A Decomposition of the Descent Algebra of a Finite Coxeter Group

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Abstract

The purpose of this paper is two-fold. First we aim to unify previous work by the first two authors, A. Garsia, and C. Reutenauer (see [1], [2], [4], [3] and [9]) on the structure of the *descent algebras* of the Coxeter groups of type A_n and B_n. But we shall also extend these results to the descent algebra of an arbitrary finite Coxeter group W. The descent algebra, introduced by Solomon in [13], is a subalgebra of the group algebra of W. It is closely related to the subring of the Burnside ring $B(W)$ spanned by the permutation representations W/W_J, where the W_J are the parabolic subgroups of W. Specifically, our purpose is to lift a basis of primitive idempotents of the *parabolic Burnside algebra* to a basis of idempotents of the descent algebra.

Introduction

Let (W,S) be a finite Coxeter system. That is to say, W is a finite group generated by a set S subject to the fining relations

$$(sr)^{m_{sr}} = 1 \qquad \text{for all } s, r \in S,$$

ıere the m_{sr} are positive integers and $m_{ss} = 1$ for all $s \in S$.

As is well known, W is faithfully represented in the orthogonal group of an inner product space V which has basis $\Pi = \{\, \alpha_s \mid s \in S \,\}$ in bijective correspondence with S. The inner product is given by

$$(\alpha_s, \alpha_r) = -\cos(\pi/m_{sr}),$$

ıd the action of W by

$$s(v) = v - 2(\alpha_s, v)\alpha_s$$

r all r, $s \in S$ and $v \in V$. Thus s acts as the reflection in the hyperplane orthogonal to α_s, and as a consequence is called the *reflection representation* of W. One easily checks that for all $s, r \in S$ we have $\alpha_r = \pm w(\alpha_s)$ in V if ıd only if $r = wsw^{-1}$ in W.

We call the set $\Phi = \{\, w(\alpha) \mid w \in W, \alpha \in \Pi \,\}$ the *root system* of W, and Π the set of *fundamental* roots. is well known (see [6]) that Φ can be decomposed as $\Phi = \Phi^+ \uplus \Phi^-$, where every element of Φ^+ (resp. Φ^-) is linear combination of fundamental roots with coefficients all non-negative (resp. all non-positive). Moreover, if $\in W$ and $\ell(w)$ denotes the length of a minimal expression for w in terms of elements of S, then $\ell(w)$ equals the rdinality of the set $N(w)$, where

$$N(w) = \{\, \alpha \in \Phi^+ \mid w(\alpha) \in \Phi^- \,\}.$$

ote that $\ell(vw) = \ell(v) + \ell(w)$ if and only if $N(vw) = w^{-1}(N(v)) \uplus N(w)$.

For each $J \subseteq \Pi$ the *standard parabolic subgroup* W_J is the subgroup of W generated by

$$S_J = \{\, s \in S \mid \alpha_s \in J \,\}.$$

Then (W_J, S_J) is also a Coxeter system. If V_J is the subspace of V spanned by J, then the W-action on V yields a W_J-action on V_J, which can be identified with the reflection representation of W_J. The root system of W_J is $\Phi_J = \Phi \cap V_J$; and we write Φ_J^+ for $\Phi^+ \cap V_J$ and Φ_J^- for $\Phi^- \cap V_J$. It is easily shown that $N(w) \subseteq \Phi_J^+$ if and only if $w \in W_J$.

In this paper we study the *descent algebra* (or *Solomon algebra*) $\Sigma(W)$ of a Coxeter group W. If $w \in W$, then the *descent set* of w is defined to be

$$D(w) = N(w) \cap \Pi = \{\, \alpha \in \Pi \mid w(\alpha) \in \Phi^- \,\}.$$

In terms of the generating set S this corresponds to $\{\, s \in S \mid \ell(ws) < \ell(w) \,\}$. If $J \subseteq \Pi$, let

$$X_J = \{\, w \in W \mid D(w) \cap J = \emptyset \,\} = \{\, w \in W \mid w(J) \subseteq \Phi^+ \,\}$$

and let

$$x_J = \sum_{w \in X_J} w.$$

Define $\Sigma(W)$ to be the subspace of $\mathbf{Q}(W)$ spanned by all such elements x_J.

It has been shown by Solomon [13] that $\Sigma(W)$ is a subalgebra of $\mathbf{Q}(W)$. More precisely, Solomon has shown that

$$x_J x_K = \sum_{L \subseteq K} a_{JKL} x_L, \tag{1.1}$$

where

$$a_{JKL} = |\{\, w \in X_J^{-1} \cap X_K \mid w^{-1}(J) \cap K = L \,\}|.$$

In Section 2 we shall prove these facts using techniques that will be developed further in later sections. It is easily shown (Solomon [13]) that the x_K's are linearly independent; thus they form a basis of $\Sigma(W)$.

In [9] A. Garsia and C. Reutenauer have given a decomposition of the multiplicative structure of the descent algebra of the symmetric group (the Coxeter group of type A_n). This decomposition exploits the action of the symmetric group on the free Lie algebra in a manner reminiscent of the Poincaré-Birkoff-Witt Theorem. In [1] and [4] we showed that a similar decomposition, as well as related results, also holds for the hyperoctahedral group (type B_n). The object of this paper, and ongoing work, is to extend these results to the descent algebra of any finite Coxeter group.

For a general descent algebra $\Sigma(W)$ we shall exhibit a new basis consisting of elements e_K, $K \subseteq \Pi$, defined by

$$e_K = \sum_{L \subseteq K} \beta_K^L x_L,$$

for some constants β_K^L, such that each e_K is a scalar multiple of an idempotent, and $\sum_{K \subseteq \Pi} e_K = 1$. Furthermore, for all J, $M \subseteq \Pi$, when $e_J e_M$ is expressed as a linear combination of the e_K's, the only non-zero coefficients correspond to subsets K of M that are equivalent to J, in the sense that $J = w(K)$ for some $w \in W$. As a consequence we obtain a set of idempotents $E_\lambda = \sum_{K \in \lambda} e_K$ indexed by equivalence classes λ of subsets of Π, such that

$$E_\lambda E_\mu = \begin{cases} 0 & \text{if } \lambda \neq \mu, \\ E_\lambda & \text{if } \lambda = \mu, \end{cases} \tag{1.2}$$

and $\sum_\lambda E_\lambda = 1$. In fact, the E_λ's form a decomposition of the identity into primitive idempotents. Furthermore, the E_λ's induce a decomposition of the action of $\Sigma(W)$ on $\mathbf{Q}(W)$ by left multiplication:

$$\mathbf{Q}(W) = \bigoplus_\lambda H_\lambda,$$

where $H_\lambda = E_\lambda \cdot \mathbf{Q}(W)$. From the formula for $e_J e_M$ given in Section 7 it can be seen that $e_J - e_K$ is in the radical of $\Sigma(W)$ whenever J is equivalent to K, and therefore

$$\dim(\sqrt{\Sigma(W)}) = 2^{|S|} - |\Lambda|,$$

where Λ is the set of equivalence classes of subsets of Π.

These constructions have already been carried through for all indecomposable finite Coxeter groups of type A_n (see [9]), and of type B_n (see [1] and [4]). Part of the study of the descent algebra has been carried through with extensive use of the computer algebra system Maple [2].

2. The Solomon Algebra

We start by proving some basic facts concerning the elements x_J defined in Section 1. Proofs of results we assume can be found in §2.7 of Carter [6].

If $J \subseteq \Pi$, then each element of W is uniquely expressible in the form du with $d \in X_J$ and $u \in W_J$, and here we have $\ell(du) = \ell(d) + \ell(u)$. Thus X_J is a set of representatives of the cosets wW_J in W. Likewise, if $K \subseteq J \subseteq \Pi$, then $X_K \cap W_J$ is a set of representatives of the cosets wW_K in W_J. In this situation we define

$$x_K^J = \sum_{w \in W_J \cap X_K} w$$

and note that $x_K^\Pi = x_K$. The next two lemmas provide analogues of induction and restriction for Solomon algebras. The connection with induction and restriction of permutation characters will be given in detail in Section 4.

Lemma 2.1. *If $K \subseteq J \subseteq \Pi$, then $X_K = X_J(W_J \cap X_K)$ and thus $x_K = x_J x_K^J$.*

Proof. If $d \in X_J$ and $w \in W_J \cap X_K$, then $w(K) \subseteq \Phi_J^+$, whence $dw(K) \subseteq d(\Phi_J^+) \subseteq \Phi^+$. It follows that $dw \in X_K$ and this shows that

$$\{\, dw \mid d \in X_J,\ w \in W_J \cap X_K \,\} \subseteq X_K.$$

Comparing cardinalities we see that equality holds; and, on taking sums, we have $x_K = x_J x_K^J$. ∎

Lemma 2.2. *For all $J, K \subseteq S$*

$$X_K = \biguplus_{d \in X_{JK}} (W_J \cap X_{J \cap d(K)})d,$$

where $X_{JK} = X_J^{-1} \cap X_K$; and thus

$$x_K = \sum_{d \in X_{JK}} x_{J\cap d(K)}^J d.$$

Proof. First note that if $d \in X_{JK}$ and $u \in W_J \cap X_{J\cap d(K)}$, then $d \in X_J^{-1}$ and $u \in W_J$; so an element of W can arise as a product ud in at most one way. Let $w \in X_K$ and write $w = ud$ with $d \in X_J^{-1}$ and $u \in W_J$. Since $\ell(ud) = \ell(u) + \ell(d)$ we have $N(d) \subseteq N(ud) = N(w)$, and so $d \in X_K$. Thus $d \in X_{JK}$, and furthermore

$$u(J \cap d(K)) \subseteq ud(K) = w(K) \subseteq \Phi^+,$$

so that $u \in W_J \cap X_{J\cap d(K)}$. It remains to prove that $ud \in X_K$ whenever $d \in X_{JK}$ and $u \in W_J \cap X_{J\cap d(K)}$. Since a fundamental root cannot be nontrivially expressed as a positive linear combination of positive roots we see that $K \cap d^{-1}(\Phi_J^+) = K \cap d^{-1}(J)$. But $d(K) \subseteq \Phi^+$ (since $d \in X_K$) and so $d(K) \subseteq (\Phi^+ \setminus \Phi_J^+) \cup (J \cap d(K))$. It follows that $ud(K) \subseteq u(\Phi^+ \setminus \Phi_J^+) \cup u(J \cap d(K)) \subseteq \Phi^+$, and therefore $ud \in X_K$, as required. ∎

Lemma 2.2 shows that each element of W is uniquely expressible in the form udw with $w \in W_K$, $d \in X_{JK}$ and $u \in W_J \cap X_{J\cap d(K)}$. Moreover, in this situation $\ell(udw) = \ell(u) + \ell(d) + \ell(w)$. It follows readily that each double coset $W_J w W_K$ contains a unique $d \in X_{JK}$, and that $W_J \cap dW_K d^{-1} = W_{J\cap d(K)}$.

For J, $K \subseteq \Pi$ we write $J \sim K$ whenever $w(J) = K$ for some $w \in W$ (that is, J and K are equivalent) and $J \preceq K$ whenever J is equivalent to a subset of K. The next lemma shows that this equivalence relation is the one used by Solomon in [13].

Lemma 2.3. *If J, $K \subseteq \Pi$, then $J \sim K$ if and only if W_J and W_K are conjugate, and $J \preceq K$ if and only if W_J is conjugate to a subgroup of W_K.*

Proof. Suppose that $w \in W$ satisfies $w^{-1}W_J w \subseteq W_K$. If d is the shortest element in $W_J w W_K$, then $d^{-1}W_J d \subseteq W_K$ and therefore

$$W_{J\cap d(K)} = W_J \cap dW_K d^{-1} = W_J.$$

Thus $J \cap d(K) = J$ and therefore $d^{-1}(J) \subseteq K$. All assertions of the lemma now follow.

Lemma 2.4. *If $J \subseteq \Pi$ and $d \in W$ with $d^{-1}(J) \subseteq \Pi$, then $X_J d = X_{d^{-1}(J)}$.*

Proof. For $w \in X_{d^{-1}(J)}$, it is clear that $wd^{-1} \in X_J$, and conversely for $w \in X_J$, that $wd \in X_{d^{-1}(J)}$.

Theorem 2.5. *For all $J, K \subseteq \Pi$*

$$x_J x_K = \sum_{L\subseteq K} a_{JKL} x_L$$

Proof.

$$\begin{aligned} x_J x_K &= x_J \sum_{d\in X_{JK}} x^J_{J\cap d(K)} d && \text{by Lemma 2.2} \\ &= \sum_{d\in X_{JK}} x_{J\cap d(K)} d && \text{by Lemma 2.1} \\ &= \sum_{d\in X_{JK}} x_{d^{-1}(J)\cap K} && \text{by Lemma 2.4} \\ &= \sum_L a_{JKL} x_L. \end{aligned}$$

Obviously $a_{JKL} = 0$ when $L \not\subseteq K$. Thus the theorem is proved.

Proposition 2.6. *Let a^J_{MLP} denote the structure constants of the descent algebra $\Sigma(W_J)$ corresponding to the x^J_K basis. If L, $K \subseteq \Pi$, then*

$$x_K x_L = \sum_{P\subseteq L} \Big(\sum_{M\subseteq J} a_{KJM} a^J_{MLP}\Big) x_P,$$

for all $J \subseteq \Pi$ such that $L \subseteq J$. Thus the structure constants satisfy the identities

$$a_{KLP} = \sum_{M\subseteq J} a_{KJM} a^J_{MLP},$$

for all J containing L.

Proof. We have

$$\begin{aligned} x_K x_L &= x_K x_J x^J_L \\ &= \Big(\sum_{M\subseteq J} a_{KJM} x_M\Big) x^J_L \\ &= \sum_{M\subseteq J} a_{KJM} x_J x^J_M x^J_L \\ &= \sum_{M\subseteq J} a_{KJM} x_J \Big(\sum_{P\subseteq L} a^J_{MLP} x^J_P\Big) \\ &= \sum a_{KJM} a^J_{MLP} x_P. \end{aligned}$$

This proves the first assertion of the theorem, and comparison with

$$x_K x_L = \sum_{P \subseteq L} a_{KLP} x_P$$

completes the proof. ∎

3. Reduction to indecomposable finite Coxeter groups

We shall now give a decomposition of the descent algebra of a product of two Coxeter groups. For a given Coxeter system (W, S), let W_K denote the subgroup generated by a subset K of S. This subgroup is also a Coxeter group.

One has the following

Lemma 3.1. *Let J and K be subsets of S such that all elements of J commute with all elements of K, then*

$$x_L^{J \cup K} = x_{L \cap J}^{J} x_{L \cap K}^{K} \tag{3.2}$$

Proof. We might as well suppose that $J \cup K = S$ since this does not change the argument. Hence we now want to show that $x_L = x_{L \cap J}^J x_{L \cap K}^K$. Given $w \in X_L$, there exists a unique decomposition $w = w_J w_K$, with $w_J \in W_J$ and $w_K \in W_K$. It follows immediately that

$$D(w_J) = D(w) \cap J, \quad \text{and} \quad D(w_K) = D(w) \cap K. \tag{3.3}$$

Whence $w_J \in X_L \cap W_J$ and $w_K \in X_L \cap W_K$. Moreover every pair (w_J, w_K) satisfying (3.3) gives rise to a unique w in X_L. This proves the lemma. ∎

It follows that

Proposition 3.4. *If $S = S_1 \cup S_2$, where all elements of S_1 commute with all elements of S_2, then the function*

$$\varphi : \Sigma(W_{S_1}) \otimes \Sigma(W_{S_2}) \xrightarrow{\sim} \Sigma(W_S),$$

defined as

$$\varphi(\alpha \otimes \beta) = \alpha\beta, \tag{3.5}$$

is an isomorphism of algebras.

Proof. Since the product of two basis elements in $\Sigma(W_{S_1}) \otimes \Sigma(W_{S_2})$ is by definition

$$(x_{K_1} \otimes x_{K_2})(x_{L_1} \otimes x_{L_2}) = (x_{K_1} x_{L_1}) \otimes (x_{K_2} x_{L_2}),$$

we shall prove that φ is a morphism if we show that

$$x_{K_1}^{S_1} x_{K_2}^{S_2} x_{L_1}^{S_1} x_{L_2}^{S_2} = x_{K_1}^{S_1} x_{L_1}^{S_1} x_{K_2}^{S_2} x_{K_2}^{S_2}. \tag{$*$}$$

But every element of W_{S_1} commutes with all elements of W_{S_2}, thus

$$x_{K_2}^{S_2} x_{L_1}^{S_1} = x_{L_1}^{S_1} x_{K_2}^{S_2},$$

and $(*)$ follows. Morover, φ is clearly bijective. This proves the proposition. ∎

Thus we can reduce our discussion to indecomposable finite Coxeter groups.

4. The parabolic Burnside ring

For each $J \subseteq \Pi$ we have a permutation representation of W on the set $W/_{W_J}$ of cosets $W_J w$. The orbits of W on $W/_{W_J} \times W/_{W_K}$ have representatives of the form $(W_J d, W_K)$, where $d \in X_{JK}$; and the stabilizer of $(W_J d, W_K)$ in W is $d^{-1}W_J d \cap W_K = W_{d^{-1}(J)\cap K}$. Thus

$$W/_{W_J} \times W/_{W_K} = \sum_{L \subseteq K} a_{JKL} W/_{W_L}, \tag{4.1}$$

where the a_{JKL}'s are defined as in Section 1. This proves that the representations $W/_{W_J}$ span a subring $\mathcal{PB}(W)$ of the Burnside ring of W. We call this the *parabolic Burnside ring* of W. On comparing (4.1) and (1.1) we see that there is a homomorphism $\theta : \Sigma(W) \longrightarrow \mathcal{PB}(W)$. Note that θ is not in general an isomorphism, because $W/_{W_J}$ and $W/_{W_K}$ represent the same element of $\mathcal{PB}(W)$ whenever $J \sim K$.

A subgroup of W is said to be *parabolic* if it is conjugate to a standard parabolic subgroup W_J for some $J \subseteq \Pi$. For each $v \in V$, the stabilizer in W of v,

$$\mathrm{Stab}_W(v) = \{\, w \in W \mid w(v) = v \,\},$$

is a parabolic subgroup. Indeed, the set

$$C = \{\, u \in V \mid (\alpha, u) \geq 0 \text{ for all } \alpha \in \Pi \,\}$$

is a fundamental domain for the action of W, and we may choose $t \in W$ such that $t(v) \in C$. Then (see Steinberg [14])

$$t\ \mathrm{Stab}_W(v)\, t^{-1} = \mathrm{Stab}_W(t(v)) = W_J,$$

where $J = \{\, \alpha \in \Pi \mid (\alpha, t(v)) = 0 \,\}$.

Since W_J stabilizes $J^{\perp}$ it follows that $w \in W_J$ stabilizes $v \in V$ if and only if it stabilizes the orthogonal projection of v in V_J. Hence $\mathrm{Stab}_{W_J}(v)$ is a parabolic subgroup of W_J. It follows by induction that the pointwise stabilizer, $\mathrm{Stab}_W(P)$, of an arbitrary subset P of V, is a parabolic subgroup of W. Since $\mathrm{Stab}_W(P \cup Q) = \mathrm{Stab}_W(P) \cap \mathrm{Stab}_W(Q)$ we see that the intersection of two parabolic subgroups is again parabolic; this also follows from the fact, mentioned in Section 2, that $W_J \cap dW_K d^{-1} = W_{J \cap d(K)}$ whenever $d \in X_{JK}$.

If g is an arbitrary orthogonal transformation on V, define

$$[V, g] = \{\, (1-g)(v) \mid v \in V \,\}$$

and

$$C_V(g) = \{\, v \in V \mid g(v) = v \,\},$$

and let $\tau(g) = \dim[V, g]$. It is easily checked that $[V, g]$ is the orthogonal complement of $C_V(g)$ in V. Furthermore if $0 \neq v \in V$ and r is the reflection in the hyperplane orthogonal to v, then

$$\tau(rg) = \begin{cases} \tau(g) + 1 & \text{if } v \notin [V, g] \\ \tau(g) - 1 & \text{if } v \in [V, g]. \end{cases} \tag{4.2}$$

Thus $\tau(g)$ is the length of a minimal expression for g as a product of reflections. In [6] Carter proves that every element $w \in W$ can be written as a product of $\tau(w)$ reflections in W. (We include a proof in Lemma 4.3 below.)

Following Solomon [13], for $w \in W$, we define

$$A(w) = \{\, y \in W \mid [V, y] \subseteq [V, w] \,\} = \{\, y \in W \mid C_V(w) \subseteq C_V(y) \,\}.$$

Equivalently, $A(w) = \mathrm{Stab}_W(C_V(w))$. In particular, $A(w)$ is a parabolic subgroup of W. We say that w is of *type* J if $A(w)$ is conjugate to W_J. We shall sometimes say that w is of type λ, where λ is the equivalence class of J since (by Lemma 2.3) J is determined by w only to within equivalence. It is clear that $A(twt^{-1}) = tA(w)t^{-1}$, and hence conjugate elements have the same type.

Observe that the maps $P \longmapsto \mathrm{Stab}_W(P)$ and $H \longmapsto C_V(H)$, where H is a subgroup of W, form a Galois connection between the partially ordered set of subspaces of V and the partially ordered set of subgroups of W, i the sense that $P \subseteq C_V(H)$ if and only if $H \subseteq \mathrm{Stab}_W(P)$. The parabolic subgroups are the closed subgroups of W for this Galois connection; that is, H is parabolic if and only if $H = \mathrm{Stab}_W(C_V(H))$. Thus if H is any subgroup of W, then $\mathrm{Stab}_W(C_V(H))$ is the smallest parabolic subgroup of W containing H. In particular, if $w \in W$, then $A(w)$ is the smallest parabolic subgroup containing w, and so w is of type J if and only if $J \subseteq \Pi$ is minimal subject to W_J containing a conjugate of w.

Lemma 4.3. *Let $J \subseteq \Pi$ and suppose that $w \in W$ is of type J. Then*
(1) *if $K \subseteq \Pi$ and W_K contains a conjugate of w, then $J \preceq K$,*
(2) $\tau(w) = |J|$,
(3) *w can be written as a product of $|J|$ reflections in W.*

Proof. Replacing w by a conjugate of itself, we may assume that $w \in W_J$. Since w has type J it is not contained in any proper parabolic subgroup of W_J. If $t \in W$ and $t^{-1}wt \in W_K$, then $w \in W_J \cap tW_Kt^{-1}$, a parabolic subgroup of W_J. It follows that $W_J \cap tW_Kt^{-1} = W_J$. Now Lemma 2.3 gives $J \preceq K$, proving (1). The generators of W_J all fix $J^\perp$ pointwise, and so $J^\perp \subseteq C_V(w)$. Taking orthogonal complements gives $[V, w] \subseteq V_J$. If $[V, w] \neq V_J$, we deduce that V_J contains a nonzero $v \in C_V(w)$, and hence that $w \in \mathrm{Stab}_{W_J}(v)$, a proper parabolic subgroup of W_J. This is a contradiction, and therefore $[V, w] = V_J$. Thus

$$\tau(w) = \dim[V, w] = \dim V_J = |J|,$$

proving (2).

Since $[V, w] = V_J$ it follows from (4.2) above that $\tau(sw) = \tau(w) - 1$ whenever $s \in S_J$. Hence sw has type K for some $K \subseteq \Pi$ with $|K| = |J| - 1$. Arguing by induction we deduce that sw is a product of $|J| - 1$ reflections in W, and therefore $w = s(sw)$ is a product of $|J|$ reflections. ∎

For $J \subseteq S$, let c_J be the product of the reflections s, $s \in S_J$, taken in some fixed order. The conjugacy class of c_J in W_J is independent of the order, and the elements of this class are called the *Coxeter elements* of W_J. Since J is a linearly independent set it is clear that $[V, c_J] = V_J$, and so c_J has type J. We note as a consequence that the parabolic subgroups of W are precisely the subgroups $A(w)$.

Proposition 4.4. *If J, $K \subseteq \Pi$, then c_J is conjugate to c_K if and only if $J \sim K$.*

Proof. If c_J and c_K are conjugate, then they are of the same type—that is, $J \sim K$. Conversely, if $J = d(K)$ for some $d \in W$, then $dS_Jd^{-1} = S_K$, and so dc_Jd^{-1}, being a product of the reflections in S_K, is conjugate to c_K. ∎

Let $\varphi_J = \mathrm{Ind}_{W_J}^W 1$, the character of W induced from the trivial character of W_J. In other words, φ_J is the character corresponding to the permutation representation $W/_{W_J}$.

Theorem 4.5. *The assignment $W/_{W_J} \mapsto \varphi_J$ defines an isomorphism Θ from $\mathcal{PB}(W)$ to the ring of* **Q***-linear combinations of the φ_J. Thus we may identify $\mathcal{PB}(W)$ with this ring of class functions.*

Proof. If $J \sim K$, the representations $W/_{W_J}$ and $W/_{W_K}$ are equal in $\mathcal{PB}(W)$ and hence

$$\varphi_J = \Theta(W/_{W_J}) = \Theta(W/_{W_K}) = \varphi_K.$$

This makes it legitimate to write φ_λ instead of φ_J where λ is the equivalence class of J. For each equivalence class μ choose an element c_μ of type μ: for example, a Coxeter element. Since W_J contains an element of type K if and only if $K \preceq J$ it is clear that $\varphi_\lambda(c_\mu) \neq 0$ if and only if $\mu \preceq \lambda$. For a suitable ordering of the rows and columns, the matrix $(\varphi_\lambda(c_\mu))_{\lambda,\mu}$ is upper triangular with non-zero diagonal entries. Therefore the φ_λ are linearly independent. ∎

Induction and restriction of characters give rise to maps between $\mathcal{PB}(W_J)$ and $\mathcal{PB}(W)$. The permutation representation W_J/W_K induced to $\mathcal{PB}(W)$ is simply $W/_{W_K}$. By Lemma 2.1 the analogue of induction for the

Solomon algebras is left multiplication by x_J. The restriction of W/W_K to $\mathcal{PB}(W_J)$ is obtained by considering the orbits of W_J on the cosets $W_K d$. Thus

$$\mathrm{Res}_{W_J}(W/W_K) = \sum_{d \in X_{JK}} W_J/W_{J\cap d(K)}.$$

and the analogue of restriction for $\Sigma(W)$ is given by Lemma 2.2. Combining these two observations we see that Theorem 2.5 is the Solomon algebra analogue of the Mackey formula for the product of induced characters.

5. Dihedral groups

We shall now study in particular the descent algebra of dihedral groups $W = I_2(p)$, that is, Coxeter groups with only two generators $S = \{s, r\}$ satisfying

$$(sr)^p = 1.$$

The corresponding descent algebra is of (linear) dimension 4. Its generators are

$$\begin{aligned} x_{\{s,r\}} &= 1 \\ x_{\{s\}} &= 1 + r + sr + rsr + srsr + \dots \\ x_{\{r\}} &= 1 + s + rs + srs + rsrs + \dots \\ x_{\emptyset} &= \sum_w w. \end{aligned}$$

The summation for $x_{\{s\}}$ (resp. $x_{\{r\}}$) is over the set of all $w \in W$ with only one reduced expression, this unique expression must also end in r (resp. s). In order to simplify notation, we shall write x_{sr} (resp. x_s, x_r) instead of $x_{\{s,r\}}$ (resp. $x_{\{s\}}$, $x_{\{r\}}$). The multiplication table for $\Sigma(W)$ is easy to compute explicitly in this case. It is as follows

	x_{sr}	x_s	x_r	$x_\emptyset$
x_{sr}	x_{sr}	x_s	x_r	$x_\emptyset$
x_s	x_s	$2x_s + \frac{p-2}{2}x_\emptyset$	$\frac{p}{2}x_\emptyset$	$px_\emptyset$
x_r	x_r	$\frac{p}{2}x_\emptyset$	$2x_r + \frac{p-2}{2}x_\emptyset$	$px_\emptyset$
$x_\emptyset$	$x_\emptyset$	$px_\emptyset$	$px_\emptyset$	$2px_\emptyset$

Table 1, p even

when p is even. Whereas for p odd it is

	x_{sr}	x_s	x_r	$x_\emptyset$
x_{sr}	x_{sr}	x_s	x_r	$x_\emptyset$
x_s	x_s	$x_s + \frac{p-1}{2}x_\emptyset$	$x_r + \frac{p-1}{2}x_\emptyset$	$px_\emptyset$
x_r	x_r	$x_s + \frac{p-1}{2}x_\emptyset$	$x_r + \frac{p-1}{2}x_\emptyset$	$px_\emptyset$
$x_\emptyset$	$x_\emptyset$	$px_\emptyset$	$px_\emptyset$	$2px_\emptyset$

Table 2, p odd

Using these tables, one can verify that for even p

$$\begin{aligned} e_{sr} &= x_{sr} - \frac{1}{2}x_s - \frac{1}{2}x_r + \frac{p-1}{2p}x_\emptyset \\ e_s &= \frac{1}{2}(x_s - \frac{1}{2}x_\emptyset) \\ e_r &= \frac{1}{2}(x_r - \frac{1}{2}x_\emptyset) \\ e_\emptyset &= \frac{1}{2p}x_\emptyset, \end{aligned} \tag{5.1}$$

are idempotents such that $e_K e_L = 0$, for all K, L distinct subsets of $S = \{s, r\}$. Since the equivalence classes of subsets of Π coincide in this case with subsets of Π, we obtain a set of idempotents

$$\begin{aligned} E_{\lambda(sr)} &= e_{sr} \\ E_{\lambda(s)} &= e_s \\ E_{\lambda(r)} &= e_r \\ E_{\lambda(\emptyset)} &= e_\emptyset. \end{aligned}$$

satisfying condition 1.2, moreover the sum of these idempotents is 1.

In the p odd case, the following are idempotents

$$\begin{aligned} e_{sr} &= x_{sr} - \frac{1}{2}x_s - \frac{1}{2}x_r + \frac{p-1}{2p}x_\emptyset \\ e_s &= x_s - \frac{1}{2}x_\emptyset \\ e_r &= x_r - \frac{1}{2}x_\emptyset \\ e_\emptyset &= \frac{1}{2p}x_\emptyset. \end{aligned} \tag{5.2}$$

But there are now only three conjugacy classes of subsets of S: $\{\{s, r\}\}$, $\{\{s\}, \{r\}\}$ and $\{\emptyset\}$. The non trivial products between two different e_K's are

$$e_s e_r = e_r \quad \text{and} \quad e_r e_s = e_s.$$

Hence we can set

$$\begin{aligned} E_{\lambda(sr)} &= e_{sr} \\ E_{\lambda(s)} = E_{\lambda(r)} &= \frac{1}{2}(e_s + e_r) \\ E_{\lambda(\emptyset)} &= e_\emptyset. \end{aligned}$$

These also satisfy condition 1.2 and sum to 1. In this case, the radical of the descent algebra is generated by the nilpotent $e_s - e_r$.

In preparation for Section 7, we shall now reconsider part of this construction in the context of a general Coxeter system (W, S). For two elements s and r of S, let us compute the product $x_s x_r$. A direct application of (1.1) gives

$$x_s x_r = \alpha_s^r x_r + \beta_s^r x_\emptyset, \tag{5.3}$$

where $\alpha_s^r = |\{w \mid w^{-1} \in x_s,\ w \in x_r,\ w^{-1}sw = r\}|$. Observe that for any $\aleph = \sum_w \aleph_w w$ in $\mathbb{Q}(W)$, one has $\aleph x_\emptyset = x_\emptyset \aleph = (\sum_w \aleph_w) x_\emptyset$. From Lemma 2.1 it follows that

$$\frac{\alpha_s^r}{2} + \beta_s^r = \frac{|W|}{4},$$

since $x_\emptyset^r = 1 + r$. Thus we obtain

$$x_s x_r = \alpha_s^r (x_r - \frac{1}{2} x_\emptyset) + \frac{|W|}{4} x_\emptyset. \tag{5.4}$$

Identity (5.4) suggests that we set for any Coxeter group

$$e_s = \frac{1}{\alpha_s^s}(x_s - \frac{1}{2} x_\emptyset),$$

for then e_s is clearly an idempotent since

$$\begin{aligned}(x_s - \frac{1}{2} x_\emptyset)^2 &= x_s^2 - 2x_s x_\emptyset + x_\emptyset^2 \\ &= \alpha_s^s (x_s - \frac{1}{2} x_\emptyset).\end{aligned}$$

Moreover a similar computation implies that

$$e_s e_r = \frac{\alpha_s^r}{\alpha_s^s} e_r.$$

Clearly if s and r are not conjugate, $\alpha_s^r = 0$. But if they are conjugate then we maintain that $\alpha_s^r = \alpha_s^s$. In fac this results from the fact that both these quantities are equal to half the cardinality of the centralizer $C(s) = \{w \in W \mid w^{-1} s w = s\}$. This last assertion results from the observation that for $w \in C(s)$, either $w \in X_s^{-1} \cap X_s$ o $ws \in X_s^{-1} \cap X_s$, since evidently $\ell(ws) = \ell(sw)$. Whence

$$e_s e_r = \begin{cases} e_r & \text{if } s \text{ and } r \text{ are conjugate} \\ 0 & \text{otherwise.} \end{cases}$$

From this we conclude that

Proposition 5.5. *In any Coxeter group, for all $s \in S$, the*

$$E_{\lambda(s)} = \frac{1}{|\lambda(s)|} \sum_{r \in \lambda(s)} e_r,$$

are idempotents, and if s and r are not conjugate, then

$$E_{\lambda(s)} E_{\lambda(r)} = 0.$$

We shall generalize this result to all descent algebras in Section 7.

6. Idempotents in the parabolic Burnside ring

The Q-algebra $\mathcal{PB}(W)$ is isomorphic to an algebra of functions, and therefore it has a basis of idempoter elements. Specifically, if we define

$$\xi_\lambda = \sum_\mu \nu_{\lambda\mu} \varphi_\mu,$$

where the coefficient matrix $(\nu_{\lambda\mu})$ is the inverse of the matrix $(\varphi_\lambda(c_\mu))$ which appears in the proof of Theorem 4. then

$$\xi_\lambda(c_\mu) = \begin{cases} 0 & \text{if } \lambda \neq \mu \\ 1 & \text{if } \lambda = \mu, \end{cases} \tag{6.1}$$

and it follows that ξ_λ is idempotent. The next theorem shows that (6.1) holds when c_μ is an arbitrary element type μ.

Theorem 6.2. *Let J, $K \subseteq \Pi$ and let $c \in W$ be any element of type J. Then $\varphi_K(c) = a_{KJJ}$, the number of $d \in X_{KJ}$ such that $d(J) \subseteq K$.*

Proof. Without loss of generality we may suppose that $c \in W_J$. By Mackey's formula, the restriction of φ_K to W_J is

$$\mathrm{Res}_{W_J}(\mathrm{Ind}_{W_K}^{W} 1) = \sum_{d \in X_{KJ}} \mathrm{Ind}_{W_{d^{-1}(K) \cap J}}^{W_J} 1.$$

But since c is not contained in any proper parabolic subgroup of W_J, the character $\mathrm{Ind}_{W_{d^{-1}(K)\cap J}}^{W_J} 1$ vanishes on c unless $d^{-1}(K) \cap J = J$, in which case it takes the value 1. ∎

For $J \subseteq \Pi$, let $N_J = \{ w \in W \mid w(J) = J \}$. Then N_J is the intersection of X_J and the normalizer of W_J, whence $|N_J| = a_{JJJ}$ is the index of W_J in its normalizer.

For convenience we define $\xi_J = \xi_\lambda$ and $\nu_{JK} = \nu_{\lambda\mu}$ whenever $J \in \lambda$ and $K \in \mu$. For $J \subseteq K \subseteq \Pi$, let ξ_J^K be the primitive idempotent of $\mathcal{PB}(W_K)$ that takes the value 1 on elements of type J relative to W_K.

The next two propositions describe the effect of the restriction and induction maps on these idempotents.

Proposition 6.3. *Let J, $K \subseteq \Pi$ and let $J_1, J_2, \ldots, J_h$ be representatives of the W_K-equivalence classes of subsets of K which are W-equivalent to J. Then*

$$\mathrm{Res}_{W_K} \xi_J = \sum_{i=1}^{h} \xi_{J_i}^K.$$

In particular, $\mathrm{Res}_{W_K} \xi_J = 0$ if J is not equivalent to any subset of K. ∎

Proposition 6.4. *If $J \subseteq K \subseteq \Pi$, then*

$$\mathrm{Ind}_{W_K}^{W} \xi_J^K = \frac{|N_J|}{|W_K \cap N_J|} \xi_J.$$

Proof. Suppose at first that $J = K$ and that $c \in W_J$ is an element of type J. Then $A(c) = W_J$ and therefore $x^{-1}cx \in W_J$ if and only if x is in the normalizer of W_J. So $(\mathrm{Ind}_{W_J}^{W} \xi_J^J)(c) = |N_J|$. It is clear that $\mathrm{Ind}_{W_J}^{W} \xi_J^J$ vanishes everywhere except at elements of type J, and therefore $\mathrm{Ind}_{W_J}^{W} \xi_J^J = |N_J| \xi_J$.

In general, we have

$$\mathrm{Ind}_{W_K}^{W} \xi_J^K = \mathrm{Ind}_{W_K}^{W} \Big(\frac{1}{|W_K \cap N_J|} \mathrm{Ind}_{W_J}^{W_K} \xi_J^J\Big) = \frac{|N_J|}{|W_K \cap N_J|} \xi_J.$$

∎

For the purposes of calculation, the following theorem is sometimes more useful than Theorem 6.2. The quantities $|N_J|/|W_K \cap N_{J_i}|$ can be obtained from the tables in Howlett [10].

Theorem 6.5. *Let $J \preceq K \subseteq \Pi$ and let $J_1, J_2, \ldots, J_h$ be representatives of the W_K-equivalence classes of subsets of K which are W-equivalent to J. If $c \in W$ is of type J, then*

$$\varphi_K(c) = \sum_{i=1}^{h} \frac{|N_J|}{|W_K \cap N_{J_i}|}.$$

Proof. By definition, $\sum \xi_L^K = 1$, where L runs through representatives of the W_K-eqivalence classes of subsets of K. Inducing to W and using Proposition 6.4 gives

$$\varphi_J = \sum_{L} \frac{|N_L|}{|W_K \cap N_L|} \xi_L,$$

Since $\xi_L(c) = 1$ if and only if $L \sim J_i$ for some i, evaluation at c completes the proof.

This theorem is also a consequence of the fact that $|N_J|/|W_K \cap N_{J_i}|$ is the number of $d \in X_{KJ}$ such t $d(J) \subseteq K$ and $d(J)$ is W_K-equivalent to J_i.

Let $\mathcal{C}(J)$ be the set of elements of type J and note that $\mathcal{C}(J)$ depends only on the equivalence class of J.

The main result of this section yields a remarkable formula for the coefficients ν_{JK} in the case $K = \emptyset$.

Theorem 6.6. *If $m_1, m_2, \ldots, m_n$ are the exponents of W, then*

$$\nu_{\Pi\emptyset} = (-1)^n \frac{m_1 m_2 \cdots m_n}{|W|}.$$

Proof. If ε is the sign character of W, then by Frobenius reciprocity

$$(\varphi_J, \varepsilon) = \begin{cases} 1 & J = \emptyset \\ 0 & J \neq \emptyset \end{cases}$$

and therefore $\nu_{\Pi\emptyset} = (\xi_\Pi, \varepsilon)$. By definition of the inner product,

$$\begin{aligned}(\xi_\Pi, \varepsilon) &= |W|^{-1} \sum_{w \in W} (-1)^{\tau(w)} \xi_\Pi(w) \\ &= (-1)^n |\mathcal{C}(\Pi)|/|W|.\end{aligned}$$

A well-known formula of Shephard and Todd **[11]** (see also Solomon **[12]**) states that

$$\sum_{w \in W} t^{\tau(w)} = (1 + m_1 t)(1 + m_2 t) \cdots (1 + m_n t).$$

Lemma 4.3 (2) shows that $\tau(w) = n$ if and only if w is of type Π. Thus, $m_1 m_2 \cdots m_n$ is the number of eleme of type Π in W. This completes the proof.

Corollary 6.7. *Let $J \subset \Pi$ with $|J| = k$ and let $m_1, m_2, \ldots, m_k$ be the exponents of W_J. Then*

$$\nu_{J\emptyset} = (-1)^k \frac{m_1 m_2 \cdots m_k}{|N_W(W_J)|},$$

where $N_W(W_J)$ is the normalizer in W of W_J.

Proof. To see this, apply Theorem 6.6 to W_J and then use Proposition 6.4.

It is also interesting to observe that

$$|W| \sum_\mu \nu_{J\mu} = |\mathcal{C}(J)|.$$

The proof is obtained by taking the inner product of $\xi_J = \sum \nu_{J\mu} \varphi_\mu$ with the trivial character and using fact that $(\varphi_\mu, 1) = 1$ for all μ.

A similar calculation, but taking the inner product of ξ_J with the sign character of W_L induced to W give

$$\sum_\mu \nu_{J\mu} a_{L\mu\emptyset} = (-1)^{|J|} \frac{|\mathcal{C}(J) \cap W_L|}{|W_L|}.$$

7. IDEMPOTENTS IN THE SOLOMON ALGEBRA

The parabolic Burnside ring is commutative and semisimple and consequently it has a unique set of primitive idempotents which sum to 1. These are the ξ_λ determined in the previous section. The descent algebra $\Sigma(W)$ is not semisimple but we have $\Sigma(W)/\sqrt{\Sigma(W)} \simeq \mathcal{PB}(W)$ and therefore we may find primitive idempotents of $\Sigma(W)$ by lifting the ξ_λ.

We begin by defining certain constants μ_K^J for all $K \subseteq J \subseteq \Pi$

$$\mu_K^J = |\{w \in X_J \mid w(K) \subseteq \Pi\}|. \tag{7.1}$$

This implies that, for all $K \subseteq \Pi$, $\mu_K^\Pi = 1$ and $\mu_\emptyset^K = |X_K|$. Moreover

Lemma 7.2. *For $K \subseteq J \subseteq L \subseteq \Pi$ we have*

$$\mu_K^J = \sum_{\substack{w \in W_L \cap X_J \\ w(K) \subseteq L}} \mu^\Pi_{w(K)L}.$$

Proof. Using the definit'on of $\mu^L_{w(K)}$, the above expression becomes the cardinatity of the set

$$\biguplus_{\substack{w \in W_L \cap X_J \\ w(K) \subseteq L}} \{v \in X_L \mid vw(K) \subseteq \Pi\}.$$

If $v \in X_L$, $w \in W_L \cap X_J$ and $vw(K) \subseteq \Pi$, then $w(K) \subseteq w(J) \subseteq \Phi_L^+$ and so $w(K) \subseteq L$. Thus by Lemma 2.2 this set is

$$\{w \in X_J \mid w(K) \subseteq \Pi\},$$

whose cardinality is μ_K^J. ∎

Lemma 7.3. *If $K \subseteq J$ and $d(J) \subseteq \Pi$, then $\mu^{d(J)}_{d(K)} = \mu^J_K$.*

Proof. Lemma 2.4 states that $w \in X_{d(J)}$ is equivalent to wd/inX_J, hence multiplication (on the right) by d establishes a bijection between the sets $\{w \in X_J \mid w(K) \subseteq \Pi\}$ and $\{w \in X_{d(J)} \mid wd(K) \subseteq \Pi\}$. This proves the lemma. ∎

Now, for each $J \subseteq \Pi$, define $e^J \in \Sigma(W_J)$ recursively by

$$\mu_J^J e^J = 1 - \sum_{K \subset J} \mu_K^J x_K^J e^K. \tag{7.4}$$

By induction we have

$$x_K^J e^K \in \Sigma(W_J).$$

Lemma 7.5. *If $v^{-1}(J) \subseteq \Pi$, then $e^J v = v e^{v^{-1}(J)}$.*

Proof. We argue by induction on $|J|$, the case $J = \emptyset$ being trivial. First observe that $v^{-1} W_J v = W_{v^{-1}(J)}$ and if $K \subseteq J$ and $d \in W_J \cap X_K$, then

$$(v^{-1}dv)v^{-1}(K) \subseteq v^{-1}(\Phi_J^+) \subseteq \Phi^+,$$

whence $v^{-1}dv \in W_{v^{-1}(J)} \cap X_{v^{-1}(K)}$. Thus $x_K^J v = v x^{v^{-1}(J)}_{v^{-1}(K)}$.

Now suppose that $J \neq \emptyset$. Then

$$\begin{aligned}
\mu_J^J e^J v &= v - \sum_{K \subset J} \mu_K^J x_K^J e^K v && \text{by (7.4)} \\
&= v - \sum_{K \subset J} \mu_K^J x_K^J v e^{v^{-1}(K)} && \text{by induction} \\
&= v - \sum_{K \subset J} \mu_K^J v x_{v^{-1}(K)}^{v^{-1}(J)} e^{v^{-1}(K)} \\
&= v - \sum_{v(K) \subset J} \mu_K^{v^{-1}(J)} v x_K^{v^{-1}(J)} e^K && \text{by Lemma 7.3} \\
&= \mu_J^J v e^{v^{-1}(J)}.
\end{aligned}$$

This proves the lemma.

Theorem 7.6. *If J, $M \subseteq \Pi$, then*

$$x_J e^J x_M = \sum_{N \subseteq M} \gamma_N x_N e^N,$$

where $\gamma_N = |\{\, v \in X_M \mid v(N) = J \,\}|$.

Proof. We argue by induction on $|J|$. If $J = \emptyset$, it is straightforward to check that both sides of the equality reduc to $(\mu_\emptyset^\emptyset)^{-1} |X_M| x_\emptyset$. Suppose that $J \neq \emptyset$. Then by (7.4) and Lemma 2.1

$$\mu_J^J x_J e^J x_M = x_J x_M - \sum_{K \subset J} \mu_K^J x_K e^K x_M.$$

and by Lemma 2.2 we have

$$\mu_J^J x_J e^J x_M = x_J x_M - \sum_{K \subset J} \mu_K^J x_K e^K \sum_{v \in X_{KM}} x_{K \cap v(M)}^K v.$$

If $K \cap v(M) \neq K$, then $x_K e^K x_{K \cap v(M)}^K = 0$ by induction and therefore the expression becomes

$$\mu_J^J x_J e^J x_M = x_J x_M - \sum_{K \subset J} \mu_K^J x_K e^K \sum_{\substack{v \in X_M \\ v^{-1}(K) \subseteq M}} v.$$

By Lemma 7.5 and the fact that $x_K v = x_{v^{-1}(K)}$ this can be written as

$$\mu_J^J x_J e^J x_M = x_J x_M - \sum_{K \subset J} \mu_K^J \sum_{\substack{v \in X_M \\ v^{-1}(K) \subseteq M}} x_{v^{-1}(K)} e^{v^{-1}(K)}.$$

Writing $N = v^{-1}(K)$, and rearranging, this becomes

$$\mu_J^J x_J e^J x_M = x_J x_M - \sum_{N \subseteq M} \left(\sum_{\substack{v \in X_M \\ v(N) \subset J}} \mu_{v(N)}^J \right) x_N e^N. \tag{7.7}$$

In order to evaluate the inner sum we apply Lemma 2.2 to obtain

$$\sum_{\substack{v \in X_M \\ v(N) \subseteq J}} \mu_{v(N)}^J = \sum_{\substack{t \in X_{JM} \\ t(N) \subseteq J}} \sum_{\substack{u \in W_J \cap X_{J \cap t(M)} \\ ut(N) \subseteq \Pi}} \mu_{ut(N)}^J.$$

Using Lemma 7.2 and 7.3 this last expression gives

$$\sum_{\substack{v\in X_M\\ v(N)\subseteq J}} \mu^J_{v(N)} = \sum_{\substack{t\in X_{JM}\\ t(N)\subseteq J}} \mu^{J\cap t(M)}_{t(N)}$$

$$= \sum_{\substack{t\in X_{JM}\\ N\subseteq J}} \mu^{t^{-1}(J)\cap M}_N.$$

Writing P for $t^{-1}(J)\cap M$ and using the definition of a_{JMP}, this last identity becomes

$$\sum_{\substack{v\in X_M\\ v(N)\subseteq J}} \mu^J_{v(N)} = \sum_{N\subseteq P} \sum_{\substack{t\in X_{JM}\\ t^{-1}(J)\cap M=P}} \mu^P_N$$

$$= \sum_{N\subseteq P} a_{JMP}\mu^P_N.$$

Now using Theorem 7.6 applied to W_L we have

$$x_Jx_M - \sum_{N\subseteq M}\sum_{\substack{v\in X_M\\ v(N)\subseteq J}} \mu^J_{v(N)} x_N e^N = \sum_{P\subseteq M} a_{JMP}x_P - \sum_{N\subseteq M}\sum_{N\subseteq P} a_{JMP}\mu^P_N x_N e^N$$

$$= \sum_{P\subseteq M} a_{JMP}x_P\left(1-\sum_{N\subseteq P}\mu^P_L x^P_N e^N\right)$$

$$= 0 \qquad\qquad \text{by (7.4).}$$

Returning to (7.7) we find that

$$\mu^J_J x_J e^J x_M = \sum_{N\subseteq M}\sum_{\substack{v\in X_M\\ v(N)=J}} \mu^J_{v(N)} x_N e^N,$$

whence

$$x_J e^J x_M = \sum_{N\subseteq M} |\{\, v\in X_M \mid v(N)=J\,\}| x_N e^N,$$

as required. ∎

Let us write e_J for x_Je^{Π}, and set $\mu^J_{KN} = |\{\, v\in X_J \mid v(K)=N\,\}|$. We further need the inverse (β^J_K) of the matrix (μ^J_K). Notice that

$$x_J = \sum_{K\subseteq J} \mu^J_K e_K$$

and also

$$\mu^J_K = \sum_{N\sim K} \mu^J_{KN}$$

whence

$$\sum_{\substack{J,N\\ J\sim P}} \beta^M_N \mu^N_{PJ} = \delta_{MP}.$$

Then

Proposition 7.8. *For all J, $M\subseteq \Pi$*

$$e_Je_M = \sum_{P\subseteq M}\left(\sum_{P\subseteq N\subseteq M} \beta^M_N \mu^N_{PJ}\right) e_P.$$

Proof. Simply write $e_M = \sum_{N \subseteq M} \beta_N^M x_N$ and use the previous theorem. ∎

Notice that when $J \sim K$, this propsition implies that

$$e_J e_K = \frac{1}{|\lambda(K)|} e_K. \tag{7.9}$$

This suggests that we should define

$$E_\lambda = \sum_{J \in \lambda} e_J,$$

for then

$$E_\lambda e_M = \begin{cases} e_K & \text{if } K \in \lambda \\ 0 & \text{otherwise.} \end{cases}$$

And now we have (as announced in (1.2))

(1) $1 = \sum_\lambda E_\lambda$,

(2) $E_\lambda E_\mu = \delta_{\lambda\mu} E_\lambda$.

These E_λ's clearly correspond (through $\theta : \Sigma(W) \to \mathcal{PB}(W)$) to the primitive idempotents ξ_λ of $\mathcal{PB}(W)$. Moreover, for each conjugacy class λ and any choice of constants b_K, $K \in \lambda$, such that $\sum_{K \in \lambda} b_K = 0$, it follows from 7.9 that $\sum_{K \in \lambda} b_K e_K$ is in the radical of $\Sigma(W)$.

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