for for $1 \le i \le r$, α_i belongs to C_{λ^i}. As a corollary of Theorem 1 we have:

Theorem 2 (Corollary)

$$\frac{1}{(n-1)!^{r-1}} \sum_{\lambda^1,\lambda^2,\dots,\lambda^r \vdash n} k^n_{\lambda^1,\lambda^2,\dots,\lambda^r} \prod_{1 \le i \le r} m_{\lambda^i}(\mathbf{x}^i) = \sum_{\lambda^1,\lambda^2,\dots,\lambda^r \vdash n} \frac{\sum_{\mathbf{a}} \Delta_r(\mathbf{a}) \binom{n}{\mathbf{a}}}{\prod_i \binom{n-1}{\ell(\lambda_i)-1}} \prod_{1 \le i \le r} p_{\lambda^i}(\mathbf{x}^i) \quad (3)$$

where $\ell(\lambda^i)$ *is substituted to* p_i *in the definition of* $\mathbf{a}$ *in (2).*

1.1 Background

Despite the attention the problem received over the past twenty years no closed formulas are known for the coefficients $k^n_{p_1,\dots,p_r}$ and $k^n_{\lambda^1,\dots,\lambda^r}$ except for very special cases. Using characters of the symmetric group and a combinatorial development, Goupil and Schaeffer [4] derived an expression for $k^n_{\lambda^1,\lambda^2}$ $(r = 2)$ as a sum of positive terms. This work has been later generalized by Poulalhon and Schaeffer [8] and Irving [5] but, as a rule, the formulas obtained are rather complicated. Using the theory of the irreducible characters of the symmetric group, Jackson [6] computed an elegant expression for the generating series in the LHS of (1) for arbitrary r and an arbitrary permutation of S_n instead of γ_n. This later result shows that the coefficients in the expansion of this generating series in the basis of $\binom{x_i}{p_i}$ can be derived as closed form formulas but fails to provide a combinatorial interpretation. Schaeffer and Vassilieva in [9], Vassilieva in [10] and Morales and Vassilieva in [7] provided the first purely bijective computations of the generating series in (1) and (3) for $r = 2, 3$. In a recent paper, Bernardi and Morales [1] addressed the problem of finding a general combinatorial proof of Jackson's formula for the factorizations of γ_n. Using an argument based on several successive bijections and a probabilistic puzzle, they provide a complete proof for the cases $r = 2, 3$ and a sketch for $r = 4$. In the present paper we generalize and put together all the ideas developed in our previous articles ([9], [10] and [7]) and make them work in the general context of r-factorizations of γ_n. We prove theorems (1) and (2) thanks to a direct (single step) bijection. The combinatorial ingredients we use and the bijection itself are described in sections 2 and 3. Section (4) proves that the bijection is indeed one-to-one. While (1) is similar to Jackson's formula in [6], the two expressions are different. We address their equivalence in section 5.

2 Cacti, partitioned cacti and cactus trees

2.1 Cacti

Factorizations of γ_n can be represented as r-*cacti* (short *cacti*), i.e $2-$cell decompositions of an oriented surface of arbitrary genus into a finite number of vertices ($0-$cells), edges ($1-$cells) and faces ($2-$cells) homeomorphic to open discs, with n black and one white face such that all the black faces are r-gons and not adjacent to each other. They are defined up to an homeomorphism of the surface that preserves its orientation, the type of cells and incidences in the graph. We consider *rooted* cacti, i.e. cacti with a marked black face. We assume as well that within each r-gon the r vertices are colored with r distinct colors so that moving around the r-gons counter-clockwise the vertex of color $i+1$ (modulo r) follows the vertex of color i. Each black r-gon is labeled with an index in $\{1, 2, \ldots, n\}$ such that the marked r-gon is labeled 1 and that moving around the white face starting from the edge linking the vertex of color 1 and the vertex of color r in this marked r-gon, the i-th edge connecting a vertex of color 1 and a vertex of color r belongs to the black r-gon of index i.

Proposition 1 ([10]) *Cacti as defined above are in bijection with r-tuples of permutations $(\alpha_1, \alpha_2, \ldots, \alpha_r)$ such that $\alpha_1\alpha_2 \ldots \alpha_r = \gamma_n$. Each vertex of color i corresponds to a cycle of α_i defined by the sequence of the indices of the r-gons incident to this vertex.*

As a consequence, r-cacti with p_i vertices of color i are counted by $k^n_{p_1,p_2,\ldots,p_r}$.

Example 1 *Figure 1 depicts a 5-cactus corresponding to the factorizations of $\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5 = \gamma_6$ with $\alpha_1 = (12)(3)(4)(5)(6)$, $\alpha_2 = (1)(24)(3)(56)$, $\alpha_3 = (1)(2)(3)(4)(5)(6)$, $\alpha_4 = (1)(23)(46)(5)$, $\alpha_5 = (1)(2)(3)(4)(5)(6)$ and a 4-cactus described by $\alpha_1 = (12)(3)$, $\alpha_2 = (13)(2)$, $\alpha_3 = (12)(3)$, $\alpha_4 = (13)(2)$.*

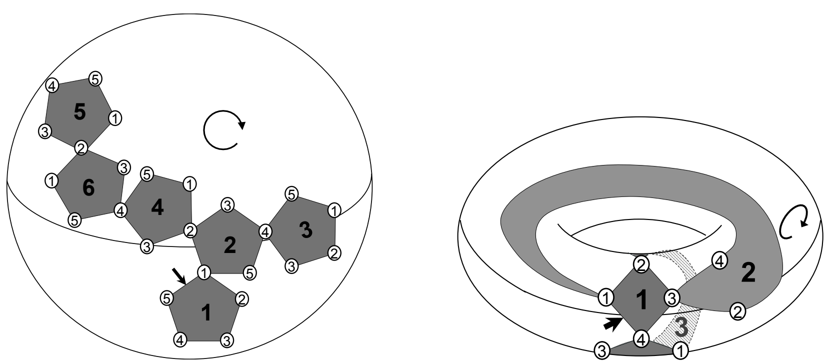

Fig. 1: A 5-cactus embedded on a surface of genus 0 (left) and a 4-cactus embedded on a surface of genus 1 (right)

Remark 1 *Alternatively, a cactus can be seen as a set of n r-tuples of integers such that each integer of $1 \ldots n$ is used exactly once in the i-th positions of the r-tuples ($1 \leq i \leq r$). Moving around the white face of the cactus according to the surface orientation and starting with the edge linking the vertices of color*

1 and r of the root r-gon, we define a labeling of the edges. We assign label i to the i-th edge linking vertices of color 1 and r during the traversal. The j-th edge linking vertices of color r and $r-1$ during the same traversal is indexed by j and so on for all the colors. The n r-tuples defined by the edge labeling of the n r-gons is an equivalent description of the initial cactus.

Lemma 1 *One can show that if the label of an r-gon of a cacti described by $(\alpha_1, \alpha_2, \ldots, \alpha_r)$ is i then the index (as defined in Remark 1) of the edge linking the vertices of color l and $l-1$ is $\alpha_r^{-1}\alpha_{r-1}^{-1}\ldots\alpha_l^{-1}(i)$ for $2 \leq l \leq r$ and i for $l = 1$.*

Example 2 *The edge labeling defined above for the 5-cactus of Example 1 is shown on Figure 2.*

2.2 Partitioned cacti

Cacti are non planar non recursive objects intractable to compute directly in the general case. Our bijective construction relies on the use of *partitioned cacti* that we define as follows:

Definition 1 (Partitioned cacti) *Let $C^n(p_1, p_2, \ldots, p_r)$ be the set of couples composed of a cactus (as defined in Section 2.1) and a r-tuple of partitions $(\tilde{\pi}_1, \ldots, \tilde{\pi}_r)$ such that $\tilde{\pi}_i$ is a partition with p_i blocks on the set of vertices of color i.*

Remark 2 *Using Proposition 1, we state that partitioned cacti in $C^n(p_1, p_2, \ldots, p_r)$ are in bijection with the $2r$-tuples $(\alpha_1, \ldots, \alpha_r, \pi_1, \ldots, \pi_r)$ where $\alpha_1 \ldots \alpha_r = \gamma_n$ and π_i is a partition on the set of integers $1..n$ composed of exactly p_i blocks stable by α_i. (As such the blocks of π_i are unions of cycles of α_i). Two vertices u and v of color i in the partitioned cactus belong to the same block if and only if the corresponding cycles of α_i belong to the same block of π_i.*

Example 3 *We use geometric shapes in Figure 2 to represent an example of partitions on the sets of vertices of the 5-cactus in Example 1. The equivalent numeric set partitions are $\pi_1 = \{1,2,6\}\{3\}\{4,5\}$, $\pi_2 = \{1,2,3,4\}\{5,6\}$, $\pi_3 = \{1,2,4\}\{3\}\{5,6\}$, $\pi_4 = \{1,2,3,5\}\{4,6\}$, and $\pi_5 = \{1,2,3\}\{4\}\{5\}\{6\}$.*

Similarly to [10], the numbers $k^n_{p_1,p_2,\ldots,p_r}$ and the cardinalities $\mid C^n(p_1, p_2, \ldots, p_r) \mid$ are linked by the relation:

$$\sum_{p_1,p_2,\ldots,p_r} k^n_{p_1,p_2,\ldots,p_r} \prod_{1\leq i\leq r} x_i^{p_i} = \sum_{p_1,p_2,\ldots,p_r} \mid C^n(p_1,p_2,\ldots,p_r) \mid \prod_{1\leq i\leq r} (x_i)_{p_i} \tag{4}$$

where $(x)_p = x(x-1)\ldots(x-p+1)$. Partitioned cacti are actually one-to-one with decorated recursive tree structures that we define in the next section.

2.3 Cactus trees

We look at non classical tree-like structures with colored vertices and various types of children. More specifically we work with recursive non cyclic graphs rooted in a given vertex such that all the vertices are colored with $1, 2, \ldots, r$. The ordered set of children of a given vertex v of color i may contain:

- half edges (later called 1-*gons*) linking v to no other vertex.
- full edges (later called 2-*gons*) linking v to a vertex of color $i+1$ (modulo r). This later vertex is the root of a descending subtree.

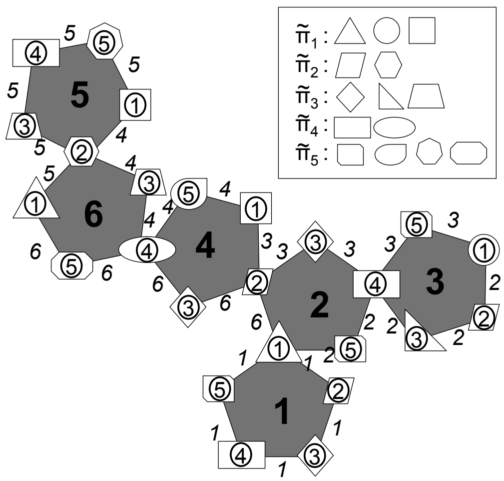

Fig. 2: A partitioned cactus with the additional edge labeling defined in Remark 1.

- j-gons linking v to $j-1$ vertices $v_1, v_2, \dots, v_{j-1}$ of respective colors $i+1, i+2, \dots, i+j-1$ (modulo r). Each v_k is the root of a descending subtree. When the size j of the j-gons is not determined, we simply call them *polygons*.

Now we are ready to give the full definition of the considered structure:

Definition 2 (Cactus trees) *For any sequence $\mathbf{a} = (a_t)$ of $2^r - 1$ non-negative integers a_t whith the index t being any non empty strictly increasing subsequence of integers of $1 \dots r$, we define the set $T(\mathbf{a})$ of* **cactus trees** *with vertices of r distinct colors as follows:*

(i) the root vertex of the cactus tree is of color 1,

(ii) the ordered set of children of a given vertex v of color i is composed of j-gons ($1 \le j \le r$) linking v to $j-1$ vertices (and subsequent subtrees) of respective colors $i+1, i+2, \dots, i+j-1$ (modulo r),

(iii) symbolic labels $\beta_1, \beta_2, \dots, \beta_n$ (where $n = \sum_t a_t$) are assigned to the polygons such that the set of polygons indexed with the same given label contains exactly one vertex of each color and that all the polygons in the tree are labelled,

(iv) for $t = (i_1, i_2, \dots, i_l)$ ($1 \le i_1 < i_2 < \dots < i_l \le r$), a_t is the number of those sets composed of a $(i_2 - i_1)$-gon child of a vertex of color i_1, a $(i_3 - i_2)$-gon child of a vertex of color i_2, ..., a $(i_l - i_{l-1})$-gon child of a vertex of color i_{l-1} and a $(r - i_l + i_1)$-gon child of a vertex of color i_l with the same symbolic label.
(One can easily check that exactly one vertex of each color is contained in such sets.)

Example 4 *The cactus tree depicted on the left hand side of Figure 3 has 10 1-gons, 4 2-gons, 1 3-gon, 1 4-gon and 1 5-gon. The corresponding non zero parameters (a_t) are $a_2 = 1$, $a_{1,2} = 1$, $a_{2,4} = 1$,*

$a_{1,3,4,5} = 1$, $a_{1,3,4} = 1$, $a_{1,2,3,4,5} = 1$. *The cactus tree on the right hand side has* 4 *1-gons,* 1 *2-gon and* 2 *3-gons. The* (a_t) *non equal to zero are* $a_{1,4} = 2$, $a_{1,2,3} = 1$.

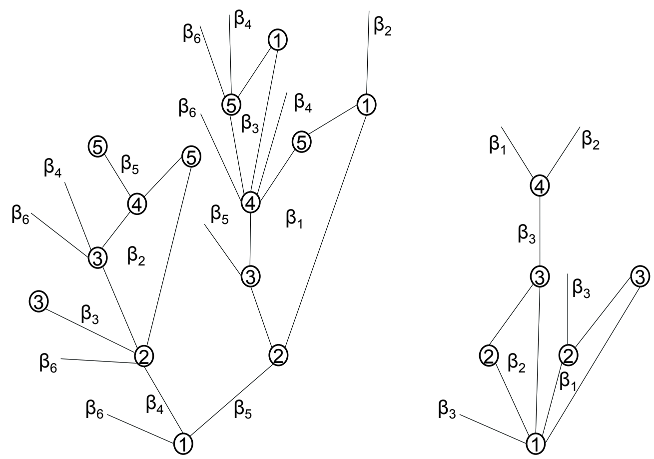

Fig. 3: Two examples of cactus trees.

Remark 3 *Point (iii) in Definition 2 restricts the number of possible cactus trees. In this paper, we consider only the cactus trees for which such a labeling is possible.*

Remark 4 *If we note* p_i *the number of vertices of color* i $(1 \leq i \leq r)$ *in a given cactus tree of* $T(\mathbf{a})$, *we have* $\sum_{t;l \notin t} a_t = p_l$ *for* $2 \leq l \leq r$ *and* $\sum_{t;1 \notin t} a_t = p_1 - 1$.

Proposition 2 *Let* $\mathbf{a} = (a_t)$, n, $(p_i)_{1 \leq i \leq r}$ *be such that* $n = \sum_t a_t$ *and* $\sum_{t;i \notin t} a_t = p_i - \delta_{i,1}$. *The number* $\mid T(\mathbf{a}) \mid$ *of cactus trees is given by:*

$$\mid T(\mathbf{a}) \mid = \frac{(n-1)!^{r-1}}{\prod_{1 \leq i \leq r} p_i!} \Delta_r(\mathbf{a}) \binom{n}{\mathbf{a}} \tag{5}$$

The proof of Proposition 2 can be obtained by using the Lagrange theorem in order to compute the number of cactus trees without the labeling and the 1-gons. Then, counting the number of ways to add the 1-gons and the symbolic labeling leads to the desired result. Combining Equations (1), (4) and (5), we notice that Theorem 1 is equivalent to the following statement:

Theorem 3 *The set of partitioned cacti* $C^n(p_1, p_2, \dots, p_r)$ *is in bijection with the union of sets of cactus trees* $T(\mathbf{a})$ *with* $\mathbf{a}$ *verifying the properties* $n = \sum_t a_t$ *and* $\sum_{t;i \notin t} a_t = p_i - \delta_{i,1}$.

According to the symmetry property proved in [1], Theorem 2 is implied by Theorem 1. As a result, Theorem 2 is also a consequence of Theorem 3.

3 Bijection between partitioned cacti and cactus trees

We start with a partitioned cactus κ of $C^n(p_1, p_2, \ldots, p_r)$. The edges of the black r-gons in κ are labeled according to Remark 1. We note $\tilde{\pi}_1, \ldots, \tilde{\pi}_r$ the partitions of the vertices in κ. Moreover, let $(\alpha_1, \ldots, \alpha_r, \pi_1, \ldots, \pi_r)$ be the $2r$-tuple corresponding to κ within the bijection described in Remark 2. We proceed with the following construction.

First, we define a set containing p_i *tree-vertices* (not to be confused with the vertices of κ) of color i ($1 \leq i \leq r$) with an ordered set of children composed of labeled half edges. Each tree-vertex of color i is associated to a block of $\tilde{\pi}_i$ (or equivalently π_i). The children are indexed by the labels of the edges linking a vertex of color i belonging to the considered block of $\tilde{\pi}_i$ in κ and a vertex of color $i-1$ (modulo r) *sorted in ascending order*. We assume that for all tree-vertices, the resulting labels of the half edges are increasing when we traverse them from *left to right*.

Then, we look at maximum length sequences of labels m_i, m_{i+1}, …, m_{i+l} (indices are taken modulo r) such that :

(i) m_t is the greatest label (and therefore the rightmost) of a half edge child of a tree-vertex of color t and

(ii) m_t's for $i \leq t \leq i + l$ are also the respective labels of the edges linking the vertices of color t and $t-1$ *in the same* r-gon of κ. If such a sequence contains a label m_1, maximum index around the tree-vertex of color 1 that is also parent of a half edge labeled by 1, we split the sequence into two subsequences m_i, …, m_r and m_2, …, m_{i+l}. If the initial sequence was the singleton m_1, we simply remove it.

Lemma 2 *The maximum length of the sequences defined above is* $r-1$.

Next we build a cactus tree by connecting the tree-vertices using j-gons ($j \geq 2$). The tree-vertex of color 1 connected to the half-edge labeled 1 is the root of the cactus tree. For any sequence m_i, m_{i+1}, …, m_{i+l} corresponding to the same r-gon in κ, let t be the label of the edge linking the vertex of colors $i-1$ and $i-2$ in this r-gon. By definition, t is not the maximum label around a tree-vertex of color $i-1$. We connect the tree-vertices with maximum child labels m_i, m_{i+1}, …, m_{i+l} thanks to a $l+2$-gon. We substitute this $l+2$-gon to the half edge labeled t in the children set of the corresponding tree-vertex of color $i-1$. We assign the labels m_i, m_{i+1}, …, m_{i+l}, t to it so that the edge linking the tree-vertices of colors i and $i-1$ is labeled m_i, the edge linking the tree-vertices of colors $i+1$ and i is labeled m_{i+1}, and so on, the edge linking the tree-vertices of colors $i+l$ and $i+l-1$ is labeled m_{i+l} and the edge linking the tree-vertices of colors $i+l$ and $i-1$ is labeled t. In what follows, we use the term u-*color numeric label* of the considered $l+2$-gon for m_u ($i \leq u \leq i+l$) and $i-1$-*color numeric label* for t.

Finally, we allocate a symbolic label from $\beta_1, \ldots, \beta_n$ to each polygon and each of the remaining half edges (1-gons) such that all the j-gons ($1 \leq j \leq r$) with numeric labels corresponding to the same r-gon of κ have the same symbolic label. At this stage, we remove all the numeric labels.

Example 5 *We apply the construction above to the partitioned cactus depicted on Figure 2. The definition of the set of tree-vertices and their connections by polygons is shown on Figure 4. The final step of symbolic labeling leads to the cactus on the left hand side of Figure 3.*

Lemma 3 *The construction above defines a cactus tree* τ *in* $T(\mathbf{a})$ *for some* $\mathbf{a}$ *verifying the properties* $n = \sum_t a_t$ *and* $\sum_{t; i \notin t} a_t = p_i - \delta_{i,1}$.

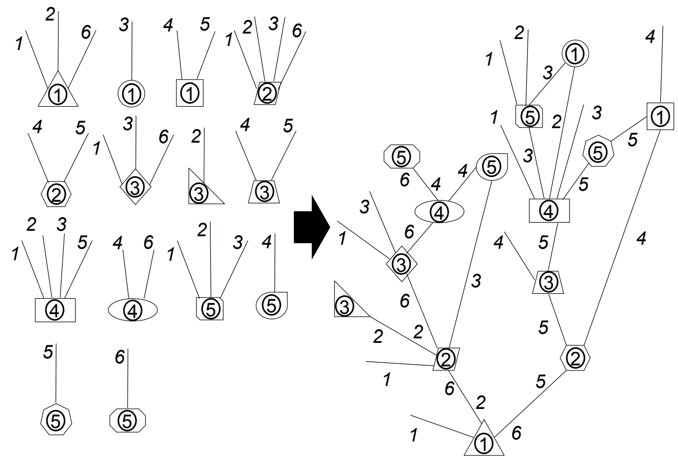

Fig. 4: Application of the bijective construction to the partitioned cactus of Figure 2

4 Proof of the bijection

4.1 Injectivity

Assume τ is a cactus tree of $T(\mathbf{a})$ obtained by the construction of Section 3. We show that at most one $2r$-tuple $(\alpha_1, \ldots, \alpha_r, \pi_1, \ldots, \pi_r)$ (or equivalently at most one partitioned cactus) is mapped to τ. To this purpose we show by induction that the numeric labels removed at the end of the procedure can be uniquely recovered.

First step is to notice that 1 is necessarily (one of) the numeric label(s) of the leftmost child of the root of τ (of color 1).

Now assume that the u-color (resp. 1-color) numeric labels $1, \ldots, i-1$ (resp. $1, \ldots, i$) have been recovered for $u = 2, 3, \ldots, r$ and $i < n$.

- Let β be the symbolic label of the polygon in τ with recovered numeric 1-color label i (this polygon is by assumption connected to a vertex of color 1). Then, β is also the symbolic label of exactly one polygon (possibly the one with recovered numeric 1-color label i) connected to a vertex v of color r. But as noticed in Lemma 1, if the 1-color numeric label is i, the r-color numeric label corresponding to the same r-gon in the initial partitioned cactus is $\alpha_r^{-1}(i)$. As a result, $\alpha_r^{-1}(i)$ is the numeric r-color label of the polygon connected to v. The blocks of π_r are stable by α_r so $\alpha_r(i)$ belongs to the same block of π_r. According to the order of the r-color labels around v, $i = \alpha_r^{-1}\alpha_r(i)$ is necessarily the r-color numeric label of the leftmost polygon connected to v with non recovered r-color label.

- Assume that we have recovered the polygon with u-color label i ($2 < u$). The index of the corresponding black r-gon in the partitioned cactus is necessarily $\alpha_u\alpha_{u+1}\ldots\alpha_r(i)$. Integers $\alpha_u\alpha_{u+1}\ldots\alpha_r(i)$ and $\alpha_{u-1}\alpha_u\alpha_{u+1}\ldots\alpha_r(i)$ belong to the same block of π_{u-1}. As the $(u-1)$-color labels have been sorted according to $\alpha_r^{-1}\ldots\alpha_{u-1}^{-1}$ around the vertices of color $(u-1)$, i is necessarily the $(u-1)$-color numeric label of the leftmost polygon (for which such a label has not been recovered yet) around the vertex of color $u-1$ incident to the polygon with the same symbolic label as the one incident to the polygon of u-color label i.

- Finally assume that we have recovered the polygon with 2-color label i. The index of the corresponding black r-gon in the partitioned cactus is necessarily $\alpha_2\alpha_3\ldots\alpha_r(i)$. We use the symbolic label to identify the vertex v of color 1 incident to the polygon of 1-color label $\alpha_2\alpha_3\ldots\alpha_r(i)$. With a similar argument as above $i+1 = \alpha_1\alpha_2\alpha_3\ldots\alpha_r(i)$ is necessarily the 1-color label of the leftmost polygon (with no recovered 1-color numeric label) incident to v.

The knowledge of τ uniquely determines the numeric labels of the polygons. But it is easy to see that combining symbolic and numeric labels uniquely determines $\alpha_r^{-1},\ldots,\alpha_r^{-1}\ldots\alpha_u^{-1},\ldots,\alpha_r^{-1}\ldots\alpha_2^{-1}$ and the α_i themselves. Then the knowledge of the numeric labels around the same vertices of τ uniquely determines the partitions $\pi_1,\ldots,\pi_r$. As a result, the partitioned cactus is uniquely determined.

Example 6 *We apply this inverse procedure to the cactus tree on the right hand side of Figure 3. Figure 5 shows how the numeric labels are iteratively recovered. The resulting numerically labeled cactus tree corresponds to the cactus on the right hand side of Figure 1 with partitions* $\pi_1 = \{1,2,3\}$, $\pi_2 = \{1,3\}\{2\}$, $\pi_3 = \{1,2\}\{3\}$ *and* $\pi_4 = \{1,2,3\}$.

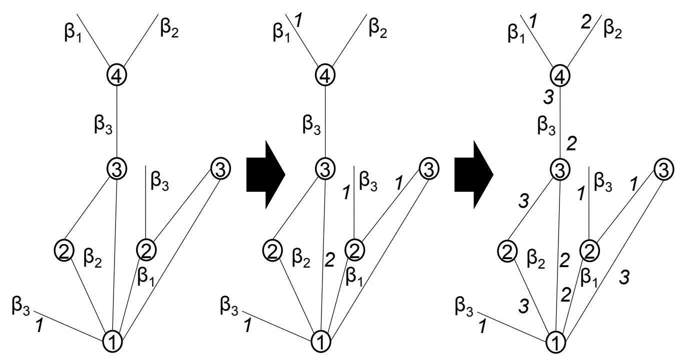

Fig. 5: Application of the inverse procedure to the cactus tree on the right hand side of Figure 3.

4.2 Surjectivity

We show that the reconstruction procedure defined in Section 4.1 always ends with a valid output. The procedure would be interrupted before the full recovery of the numeric labels if and only if no leftmost polygon around a vertex v of color i with non recovered i-color numeric label is available. Two cases are to be considered.

(i) If v is not the root, this situation is clearly impossible. If the number of polygons incident to v is c, we traverse exactly c times the vertex v to allocate i-color labels (the symbolic labels link the polygons around v to exactly c polygons incident to vertices of color $i+1$).

(ii) If v is the root, then an additional difficulty occurs as 1-color label 1 is recovered out of the main procedure. However for any vertex u of color $j+1$ in the cactus tree, the rightmost polygon connecting it to a vertex w of color j is obviously the last one to be recovered. The symbolic label of this polygon naturally links u and w. As a result, all the j-color labels of w are recovered necessarily *after* all the $j+1$ - labels are recovered around u. This extends to all the children of w and the vertex of color $j+1$ in the rightmost polygon of w if any. Since v is the root all the 1-color labels of v are recovered necessarily after all the labels of all the other vertices in the cactus tree are recovered. Finally the procedure ends in the proper way.

5 Equivalence of the main theorem and Jackson's formula

A natural question is the equivalence between the formula of Theorem 1 and Jackson's formula of [6] addressing the factorizations of γ_n. The result of Jackson for the factorizations of the long cycle can be stated as follows:

$$\frac{1}{(n!)^{r-1}}\sum_{p_1,p_2,\ldots,p_r} k^n_{p_1,p_2,\ldots,p_r}\prod_{1\le i\le r} x_i^{p_i} = \phi\left(\prod_{1\le i\le r} x_i\left(\prod_{1\le i\le r}(1+x_i)-\prod_{1\le i\le r}(x_i)\right)^{n-1}\right) \tag{6}$$

where ϕ is the mapping defined by $\phi(\prod_i x_i^{p_i}) = \prod_i \binom{x_i}{p_i}$ and extended linearly. One can show that this formula is equivalent to:

$$\frac{1}{(n!)^{r-1}}\sum_{p_1,p_2,\ldots,p_r} k^n_{p_1,p_2,\ldots,p_r}\prod_{1\le i\le r} x_i^{p_i} = \sum_{p_1,p_2,\ldots,p_r}\sum_{\mathbf{a}}\binom{n-1}{\mathbf{a}}\prod_{1\le i\le r}\binom{x_i}{p_i} \tag{7}$$

where the sequences $\mathbf{a}=(a_t)$ of 2^r-1 non-negative integers (a_t) satisfy $\sum_t a_t = n-1$, $\sum_{t;l\in t} a_t = p_l - 1$ for $1\le l\le r$.

For $r=2$ the equivalence is obvious as in Equation (1) the only sequence $\mathbf{a}$ fitting the conditions is $a_1 = p_2$, $a_2 = p_1-1$, $a_{1,2} = n+1-p_1-p_2$ and $\Delta_2(\mathbf{a}) = p_2$ in this case. The summand for indices p_1 and p_2 in our formula reads

$$(n-1)!\,p_2\binom{n}{p_1-1,p_2} = n!\binom{n-1}{p_1-1,p_2-1} \tag{8}$$

This shows that the two results are identical in this case.

For $r=3$, the determinant is $\Delta_3(\mathbf{a}) = p_2p_3 - a_3(p_3-a_1)$. For given p_1, p_2 and p_3 the equivalence between the two formulas can be shown in two steps. First we have:

$$\begin{aligned}
&(p_2p_3 - a_3(p_3 - a_1))\binom{n}{a_1, a_2, a_3, p_1 - 1 - a_2 - a_3, p_2 - a_3 - a_1, p_3 - a_1 - a_2}\\
&= ((p_2 - a_3 - a_1)(p_3 - a_2 - a_1) + a_1(p_2 + p_3 - a_1) + a_2(p_2 - a_3 - a_1))\\
&\qquad \times \binom{n}{a_1, a_2, a_3, p_1 - 1 - a_2 - a_3, p_2 - a_3 - a_1, p_3 - a_1 - a_2}\\
&= n(n + 2 + a_1 + a_2 + a_3 - p_1 - p_2 - p_3)\\
&\qquad \times \binom{n-1}{a_1, a_2, a_3, p_1 - 1 - a_2 - a_3, p_2 - 1 - a_3 - a_1, p_3 - 1 - a_1 - a_2}\\
&\qquad + n(p_1 - a_3 - a_2)\binom{n-1}{a_1, a_2 - 1, a_3, p_1 - a_2 - a_3, p_2 - 1 - a_3 - a_1, p_3 - a_1 - a_2}\\
&\qquad + n(p_2 + p_3 - a_1)\binom{n-1}{a_1 - 1, a_2, a_3, p_1 - 1 - a_2 - a_3, p_2 - a_3 - a_1, p_3 - a_1 - a_2}
\end{aligned}$$

Then summing over a_1, a_2, a_3 with the proper shifts of variable to get the same multinomial coefficient brings us to

$$\begin{aligned}
&(n-1)!^2 \sum_{a_1,a_2,a_3} (p_2p_3 - a_3(p_3 - a_1))\binom{n}{a_1, a_2, a_3, p_1 - 1 - a_2 - a_3, p_2 - a_3 - a_1, p_3 - a_1 - a_2}\\
&= n!^2 \sum_{a_1,a_2,a_3} \binom{n-1}{a_1, a_2, a_3, p_1 - 1 - a_2 - a_3, p_2 - 1 - a_3 - a_1, p_3 - 1 - a_1 - a_2}
\end{aligned}$$

that proves the equivalence of the two formulas also in the case $r = 3$.

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Evaluations of Hecke algebra traces at Kazhdan-Lusztig basis elements

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Abstract. For irreducible characters $\{\chi_q^\lambda \,|\, \lambda \vdash n\}$ and induced sign characters $\{\epsilon_q^\lambda \,|\, \lambda \vdash n\}$ of the Hecke algebra $H_n(q)$, and Kazhdan-Lusztig basis elements $C'_w(q)$ with w avoiding the pattern 312, we combinatorially interpret the polynomials $\chi_q^\lambda(q^{\frac{\ell(w)}{2}} C'_w(q))$ and $\epsilon_q^\lambda(q^{\frac{\ell(w)}{2}} C'_w(q))$. This gives a new algebraic interpretation of q-chromatic symmetric functions of Shareshian and Wachs. We conjecture similar interpretations and generating functions corresponding to other $H_n(q)$-traces.

Résumé. Pour les caractères irreductibles $\{\chi_q^\lambda \,|\, \lambda \vdash n\}$ et les caractères induits du signe $\{\epsilon_q^\lambda \,|\, \lambda \vdash n\}$ du algèbre de Hecke, et les éléments $C'_w(q)$ du base Kazhdan-Lusztig avec w qui évite le motif 312, nous interprétons les polinômes $\chi_q^\lambda(q^{\frac{\ell(w)}{2}} C'_w(q))$ et $\epsilon_q^\lambda(q^{\frac{\ell(w)}{2}} C'_w(q))$ de manière combinatorielle. Cette donne une nouvelle interprétation aux fonctionnes symétriques q-chromatiques de Shareshian et Wachs. Nous conjecturons des interprétations semblables et des foncionnes generatrices qui correspondent aux autres applications centrales de $H_n(q)$.

Keywords: Hecke algebra, trace, Kazhdan-Lusztig basis, tableau

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1 Introduction

The *symmetric group algebra* $\mathbb{C}\mathfrak{S}_n$ and the *(Iwahori-) Hecke algebra* $H_n(q)$ have similar presentations as algebras over $\mathbb{C}$ and $\mathbb{C}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ respectively, with multiplicative identity elements e and T_e, generators $s_1, \ldots, s_{n-1}$ and $T_{s_1}, \ldots, T_{s_{n-1}}$, and relations

$$\begin{aligned} s_i^2 &= e & T_{s_i}^2 &= (q-1)T_{s_i} + qT_e & &\text{for } i = 1, \ldots, n-1, \\ s_i s_j s_i &= s_j s_i s_j & T_{s_i}T_{s_j}T_{s_i} &= T_{s_j}T_{s_i}T_{s_j} & &\text{for } |i-j| = 1, \\ s_i s_j &= s_j s_i & T_{s_i}T_{s_j} &= T_{s_j}T_{s_i} & &\text{for } |i-j| \geq 2. \end{aligned}$$

Analogous to the natural basis $\{w \,|\, w \in \mathfrak{S}_n\}$ of $\mathbb{C}\mathfrak{S}_n$ is the natural basis $\{T_w \,|\, w \in \mathfrak{S}_n\}$ of $H_n(q)$, where we define $T_w = T_{s_{i_1}} \cdots T_{s_{i_\ell}}$ whenever $s_{i_1} \cdots s_{i_\ell}$ is a reduced expression for w in $\mathfrak{S}_n$. We call ℓ the *length* of w and write $\ell = \ell(w)$. (See [Hum90].) The specialization of $H_n(q)$ at $q^{\frac{1}{2}} = 1$ is isomorphic to $\mathbb{C}\mathfrak{S}_n$. In addition to the natural bases of $\mathbb{C}\mathfrak{S}_n$ and $H_n(q)$, we have the (signless) *Kazhdan-Lusztig bases* [KL79] $\{C'_w(1) \,|\, w \in \mathfrak{S}_n\}$, $\{C'_w(q) \,|\, w \in \mathfrak{S}_n\}$, defined in terms of certain *Kazhdan-Lusztig* *polynomials* $\{P_{u,v}(q) \,|\, u,v \in \mathfrak{S}_n\}$ in $\mathbb{N}[q]$ by

$$C'_w(1) = \sum_{v \leq w} P_{v,w}(1)v, \qquad C'_w(q) = q_{e,w}^{-1} \sum_{v \leq w} P_{v,w}(q) T_v, \tag{1}$$

where $\leq$ denotes the Bruhat order and we define $q_{v,w} = q^{\frac{\ell(w)-\ell(v)}{2}}$. (See, e.g., [BB96].)

Representations of $\mathbb{C}\mathfrak{S}_n$ and $H_n(q)$ are often studied in terms of *characters*. The $\mathbb{C}$-span of the $\mathfrak{S}_n$-characters is called the space of $\mathfrak{S}_n$-*class functions*, and has dimension is equal to the number of integer partitions of n. (See [Sag01].) Three well-studied bases are the irreducible characters $\{\chi^\lambda \,|\, \lambda \vdash n\}$, induced sign characters $\{\epsilon^\lambda \,|\, \lambda \vdash n\}$, and induced trivial characters $\{\eta^\lambda \,|\, \lambda \vdash n\}$, where $\lambda \vdash n$ denotes that λ is a partition of n. The $\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-span of the $H_n(q)$-characters, called the space of $H_n(q)$-*traces*, has the same dimension and analogous character bases $\{\chi_q^\lambda \,|\, \lambda \vdash n\}$, $\{\epsilon_q^\lambda \,|\, \lambda \vdash n\}$, $\{\eta_q^\lambda \,|\, \lambda \vdash n\}$, specializing at $q^{\frac{1}{2}} = 1$ to the $\mathfrak{S}_n$-character bases. Each of the two spaces has a fourth basis consisting of *monomial* class functions $\{\phi^\lambda \,|\, \lambda \vdash n\}$ or traces $\{\phi_q^\lambda \,|\, \lambda \vdash n\}$, and a fifth basis consisting of *power sum* class functions $\{\psi^\lambda \,|\, \lambda \vdash n\}$ or traces $\{\psi_q^\lambda \,|\, \lambda \vdash n\}$. These are defined via the inverse Kostka numbers $\{K_{\lambda,\mu}^{-1} \,|\, \lambda, \mu \vdash n\}$ and the numbers $\{L_{\lambda,\mu}^{-1} \,|\, \lambda, \mu \vdash n\}$ of row-constant Young tableaux of shape λ and content μ by

$$\phi^\lambda \underset{\text{def}}{=} \sum_\mu K_{\lambda,\mu}^{-1} \chi^\mu, \quad \phi_q^\lambda \underset{\text{def}}{=} \sum_\mu K_{\lambda,\mu}^{-1} \chi_q^\mu, \qquad \psi^\lambda \underset{\text{def}}{=} \sum_\mu L_{\lambda,\mu} \phi^\mu, \quad \psi_q^\lambda \underset{\text{def}}{=} \sum_\mu L_{\lambda,\mu} \phi_q^\mu. \tag{2}$$

These functions are not characters. (See [BRW96], [Hai93], [Ste92].) In each space, the five bases are related to one another by the same transition matrices which relate the Schur, elementary, complete homogeneous, monomial, and power sum bases of the homogeneous degree n symmetric functions. (See, e.g., [Sta99].)

It is known that irreducible $\mathfrak{S}_n$-characters $\{\chi^\lambda \,|\, \lambda \vdash n\}$ satisfy $\chi^\lambda(w) \in \mathbb{Z}$ for all $w \in \mathfrak{S}_n$. Thus for any integer linear combination θ of these and any element $z \in \mathbb{Z}\mathfrak{S}_n$, we have $\theta(z) \in \mathbb{Z}$ as well. In some cases, we may associate sets R, S to the pair (θ, z) to combinatorially interpret the integer $\theta(z)$ as $(-1)^{|S|}|R|$. We summarize known results and open problems in the following table.

θ	$\theta(w) \in \mathbb{N}$?	interpretation of $\theta(w)$ as $(-1)^{\lvert S\rvert}\lvert R\rvert$?	$\theta(C'_w(1)) \in \mathbb{N}$?	interpretation of $\theta(C'_w(1))$ as $\lvert R\rvert$ for w avoiding 312?
η^λ	yes	yes	yes	yes
ϵ^λ	no	yes	yes	yes
χ^λ	no	open	yes	yes
ψ^λ	yes	yes	yes	yes
ϕ^λ	no	yes	conj. by Stembridge, Haiman	open

For known combinatorial interpretations of $\theta(w)$, see [BRW96]. The number $\chi^\lambda(w)$ may be computed by the well-known algorithm of Murnaghan and Nakayama. (See, e.g., [Sta99].) Otherwise, $\chi^\lambda(w)$ has no conjectured expression of the type stated above. Interpretations of $\theta(C'_w(1))$ are not known for general $w \in \mathfrak{S}_n$, but nonnegativity follows from work of Haiman [Hai93] and Stembridge [Ste91]. Interpretations of $\eta^\lambda(C'_w(1))$, $\epsilon^\lambda(C'_w(1))$, $\chi^\lambda(C'_w(1))$ for w avoiding the pattern 312 follow via straightforward arguments from results of various authors, notably Gasharov [Gas96], Karlin-MacGregor [KM59], Lindström [Lin73], Littlewood [Lit40], Merris-Watkins [MW85], Stanley-Stembridge [SS93], [Ste91]. These will be discussed in Section 3. There is no conjectured combinatorial interpretation of $\phi^\lambda(C'_w(1))$, even for w avoiding the pattern 312 although interpretations have been given for particular partitions λ by Stembridge [Ste92] and several of the authors [CSS11].

It is known that irreducible $H_n(q)$-characters $\{\chi_q^\lambda \,|\, \lambda \vdash n\}$ satisfy $\chi_q^\lambda(T_w) \in \mathbb{Z}[q]$ for all $w \in \mathfrak{S}_n$. Thus for any integer linear combination θ_q of these and any element $z \in \mathrm{span}_{\mathbb{Z}[q]}\{T_w \,|\, w \in \mathfrak{S}_n\}$, we have $\theta_q(z) \in \mathbb{Z}[q]$ as well. In some cases, we may associate sequences $(S_k)_{k\geq 0}$, $(R_k)_{\geq 0}$ of sets to the pair (θ_q, z) to combinatorially interpret $\theta_q(z)$ as $\sum_k (-1)^{|S_k|}|R_k|q^k$. We summarize known results and open problems in the following table.

θ_q	$\theta_q(T_w) \in \mathbb{N}[q]$?	interpretation of $\theta_q(T_w)$ as $\sum_k (-1)^{\lvert S_k\rvert}\lvert R_k\rvert q^k$?	$\theta(q_{e,w}C'_w(q)) \in \mathbb{N}[q]$?	interpretation of $\theta_q(q_{e,w}C'_w(q))$ as $\sum_k \lvert R_k\rvert q^k$ for w avoiding 312?
η_q^λ	no	open	yes	conj. in Section 4
ϵ_q^λ	no	open	yes	stated in Section 4
χ_q^λ	no	open	yes	stated in Section 4
ψ_q^λ	no	open	conj. by Haiman	conj. in Section 4
ϕ_q^λ	no	open	conj. by Haiman	open

The polynomial $\chi_q^\lambda(T_w)$, and therefore all polynomials $\theta_q(T_w)$, may be computed via a q-extension of the Murnaghan-Nakayama algorithm. (See, e.g., [KV84], [KW92], [Ram91].) Otherwise, $\theta_q^\lambda(w)$ has no conjectured expression of the type stated above. Interpretations of $\theta_q(q_{e,w}C'_w(q))$ are not known for general $w \in \mathfrak{S}_n$, but results concerning containment in $\mathbb{N}[q]$ follow principally from work of Haiman [Hai93]. For w avoiding the pattern 312, a formula for $\epsilon_q^\lambda(q_{e,w}C'_w(q))$ is given by the authors in Section 4. Work of Gasharov [Gas96] and Shareshian-Wachs [SW12] then implies a formula for $\chi_q^\lambda(q_{e,w}C'_w(q))$. Conjectures for $\psi_q^\lambda(q_{e,w}C'_w(q))$ are due to the authors and Shareshian-Wachs. These results and conjectures will also be discussed in Section 4. There is no conjectured combinatorial interpretation of $\phi_q^\lambda(q_{e,w}C'_w(q))$, even for w avoiding the pattern 312.

Another way to understand the evaluations $\theta(w)$ is to define a generating function $\mathrm{Imm}_\theta(x)$ in the polynomial ring $\mathbb{C}[x_{1,1}, \ldots, x_{n,n}]$ for $\{\theta(w) \,|\, w \in \mathfrak{S}_n\}$. Similarly, we may define a generating function $\mathrm{Imm}_{\theta_q}(x)$ in a certain noncommutative ring $\mathcal{A}(n;q)$ for $\{\theta(T_w) \,|\, w \in \mathfrak{S}_n\}$. In some cases these generating functions have simple forms. We summarize known results in the following tables.

θ	nice expression for $\mathrm{Imm}_\theta(x)$?
η^λ	yes
ϵ^λ	yes
χ^λ	open
ψ^λ	yes
ϕ^λ	open

θ_q	nice expression for $\mathrm{Imm}_{\theta_q}(x)$?
η_q^λ	yes
ϵ_q^λ	yes
χ_q^λ	open
ψ_q^λ	conj. in Section 2
ϕ_q^λ	open

Nice expressions for $\mathrm{Imm}_{\eta^\lambda}(x)$ and $\mathrm{Imm}_{\epsilon^\lambda}(x)$ are due to Littlewood [Lit40] and Merris-Watkins [MW85], and a nice expression for $\mathrm{Imm}_{\psi^\lambda}(x)$ follows immediately from the usual definition of ψ. An expression for $\mathrm{Imm}_{\chi^\lambda}(x)$ as a coefficient of a generating function in two sets of variables was given by Goulden-Jackson [GJ92]. There is no conjectured nice formula for $\mathrm{Imm}_{\phi^\lambda}(x)$, although a nice formula for particular partitions λ was stated by Stembridge [Ste92]. These results will be discussed in Section 2. Nice expressions for $\mathrm{Imm}_{\eta_q^\lambda}(x)$ and $\mathrm{Imm}_{\epsilon_q^\lambda}(x)$ are due to the fourth author and Konvalinka [KS11], as is an expression for $\mathrm{Imm}_{\chi_q^\lambda}(x)$ as a coefficient in a generating function in two sets of variables. A nice expression for $\mathrm{Imm}_{\psi_q^\lambda}(x)$ is conjectured by the authors. These results and conjecture will be discussed in Section 2.

In Section 2 we discuss known descriptions of the class functions in terms of generating functions in the ring $\mathbb{C}[x_{1,1},\ldots,x_{n,n}]$ and in a certain quantum analog $\mathcal{A}(n;q)$ of $\mathbb{C}[x_{1,1},\ldots,x_{n,n}]$ known as the *quantum matrix bialgebra*. We also give a combinatorial interpretation of the entries of the transition matrices relating certain bases of $\mathcal{A}(n;q)$. In Section 3 we give combinatorial interpretations, using results in the previous section. In Section 4 we give new descriptions of the class functions in terms of generating functions in the rings $\mathbb{C}\otimes\Lambda$ and $\mathbb{C}[q^{\frac{1}{2}},q^{\frac{-1}{2}}]\otimes\Lambda$ of symmetric functions having coefficients in $\mathbb{C}$ and $\mathbb{C}[q^{\frac{1}{2}},q^{\frac{-1}{2}}]$. Finally, in Section 5 we draw connections to posets and to the chromatic quasisymmetric functions of Shareshian and Wachs.

2 Generating functions for $\theta(w)$ and $\theta_q(T_w)$ when θ is fixed

For a fixed $\mathfrak{S}_n$-class function θ, we create a generating function for $\{\theta(w)\,|\,w\in\mathfrak{S}_n\}$ by writing $x=(x_{i,j})$, $\mathbb{C}[x]\underset{\mathrm{def}}{=}\mathbb{C}[x_{1,1},\ldots,x_{n,n}]$, and defining

$$\mathrm{Imm}_\theta(x)\underset{\mathrm{def}}{=}\sum_{w\in\mathfrak{S}_n}\theta(w)x_{1,w_1}\cdots x_{n,w_n}\ \in\ \mathbb{C}[x].$$

We call this polynomial the θ-*immanant*. The sign character ($w\mapsto(-1)^{\ell(w)}$) immanant and trivial character ($w\mapsto 1$) immanant are the determinant and permanent. Nice formulas for the ϵ^λ-immanants and η^λ-immanants employ determinants and permanents of submatrices of x,

$$x_{I,J}\underset{\mathrm{def}}{=}(x_{i,j})_{i\in I,j\in J},\qquad I,J\subset[n]\underset{\mathrm{def}}{=}\{1,\ldots,n\}.$$

In particular, for $\lambda=(\lambda_1,\ldots,\lambda_r)\vdash n$ we have Littlewood-Merris-Watkins identities [Lit40], [MW85]

$$\mathrm{Imm}_{\epsilon^\lambda}(x)=\sum_{(I_1,\ldots,I_r)}\det(x_{I_1,I_1})\cdots\det(x_{I_r,I_r}),\qquad \mathrm{Imm}_{\eta^\lambda}(x)=\sum_{(I_1,\ldots,I_r)}\mathrm{per}(x_{I_1,I_1})\cdots\mathrm{per}(x_{I_r,I_r}),\quad(3)$$

where the sums are over all sequences of pairwise disjoint subsets of $[n]$ satisfying $|I_j|=\lambda_j$. A formula for the ψ^λ-immanant relies upon a sum over all permutations of cycle type λ,

$$\mathrm{Imm}_{\psi^\lambda}(x)=z_\lambda\sum_{\substack{w\\ \mathrm{cyc}(w)=\lambda}}x_{1,w_1}\cdots x_{n,w_n},$$

where z_λ is the product $1^{\alpha_1}2^{\alpha_2}\cdots n^{\alpha_n}\alpha_1!\cdots\alpha_n!$, and λ has α_i parts equal to i for $i=1,\ldots,n$. No such nice formulas are known for the χ^λ-immanants or ϕ^λ-immanants in general, although we do have a formula [Ste92, Thm. 2.8] for $\mathrm{Imm}_{\phi^\lambda}(x)$ when $\lambda_1=\cdots=\lambda_r=k$,

$$\mathrm{Imm}_{\phi^{k^r}}(x)=\sum_{(I_1,\ldots,I_k)}\det(x_{I_1,I_2})\det(x_{I_2,I_3})\cdots\det(x_{I_k,I_1}),$$

where the sum is over all sequences of pairwise disjoint subsets of $[n] = [kr]$ satisfying $|I_j| = r$.

For a fixed $H_n(q)$-trace θ_q, we create a generating function for $\{\theta_q(T_w) \,|\, w \in \mathfrak{S}_n\}$ as before, but interpreting polynomials in $x = (x_{i,j})$ as elements of the *quantum matrix bialgebra* $\mathcal{A}(n; q)$, the non-commutative $\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-algebra generated by n^2 variables $x = (x_{1,1}, \dots, x_{n,n})$, subject to the relations

$$\begin{aligned} x_{i,\ell}x_{i,k} &= q^{\frac{1}{2}} x_{i,k}x_{i,\ell}, \qquad & x_{j,k}x_{i,\ell} &= x_{i,\ell}x_{j,k} \\ x_{j,k}x_{i,k} &= q^{\frac{1}{2}} x_{i,k}x_{j,k} \qquad & x_{j,\ell}x_{i,k} &= x_{i,k}x_{j,\ell} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x_{i,\ell}x_{j,k}, \end{aligned} \tag{4}$$

for all indices $1 \leq i < j \leq n$ and $1 \leq k < \ell \leq n$. As a $\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-module, $\mathcal{A}(n; q)$ has a basis of monomials $x_{\ell_1,m_1} \cdots x_{\ell_r,m_r}$ in which index pairs appear in lexicographic order. The relations (4) allow one to express other monomials in terms of this natural basis.

As a generating function for $\{\theta_q(T_w) \,|\, w \in \mathfrak{S}_n\}$, we define

$$\mathrm{Imm}_{\theta_q}(x) \underset{\mathrm{def}}{=} \sum_{w \in \mathfrak{S}_n} \theta_q(T_w) q_{e,w}^{-1} x_{1,w_1} \cdots x_{n,w_n}$$

in $\mathcal{A}(n; q)$, and call this the θ_q-*immanant*. The $H_n(q)$ sign character ($T_w \mapsto (-1)^{\ell(w)}$) immanant and trivial character ($T_w \mapsto q^{\ell(w)}$) immanant are called the *quantum determinant* and *quantum permanent*,

$$\det_q(x) = \sum_{w \in \mathfrak{S}_n} (-q^{\frac{-1}{2}})^{\ell(w)} x_{1,w_1} \cdots x_{n,w_n}, \qquad \mathrm{per}_q(x) = \sum_{w \in \mathfrak{S}_n} (q^{\frac{1}{2}})^{\ell(w)} x_{1,w_1} \cdots x_{n,w_n}.$$

Specializing $\mathcal{A}(n; q)$, $\det_q(x)$, and $\mathrm{per}_q(x)$ at $q^{\frac{1}{2}} = 1$, we obtain the commutative polynomial ring $\mathbb{C}[x]$ and the classical determinant $\det(x)$ and permanent $\mathrm{per}(x)$.

Nice formulas for the ϵ_q^λ-immanants and η_q^λ-immanants employ quantum determinants and quantum permanents of submatrices of x. In particular, the fourth author and Konvalinka [KS11, Thm. 5.4] proved quantum analogs of the Littlewood-Merris-Watkins identities in (3),

$$\mathrm{Imm}_{\epsilon_q^\lambda}(x) = \sum_{(I_1,\dots,I_r)} \det_q(x_{I_1,I_1}) \cdots \det_q(x_{I_r,I_r}), \quad \mathrm{Imm}_{\eta_q^\lambda}(x) = \sum_{(I_1,\dots,I_r)} \mathrm{per}_q(x_{I_1,I_1}) \cdots \mathrm{per}_q(x_{I_r,I_r}), \tag{5}$$

where the sums are as in (3).

To state a nice form for the ψ_q^λ-immanant, we introduce the following definitions. Given a sequence $c = (i_1, \dots, i_k)$ of distinct elements of $[n]$ with $i_1 = \min\{i_1, \dots, i_k\}$, define the element $d_{(i_1,\dots,i_k)}(x)$ of $\mathcal{A}(n; q)$ to be the sum of all cyclic rearrangements of the monomial $x_{i_1,i_2}x_{i_2,i_3} \cdots x_{i_k,i_1}$, each weighted by $q^{j-(k+1)/2}$, where x_{i_1,i_2} appears in position j,

$$d_c(x) = q^{\frac{-(k-1)}{2}} x_{i_1,i_2}x_{i_2,i_3} \cdots x_{i_k,i_1} + q^{\frac{-(k-3)}{2}} x_{i_k,i_1}x_{i_1,i_2} \cdots x_{i_{k-1},i_k} + \cdots + q^{\frac{(k-1)}{2}} x_{i_2,i_3} \cdots x_{i_k,i_1}x_{i_1,i_2}.$$

For $w \in \mathfrak{S}_n$ having cycle type $\lambda = (\lambda_1, \dots, \lambda_r)$, define the polynomial $g_w(x)$ to be the sum, over all cycle decompositions $(c_1, \dots, c_r)$ of w with $|c_j| = \lambda_j$, of $d_{c_1}(x) \cdots d_{c_r}(x)$. For example, the permutation $w = (1, 4, 3)(2, 7)(5, 6) = (1, 4, 3)(5, 6)(2, 7)$ with its (exactly) two admissible cycle decompositions leads to the element $g_w(x) =$

$$(q^{-1}x_{1,4}x_{4,3}x_{3,1} + x_{3,1}x_{1,4}x_{4,3} + qx_{4,3}x_{3,1}x_{1,4})(q^{\frac{-1}{2}}x_{2,7}x_{7,2} + q^{\frac{1}{2}}x_{7,2}x_{2,7})(q^{\frac{-1}{2}}x_{5,6}x_{6,5} + q^{\frac{1}{2}}x_{6,5}x_{5,6}) +$$
$$(q^{-1}x_{1,4}x_{4,3}x_{3,1} + x_{3,1}x_{1,4}x_{4,3} + qx_{4,3}x_{3,1}x_{1,4})(q^{\frac{-1}{2}}x_{5,6}x_{6,5} + q^{\frac{1}{2}}x_{6,5}x_{5,6})(q^{\frac{-1}{2}}x_{2,7}x_{7,2} + q^{\frac{1}{2}}x_{7,2}x_{2,7}).$$

Conjecture 2.1 *Fix $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$. Then in $\mathcal{A}(n;q)$ we have* $\mathrm{Imm}_{\psi_q^\lambda}(x) = \sum_{\substack{w \\ \mathrm{cyc}(w)=\lambda}} g_w(x)$.

For example, when $n = 5$ and $\lambda = (3, 2)$, we represent each permutation having cycle type $(3, 2)$ as a product of a 3-cycle with least letter written first and a 2-cycle with least letter written first, $(1, 2, 3)(4, 5)$, $(1, 4, 2)(3, 5), \ldots, (3, 5, 4)(1, 2)$, and we have

$$\begin{aligned}\mathrm{Imm}_{\psi_q^{32}}(x) = {} & (q^{-1}x_{1,2}x_{2,3}x_{3,1} + x_{3,1}x_{1,2}x_{2,3} + qx_{2,3}x_{3,1}x_{1,2})(q^{\frac{-1}{2}}x_{4,5}x_{5,4} + q^{\frac{1}{2}}x_{5,4}x_{4,5}) \\ & + (q^{-1}x_{1,4}x_{4,2}x_{2,1} + x_{2,1}x_{1,4}x_{4,2} + qx_{4,2}x_{2,1}x_{1,4})(q^{\frac{-1}{2}}x_{3,5}x_{5,3} + q^{\frac{1}{2}}x_{5,3}x_{3,5}) \\ & + \cdots + (q^{-1}x_{3,5}x_{5,4}x_{4,3} + x_{4,3}x_{3,5}x_{5,4} + qx_{5,4}x_{4,3}x_{3,5})(q^{\frac{-1}{2}}x_{1,2}x_{2,1} + q^{\frac{1}{2}}x_{2,1}x_{1,2}).\end{aligned}$$

No such nice formulas are known for the χ_q^λ- or ϕ_q^λ- immanants.

To obtain values of $\epsilon_q^\lambda(T_w)$ and $\eta_q^\lambda(T_w)$ from (5), one must use the relations (4) to expand in the natural basis of (the zero-weight space $\mathrm{span}\{x_{1,w_1} \cdots x_{n,w_n} \,|\, w \in \mathfrak{S}_n\}$ of) $\mathcal{A}(n;q)$. For this purpose, it is helpful to combinatorially interpret the coefficients arising as entries in the transition matrix relating the bases $\mathcal{B}_u = \{x_{u_1,v_1} \cdots x_{u_n,v_n} \,|\, v \in \mathfrak{S}_n\}$ and the natural basis $\{x_{1,w_1} \cdots x_{n,w_n} \,|\, w \in \mathfrak{S}_n\}$. These were obtained by Lambright and the fourth author in [LS10]. To combinatorially interpret the evaluations $\{\epsilon_q^\lambda(q_{e,w}C'_w(q)) \,|\, \lambda \vdash n\}$ when w avoids the pattern 312, we prove a stronger result.

Theorem 2.2 *Fix $u, w \in \mathfrak{S}_n$ with $u \leq w$, and let $s_{i_1} \cdots s_{i_\ell}$ be the right-to-left lexicographically greatest reduced expression for u. Choose an index $k \leq \ell + 1$ and define $u' = s_{i_{k-1}} \cdots s_{i_1} u$, $w' = s_{i_{k-1}} \cdots s_{i_1} w$. Then we have*

$$x_{u_1,w_1} \cdots x_{u_n,w_n} = \sum_{v \in \mathfrak{S}_n} t_{u,w',v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}) x_{u'_1,v_1} \cdots x_{u'_n,v_n},$$

where $\{t_{u,w',v}(q) \,|\, v \in \mathfrak{S}_n\}$ are polynomials in $\mathbb{N}[q]$. Moreover, the coefficient of q^b in $t_{u,w',v}(q)$ is equal to the number of sequences $(\pi^{(0)}, \ldots, \pi^{(k-1)})$ of permutations satisfying

1. *$\pi^0 = w$, $\pi^{(k-1)} = v$,*
2. *$\pi^{(j)} \in \{s_{i_j}\pi^{(j-1)}, \pi^{(j-1)}\}$ for $j = 1, \ldots, k-1$,*
3. *$\pi^{(j)} = s_{i_j}\pi^{(j-1)}$ if $s_{i_j}\pi^{(j-1)} > \pi^{(j-1)}$,*
4. *$\pi^{(j)} = \pi^{(j-1)}$ for exactly b values of j.*

Proof: Omitted. □

We may think of each sequence $(\pi^{(0)}, \ldots, \pi^{(k-1)})$ in the above proof as a $(k-1)$-step walk from w to v in the weak order on $\mathfrak{S}_n$. After visiting $\pi^{(j)} \in \mathfrak{S}_n$, we may either revisit this permutation or move to $s_{i_j}\pi^{(j)}$, with the latter option being mandatory if s_{i_j} is a left ascent for $\pi^{(j)}$.

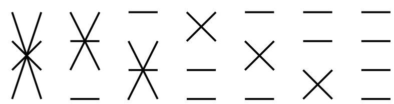
3 Descending star networks and interpretations of class functions

Call a directed planar graph G a *planar network of order* n if it is acyclic and may be embedded in a disc so that $2n$ boundary vertices labeled clockwise as *source* $1, \ldots,$ *source* n (with indegrees of 0) and *sink* $n, \ldots,$ *sink* 1 (with outdegrees of 0). In figures, we will draw sources on the left and sinks on the right, implicitly labeled $1, \ldots, n$ from bottom to top. Given a planar network G, define the *path matrix* $B = B(G) = (b_{i,j})$ of G by

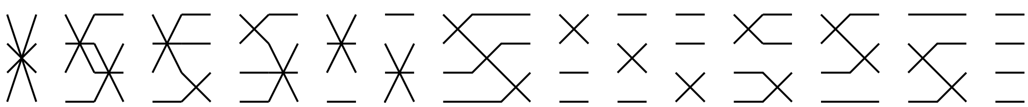
$$b_{i,j} = \text{ number of paths in } G \text{ from source } i \text{ to sink } j. \tag{6}$$

It is known that the path matrix of any planar network is totally nonnegative (TNN), i.e., that every minor of this matrix is nonnegative. This fact is known as *Lindström's Lemma*.

Call a sequence $\pi = (\pi_1, \ldots, \pi_n)$ of source-to-sink paths in a planar network a *bijective path family* if for some $w \in \mathfrak{S}_n$ with one-line notation $w_1 \cdots w_n$, each component path π_i begins at source i and terminates at sink w_i. We will say also that π has *type* w. Call a planar network a *bijective skeleton* if it is a union of n source-to-sink paths. Clearly a bijective path family can cover an entire planar network G only if G is a bijective skeleton. For $[a, b]$ a subinterval of $[n]$, let $G_{[a,b]}$ be the bijective skeleton consisting of $a-1$ horizontal edges, a "star" of $b-a+1$ edges from sources $a, \ldots, b$ to an intermediate vertex, and $b-a+1$ more edges from this vertex to sinks $a, \ldots, b$, and $n-b$ more horizontal edges. For $n = 4$, there are seven such networks: $G_{[1,4]}$, $G_{[2,4]}$, $G_{[1,3]}$, $G_{[3,4]}$, $G_{[2,3]}$, $G_{[1,2]}$, $G_{[1,1]} = \cdots = G_{[4,4]}$, respectively,

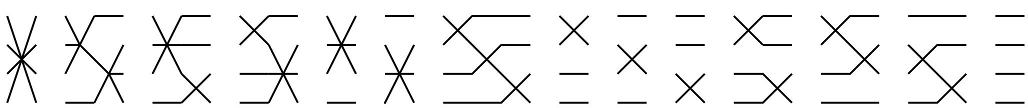


Define $G_I \circ G_J$ to be the concatenation of planar networks G_I and G_J, and consider a sequence $([c_1, d_1], \ldots, [c_r, d_r])$ of subintervals of $[n]$ satisfying $c_1 > \cdots > c_r$ and $d_1 > \cdots > d_r$, and the concatenation $G_{[c_1,d_1]} \circ \cdots \circ G_{[c_r,d_r]}$ of corresponding star networks. For $n = 4$, these are



(7)

For each such planar network G, we define a related planar network F by modifying G as follows. For $i = 1, \ldots, r-1$, if the intersection $[c_{i+1}, d_{i+1}] \cap [c_i, d_i]$ has cardinality $k > 1$, then collapse the k paths from the central vertex of $G_{[c_{i+1},d_{i+1}]}$ to the central vertex of $G_{[c_i,d_i]}$, creating a single path between these vertices. Call F a *descending star network*. For $n = 4$, the decending star networks are



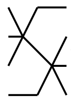
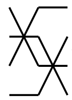
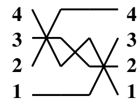
(8)

Proposition 3.1 *There are* $\frac{1}{n+1}\binom{2n}{n}$ *descending star networks of order* n.

Proof: (Idea.) Let F be the descending star network which corresponds as before (7) to the concatenation $G = G_{[c_1,d_1]} \circ \cdots \circ G_{[c_r,d_r]}$. Modify G to create the related network

$$G' \underset{\text{def}}{=} G_{[c_1,d_1]} \circ G_{[c_1,d_1] \cap [c_2,d_2]} \circ G_{[c_2,d_2]} \circ \cdots \circ G_{[c_{r-1},d_{r-1}]} \circ G_{[c_{r-1},d_{r-1}] \cap [c_r,d_r]} \circ G_{[c_r,d_r]}$$

by inserting $G_{[c_i,d_i] \cap [c_{i+1},d_{i+1}]}$ between $G_{[c_i,d_i]}$ and $G_{[c_{i+1},d_{i+1}]}$ for $i = 1, \ldots, r-1$. Now visually follow paths from sources to sinks, passing "straight" through each intersection, to complete a bijection to 312-avoiding permutations in $\mathfrak{S}_n$. For example, when $n = 4$ and F corresponds to $G = G_{[2,4]} \circ G_{[1,3]}$, we construct $G' = G_{[2,4]} \circ G_{[2,3]} \circ G_{[1,3]}$ and obtain the 312-avoiding permutation $w = w(F) = 3421$:

$F =$ , $G =$ , $G' =$ , $w = \begin{pmatrix} 1234 \\ 3421 \end{pmatrix}$.

□

For each 312-avoiding permutation $w \in \mathfrak{S}_n$, let F_w denote the descending star network corresponding to w by the bijection in the proof of Proposition 3.1. Every descending star network F_w is a bijective skeleton, and for every $v \leq w$ in the Bruhat order, there is exactly one bijective path family π of type v which covers F_w.

In a planar network G of order n, the source-to-sink paths have a natural partial order $Q = Q(G)$. If π_i is a path originating at source i, and ρ_j is a path originating at source j, then we define $\pi_i <_Q \rho_j$ if $i < j$ and π_i and ρ_j never intersect. Observe that these conditions imply the index of the sink of π_i to be less than the index of the sink of ρ_j. Let $P(G)$ be the subposet of $Q(G)$ induced by paths whose source and sink indices are equal. For each descending star network F_w, the poset $P(F_w)$ has exactly n elements: there is exactly one path from source i to sink i, for $i = 1, \ldots, n$.

To combinatorially interpret evaluations of $\mathfrak{S}_n$-class functions and $H_n(q)$-traces, we will fill (French) Young diagrams with path families $(\pi_1, \ldots, \pi_n)$ covering a descending star network F_w, and will call the resulting structures F*-tableaux*. If an F_w-tableau U contains a path family π of type v, then we also say that U has *type* v. We say that an F_w-tableau U has *shape* λ for some partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ if it has λ_i cells in row i for all i. If U has λ_i cells in *column* i for all i, we say that U has shape $\lambda^\top$. In this case we define $\lambda^\top$ to be the partition whose ith part is equal to the number of cells in row i of U. Let $L(U)$ and $R(U)$ be the Young tableaux of integers obtained from U by replacing paths $\pi_1, \ldots, \pi_n$ with their corresponding source and sink indices, respectively.

We define several properties of an F-tableau in terms of the poset Q and the tableaux $L(U)$ and $R(U)$.

1. Call U *column-strict* if whenever paths $\pi_{i_1}, \ldots, \pi_{i_r}$ appear from bottom to top in a column, then we have $\pi_{i_1} <_Q \cdots <_Q \pi_{i_r}$.

2. Call U *row-semistrict* if whenever paths π_{i_1}, π_{i_2} appear consecutively (from left to right) in a row, we have $\pi_{i_1} <_Q \pi_{i_2}$ or π_{i_1} is incomparable to π_{i_2} in Q.

3. Call U *cyclically row-semistrict* if it is row-semistrict and the condition above applies also to paths π_{i_1}, π_{i_2} appearing last and first (respectively) in the same row.

4. Call U *standard* if it is column-strict and row-semistrict.

5. Call U *cylindrical* if for each row of $L(U)$ containing indices $i_1, \ldots, i_k$ from left to right, the corresponding row of $R(U)$ contains $i_2, \ldots, i_k, i_1$ from left to right.

6. Call U *row-closed* if $L(U)$ is row-strict (entries increase to the right) and if each row of $R(U)$ is a permutation of the corresponding row of $L(U)$.

For some $\mathfrak{S}_n$-class functions θ, and all 312-avoiding permutations w, we may combinatorially interpret $\theta(C'_w(1))$ in terms of a star network F_w as follows.

Proposition 3.2 *Let w avoid the pattern 312, and let F_w be the corresponding descending star network.*

1. *$\eta^\lambda(C'_w(1))$ equals the number of row-semistrict F_w-tableaux of type e and shape λ. It also equals the number of row-closed F_w-tableaux of shape λ.*

2. *$\epsilon^\lambda(C'_w(1))$ equals the number of column-strict F_w-tableaux of type e and shape $\lambda^\top$.*

3. *$\chi^\lambda(C'_w(1))$ equals the number of semistandard F_w-tableaux of type e and shape λ.*

4. *$\psi^\lambda(C'_w(1))$ equals the number of cyclically row-semistrict F_w-tableaux of type e and shape λ. It also equals the number of cylindrical F_w-tableaux of shape λ.*

5. *For $\lambda_1 \leq 2$, $\phi^\lambda(C'_w(1))$ equals zero if there exists a column-strict F_w-tableaux of type e and shape $\mu \prec \lambda$; otherwise it equals the number of column-strict F_w-tableaux of type e and shape λ.*

6. *For $\lambda = k^r$, $\phi^\lambda(C'_w(1))$ equals the number of column-strict cylindrical F_w-tableaux of shape r^k.*

Proof: (Idea.) For w avoiding 312, the path matrix $B = (b_{i,j})$ of F_w satisfies $\theta(C'_w(1)) = \mathrm{Imm}_\theta(B)$. □

Haiman [Hai93] and Stembridge [Ste91] have shown that we have $\chi^\lambda(C'_w(1)) \geq 0$ for all $\lambda \vdash n$ and all $w \in \mathfrak{S}_n$. However, there is no conjectured combinatorial interpretation for $\chi^\lambda(C'_w(1))$ unless w avoids 312. Haiman [Hai93] and Stembridge [Ste92] have also conjectured that we have $\phi^\lambda(C'_w(1)) \geq 0$ for all $\lambda \vdash n$ and all $w \in \mathfrak{S}_n$. There is no general conjectured combinatorial interpretation for $\phi^\lambda(C'_w(1))$, even in the case that w avoids 312, unless λ has the special form stated in Proposition 3.2.

4 Statistics on F-tableaux and interpretations of $H_n(q)$-traces

For θ an $\mathfrak{S}_n$-class function and w avoiding 312, Proposition 3.2 interprets $\theta(C'_w(1))$ as the cardinality of a set of certain F_w-tableaux. For each of these sets of F_w-tableaux, we define a statistic mapping tableaux to nonnegative integers, and show (or conjecture) that $\theta_q(q_{e,w}C'_w(q))$ is a generating function for tableaux on which the statistic takes the values $k = 0, 1, \ldots$. In each case, our statistic is based upon the number of inversions of a permutation in $\mathfrak{S}_n$. Specifically, let F be a descending star network, and let U be an F-tableau containing path family $\pi = (\pi_1, \ldots, \pi_n)$ of type w. (Thus π_i begins at source vertex i and terminates at sink vertex w_i for $i = 1, \ldots, n$.) Let (π_i, π_j) be a pair of intersecting paths in F such that π_i appears in a column of U to the left of the column containing π_j. Call (π_i, π_j) a *left inversion* in U if we have $i > j$ and a *right inversion* in U if we have $w_i > w_j$. Let $\mathrm{INV}(U)$ denote the number of left inversions in U, and let $\mathrm{RINV}(U)$ denote the number of right inversions in U.

Proofs of the validity of the tableaux interpretations in Proposition 3.2 depend upon a relationship between immanants and path matrices. To state a q-analog of this relationship, we define a map for each $n \times n$ complex matrix B by

$$\begin{aligned} \sigma_B : \mathcal{A}(n;q) &\to \mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}] \\ x_{1,w_1} \cdots x_{n,w_n} &\mapsto q_{e,w} b_{1,w_1} \cdots b_{n,w_n}. \end{aligned}$$

Proposition 4.1 *Let θ_q be an $H_n(q)$-trace and let $w \in \mathfrak{S}_n$ avoid the pattern 312. Then the path matrix B of F_w satisfies $\theta_q(q_{e,w}C'_w(q)) = \sigma_B(\mathrm{Imm}_{\theta_q}(x))$.*

Proof: Omitted. □

Theorem 4.2 *Let $w \in \mathfrak{S}_n$ avoid the pattern* 312*. For $\lambda \vdash n$ we have*

$$\epsilon_q^\lambda(q_{e,w}C'_w(q)) = \sum q^{\mathrm{INV}(U)}, \tag{9}$$

where the sum is over all column-strict F_w-tableaux U of type e and shape $\lambda^\top$. We also have

$$\chi_q^\lambda(q_{e,w}C'_w(q)) = \sum q^{\mathrm{INV}(U)}, \tag{10}$$

where the sum is over all standard F_w-tableaux U of type e and shape λ.

Proof: Omitted. The proof of (10) depends upon a result of Shareshian and Wachs [SW12]. □

Let U be an F-tableau of shape $\lambda = (\lambda_1, \ldots, \lambda_r)$ containing a path family π, and let U_i be the ith row of U. Let $U_1 \circ \cdots \circ U_r$ and $U_r \circ \cdots \circ U_1$ be the F-tableaux of shape n consisting of the rows of U concatenated in increasing and decreasing order, respectively.

Conjecture 4.3 *Let $w \in \mathfrak{S}_n$ avoid the pattern* 312*. For $\lambda \vdash n$ we have*

$$\eta_q^\lambda(q_{e,w}C'_w(q)) = \sum q^{\mathrm{RINV}(U_1 \circ \cdots \circ U_r)}, \tag{11}$$

where the sum is over all row-closed F_w-tableaux U of shape λ. We also have

$$\psi_q^\lambda(q_{e,w}C'_w(q)) = \sum q^{\mathrm{INV}(U_r \circ \cdots \circ U_1)}, \tag{12}$$

where the sum is over all cylindrical F_w-tableaux U of shape λ.

Haiman [Hai93] has shown that we have $\chi_q^\lambda(q_{e,w}C'_w(q)) \in \mathbb{N}[q]$ for all $\lambda \vdash n$ and all $w \in \mathfrak{S}_n$. He has also conjectured that we have $\phi_q^\lambda(q_{e,w}C'_w(q)) \in \mathbb{N}[q]$ for all $\lambda \vdash n$ and all $w \in \mathfrak{S}_n$. There is no general conjectured combinatorial interpretation for $\phi_q^\lambda(q_{e,w}C'_w(q))$, even in the case that w avoids 312.

5 Generating functions for $\theta(C'_w(1))$, $\theta_q(q_{e,w}C'_w(q))$ when w is fixed

For each $w \in \mathfrak{S}_n$, we define a symmetric generating function for values of $\theta(C'_w(1))$ by

$$X_w = \sum_{\lambda \vdash n} \epsilon^\lambda(C'_w(1)) m_\lambda \;\in\; \Lambda_n \underset{\mathrm{def}}{=} \mathrm{span}_{\mathbb{Z}}\{m_\lambda \,|\, \lambda \vdash n\}. \tag{13}$$

Expanding X_w in various bases of the space of homogeneous degree-n symmetric functions, including the forgotten basis $\{f_\lambda \,|\, \lambda \vdash n\}$, we have

$$X_w = \sum_{\lambda \vdash n} \eta^\lambda(C'_w(1)) f_\lambda = \sum_{\lambda \vdash n} \chi^{\lambda^\top}(C'_w(1)) s_\lambda = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} \psi^\lambda(C'_w(1)) \frac{p_\lambda}{z_\lambda} = \sum_{\lambda \vdash n} \phi^\lambda(C'_w(1)) e_\lambda,$$

where $\ell(\lambda)$ is the number of (nonzero) parts of λ.

The function X_w is related to the *chromatic symmetric functions* $\{X_P \,|\, P \text{ a poset}\}$ of Stanley and Stembridge [Sta95], [SS93]: if w avoids the pattern 312, then X_w is equal to the Stanley-Stembridge chromatic symmetric function $X_{P(F_w)}$. On the other hand, not all chromatic symmetric functions X_P can be expressed as X_w for appropriate $w \in \mathfrak{S}_n$, nor can all generating functions X_w be expressed as X_P for an appropriate poset P. Stanley and Stembridge [Sta95], [SS93] have conjectured that X_P is elementary nonnegative when P has no induced subposet isomorphic to the disjoint union $(\mathbf{3}+\mathbf{1})$ of a three element chain and a single element. Call such a poset $(\mathbf{3}+\mathbf{1})$-*free*. A special case of this conjecture is that X_w is elementary nonnegative for w avoiding 312. Haiman [Hai93] conjectured that X_w is elementary nonnegative for all $w \in \mathfrak{S}_n$.

For each $w \in \mathfrak{S}_n$, we define a $\mathbb{Z}[q]$-symmetric generating function for values of $\theta_q(q_{e,w}C'_w(q))$ by

$$X_{T_w} = \sum_{\lambda \vdash n} \epsilon_q^\lambda(q_{e,w}C'_w(q)) m_\lambda \ \in\ \mathbb{Z}[q] \otimes \Lambda_n = \mathrm{span}_{\mathbb{Z}[q]}\{m_\lambda \,|\, \lambda \vdash n\}. \tag{14}$$

Expanding X_{T_w} in various bases of the homogeneous degree-n graded component of $\mathbb{Z}[q] \otimes \Lambda_n$, we have

$$X_{T_w} = \sum_{\lambda \vdash n} \eta_q^\lambda(q_{e,w}C'_w(q)) f_\lambda = \sum_{\lambda \vdash n} \chi_q^{\lambda^\top}(q_{e,w}C'_w(q)) s_\lambda = \sum_{\lambda \vdash n} \frac{\psi_q^\lambda(q_{e,w}C'_w(q))}{(-1)^{n-\ell(\lambda)}} \frac{p_\lambda}{z_\lambda} = \sum_{\lambda \vdash n} \phi_q^\lambda(q_{e,w}C'_w(q)) e_\lambda.$$

The function X_{T_w} specializes at $q = 1$ to X_w, and is related to the *chromatic quasisymmetric functions* $\{X_{P,q} \,|\, P \text{ a labeled poset }\}$ of Shareshian and Wachs [SW12], which specialize at $q = 1$ to X_P. The function $X_{P,q}$ is itself symmetric (i.e., it belongs to $\mathbb{Z}[q] \otimes \Lambda_n$) when P is $(\mathbf{3}+\mathbf{1})$-free, $(\mathbf{2}+\mathbf{2})$-free, and labeled strategically. If w avoids the pattern 312, then by Theorem 4.2, X_{T_w} is equal to the Shareshian-Wachs chromatic symmetric function $X_{P(F_w),q}$, with each element of $P(F_w)$ labeled according to the source and sink of the path in F_w it represents. Again, not all chromatic symmetric functions $X_{P,q}$ can be expressed as X_{T_w} for appropriate $w \in \mathfrak{S}_n$, nor can all generating functions X_{T_w} be expressed as $X_{P,q}$ for an appropriate labeled poset P. Shareshian and Wachs [SW12] conjectured that $X_{P,q}$ belongs to $\mathrm{span}_{\mathbb{N}[q]}\{e_\lambda \,|\, \lambda \vdash n\}$ when P is $(\mathbf{3}+\mathbf{1})$-free, $(\mathbf{2}+\mathbf{2})$-free, and labeled appropriately. By Theorem 4.2, this is equivalent to the conjecture that X_{T_w} belongs to $\mathrm{span}_{\mathbb{N}[q]}\{e_\lambda \,|\, \lambda \vdash n\}$ for w avoiding 312. Haiman [Hai93] conjectured that X_{T_w} belongs to $\mathrm{span}_{\mathbb{N}[q]}\{e_\lambda \,|\, \lambda \vdash n\}$ for all $w \in \mathfrak{S}_n$.

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Counting words with Laguerre polynomials

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Abstract. We develop a method for counting words subject to various restrictions by finding a combinatorial interpretation for a product of formal sums of Laguerre polynomials. We use this method to find the generating function for k-ary words avoiding any vincular pattern that has only ones. We also give generating functions for k-ary words cyclically avoiding vincular patterns with only ones whose runs of ones between dashes are all of equal length, as well as the analogous results for compositions.

Résumé. Nous développons une méthode pour compter des mots satisfaisants certaines restrictions en établissant une interprétation combinatoire utile d'un produit de sommes formelles de polynômes de Laguerre. Nous utilisons cette méthode pour trouver la série génératrice pour les mots k-aires évitant les motifs vinculars consistant uniquement de uns. Nous présentons en suite les séries génératrices pour les mots k-aires évitant de façon cyclique les motifs vinculars consistant uniquement de uns et dont chaque série de uns entre deux tirets est de la même longueur. Nous présentons aussi les résultats analogues pour les compositions.

Keywords: Laguerre polynomial, orthogonal polynomial, combinatorics on words, vincular pattern

1 Introduction

Define a *factorization* of a word W to be an ordered list of words that, when concatenated, give W. Given a set of factoriations A and a weight w, we will define a power series $f_{A,w}(t)$, the associated *Laguerre series* for A, in terms of the generalized Laguerre polynomials with parameter $\alpha = -1$. The key fact we will use is the rule (Theorem 2.4)

$$f_{A_1 * A_2, w}(t) = f_{A_1,w}(t) \cdot f_{A_2,w}(t).$$

Here A_1 and A_2 are factorizations with disjoint alphabets and $*$ is a combinatorial operation that, roughly speaking, interlaces the factorizations in A_1 and A_2. For example, if $\phi_1 = (aaa)(a)(a) \in A_1$ and $\phi_2 = (bb)(b)(b)(b) \in A_2$ then $\phi = (aaabb)(babab) \in A_1 * A_2$.

Let Φ denote the linear operator on $\mathbb{R}[t]$ mapping t^k to $k!$. It has the integral representation $\Phi(f(t)) = \int_0^\infty e^{-t} f(t)\, dt$. We will show in Proposition 2.3 that $\Phi(f_A(t))$ gives the weight of all words in A, which we define to be factorizations with one or no parts. By applying Φ to a product of Laguerre series we may count a variety of sets of restricted words, especially when the restrictions are on the length of runs of particular letters. For example, the number of arrangements of the word "WALLAWALLA" with no LLL, AAA or WW as consecutive subwords is

$$\int_0^\infty e^{-t} \left(\frac{1}{24} t^4 - t^2 + t \right)^2 \left(\frac{1}{2} t^2 - t \right) dt = 1584$$

as we will see.

In Section 3, we describe the transformation T that turns certain ordinary generating functions into Laguerre series. The transformation can be described in terms of the Laplace transform, and so can be easily implemented in mathematical software packages. We can use T to determine the Laguerre series for a variety of sets of factorizations A, and use them to derive formulas and generating functions to count words that obey various restrictions.

In particular, we use this technique to analyze certain pattern avoidance problems. A vincular, or generalized, pattern is a pattern with dashes such as 13-2. This is a generalization of classical permutation patterns where the dashes are used to indicate that the numbers on either side are not required to be adjacent, but all others are. We will only study patterns that have only ones, such as $111-1-11$, so we define pattern avoidance only in this context. A word $W = s_1 \cdots s_l$, with each s_i in some alphabet S, contains a vincular pattern $\tau = 1^{k_1}\text{-}\cdots\text{-}1^{k_n}$ if there is a subsequence of W consisting of $m = k_1 + \ldots + k_n$ identical letters of which the first k_1 are consecutive, the next k_2 are consecutive, and so on. Formally, we require that there are indices $1 \leq i_1 < i_2 < \ldots < i_m \leq l$ with $s_{i_1} = \ldots = s_{i_m}$ and $i_{j+1} - i_j = 1$ for $j \neq k_1, k_1 + k_2, \ldots, k_1 + \ldots + k_{n-1}$. Otherwise, we say that W avoids τ. These patterns were first studied by Babson and Steingrímsson (Babson and Steingrímsson, 2000), who showed that many statistics of interest can be classified in terms of vincular patterns. The term *vincular* itself was coined by Claesson in (Bousquet-Mélou et al., 2010), from the Latin *vinculare*, to bind. Words avoiding vincular patterns are studied in (Bernini, Ferrari, and Pinzani, 2009; Burstein, 1998; Burstein and Mansour, 2003a; Heubach and Mansour, 2009; Burstein and Mansour, 2003b; Mansour, 2006). In this paper we will study vincular patterns with all ones, such as $\tau = 111\text{-}11$. A word avoids this pattern if it does not have five appearances of the same letter in the word, of which the first three and the last two are consecutive. Although such patterns are useless in the context of permutations, where only the pattern 1 can be contained, they are meaningful in the context of general words on the alphabet $\mathbb{N}$ where letters may be repeated.

In Section 4, we give a formula to calculate the generating function for the number of words avoiding any such vincular pattern with only ones. This formula involves the use of the maps T and Φ, but these can be easily calculated. For example, we can use Sage to compute the the generating function $\sum_W x^{\mathrm{len}(W)}$ where the sum is taken over all ternary words W avoiding the pattern 11-11, where $\mathrm{len}(W)$ is the *length* of W, the number of letters counting multiplicity:

$$\frac{6\,x^7 - 6\,x^6 + 6\,x^5 - 2\,x^4 - 5\,x^3 + 9\,x^2 - 5\,x + 1}{16\,x^4 - 32\,x^3 + 24\,x^2 - 8\,x + 1} = 1 + 3x + 9x^2 + 27x^3 + 78x^4 + 222x^5 + \ldots.$$

Finally, we give a cyclic version of this result for the case of patterns $1^m\text{-}1^m\text{-}\cdots\text{-}1^m$, where all runs of ones are the same length. This gives the generating functions for words so that any cyclic permutation of their letters avoids such a pattern. This generalizes a result of Burstein and Wilf (Burstein and Wilf, 1997) who give the generating function for the number of words cyclically avoiding 1^m.

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2 Laguerre series

Define the polynomials $l_k(t)$ by their generating function

$$\sum_{k=0}^{\infty} l_k(t)x^k = e^{\frac{tx}{1+x}}.$$

These polynomials are a form of Laguerre polynomial. Specifically, $l_k(t) = (-1)^k L_k^{(-1)}(t)$ where

$$L_k^{(\alpha)}(t) = \sum_{i=0}^{k}(-1)^i \binom{k+\alpha}{k-i}\frac{t^i}{i!}$$

defines the generalized Laguerre polynomials. They have been found to have a number of interesting combinatorial properties, beginning with their use by Even and Gillis to count generalized derangements when α is set to 0 in (Even and Gillis, 1976). This was later extended by Foata and Zeilberger who use α to keep track of the number of cycles (Foata and Zeilberger:i, 1988). For our purpose, we may assume $\alpha = -1$.

Define a *word* W on an alphabet S to be an ordered list $s_1 \cdots s_n$ of letters $s_i \in S$. A subword of W is a word $s_k s_{k+1} \cdots s_{k+m}$. Note that we require the indices in a subword to be consecutive, while some authors do not. A word using letters from the alphabet $[k] = \{1, 2, \ldots, k\}$ is called k-ary, and a word in which no two adjacent letters are the same is called a Carlitz word, after Leonard Carlitz.

Our work is based on the following remarkable result of Ira Gessel (Gessel, 1989, Section 6), which he found in the context of a generalization of rook theory. We present an unlabeled version.

Theorem 2.1. *Let Φ be the linear functional on polynomials in t given by $\Phi(t^n) = n!$. Given nonnegative integers $k_1, \ldots, k_m$, the number of Carlitz words on an alphabet of m symbols with the ith symbol used k_i times is*

$$\Phi\left(\prod_i l_{k_i}(t)\right).$$

For example, in (Blom et al., 1998) the authors consider the "Mississippi Problem". How many arrangements of the letters in the word "MISSISSIPPI" have no adjacent letters the same? We can use the preceding theorem to calculate this directly. There is one M, four I's, four S's, and two P's. So the solution is

$$\Phi(l_1(t)l_4(t)l_4(t)l_2(t)) = \int_0^{\infty} e^{-t}\left(t\right)\left(\frac{1}{24}t^4 - \frac{1}{2}t^3 + \frac{3}{2}t^2 - t\right)^2\left(\frac{1}{2}t^2 - t\right)dt = 2016.$$

Using Theorem 2.1, it is easy to see combinatorially that

$$\Phi(l_i(t)l_j(t)) = \begin{cases} 2 & \text{if } i = j \\ 1 & \text{if } |i-j| = 1 \\ 0 & \text{if } |i-j| > 1 \end{cases} \tag{1}$$

and so the polynomials $l_k(t)$ are "almost" orthogonal with respect to Φ.

Note that $l_k(t)$ is a polynomial of degree k; so the matrix of l_k's expanded into powers of t is triangular with no zeroes on the diagonal, and so $\{l_k\}_k$ forms a basis of $\mathbb{R}[t]$. It is natural to ask, then, what is the expansion of $l_i(t)l_j(t)$ in this basis? These are known as *linearization coefficients*. The linearization coefficients of general Laguerre polynomials, with α indeterminate, is known (Foata and Zeilberger:i, 1988; Zeng, 1992), but we will need a combinatorial interpretation of the case $\alpha = -1$. To give it, we need a few more definitions.

Definition. A *factorization* ϕ is a finite, ordered list of nonempty words $(\phi_1)\cdots(\phi_n)$, and $\phi_1, \ldots, \phi_n$ are the *parts* or *factors* of ϕ. If the concatenation of these words is a word W, we say that ϕ is a factorization of W. We take the convention that the empty word has exactly one factorization, namely the *empty* factorization which has no factors. Abusing notation, we identify a word W with the factorization (W) in one part, and the empty word with the empty factorization, writing $\emptyset$ for both. We write $\mathrm{par}(\phi) = n$, the number of factors, and $\mathrm{len}(\phi)$ for the *length* of ϕ, that is, the length of the word W when ϕ is a factorization of W.

For example, $\phi = (MISS)(IS)(IPPI)$ is a factorization of "MISSISSIPPI", with $\mathrm{par}(\phi) = 3$ and $\mathrm{len}(\phi) = 11$. Frequently, we will be interested in the factorization itself without thinking of it as a factorization of a particular word. Rather, the spaces between the factors should be thought of as slots to be filled with nonempty words.

Denote by $n_{i,j,k}$ the number of factorizations over the alphabet $\{a, b\}$ with k parts and exactly i a's and j b's so that each part is Carlitz. For example, $n_{2,5,3} = 6$: the possibilities are $(bab)(bab)(b)$, $(babab)(b)(b)$ and the different permutations of these sets of factors.

Lemma 2.2. *We have, for all* $i, j \in \mathbb{N}$,

$$l_i(t)l_j(t) = \sum_k n_{i,j,k} l_k(t).$$

Proof. Note that if $p(t) = a_0 + a_1 t + \ldots + a_n t^n$ is a polynomial and $\Phi(t^m p(t)) = 0$ for all m, then

$$a_0 m! + a_1(m+1)! + \ldots + a_n(n+m)! = 0.$$

This is a homogenous linear recurrence relation with constant coefficients for the factorial sequence, which is impossible unless $a_0 = a_1 = \ldots = a_n = 0$ since it grows superexponentially. Since $\{l_k(t)\}_k$ forms a basis for $\mathbb{R}[t]$, if $\Phi(p(t)l_k(t)) = 0$ for all k then we can still conclude $p(t) = 0$. So it is enough to show that

$$\Phi\left(l_i(t)l_j(t)l_m(t)\right) = \phi\left(\sum_k n_{i,j,k} l_k(t) l_m(t)\right).$$

We know that the left hand side counts the number of Carlitz arrangements of i a's, j b's, and m c's, while the right hand side gives the total number of pairs (ϕ, W) where ϕ is a factorization in k parts with i a's and j b's with each part Carlitz, and W is a Carlitz word with k x's and m c's. There is a simple bijection between these sets. Given such a pair (ϕ, W), we can get a Carlitz arrangement of i a's, j b's and m c's by replacing the ith x of W with the ith part of ϕ. For example, if $\phi = (ab)(bab)$ and $W = cxcx$, we get the Carlitz word *cabcbab*. This process is reversible: given a Carlitz word on a, b, c we replace the c's by parentheses to make a factorization ϕ with only the letters a and b, and to get W we replace each maximal subword that does not contain c by a single x, getting a word with only c's and x's. For example, given the word $abcbcab$, we get the pair $W = xcxcx$, and $\phi = (ab)(b)(ab)$. The maximality condition guarantees that W will be Carlitz. $\square$

Let S be an alphabet, not necessarily finite, and let A be any set of factorizations of words on S. We say that a word W is an *allowed word* of A if the factorization of W in one part (or zero for the empty word) is in A. We think of A as some set of factorizations on S we are interested in investigating. For example, in the above proof we might have defined A to be those factorizations of words on $S = \{a, b\}$ so that each part is Carlitz.

Definition. Let ϕ be a factorization of a word W on an alphabet S. For $T \subseteq S$ and a word W on S, let $\phi|_T$ be the factorization created from ϕ whose parts are the maximal subwords in each part of ϕ that have only letters in T listed in the same order they appeared in ϕ. The resulting factorization will have only letters from T, with parts of ϕ possibly split up into multiple parts of $\phi|_T$. We call $\phi|_T$ the *restriction* of the factorization to T. For example, if $S = \{a, b\}$ and $T = \{a\}$, then the restriction of the factorization $(aabba)(aab)(b)(aaab)$ to T is $\phi|_T = (aa)(a)(aa)(aaa)$. If ϕ contains no letters from T, we define $\phi|_T$ to be the empty factorization.

Definition. Let A_1 and A_2 be two sets of factorizations so that the alphabets of symbols S_1, S_2 used in A_1 and A_2, respectively, are disjoint. Let $S = S_1 \cup S_2$, and denote by $A_1 * A_2$ the set of factorizations ϕ of words on S so that $\phi|_{S_1} \in A_1$ and $\phi|_{S_2} \in A_2$.

Thus the factorizations in $A_1 * A_2$ are obtained by interlacing the parts of factorizations in A_1 and A_2. The factors of ϕ_1 and ϕ_2 must appear in the correct order in ϕ, but a factor of ϕ may be a word that is concatenated from factors that alternate between ϕ_1 and ϕ_2. For example, if $(a)(aa)(a) \in A_1$ and $(b)(bb) \in A_2$ then $\phi = (abaa)(abb) \in A_1 * A_2$ since its restrictions to $\{a\}$ and $\{b\}$ are $\phi|_{\{a\}} = (a)(aa)(a)$ and $\phi|_{\{b\}} = (b)(bb)$.

It is easy to see that $*$ is associative and commutative. Note also that $A_1, A_2 \subseteq A_1 * A_2$ if A_1 and A_2 contain the empty factorization. If $\phi \in A_1$, for example, then $\phi|_{S_1} = \phi \in A_1$ and $\phi|_{S_2} = \emptyset \in A_2$ when S_1, S_2 are the disjoint alphabets of A_1, A_2. The allowed words in $A_1 * A_2$ are often of interest; for example, if we have singleton alphabets $S_i = \{i\}$ for $i = 1, \ldots, n$, and A_i consists of those factorizations with all parts having length one, then the factorizations in $A_1 * \cdots * A_n$ are those so that each factor is Carlitz, and the words of $A_1 * \cdots * A_n$ are exactly the Carlitz words.

Definition. Given a set of factorizations A on an alphabet S, a *weight* is a function w from A and all of the restrictions of factorizations in A into a polynomial ring $\mathbb{R}[x_1, x_2, \ldots]$ that commutes with restriction in the sense that if $\phi \in A$ and $T \subseteq S$, then $w(\phi) = w(\phi|_T)\, w(\phi|_{S \setminus T})$.

Note that in particular, if $A = A_1 * A_2$ for some sets of factorizations A_1, A_2 then w is also a weight on A_1 and A_2. Also note that taking T to be empty forces $w(\emptyset) = 1$. Typically we will take the weight $w(\phi)$ to be a monomial $x_1^{n_1(\phi)} x_2^{n_2(\phi)} \cdots x_m^{n_m(\phi)}$ where each $n_i(\phi)$ is a statistic so that $x_i^{n_i(\phi)}$ is multiplicative in the above sense. Examples include the length of ϕ, $\operatorname{len}(\phi)$; the number of distinct symbols in ϕ; the number of appearances of a particular symbol; the sum of ϕ, $\operatorname{sum}(\phi)$, if the symbols in ϕ are nonnegative integers; or simply $w = 1$ if we wish to enumerate a finite set. We will write $\operatorname{par}(\phi)$ for the number of parts of ϕ; but $x^{\operatorname{par}(\phi)}$ is *not* a weight.

Definition. Let A be a set of factorizations on an alphabet S and w be a weight on A. Define the *Laguerre series* of A with respect to w to be the formal power series

$$f_{A,w}(t) = \sum_{\phi \in A} w(\phi) l_{\mathrm{par}(\phi)}(t)$$

when this sum is well-defined as a formal power series. For convenience we will omit the w in the subscript when $w = 1$, writing $f_{A,1}(t)$ as $f_A(t)$.

Note that our definition of Laguerre series uses a different normalization of Laguerre polynomials than the sum

$$\sum_n \lambda_n^{(\alpha)} L_n^{(\alpha)}(t)$$

as defined in, e.g., Pollard (1948); Szász and Yeardley (1948); Weniger (2008).

Proposition 2.3. *Assume A is a set of factorizations and w is a weight on A. Let Φ be the linear operator so that $\Phi(t^n) = n!$ and Φ fixes any other variables. Then*

$$\Phi(f_{A,w}(t)) = \sum_W w(W)$$

when both sides are defined, where the sum is over allowed words $W \in A$ (factorizations with one or no parts.)

Proof. We have

$$\Phi(f_{A,w}(t)) = \sum_{\phi \in A} w(\phi) \Phi(l_{\mathrm{par}(\phi)}(t))$$

and $\Phi(l_{\mathrm{par}(\phi)}(t))$ is 1 when ϕ has 0 or 1 part and is 0 otherwise by (1) since $l_0(t) = 1$ and $l_1(t) = t$. $\square$

Now we are ready to state our main theorem on the combinatorial properties of Laguerre series.

Theorem 2.4.

1. *Let A_1 and A_2 be disjoint sets of allowed factorizations on a common alphabet S, and let w be a weight on $A_1 \cup A_2$. Then*
$$f_{A_1 \cup A_2, w}(t) = f_{A_1,w}(t) + f_{A_2,w}(t).$$

2. *Let S_1 and S_2 be disjoint alphabets with sets of allowed factorizations A_1, A_2 respectively, and let w be a weight on $A_1 * A_2$ (and hence on A_1 and A_2.) Then*
$$f_{A_1 * A_2, w}(t) = f_{A_1,w}(t) \cdot f_{A_2,w}(t).$$

Proof. The proof of the first part is evident from the definition. We will prove the second. By Lemma 2.3,

$$f_{A_1,w}(t)\cdot f_{A_2,w}(t)=\sum_{\phi_1\in A_1}\sum_{\phi_2\in A_2} w(\phi_1)w(\phi_2)l_{\mathrm{par}(\phi_1)}(t)l_{\mathrm{par}(\phi_1)}(t)$$
$$=\sum_{\phi_1\in A_1,\phi_2\in A_2,k\geq 0} n_{\mathrm{par}(\phi_1),\mathrm{par}(\phi_2),k}w(\phi_1)w(\phi_2)l_k(t).$$

Fix $\phi_1\in A_1,\phi_2\in A_2$. It is enough to show that $n_{\mathrm{par}(\phi_1),\mathrm{par}(\phi_2),k}$ is the number of factorizations with k parts on $S_1\cup S_2$ whose restrictions to S_1 and S_2 are ϕ_1 and ϕ_2, respectively. Then each allowed word of $A_1 * A_2$ will then be represented exactly once in the series $f_{A_1,w}(t)\cdot f_{A_2,w}(t)$, giving

$$f_{A_1*A_2,w}(t)=\sum_{\phi\in A_1*A_2} w(\phi)l_{\mathrm{par}(\phi)}=f_{A_1,w}(t)\cdot f_{A_2,w}(t).$$

For fixed k, we will construct a simple bijection from the set of triples (ϕ,ϕ_1,ϕ_2) where ϕ_1, ϕ_2 are factorizations in A_1, A_2 respectively and ϕ is a factorization on the alphabet $\{a,b\}$ with $\mathrm{par}(\phi_1)$ a's and $\mathrm{par}(\phi_1)$ b's so that each part is Carlitz, and the set of factorizations ϕ_3 of $A_1 * A_2$ with k parts. Let ϕ_3 be the factorization created by replacing the nth a in ϕ with the nth part of ϕ_1, and the nth b with the nth part of ϕ_2. Then by construction $\phi_3\in A_1*A_2$: its restrictions are ϕ_1 and ϕ_2. Furthermore, given an allowed factorization $\phi_3\in A_1*A_2$ with k parts so that $\phi_{S_1}=\phi_1$, $\phi_{S_2}=\phi_2$, we can reconstruct the factorization ϕ of a word on $\{a,b\}$ by replacing each subword of a factor of ϕ_3 that uses only the letters of S_1, and is maximal with respect to this condition, by an a and each maximal subword using only letters of S_2 by a b. For example, if $S_1=\{1,2\}$ and $S_2=\{3,4\}$, with $\phi_3=(123,2213,34413)$, we get the word $\phi=(ab,ab,bab)$. No part of ϕ can have aa or bb by the maximality condition. These two algorithms are inverse to each other, establishing the theorem. □

Inductively, we see that if $A_1,\ldots,A_n$ are sets of factorizations on disjoint alphabets and w is a weight on $A_1*\cdots*A_n$, then $f_{A_1*\cdots*A_n,w}(t)=f_{A_1,w}(t)\cdots f_{A_n,w}(t)$.

3 Computing Laguerre series

The Laguerre series for a set of factorizations would not be especially useful if it was difficult to compute. Fortunately, there is an efficient method to calculate them in some situations. It may be difficult to find a convenient formula for the coefficients of $l_k(t)$ in a given Laguerre series $f_{A,w}(t)$, but this is not needed to find an expression for $f_{A,w}(t)$. It is enough to find the ordinary generating function. Specifically, we define

$$g_{A,w}(u)=\sum_{\phi\in A} w(\phi)u^k.$$

If a nice form of $g_{A,w}(u)$ is known, we may obtain the Laguerre series $f_{A,w}(t)$ by applying the linear transformation T that sends u^k to $l_k(t)$. As it happens, T can be easily computed in many situations using the inverse Laplace transform. We have

$$\mathcal{L}\{l_k(t)\}=\frac{1}{s(1-s)}\left(\frac{1-s}{s}\right)^k$$

for $k \geq 1$, where $\mathcal{L}$ is the Laplace transform; this is easily proved from the formula for $l_k(t)$ in terms of the generalized Laguerre polynomials, the fact that $\mathcal{L}\{t^i\} = \frac{i!}{s^{i+1}}$, and the binomial theorem.

Therefore, if $g_{A,w}(0) = 0$, we have by linearity

$$T\{g_{A,w}(u)\} = f_{A,w}(t) = \mathcal{L}^{-1}\left\{\frac{g_{A,w}(\frac{1-s}{s})}{s(1-s)}\right\}$$

when the right-hand side is well-defined. If $g_{A,w}(0) \neq 0$, we can calculate

$$f_{A,w}(t) = T\{g_{A,w}(u) - g_{A,w}(0)\} + g_{A,w}(0) = \mathcal{L}^{-1}\left\{\frac{g_{A,w}(\frac{1-s}{s}) - g_{A,w}(0)}{s(1-s)}\right\} + g_{A,w}(0)$$

since $l_0(t) = 1$. The use of the inverse Laplace transform here is not central to the theory, but it is convenient since many software packages provide symbolic calculation of the inverse Laplace transform, making it easy to implement the transformation T. However, the function T itself should not be thought of as an integral transform as we do not consider questions of convergence.

For example, consider the problem of counting words that have no subword consisting of m identical letters. These are words that avoid the subword pattern 1^m, and are sometimes called m-Carlitz words; when $m = 2$ we have the ordinary Carlitz words. To find the generating function, let A be the set of factorizations on a one-letter alphabet with each part having length smaller than m, and again let $w(\phi) = x^{\text{len}(\phi)}$. We see that

$$g_{A,w}(u) = \sum_{n=0}^{\infty} u^n (x + \ldots + x^{m-1})^n = \frac{1-x}{1-x-u(x-x^m)}$$

and so we compute

$$f_{A,w}(t) = T\{g_{A,w}(u)\} = \exp\left(t \cdot \frac{x-x^m}{1-x^m}\right). \tag{2}$$

Taking the coefficient of x^n in (2) gives the Laguerre series for the set of factorizations with length n so that each part is smaller than m. This gives a generalization of Theorem 2.1. If $m_1, \ldots, m_k, n_1, \ldots, n_k$ are nonnegative integers, and $p_{m,n}(t)$ are polynomials defined by $\sum_{n=0}^{\infty} p_{m,n}(t)\, x^n = \exp\left(\frac{t(x-x^m)}{1-x^m}\right)$, we see that

$$\Phi\left(\prod_{i=1}^{k} p_{m_i,n_i}(t)\right)$$

is the total number of k-ary words that use the letter i exactly n_i times and do not contain the subwords i^{m_i}. Thus the number of arrangements of the word "WALLAWALLA" with no LLL, AAA or WW as consecutive subwords is

$$\begin{aligned}\int_0^{\infty} e^{-t} p_{3,4}(t) \cdot p_{3,4}(t) \cdot p_{2,2}(t)\, dt &= \int_0^{\infty} e^{-t}\left(\frac{1}{24}t^4 - t^2 + t\right)^2 \cdot \left(\frac{1}{2}t^2 - t\right) dt \\ &= 1584.\end{aligned}$$

Recalling again the formula

$$\Phi(e^{tf}) = \frac{1}{1-f},$$

we see that the generating function for the number of k-ary m-Carlitz words of length n is given by

$$\Phi\left(\exp\left(kt\cdot\frac{x-x^m}{1-x^m}\right)\right) = \frac{1-x^m}{1-kx-(k-1)x^m}.$$

Another derivation of this formula is given by Burstein and Mansour (Burstein and Mansour, 2003a, Example 2.2).

More generally, we might count the number of times the pattern 1^m occurs. We define the weight $w(W)$ of a word W to be $x^n y^l$, where n is the length of W and l is the number of times W contains the pattern 1^m. Letting A be the set of all factorizations on a one-letter alphabet, it is not difficult to compute the generating function $g_{A,w}$. A formula for the generating function for k-ary words by the number of times they avoid a pattern 1^m can also be found in Burstein and Mansour (2003a).

4 Vincular patterns

We are now ready to state a general formula for k-ary words avoiding vincular patterns with ones. We say that a k-ary factorization ϕ contains a vincular pattern τ with only ones if the word made from ϕ by inserting a single 0 between each pair of adjacent factors contains τ, and this copy of τ does not use 0. Using the transformations T and Φ, we can reduce the problem to finding ordinary generating functions for factorizations that only use one symbol and avoid the given vincular pattern.

Theorem 4.1. *Let $k_1, \ldots, k_n$ be positive integers. Let w be the weight on k-ary words with $w(a_1\cdots a_l) = x_{a_1}x_{a_2}\cdots x_{a_l}$, so that the power of x_i represents the number of times i appears in W, and let A be the set of k-ary words avoiding the pattern $\tau = 1^{k_1}\text{-}1^{k_2}\text{-}\cdots\text{-}1^{k_n}$. Then*

$$\sum_{W\in A} w(W) = \Phi\left(\prod_{i=1}^{k}\left[e^{tx_i} - T\left\{G_\tau(x_i,u)\right\}\right]\right)$$

where T is the operator defined in Section 3, and

$$G_\tau(x,u) = \frac{ux^{k_1}(1-x)}{(1-x-u(x-x^{k_i}))(1-x-ux)}\prod_{i=2}^{n}\left[x^{k_i} + \frac{ux^{k_i}(1-x^{k_i})}{1-x-u(x-x^{k_i})}\right]. \tag{3}$$

We say that a word W cyclically avoids a vincular pattern τ if W avoids τ no matter how its letters are cycled. More formally, let r be the function that cycles W, moving the last letter into the first position: $r(a_1\cdots a_n) = a_n a_1\cdots a_{n-1}$. Then W cyclically avoids τ if $r^k(W)$ avoids τ for each k.

In order to find the generating function for the number of words cyclically avoiding the pattern $\tau = 1^m\text{-}1^m\text{-}\cdots\text{-}1^m$, we will need a little more information than provided by the generating function $G_\tau(x,u)$ defined by (3). Let $H(x,u,v) = g_{A,w}(u)$ where A is the set of factorizations on the alphabet $\{1\}$ *avoiding* the pattern τ, where w is the weight $w(\phi) = x^{\mathrm{len}(\phi)}u^{\mathrm{par}(\phi)}v^{\mathrm{fst}(\phi)}$ where $\mathrm{fst}(\phi)$ is the size of the first factor of ϕ. Note that in this case w is trivially a weight by our definition since we are using a singleton alphabet, but generally is not. We will find a closed-form expression for $H(x,u,v)$, although it is rather unwieldy.

Lemma 4.2. *The generating function $H(x,u,v)$ is given by*

$$H(x,u,v) = 1 + \left[\frac{1-x}{(1-vx)(1-x-ux)}\right]\left[u(vx-(vx)^{mn}) + \frac{u^2x^m\left((1-vx)(z-(vx)^m)z^{n-1}-(1-(vx)^m)(z^n-(vx)^{mn})\right)}{(z-(vx)^m)(1-x-u\left(x-x^m\right))}\right]$$

where

$$z = x^m + \frac{ux^m(1-x^m)}{1-x-u(x-x^m)}.$$

Theorem 4.3. *Let A be the set of words cyclically avoiding the pattern 1^m-1^m-$\cdots$-1^m, with $n-1$ dashes, and let w be the weight on k-ary words with $w(s_1\dots s_l) = x_{s_1}x_{s_2}\cdots x_{s_l}$. Then the generating function $\sum_{W\in A} w(W)$ is given by*

$$1 + \sum_{i=1}^{k}\Phi\left(t^{-1}\cdot T\left\{u\frac{d^2}{dv\,du}\bigg|_{v=1} H(x_i,u,v)\right\}\left(-1+\prod_{\substack{j=1\\ j\neq i}}^{k} T\{H(x_j,u,1)\}\right)\right) + \sum_{i=1}^{k}\frac{x_i-x_i^{mn}}{1-x}.$$

In particular, letting $x_i = x$ for each i gives:

$$\sum_{W\in A} x^{\mathrm{len}(W)} = 1 + k\cdot\Phi\left(t^{-1}\cdot T\{u\frac{d^2}{dv\,du}\bigg|_{v=1} H(x,u,v)\}\left((T\{H(x,u,1)\})^{k-1}-1\right)\right) + \frac{k(x-x^{mn})}{1-x}.$$

If we set $n=1$, considering k-ary words that cyclically avoid 1^m, the formula simplifies considerably. After some computation, which we omit here, we arrive at the following.

Corollary 4.4. *Let A be the set of nonempty k-ary words avoiding 1^m. As above, let w be the weight $w(a_1\cdots a_l) = x_{a_1}x_{a_2}\cdots x_{a_l}$. Then*

$$\sum_{W\in A} w(W) = \sum_{i=1}^{k}\frac{x_i^{2m}-mx_i^{m+1}+(m-1)x_i^m}{(x_i^m-1)(x_i-1)} + \frac{\sum_{i=1}^{k}\frac{(m-1)x_i^{m+1}-mx_i^m+x_i}{(x_i^m-1)^2}}{1-\sum_{i=1}^{k}\frac{x_i^m-x_i}{x_i^m-1}}.$$

In the book by Heubach and Mansour (Heubach and Mansour, 2009), the authors define a cyclic Carlitz composition as a Carlitz composition so that the first and last parts are not equal; they ask (Research Direction 3.3) for the generating function for the number of cyclic Carlitz compositions. If we let k approach infinity, $m=2$, and $x_i = x^i$, we get the following.

Corollary 4.5. *Let A be the set of cyclic Carlitz compositions. Then*

$$\sum_{W\in A} x^{\mathrm{sum}(W)} = \frac{\sum_{i=1}^{\infty}\frac{x^i}{(1+x^i)^2}}{1-\sum_{i=1}^{\infty}\frac{x^i}{1+x^i}} + \sum_{i=1}^{\infty}\frac{x^{2i}}{1+x^i}.$$

Setting $x_i = x$ in Corollary 4.4 and simplifying gives the following formula.

Corollary 4.6. *Let A be the set of nonempty k-ary words that cyclically avoid 1^m. Then*

$$\sum_{W \in A} x^{\mathrm{len}(W)} = \frac{1 - x^{m-1}}{1 - x} \left(kx + (k-1)x \left(\frac{m - (m-1)kx}{1 - kx + (k-1)x^m} - \frac{m}{1 - x^m} \right) \right).$$

This was found by Burstein and Wilf (Burstein and Wilf, 1997). They go on to show that the number of k-ary words of length n cyclically avoiding 1^m is asymptotically β^n, where β is the positive root of $x^{m+1} = (k-1)(1 + x + x^2 + \ldots + x^m)$; in fact, they extract an explicit formula when n is sufficiently large.

We can also give a cyclic version of Theorem 2.1, which can be derived by extracting the coefficient of a monomial $x_1^{n_1} \cdots x_k^{n_k}$ in the generating function from Theorem 4.2 for words cyclically avoiding 1^m.

Corollary 4.7. *Let $n_1, \ldots, n_k$ be positive integers, and let $p_{m,n}(t)$ be defined as before by $\sum_{n=0}^{\infty} p_{m,n}(t)\, x^n = \exp\left(\frac{t(x - x^m)}{1 - x^m}\right)$. Then*

$$N \cdot \Phi \left(t^{-1} \cdot \prod_{i=1}^{k} p_{m,n_i}(t) \right)$$

is the total number of k-ary words that use the letter i exactly n_i times and cyclically avoid 1^m, where $N = \sum_{i=1}^{k} n_i$ is the total number of letters counted with multiplicity.

There are other variations. For example, if we would like to keep track of the length (number of parts) of a composition instead of just the sum, we can replace each x^i in Corollary 5.4 by yx^i, so that the power of y represents the number of parts. Furthermore, if we are interested in only words or compositions whose symbols lie in a given set other than $\{1, \ldots, k\}$ or $\mathbb{N}$, then we may sum over that set instead in the above formulas.

5 Questions and future directions

There are a number of combinatorial applications of Laguerre series that might be pursued in the future. One direction is to extend the work of Section 4, finding generating functions for words avoiding other cyclic patterns. Another possibility would be to count the number of occurrences of a given pattern of ones, which would amount to finding the appropriate generating function for factorizations on a single-letter alphabet by the number of occurrences of this pattern. One might also look for a combinatorial interpretation of some form of *composition* of Laguerre series; empirically, it seems that $l_i(l_j(t))$ has nonnegative integer coefficients in the l_k-basis for $j > 0$. Finally, it would be useful to develop bijections from sets of words with restrictions to other combinatorial objects that are not obviously described in terms of words, using the methods outlined here to count sets that may be otherwise difficult to enumerate.

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The number of k-parallelogram polyominoes

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Abstract. A convex polyomino is *k-convex* if every pair of its cells can be connected by means of a *monotone path*, internal to the polyomino, and having at most k changes of direction. The number k-convex polyominoes of given semi-perimeter has been determined only for small values of k, precisely $k = 1, 2$. In this paper we consider the problem of enumerating a subclass of k-convex polyominoes, precisely the *k-convex parallelogram* polyominoes (briefly, *k-parallelogram* polyominoes). For each $k \geq 1$, we give a recursive decomposition for the class of k-parallelogram polyominoes, and then use it to obtain the generating function of the class, which turns out to be a rational function. We are then able to express such a generating function in terms of the *Fibonacci polynomials*.

Résumé. Un polyomino convexe est dit *k-convexe* lorsqu'on peut relier tout couple de cellules par un chemin monotone ayant au plus k changements de direction. Le nombre de polyominos k-convexes n'est connu que pour les petites valeurs de $k = 1, 2$. Dans cet article, nous énumérons la sous classes des polyominos k-convexes qui sont également parallélogramme, que nous appelons *k-parallelogrammes*. Nous donnons une décomposition récursive de la classe des polyominos k-parallélogrammes pour chaque k, et en déduisons la fonction génératrice, rationnelle, selon le demi-périmètre. Nous donnons enfin une expression de cette fonction génératrice en terme des *polynômes de Fibonacci*.

Keywords: Convex polyominoes, L-convex polyominoes, Rational generating functions, Fibonacci polynomials

1 Introduction

In the plane $\mathbb{Z} \times \mathbb{Z}$ a cell is a unit square and a polyomino is a finite connected union of cells having no cut point. Polyominoes are defined up to translations. A *column (row)* of a polyomino is the intersection between the polyomino and an infinite strip of cells whose centers lie on a vertical (horizontal) line. A polyomino is said to be *column-convex (row-convex)* when its intersection with any vertical (horizontal) line is convex. A polyomino is *convex* if it is both column and row convex (see Figure 1 (a)). In a convex polyomino the semi-perimeter is given by the sum of the number of rows and columns, while the area is the number of its cells. For more definitions on polyominoes, we address the reader to [1].

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The number f_n of convex polyominoes with semi-perimeter $n \geq 2$ was obtained by Delest and Viennot in [7], and it is:

$$f_{n+2} = (2n+11)4^n - 4(2n+1)\binom{2n}{n}, \qquad n \geq 0; \qquad f_0 = 1, \qquad f_1 = 2.$$

The study of this paper originates from the work on k-convex polyominoes by Castiglione and Restivo in [5]. Their idea is to consider paths internal to polyominoes, where a *path* is simply a self-avoiding sequence of unit steps of four types: north $n = (0,1)$, south $s = (0,-1)$, east $e = (1,0)$, and west $w = (-1,0)$, entirely contained in the polyomino. A path connecting two distinct cells, A and B, starts from the center of A, and ends at the center of B (see Figure 1 (b)). We say that a path is monotone if it is constituted only of steps of two types (see Figure 1 (c)). Given a path $w = u_1 \ldots u_k$, each pair of steps $u_i u_{i+1}$ such that $u_i \neq u_{i+1}$, $0 < i < k$, is called a *change of direction*.

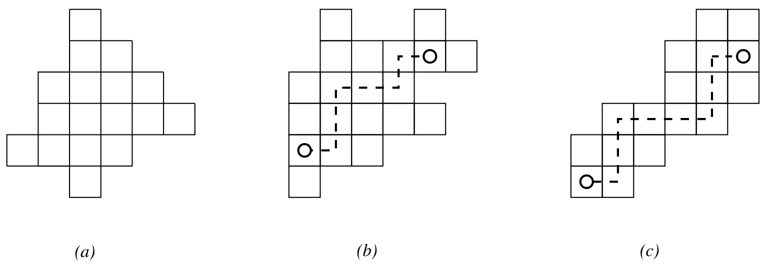

Figure 1: (a) a convex polyomino; (b) a monotone path between two cells of the polyomino with four changes of direction; (c) a 4-parallelogram (non 3-parallelogram) polyomino.

In [5] the authors observed that convex polyominoes have the property that every pair of cells is connected by a monotone path, and proposed a classification of convex polyominoes based on the number of changes of direction in the paths connecting any two cells of a polyomino. More precisely, a convex polyomino is *k-convex* if every pair of its cells can be connected by a monotone path with at most k changes of direction, and k is called the *convexity degree* of the polyomino.

For $k = 1$ we have the L-convex polyominoes, where any two cells can be connected by a path with at most one change of direction. In recent literature L-convex polyominoes have been considered from several points of view: in [5] it is shown that they are a well-ordering according to the sub-picture order; in [2] the authors have investigated some tomographical aspects, and have discovered that L-convex polyominoes are uniquely determined by their horizontal and vertical projections. Finally, in [3, 4] it is proved that the number l_n of $L-$convex polyominoes having semi-perimeter equal to $n + 2$ satisfies the recurrence relation

$$l_{n+2} = 4l_{n+1} - 2l_n, \qquad n \geq 1, \qquad l_0 = 1, \qquad l_1 = 2 \qquad l_2 = 7.$$

For $k = 2$ we have 2-convex (or Z-convex) polyominoes, such that each two cells can be connected by a path with at most two changes of direction. Unfortunately, Z-convex polyominoes do not inherit most of the combinatorial properties of L-convex polyominoes. In particular, their enumeration resisted standard enumeration techniques and it was obtained in [8] by applying the so-called *inflation method*. The authors proved that the generating function is algebraic and that the sequence asymptotically grows as $n4^n$, that is the same growth of the whole class of the convex polyominoes.

However, the solution found for 2-convex polyominoes seems to be not easily generalizable to a generic k, hence the problem of enumerating k-convex polyominoes for $k > 2$ is still open and difficult to solve. Recently, some efforts in the study of the asymptotic behavior of k-convex polyominoes have been made by Micheli and Rossin in [9].

In order to probe further, in this paper we tackle the problem of enumerating a remarkable subclass of k-convex polyominoes, precisely the k-convex polyominoes which are also *parallelogram polyominoes*, called for brevity *k-parallelogram polyominoes*.

We recall that a parallelogram polyomino is a polyomino whose boundary can be decomposed in two paths, the upper and the lower paths, which are made of north and east unit step and meet only at their starting and final points. Figure 1 (c) depicts a 4-parallelogram (non 3-convex) polyomino.

Moreover, it is well known [11] that the number of parallelogram polyominoes with semi-perimeter $n \geq 2$ is equal to the $(n-1)$th Catalan number, where *Catalan numbers* are defined by

$$c_n = \frac{1}{n+1}\binom{2n}{n}.$$

The class of k-parallelogram polyominoes can be treated in a simpler way than k-convex polyominoes, since we can use the simple fact that a parallelogram polyomino is k-convex if and only if every cell can be reached from the lower leftmost cell by at least one monotone path having at most k-changes of direction.

Using such a property, we will be able to enumerate k-parallelogram polyominoes according to the semi-perimeter, for all k. More precisely, in the next sections we will partition the class of k-parallelogram polyominoes into three subclasses, namely the *flat*, *right*, and *up* k-parallelogram polyominoes. We will provide an unambiguous decomposition for each of the three classes, so we will use these decompositions in order to obtain the generating function of the three classes and then of k-parallelogram polyominoes. An interesting fact is that, while the generating function of parallelogram polyominoes is algebraic, for every k the generating function of k-parallelogram polyominoes is rational. Moreover, we will be able to express such generating function in terms of the known *Fibonacci polynomials* [6].

To our opinion, this work is a first step towards the enumeration of k-convex polyominoes, since it is possible to apply our decomposition strategy to some larger classes of k-convex polyominoes (such as, for instance, directed k-convex polyominoes).

2 Classification and decomposition of k-parallelogram polyominoes

In this section we present some basic definitions which will be useful for the rest of the paper, we provide a classification of k-parallelogram polyominoes, and then an unique decomposition for each class.

We start noting that to find out the convexity degree of a parallelogram polyomino P it is sufficient to check the changes of direction required to any path running from the lowest leftmost cell (denoted by S) to the upper rightmost cell (denoted by E).

In order to use such a property, we define the *vertical (horizontal)* path $v(P)$ (respectively $h(P)$) as the path - if it exists - internal to P, running from S, and starting with a north step n (respectively e), where every side has maximal length (see Figure 2). In the graphical representation, we use a dashed line to represent $v(P)$, and a solid line to represent $h(P)$. We remark that our definition does not work if the first column (resp. row) of P is made of one cell, and then in this case we set by definition that $v(P)$ and $h(P)$ coincide (Figure 2 (d)).

Henceforth, if there are no ambiguities we will write v (resp. h) in place of $v(P)$ (resp. $h(P)$). As the reader can easily check the numbers of changes of direction that h and v require to run from S to E may differ at most by one. The following property is straightforward.

Proposition 1 *A polyomino P is k-parallelogram if and only if at least one among $v(P)$ and $h(P)$ has at most k changes of direction.*

In our study we will deal with the class $\mathbb{P}_k$ of k-parallelogram polyominoes where the convexity degree is exactly equal to $k \geq 0$, then enumeration of k-parallelogram polyominoes will readily follow. According to our definition, $\mathbb{P}_0$ is made of horizontal and vertical bars of any length. We further notice that, in the given parallelogram polyomino P, there may exist a cell starting from which the two paths h and v are superimposed (see Figure 2 (b), (c)). In this case, we denote such a cell by $C(P)$ (briefly, C). Clearly C may even coincide with S (see Figure 2 (d)). If such cell does not exist, we assume that C coincides with E (see Figure 2 (a)).

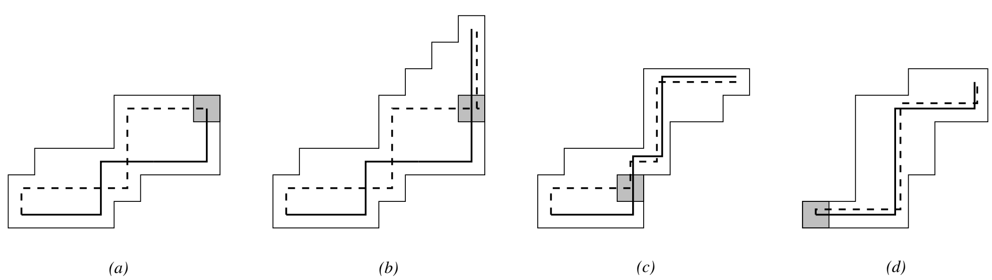

Figure 2: The paths h (solid line) and v (dashed line) in a parallelogram polyomino, where the cell C has been highlighted (a) A polyomino in $\overline{\mathbb{P}}_3$; (b) A polyomino in $\mathbb{P}_3^U$; (c) A polyomino in $\mathbb{P}_4^R$; (d) A polyomino in $\mathbb{P}_3^U$ where C coincides with S.

From now on, unless otherwise specified, we will always assume that $k \geq 1$. Let us give a classification of the polyominoes in $\mathbb{P}_k$, based on the position of the cell C inside the polyomino:

1. A polyomino P in $\mathbb{P}_k$ is said to be a *flat* k-parallelogram polyomino if $C(P)$ coincides with E (see Figure 2 (a)). The class of these polyominoes will be denoted by $\overline{\mathbb{P}}_k$. According to this definition all rectangles having width and height greater than one belong to $\overline{\mathbb{P}}_1$.

2. A polyomino P in $\mathbb{P}_k$ is said to be *up* (resp. *right*) k-parallelogram polyomino, if the cell $C(P)$ is distinct from E and h and v end with a north (resp. east) step. The class of up (resp. right) k-parallelogram polyominoes will be denoted by $\mathbb{P}_k^U$ (resp. $\mathbb{P}_k^R$). Figures 2 (b), (c), and (d) depict polyominoes in $\mathbb{P}_k^U$ (resp. $\mathbb{P}_k^R$).

The reader can easily check that up (resp. right) k-parallelogram polyominoes where the cell $C(P)$ is distinct from S can be characterized as those parallelogram polyominoes where h (resp. v) has k changes of direction and v (resp. h) has $k+1$ changes of direction.

Now we present a unique decomposition of polyominoes in $\mathbb{P}_k$, based on the following idea: given a polyomino P, we are able to detect – using the paths h and v – a set of paths on the boundary of P, that uniquely identify the polyomino itself.

More precisely, let P be a polyomino of $\mathbb{P}_k$; the path h (resp. v), when encountering the boundary of P, determines m (resp. m') steps where m (resp. m') is equal to the number of changes of directions of h (resp. v) plus one. To refer to these steps we agree that the step encountered by h (resp. v) for the ith time is called X_i or Y_i according if it is a horizontal or vertical one (see Fig. 3). We point out that if P is flat all steps X_i and Y_i are distinct, otherwise there may be some indices i for which $X_i = X_{i+1}$ (or $Y_i = Y_{i+1}$), and this happens precisely with the steps determined after the cell $C(P)$ (see Fig. 4 (b), (c)). The case $C(P) = S$ can be viewed as a degenerate case where the initial sequence of north (resp. east) steps of v (resp. h) has length zero. Therefore the definition of the steps X_i, Y_i can be given as follows: if the first column (resp. the lowest row) is made of one cell, then we set the step X_1 (resp. Y_1) to be equal to the leftmost east (resp. lowest north) step of the upper (resp. lower) path of P, and all the other steps X_i and Y_i are determined as before (see Fig. 4 (b)).

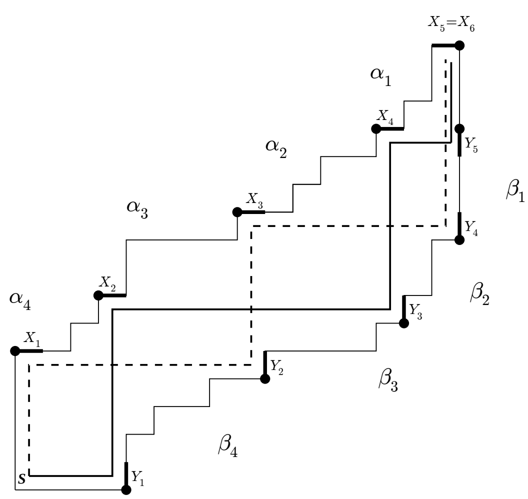

Figure 3: Decomposition of a polyomino of $\mathbb{P}_4^U$.

Now we decompose the upper (resp. lower) path of P in k (possibly empty) subpaths $\alpha_1, \ldots, \alpha_k$ (resp. $\beta_1, \ldots, \beta_k$) using the following rule: α_1 (resp. β_1) is the path running from the beginning of X_k to end of X_{k+1} (resp. from the beginning of Y_k to Y_{k+1}); let us consider now the $k-1$ (possibly empty) subpaths, α_i (resp. β_i) from the beginning of X_{k+1-i} (resp. Y_{k+1-i}) to the beginning of X_{k+2-i} (resp. Y_{k+2-i}), for $i = 2 \cdots k$. We observe that these paths are ordered from the right to the left of P. For simplicity we say that a path is *flat* if it is composed of steps of just one type.

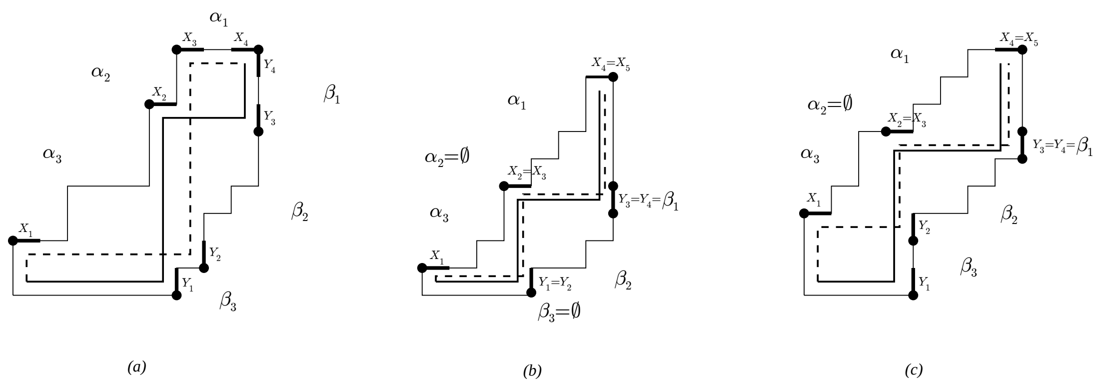

Figure 4: (a) A polyomino $P \in \overline{\mathbb{P}}_3$ in which α_1 and β_1 are flat and each other path is non empty and non flat. (b) A polyomino $P \in \mathbb{P}_3^U$ where: β_3 is empty, α_2 is empty and β_1 is equal to a north unit step. (c) A polyomino $P \in \mathbb{P}_3^U$ where β_3 is flat, α_2 is empty and β_1 is equal to a north unit step.

The following proposition provides a characterization of the polyominoes of $\mathbb{P}_k$ in term of the paths $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k$.

Proposition 2 *A polyomino P in $\mathbb{P}_k$ is uniquely determined by a sequence of (possibly empty) paths $\alpha_1, \ldots, \alpha_k$, $\beta_1, \ldots, \beta_k$, each of which made by north and east unit steps. Moreover, these paths have to satisfy the following properties:*

- *α_i and β_{i+1} must have the same width, for every $i \neq 1$; if $i = 1$, we have that α_1 is always non empty and the width of α_1 is equal to the width of β_2 plus one;*
- *β_i and α_{i+1} must have the same height, for every $i \neq 1$; if $i = 1$, we have that β_1 is always non empty and the height of β_1 is equal to the width of α_2 plus one;*
- *if α_i (β_i) is non empty then it starts with an east (north) step, $i \geqslant 1$. In particular, for $i = 1$, if α_1 (β_1) is different from the east (north) unit step, then it must start and end with an east (north) step.*

The semi-perimeter of P is obtained as the sum $|\alpha_1| + |\alpha_2|_e + \ldots + |\alpha_k|_e + |\beta_1| + |\beta_2|_n + \ldots + |\beta_k|_n$.

The reader can easily check the decomposition of a polyomino of $\mathbb{P}_4^U$ in Figure 3. For clarity sake, we need to remark the following consequence of Proposition 2:

Corollary 1 *Let $P \in \mathbb{P}_k$ be encoded by the paths $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k$. We have:*

- for every $i > 1$, we have that α_i (β_i) is empty if and only if β_{i+1} (α_{i+1}) is empty or flat;

- α_1 (β_1) is equal to the east (north) unit step if and only if β_2 (α_2) is empty or flat.

Figure 4 (a) shows the decomposition of a flat polyomino, (b) shows the case in which $C(P) = S$, so we have that β_3 is empty, then α_2 is empty, hence β_1 is a unit north step. Figure 4 (c) shows the case in which h and v coincide after the first change of direction and so we have that β_3 is flat, then α_2 is empty and β_1 is a unit north step.

Now we provide another characterization of the classes of flat, up, and right polyominoes of $\mathbb{P}_k$ which directly follows from Corollary 1 and will be used for the enumeration of these objects.

Proposition 3 *Let P be a polyomino in $\mathbb{P}_k$. We have:*

i) P is flat if and only if α_1 and β_1 are flat and they have length greater than one. It follows that all α_i and β_i are non empty paths, $i = 2, \ldots, k$.

ii) P is up (right) if and only if β_1 (α_1) is flat and α_1 (β_1) is non flat.

The reader can see examples of the statement of Proposition 3 i) in Figure 4 (a), and of Proposition 3 ii) in Figure 4 (b) and (c).

As a consequence of Proposition 2, from now on we will encode every polyomino $P \in \mathbb{P}_k$ in terms of the two sequences:

$$\mathcal{A}(P) = (\alpha_1, \beta_2, \alpha_3, \ldots, \theta_k),$$

with $\theta = \alpha$ if k is odd, otherwise $\theta = \beta$, and

$$\mathcal{B}(P) = \left(\beta_1, \alpha_2, \beta_3, \ldots, \overline{\theta}_k\right),$$

where $\overline{\theta} = \alpha$ if and only if $\theta = \beta$. The dimension of $\mathcal{A}$ (resp. $\mathcal{B}$) is given by $|\alpha_1| + |\beta_2|_n + |\alpha_3|_e + \ldots$ (resp. $|\beta_1| + |\alpha_2|_e + |\beta_3|_n + \ldots$). In particular, if $C(P) = S$ and P is an up (resp. right) polyomino then $\mathcal{B}(P) = (\beta_1, \emptyset, \ldots, \emptyset)$, (resp. $\mathcal{A}(P) = (\alpha_1, \emptyset, \ldots, \emptyset)$) where β_1 (resp. α_1) is the north (resp. east) unit step.

3 Enumeration of k-parallelogram polyominoes

This section is organized as follows: first, we furnish a method to pass from the generating function of the class $\mathbb{P}_k$ to the generating function of $\mathbb{P}_{k+1}$, $k > 1$. Then, we provide the enumeration of the trivial cases, i.e. $k = 0, 1$, and finally apply the inductive step to determine the generating function of $\mathbb{P}_k$. The enumeration of k-parallelogram polyominoes is readily obtained by summing all the generating functions of the classes $\mathbb{P}_s$, $s \leq k$.

3.1 Generating function of k-parallelogram polyominoes

The following theorem establishes a criterion for translating the decomposition of Proposition 2 into generating functions.

Theorem 1

i) A polyomino P belongs to $\mathbb{P}_2$ if and only if it is obtained from a polyomino of $\mathbb{P}_1$ by adding two new paths α_2 and β_2, which cannot be both empty, where the height of α_2 is equal to the height of β_1 minus one, and the width of β_2 is equal to the width of α_1 minus one.

ii) A polyomino P belongs to $\mathbb{P}_k$, $k > 2$, if and only if it is obtained from a polyomino of $\mathbb{P}_{k-1}$ by adding two new paths α_k and β_k, which cannot be both empty, where α_k has the same height of β_{k-1} and β_k has the same width of α_{k-1}.

The proof of Theorem 1 directly follows from our decomposition in Proposition 2, where the difference between the case $k = 2$ and the other cases is clearly explained. We would like to point out that if P belongs to $\overline{\mathbb{P}}_k$, then neither α_k nor β_k can be empty or flat. Following the statement of Theorem 1, to pass from $k \geq 1$ to $k + 1$ we need to introduce following generating functions:

i) the generating function of the sequence $\mathcal{A}(P)$. Such a function is denoted by $A_k(x,y,z)$ for up, and by $\overline{A}_k(x,y,z)$ for flat k-parallelogram polyominoes, respectively, and, for each function, $x+z$ keeps track of the dimensions of $\mathcal{A}(P)$, and z keeps track of the width/height of θ_k alternately, according to the parity of k.

ii) the generating function of the sequence $\mathcal{B}(P)$. Such a function is denoted by $B_k(x,y,t)$ for up, and by $\overline{B}_k(x,y,t)$ for flat k-parallelogram polyominoes, respectively, and here $y+t$ keeps track of the dimensions of $\mathcal{B}(P)$, and the variable t keeps track of the height/width of $\overline{\theta}_k$ alternately, according to the parity of k.

Now the generating functions $Gf_k^U(x,y,z,t)$, $Gf_k^R(x,y,z,t)$ and $\overline{Gf}_k(x,y,z,t)$, of the classes $\mathbb{P}_k^U$, $\mathbb{P}_k^R$, and $\overline{\mathbb{P}}_k$, respectively, are clearly obtained as follows:

$$
\begin{aligned}
Gf_k^U(x,y,z,t) &= A_k(x,y,z)\cdot B_k(x,y,t) && (1)\\
\overline{Gf}_k(x,y,z,t) &= \overline{A}_k(x,y,z)\cdot \overline{B}_k(x,y,t) && (2)\\
Gf_k(x,y,z,t) &= Gf_k^U(x,y,z,t)+Gf_k^R(y,x,t,z)+\overline{Gf}_k(x,y,z,t). && (3)
\end{aligned}
$$

Then, setting $z=t=y=x$, we have the generating functions according to the semi-perimeter. Since $Gf_k^U(x,y,z,t)=Gf_k^R(y,x,t,z)$, for all k, then starting from now, we will study only the flat and the up classes.

The case $k=0$. The class $\mathbb{P}_0$ is simply made of horizontal and vertical bars of any length. We keep this case distinct from the others since it is not useful for the inductive step, so we simply use the variables x and y, which keep track of the width and the height of the polyomino, respectively. The generating function is trivially equal to

$$Gf_0(x,y)=xy+\frac{x^2y}{1-x}+\frac{xy^2}{1-y},$$

where the term xy corresponds to the unit cell, and the other terms to the horizontal and vertical bars, respectively.

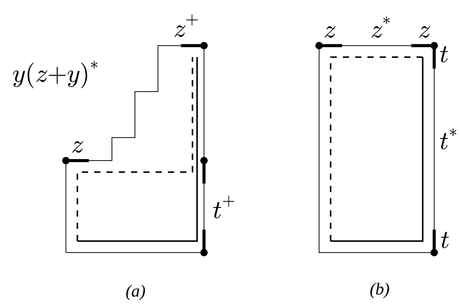

Figure 5: (a) A polyomino $\in \mathbb{P}_1^U$ and (b) a polyomino in $\overline{\mathbb{P}}_1$.

The case $k = 1$. Following our decomposition and Figure 5, we easily obtain

$$A_1(x, y, z) = \frac{z^2 y}{(1 - z - y)(1 - z)}, \qquad B_1(x, y, t) = t + \frac{t^2}{1 - t}.$$

We point out that we have written B_1 as the sum of two terms because, according to Corollary 1, we have to treat the case when β_1 is made by a north unit step separately from the other cases. To this aim, we set $\hat{B}_1(x, y, t) = \frac{t^2}{1-t}$. Moreover, we have

$$\overline{A}_1(x, y, z) = \frac{z^2}{1 - z}, \qquad \overline{B}_1(x, y, t) = \frac{t^2}{1 - t}.$$

According to (1) and (2), we have that

$$Gf_1^U(x, y, z, t) = \frac{tyz^2}{(1 - t)(1 - z)(1 - y - z)} \qquad \overline{Gf}_1(x, y, z, t) = \frac{t^2 z^2}{(1 - t)(1 - z)}.$$

Now, according to (3), and setting all variables equal to x, we have the generating function of 1-parallelogram polyominoes

$$Gf_1(x) = \frac{x^4(2x - 3)}{(1 - x)^2(1 - 2x)}.$$

The case $k = 2$. Now we can use the inductive step, recalling that the computation of the case $k = 2$ will be slightly different from the other cases, as explained in Theorem 1. Using the pictures in Figure 6 we can calculate the generating functions

$$A_2(x, y, z) = z \cdot A_1\left(x, y, \frac{x}{1 - z}\right) = \frac{x^2 yz}{(1 - x - y - z - yz)(1 - x - z)}$$

$$B_2(x, y, t) = \frac{y}{1 - t} + t \cdot \hat{B}_1\left(x, y, \frac{y}{1 - t}\right) = \frac{y - y^2}{(1 - y - t)} = y + \frac{yt}{1 - y - t}$$

$$\overline{A}_2(x, y, z) = z \cdot \overline{A}_1\left(x, y, \frac{x}{1 - z}\right) = \frac{x^2 z}{(1 - z)(1 - x - z)}$$

$$\overline{B}_2(x, y, t) = t \cdot \overline{B}_1\left(x, y, \frac{y}{1 - t}\right) = \frac{y^2 t}{(1 - t)(1 - y - t)}.$$

We observe that the performed substitutions allow us to add the contribution of the terms α_2 and β_2 from the generating functions obtained for $k = 1$. Then, using formulas (1), (2) and (3), and setting all variables equal to x, it is straightforward to obtain the generating function according to the semi-perimeter:

$$Gf_2(x) = \frac{x^5(2 - 5x + 3x^2 - x^3)}{(1 - x)^2(1 - 2x)^2(1 - 3x + x^2)}$$

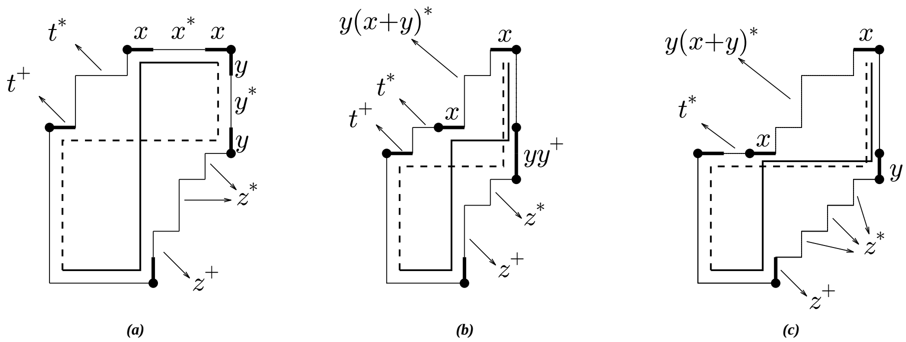

Figure 6: (a) A polyomino in $\overline{\mathbb{P}}_2$, (b) a polyomino in $\mathbb{P}_2^U$ in which β_1 has at least two north steps and (c) a polyomino in $\mathbb{P}_2^U$ in which β_1 is equal to an unit north step.

The case $k > 2$. The generating functions for the case $k > 2$ are obtained in a similar way. Here, for simplicity sake, we set $\hat{B}_k(x,y,t) = B_k(x,y,t) - y$; this trick will help us treat separately the case when β_1 is made by a north unit step. Then we have

$$
\begin{aligned}
A_k(x,y,z) &= \frac{z}{1-z} \cdot A_{k-1}\left(x, y, \frac{x}{1-z}\right) && (4)\\
B_k(x,y,t) &= \frac{y}{1-t} + \frac{t}{1-t} \cdot \hat{B}_{k-1}\left(x, y, \frac{y}{1-t}\right) && (5)\\
\overline{A}_k(x,y,z) &= \frac{z}{1-z} \cdot \overline{A}_{k-1}\left(x, y, \frac{x}{1-z}\right) && (6)\\
\overline{B}_k(x,y,t) &= \frac{t}{1-t} \cdot \overline{B}_{k-1}\left(x, y, \frac{y}{1-t}\right) \ . && (7)
\end{aligned}
$$

We remark that (4), (5), (6) and (7) slightly differ from the respective formulas for $k = 2$, according to the statement of Theorem 1. Using the previous formulas we are now able to obtain an expression for $F_k(x)$, for all $k > 2$.

3.2 A formula for the number of k-parallelogram polyominoes

In this section we show a simpler way to express the generating function of $\mathbb{P}_k$. First we need to define the following polynomials:

$$
\begin{cases}
F_0(x,z) = F_1(x,z) & = \ 1\\
F_2(x,z) & = \ 1 - z\\
F_k(x,z) & = \ F_{n-1}(x,z) - xF_{n-2}(x,z) \ .
\end{cases}
$$

These objects are already known as *Fibonacci polynomials* [6].

In the sequel, unless otherwise specified, we will denote $F_k(x,x)$ with F_k. The closed Formula of F_k is:

$$F_k = \frac{b(x)^{k+1} - a(x)^{k+1}}{\sqrt{1-4x}} \ .$$

where $a(x)$ and $b(x)$ are the solutions of the equation $X^2 - X + x = 0$, $a(x) = \left(\frac{1-\sqrt{1-4x}}{2}\right)$ and $b(x) = \left(\frac{1+\sqrt{1-4x}}{2}\right)$.

Our aim is to express the functions A_k, B_k, $\overline{A}_k$, and $\overline{B}_k$ in terms of the previous polynomials. In order to do this we need to prove the following lemma:

Lemma 1 *For every k*

$$F_k\left(x, \frac{x}{1-z}\right) = \frac{F_{k+1}(x,z)}{1-z}.$$

The proof is easily obtained by induction. Placing $y = x$, we can write $A_1(x,z) = \frac{xz^2}{F_2(x,z)F_3(x,z)}$ and iterating (4), and using Lemma 1, we obtain

$$A_k(x,z) = \frac{zx^{k+1}}{F_{k+1}(x,z)F_{k+2}(x,z)} \quad .$$

Performing the same calculations on the other functions we obtain:

$$\begin{aligned} B_k(x,z) &= \frac{xF_k}{F_{k+1}(x,z)} \\ \overline{A}_k(x,z) &= \overline{B}_k(x,z) = \frac{zx^k}{F_k(x,z)\cdot F_{k+1}(x,z)} \quad . \end{aligned}$$

From these new functions, by setting all variables equal to x, we can calculate the generating function of the class $\mathbb{P}_k$ in an easier way:

$$Gf_k(x) = 2A_k(x,x)B_k(x,x) + (\overline{A}_k)^2(x,x).$$

Then we have the following:

Theorem 2 *The generating function of the whole class of k-parallelogram polyominoes is given by*

$$P_k(x) = \sum_{n=0}^{k} Gf_n(x) = x^2 \cdot \left(\frac{F_{k+1}}{F_{k+2}}\right)^2 - x^2 \cdot \left(\frac{F_{k+1}}{F_{k+2}} - \frac{F_k}{F_{k+1}}\right)^2 .$$

As an example, the generating functions of $\mathbb{P}_k$ for the first values of k are:

$$P_0(x) = \frac{x^2(1+x)}{1-x} \qquad P_1(x) = \frac{x^2(1-2x+2x^2)}{(-1+x)^2(1-2x)}$$

$$P_2(x) = \frac{x^2(1-x)(1-4x+4x^2+x^3)}{(1-2x)^2(1-3x+x^2)} \qquad P_3(x) = \frac{x^2(1-2x)(1-6x+11x^2-6x^3+2x^4)}{(1-x)(1-3x)(1-3x+x^2)^2}$$

In [6] it is proved that xF_k/F_{k+1} is the generating function of plane trees having height less than or equal to $k+1$. Hence, $P_k(x)$ is the difference between the generating functions of pairs of trees having height at most $k+3$, and pairs of trees having height exactly equal to $k+2$. It would be interesting to provide a combinatorial explanation to this fact.

As one would expect we have the following corollary:

Corollary 2 *Let $C(x) = \frac{1-\sqrt{1-4x}}{2x}$ be the generating function of Catalan numbers, we have:*

$$\lim_{k\to\infty} P_k(x) = C(x)\,.$$

Proof: We have that $C(x)$ satisfies the equation $C(x) = 1 + xC^2(x)$, and $a(x)b(x) = x$, $a(x) = xC(x)$, so we can write

$$F_k = \frac{1 - x^{k+1}C^{2(k+1)}(x)}{C^{k+1}\sqrt{1-4x}}\,.$$

Now we can prove the following statements:

$$\lim_{k\to\infty} \frac{F_k}{F_{k+1}} = C(x)\,, \qquad \lim_{k\to\infty} \left(\frac{F_k}{F_{k+1}}\right)^2 = \frac{C(x)-1}{x}\,, \qquad \lim_{k\to\infty} \frac{F_k}{F_{k+2}} = \frac{C(x)-1}{x}\,.$$

From Theorem 2, and using the above results, we obtain the desired proof. □

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A t-generalization for Schubert Representatives of the Affine Grassmannian

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Abstract. We introduce two families of symmetric functions with an extra parameter t that specialize to Schubert representatives for cohomology and homology of the affine Grassmannian when $t = 1$. The families are defined by a statistic on combinatorial objects associated to the type-A affine Weyl group and their transition matrix with Hall-Littlewood polynomials is t-positive. We conjecture that one family is the set of k-atoms.

Nous présentons deux familles de fonctions symétriques dépendant d'un paramètre t et dont les spécialisations à $t = 1$ correspondent aux classes de Schubert dans la cohomologie et l'homologie des variétés Grassmanniennes affines. Les familles sont définies par des statistiques sur certains objets combinatoires associés au groupe de Weyl affine de type A et leurs matrices de transition dans la base des polynômes de Hall-Littlewood sont t-positives. Nons conjecturons qu'une de ces familles correspond aux k-atomes.

Keywords: k-Schur functions, Pieri rule, Bruhat order, Hall-Littlewood polynomials

1 Introduction

Affine Schubert calculus is a generalization of classical Schubert calculus where the Grassmannian is replaced by infinite-dimensional spaces Gr_G known as affine Grassmannians. As with Schubert calculus, topics under the umbrella of affine Schubert calculus are vast but now, it is the combinatorics of a family of polynomials called k-Schur functions that underpins the theory.

The theory of k-Schur functions came out of a study of symmetric functions over $\mathbb{Q}(q,t)$ called Macdonald polynomials. Macdonald polynomials posses remarkable properties whose proofs inspired deep work in many areas One aspect that has been intensely studied from a combinatorial, representation theoretic, and algebraic geometric perspective is the Macdonald/Schur transition matrix. In particular, in the late 1980's, Macdonald conjectured [Mac88] that the coefficients in the expansion

$$H_\mu[X;q,t] = \sum_\lambda K_{\lambda,\mu}(q,t)\, s_\lambda \tag{1}$$

are positive sums of monomials in q and t; that is, $K_{\lambda,\mu}(q,t) \in \mathbb{N}[q,t]$. These coefficients have since been a matter of great interest. For starters, they generalize the *Kostka-Foulkes polynomials*. These are given

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by $K_{\lambda,\mu}(0,t)$ and they appear in many contexts such as Hall-Littlewood polynomials [Gre55], affine Kazhdan-Lusztig theory [Lus81], and affine tensor product multiplicities [NY97]. Moreover, Kostka-Foulkes polynomials encode the dimensions of certain bigraded S_n-modules [GP92]. They were beautifully characterized by Lascoux and Schützenberger [LS78] by associating a statistic (non-negative integer) called *charge* to each tableau T so that

$$K_{\lambda,\mu}(0,t) = \sum_{\substack{weight(T)=\mu \\ \text{shape}(T)=\lambda}} t^{\text{charge}(T)} . \tag{2}$$

Despite having such concrete results for the $q = 0$ case, it was a big effort even to establish polynomiality for general $K_{\lambda,\mu}(q,t)$ [GR96, GT96, Kno97, LV98, KN96, Sah96] and the geometry of Hilbert schemes was eventually needed to prove positivity [Hai01]. A formula in the spirit of (2) still remains a mystery.

In one study of Macdonald polynomials, Lapointe, Lascoux, and Morse found computational evidence for a family of new bases

$$\{A_\mu^{(k)}[X;t]\}_{\mu_1\leq k} \tag{3}$$

for subspaces

$$\Lambda_t^{(k)} = \text{span}\{H_\lambda[X;q,t]\}_{\lambda_1\leq k}$$

in a filtration $\Lambda_t^{(1)} \subseteq \Lambda_t^{(2)} \subseteq \cdots \subseteq \Lambda_t^{(\infty)}$ of Λ. Conjecturally, the star feature of each basis was the property that Macdonald polynomials expand positively in terms of it, giving a remarkable factorization for the Macdonald/Schur transition matrices over $\mathbb{N}[q,t]$. To be precise, for any fixed integer $k > 0$ and each $\lambda \in \mathcal{P}^k$ (a partition where $\lambda_1 \leq k$),

$$H_\lambda[X;q,t] = \sum_{\mu\in\mathcal{P}^k} K_{\mu,\lambda}^{(k)}(q,t)\, A_\mu^{(k)}[X;t] \quad \text{where} \quad K_{\mu,\lambda}^{(k)}(q,t) \in \mathbb{N}[q,t] . \tag{4}$$

It was conjectured in [LLM03] that for all $k > 0$, $\{A_\mu^{(k)}[X;t]\}_{\mu_1\leq k}$ exists and forms a basis for $\Lambda_t^{(k)}$, and that for $k \geq |\mu|$, $A_\mu^{(k)}[X;t] = s_\mu$. These conjectures and the decomposition (4) strengthen Macdonald's conjecture.

A construction for $A_\mu^{(k)}[X;t]$ is given in [LLM03], but it is so intricate that these conjectures remain unproven. However, pursuant investigations of these bases led to various conjecturally equivalent characterizations. One such family of polynomials $\{s_\lambda^{(k)}\}$ was introduced in [LM05] and conjectured to be the $t = 1$ case of $A_\lambda^{(k)}[X;t]$. It has since been proven that the $s_\lambda^{(k)}$ refine the very aspects of Schur functions that make them so fundamental and wide-reaching and they are now called *k-Schur functions*.

The role of k-Schur functions in affine Schubert calculus emerged over a number of years. The springboard was a realization that the combinatorial backbone of k-Schur theory lies in the setting of the affine Weyl group. The k-Schur functions are tied to Pieri rules, tableaux, Young's lattice, sieved q-binomial identities, and Cauchy identities that are naturally described in terms of posets of elements in $\tilde{A}^k$. For example, $K_{\lambda,\mu}^{(k)}(1,1)$ is the number of reduced expressions for an element in $\tilde{A}^k$. The combinatorial exploration fused into a geometric one when the k-Schur functions were connected to the quantum cohomology of Grassmannians. Quantum cohomology originated in string theory and symplectic geometry. It has had a great impact on algebraic geometry and is intimately tied to the *Gromov-Witten invariants*. These invariants appear in the study of subtle enumerative questions such as: how many degree d plane curves of genus g contain r generic points? Lapointe and Morse [LM08] showed that each Gromov-Witten invariant for the quantum cohomology of Grassmannians exactly equals a k-Schur coefficient in the product of k-Schur functions in Λ. A basis of *dual (or affine) k-Schur functions* was also introduced in [LM08] and Lam proved [Lam08] that the Schubert bases for cohomology and homology of the affine Grassmannian $\mathrm{Gr}_{SL_{k+1}}$ are given by the dual k-Schur functions and the k-Schur functions, respectively.

Our motivation here is that the k-Schur functions $s_\lambda^{(k)}$ are parameterless and the t is needed to connect with theories outside of geometry. Unfortunately, the characterizations for generic t lack in mechanism for proofs. We introduce a new family of functions that reduce to $\{s_\lambda^{(k)}\}$ when $t = 1$. Our definition uses a combinatorial object called affine Bruhat counter-tableaux (ABC's), whose weight generating functions are the dual k-Schur functions [DM12]. We associate a statistic (a non-negative integer) to each ABC called the k-charge. From this, we use the polynomials

$$K_{\lambda,\mu}^{(k)}(t) = \sum_{\substack{\mathrm{shape}(A)=\mathfrak{c}(\lambda)\\ \mathrm{weight}(A)=\mu}} t^{k\text{-charge}(A)} \tag{5}$$

to define a t-generalization of $s_\lambda^{(k)}$. In particular, we show that the matrix $(K_{\lambda,\mu}^{(k)}(t))_{\{\lambda,\mu\in\mathcal{P}^k\}}$ is unitriangular and taking the inverse of this matrix to be $\tilde{K}^{(k)}$, a basis for Λ_t^k is given by

$$s_\lambda^{(k)}[X;t] = \sum_\mu \tilde{K}_{\lambda,\mu}^{(k)}(t)\, H_\mu[X;t]\,,$$

for all λ with $\lambda_1 \le k$. We prove that $s_\lambda^{(k)}[X;t]$ reduce to k-Schur functions when $t = 1$. When $k = \infty$, these are Schur functions, and thus (5) gives a new description for the Kostka-Foulkes polynomials. Naturally, we conjecture that these functions are the $A_\lambda^{(k)}[X;t]$.

2 Related work

A refinement of the plactic monoid to a structure on k-tableaux that can be applied to combinatorial problems involving k-Schur functions is partially given in [LLMS12] by a bijection compatible with the RSK-bijection. A deeper understanding of this intricate bijection is underway. Towards this effort, Lapointe and Pinto [LP] have recently shown that a statistic on k-tableaux is compatible with the bijection. There are now several statistics (on k-tableaux, elements of the affine symmetric group, and on ABC's) whose charge generating functions are the same. The ABC's can be used to find the image of certain elements under this bijection and we are working to put the ABC's in a context that simplifies the bijection.

3 Background

We identify each partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ with its Ferrers shape (having λ_i lattice squares in the i^{th} row, from the bottom to top). For partitions λ and μ, we say λ contains μ, denoted $\mu \subseteq \lambda$, if $\lambda_i \ge \mu_i$. A skew shape is a pair of partitions λ, μ such that $\mu \subseteq \lambda$, denoted λ/μ.

A *semistandard tableau* T is a filling of a Ferrers shape λ with positive integers that weakly decrease along rows and strictly increase up the columns. The *weight* of a semistandard tableau is the composition $(\mu_i)_{i\in\mathbb{N}}$, where μ_i is the number of cells containing i. For a partition λ and composition μ, let $SSYT(\lambda,\mu)$ be the set of semistandard tableaux of shape λ and weight μ.

The *hook-length* of a cell (i, j) of any partition is the number of cells to the right of (i, j) in row i plus the number of cells above (i, j) in column j plus 1. A p-*core* is a partition that does not contain any cell with hook-length p. The p-degree of a p-core λ, $deg^p(\lambda)$, is the number of cells in λ whose hook-length is smaller than p. Hereafter we work with a fixed integer $k > 0$ and all cores (resp. residues) are $k+1$-cores (resp. $k+1$-residues) and deg^{k+1} will simply be written as deg. We let $\mathcal{P}^k$ denote the set of all partitions λ with $\lambda_1 \leq k$. We also let $\mathcal{C}^{k+1}$ denote the set of all $k+1$-cores. We use a bijection given in [LM05] $\mathfrak{c} : \mathcal{P}^k \to \mathcal{C}^{k+1}$.

For $n \geq 0$, an n-*ribbon* R is a skew diagram λ/μ consisting of n rookwise connected cells such that there is no 2×2 shape contained in R. We refer to the southeasternmost cell of a ribbon as its *head*, and the northweasternmost cell of a ribbon as its *tail*.

A *ribbon tableau* T of shape λ/μ is a chain of partitions

$$\mu = \mu^0 \subset \mu^1 \subset \cdots \subset \mu^r = \lambda$$

such that each μ^i/μ^{i-1} is a tiling of ribbons filled with a positive integer. A *ribbon counter-tableau* A of shape λ/μ is a ribbon tableau such that each skew shape μ^i/μ^{i-1} is filled with the same positive integer $r - i + 1$. We set the cell (i, j) of a ribbon counter-tableau to be the cell in row i, column j, where row one is the topmost row and column one is the leftmost column. For more on partitions and tableaux see [Mac95], [Sta99], [Ber09].

4 Schubert representatives for $H^*(\mathrm{Gr}_{SL_{k+1}})$ and $H_*(\mathrm{Gr}_{SL_{k+1}})$

Despite the many characterizations for the Schubert representatives for the cohomology and homology of the infinite dimensional affine Grassmannian spaces for SL_{k+1} (e.g. [LM05, LM08, Lam06, LLMS10, DM12, AB12]), none have been shown to be the $t = 1$ case of functions conjectured to give a positive Macdonald expansion (4). Our goal is to present functions with a t parameter which reduce to the k-Schur functions as formulated in [DM12] when $t = 1$. The formulation is given in terms of a combinatorial structure called ABC's.

Recall that the strong (Bruhat) order on the affine Weyl group $\tilde{A}^k$ can be instead realized on $k+1$-cores by the covering relation:

$$\rho \lessdot_B \gamma \iff \rho \subseteq \gamma \text{ and } deg(\gamma) = deg(\rho) + 1.$$

An important fact about strong covers is useful in our study.

Lemma 1 *[LLMS10] Let $\rho \lessdot_B \gamma$ be cores. Then*

1. *Each connected component of ρ/γ is a ribbon.*
2. *The components are translates of each other and their heads have the same residue.*

A specific subset of ribbon counter-tableaux are those where each ribbon is of height one. An ABC will be defined as such ribbon counter-tableaux where the skew shapes are a certain strip defined in terms of strong order.

Definition 2 *For $0 < \ell \leq k$ and $k+1$-cores λ and ν, the skew shape $(k + \lambda_1, \lambda)/\nu$ is a* bottom strong $(k - \ell)$-strip *if there is a saturated chain of cores*

$$\nu = \nu^0 \lessdot_B \nu^1 \lessdot_B \cdots \lessdot_B \nu^{k-\ell} = (k + \lambda_1, \lambda)\,,$$

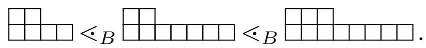
where

1. $(k+\lambda_1, \lambda)/\nu$ *is a horizontal strip*

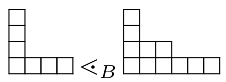
2. *The bottom rightmost cell of* ν^i *is also a cell in* ν^i/ν^{i-1}, *for* $1 \leq i \leq k-\ell$.

It turns out that if a skew shape is a bottom strong strip then there is a unique chain meeting the conditions described in Definition 2.

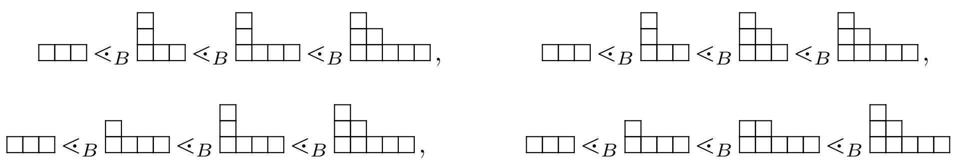
Example 3 *The skew shape* $(8,3)/(4,2)$ *of 6-cores is a bottom strong 2-strip as there is the saturated chain*



Example 4 *The skew shape* $(6,3,1,1)/(4,1,1,1)$ *of 4-cores is a bottom strong 1-strip as there is the saturated chain*



Example 5 *There are 4 saturated chains of 4-cores in the strong order from* (3) *to* $(5,2,1)$,



Since none of these give a bottom strong strip, $(5,2,1)/(3)$ *is not a bottom strong strip.*

Remark 6 *Bottom strong* $(k-\ell)$*-strips are a distinguished subset of strong strips in [LLMS10] that define the Pieri rule for the cohomology of the affine Grassmannian.*

The iteration of bottom strong strips leads to the definition of an ABC. First let us set some notation. Given a ribbon counter-tableau A, let $A^{(x)}$ denote the subtableau made up of the rows of A weakly higher than row x. Let $A_{>i}$ denote the restriction of A to letters strictly larger than i where empty cells in a skew are considered to contain ∞. With this in hand, we are now ready to define the ABC's.

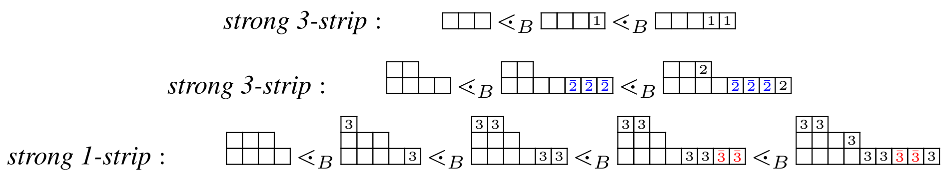
Definition 7 *For a composition* α *whose entries are not larger than* k, *a skew ribbon counter-tableau* A *is an affine Bruhat counter-tableau (or* ABC*) of* k*-weight* α *if*

$$(k+\lambda_1^{(i-1)}, \lambda^{(i-1)})/\lambda^{(i)} \ \text{ is a bottom strong } \alpha_i\text{-strip for all } 1 \leq i \leq \ell(\alpha)\,,$$

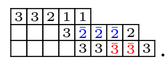
where $\lambda^{(x)} = \mathrm{shape}(A^{(x)}_{>x})$. *We define the inner shape of* A *to be* $\lambda^{(\ell(\alpha))}$.

The easiest method to construct an ABC of k-weight α is iteratively, from the empty shape $\lambda^{(0)}$, using Definition 2 to successively add bottom strong strips that are a tiling of $(k+\lambda_1^{(i-1)}, \lambda^{(i-1)})/\lambda^{(i)}$ with α_i-ribbons at each step.

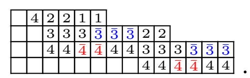
Example 8 *With $k = 5$, we construct an ABC of 5-weight $(3,3,1)$ by*



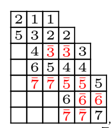
The black letters are ribbons of size one, red letters make a ribbon of size two and blue letters make a ribbon of size 3 (or for those without color the ribbons are depicted with a bar). This can be more compactly represented as



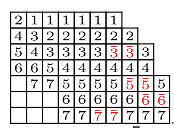
Example 9 *An example of an ABC of* 6*-weight* $(4,4,2,1)$ *with inner shape* $(8,2,2,1) = \mathfrak{c}(6,2,2,1)$ *is*



Example 10 *Two examples of ABC's of k-weight* $(1,1,1,1,1,1,1) = (1^7)$ *are*



3-*weight* (1^7)



7-*weight* (1^7)

The weight generating functions of the ABC's turn out to be the dual k-Schur functions.

Theorem 11 *[DM12] For any $\lambda \in \mathcal{C}^{k+1}$, the dual k-Schur function can be defined by*

$$\mathfrak{S}_\lambda^{(k)} = \sum_A x^A$$

where the sum is over all affine Bruhat counter-tableaux of inner shape λ, and $x^A = x^{k\text{-}weight(A)}$.

These are symmetric functions, implying that

$$\mathfrak{S}_\lambda^{(k)} = \sum_{\mu:\mu_1 \le k} K_{\lambda,\mu}^{(k)}\, m_\mu \,, \tag{6}$$

where $K_{\lambda,\mu}^{(k)}$ is the number of affine Bruhat counter-tableaux of inner shape λ and k-weight μ. Then, using the Hall-inner product defined by

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu} \,,$$

we arrive at a characterization for k-Schur functions.

$$h_\mu = \sum_\lambda K_{\lambda,\mu}^{(k)}\, s_\lambda^{(k)}. \tag{7}$$

5 Kostka-Foulkes polynomials

Our goal is to introduce polynomials $s_\lambda^{(k)}[X;t]$ that reduce to $s_\lambda^{(k)}[X]$ when $t = 1$ using (4) as an inspiration. Our approach is to introduce a statistic on ABC's. When $k = deg(\lambda)$, both $\mathfrak{S}_\lambda^{(k)}$ and $s_\lambda^{(k)}$ are simply the Schur function s_λ. One advantage of using ABC combinatorics in the theory of k-Schur functions is that known results concerning Schur functions can be reinterpreted in the ABC framework with k large and this can shed light on the smaller k cases. With this in mind, we consider a reformulation for the Kostka-Foulkes polynomials in terms of ABC's. Our results enable us to give a characterization for symmetric polynomials in an extra parameter t that reduce to $\mathfrak{S}_\lambda^{(k)}$ and $s_\lambda^{(k)}$ when $t = 1$.

Let us start by recalling the Hall-Littlewood polynomials $\{H_\lambda[X;t]\}_\lambda$. These are a basis for Λ over the polynomial ring $\mathbb{Z}[t]$, which reduces to the homogeneous basis when the parameter $t = 1$. These often are denoted by $\{Q'_\lambda[X;t]\}$ in the literature ([Mac95]). Hall-Littlewood polynomials arise and can be defined in various contexts such as the Hall Algebra, the character theory of finite linear groups, projective and modular representations of symmetric groups, and algebraic geometry. We define them here via a tableaux Schur expansion due to Lascoux and Schützenberger [LS78].

The key notion is the *charge* statistic on semistandard tableaux. This is given by defining charge on words and then defining the charge of a tableau to be the charge of its reading word. For our purposes, it is sufficient to define charge only on words whose evaluation is a partition. We begin by defining the charge of a word with weight $(1, 1, \ldots, 1)$, or a *permutation*. If w is a permutation of length n, then the charge of w is given by $\sum_{i=1}^{n} c_i(w)$ where $c_1(w) = 0$ and $c_i(w)$ is defined recursively as

$$c_i(w) = c_{i-1}(w) + \chi\,(i \text{ appears to the right of } i-1 \text{ in } w)\,.$$

Here we have used the notation that when P is a proposition, $\chi(P)$ is equal to 1 if P is true and 0 if P is false.

Example 12 *The charge,* $ch(3, 5, 1, 4, 2) = 0 + 1 + 1 + 2 + 2 = 6$.

We will now describe the decomposition of a word with partition evaluation into *charge subwords*, each of which are permutations. The charge of a word will then be defined as the sum of the charge of its charge subwords. To find the first charge subword $w^{(1)}$ of a word w, we begin at the *right* of w (i.e. at the last element of w) and move leftward through the word, marking the first 1 that we see. After marking a 1, we continue to travel to the left, now marking the first 2 that we see. If we reach the beginning of the word, we loop back to the end. We continue in this manner, marking successively larger elements, until we have marked the largest letter in w, at which point we stop. The subword of w consisting of the marked elements (with relative order preserved) is the first charge subword. We then remove the marked elements from w to obtain a word w'. The process continues iteratively, with the second charge subword being the first charge subword of w', and so on.

Example 13 *Given* $w = (5, 2, 3, 4, 4, 1, 1, 1, 2, 2, 3)$*, the first charge subword of* w *are the bold elements in* $(\mathbf{5}, \mathbf{2}, 3, 4, \mathbf{4}, 1, 1, \mathbf{1}, 2, 2, \mathbf{3})$*. If we remove the bold letters, the second charge subword is given by the bold elements in* $(\mathbf{3}, \mathbf{4}, 1, \mathbf{1}, 2, \mathbf{2})$*. It is now easy to see that the third and final charge subword is* $(\mathbf{1}, \mathbf{2})$*. Thus we get that* $ch(w) = ch(5, 2, 4, 1, 3) + ch(3, 4, 1, 2) + ch(1, 2) = 8$*. Since* w *is the reading word of the tableau* $T = \begin{array}{|c|c|c|c|c|c|c|}\hline 5 \\ \cline{1-1} 2 & 3 & 4 & 4 \\ \cline{1-4} 1 & 1 & 1 & 1 & 2 & 2 & 3 \\ \hline\end{array}$ *we find that the* $ch(T) = 8$.

Equipped with the definition of charge, Hall-Littlewood polynomials are then defined by

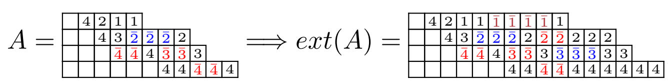
$$H_\mu[X;t] = \sum_\lambda K_{\lambda,\mu}(t)\, s_\lambda\,, \tag{8}$$

where $K_{\lambda,\mu}(t) = K_{\lambda,\mu}(0,t)$ from (2).

Our first order of business to reformulate Kostka-Foulkes polynomials is to describe the reading word of an ABC. To do so, we first modify a given ABC by lengthening the row sizes.

Definition 14 *From a given ABC A of partition k-weight μ, the extension of A, $ext(A)$, is the counter-tableau constructed from A by adding k cells with letter i to each row i, where the first $\mu_i - s_i + r_i + 1$ added cells form a ribbon for s_i the sum of the size of the ribbons filled with the letter i in row i and r_i the number of such ribbons.*

Example 15 *Consider the following extension of an ABC with 5-weight $(3,3,3,1)$.*



5.1 Reading word of standard ABC's

As with tableaux, we first define the reading word of a standard ABC (one of k-weight 1^n) and use this to describe the general reading word. Standard ABC's have a much more predictable structure than the general case. Namely, a standard ABC A has only ribbons of size 1 or 2. In fact, if a row i in A has an i-ribbon of size 2, then $\mu_i - s_i + r_i = 1$. Otherwise $\mu_i - s_i + r_i + 1 = 2$. Thus, each row i of $ext(A)$ has a unique i-ribbon of size 2.

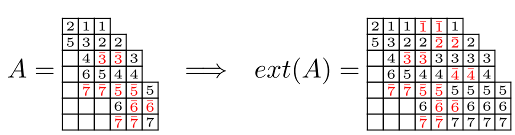
Our construction of the word of an ABC A considers only a subset of the cells in $ext(A)$. Namely,

$$V_A = \{(i,c_i) \in ext(A) : (i,c_i) \text{ is any cell in a } i\text{-ribbon of row } i \text{ that is not its tail}\}. \tag{9}$$

For standard A of k-weight 1^n, V_A is simply a set of n ribbon heads; the one in each row i of $ext(A)$ that contains i. Using V_A, we define the reading word on this standard ABC A.

Definition 16 *For a given ABC A of k-weight 1^n, iteratively construct the reading word $w(A)$ by inserting letter i directly right of letter j where $j < i$ is the largest index such that $c_j < c_i$ and $(j,c_j) \in V_A$. If there is no such j then i is placed at the beginning.*

Example 17 *Recall the ABC from example 10 of 3-weight (1^7) is*



From $ext(A)$, we see that $V_A = \{(1,5),(2,6),(3,4),(4,7),(5,5),(6,6),(7,5)\}$. From V_A, we have the iterative construction of the reading word of A as $(1) \to (1,2) \to (3,1,2) \to (3,1,2,4) \to (3,5,1,2,4) \to (3,5,6,1,2,4) \to (3,7,5,6,1,2,4)$. This tells us that $w(A) = (3,7,5,6,1,2,4)$.

5.2 Reading words of ABC

Equipped with a method to obtain the reading word (permutation) of standard ABC's, we now define a way to construct a sequence of permutations from any ABC with partition k-weight.

Definition 18 *Let A be an ABC of partition k-weight μ. For $r = 1, 2, \ldots, \mu_1$, starting with $r = 1$, we iteratively construct sets E_A^r from $ext(A)$ as follows; put $(1, k+\mu_1+2-r) \in E_A^r$, and let $(i, c_i) \in E_A^r$ if and only if $(i-1, c_{i-1}) \in E_A^r$ and*

$$(c_{i-1} - c_i + k)\ mod\ (k+1) = min\{(c_{i-1} - x + k)\ mod\ (k+1) | (i, x) \in V_A \setminus \cup_{p=1}^{r-1} E_A^p\}.$$

Example 19 *Recall the ABC A from example 15 of 5-weight $(3, 3, 3, 1)$. From its $ext(A)$, we see that*

$$V_A = \{(1,7), (1,8), (1,9), (2,6), (2,7), (2,10), (3,8), (3,11), (3,12), (4,10)\}.$$

We iteratively construct the sets E_A^r for each $r = 1, 2, 3$, using $ext(A)$ and V_A. For $r = 1$, begin by setting $E_A^1 = \{(1,9)\}$. Next, we see that $(2,7) \in E_A^1$, because $(1,9) \in E_A^1$ and

$$1 = min\{2 = (9-6+5)\ mod\ 6, 1 = (9-7+5)\ mod\ 6, 4 = (9-10+5)\ mod\ 6\}.$$

So the next iteration gives us that $E_A^1 = \{(1,10), (2,7)\}$. Next, we see that $(3,12) \in E_A^1$, because $(2,7) \in E_A^1$ and $0 = min\{4 = (7-8+5)\ mod\ 6, 1 = (7-11+5)\ mod\ 6, 0 = (7-12+5)\ mod\ 6\}$. So the next iteration gives us that $E_A^1 = \{(1,10), (2,7), (3,12)\}$. Finally since the $(4,10)$ is the only element in V_A from the fourth row of A, then we see that $E_A^1 = \{(1,9), (2,7), (3,12), (4,10)\}$.

For $r = 2$, to construct E_A^2, we begin by setting $E_A^2 = \{(1,8\}$, and repeat what we did to construct E_A^1, except this time we only consider elements from the set $V_A \backslash E_A^1 = \{(1,7), (1,8), (2,6), (2,10), (3,8), (3,11)\}$. This gives us $E_A^2 = \{(1,8), (2,6), (3,11)\}$.

Finally for $r = 3$, to construct E_A^3, we begin by setting $E_A^3 = \{(1,7)\}$, and we only consider elements from the set $V_A \setminus (E_A^1 \cup E_A^2) = \{(1,7), (2,10), (3,8)\}$, which immediately gives us $E_A^3 = \{(1,7), (2,10), (3,8)\}$.

Using each set E_A^r, we construct a sequence of reading word w_r for $1 \leq r \leq \mu_1$.

Definition 20 *Given an ABC A of partition k-weight μ, for $1 \leq r \leq \mu_1$, the r^{th} reading word of A, $w_r(A)$, is constructed using the same procedure in definition 16, where V_A is replaced by E_A^r.*

Example 21 *If we consider the ABC A from example 15, then we know from example 19 that $E_A^1 = \{(1,9), (2,7), (3,12), (4,10)\}$, $E_A^2 = \{(1,8), (2,6), (3,11)\}$, $E_A^3 = \{(1,7), (2,10), (3,8)\}$. This tells us from definition 20 that $w_1(A) = (2,1,4,3)$, $w_2(A) = (2,1,3)$ and $w_3(A) = (3,1,2)$.*

For partitions λ, μ with $|\lambda| = |\mu| = n$, an ABC A is of n-weight μ and inner shape λ, has a charge statistic associated to it.

Definition 22 *Suppose λ and μ are partitions with $|\lambda| = |\mu| = n$. For any ABC A of n-weight μ and inner shape λ, the charge of A is $ch(A) = \sum_{r=1}^{\mu_1} ch(w_r(A))$.*

Example 23 *The ABC A from example 15 has the reading words $w_1(A) = (2,1,4,3)$, $w_2(A) = (2,1,3)$ and $w_3(A) = (3,1,2)$. as described in example 18. Hence, we have that the charge of A is $ch(A) = ch((2,1,4,3)) + ch((2,1,3)) + ch((3,1,2)) = 2 + 1 + 2 = 5$.*

There is a direct connection between reading words of semi-standard Young tableaux and a certain set of ABC's.

Theorem 24 *Suppose λ and μ are partitions with $|\lambda| = |\mu| = n$. If the set*

$$ABC(\lambda, \mu) = \{A \mid A \text{ is an } ABC \text{ of } n\text{-weight } \mu \text{ and inner shape } \mathfrak{c}(\lambda)\},$$

then there is a bijection between the sets $ABC(\lambda, \mu)$ and $SSYT(\lambda, \mu)$, which is charge preserving.

From this theorem, we have the following corollary that gives the Kostka-Foulkes polynomials in the spirit of ABC's.

Corollary 25 *For partitions λ and μ, the Kostka-Foulkes polynomial*

$$K_{\lambda,\mu}(t) = \sum_{A \in ABC(\lambda,\mu)} t^{ch(A)}.$$

6 k-charge and k-Schur functions

We now look towards generalizing Corollary 25 by considering ABC's of any partition k-weight. What is needed is an extra concept of an *offset* of a given ABC. Any r-ribbon of an ABC is an *offset* if there is a lower r-ribbon filled with the same letter as R whose head has the same residue as the head of R.

Definition 26 *For any ABC A of partition k-weight μ, we set*

$$\mathit{off}^{\,k}(A) = \sum_{R:\text{ offset in } A} (\mathit{size}(R) - 1)$$

Definition 27 *Let A be an ABC of partition k-weight μ and inner shape λ, and $w_1(A), \ldots, w_{\mu_1}(A)$ be the sequence of reading words that result from definition 18. Then, the k-charge of A*

$$ch^k(A) = \sum_{r=1}^{\mu_1} ch(w_r(A)) - \mathit{off}^{\,k}(A) - \beta(A)$$

where $\beta(A)$ is the number of cells in λ whose hook-length exceeds k.

Example 28 *Consider the ABC of 3-weight 1^5* $A = \begin{array}{|c|c|c|c|c|c|} \hline 3 & 1 & 1 & & & \\ \hline & 2 & \bar{2} & \bar{2} & & \\ \hline & 4 & 3 & 3 & & \\ \hline & \bar{5} & \bar{5} & 4 & 4 & \\ \hline & & & 5 & \bar{5} & \bar{5} \\ \hline \end{array}$. *Here we see that A has only one offset* $\begin{array}{|c|c|} \hline \bar{5} & \bar{5} \\ \hline \end{array}$ *in the second row from the bottom. The only reading word for this A is $w_1(A) = (2, 5, 1, 3, 4)$. So we get $ch^3(A) = ch((2, 5, 1, 3, 4)) - 1 - 1 = 5 - 1 - 1 = 3$.*

Definition 29 *For any λ, $\mu \in \mathcal{P}^k$, we let*

$$K^{(k)}_{\lambda,\mu}(t) = \sum_{\substack{A:\ ABC \text{ of } k\text{-weight } \mu, \\ \text{inner shape } \mathfrak{c}(\lambda)}} t^{ch^k(A)}.$$

Note that when $k \geq |\lambda|$, the polynomials $K^{(k)}_{\lambda,\mu}(t)$ are the Kostka-Foulkes polynomials of 2. Definition 29 generalizes the Kostka-Foulkes polynomials, and it also helps us to define a new set of symmetric functions with parameter t. To see this, we only need the following claim.

Lemma 30 *The matrix*

$$\left[K^{(k)}_{\lambda,\mu}(t)\right]_{\{\lambda,\mu\in\mathcal{P}^k\}}$$

is unitriangular.

Taking the inverse of the matrix in theorem 30, we form a basis for the subring $\Lambda_t^{(k)}$ of the ring Λ.

Definition 31 *For $\lambda \in \mathcal{P}^k$, the k-Schur function with parameter t is*

$$s^{(k)}_\lambda[X;t] = \sum_\mu \tilde{K}^{(k)}_{\lambda,\mu}(t) H_\mu[X;t],$$

where $\tilde{K}^{(k)}_{\lambda,\mu}(t)$ are entries in the inverse of the matrix $\left[K^{(k)}_{\lambda,\mu}(t)\right]_{\{\lambda,\mu\in\mathcal{P}^k\}}$.

These new symmetric functions exhibit properties which connect them to the k-Schur and the Schur functions.

Property 32 *As λ ranges over partitions in $\mathcal{P}^k$, $s^{(k)}_\lambda[X;t]$ forms a basis for the subring $\Lambda_t^{(k)}$, $s^{(k)}_\lambda[X;1] = s^{(k)}_\lambda$, and $s^{(\infty)}_\lambda[X;1] = s_\lambda$.*

Finally we make the following conjecture which ties the functions in Definition 31 to those described in [LLM03].

Conjecture 33 *For $\mu \in \mathcal{P}^k$, $s^{(k)}_\mu[X;t] = A^{(k)}_\mu[X;t]$.*

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A lattice of combinatorial Hopf algebras: Binary trees with multiplicities

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Abstract. In a first part, we formalize the construction of *combinatorial Hopf algebras* from plactic-like monoids using *polynomial realizations*. Thank to this construction we reveal a lattice structure on those combinatorial Hopf algebras. As an application, we construct a new combinatorial Hopf algebra on binary trees with multiplicities and use it to prove a *hook length formula* for those trees.

Résumé. Dans une première partie, nous formalisons la construction d'*algèbres de Hopf combinatoires* à partir d'une *réalisation polynomiale* et de monoïdes de type *monoïde plaxique*. Grâce à cette construction, nous mettons à jour une structure de treillis sur ces algèbres de Hopf combinatoires. Comme application, nous construisons une nouvelle algèbre de Hopf sur des arbres binaires à multiplicités et on l'utilise pour démontrer une *formule des équerres* sur ces arbres.

Keywords: Combinatorial Hopf algebras, monoids, polynomial realization, hook length formula, generating series, binary trees

1 Introduction

In the past decade a large amount of work in algebraic combinatorics has been done around combinatorial Hopf algebras. Many have been constructed on various combinatorial objects such as *partitions* (symmetric functions [Mac95]), *compositions* (**NCSF** [GKL$^+$94, MR95]), *permutations* (**FQSym** [MR95, DHT02]), *set-partitions* (**WQSym** [Hiv99]), *binary trees* (**PBT** or the LODAY-RONCO Hopf algebra [LR98, HNT05]), or *parking functions* (**PQSym** [NT04, NT07]). A powerful method to construct those algebras, called *polynomial realization*, is to construct the Hopf algebra as a sub algebra of a free algebra of polynomials (commutative or not) admitting certain symmetries. Beside the contruction of *Combinatorial Hopf algebra*, several recent papers investigate toward the formalization of combinatorial applications such as *hook formulas*, or seek some structure in this large zoo.

This extended abstract, reports on a work in progress which proposes to formalize the construction of Hopf algebras by polynomial realizations: starting with one of the three Hopf algebras **FQSym**, **WQSym** or **PQSym** realized in a *free algebra*, we impose some relations on the variables. Under some simple hypotheses, the result is again a Hopf algebra (Theorem 1). Two important examples are already known,

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namely the POIRIER-REUTENAUER algebra of tableaux (**FSym** [PR95, DHT02]) obtained from the *plactic monoid* [LS81] and the *planar binary tree algebra* of LODAY-RONCO obtained from the *sylvester monoid* [LR98, HNT05].

We further observe that the construction transports the lattice structure on monoids to a lattice structure on those Hopf algebras (Theorem 2). This structure was used implicitely by GIRAUDO for constructing the Baxter Hopf algebra from the Baxter monoid as the infimum of the *sylvester monoid* and its image under SCHÜTZENBERGER involution. The supremum of those two monoids is known as the *hypoplactic monoid* which gives the algebra of *quasi symmetric functions* [Nov00].

As an application (Section 5) we take the supremum of the *sylvester monoid* and the *stalactic monoid* of [HNT08a]. The result is a monoid on binary search trees with multiplicities leading to a Hopf algebra on binary trees with multiplicities. Interestingly, there is a *hook length formula* for those trees (Theorem 3) and we prove it using the Hopf algebra as generating series.

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2 Background

In this section, we introduce some notations and three specific maps from words to words: *standardization*, *packing*, and *parkization*. These will be the main tool for polynomial realizations of Hopf algebras.

2.1 Lattice structure on Congruences

The free monoid $\mathfrak{A}^*$ on an alphabet $\mathfrak{A}$ is the set of words with concatenation as multiplication. We denote by 1 the empty word. Recall that a *monoid congruence* is an equivalence relation $\equiv$ which is left and right compatible with the product; in other words, for any monoid elements a, b, c, d, if $a \equiv b$ and $c \equiv d$ then $ac \equiv bd$. Starting with two congruences on can build two new congruences:

- the *union* $\sim \vee \approx$ of $\sim$ and $\approx$ is the transitive closure of the union $\sim$ and $\approx$; that is $u \equiv v$ if there exists $u = u_0, \ldots, u_k = v$ such that for any i, $u_i \sim u_{i+1}$ or $u_i \approx u_{i+1}$. It is the smallest congruence containing both $\sim$ and $\approx$;
- the *intersection* $\sim \wedge \approx$ of $\sim$ and $\approx$ is defined as the relation $\equiv$ with $u \equiv v$ if $u \sim v$ and $u \approx v$.

2.2 Some φ-maps

Throughout this paper we construct Hopf algebras from the equivalence classes of words given by the fibers of some map φ from the free monoid to itself. Our main examples are *standardization* and *packing* functions which can be defined for any totally ordered alphabet $\mathfrak{A}$. We could easily extend these following properties to *parkization* [NT04, NT07] *if the alphabet* $\mathfrak{A}$ *is well-ordered* (any element has a successor).

In the following, *we suppose that* $\mathfrak{A}$ *is an totally ordered infinite alphabet*. Most of the time we use $\mathfrak{A} = \mathbb{N}^{>0}$ for simplicity. For w in $\mathfrak{A}^*$, we denote by $part(w)$ the ordered set partition of positions of w

letters obtained as follows: for each letter $l \in \mathfrak{A}$ appearing in w, there is a part containing the positions of each occurrence of l in w; the parts are ordered using the order on the alphabet $\mathfrak{A}$. For example: $part(13231) = [\{1,5\},\{3\},\{2,4\}]$ and $part(1112) = [\{1,2,3\},\{4\}]$.

Standardization std computes the lexicographically smallest word w which has same length and same set of inversions. This map is used in the realization of the Hopf algebra **FQSym** of permutations [DHT02, MR95]. The image $std(\mathfrak{A}^*)$ is identified with the set $\mathfrak{S}$ of all permutations.

Algorithm 1: Standardization std

Data: $w = (a_1, \ldots, a_k)$ a word of $\mathfrak{A}^k$
Result: $\sigma = (\sigma_1, \ldots, \sigma_k) \in \mathfrak{S}_k \subseteq \mathfrak{A}^k$
$osp \leftarrow part(w)$; $i \leftarrow 1$
forall the $set \in osp$ **do**
  forall the $p \in set$ **do**
    $\sigma_p \leftarrow i$
    increment(i)
return σ

Some examples:

w	$std(w)$
$(7,2,14,3,7)$	$(3,1,5,2,4)$
$(23,14,5,92)$	$(3,2,1,4)$
$(4,2,1,3,5)$	$(4,2,1,3,5)$
$(1,5,1,1,5,5)$	$(1,4,2,3,5,6)$

Packing $pack$ computes the lexicographically smallest word w which has same ordered set partitions. This map is used in the realization of the Hopf algebra **WQSym** of ordered set partition or packed words [Hiv99]. We identify $tass(\mathfrak{A}^*)$ with the collection of ordered set partitions.

Algorithm 2: Packing $pack$

Data: $w = (a_1, \ldots, a_k)$ a word of $\mathfrak{A}^*$
Result: $c = (c_1, \ldots, c_k)$
$osp \leftarrow part(w)$; $i \leftarrow 1$
forall the $set \in osp$ **do**
  forall the $p \in set$ **do** $c_p \leftarrow i$
  increment(i)
return c

Some examples:

w	$tass(w)$
$(3,13,3,2,13)$	$(2,3,2,1,3)$
$(2,2,2,5,8,2)$	$(1,1,1,2,3,1)$
$(4,2,1,3,5)$	$(4,2,1,3,5)$
$(2,3,1,1,2)$	$(2,3,1,1,2)$

Those maps are used to realize some Hopf algebras like **FQSym**, **WQSym**, or **PQSym**. For each such map φ we say that a word w is *canonical* if $\varphi(w) = w$. For example, 1423 is canonical for std and 1121 is canonical for $pack$. The set of canonical words for the *standardization* function is the set of permutations set; for the *packing* function it is the set of *packed words*. The set $\varphi(\mathfrak{A}^*)$ of canonical words is denoted by can_φ. We call these maps the φ*-maps*.

3 Polynomial realizations and Hopf algebras

In this section we describe how, from a φ-map, one can construct a Hopf algebra such as **FQSym**, **WQSym**, or **PQSym**, using two tricks: *polynomial realization* and *alphabet doubling*. *Polynomial realizations* are a powerful trick to construct algebras as sub-algebras of a free algebra by manipulating some polynoms having certain symmetries. Futhermore the *alphabet doubling trick* defines a graded algebra morphism on a free algebra which endows it with a compatible coproduct, that is a Hopf algebra structure.

3.1 φ-polynomial realization

The notion of polynomial realizations has been introduced and implicitly used in many articles of the "*phalanstère de Marne-la-Vallée*" (France). See *e.g.* [DHT02, NT06a, HNT08a]. In the following, we call *alphabet* $\mathfrak{A}$ an infinite and totally ordered (when appropriate, we assume furthermore that the total order admits a successor function) set of symbols all of which are of weight 1. By an abuse of language, we call the free algebra the graded algebra infinite **but finite degree** sum of words.

Definition 1 (Polynomial realization): *Let $\mathcal{A} := \oplus_{n\geqslant 0}\mathcal{A}_n$ be a graded algebra. A* polynomial realization *r of $\mathcal{A}$ is a map which associates to each alphabet $\mathfrak{A}$ an injective graded algebra morphism $r_{\mathfrak{A}}$ from $\mathcal{A}$ to the free non-commutative algebra $\mathbb{K}\langle\mathfrak{A}\rangle$ such that, if $\mathfrak{A}\subset\mathfrak{B}$, then for all $x\in\mathcal{A}$ one has $r_{\mathfrak{A}}(x) = r_{\mathfrak{B}}(x)/_{\mathfrak{A}}$, where $r_{\mathfrak{B}}(x)/_{\mathfrak{A}}$ is the sub linear combination obtained from $r_{\mathfrak{B}}(x)$ by keeping only those words in $\mathfrak{A}^*$.*

When the realization is clear from the context we write $\mathcal{A}(\mathfrak{A}) := r_{\mathfrak{A}}(\mathcal{A})$ for short.

For a given φ, we consider the subspace $\mathcal{A}_\varphi$ admitting the basis $(m_u)_{u\in can_\varphi}$ defined on $\mathcal{A}_\varphi(\mathfrak{A})$:

$$r_{\mathfrak{A},\varphi}(m_u) = \sum_{w\in\mathfrak{A}^*;\varphi(w)=u} w\,. \tag{1}$$

The result does not depend on the alphabet. For $\varphi = std$, $pack$ or $park$ the linear span of $(m_u)_{u\in can_\varphi}$ is a sub-algebra of $\mathbb{K}\langle\mathfrak{A}\rangle$.

Example 1 (Realization of FQSym): If $\varphi = std$ then can_φ is in fact the set of permutations and $\mathfrak{A}_\varphi$ is the permutations Hopf algebra **FQSym** [DHT02, MR95]. It is realized by the std-polynomial realization in $\mathbb{K}\langle\mathfrak{A}\rangle$: let $\mathbb{G}_\sigma(\mathfrak{A}) := r_{\mathfrak{A},std}(\mathbb{G}_\sigma)$ such that, for example

$$\mathbb{G}_{132}(\mathbb{N}^*) = 121 + 131 + 132 + 141 + 142 + 143 + \cdots + 242 + 243 + \cdots\,.$$

The realization is an algebra morphism: $\mathbb{G}_\sigma(\mathfrak{A})\cdot\mathbb{G}_\mu(\mathfrak{A}) = r_{\mathfrak{A},std}(\mathbb{G}_\sigma\times\mathbb{G}_\mu)$ where "$\cdot$" is the classical concatenation product on words in the free algebra. For example,

$$\mathbb{G}_{213}\times\mathbb{G}_1 \quad=\quad \mathbb{G}_{2134}+\mathbb{G}_{2143}+\mathbb{G}_{3142}+\mathbb{G}_{3241}$$

which is equivalent to

$$\begin{aligned} r_{std,\mathbb{N}^*}(\mathbb{G}_{213}\times\mathbb{G}_1) &= \mathbb{G}_{213}(\mathbb{N}^*)\cdot\mathbb{G}_1(\mathbb{N}^*) = (212+213+214+\cdots)\cdot(\mathbf{1}+\mathbf{2}+\mathbf{3}+\mathbf{4}+\cdots) \\ &= 212\mathbf{1}+212\mathbf{2}+212\mathbf{3}+\cdots 213\mathbf{1}+213\mathbf{2}+213\mathbf{3}+\cdots+324\mathbf{1}+\cdots \end{aligned}$$

Proposition 1: *If $span((m_u)_{u\in can_\varphi})$ is stable under the product $\times$ then it is given by:*

$$m_u\times m_v = \sum_{\substack{w:=u'v'\in can_\varphi\\ \varphi(u')=u;\varphi(v')=v}} m_w\,. \tag{2}$$

Remark 1: Let $\mathfrak{A},\mathfrak{B}$ be two totally ordered alphabets such that any element in $\mathfrak{A}$ is strictly smaller than any element of $\mathfrak{B}$. By definition we have the following isomorphisms, where $\sqcup$ denotes the disjoint union:

$$\mathcal{A}\simeq\mathcal{A}(\mathfrak{A})\simeq\mathcal{A}(\mathfrak{B})\simeq\mathcal{A}(\mathfrak{A}\sqcup\mathfrak{B})\,. \tag{3}$$

3.2 Alphabet doubling trick

The *alphabet doubling trick* [DHT02, Hiv07] is a way to define coproducts. We consider the algebra $\mathbb{K}\langle\mathfrak{A}\sqcup\mathfrak{B}\rangle$ generated by two (infinite and totally ordered) alphabets $\mathfrak{A}$ and $\mathfrak{B}$ such that the letters of $\mathfrak{A}$ are strictly smaller than the letters of $\mathfrak{B}$. The relation $\rightleftarrows$ make the letters of $\mathfrak{A}$ commute with those of $\mathfrak{B}$. One identifies $\mathbb{K}\langle\mathfrak{A}\sqcup\mathfrak{B}\rangle/_{\rightleftarrows}$ with the algebra $\mathbb{K}\langle\mathfrak{A}\rangle\otimes\mathbb{K}\langle\mathfrak{B}\rangle$. We follow here the abuse of language allowing infinite but finite degree sum. We denote by $r_{\mathfrak{A}\sqcup\mathfrak{B}}(x)/_{\rightleftarrows}$ the image of $r_{\mathfrak{A}\sqcup\mathfrak{B}}(x)$ given by the canonical map from $\mathbb{K}\langle\mathfrak{A}\sqcup\mathfrak{B}\rangle$ to $\mathbb{K}\langle\mathfrak{A}\sqcup\mathfrak{B}\rangle/_{\rightleftarrows}$. The map $x\mapsto r_{\mathfrak{A}\sqcup\mathfrak{B}}(x)/_{\rightleftarrows}$ is always an algebra morphism from $\mathcal{A}$ to $\mathbb{K}\langle\mathfrak{A}\rangle\otimes\mathbb{K}\langle\mathfrak{B}\rangle$. Whenever its image is included in $\mathcal{A}(\mathfrak{A})\otimes\mathcal{A}(\mathfrak{B})$ this defines a coproduct on $\mathcal{A}$.

Definition 2 (Hopf polynomial realization): *A* Hopf polynomial realization r *of* $\mathcal{H}$ *is a polynomial realization such that for all* x*:*

$$r_{\mathfrak{A}\sqcup\mathfrak{B}}(x)/_{\rightleftarrows}=(r_{\mathfrak{A}}\otimes r_{\mathfrak{B}})(\Delta(x))\ . \tag{4}$$

Example 2 (Coproduct in FQSym): We denote by $\mathbb{G}_\sigma(\mathfrak{A}\sqcup\mathfrak{B})$ the std-polynomial realization of the **FQSym** element indexed by σ in the algebra $\mathbb{K}\langle\mathfrak{A}\sqcup\mathfrak{B}\rangle/_{\rightleftarrows}$. Also we denote by $1,2,3,\cdots$ the symbols of $\mathfrak{A}$ and in bold red $\mathbf{1},\mathbf{2},\mathbf{3},\boldsymbol{\cdots}$ the symbols of $\mathfrak{B}$ ordered with $1<2<3<\cdots<\mathbf{1}<\mathbf{2}<\mathbf{3}<\cdots$. Then,

$$\begin{aligned}\mathbb{G}_{132}(\mathfrak{A}\sqcup\mathfrak{B}) &= 121+131+132+\cdots+1\mathbf{1}1+1\mathbf{12}+\cdots+1\mathbf{21}+1\mathbf{31}+\cdots+\mathbf{121}+\cdots\\ &= 121+131+\cdots+11{\cdot}\mathbf{1}+11{\cdot}\mathbf{2}+\cdots+1{\cdot}\mathbf{21}+1{\cdot}\mathbf{31}+\cdots+\mathbf{123}+\mathbf{132}+\cdots\\ &= \Delta(\mathbb{G}_{132})=1\otimes\mathbb{G}_{132}+\mathbb{G}_1\otimes\mathbb{G}_{21}+\mathbb{G}_{12}\otimes\mathbb{G}_1+\mathbb{G}_{132}\otimes 1\ .\end{aligned}$$

3.3 Good Hopf algebras

We call a *Hopf algebra* $\mathcal{H}_\varphi$ *good* if it is defined by a Hopf polynomial realization r_φ. We call a *function* φ *good* if it produces a *good Hopf algebra* $\mathcal{H}_\varphi$. Currently, we know three main good Hopf algebras: **FQSym**, **WQSym** and **PQSym** are respectivly associated to the *standardization*, *packing* and *parkization* functions.

4 Good monoids

In the previous section (Section 3), we realized some Hopf algebras in free algebras. In this section, we give sufficient conditions on a congruence $\equiv$ to build a combinatorial quotient of a *good Hopf algebra*. We call a *monoid good* if it statisfies these conditions. We give a sufficient compatibility between φ and $\equiv$ to ensure the product is carried to the quotient. The second condition ensures that the *alphabet doubling trick* map. It is used to project the coproduct in the quotient. Under these conditions, a monoids is guaranted to produce a Hopf algebra quotient (Theorem 1). Furthermore, these conditions on monoid are preserved under taking infimum and supremum (Theorem 2).

4.1 Definition

The notion of *Good monoids* has been introduced by Hivert-Nzeutchap [HN07] to build quotients (sub-algebras) of **FQSym**. We could also mention PhD thesis.

A *good monoid* is a monoid which is similar to the *plactic monoid* [LS81, Knu73]. We consider a free monoid $\mathfrak{A}^*$ with concatenation product "$\cdot$", a congruence $\equiv$ on $\mathfrak{A}^*$ and a map $\varphi:\mathfrak{A}^*\to\mathfrak{A}^*$. We define the *evaluation* $ev(w)$ of a word w as its number of occurrences of each letter of w. For example, the words `ejajv` and `jjaev` have the same evaluation: both have one `a`, one `e`, one `v` and two `j`. The free monoid $\mathfrak{A}^*/\equiv$ is a φ*-good monoid* if it has the following properties:

Definition 3 (φ-congruence): *The congruence* $\equiv$ *is a* φ-congruence *if for all* $u,v\in\mathfrak{A}^*$*,* $u\equiv v$ *if and only if* $\varphi(u)\equiv\varphi(v)$ *and* $ev(u)=ev(v)$*.*

This first compatibility is sufficient to build a quotient algebra of $\mathcal{A}_\varphi$.

Definition 4 (Compatibility with restriction to alphabet intervals): *The congruence $\equiv$ is compatible with the* restriction to alphabet intervals *if, for all $u, v \in \mathfrak{A}^*$ such that $u \equiv v$ one has $u_{|I} \equiv v_{|I}$ for any I interval of $\mathfrak{A}$, where $w_{|\mathfrak{A}}$ is word restricted to the alphabet $\mathfrak{A}$.*

This second compatibility in association with the first ensures that *alphabet doubling trick* defines a quotient coproduct. Both compatibilities give us an extended definition of a HIVERT-NZEUTCHAP's *good monoid* which one is defined only with φ the standardization map:

Definition 5 (φ-good monoid): *A quotient $\mathfrak{A}^*/\equiv$ of the free monoid is a φ-*good monoid *if $\equiv$ is a φ-congruence and is compatible with restriction to alphabet intervals. We call such a congruence a φ-*good congruence.

In the following examples, we denote words of $\mathfrak{A}^*$ by u, v, w and the letters by a, b, c.

Example 3 (sylvester and stalactic monoids): The sylvester congruence: $\equiv_{sylv}$, defined by

$$u \cdot ac \cdot w \cdot b \cdot v \equiv_{sylv} u \cdot ca \cdot w \cdot b \cdot v \text{ whenever } a \leqslant b < c \,, \tag{5}$$

is std-compatible and compatible with the restriction to alphabet intervals. Thanks to the binary search tree insertion algorithm the equivalence classes are in natural bijection with binary search trees. The quotient monoid is a monoid on binary search trees called the sylvester monoid in [HNT05].

The stalactic congruence [HNT08a]: $\equiv_{stal}$, defined by

$$u \cdot ba \cdot v \cdot b \cdot w \equiv_{stal} u \cdot ab \cdot v \cdot b \cdot w \,, \tag{6}$$

is compatible with *packing* but not with *standardization*. The quotient monoid is the *stalactic monoid*. It is clear that any stalactic class contains a word of the form $a_1^{m_1} a_2^{m_2} \dots a_k^{m_k}$, where the a_i are distinct. We call these words canonical.

$$515431511145312455 \equiv_{stal} 3^2 1^5 2^1 4^3 5^6$$

4.2 Hopf algebra quotient

These differents *good monoids* tools was used to (re-)define several Hopf algebra quotients: **FSym** the Free Symmetric functions Hopf algebra [DHT02], **PBT** [LR98, HNT05] or Baxter Hopf algebra [Gir11a, Gir12]; the Hopf algebra associated with the stalactic monoid [HNT08a]; or **CQSym** [NT04, NT07] (a **PQSym** quotient).

Lemma 1 (Algebra quotient): *Let $\mathcal{H}_\varphi$ be a* good Hopf algebra *and $\equiv$ be a φ-good congruence such that its free monoid quotient is a φ-*good monoid*. Then, the quotient $\mathcal{H}_\varphi/_\equiv$ is an algebra quotient whose bases are indexed by $can_\varphi/_\equiv$, identifying basis elements m_u and m_v whenever $u \equiv v$.*

Example 4 (PBT and Hopf algebra stalactic): We go back to Example 3. The sylvester quotient of **FQSym** is the Hopf algebra **PBT** [LR98, HNT05].

The stalactic monoid gives a quotient of **WQSym**. Let π be the projection of **WQSym** in **WQSym**$/_{\equiv_{stal}}$ and $u := 112$ and $v := 11$ two (packed) words. We denote by π the projection of $\mathbb{M}_u$ by $\mathbb{Q}_s$, with s the planar diagram associated to the stalactic class of u.

$$\begin{array}{rcl} \pi(\mathbb{M}_{112} \times \mathbb{M}_{11}) & = & \pi(\mathbb{M}_{11211} + \mathbb{M}_{11222} + \mathbb{M}_{11233} + \mathbb{M}_{11322} + \mathbb{M}_{22311}) \\ = \mathbb{Q}_{1^22} \times \mathbb{Q}_{1^2} & = & \mathbb{Q}_{21^4} + \mathbb{Q}_{1^22^3} + \mathbb{Q}_{1^223^2} + \mathbb{Q}_{1^232^2} + \mathbb{Q}_{2^231^2} \end{array}$$

Lemma 2 (Coalgebra quotient): *The quotient $\mathcal{H}/_\equiv$ is a coalgebra quotient.*

SKETCH OF THE PROOF: The relation $\equiv$ is compatible with the restriction to alphabet intervals, hence the alphabet doubling trick ensures that coproduct projects to the quotient. ∎

Example 5:

$$\begin{aligned}\pi(\Delta(\mathbb{M}_{332122})) &= \pi(1\otimes\mathbb{M}_{332122}+\mathbb{M}_1\otimes\mathbb{M}_{22111}+\mathbb{M}_{2122}\otimes\mathbb{M}_{11}+\mathbb{M}_{332122}\otimes 1)\\ =\Delta(\mathbb{Q}_{3^21 2^3}) &= 1\otimes\mathbb{Q}_{3^21 2^3}+\mathbb{Q}_1\otimes\mathbb{Q}_{2^21^3}+\mathbb{Q}_{12^3}\otimes\mathbb{Q}_{1^2}+\mathbb{Q}_{3^21 2^3}\otimes 1\end{aligned}$$

Theorem 1 (Good monoid and good Hopf algebra): *Let $\mathcal{H}_\varphi$ be a good Hopf algebra and $\equiv$ be a φ-good congruence. The quotient $\mathcal{H}/_\equiv$ is a Hopf algebra quotient.*

Corollary 1: *The dual Hopf algebra $(\mathcal{H}/_\equiv)^\#$ is a sub-algebra of the dual Hopf algebra $\mathcal{H}^\#$, with basis given by:*

$$\overline{M}^\#_{U\in can_\varphi/_\equiv}=\textstyle\sum_{u\in U} m^\#_u\ . \tag{7}$$

4.3 Operations

Previously we introduced some *good functions* φ: std, $pack$ (and $park$). It is interesting to investigate the connections between them:

Definition 6 (refinement): *Let φ and π be two functions. We say that π refines φ, written $\varphi\prec\pi$ if $\varphi(\pi(u))=\varphi(u)$ for all $u\in\mathfrak{A}^*$.*

It is clear that *refinement* is an order.

Proposition 2 (std, $tass$, $park$ and refinement): *For these three functions:* standardization std, packing $pack$ *and* parking $park$ *we have the relation: $std\prec pack\prec park$.*

Proposition 3 (Good functions and refinement): *Let φ and π be two* good functions *such that $\varphi\prec\pi$. Then any* φ-good monoid *is a* π-good monoid.

Propositions 2 and 3 give us, for example, that any std-good monoid is $pack$-good. Furthermore operations on two *good congruences* give *good congruences*.

Theorem 2 ($\vee$, $\wedge$ and good congruences): *The union and intersection of two* φ-good congruences $\sim$ *and* $\approx$ *are* φ-good congruences.

As an intriguing consequence the lattice structure on monoids is transported to Hopf algebras. Several examples of this are know.

Example 6: The intersection ($\equiv_{sylv}\ \wedge\ \equiv_{\#sylv}$) of the sylvester relation (5) and its image under the SCHÜTZENBERGER involution gives std-good monoid: the *Baxter monoid* [Gir11a, Gir11b].

The union ($\equiv_{sylv}\vee\equiv_{\#sylv}$) of those relations gives the *hypoplactic monoid* [Nov00].

In the sequel, we study in detail another example.

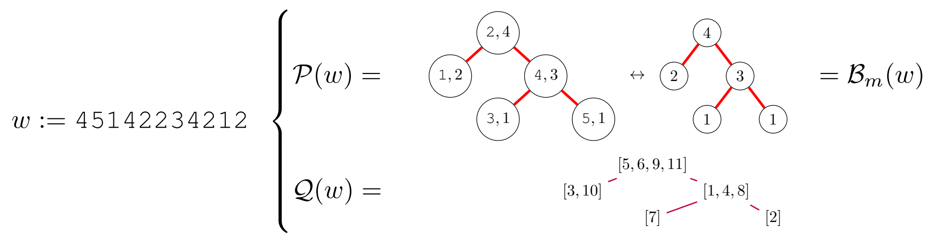


Figure 1: We start by considering the packed word `45142234212`, and insert it in a BSTM by the algorithm $\mathcal{P}$; that give us $\mathcal{P}(\texttt{45142234212})$ above in the middle. On the right, there is the BTm ($\mathcal{B}_m(w)$) associated with the BSTM ($\mathcal{P}(w)$) of **WQSym**$/_{\equiv_t}$. At the top of the figure there is the P-symbol given by $\mathcal{P}$ or $\mathcal{B}_m$ and below the Q-symbol is given by $\mathcal{Q}$.

5 The union of the sylvester and the stalactic congruences

As an application of the preceding construction, we consider the union ($\equiv_{sylv} \vee \equiv_{stal}$) of the sylvester congruence (5) and the stalactic congruence (6); we call it the *taïga relation* $\equiv_t$,

$$\begin{aligned} u \cdot ac \cdot v \cdot b \cdot w &\equiv_t u \cdot ca \cdot v \cdot b \cdot w \quad \text{for } a \leqslant b < c\,, \\ u \cdot ba \cdot v \cdot b \cdot w &\equiv_t u \cdot ab \cdot v \cdot b \cdot w \end{aligned} \tag{8}$$

From Proposition 3 we know that the sylvester congruence (5) is a *pack-good congruence* and from Theorem 2 we deduce that the *taïga monoid* is a *pack-good monoid*.

5.1 Algorithm and taïga monoid

The taïga congruence can be calculated using an insertion algorithm similar to the binary search tree insertion (see Algorithm 3 for a definition). This insertion algorithm uses a search tree structure:

Definition 7 (Binary search tree with multiplicity): *A (planar)* binary search tree with multiplicity *(BSTM) is a binary tree T where each node is labelled by a letter l and a non-negative integer k, called the multiplicity, so that T is a binary search tree if we drop the multiplicities and such that each letter appears at most once in T. We denote by (l, k) a node label and for any node n, by $l(n)$ its letter and by $m(n)$ its multiplicity.*

We denote by $\mathcal{P}(w)$ the result of the insertion using Algorithm 3 of w from *the right to the left* in the empty tree (*cf.* the left part of the figure 1).

Proposition 4: *The taïga classes are the fibers of $\mathcal{P}$. That is for u and v two words: $u \equiv_t v$ if and only if $\mathcal{P}(u) = \mathcal{P}(v)$.*

The Q-symbol of w is the tree $\mathcal{Q}(w)$ of same shape as $\mathcal{P}(w)$ which records the positions of each inserted letter. This gives us a ROBINSON-SCHENSTED *like correspondance* [LS81] (*cf.* Figure 1). As a corollary of Theorem 2 we get the taïga monoid is a *tass-good monoid*.

5.2 Quotient of *WQSym*: *PBTm*

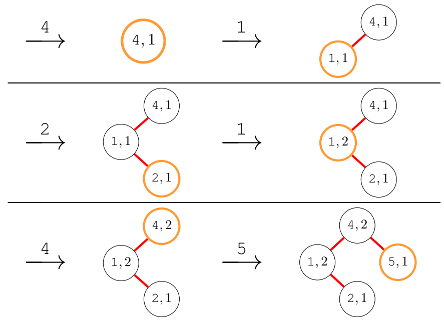
As in [HNT05], we consider a binary trees with multiplicities without letters.

Algorithm 3: insertion in a BSTM

Data: t a BSTM with L_t and R_t its left and right subtrees, and l a letter of $\mathfrak{A}$
Result: t with l inserted

if *t is empty tree* **then**
  $t \leftarrow$ node labelled by $(l, 1)$
else
  if $l(t) = l$ **then**
    increment $m(t)$
  else
    if $l(t) < l$ **then** insert recursively l in L_t
    else insert recursively l in R_t
return t

Insertion of word `541214` from *the right to the left* in the empty tree :



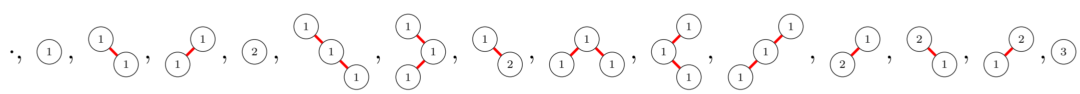
Definition 8 (BTM): *A* binary tree with multiplicities *(BTM) is a (planar) binary tree labelled by non-negative integers on its nodes. The size of a BTM T denoted by $|T|$ is the sum of the multiplicities.*

Let T_w be a BSTM associated to a packed word w, and T be the BTM obtained by removing its letters. One can recover uniquely T_w from T: indeed each letter of T_w is deduced by a left infix reading of T. We identify the set of words in $pack(\mathfrak{A}^*)/_{\equiv_t}$ of size k (for $k \geqslant 0$) with the set of BTM of size k. We denote $\mathcal{B}_m$ the algorithm which computes the BTM associated to the BSTM computed by $\mathcal{P}$ (*cf.* Figure 1).

Let us denote by $S(t)$ the generating series of these trees counted by size. The generating serie statisfies the following functional equation $S(t) = 1 + S(t)^2(1-t)^{-1}$ (see `A002212` of OEIS):

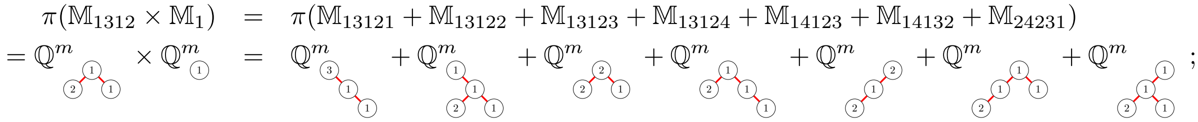
$$S(t) \;=\; \frac{1-t-\sqrt{5t^2-6t+1}}{2t} \;=\; 1 + t + 3t^2 + 10t^3 + 36t^4 + 137t^5 + 543t^6 + 2219t^7 + \dots \qquad (9)$$

This structure is in bijection with *binary unary tree* structure. Here is the list of trees of size 0,1,2 and 3:

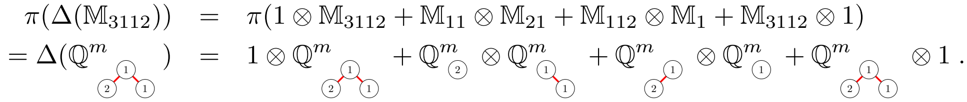


With Lemma 1 and Theorem 1 we know that the quotient of $\mathbf{WQSym}(\mathfrak{A})$ by the taïga relations has a natural basis indexed by $tass(\mathfrak{A}^*)/_{\equiv_t}$ identified by BTM. We call **PBTm** (planar binary tree with multiplicities) that quotient. More precisely, we consider the basis $(\mathbb{M}_u)_u$ of **WQSym** obtained by the Hopf polynomial realization r_{tass}. We denote by $(\mathbb{Q}^m_t)_t$ the canonical projection by the map π of $(\mathbb{M}_u)_u$ in **PBTm** such that $\pi(\mathbb{M}_u) := \mathbb{Q}^m_t$ if $t = \mathcal{B}_m(u)$. The product and coproduct are given by some explicit algorithms. For brevity, we only give here some examples:

$$\pi(\mathbb{M}_{1312} \times \mathbb{M}_1) \;=\; \pi(\mathbb{M}_{13121} + \mathbb{M}_{13122} + \mathbb{M}_{13123} + \mathbb{M}_{13124} + \mathbb{M}_{14123} + \mathbb{M}_{14132} + \mathbb{M}_{24231})$$



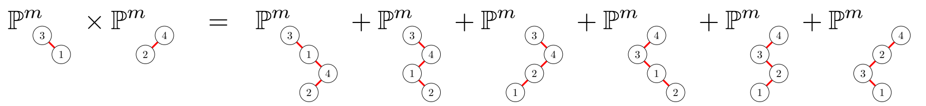
$$\pi(\Delta(\mathbb{M}_{3112})) \;=\; \pi(1 \otimes \mathbb{M}_{3112} + \mathbb{M}_{11} \otimes \mathbb{M}_{21} + \mathbb{M}_{112} \otimes \mathbb{M}_1 + \mathbb{M}_{3112} \otimes 1)$$



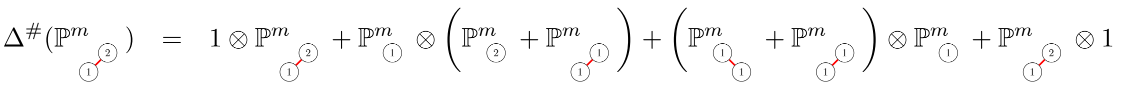
We consider $\mathbf{PBTm}^{\#}$ the dual of **PBTm**. This is a sub-algebra of $\mathbf{WQSym}^{\#}$. We denote by $(\mathbb{P}_t^m)_t := (\mathbb{Q}_t^m)^{\#}$ its dual basis: $\langle \mathbb{Q}_t^m, \mathbb{P}_{t'}^m \rangle = \delta_{t,t'}$. The product is given by:

$$\mathbb{P}_{t'}^m \times \mathbb{P}_{t''}^m \quad = \quad \textstyle\sum_t \langle \Delta(\mathbb{Q}_t^m), \mathbb{P}_{t'}^m \times \mathbb{P}_{t''}^m \rangle \mathbb{P}_t^m \ . \tag{10}$$

Here is an example,



If we consider only shape tree, the product is exactly the product of $(\mathbb{P}_t)_t$ basis in **PBT** [HNT05]. Hence this product is a shifted shuffle on trees. The coproduct is given by: $\Delta^{\#}(\mathbb{P}_t^m) = \sum_{t',t''} \langle \mathbb{Q}_{t'}^m \times \mathbb{Q}_{t''}^m, \mathbb{P}_t^m \rangle \, \mathbb{P}_{t'}^m \otimes \mathbb{P}_{t''}^m$. Here is an example:



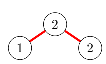
6 The hook length formula

Its well known from [Knu73] (§5.14 ex. 20) that the number of decreasing labelling of a binary tree is given by a simple product formula. [HNT05] remarks that this is also the number of permutations given upon a tree by the binary search tree insertion. In this section we generalize this formula for trees with multiplicities.

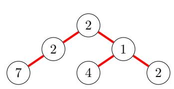
Proposition 5: *The cardinal $f(T)$ of the taïga class associated to T (i.e. the set of packed words giving the tree T by the insertion algorithm $\mathcal{B}_m$) is given by*

$$f(T) \quad = \quad |T|! \left(\textstyle\prod_{t \in T} |t| \, (m(t) - 1)! \right)^{-1} \ . \tag{11}$$

where t ranges throwgh all the subtrees of T and $|T|$ denotes the size of T (the sum of the multiplicities).

Example 7: The taïga class of $T :=$  contains 12 packed words w:

$$23132, 33122, 31232, 32312, 13232, 33212, 23312, 32132, 21332, 31322, 12332, 13322 \ .$$

The class of  contains $23,337,600 = \frac{18!}{(18\cdot 9\cdot 7\cdot 7\cdot 4\cdot 2)(1!1!6!0!3!1!)}$ packed words.

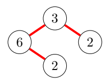
This formula is easily proven by induction. However, we prefer to give a generating series proof as in [HNT08b]. Let $\mathcal{A}$ be an associative algebra, and consider the functional equation for power series $x \in \mathcal{A}[[z]]$:

$$x \quad = \quad a + \textstyle\sum_{k \geqslant 1} B_k(x, x) \ , \tag{12}$$

where $a \in \mathcal{A}$ and for any $k > 0$, $B_k(x, y)$ is a bilinear map with values in $\mathcal{A}[[z]]$. We suppose such that the valuation of $B_k(x, y)$ is strictly greater than the sum of the valuations of x and y (plus k). Then

Equation 12 has a unique solution:

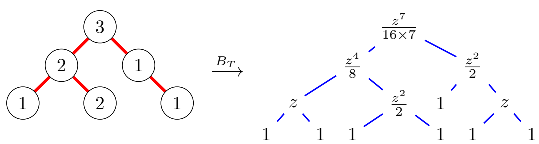
$$\begin{aligned} x &= a + \textstyle\sum_{k\geqslant 1}\Big(B_k(a,a) + B_k(a, \sum_{k'\geqslant 1} B_{k'}(a,a)) + B_k(\sum_{k'\geqslant 1} B_{k'}(a,a)), a) + \dots\Big) \quad (13)\\ &= \textstyle\sum_{T\in \mathbf{BTM}} B_T(a)\ , \end{aligned}$$

where for a tree T, $B_T(a)$ is the result of evaluating the expression formed by labelling by a the leaves of the complete tree associated to T and by B_k its internal node labelled by k.
For example: B  $(a) = B_3(B_6(a, B_2(a,a)), B_2(a,a))$.

So if we try to solve the fixed point problem:

$$x = 1 + \int_0^z e^s x(s)^2 ds = 1 + \textstyle\sum_{k\geqslant 1} \int_0^z \frac{s^{k-1}}{(k-1)!} x(s)^2 ds \;\; = \;\; 1 + \sum_{k\geqslant 1} B_k(x,x)\ , \qquad (14)$$

where $B_k(x,y) = \int_0^z \frac{s^{k-1}}{(k-1)!} x(s)y(s)ds$. Then for a binary tree of non-negative integer T, $B_T(1)$ is the monomial obtained by putting 1 on each leaf and integrating at each node n the product of the evaluations of its subtrees and $s^k/k!$ with $m(n) = k+1$.

For example:  .

One can observe that $B_T(1) = f(T)\frac{z^n}{n!}$, where $n = |T|$.

To prove the *hook length formula*, following the same technique as in [HNT08b], we want to lift in $\mathbf{WQSym}^{\#}$ the fixed point computation of Equation 14. From the multiplication rule [Hiv99] of the dual basis $\mathbb{S}_u$ ($\mathbb{M}_u^{\#} := \mathbb{S}_u$), one easily sees that the linear map $\phi : \mathbb{S}_u \mapsto \frac{z^n}{n!}$ with n the length of u is a morphism of algebras from $\mathbf{WQSym}^{\#}$ to $\mathbb{K}[[z]]$. For u, v two packed words of respective size $n-1$ and m, set $B_k(\mathbb{S}_u, \mathbb{S}_v) := \sum_{w\in(u\overline{\sqcup\!\sqcup}1^{k-1}\overline{\sqcup\!\sqcup}v)\cdot n} \mathbb{S}_w$. The crucial observation which allows to express the *hook length formula* in a generating series way is the following theorem:

Theorem 3: *For any binary tree with multiplicities $B_T(1) = \sum_{\mathcal{B}_m(u)=T} \mathbb{S}_u$.*

*In particular, $B_T(1)$ coincide with $\mathbb{P}_T^m$, the natural basis of **PBTm**$^{\#}$.*

Corollary 2: *The number of packed words u such that $\mathcal{B}_m(u) = T$ is computed by $f(T)$.*

7 Conclusion, work in progress and perspectives

In this paper, we unraveled some new combinatorics on binary trees with multiplicities from the union of the *sylvester* and *stalactic* monoids. Using the machinery of *realizations*, we built a Hopf algebra on those trees, allowing us to give a generating series proof of a new *hook length formula*. Following [HNT08b], it is very likely that we will also be able to prove a *q-hook length formula*. On the other hand, the usual case of the LODAY-RONCO algebra has a lot of nice properties. For example, the product and coproduct can be expressed by the means of an order on the trees called the *Tamari Lattice* [LR98]. It would be good to know if such a lattice exists for trees with multiplicities. This should also relate to N. READING work on lattice congruences [Rea05]. Also it could be interesting to study some other combinations in the lattice of *good monoids*. For example, the union of the *plactic monoid* and the *stalactic monoid* should give a Hopf algebra of *tableaux* with multiplicties. Finally, in our construction, it seems that std, $tass$ and $park$ play some canonical role from which everything else is built. Are there some more examples? Is there a definition for such a φ-map? Could we except to always have a *hook formula* as soon as we have a *good monoid*?

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Top degree coefficients of the Denumerant

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For a given sequence $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_N, \alpha_{N+1}]$ of $N+1$ positive integers, we consider the combinatorial function $E(\boldsymbol{\alpha})(t)$ that counts the nonnegative integer solutions of the equation $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_N x_N + \alpha_{N+1} x_{N+1} = t$, where the right-hand side t is a varying nonnegative integer. It is well-known that $E(\boldsymbol{\alpha})(t)$ is a quasipolynomial function of t of degree N. In combinatorial number theory this function is known as the *denumerant*. Our main result is a new algorithm that, for every fixed number k, computes in polynomial time the highest $k+1$ coefficients of the quasi-polynomial $E(\boldsymbol{\alpha})(t)$ as step polynomials of t. Our algorithm is a consequence of a nice poset structure on the poles of the associated rational generating function for $E(\boldsymbol{\alpha})(t)$ and the geometric reinterpretation of some rational generating functions in terms of lattice points in polyhedral cones. Experiments using a `MAPLE` implementation will be posted separately.

Considérons une liste $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_{N+1}]$ de $N+1$ entiers positifs. Le dénumérant $E(\boldsymbol{\alpha})(t)$ est la fonction qui compte le nombre de solutions en entiers positifs ou nuls de l'équation $\sum_{i=1}^{N+1} x_i \alpha_i = t$, où t varie dans les entiers positifs ou nuls. Il est bien connu que cette fonction est une fonction quasi-polynomiale de t, de degré N. Nous donnons un nouvel algorithme qui calcule, pour chaque entier fixé k (mais N n'est pas fixé), les $k+1$ plus hauts coefficients du quasi-polynôme $E(\boldsymbol{\alpha})(t)$ en termes de fonctions en dents de scie. Notre algorithme utilise la structure d'ensemble partiellement ordonné des pôles de la fonction génératrice de $E(\boldsymbol{\alpha})(t)$. Les $k+1$ plus hauts coefficients se calculent à l'aide de fonctions génératrices de points entiers dans des cônes polyèdraux de dimension inférieure ou égale à k.

Keywords: Denumerants, Ehrhart quasi-polynomials, partitions, polynomial-time algorithm

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1 Introduction

Let $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_N, \alpha_{N+1}]$ be a sequence of positive integers. If t is a non-negative integer, we denote by $E(\boldsymbol{\alpha})(t)$ the number of solutions in nonnegative integers of the equation $\sum_{i=1}^{N+1} \alpha_i x_i = t$. In other words, $E(\boldsymbol{\alpha})(t)$ is the same as the number of partitions of the number t using the parts $\alpha_1, \alpha_2, \ldots, \alpha_N, \alpha_{N+1}$ (with repetitions allowed). This paper presents a new algorithm to compute individual coefficients of this quasipolynomial function and uncovers new structure that allows to describe their periodic nature. Let us begin with some background and history before stating the precise results:

The combinatorial function $E(\boldsymbol{\alpha})(t)$ was called by J. Sylvester the *denumerant*. The denumerant $E(\boldsymbol{\alpha})(t)$ has a beautiful structure and it has been known since the times of Cayley and Sylvester that $E(\boldsymbol{\alpha})(t)$ is in fact a *quasi-polynomial*, i.e., it can be written in the form $E(\boldsymbol{\alpha})(t) = \sum_{i=0}^{N} E_i(t)t^i$, where $E_i(t)$ is a periodic function of t (a more precise description of the periods of the coefficients $E_i(t)$ will be given later). In other words, there exists a positive integer Q such that for t in the coset $q + Q\mathbb{Z}$, the function $E(\boldsymbol{\alpha})(t)$ coincides with a polynomial function of t. The study of the coefficients $E_i(t)$, in particular determining their periodicity, is a problem that has occupied various authors and it is the key focus of our investigations here.

Sylvester and Cayley first showed that the function can be written in the form $A(t) + U(t)$, where $A(t)$ is a polynomial in t of degree N and $U(t)$ is a periodic function of period the least common multiple of $a_1, \ldots, a_r$ (see [6, 7] and references therein). In 1943, E.T. Bell gave a simpler proof and remarked that the period Q is in the worst case given by the least common multiple of the a_i, but in general it can be smaller. A classical observation that goes back to I. Schur is that when the list $\boldsymbol{\alpha}$ consist of relatively prime numbers, then asymptotically $E(\boldsymbol{\alpha})(t) \approx \frac{t^N}{N!\alpha_1\alpha_2\cdots\alpha_{N+1}}$ as the number $t \to \infty$.

Thus, in particular, there is a large enough integer F such that for any $t \geq F$, $E(\boldsymbol{\alpha})(t) > 0$ and there is a largest t for which $E(\boldsymbol{\alpha})(t) = 0$. Let us give a simple example:

Example 1.1. Let $\boldsymbol{\alpha} = [6, 2, 3]$. Then on each of the cosets $q + 6\mathbb{Z}$, the function $E(\boldsymbol{\alpha})(t)$ coincides with a polynomial $E^{[q]}(t)$. Here are the corresponding polynomials.

$$E^{[0]}(t) = \tfrac{1}{72}t^2 + \tfrac{1}{4}t + 1, \qquad E^{[1]}(t) = \tfrac{1}{72}t^2 + \tfrac{1}{18}t - \tfrac{5}{72}, \qquad E^{[2]}(t) = \tfrac{1}{72}t^2 + \tfrac{7}{36}t + \tfrac{5}{9},$$
$$E^{[3]}(t) = \tfrac{1}{72}t^2 + \tfrac{1}{6}t + \tfrac{3}{8}, \qquad E^{[4]}(t) = \tfrac{1}{72}t^2 + \tfrac{5}{36}t + \tfrac{2}{9}, \qquad E^{[5]}(t) = \tfrac{1}{72}t^2 + \tfrac{1}{9}t + \tfrac{7}{72}.$$

Naturally, the function $E(\boldsymbol{\alpha})(t)$ is equal to 0 if t does not belong to the lattice $\sum_{i=1}^{N+1} \mathbb{Z}\alpha_i \subset \mathbb{Z}$ generated by the integers α_i. So if g is the greatest common divisor of the α_i (which can be computed in polynomial time), and $\boldsymbol{\alpha}/g = [\frac{\alpha_1}{g}, \frac{\alpha_2}{g}, \ldots, \frac{\alpha_{N+1}}{g}]$ the formula $E(\boldsymbol{\alpha})(gt) = E(\boldsymbol{\alpha}/g)(t)$ holds, and we may assume that the numbers α_i span $\mathbb{Z}$ without changing the complexity of the problem. In other words, we may assume that the greatest common divisor of the α_i is equal to 1.

Our primary concern is how to compute $E(\boldsymbol{\alpha})(t)$. This problem has received a lot of attention. Computing the denumerant $E(\boldsymbol{\alpha})(t)$ as a close formula or evaluating it for specific t is relevant in several other areas of mathematics. In the combinatorics literature the denumerant has been studied extensively (see e.g., [6, 8, 14, 16] and the references therein). In combinatorial number theory and the theory of partitions, the problem appears in relation to the *Frobenius problem* or the *coin-change problem* of finding the largest value of t with $E(\boldsymbol{\alpha})(t) = 0$ (see [12, 13, 15] for details and algorithms). Authors in the theory of numerical semigroups have also investigated the so called *gaps* of the function, which are values of t for which $E(\boldsymbol{\alpha})(t) = 0$, i.e., those positive integers t which cannot be represented by the α_i. For $N = 1$ the number of gaps is $(\alpha_1 - 1)(\alpha_2 - 1)/2$ but for larger N the problem is quite difficult.

Unfortunately, computing $E(\boldsymbol{\alpha})(t)$ or evaluating it are very challenging computational problems. Even deciding whether $E(\boldsymbol{\alpha})(t) > 0$ for a given t, is a well-known (weakly) NP-hard problem. Computing $E(\boldsymbol{\alpha})(t)$, i.e., determining the number of solutions for a given t, is $\#P$-hard. Computing the Frobenius number is also known to be NP-hard [15]. Likewise, for a given coset $q + Q\mathbb{Z}$, computing the polynomial $E^{[q]}(t)$ is NP-hard. Despite the difficulty to compute the function, in some special cases one can compute information efficiently. For example, the Frobenius number can be computed in polynomial time when $N + 1$ is fixed [13]. At the same time for *fixed* $N + 1$ one can compute $E(\boldsymbol{\alpha})(t)$ in polynomial time as a special case of a well-known result of Barvinok [3]. There are several papers exploring the practical computation of the Frobenius numbers (see e.g., [12] and the many references therein).

These wonderful results for fixed N were achieved using a powerful geometric interpretation of $E(\boldsymbol{\alpha})(t)$ (which was the original way we encountered the problem): The function $E(\boldsymbol{\alpha})(t)$ can also be thought of as the number of integral points in the N-dimensional simplex in $\mathbb{R}^{N+1}$ defined by

$$\Delta_{\boldsymbol{\alpha}} = \{[x_1, x_2, \ldots, x_N, x_{N+1}] : x_i \geq 0, \sum_{i=1}^{N+1} \alpha_i x_i = t\}$$

with rational vertices $\mathbf{s}_i = [0, \ldots, 0, \frac{t}{\alpha_i}, 0, \ldots, 0]$. In this context, $E(\boldsymbol{\alpha})(t)$ is a very special case of the *Ehrhart function* (in honor of French mathematician Eugène Ehrhart who started its study [11]). Ehrhart functions count the lattice points inside a convex polytope P as it is dilated t times. All of the results we mentioned about $E(\boldsymbol{\alpha})(t)$ are in fact special cases of theorems from Ehrhart theory [5]. For example, the asymptotic result of I. Schur can be recovered from seeing that the highest-degree coefficient of $E(\boldsymbol{\alpha})(t)$ is just the normalized N-dimensional volume of the simplex $\Delta_{\boldsymbol{\alpha}}$. Our coefficients are just special cases of Ehrhart coefficients.

This paper is about the computation of $E(\boldsymbol{\alpha})(t)$ and in particular its coefficients. Here are our main results:

It is clear that the leading coefficient is given by Schur's result. Our main theorem recovers explicit formulas for other coefficients.

Theorem 1.2. *Given any fixed integer k, there is a polynomial time algorithm to compute the highest $k + 1$ degree terms of the quasi-polynomial $E(\boldsymbol{\alpha})(t)$, that is*

$$\mathrm{Top}_k E(\boldsymbol{\alpha})(t) = \sum_{i=0}^{k} E_{N-i}(t) t^{N-i}.$$

The coefficients are recovered as step polynomial *functions of t.*

Note that the number Q of cosets for $E(\boldsymbol{\alpha})(t)$ can be exponential in the binary encoding size of the problem, and thus it is impossible to list, in polynomial time, the polynomials $E^{[q]}(t)$ for all the cosets $q + Q\mathbb{Z}$. That is why to obtain a polynomial time algorithm, the output is presented in the format of *step polynomials*, which we now explain:

(i) We first define the function $\{s\} = s - \lfloor s \rfloor \in [0, 1)$ for $s \in \mathbb{R}$, where $\lfloor s \rfloor$ denotes the largest integer smaller or equal to s. The function $\{s + 1\} = \{s\}$ is a periodic function of s modulo 1.

(ii) If r is rational with denominator q, the function $T \mapsto \{rT\}$ is a function of $T \in \mathbb{R}$ periodic modulo q. A function of the form $T \mapsto \sum_i c_i \{r_i T\}$ will be called a *step linear function*. If all the r_i have a common denominator q, this function is periodic modulo q.

(iii) Then consider the algebra generated over $\mathbb{Q}$ by such functions on $\mathbb{R}$. An element ϕ of this algebra can be written (not in an unique way) as

$$\phi(T) = \sum_{l=1}^{L} c_l \prod_{j=1}^{J_l} \{r_{l,j}T\}^{n_{l,j}}.$$

Such a function $\phi(T)$ will be called a *step polynomial*.

(iv) We will say that the step polynomial ϕ is of *degree* u if $\sum_j n_j \leq u$ for all set of indices I occurring in the formula for ϕ. We will say that ϕ is of *period* q if all the rational numbers r_j have common denominator q.

It must be stress that evaluating these expressions can be done very fast. Moreover, one can also see that the *step polynomial* representation is much more economical than writing the individual polynomials for each coset of the period. For example instead of six polynomial "pieces" for $E(\boldsymbol{\alpha})(t)$ we can simply write a single step polynomial:

$$\frac{1}{72}t^2 + \left(\frac{1}{4} - \frac{\{-\frac{t}{3}\}}{6} - \frac{\{\frac{t}{2}\}}{6}\right)t + \left(1 - \frac{3}{2}\{-\tfrac{t}{3}\} - \frac{3}{2}\{\tfrac{t}{2}\} + \frac{1}{2}\left(\{-\tfrac{t}{3}\}\right)^2 + \{-\tfrac{t}{3}\}\{\tfrac{t}{2}\} + \frac{1}{2}\left(\{\tfrac{t}{2}\}\right)^2\right)$$

We must remark our results come after an earlier result of Barvinok [4] who first proved a similar theorem valid for all simplices. Also in [2], the authors presented a polynomial-time algorithm of to compute the coefficient functions of $\mathrm{Top}_k E(P)(t)$ for any simple polytope P (given by its rational vertices) in the form of *step polynomials* defined as above. We note that both of these earlier papers use the geometry of the problem very strongly; instead our algorithm is different as it uses more of the number-theoretic structure of the special case at hand. We must stress a marked advantage of our algorithms over the work in [4]: We compute using the step polynomials all the possibilities of $E^{[q]}(t)$ while [4] recovers a single piece for given q. More important, our new algorithm is much easier to implement.

The new algorithm uses directly the residue theorem in one complex variable, which can be applied more efficiently as a consequence of a rich poset structure on the set of poles of the associated rational generating function for $E(\boldsymbol{\alpha})(t)$ (see Subsection 2.2). The other important ingredient used in the efficient computation of the top coefficients is the reinterpretation of some generating functions in terms of lattice points in cones. This allows us to apply the polynomial-time signed cone decomposition of Barvinok for simplicial cones of fixed dimension k [3].

2 The Residue formula for $E(\boldsymbol{\alpha})(t)$

Let us begin by fixing some notation. If $\omega(z)\,\mathrm{d}z$ is a meromorphic one form on $\mathbb{C}$, with a pole at $z = \zeta$, we write

$$\mathrm{Res}_{z=\zeta}\,\omega(z)\,\mathrm{d}z = \frac{1}{2i\pi}\int_{C_\zeta} \omega(z)\,\mathrm{d}z$$

where C_ζ is a small circle around the pole ζ. If $\phi(z) = \sum_{k\geq k_0} \phi_k z^k$ is a Laurent series in z, we denote by $\mathrm{res}_{z=0}$ the coefficient of z^{-1} of $\phi(z)$. Cauchy's formula implies that $\mathrm{res}_{z=0}\,\phi(z) = \mathrm{Res}_{z=0}\,\phi(z)\,\mathrm{d}z$.

2.1 A residue formula for $\boldsymbol{E}(\boldsymbol{\alpha})(\boldsymbol{t})$.

Let $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \dots, \alpha_{N+1}]$ be our list of integers. Define

$$F(\boldsymbol{\alpha})(z) := \frac{1}{\prod_{i=1}^{N+1}(1 - z^{\alpha_i})}.$$

Denote by $\mathcal{P} = \bigcup_{i=1}^{N+1}\{\,\zeta \in \mathbb{C} : \zeta^{\alpha_i} = 1\,\}$ the set of poles of the meromorphic function $F(\boldsymbol{\alpha})$ and by $p(\zeta)$ the order of the pole ζ for $\zeta \in \mathcal{P}$.

Note that because the α_i have greatest common divisor 1, we have $\zeta = 1$ as a pole of order $N+1$, and the other poles have order strictly less.

Theorem 2.1. *Let $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \dots, \alpha_{N+1}]$ be a list of integers with greatest common divisor equal to 1, and let*

$$F(\boldsymbol{\alpha})(z) := \frac{1}{\prod_{i=1}^{N+1}(1 - z^{\alpha_i})}.$$

If t is a non-negative integer, then

$$E(\boldsymbol{\alpha})(t) = -\sum_{\zeta \in \mathcal{P}} \operatorname{Res}_{z=\zeta} z^{-t-1} F(\boldsymbol{\alpha})(z)\,\mathrm{d}z \tag{2.1}$$

and the ζ-term of this sum is a quasi-polynomial function of t with degree less than or equal to $p(\zeta) - 1$.

Proof. For $|z| < 1$, we write $\frac{1}{1-z^{\alpha_i}} = \sum_{u=0}^{\infty} z^{u\alpha_i}$ so that $F(\boldsymbol{\alpha})(z) = \sum_{t \geq 0} E(\boldsymbol{\alpha})(t) z^t$.

For a small circle $|z| = \epsilon$ of radius ϵ around 0, the integral of $z^k\,\mathrm{d}z$ is equal to 0 except if $k = -1$, when it is $2i\pi$. Thus

$$E(\boldsymbol{\alpha})(t) = \frac{1}{2i\pi}\int_{|z|=\epsilon} z^{-t} F(\boldsymbol{\alpha})(z)\frac{\mathrm{d}z}{z} = \frac{1}{2i\pi}\int_{|z|=\epsilon} z^{-t}\prod_{i=1}^{N+1}\frac{1}{(1 - z^{\alpha_i})}\frac{dz}{z}.$$

Because the α_i are positive integers, and t a non-negative integer, there are no residues at $z = \infty$ and we obtain equation (2.1) by applying the residue theorem.

Write $E_\zeta(t) := -\operatorname{Res}_{z=\zeta} z^{-t} F(\boldsymbol{\alpha})(z)\frac{\mathrm{d}z}{z}$; then the dependence in t of $E_\zeta(t)$ comes from the expansion of z^{-t} near $z = \zeta$. We write $z = \zeta + y$, so that $E_\zeta(t) = -\operatorname{Res}_{y=0}(\zeta + y)^{-t} F(\boldsymbol{\alpha})(\zeta + y)\frac{\mathrm{d}y}{\zeta + y}$. As the pole of $F(\boldsymbol{\alpha})(\zeta + y)$ at $y = 0$ is of order $p(\zeta)$, to compute the residue at $y = 0$, we only need to expand in y the function $(\zeta + y)^{-t-1}$ and take the coefficient of $y^{p(\zeta)-1}$. Now for $k = t+1$ the function $(\zeta + y)^{-k} = \zeta^{-k} - k\zeta^{-k-1}y + \cdots$ and we can easily check that the dependence in t of our residue is quasi-polynomial with degree less than or equal to $p(\zeta) - 1$. We thus obtain the result. □

2.2 The poset of the high-order poles

Given an integer $0 \leq k \leq N$, we partition the set of poles $\mathcal{P}$ in two disjoint sets according to the order of the pole:

$$\mathcal{P}_{>N-k} = \{\,\zeta : p(\zeta) \geq N + 1 - k\,\}, \qquad \mathcal{P}_{\leq N-k} = \{\,\zeta : p(\zeta) \leq N - k\,\}.$$

According to the disjoint decomposition $\mathcal{P} = \mathcal{P}_{\leq N-k} \cup \mathcal{P}_{>N-k}$, we write

$$E_{\mathcal{P}_{>N-k}}(t) = -\sum_{\zeta\in\mathcal{P}_{>N-k}} \mathrm{Res}_{z=\zeta}\, z^{-t-1}F(\boldsymbol{\alpha})(z)\,\mathrm{d}z$$

and

$$E_{\mathcal{P}_{\leq N-k}}(t) = -\sum_{\zeta\in\mathcal{P}_{\leq N-k}} \mathrm{Res}_{z=\zeta}\, z^{-t-1}F(\boldsymbol{\alpha})(z)\,\mathrm{d}z.$$

The following proposition is a direct consequence of Theorem 2.1.

Proposition 2.2. *We have*

$$E(\boldsymbol{\alpha})(t) = E_{\mathcal{P}_{>N-k}}(t) + E_{\mathcal{P}_{\leq N-k}}(t),$$

where the function $E_{\mathcal{P}_{\leq N-k}}(t)$ is a quasi-polynomial function of t of degree in t strictly less than $N-k$.

Thus for the purpose of computing $\mathrm{Top}_k E(\boldsymbol{\alpha})(t)$ it is sufficient to compute the function $E_{\mathcal{P}_{>N-k}}(t)$. This function is computable in polynomial time, as stated in the following theorem that implies is Theorem 1.2

Theorem 2.3. *Let k be a fixed number. Then the coefficient functions of the quasi-polynomial function $E_{\mathcal{P}_{>N-k}}(t)$ are computable in polynomial time as step polynomials of t.*

We prove the theorem in the rest of this section and the next.

We first rewrite our set $\mathcal{P}_{>N-k}$. Note that if ζ is a pole of order $\geq p$, this means that there exist at least p elements α_i in the list $\boldsymbol{\alpha}$ so that $\zeta^{\alpha_i}=1$. But if $\zeta^{\alpha_i}=1$ for a set $I \subseteq \{1,\dots,N+1\}$ of indices i, this is equivalent to the fact that $\zeta^f = 1$, for f the greatest common divisor of the elements $\alpha_i, i \in I$.

Now let $\mathcal{I}_{>N-k}$ be the set of index sets that correspond to sublists of $\boldsymbol{\alpha}$ of length greater than $N-k$. Note that when k is fixed, the cardinality of $\mathcal{I}_{>N-k}$ is a polynomial function of N. For each subset $I \in \mathcal{I}_{>N-k}$, define f_I to be the greatest common divisor of the sublist α_i, $i \in I$. Let $\mathcal{G}_{>N-k}(\boldsymbol{\alpha}) = \{\, f_I : I \in \mathcal{I}_{>N-k} \,\}$ be the set of integers so obtained. Because $\mathcal{I}_{>N-k}$ is stable by the operation of taking supersets, the set $\mathcal{G}_{>N-k}(\boldsymbol{\alpha})$ is a set of integers stable by the operation of taking greatest common divisors. Thus, $\mathcal{G}_{>N-k}(\boldsymbol{\alpha})$ can be considered as a poset (partially ordered set), where $f \preceq f'$ if f divides f'.

Using the group $G(f) \subset \mathbb{C}^\times$ of f-th roots of unity,

$$G(f) = \{\, \zeta \in \mathbb{C} : \zeta^f = 1 \,\},$$

we have thus $\mathcal{P}_{>N-k} = \bigcup_{f\in\mathcal{G}_{>N-k}(\boldsymbol{\alpha})} G(f)$; this is, of course, not a disjoint union. Then using the inclusion–exclusion principle, we can write the characteristic function of the set $\mathcal{P}_{>N-k}$ as a linear combination of characteristic functions of the sets $G(f)$:

$$[\mathcal{P}_{>N-k}] = \sum_{f\in\mathcal{G}_{>N-k}(\boldsymbol{\alpha})} \mu(f)[G(f)],$$

where $\mu(f)$ are integers computed recursively. Such a function μ will be called as always a *Möbius function* for the poset (see Chapter 3 [18] for details on posets).

For fixed k, all the data above can be computed in polynomial time in function of the data $\boldsymbol{\alpha}$. The greatest common divisor of a set of integers is computed in polynomial time. Finally the Möbius function

$\mu(f)$ is computed in polynomial time, because there are polynomially many levels of the poset being considered.

Let us define for any positive integer f

$$E(\boldsymbol{\alpha}, f)(t) = -\sum_{\zeta^f=1} \operatorname{Res}_{z=\zeta} z^{-t-1} F(\boldsymbol{\alpha})(z)\,\mathrm{d}z.$$

Proposition 2.4. *Let k be a fixed integer, then*

$$E_{\mathcal{P}_{>N-k}}(t) = -\sum_{f \in \mathcal{G}_{>N-k}(\boldsymbol{\alpha})} \mu(f) E(\boldsymbol{\alpha}, f)(t). \tag{2.2}$$

Thus we have reduced the computation to the fast computation of $E(\boldsymbol{\alpha}, f)(t)$. We will return to that in a moment but before we continue with the proof of Theorem 1.2, there are some interesting consequences for the classical theory of Denumerants.

Equation (2.2) provides explicit expressions for the coefficients of the denumerant $E(\boldsymbol{\alpha})(t)$. In the past, researchers have discussed $E(\boldsymbol{\alpha})(t)$ in terms of its generating function (which belongs to the well-known clan of rational generating functions [18]), formulas for $E(\boldsymbol{\alpha})(t)$ in terms of binomial coefficients can be obtained using partial fraction decomposition. In [17] the authors propose another way to recover the coefficients of the quasipolynomial by a method they named *rigorous guessing*. In [17] quasipolynomials are represented as a function $f(t)$ given by q polynomials $f^{[1]}(t), f^{[2]}(t), \ldots, f^{[q]}(t)$ such that $f(t) = f^{[i]}(t)$ when $t \equiv i \pmod{q}$. To find the coefficients of the $f^{[i]}$ their method finds the first few terms of the Maclaurin expansion of the partial fraction decomposition to find enough evaluations of those polynomials and then recovers the coefficients of the $f^{[i]}$ as a result of solving a linear system. Our approach appeals instead to the number theoretic and polyhedral geometric nature of the problem and instead of $f^{[i]}$'s we have a single expression whose coefficients are products of step polynomials.

3 Polyhedral reinterpretation of the generating function $E(\boldsymbol{\alpha}, f)(t)$

To complete the proof of Theorem 1.2 we need only to prove the following proposition.

Proposition 3.1. *For any integer $f \in \mathcal{G}_{>N-k}(\boldsymbol{\alpha})$, the coefficient functions of the quasi-polynomial function $E(\boldsymbol{\alpha}, f)(t)$ and hence $E_{\mathcal{P}_{>N-k}}(t)$ are computed in polynomial time as step polynomials of t.*

By the previous proposition we know we need to compute the value of $E(\boldsymbol{\alpha}, f)(t)$. Our goal now is to demonstrate that this function can be thought of as the generating function of the lattice points inside a convex cone. This is a key point to guarantee good computational bounds. Before we can do that we review some preliminaries on generating functions of cones. We recall the notion of generating functions of cones; see also [2].

Let $V = \mathbb{R}^r$ provided with a lattice Λ, and let V^* denote the dual space. A (rational) simplicial cone $\mathfrak{c} = \mathbb{R}_{\geq 0}\mathbf{w}_1 + \cdots + \mathbb{R}_{\geq 0}\mathbf{w}_r$ is a cone generated by r linearly independent vectors $\mathbf{w}_1, \ldots, \mathbf{w}_r$ of Λ. We consider the semi-rational affine cone $\mathbf{s} + \mathfrak{c}$, $\mathbf{s} \in V$. Let $\boldsymbol{\xi} \in V^*$ be a dual vector such that $\langle \boldsymbol{\xi}, \mathbf{w}_i \rangle < 0,\ 1 \leq i \leq r$. Then the sum

$$S(\mathbf{s} + \mathfrak{c}, \Lambda)(\boldsymbol{\xi}) = \sum_{\mathbf{n} \in (\mathbf{s}+\mathfrak{c}) \cap \Lambda} \mathrm{e}^{\langle \boldsymbol{\xi}, \mathbf{n} \rangle}$$

is summable and defines an analytic function of $\boldsymbol{\xi}$. It is well known that this function extends to a meromorphic function of $\boldsymbol{\xi} \in V_{\mathbb{C}}^*$. We still denote this meromorphic extension by $S(\mathbf{s}+\mathfrak{c},\Lambda)(\boldsymbol{\xi})$.

Recall the following result.

Theorem 3.2. *The series $S(\mathbf{s}+\mathfrak{c},\Lambda)(\boldsymbol{\xi})$ is a meromorphic function of $\boldsymbol{\xi}$ such that $\prod_{i=1}^{r}\langle\boldsymbol{\xi},\mathbf{w}_i\rangle S(\mathbf{s}+\mathfrak{c},\Lambda)(\boldsymbol{\xi})$ is holomorphic in a neighborhood of $\mathbf{0}$.*

Let $\mathbf{t}\in\Lambda$. Consider the translated cone $\mathbf{t}+\mathbf{s}+\mathfrak{c}$ of $\mathbf{s}+\mathfrak{c}$ by $\mathbf{t}$. Then we have the covariance formula

$$S(\mathbf{t}+\mathbf{s}+\mathfrak{c},\Lambda)(\boldsymbol{\xi}) = \mathrm{e}^{\langle\boldsymbol{\xi},\mathbf{t}\rangle} S(\mathbf{s}+\mathfrak{c},\Lambda)(\boldsymbol{\xi}). \tag{3.1}$$

Because of this formula, it is convenient to introduce the following function.

Definition 3.3. Define the function

$$M(\mathbf{s},\mathfrak{c},\Lambda)(\boldsymbol{\xi}) = \mathrm{e}^{-\langle\boldsymbol{\xi},\mathbf{s}\rangle} S(\mathbf{s}+\mathfrak{c},\Lambda)(\boldsymbol{\xi}).$$

Thus the function $\mathbf{s}\mapsto M(\mathbf{s},\mathfrak{c},\Lambda)(\boldsymbol{\xi})$ is a function of $\mathbf{s}\in V/\Lambda$ (a periodic function of $\mathbf{s}$) whose values are meromorphic functions of $\boldsymbol{\xi}$.

The function is easy to write down for a *unimodular* cone, that is a cone $\mathfrak{u}$ whose primitive generators $\mathbf{g}_i^{\mathfrak{u}}$ form a basis of the lattice Λ. We introduce the following notation.

Definition 3.4. Let $\mathfrak{u}$ be a unimodular cone with primitive generators $\mathbf{g}_i^{\mathfrak{u}}$ and let $\mathbf{s}\in V$. Then, write $\mathbf{s}=\sum_i s_i\mathbf{g}_i^{\mathfrak{u}}$, with $s_i\in\mathbb{R}$, and define

$$\{-\mathbf{s}\}_{\mathfrak{u}} = \sum_i \{-s_i\}\mathbf{g}_i^{\mathfrak{u}}.$$

Thus $\mathbf{s}+\{-\mathbf{s}\}_{\mathfrak{u}} = \sum_i \lceil s_i\rceil \mathbf{g}_i^{\mathfrak{u}}$. Note that if $\mathbf{t}\in\Lambda$, then $\{-(\mathbf{s}+\mathbf{t})\}_{\mathfrak{u}} = \{-\mathbf{s}\}_{\mathfrak{u}}$. Thus, $\mathbf{s}\mapsto\{-\mathbf{s}\}_{\mathfrak{u}}$ is a function on V/Λ with value in V. For any $\boldsymbol{\xi}\in V^*$, we then find

$$S(\mathbf{s}+\mathfrak{u},\Lambda)(\boldsymbol{\xi}) = \mathrm{e}^{\langle\boldsymbol{\xi},\mathbf{s}\rangle}\mathrm{e}^{\langle\boldsymbol{\xi},\{-\mathbf{s}\}_{\mathfrak{u}}\rangle}\frac{1}{\prod_j(1-\mathrm{e}^{\langle\boldsymbol{\xi},\mathbf{g}_j^{\mathfrak{u}}\rangle})}$$

and thus

$$M(\mathbf{s},\mathfrak{u},\Lambda)(\boldsymbol{\xi}) = \mathrm{e}^{\langle\boldsymbol{\xi},\{-\mathbf{s}\}_{\mathfrak{u}}\rangle}\frac{1}{\prod_j(1-\mathrm{e}^{\langle\boldsymbol{\xi},\mathbf{g}_j^{\mathfrak{u}}\rangle})}. \tag{3.2}$$

For a general cone $\mathfrak{c}$, we can decompose its characteristic function $[\mathfrak{c}]$ as a signed sum of characteristic functions of unimodular cones, $\sum_{\mathfrak{u}}\epsilon_{\mathfrak{u}}[\mathfrak{u}]$, modulo characteristic functions of cones containing lines. As shown by Barvinok, if the dimension r of V is fixed, this decomposition can be computed in polynomial time. Then we can write

$$S(\mathbf{s}+\mathfrak{c},\Lambda)(\boldsymbol{\xi}) = \sum_{\mathfrak{u}}\epsilon_{\mathfrak{u}}\, S(\mathbf{s}+\mathfrak{u},\Lambda)(\boldsymbol{\xi}).$$

Thus we obtain, using Formula (3.2),

$$M(\mathbf{s},\mathfrak{c},\Lambda)(\boldsymbol{\xi}) = \sum_{\mathfrak{u}}\epsilon_{\mathfrak{u}}\,\mathrm{e}^{\langle\boldsymbol{\xi},\{-\mathbf{s}\}_{\mathfrak{u}}\rangle}\frac{1}{\prod_j(1-\mathrm{e}^{\langle\boldsymbol{\xi},\mathbf{g}_i^{\mathfrak{u}}\rangle})}. \tag{3.3}$$

Here $\mathfrak{u}$ runs through all the unimodular cones occurring in the decomposition of $\mathfrak{c}$, and the $\mathbf{g}_i^{\mathfrak{u}}\in\Lambda$ are the generators of the unimodular cone $\mathfrak{u}$.

Remark 3.5. For computing explicit examples, it is convenient to make a change of variables that leads to computations in the standard lattice $\mathbb{Z}^r$. Let B be the matrix whose columns are the generators of the lattice Λ; then $\Lambda = B\mathbb{Z}^r$.

$$\begin{aligned} M(\mathbf{s},\mathfrak{c},\Lambda)(\boldsymbol{\xi}) &= \mathrm{e}^{-\langle \boldsymbol{\xi},\mathbf{s}\rangle} \sum_{\mathbf{n}\in(\mathbf{s}+\mathfrak{c})\cap B\mathbb{Z}^r} \mathrm{e}^{\langle \boldsymbol{\xi},\mathbf{n}\rangle} \\ &= \mathrm{e}^{-\langle B^\top\boldsymbol{\xi},B^{-1}\mathbf{s}\rangle} \sum_{\mathbf{x}\in(B^{-1}(\mathbf{s}+\mathfrak{c})\cap\mathbb{Z}^r} \mathrm{e}^{\langle B^\top\boldsymbol{\xi},\mathbf{x}\rangle} = M(B^{-1}\mathbf{s},B^{-1}\mathfrak{c},\mathbb{Z}^r)(B^\top\boldsymbol{\xi}). \end{aligned}$$

3.1 Back to the computation of $E(\boldsymbol{\alpha},f)(t)$

After the preliminaries we will see how to rewrite $E(\boldsymbol{\alpha},f)(t)$ in terms of lattice points of cones. This will require some suitable manipulation of the initial form of $E(\boldsymbol{\alpha},f)(t)$. So we introduce some notation. Let k be fixed. For $f\in\mathcal{G}_{>N-k}(\boldsymbol{\alpha})$, define $\mathcal{F}(\boldsymbol{\alpha},f,T)(x)=\sum_{\zeta^f=1}\frac{\zeta^{-T}}{\prod_{i=1}^{N+1}(1-\zeta^{\alpha_i}\mathrm{e}^{\alpha_i x})}$, $\mathcal{E}(\boldsymbol{\alpha},f)(t,T) = -\operatorname{res}_{x=0}\mathrm{e}^{-tx}\mathcal{F}(\boldsymbol{\alpha},f,T)(x)$, and $E_i(f)(T)=\operatorname{res}_{x=0}\frac{(-x)^i}{i!}\mathcal{F}(\boldsymbol{\alpha},f,T)(x)$. Writing $z=\zeta\mathrm{e}^x$ and changing coordinates in residues, we obtain immediately:

$$E(\boldsymbol{\alpha},f)(t)=\mathcal{E}(\boldsymbol{\alpha},f)(t,T)\big|_{T=t}. \tag{3.4}$$

The dependence in T of $\mathcal{F}(\boldsymbol{\alpha},f,T)(x)$ is through ζ^T. As $\zeta^f=1$, the function $\mathcal{F}(\boldsymbol{\alpha},f,T)(x)$ is a periodic function of T modulo f whose values are meromorphic functions of x. Since the pole in x is of order at most $N+1$, we can rewrite $\mathcal{E}(\boldsymbol{\alpha},f)(t,T)$ in terms of $E_i(f)(T)$ and prove:

Theorem 3.6. *Let k be fixed. Then for $f\in\mathcal{G}_{>N-k}(\boldsymbol{\alpha})$ we can write*

$$\mathcal{E}(\boldsymbol{\alpha},f)(t,T)=\sum_{i=0}^{N} t^i E_i(f)(T)$$

with $E_i(f)(T)$ a step polynomial of degree less than or equal to $N-i$ and periodic of T modulo f. This step polynomial can be computed in polynomial time.

For example E_N is independent of T, thus it is a constant.

It is now clear that once we have proved Theorem 3.6, then the proof of Theorem 1.2 will follow. So we now concentrate on writing the function $\mathcal{F}(\boldsymbol{\alpha},f,T)(x)$ more explicitly.

Definition 3.7. For a list $\boldsymbol{\alpha}$ and integers f and T, define meromorphic functions of $x\in\mathbb{C}$ by:

$$\mathcal{B}(\boldsymbol{\alpha},f)(x):=\frac{1}{\prod_{i:\ f\mid\alpha_i}(1-\mathrm{e}^{\alpha_i x})},\quad \mathcal{S}(\boldsymbol{\alpha},f,T)(x):=\sum_{\zeta:\ \zeta^f=1}\frac{\zeta^{-T}}{\prod_{i:\ f\nmid\alpha_i}(1-\zeta^{\alpha_i}\mathrm{e}^{\alpha_i x})}.$$

Thus

$$\mathcal{F}(\boldsymbol{\alpha},f,T)(x)=\mathcal{B}(\boldsymbol{\alpha},f)(x)\,\mathcal{S}(\boldsymbol{\alpha},f,T)(x).$$

The expression we obtained will allow us to compute $\mathcal{F}(\boldsymbol{\alpha},f,T)$ by relating $S(\boldsymbol{\alpha},f,T)$ to a generating function of a cone. This cone will have fixed dimension when k is fixed.

3.2 $E(\boldsymbol{\alpha}, f)(t)$ as the generating function of a cone in fixed dimension

To this end, let f be an integer from $\mathcal{G}_{>N-k}(\boldsymbol{\alpha})$. By definition, f is the greatest common divisor of a sublist of $\boldsymbol{\alpha}$. Thus the greatest common divisor of f and the elements of $\boldsymbol{\alpha}$ which are *not* a multiple of f is still equal to 1. Let $I = I(\boldsymbol{\alpha}, f)$ be the set of indices $i \in \{1, \dots, N+1\}$ such that α_i is indivisible by f, i.e., $f \nmid \alpha_i$. Note that f by definition is the greatest common divisor of all except at most k of the integers α_j. Let r denote the cardinality of I; then $r \leq k$. Let $V_I = \mathbb{R}^I$ and let V_I^* denote the dual space. We also define the sublist $\boldsymbol{\alpha}_I = [\alpha_i]_{i \in I}$ of elements of $\boldsymbol{\alpha}$ indivisible by f and view it as a vector in V_I^*.

Definition 3.8. For an integer T, define the meromorphic function of $\boldsymbol{\xi} \in V_I^*$,

$$Q(\boldsymbol{\alpha}, f, T)(\boldsymbol{\xi}) = \sum_{\zeta\colon \zeta^f = 1} \frac{\zeta^{-T}}{\prod_{j \in I(\boldsymbol{\alpha}, f)} (1 - \zeta^{\alpha_j} \mathrm{e}^{\xi_j})}.$$

Remark 3.9. Observe that $Q(\boldsymbol{\alpha}, f, T)$ can be restricted at $\boldsymbol{\xi} = \boldsymbol{\alpha}_I x$, for $x \in \mathbb{C}$ generic, to give $\mathcal{S}(\boldsymbol{\alpha}, f, T)(x)$.

We find that $Q(\boldsymbol{\alpha}, f, T)(\boldsymbol{\xi})$ is the discrete generating function of an affine shift of the standard cone relative to a certain lattice in V_I, which we define as:

$$\Lambda(\boldsymbol{\alpha}, f) = \bigl\{\, \mathbf{y} \in \mathbb{Z}^I : \langle \boldsymbol{\alpha}_I, \mathbf{y} \rangle = \sum_{j \in I} y_j \alpha_j \in \mathbb{Z} f \,\bigr\}. \tag{3.5}$$

Consider the map $\phi\colon \mathbb{Z}^I \to \mathbb{Z}/\mathbb{Z}f$, $\mathbf{y} \mapsto \langle \boldsymbol{\alpha}, \mathbf{y} \rangle + \mathbb{Z}f$. Its kernel is the lattice $\Lambda(\boldsymbol{\alpha}, f)$. Because the greatest common divisor of f and the elements of $\boldsymbol{\alpha}_I$ is 1, by Bezout's theorem there exist $s_0 \in \mathbb{Z}$ and $\mathbf{s} \in \mathbb{Z}^I$ such that $1 = \sum_{i \in I} s_i \alpha_i + s_0 f$. Therefore, the map ϕ is surjective, and therefore the index $|\mathbb{Z}^I : \Lambda(\boldsymbol{\alpha}, f)|$ equals f.

Theorem 3.10. *Let $\boldsymbol{\alpha} = [\alpha_1, \dots, \alpha_{N+1}]$ be a list of positive integers and f be the greatest common divisor of a sublist of $\boldsymbol{\alpha}$. Let $I = I(\boldsymbol{\alpha}, f) = \{\, i : f \nmid \alpha_i \,\}$. Let $s_0 \in \mathbb{Z}$ and $\mathbf{s} \in \mathbb{Z}^I$ such that $1 = \sum_{i \in I} s_i \alpha_i + s_0 f$ using Bezout's theorem. Let T be an integer, and $\boldsymbol{\xi} \in V_I^*$. Then*

$$Q(\boldsymbol{\alpha}, f, T)(\boldsymbol{\xi}) = f\, M\bigl(-T\mathbf{s}, \mathbb{R}^I_{\geq 0}, \Lambda(\boldsymbol{\alpha}, f)\bigr)(\boldsymbol{\xi}).$$

Remark 3.11. The function $Q(\boldsymbol{\alpha}, f, T)(\boldsymbol{\xi})$ is a function of T periodic modulo f. Since $f\mathbb{Z}^I$ is contained in $\Lambda(\boldsymbol{\alpha}, f)$, the element $f\mathbf{s}$ is in the lattice $\Lambda(\boldsymbol{\alpha}, f)$, and we see that the right hand side is also a periodic function of T modulo f.

of Theorem 3.10. Consider $\boldsymbol{\xi} \in V_I^*$ with $\xi_j < 0$. Then we can write the equality

$$\frac{1}{\prod_{j \in I} (1 - \zeta^{\alpha_j} \mathrm{e}^{\xi_j})} = \prod_{j \in I} \sum_{n_j = 0}^{\infty} \zeta^{n_j \alpha_j} \mathrm{e}^{n_j \xi_j}. \quad \text{So} \quad Q(\boldsymbol{\alpha}, f, T)(\boldsymbol{\xi}) = \sum_{\mathbf{n} \in \mathbb{Z}^I_{\geq 0}} \Bigl(\sum_{f\colon \zeta^f = 1} \zeta^{\sum_j n_j \alpha_j - T} \Bigr) \mathrm{e}^{\sum_{j \in I} n_j \xi_j}.$$

We note that $\sum_{f\colon \zeta^f = 1} \zeta^m$ is zero except if $m \in \mathbb{Z}f$, when this sum is equal to f. Then we obtain that $Q(\boldsymbol{\alpha}, f, T)$ is the sum over $\mathbf{n} \in \mathbb{Z}^I_{\geq 0}$ such that $\sum_j n_j \alpha_j - T \in \mathbb{Z}f$. The equality $1 = \sum_{j \in I} s_j \alpha_j + s_0 f$ implies that $T \equiv \sum_j t s_j \alpha_j$ modulo f, and the condition $\sum_j n_j \alpha_j - T \in \mathbb{Z}f$ is equivalent to the condition $\sum_j (n_j - T s_j) \alpha_j \in \mathbb{Z}f$.

We see that the point $\mathbf{n} - T\mathbf{s}$ is in the lattice $\Lambda(\boldsymbol{\alpha}, f)$ as well as in the cone $-T\mathbf{s} + \mathbb{R}^I_{\geq 0}$ (as $n_j \geq 0$). Thus our function $Q(\boldsymbol{\alpha}, f, T)(\boldsymbol{\xi})$ is equal to $f\,\mathrm{e}^{\langle \boldsymbol{\xi}, T\mathbf{s}\rangle} S\big(-T\mathbf{s} + \mathbb{R}^I_{\geq 0}, \Lambda(\boldsymbol{\alpha}, f)\big)(\boldsymbol{\xi}) = f\,M\big(-T\mathbf{s} + \mathbb{R}^I_{\geq 0}, \Lambda(\boldsymbol{\alpha}, f)\big)(\boldsymbol{\xi})$. □

3.3 Unimodular decomposition in the dual space

The cone $\mathbb{R}^I_{\geq 0}$ is in general not unimodular with respect to the lattice $\Lambda(\boldsymbol{\alpha}, f)$. By decomposing $\mathbb{R}^I_{\geq 0}$ in cones $\mathfrak{u}$ that are unimodular with respect to $\Lambda(\boldsymbol{\alpha}, f)$, modulo cones containing lines, we can write $M\big(-T\mathbf{s}, \mathbb{R}^I_{\geq 0}, \Lambda(\boldsymbol{\alpha}, f)\big) = \sum_{\mathfrak{u}} \epsilon_{\mathfrak{u}} M(-T\mathbf{s}, \mathfrak{u}, \Lambda)$, where $\epsilon_{\mathfrak{u}} \in \{\pm 1\}$. This decomposition can be computed using Barvinok's algorithm in polynomial time for fixed k because the dimension $|I|$ is at most k.

Remark 3.12. Although we know that the meromorphic function $M\big(-T\mathbf{s}, \mathbb{R}^I_{\geq 0}, \Lambda(\boldsymbol{\alpha}, f)\big)(\boldsymbol{\xi})$ restricts via $\boldsymbol{\xi} = \boldsymbol{\alpha}_I x$ to a meromorphic function of a single variable x, it may happen that the individual functions $M\big(-T\mathbf{s}, \mathfrak{u}, \Lambda(\boldsymbol{\alpha}, f)\big)(\boldsymbol{\xi})$ do not restrict. In other words, the line $\boldsymbol{\alpha}_I x$ may be entirely contained in the set of poles. If this is the case, we can compute (in polynomial time) a regular vector $\boldsymbol{\beta} \in \mathbb{Q}^I$ so that all functions $M\big(-T\mathbf{s} + \mathfrak{u}, \Lambda(\boldsymbol{\alpha}, f)\big)(\boldsymbol{\xi})$ occurring can be evaluated on $(\boldsymbol{\alpha}_I + \epsilon\boldsymbol{\beta})x$.

Finally let us analyze the dependence in T of the functions $M(-T\mathbf{s}, \mathfrak{u}, \Lambda(\boldsymbol{\alpha}, f))$, where $\mathfrak{u}$ is a unimodular cone. Let the generators be $\mathbf{g}_i^{\mathfrak{u}}$, so the elements $\mathbf{g}_i^{\mathfrak{u}}$ form a basis of the lattice $\Lambda(\boldsymbol{\alpha}, f)$. Recall that the lattice $f\mathbb{Z}^r$ is contained in $\Lambda(\boldsymbol{\alpha}, f)$. Thus as $\mathbf{s} \in \mathbb{Z}^r$, we have $\mathbf{s} = \sum_i s_i \mathbf{g}_i^{\mathfrak{u}}$ with $f s_i \in \mathbb{Z}$ and hence $\{-T\mathbf{s}\}_{\mathfrak{u}} = \sum_i \{-T s_i\} \mathbf{g}_i^{\mathfrak{u}}$ with $\{-T s_i\}$ a function of T periodic modulo f.

Thus the function $T \mapsto \{-T\mathbf{s}\}_{\mathfrak{u}}$ is a step linear function, modulo f, with value in V. We then write $M(-T\mathbf{s}, \mathfrak{u})(\boldsymbol{\xi}) = \mathrm{e}^{\langle \boldsymbol{\xi}, \{T\mathbf{s}\}_{\mathfrak{u}}\rangle} \prod_{j=1}^r \frac{1}{(1 - \mathrm{e}^{\langle \boldsymbol{\xi}, \mathbf{g}_j \rangle})}$, and hence finally

$$\mathcal{F}(\boldsymbol{\alpha}, f, T)(x) = f\,M\big(-T\mathbf{s}, \mathbb{R}^I_{\geq 0}, \Lambda(\boldsymbol{\alpha}, f)\big)(\boldsymbol{\alpha}_I x) \prod_{j\,:\, f \mid \alpha_j} \frac{1}{(1 - \mathrm{e}^{\alpha_j x})}.$$

This is a meromorphic function of the variable x. Near $x = 0$, it is of the form $\sum_{\mathfrak{u}} \exp\{l_{\mathfrak{u}}(T)x\} h(x)/x^{N+1}$ where $h(x)$ is holomorphic in x and $l_{\mathfrak{u}}(T)$ is a step linear function of T, modulo f. Thus to compute

$$E_i(f)(T) = \mathrm{res}_{x=0} \frac{(-x)^i}{i!} \mathcal{F}(\boldsymbol{\alpha}, f, T)(x)$$

we only have to expand the function $x \mapsto \exp\{l_{\mathfrak{u}}(T)x\}$ up to the power x^{N-i}. This expansion can be done in polynomial time. We thus see that as stated in Theorem 3.6, $E_i(f)(T)$ is a step polynomial of degree less than or equal to $(N - i)$, which is periodic of T modulo f. This completes the proof of Theorem 3.6 and thus the proof of Theorem 1.2.

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Combinatorial topology of toric arrangements

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Abstract. We prove that the complement of a complexified toric arrangement has the homotopy type of a minimal CW-complex, and thus its homology is torsion-free.
To this end, we consider the toric Salvetti complex, a combinatorial model for the arrangement's complement. Using diagrams of acyclic categories we obtain a stratification of this combinatorial model that explicitly associates generators in homology to the "local no-broken-circuit sets" defined in terms of the incidence relations of the arrangement. Then we apply a suitably generalized form of Discrete Morse Theory to describe a sequence of elementary collapses leading from the full model to a minimal complex.

Résumé. On démontre que l'espace complementaire d'un arrangement torique complexifié a le type d'homotopie d'un complexe CW minimal, donc que ses groupes d'homologie sont libres. On considère d'abord un modèle combinatoire du complementaire de l'arrangement: le complexe de Salvetti torique. On obtient une stratification de ce complexe qui fait correspondre explicitement les génerateurs d'homologie aux "circuits-non-rompus locaux" associés aux relations d'incidence de l'arrangement. On applique une forme generalisée de la théorie de Morse discrète pour obtenir une suite de collapsements elementaires qui conduit à un complexe minimale.

Keywords: Combinatorial topology, Toric arrangements, Discrete Morse theory, Torsion-freeness in homology.

1 Introduction

A *toric arrangement* is a finite collection $\mathscr{A} = \{K_1, \dots, K_n\}$ of level sets of characters of the complex torus, i.e., for all i there is a character $\chi_i \in \mathrm{Hom}((\mathbb{C}^*)^d, \mathbb{C}^*)$ and a 'level' $a_i \in \mathbb{C}^*$ so that $K_i = \chi_i^{-1}(a_i)$.

Toric arrangements play a prominent role in recent work of De Concini, Procesi and Vergne on the link between partition functions and box splines (see e.g. De Concini and Procesi (2010)). A combinatorial framework for this context (in the case where $a_i = 1$ for all i) is given by the theory of arithmetic matroids, studied by D'Adderio and Moci (2011) and Brändén and Moci (2012), leading to nice theoretical constructions and strong enumerative results.

With the aim of improving these enumerative results towards a more structural description, we look at the combinatorial topology of the complement $M(\mathscr{A}) := (\mathbb{C}^*)^d \setminus \bigcup \mathscr{A}$.

We consider the case of complexified toric arrangements, allowing $|a_i| = 1$ for all i. It is known that the Poincaré polynomial of $M(\mathscr{A})$ can be recovered from the associated arithmetic matroid. Moreover, De Concini and Procesi (2005) computed the algebra structure of the cohomology with complex coefficients in the unimodular case. We prove that for any complexified toric arrangement $\mathscr{A}$,

- the space $M(\mathscr{A})$ is *minimal*, i.e., it has the homotopy type of a CW-complex whose cells in every dimension k are counted by the k-th Betti number.

- Hence, the space $M(\mathscr{A})$ is *torsion-free*, that is, the modules $H_k(M(\mathscr{A}),\mathbb{Z}), H^k(M(\mathscr{A}),\mathbb{Z})$ are torsion-free for every k.

The second item is the analogue for toric arrangements of the celebrated theorem by Brieskorn (1971) paving the way for a combinatorial study of the "Orlik-Solomon Algebra" associated to hyperplane arrangements. In this respect, our result is a step towards a "toric Orlik-Solomon algebra".

From a combinatorial point of view, our core data is the *face category* $\mathcal{F}(\mathscr{A})$, which encodes the incidence relations of the induced stratification of the 'real torus' $(S^1)^d \subseteq (\mathbb{C}^*)^d$. As we explain in Section 4, from $\mathcal{F}(\mathscr{A})$ one can construct a combinatorial model for the homotopy type of $M(\mathscr{A})$ which we call toric Salvetti complex because of its relation to the Salvetti complex of a complexified hyperplane arrangement (see Moci and Settepanella (2011); d'Antonio and Delucchi (2011)). A presentation of the fundamental group can also be obtained from $\mathcal{F}(\mathscr{A})$ (d'Antonio and Delucchi (2011)).

We prove minimality by exhibiting a sequence of elementary collapses on the toric Salvetti complex that leads to a minimal complex. To this end, we need to mildly generalize some elements of Discrete Morse Theory in order to be able to work with nonregular CW-complexes or, correspondingly, face categories that are not posets (see Section 5.3). Once this is done, we are left with finding an "acyclic matching" of the face category of the toric Salvetti complex (the so-called *Salvetti category*) with the minimum number of critical cells.

The construction of this matching is the bulk of our work. We use the fact that lower intervals in $\mathcal{F}(\mathscr{A})$ are face posets of real arrangements and call this the "local" structure of $\mathcal{F}(\mathscr{A})$. Correspondingly, the Salvetti category is covered by face posets of Salvetti complexes of these 'local' arrangements.

In Delucchi (2008) it is shown how a particular total ordering of the topes of any oriented matroid leads to a nice stratification of the associated "classical" Salvetti complex with explicitly described strata that each admit a perfect acyclic matching. Here we construct a decomposition of the Salvetti category - indexed by a special total ordering of the "local no-broken-circuits" (see Section 3.1) - which, on each 'local' piece, restricts to the above stratification of the "classical" Salvetti complex. We use diagrams over acyclic categories to prove that every piece of this decomposition is in fact isomorphic to the face category of the stratification of a (smaller dimensional) real torus by a suitable (real) toric arrangement. This construction is explained in Section 5.2.

We are then left to prove that, for any complexified toric arrangement $\mathscr{A}$, the face category $\mathcal{F}(\mathscr{A})$ admits a perfect acyclic matching, as is explained in Section 5.4.

The final step is to patch together the acyclic matchings of the different pieces making sure that they add up to an acyclic matching with the required number of critical cells (Proposition 54).

2 Basics

Definition 1 *Let* $\Lambda \cong \mathbb{Z}^d$ *a finite rank lattice. The corresponding* complex torus *is* $T_\Lambda = \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C}^*)$. *The* compact (or real) torus *corresponding to* Λ *is* $T^c_\Lambda = \mathrm{Hom}_{\mathbb{Z}}(\Lambda, S^1)$*, where* $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$.

Remark 2 Consider a finite rank lattice Λ and the corresponding torus T_Λ. Every $\lambda \in \Lambda$ defines a *character* of T_Λ, i.e. the function $\chi_\lambda : T_\Lambda \to \mathbb{C}^*$, $\chi_\lambda(\varphi) = \varphi(\lambda)$. Under pointwise multiplication, characters form a lattice which is naturally isomorphic to Λ. Therefore in the following we will identify the character lattice of T_Λ with Λ.

Definition 3 *Consider a finite rank lattice Λ, a* toric arrangement *in T_Λ is a finite set of pairs*

$$\mathscr{A} = \{(\chi_1, a_1), \dots (\chi_n, a_n)\} \subset \Lambda \times \mathbb{C}^*.$$

A toric arrangement $\mathscr{A}$ is called complexified *if $\mathscr{A} \subset \Lambda \times S^1$. Correspondingly, a* real *toric arrangement in T_Λ^c is a finite set of pairs $\mathscr{A}^c = \{(\chi_1, a_1), \dots (\chi_n, a_n)\} \subset \Lambda \times S^1$.*

Remark 4 The abstract definition is clearly equivalent to the one given in the introduction via the canonical isomorphism of Remark 2 and by $K_i := \chi_i^{-1}(a_i)$. Accordingly, we have $M(\mathscr{A}) := T_\Lambda \setminus \bigcup\{K_1, \dots, K_n\}$ and, for a real toric arrangement $\mathscr{A}^c$, $M(\mathscr{A}^c) := T_\Lambda^c \setminus \bigcup\{K_1, \dots, K_n\}$.

Definition 5 *Let Λ be a rank d lattice and let $\mathscr{A}$ be a toric arrangement on T_Λ. The* rank *of $\mathscr{A}$ is rk$(\mathscr{A}) :=$ rk $\langle \chi \mid (\chi, a) \in \mathscr{A}\rangle$. A character $\chi \in \Lambda$ is called* primitive *if, for all $\psi \in \Lambda$, $\chi = \psi^k$ only if $k \in \{-1, 1\}$. The toric arrangement $\mathscr{A}$ is called* primitive *if for each $(\chi, a) \in \mathscr{A}$, χ is primitive. The toric arrangement $\mathscr{A}$ is called* essential *if rk$(\mathscr{A}) = d$.*

Remark 6 For every non primitive arrangement there is a primitive arrangement which has the same complement. Furthermore, if $\mathscr{A}$ is a non essential arrangement, then there exist an essential arrangement $\mathscr{A}'$ such that

$$M(\mathscr{A}) \cong (\mathbb{C}^*)^{d-l} \times M(\mathscr{A}') \text{ where } l = \text{rk}(\mathscr{A}').$$

Therefore the topology of $M(\mathscr{A})$ can be derived from the topology of $M(\mathscr{A}')$.

Assumption. From now on we assume every toric arrangement to be *primitive* and *essential*.

2.1 Layers

Let $\mathscr{A} = \{(\chi_1, a_1), \dots, (\chi_n, a_n)\}$ be a toric arrangement on T_Λ. Following De Concini and Procesi (2010) we call *layer* of $\mathscr{A}$ any connected component of a nonempty intersection of some of the subtori K_i (defined in Remark 4). The set of all layers of $\mathscr{A}$ ordered by reverse inclusion is the *poset of layers* of the toric arrangement, denoted by $\mathcal{C}(\mathscr{A})$.

Definition 7 *Let Λ be a finite rank lattice and $\mathscr{A}$ be a toric arrangement in T_Λ. For every sublattice $\Gamma \subseteq \Lambda$ we define the arrangement $\mathscr{A}_\Gamma = \{(\chi, a) \in \mathscr{A} \mid \chi \in \Gamma\}$ and for every layer $X \in \mathcal{C}(\mathscr{A})$ the sublattice $\Gamma_X := \{\chi \in \Lambda \mid \chi$ is constant on $X\} \subseteq \Lambda$. Then, we can define toric arrangements*

$$\mathscr{A}_X := \mathscr{A}_{\Gamma_X} \text{ on } T_{\Gamma_X}, \quad \mathscr{A}^X := \{K_i \cap X \mid X \not\subseteq K_i\} \text{ on the torus } X.$$

Remark 8 Notice that for a layer $X \in \mathcal{C}(\mathscr{A})$ and a hypersurface K of $\mathscr{A}$, the intersection $K \cap X$ is not necessarily connected. In general $K \cap X$ consist of several connected components, each of which is a level set of a character in the torus X. Thus, $\mathscr{A}^X$ is a toric arrangement in the sense of Definition 3.

2.2 Face category

To any complexified toric arrangement is associated the stratification of the real torus T_Λ^c into *chambers* and *faces* induced by the associated ‘real’ arrangement $\mathscr{A}^c$, as follows.

Definition 9 *Consider a complexified toric arrangement* $\mathscr{A} = \{(\chi_1, a_1), \ldots, (\chi_n, a_n)\}$*, its* chambers *are the connected components of* $M(\mathscr{A}^c)$*. We denote the set of chambers of* $\mathscr{A}$ *by* $\mathcal{T}(\mathscr{A})$*.*

The faces *of* $\mathscr{A}$ *are the connected components of the intersections* $\overline{C} \cap X$ *where* $C \in \mathcal{T}(\mathscr{A})$*,* $X \in \mathcal{C}(\mathscr{A})$*. They are the (closed) cells of a polyhedral complex, which we denote by* $\mathcal{D}(\mathscr{A})$*.*

The incidence structure of a (possibly non regular) polyhedral complex X is encoded in a category with one object for every cell, and a morphism for every 'face-relation' among cells. This is called the *face category* of the complex and is denoted by $\mathcal{F}(X)$ (see (d'Antonio and Delucchi, 2011, §2.2.2) for some details on face categories). It is an *acyclic category* in the sense of Kozlov (2008).

Definition 10 *The face category of a complexified toric arrangement* $\mathscr{A}$ *is* $\mathcal{F}(\mathscr{A}) = \mathcal{F}(\mathcal{D}(\mathscr{A}))$*, i.e., the face category of the polyhedral complex* $\mathcal{D}(\mathscr{A})$*.*

2.3 Hyperplane arrangements

Throughout this section let V be a finite dimensional vector space over a field $\mathbb{K}$. An *affine hyperplane* H in V is a level set of a linear functional on V. A set of hyperplanes is called *dependent* or *independent* according to whether the corresponding set of functionals is linearly dependent in V^* or not.

Definition 11 *A* arrangement of hyperplanes *in* V *is a collection* $\mathscr{B}$ *of affine hyperplanes in* V*.*

A hyperplane arrangement $\mathscr{B}$ is called *central* if every hyperplane $H \in \mathscr{B}$ is a linear subspace of V; *finite* if $\mathscr{B}$ is finite; *locally finite* if for every $p \in V$ the set $\{H \in \mathscr{B} \mid p \in H\}$ is finite; *real* (or *complex*) if V is a real (or complex) vector space.

For every central hyperplane arrangement $\mathscr{B}$, the set $\mathcal{L}(\mathscr{B})$ of all nonempty intersections of hyperplanes, ordered by reverse inclusion, is a geometric lattice and defines the *matroid* associated to $\mathscr{B}$.

Definition 12 *An arrangement* $\mathscr{B}$ *in* $\mathbb{C}^d$ *is called* complexified *if every hyperplane* $H \in \mathscr{B}$ *is the complexification of a real hyperplane, i.e., if* $H = \alpha_H^{-1}(a_H)$ *for* $a_H \in \mathbb{R}$ *and* $\alpha_H \in (\mathbb{R}^d)^* \subset (\mathbb{C}^d)^*$*. The real part of a complexified hyperplane arrangement* $\mathscr{B}$ *is* $\mathscr{B}_\mathbb{R} = \{H \cap \mathbb{R}^d \mid H \in \mathscr{B}\}$*, an arrangement of hyperplanes in* $\mathbb{R}^d$*.*

A real hyperplane arrangement $\mathscr{B}$ induces a polyhedral decomposition $\mathcal{D}(\mathscr{B})$ of the real ambient space. The face category of this polyhedral complex is denoted $\mathcal{F}(\mathscr{B}) := \mathcal{F}(\mathcal{D}(\mathscr{B}))$. The top cells of this decomposition are called *chambers* of $\mathscr{B}$, the set of chambers is denoted $\mathcal{T}(\mathscr{B})$.

If $\mathscr{B}$ is a complexified hyperplane arrangement, we write $\mathcal{F}(\mathscr{B}) := \mathcal{F}(\mathscr{B}_\mathbb{R})$ and $\mathcal{T}(\mathscr{B}) := \mathcal{T}(\mathscr{B}_\mathbb{R})$.

2.4 Covering space

The preimage of a toric arrangement $\mathscr{A}$ under the covering map $p : \mathbb{C}^d \cong \mathrm{Hom}_\mathbb{Z}(\Lambda, \mathbb{C}) \to \mathrm{Hom}_\mathbb{Z}(\Lambda, \mathbb{C}^*) = T_\Lambda,\ \varphi \mapsto \exp \circ \varphi$ is a locally finite affine hyperplane arrangement on $\mathrm{Hom}_\mathbb{Z}(\Lambda; \mathbb{C})$. Choosing coordinates we can associate to the character χ_i an integer vector $\alpha_i = \alpha(\chi_i) \in \mathbb{Z}^d$ so that $\chi_i(x) = x_1^{\alpha_{i,1}} \cdots x_d^{\alpha_{i,d}}$ and then let

$$\mathscr{A}^\upharpoonright := \{H_{\chi,a'} \mid (\chi, e^{2\pi i a'}) \in \mathscr{A}\} \quad \text{where} \quad H_{\chi,a'} = \{x \in \mathbb{C}^n \mid \langle \alpha(\chi), x\rangle = a'\}.$$

Remark 13 If the toric arrangement $\mathscr{A}$ is complexified, so is the hyperplane arrangement $\mathscr{A}^\upharpoonright$.

The lattice Λ acts on $\mathbb{C}^n$ and on $\mathbb{R}^n$ as the group of automorphisms of the covering map p. Consider now the map $q : \mathcal{F}(\mathscr{A}^\upharpoonright) \to \mathcal{F}(\mathscr{A})$ induced by p.

Proposition 14 ((d'Antonio and Delucchi, 2011, Lemma 4.8)) *Let $\mathscr{A}$ be a complexified toric arrangement. The map $q : \mathcal{F}(\mathscr{A}^{\uparrow}) \to \mathcal{F}(\mathscr{A})$ induces an isomorphism of acyclic categories $\mathcal{F}(\mathscr{A}) \cong \mathcal{F}(\mathscr{A}^{\uparrow})/\Lambda$.*

s

3 Combinatorics

In this section we define local no-broken-circuit sets and prove some combinatorial results about chambers of real hyperplane arrangements.

Lemma 15 *Let $\mathscr{A}$ be a toric arrangement, $X \in \mathcal{C}(\mathscr{A})$ a layer. Then the subposet $\mathcal{C}(\mathscr{A})_{\leq X}$ is the intersection poset of a central hyperplane arrangement $\mathscr{A}[X]$. If $\mathscr{A}$ is complexified, then $\mathscr{A}[X]$ is, too.*

Proof: This is implicit in much of De Concini and Procesi (2005).The proof follows by lifting the layer X to $\mathscr{A}^{\uparrow}$. A precise definition of $\mathscr{A}[Y]$ can also be found in Section 5.2.1 below. □

3.1 No-broken-circuit sets, local and global

Recall the terminology of Section 2.3.

Definition 16 *Let $\mathscr{B}$ be a central arrangement of hyperplanes with an arbitrary (but fixed) total order. A* circuit *is a minimal dependent subset $C \subseteq \mathscr{A}$. A* broken circuit *is a subset of the form $C \setminus \{min\, C\} \subseteq \mathscr{B}$ obtained from a circuit removing its least element. A* no-broken-circuit set *(or, for short, an* nbc *set) is a subset $N \subseteq \mathscr{B}$ which does not contain any broken circuit. We will write $\mathrm{nbc}(\mathscr{B})$ for the set of no-broken-circuit sets of $\mathscr{B}$ and $\mathrm{nbc}_k(\mathscr{B}) = \{N \in \mathrm{nbc}(\mathscr{B}) \mid |N| = k\}$ for the set of all no-broken-circuit sets of cardinality k.*

Remark 17 For all $k = 0, \ldots, d$, the cardinality $|\,\mathrm{nbc}_k(\mathscr{B})|$ does not depend on the chosen total ordering.

Definition 18 (De Concini and Procesi (2005)) *Let $\mathscr{A}$ be a toric arrangement of rank d and let us fix a total ordering on $\mathscr{A}$. A* local no-broken-circuit set *of $\mathscr{A}$ is a pair*

$$(X, N) \text{ with } X \in \mathcal{C}(\mathscr{A}), N \in \mathrm{nbc}_k(\mathscr{A}[X]) \text{ where } k = d - \dim X$$

We will write $\mathscr{N}$ for the set of local non broken circuits, and partition it into subsets

$$\mathscr{N}_j = \{(X, N) \in \mathscr{N} \mid \dim X = d - j\}.$$

Local no-broken-circuit sets can be used to express the Poincaré polynomial of $M(\mathscr{A})$. The following result was obtained in De Concini and Procesi (2005) by computing de Rham cohomology, in Looijenga (1993) via spectral sequence computations.

Theorem 19 (see (De Concini and Procesi, 2005, Theorem 4.2)) *Consider a toric arrangement $\mathscr{A}$. The Poincaré polynomial of $M(\mathscr{A})$ can be expressed as follows:*

$$P_{\mathscr{A}}(t) = \sum_{j=0}^{\infty} \dim H^j(M(\mathscr{A}); \mathbb{C})\, t^j = \sum_{j=0}^{\infty} |\mathscr{N}_j|\, (t+1)^{k-j}\, t^j.$$

3.2 Combinatorics of real hyperplane arrangements

In this section we will discuss some of the combinatorics of affine arrangements of hyperplanes in real space. Again, we refer the reader to standard references such as Björner et al. (1999); Orlik and Terao (1992) for the basics.

If $\mathscr{B}$ is an arrangement in a real space V, then every hyperplane H is the locus where a linear form $\alpha_H \in V^*$ takes the value a_H. This way we can associate to each $H \in \mathscr{B}$, its *positive* and *negative halfspace*: $H^\epsilon := \{x \in V \mid \operatorname{sgn}(\alpha_H(x) - a_H) = \epsilon\}$ for $\epsilon \in \{+, 0, -\}$.

Definition 20 *Consider a complexified locally finite arrangement $\mathscr{B}$ with any choice of 'sides' for every $H \in \mathscr{B}$. The* sign vector *of a face $F \in \mathcal{F}(\mathscr{B})$ is the function $\gamma_F : \mathscr{B} \to \{-, 0+\}$ defined as: $\gamma_F(H) := \epsilon$ if* relint $F \subseteq H^\epsilon$.

Notice that chambers are precisely those faces whose sign vector maps $\mathscr{B}$ to $\{-, +\}$.

Definition 21 *Let C_1 and $C_2 \in \mathcal{T}(\mathscr{B})$ be chambers of a real arrangement, and let $B \in \mathcal{T}(\mathscr{B})$ be a distinguished chamber. We will write $S(C_1, C_2) := \{H \in \mathscr{B} \mid \gamma_{C_1}(H) \neq \gamma_{C_2}(H)\}$ for the set of hyperplanes of $\mathscr{B}$ which separate C_1 and C_2.*
For all $C_1, C_2 \in \mathcal{T}(\mathscr{B})$ write $C_1 \leq C_2$ if and only if $S(C_1, B) \subseteq S(C_2, B)$. This turns $\mathcal{T}(\mathscr{B})$ into a poset $\mathcal{T}(\mathscr{B})_B$, the poset of regions *of the arrangement $\mathscr{B}$ with base chamber B.*

Remark 22 Let $\mathscr{B}_0$ be a real arrangement and $B \in \mathcal{T}(\mathscr{B}_0)$. Given a subarrangement $\mathscr{B}_1 \subseteq \mathscr{B}_0$, for every chamber $C \in \mathcal{T}(\mathscr{B}_0)$ there is a unique chamber $\widehat{C} \in \mathcal{T}(\mathscr{B}_1)$ with $C \subseteq \widehat{C}$.

Definition 23 *Let $\mathscr{B}_0$ be a real arrangement and let $\succ_0$ denote any total ordering of $\mathcal{T}(\mathscr{B}_0)$. Consider a subarrangement $\mathscr{B}_1 \subseteq \mathscr{B}_0$. The function*

$$\mu[\mathscr{B}_1, \mathscr{B}_0] : \mathcal{T}(\mathscr{B}_1) \to \mathcal{T}(\mathscr{B}_0), \quad C \mapsto \min_{\succ_0}\{K \in \mathcal{T}(\mathscr{B}_0) \mid K \subseteq C\}$$

defines a total ordering $\succ_{0,1}$ on $\mathcal{T}(\mathscr{B}_1)$ by $C \succ_{0,1} D \iff \mu[\mathscr{B}_1, \mathscr{B}_0](C) \succ_0 \mu[\mathscr{B}_1, \mathscr{B}_0](D)$ that we call induced by $\succ_0$.

Proposition 24 (Proposition 11 of d'Antonio and Delucchi (2012)) *Let a base chamber B of $\mathscr{B}_0$ be chosen. If $\succ_0$ is a linear extension of $\mathcal{T}(\mathscr{B}_0)_B$, then $\succ_{0,1}$ is a linear extension of $\mathcal{T}(\mathscr{B}_1)_{\widehat{B}}$.*

4 A combinatorial model for the topology of toric arrangements

In this Section we explain the construction of a combinatorial model for the homotopy type of the complement $M(\mathscr{A})$ of a given complexified toric arrangement.

4.1 The homotopy type of complexified hyperplane arrangements

If $\mathscr{B}$ is a complexified hyperplane arrangement, one can use the combinatorial structure of $\mathscr{B}_{\mathbb{R}}$ to study the topology of $M(\mathscr{B})$. In fact, using combinatorial data about $\mathscr{B}_{\mathbb{R}}$, Salvetti defined a cell complex which embeds in the complement $M(\mathscr{B})$ as a deformation retract (see Salvetti (1987)). We explain this construction.

Definition 25 *Given a face $F \in \mathcal{F}(\mathscr{B})$ and a chamber $C \in \mathcal{T}(\mathscr{B})$, define $C_F \in \mathcal{T}(\mathscr{B})$ as the unique chamber such that*

$$\gamma_{C_F}(H) = \begin{cases} \gamma_F(H) & \text{if } \gamma_F(H) \neq 0 \\ \gamma_C(H) & \text{if } \gamma_F(H) = 0 \end{cases}$$

The reader may think of C_F as the one, among the chambers adjacent to F, that "faces" C.

Definition 26 *Consider an affine complexified locally finite arrangement $\mathscr{B}$ and define the* Salvetti poset *as follows:*

$$\mathrm{Sal}(\mathscr{B}) = \{[F, C] \mid F \in \mathcal{F}(\mathscr{B}), C \in \mathcal{T}(\mathscr{B})\, F \leq C\},$$

with the relation $[F_1, C_1] \leq [F_2, C_2] \iff F_2 \leq F_1$ and $(C_2)_{F_1} = C_1$.

Let $\mathscr{B}$ be an affine complexified locally finite hyperplane arrangement. Its Salvetti complex *is the order complex $\mathcal{S}(\mathscr{B}) = \Delta(\mathrm{Sal}(\mathscr{B}))$, i.e., the simplicial complex of all chains.*

Theorem 27 (Salvetti (1987)) *The complex $\mathcal{S}(\mathscr{B})$ is homotopically equivalent to the complement $M(\mathscr{B})$. More precisely $\mathcal{S}(\mathscr{B})$ embeds in $M(\mathscr{B})$ as a deformation retract.*

Remark 28 In fact, the poset $\mathrm{Sal}(\mathscr{B})$ is the face poset of a regular cell complex (of which $\mathcal{S}(\mathscr{B})$ is the barycentric subdivision) whose maximal cells correspond to the pairs $[P, C]$ with $P \in \min \mathcal{F}(\mathscr{B})$, $C \in \mathcal{T}(\mathscr{B})$. It is this complex which is described in Salvetti (1987).

4.2 The toric Salvetti category

In order to define the toric Salvetti category, we need an analogue of Definition 25 for toric arrangements.

Proposition 29 ((d'Antonio and Delucchi, 2011, Proposition 3.12)) *Let Λ be a finite rank lattice, Γ a sublattice of Λ. Let $\mathscr{A}$ a complexified toric arrangement on T_Λ and recall the arrangement $\mathscr{A}_\Gamma$ from Definition 7. The projection $\pi_\Gamma : T_\Lambda \to T_\Gamma$ induces a morphism of acyclic categories $\pi_\Gamma : \mathcal{F}(\mathscr{A}) \to \mathcal{F}(\mathscr{A}_\Gamma)$.*

Consider now a face $F \in \mathcal{F}(\mathscr{A})$. We associate to it the sublattice $\Gamma_F = \{\chi \in \Lambda \mid \chi \text{ is constant on } F\} \subseteq \Lambda$.

Definition 30 *Consider a toric arrangement $\mathscr{A}$ on T_Λ and a face $F \in \mathcal{F}(\mathscr{A})$. The* restriction *of $\mathscr{A}$ to F is the arrangement $\mathscr{A}_F := \mathscr{A}_{\Gamma_F}$ on T_{Γ_F}.*

We will write $\pi_F = \pi_{\Gamma_F} : \mathcal{F}(\mathscr{A}) \to \mathcal{F}(\mathscr{A}_F)$.

Definition 31 ((d'Antonio and Delucchi, 2011, Definition 4.1)) *Let $\mathscr{A}$ be a toric arrangement on the complex torus T_Λ. The* Salvetti category *of $\mathscr{A}$ is the category $\mathrm{Sal}\,\mathscr{A}$ defined as follows.*

(a) The objects are the morphisms in $\mathcal{F}(\mathscr{A})$ between faces and chambers:

$$\mathrm{Obj}(\mathrm{Sal}\,\mathscr{A}) = \{m : F \to C \mid m \in \mathrm{Mor}(\mathcal{F}(\mathscr{A})),\, C \in \mathcal{T}(\mathscr{A})\}.$$

(b) The morphisms are the triples $(n, m_1, m_2) : m_1 \to m_2$, where $m_1 : F_1 \to C_1, m_2 : F_2 \to C_2 \in \mathrm{Obj}(\mathrm{Sal}\,\mathscr{A})$, $n : F_2 \to F_1 \in \mathrm{Mor}(\mathcal{F}(\mathscr{A}))$ and m_1, m_2 satisfy the condition $\pi_{F_1}(m_1) = \pi_{F_1}(m_2)$.

(c) Composition of morphisms is defined as $(n', m_2, m_3) \circ (n, m_1, m_2) = (n \circ n', m_1, m_3)$, whenever n and n' are composable.

Remark 32 The Salvetti category is an acyclic category in the sense of Kozlov (2008).

Definition 33 *Let $\mathscr{A}$ be a complexified toric arrangement; its* Salvetti complex *is the nerve* $\mathcal{S}(\mathscr{A}) = \Delta(\mathrm{Sal}\,\mathscr{A})$.

The following result generalizes (Moci and Settepanella, 2011).

Theorem 34 ((d'Antonio and Delucchi, 2011, Theorem 4.3)) *Let $\mathscr{A}$ be a complexified toric arrangement. The Salvetti complex $\mathcal{S}(\mathscr{A})$ embeds in the complement $M(\mathscr{A})$ as a deformation retract.*

Remark 35 As for the case of affine arrangements, the Salvetti category is the face category of a polyhedral complex, of which the toric Salvetti complex is a subdivision.

For the local structure of the toric Salvetti complex see Remark 40 below.

5 Minimality and torsion-freeness

5.1 'Local' minimality

In the case of complexified arrangements, explicit constructions of a minimal CW-complex for $M(\mathscr{B})$ were given in Salvetti and Settepanella (2007) and in Delucchi (2008). We review the material of (Delucchi, 2008, §4) that will be useful for our later purposes.

Lemma 36 ((Delucchi, 2008, Theorem 4.13)) *Let $\mathscr{B}$ be a central arrangement of real hyperplanes, let $B \in \mathcal{T}(\mathscr{A})$ and let $\preceq$ be any linear extension of the poset $\mathcal{T}(\mathscr{B})_B$. The subset of all $X \in \mathcal{L}(\mathscr{B})$ such that*

$$S(C, C') \cap \mathscr{B}_X \neq \emptyset \quad \textit{for all } C' \prec C$$

is an order ideal of $\mathcal{L}(\mathscr{B})$. In particular, it has a well defined and unique minimal element we will call X_C.

Now recall the (cellular) Salvetti complex of Definition 26 and Remark 28. In particular, its maximal cells correspond to the pairs $[P, C]$ where P is a point and C is a chamber. When $\mathscr{B}$ is a central arrangement, the maximal cells correspond to the chambers in $\mathcal{T}(\mathscr{B})$. In this case we can stratify the Salvetti complex assigning to each chamber $C \in \mathcal{T}(\mathscr{B})$ the corresponding maximal cell of $\mathcal{S}(\mathscr{B})$, together with its faces.

Definition 37 *Let $\mathscr{B}$ be a central complexified hyperplane arrangement and write* $\min \mathcal{F}(\mathscr{B}) = \{P\}$. *Define a stratification of the cellular Salvetti complex $\mathcal{S}(\mathscr{B}) = \bigcup_{C \in \mathcal{T}(\mathscr{B})} \mathcal{S}_C$ through*

$$\mathcal{S}_C := \bigcup \{[F, K] \in \mathrm{Sal}(\mathscr{B}) \mid [F, K] \leq [P, C]\}.$$

Given an arbitrary linear extension $(\mathcal{T}(\mathscr{B}), \preceq)$ of $\mathcal{T}(\mathscr{B})_B$, for all $C \in \mathcal{T}(\mathscr{B})$ define

$$\mathcal{N}_C := \mathcal{S}_C \setminus \left(\bigcup_{D \prec C} \mathcal{S}_D \right), \quad \text{so that} \quad \mathrm{Sal}(\mathscr{B}) = \bigsqcup_{C \in \mathcal{T}(\mathscr{B})} \mathcal{N}_C(\mathscr{B}).$$

Theorem 38 ((Delucchi, 2008, Lemma 4.18)) *There is an isomorphism of posets*

$$\mathcal{N}_C \cong \mathcal{F}(\mathscr{B}^{X_C})^{op}$$

where X_C is the intersection defined via Lemma 36 by the same choice of base chamber and of linear extension of $\mathcal{T}(\mathscr{B})_B$ used to define the subposets $\mathcal{N}_C$, while $\mathscr{B}^{X_C} = \{H \cap X_C \mid H \in \mathscr{B}\}$ denotes the arrangement in the subspace X_C determined by restriction of $\mathscr{B}$.

5.2 Stratification of the toric Salvetti category

We now work our way toward proving the minimality of complements of toric arrangements. We start by defining a stratification of the toric Salvetti complex, in which each stratum corresponds to a local non broken circuit.

5.2.1 Local geometry of complexified toric arrangements

Consider a rank d complexified toric arrangement $\mathscr{A} = \{(\chi_1, a_1), \ldots, (\chi_n, a_n)\}$. Choose coordinates and, as usual, write $\chi_i(x) = x^{\alpha_i}$ for $\alpha_i \in \mathbb{Z}^d$ and $K_i = \{x \in T_\Lambda \mid \chi_i(x) = a_i\}$.

We introduce some central hyperplane arrangements we will work with. Consider the arrangement

$$\mathscr{A}_0 = \{H_i = \ker \langle \alpha_i, \cdot \rangle \mid i = 1, \ldots, n\}$$

in $\mathbb{R}^d$ and, from now on, fix a chamber $B \in \mathcal{T}(\mathscr{A}_0)$ and a linear extension $\prec_0$ of $\mathcal{T}(\mathscr{A}_0)_B$.

Definition 39 *For every face $F \in \mathcal{F}(\mathscr{A})$ and every layer $Y \in \mathcal{C}(\mathscr{A})$ define the arrangements*

$$\mathscr{A}[F] = \{H_i \in \mathscr{A}_0 \mid \chi_i(F) = a_i\}, \qquad \mathscr{A}[Y] = \{H_i \in \mathscr{A}_0 \mid Y \subseteq K_i\},$$

and let $B_F \in \mathcal{T}(\mathscr{A}[F])$, resp. $B_Y \in \mathcal{T}(\mathscr{A}[Y])$, be such that that $B \subseteq B_F$, resp. $B \subseteq B_Y$.

Remark 40 The Salvetti category is the colimit of a diagram over the index category $\mathcal{F}(\mathscr{A})$, which associates to every $F \in \mathcal{F}(\mathscr{A})$ the poset $\mathrm{Sal}(\mathscr{A}[F])$ (d'Antonio and Delucchi, 2012, Lemma 77).

Remark 41 The linear extension $\prec_0$ of $\mathcal{T}(\mathscr{A}_0)_B$ induces, as in Proposition 24, linear extensions $\prec_F$ of $\mathcal{T}(\mathscr{A}[F])_{B_F}$ and $\prec_Y$ of $\mathcal{T}(\mathscr{A}[Y])_{B_Y}$, for every $F \in \mathcal{F}(\mathscr{A})$ and every $Y \in \mathcal{C}(\mathscr{A})$.

Definition 42 *Given $X \in \mathcal{C}(\mathscr{A})$ let $\widetilde{X} \in \mathcal{L}(\mathscr{A}_0)$ be defined as*

$$\widetilde{X} := \bigcap_{X \subseteq K_i} H_i.$$

Definition 43 *Let $Y \in \mathcal{C}(\mathscr{A})$ be a layer of $\mathscr{A}$. For $C \in \mathcal{T}(\mathscr{A}[Y])$ let $X(Y,C) \supseteq Y$ be the layer determined by the intersection defined by Lemma 36 from $\prec_Y$. Analogously, for $C \in \mathcal{T}(\mathscr{A}[F])$ let $X(F,C)$ be defined with respect to $\prec_F$.*

Let then

$$\mathscr{Y} := \{(Y,C) \mid Y \in \mathcal{C}(\mathscr{A}),\, C \in \mathcal{T}(\mathscr{A}[Y]),\, X(Y,C) = Y\}.$$

Moreover, for $i = 0, \ldots, d$ let $\mathscr{Y}_i := \{(Y,C) \in \mathscr{Y} \mid \dim(Y) = i\}$.

Lemma 44 *Let $\mathscr{A}$ be a rank d toric arrangement. For all $i = 0, \ldots d$, we have $|\mathscr{Y}_i| = |\mathscr{N}_i|$.*

As a last preparation, we need to be able to map morphisms $m : F \to G$ of $\mathcal{F}(\mathscr{A})$ to the corresponding face of the arrangement $\mathscr{A}[F]$.

Definition 45 *Consider a toric arrangement $\mathscr{A}$ on $T_\Lambda \cong (\mathbb{C}^*)^k$ and a morphism $m : F \to G$ of $\mathcal{F}(\mathscr{A})$. We associate to m a face $F_m \in \mathcal{F}(\mathscr{A}[F])$ as follows.*

First, fix an $F^\upharpoonright \in \mathcal{F}(\mathscr{A}^\upharpoonright)$ such that $q(F^\upharpoonright) = F$. From Proposition 29 and from the freeness of the action of Λ it follows that there is a unique $G^\upharpoonright \in \mathcal{F}(\mathscr{A}^\upharpoonright)$ such that $q(F^\upharpoonright \leq G^\upharpoonright) = m$. Then, consider the arrangement $\mathscr{A}^\upharpoonright_{F^\upharpoonright} = \{H \in \mathscr{A}^\upharpoonright \mid F^\upharpoonright \in H\}$. Clearly, up to translation, $\mathscr{A}^\upharpoonright_{F^\upharpoonright} = \mathscr{A}[F]$ and we can identify the two arrangements. Now define F_m as the face of $\mathscr{A}[F]$ which contains $G^\upharpoonright$. That is, in terms of sign vectors and identifying each $H \in \mathscr{A}[F]$ with its unique translate which contains $G^\upharpoonright$: $\gamma_{F_m} = \gamma_{G^\upharpoonright|\mathscr{A}[F]}$. In particular, when G is a chamber, then F_m also is.

5.2.2 Definition of the strata

Definition 46 *Recall Definition 23. The assignment $(Y, C) \mapsto \mu[\mathscr{A}[Y], \mathscr{A}_0](C)$ defines a function $\xi_0 : \mathscr{Y} \to \mathcal{T}(\mathscr{A}_0)_B$. Choose, and fix, a total order $\dashv$ on $\mathscr{Y}$ that makes this function order preserving.*

Definition 47 *Define the map $\theta : \mathrm{Sal}(\mathscr{A}) \to \mathscr{Y};\ (m : F \to C) \mapsto (X(F, F_m),\, \sigma_{\mathscr{A}[X(F,F_m)]}(F_m))$.*

Through θ we can now define a filtration of $\mathrm{Sal}(\mathscr{A})$.

Definition 48 *Given a complexified toric arrangement $\mathscr{A}$ on $(\mathbb{C}^*)^d$, we consider the following stratification of Sal($\mathscr{A}$) indexed by $\mathscr{Y}$: we write Sal($\mathscr{A}$) $= \cup_{(Y,C)\in\mathscr{Y}} \mathcal{S}_{(Y,C)}$ where $\mathcal{S}_{(Y,C)}$ is the induced subcategory with $\mathrm{Ob}(\mathcal{S}_{(Y,C)}) = \{m \in \mathrm{Ob}(\mathrm{Sal}(\mathscr{A})) \mid \exists (m \to n) \in \mathrm{Mor}(\mathrm{Sal}(\mathscr{A})),\, n \in \theta^{-1}(Y, C)\}$. Moreover, recall the total ordering $\vdash$ on $\mathscr{Y}$ and define*

$$\mathcal{N}_y := \mathcal{S}_y \setminus \bigcup_{y' \dashv y} \mathcal{S}_{y'}.$$

We now come to the gist of our construction: everything has been arranged so that every stratum, as a category, is isomorphic to the face category of a real toric arrangement.

Theorem 49 *Consider a complexified toric arrangement $\mathscr{A}$ and for $(Y, C) \in \mathscr{Y}$ let $\mathcal{N}_{(Y,C)}$ be as in Definition 48. Then there is an isomorphism of acyclic categories*

$$\mathcal{N}_{(Y,C)} \cong \mathcal{F}(\mathscr{A}^Y)^{op}.$$

The details of the proof are very technical and quite lengthy. We believe that it is in the best interest of the clarity of this extended abstract to refer the interested reader to the full treatment given in d'Antonio and Delucchi (2012).

5.3 Discrete Morse Theory for acyclic categories

Our proof of minimality will consist in describing a sequence of cellular collapses on the toric Salvetti complex, which is not necessarily a regular cell complex. We need thus to extend discrete Morse theory from posets to acyclic categories. The setup used in the textbook of Kozlov (2008) happens to lend itself very nicely to such a generalization - in fact, once the right definitions are made, even the proofs given there just need some minor additional observation.

We will omit the technicalities in this extended abstract, and refer to (d'Antonio and Delucchi, 2012, §3) for a more detailed account. We will only say that the notion of acyclic matching extends easily to acyclic categories so that an acyclic matching on the face category of a CW-complex X defines a sequence of cellular collapses on X that preserve the homotopy type and leads to a complex with as many cells in each dimension as there are corresponding critical (unmatched) cells in the original matching (d'Antonio and Delucchi, 2012, Definition 50, Theorem 53). We will call an acyclic matching *perfect* if its number of critical cells in dimension k is the k-th Betti number of X.

Moreover, the well-known Patchwork Lemma (Kozlov, 2008, Theorem 11.10) generalizes.

Lemma 50 ("Patchwork Lemma", Lemma 52 of d'Antonio and Delucchi (2012)) *Consider a functor of acyclic categories $\varphi : \mathcal{C} \to \mathcal{C}'$ and suppose that for each object c of $\mathcal{C}'$ an acyclic matching $\mathfrak{M}_c$ of $\varphi^{-1}(c)$ is given. Then the matching $\mathfrak{M} := \bigcup_{c \in \mathrm{Ob}\,\mathcal{C}'} \mathfrak{M}_c$ of $\mathcal{C}$ is acyclic.*

5.4 Perfect acyclic matchings for compact tori

Let $\mathscr{A}$ be a complexified toric arrangement in T_Λ and let $(\chi_1, a_1), \ldots, (\chi_d, a_d) \in \mathscr{A}$ be such that $\alpha_1, \ldots, \alpha_d$ (see Section 2.4) are ($\mathbb{Q}$-) linearly independent. Then $P = \cap_i K_i \in \max \mathcal{C}(\mathscr{A})$. Up to a biholomorphic transformation we may suppose that P is the origin of the torus. For $i = 1, \ldots, d$ let H_i^1 denote the hyperplane of $\mathscr{A}^\upharpoonright$ lifting K_i at the origin of $\mathrm{Hom}(\Lambda, \mathbb{R}) \simeq \mathbb{R}^d$. We identify for ease of notation $\Lambda \simeq \mathbb{Z}^d \subseteq \mathbb{R}^d$, and in particular think of α_i as the normal vector to H_i^1.

For $j \in [d]$ we consider the rank $j-1$ lattice $\Lambda_j := \mathbb{Z}^d \cap \bigcap_{i \geq j} H_i^1$. It is a standard exercise in algebra to find a basis $u_1, \ldots, u_d$ of Λ such that for all $i = 1, \ldots, d$, the elements $u_1, \ldots, u_{i-1}$ are a basis of Λ_i.

In particular, $u_i \notin H_i^1$, hence $u_i(H_i^1) \neq H_i^1$. Moreover, without loss of generality we may suppose $u_i \in (H_i^1)^+ := \{x \in \mathbb{R}^d \mid \langle x, \alpha_i \rangle \geq 0\}$.

For $i = 1, \ldots, d$, let $(H_i^2)^+ := u_i((H_i^1)^+)$, and define $Q := \bigcap_{i=1}^d [(H_i^1)^+ \setminus (H_i^2)^+]$.

Then, Q is a fundamental region for the action of Λ on $\mathbb{R}^d$ (d'Antonio and Delucchi, 2012, Lemma 86).

Definition 51 *Let $\mathscr{A}$ be a rank d toric arrangement, and let $\mathcal{B}_d$ be the 'boolean poset on d elements', i.e., the acyclic category on the subsets of $[d]$ with the inclusion morphisms. Since $\mathcal{B}_d$ is a poset, the function $\mathrm{Ob}(\mathcal{F}(\mathscr{A})) \to \mathrm{Ob}(\mathcal{B}_d)$, $F \mapsto \{i \in [d] \mid F \subseteq K_i\}$, induces a well defined functor of acyclic categories $\mathcal{I} : \mathcal{F}(\mathscr{A}) \to \mathcal{B}_d^{op}$.*

For every $I \subseteq [d]$ define the category $\mathcal{F}_I := \mathcal{I}^{-1}(I)$.

Lemma 52 (Lemma 89 of d'Antonio and Delucchi (2012)) *For all $I \subseteq [d]$, the subcategory $\mathcal{F}_I$ is a poset admitting an acyclic matching with only one critical element (in top rank).*

Proposition 53 *For any complexified toric arrangement $\mathscr{A}$, the acyclic category $\mathcal{F}(\mathscr{A})$ admits a perfect acyclic matching.*

Proof: Let $\mathscr{A}$ be of rank d. The proof is a straightforward application of the Patchwork Lemma (Lemma 50) in order to merge the 2^d acyclic matchings described in Lemma 52 along the map $\mathcal{I}$ of Definition 51. The resulting 'global' acyclic matching has 2^d critical elements and is thus perfect. □

5.5 Minimality

Let $\mathscr{A}$ be a (complexified) toric arrangement.

Proposition 54 *The Salvetti category $\mathrm{Sal}\,\mathscr{A}$ admits a perfect acyclic matching.*

Proof: Let P denote the acyclic category given by the $|\mathscr{Y}|$-chain. We define a functor of acyclic categories

$$\varphi : \operatorname{Sal}\mathscr{A} \to P; \quad m \mapsto (Y,C) \text{ for } m \in \mathcal{N}_{(Y,C)}$$

and with Theorem 49 we have an isomorphism of acyclic categories $\varphi^{-1}((Y,C)) = \mathcal{N}_{(Y,C)} \simeq \mathcal{F}(\mathscr{A}_Y)$. Then, by Proposition 53, $\varphi^{-1}((Y,C))$ has an acyclic matching with $2^{d-\operatorname{rk} X}$ critical cells.

An application of the Patchwork Lemma 50 gives then an acyclic matching on $\operatorname{Sal}(\mathscr{A})$ with $\sum_j |\mathscr{Y}_j| 2^{d-j} = \sum_j |\mathcal{N}_j| 2^{d-j} = P_{\mathscr{A}}(1)$ critical cells, where the first equality is given by Lemma 44. This matching is thus perfect. □

Corollary 55 *The complement $M(\mathscr{A})$ is a minimal space.*

Corollary 56 *The groups $H_k(M(\mathscr{A}),\mathbb{Z})$, $H^k(M(\mathscr{A}),\mathbb{Z})$ are torsion free for all k.*

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