# The Möbius function of factor order

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**Abstract.** Intervals in the factor ordering of a free monoid are investigated. It was shown by Farmer [6] that such intervals $(\beta, \alpha)$ are contractible or homotopy spheres in case $\beta$ is the empty word. We observe here that the same is true in general. This implies that the Möbius function of factor order takes values in $\{0, +1, -1\}$. A recursive rule for this Möbius function is given, which allows efficient computation.

The Möbius function of subword order was studied in [2]. We give here a simpler proof (a parity-changing involution) for its combinatorial interpretation.

## 1. Introduction.

Let $A^*$ denote the free monoid over an alphabet $A$. The elements of $A^*$ are finite strings of elements from $A$ called *words*. The *length* $|\alpha|$ of a word $\alpha = a_1 a_2 \cdots a_n$ is the number of letters $n$. There is a unique word $\lambda \in A^*$ of length zero, the *empty word*.

We will say that $\beta$ is a *factor* of $\alpha$ if $\alpha = \gamma\beta\delta$, for some $\gamma,\ \delta \in A^*$. Furthermore, $\beta$ is a *left factor* (or prefix) in $\alpha$ if $\delta = \lambda$ and a *right factor* (or suffix) if $\gamma = \lambda$. The relation of being a factor, written $\beta \leq \alpha$, gives a partial ordering of $A^*$. As an ordered set $A^*$ has unique least element $\lambda$, and all maximal chains in an interval $[\beta, \alpha] = \{\gamma \in A^* : \beta \leq \gamma \leq \alpha\}$ have length $l(\beta, \alpha) := |\alpha| - |\beta|$.

Let $\alpha = a_1 a_2 \cdots a_n \in A^*$. We will say that $\beta$ is a *subword* of $\alpha$ if $\beta = a_{i_1} a_{i_2} \cdots a_{i_k}$ for some sequence $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. So, a factor is a particular kind of subword. The subword ordering of $A^*$ is discussed in Section 3. See [10] for further general information concerning words.

To be able to state the rule for computing the Möbius function of factor order we need a few more definitions. Let $\alpha = a_1 a_2 \cdots a_n$, $n \geq 2$. Then $i\alpha = a_2 a_3 \cdots a_{n-1}$ is the *dominant inner factor* in $\alpha$. All factors of $i\alpha$ are called *inner factors* in $\alpha$. The *dominant outer factor* $\varphi\alpha$ is the longest $\beta \neq \alpha$ which is both a left factor and a right factor of $\alpha$ (possibly $\varphi\alpha = \lambda$). The word $\alpha$ is *trivial* if $a_1 = a_2 = \cdots = a_n$.

As an illustration of these definitions, let $\alpha = aabcabb$. Then, $i\alpha = abcab$, $\varphi\alpha = \lambda$, $\varphi i\alpha = ab$. Note that $l(\varphi\alpha, \alpha) = 1$ iff $\alpha$ is trivial, and $l(\varphi\alpha, \alpha) = 2$ iff $\alpha = (ab)^k$ or $\alpha = (ab)^k a$ for some $a,\ b \in A$.

The Möbius function of a poset with finite intervals $[x, y]$ is the $\mathbb{Z}$-valued function on intervals recursively defined by

$$\sum_{x \leq z \leq y} \mu(x, z) = \begin{cases} 1, & \text{if } x = y; \\ 0, & \text{if } x < y. \end{cases}$$

For general information concerning Möbius functions see Rota [12] and Stanley [13].

**Theorem 1.** *The Möbius function of factor order is for all $\beta \leq \alpha$ in $A^*$ given by:*

$$\mu(\beta,\alpha) = \begin{cases} \mu(\beta,\varphi\alpha) & , \text{ if } l(\beta,\alpha) > 2 \text{ and } \beta \leq \varphi\alpha \not\leq i\alpha, \\ 1 & , \text{ if } l(\beta,\alpha) = 2, \ \alpha \text{ is nontrivial and } \beta = i\alpha \text{ or } \beta = \varphi\alpha, \\ (-1)^{l(\beta,\alpha)} & , \text{ if } l(\beta,\alpha) < 2, \\ 0 & , \text{ in all other cases.} \end{cases}$$

**Corollary.** $\mu(\beta,\alpha) \in \{0,+1,-1\}$, *for all* $\beta \leq \alpha$ *in* $A^*$.

Other classes of posets (actually, lattices) whose Möbius function has the $\{0,+1,-1\}$-property have been studied by Björner [1], Greene [7] and Kahn [8]. Note that factor order is not a lattice.

We exemplify the the rule with the following computations using $\alpha = abracadabra$ and $\varphi\alpha = abra$:

$$\begin{aligned} \mu(a,\alpha) &= \mu(a,\varphi\alpha) = \mu(a,a) = 1 \\ \mu(b,\alpha) &= \mu(b,\varphi\alpha) = 0 \\ \mu(br,\alpha) &= \mu(br,\varphi\alpha) = 1 \\ \mu(bra,\alpha) &= \mu(bra,\varphi\alpha) = -1. \end{aligned}$$

The pattern matching algorithm of Knuth, Morris and Pratt [9] shows that $\beta \leq \alpha$ can be decided in $O(|\alpha|)$ time. Their algorithm contains a preprocessing step which gives a linear time algorithm for computing $\varphi\alpha$ (for this, see also p. 14 of [10]). Hence, Theorem 1 shows that $\mu(\beta,\alpha)$ can be computed in quadratic time using these algorithms.

**Corollary.** *$\mu(\beta,\alpha)$ can be computed in $O(|\alpha|^2)$ steps.*

Theorem 1 is implied by the next result, which describes the topology of open intervals in factor order up to homotopy type. The proof given in Section 2 is easy to convert to a purely combinatorial proof of Theorem 1, see Remark 4A.

From now on we will assume some familiarity with the topology of posets, see e. g. [3] for background. All topological statements about a poset $P$ will refer to its *order complex*, i.e., the simplicial complex of chains (totally ordered subsets), although stronger statements are possible (see Remark 4B).

Define a function $s$ from the intervals $\beta \leq \alpha$ of factor order to the set $\{-\infty,-2,-1,0,1,2,3,\cdots$ by the following recursive rule:

(i) $l(\beta,\alpha) = 0 \iff s(\beta,\alpha) = -2,$

(ii) $l(\beta,\alpha) = 1 \iff s(\beta,\alpha) = -1,$

(iii) $l(\beta,\alpha) = 2 \Longrightarrow s(\beta,\alpha) = \begin{cases} 0, & \text{if } \alpha \text{ is nontrivial and } \beta = i\alpha \text{ or } \beta = \varphi\alpha, \\ -\infty, & \text{otherwise,} \end{cases}$

(iv) $l(\beta,\alpha) > 2 \Longrightarrow s(\beta,\alpha) = \begin{cases} 2 + s(\beta,\varphi\alpha), & \text{if } \beta \leq \varphi\alpha \not\leq i\alpha, \\ -\infty, & \text{otherwise.} \end{cases}$

For instance, using our previous example $\alpha = abracadabra$ we compute: $s(a,\alpha) = 2$, $s(b,\alpha) = -\infty$, $s(br,\alpha) = 2$, $s(bra,\alpha) = 1$.

**Theorem 2.** *For all $\beta < \alpha$ in factor order, the open interval $(\beta,\alpha) = \{\gamma \in A^* : \beta < \gamma < \alpha\}$ has the homotopy type of the $s(\beta,\alpha)$-dimensional sphere if $s(\beta,\alpha) \geq 0$, and is contractible otherwise.*

For the case when $\beta$ is the empty word this was shown by Farmer [6], and the proof given in the next section is a straight-forward extension. Since the Möbius function $\mu(\beta,\alpha)$ is the reduced Euler characteristic of the open interval $(\beta,\alpha)$ we deduce the following corollary, of which Theorem 1 is a simplified restatement.

**Corollary.** *For all $\beta \leq \alpha$ in $A^*$:*

$$\mu(\beta,\alpha) = \begin{cases} (-1)^{s(\beta,\alpha)} & , \quad \textit{if } s(\beta,\alpha) \geq -2 \\ 0 & , \quad \textit{otherwise.} \end{cases}$$

## 2. Proofs.

The analysis of the structure of lower intervals $[\lambda, \alpha]$ to be given here is implicit in Farmer [6]. The general case will follow by restricting attention to an upper part $[\beta, \alpha]$ of such a lower interval.

For a trivial word $\alpha = aa \cdots a$ the lower interval $[\lambda, \alpha]$ is a chain of length $|\alpha|$. If $\alpha$ is nontrivial then it covers exactly two words $\alpha'$ and $\alpha''$, the left and right factors of length $|\alpha| - 1$. (Clearly: every nontrivial word covers 2 elements and is covered by $2|A|$ elements.) More generally, we have:

**Lemma 1.** *Suppose $\alpha$ is nontrivial. Then $[\lambda, \alpha] = [\lambda, i\alpha] \cup [\varphi\alpha, \alpha]$. Furthermore,*
**Case 1:** *If $\varphi\alpha \not\leq i\alpha$, then $[\lambda, i\alpha] \cap [\varphi\alpha, \alpha] = \emptyset$ and $(\varphi\alpha, \alpha)$ consists of two nonempty disjoint chains with no crosswise relations (see Figure 1a).*
**Case 2:** *If $\varphi\alpha \leq i\alpha$, then $(\lambda, \alpha) \setminus (\lambda, i\alpha]$ consists of two nonempty disjoint chains with no crosswise relations (see Figure 1b).*

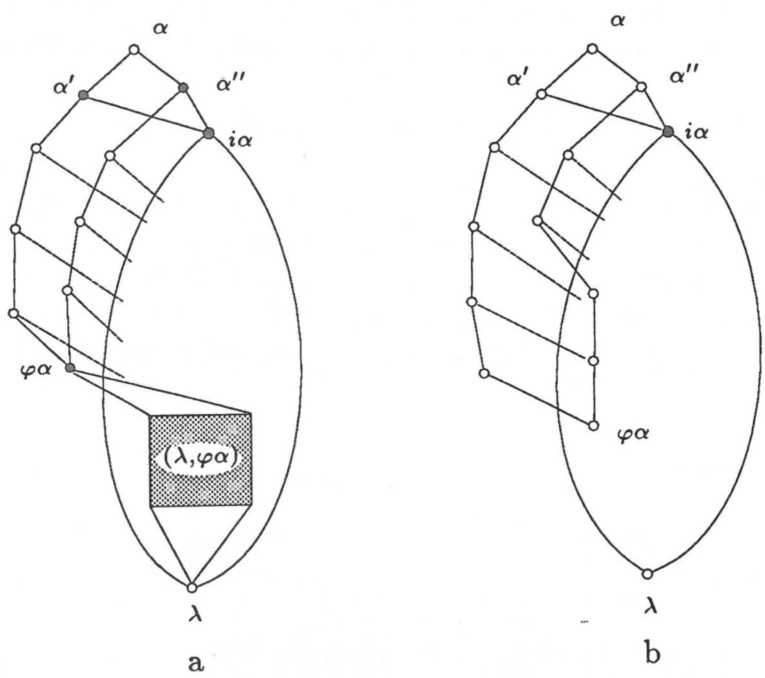


**Figure 1.**

**Proof.** Suppose that $\beta \leq \alpha$ is not an inner factor. Then $\beta$ is a left or right factor in $\alpha$, let's say a left factor. If $|\beta| < |\varphi\alpha|$, then $\beta$ is a proper left factor in $\varphi\alpha$, which (using the right factor embedding of $\varphi\alpha$ in $\alpha$) would make $\beta$ an inner factor in $\alpha$. Hence, $|\beta| \geq |\varphi\alpha|$, which implies that $\varphi\alpha \leq \beta$.

Let $\varphi\alpha = \gamma_k < \gamma_{k+1} < \cdots < \gamma_{n-1} = a_1a_2\cdots a_{n-1} = \alpha'$ and $\varphi\alpha = \delta_k < \delta_{k+1} < \cdots < \delta_{n-1} = a_2a_3\cdots a_n = \alpha''$ be the two unique chains of proper left and right factors of $\alpha = a_1a_2\cdots a_n$ ascending from $\varphi\alpha$, $|\gamma_j| = |\delta_j| = j$. Then the two chains $\gamma_{k+1} < \cdots < \gamma_{n-1}$ and $\delta_{k+1} < \cdots < \delta_{n-1}$ satisfy the description in Case 1. In Case 2 one must take the portions of these chains that are outside $[\lambda, i\alpha]$.

□

An element $x$ of a poset $P$ is called *irreducible* if either $x$ is covered by exactly one element or $x$ covers exactly one element. After removing some irreducibles, elements that previously were not irreducible may become so, and conversely. We will say that $P$ is *dismantlable to* a subposet $Q$ if $Q$ can be obtained by successive removal of irreducibles from $P$. This terminology is due to Rival [11]. A poset with a unique least or a unique greatest element (a *cone*) is clearly dismantlable to a point.

**Lemma 2.** *Let $\beta < \alpha$, with $l(\beta,\alpha) \geq 3$ and $\alpha$ nontrivial.*
**Case 1:** *$\beta \not\leq \varphi\alpha$. Then $(\beta,\alpha)$ is dismantlable to a point.*
**Case 2:** *$\varphi\alpha \leq i\alpha$. Same conclusion as in Case 1.*
**Case 3:** *$\beta = \varphi\alpha \not\leq i\alpha$. Then $(\beta,\alpha)$ is dismantlable to the subposet $\{\alpha',\alpha''\}$.*
**Case 4:** *$\beta < \varphi\alpha \not\leq i\alpha$. Then $(\beta,\alpha)$ is dismantlable to the subposet $(\beta,\varphi\alpha)\cup\{\varphi\alpha, i\alpha, \alpha', \alpha''\}$.*

**Proof.** We begin with Case 2 (see Figure 1b). If $\beta \not\leq i\alpha$ then by Lemma 1 the interval $(\beta,\alpha)$ is a chain, and the conclusion is obvious. Suppose that $\beta \leq i\alpha$. From Lemma 1 we deduce that $(\beta,\alpha)\setminus(\beta,i\alpha]$ consists of two unrelated chains. These can be removed by deleting irreducible elements from bottom to top. Hence $(\beta,\alpha)$ is dismantlable to $(\beta,i\alpha]$, which (being a cone) is further dismantlable to a point.

For the remainder of the proof we assume that $\varphi\alpha \not\leq i\alpha$ (see Figure 1a). If $\beta \not\leq \varphi\alpha$ (i.e., Case 1), then either (i) $\beta > \varphi\alpha$, or (ii) $\beta \in [\lambda, i\alpha]\setminus[\lambda,\varphi\alpha]$. In subcase (i) the interval $(\beta,\alpha)$ is a chain, and in subcase (ii) one sees from Lemma 1 that $(\beta,\alpha)\setminus(\beta,i\alpha]$ consists of two unrelated chains. Hence, in Case 1 irreducibles can be removed in exactly the same way as was described for Case 2.

Case 3 is easy, since $(\beta,\alpha) = (\varphi\alpha,\alpha)$ consists of two unrelated chains with $\alpha'$ and $\alpha''$ at the top.

Consider finally Case 4. The elements on the two chains strictly between $\varphi\alpha$ and $\alpha'$, resp. $\alpha''$, are irreducible and can be removed in any order. After their removal, the maximal elements of $(\beta,i\alpha)\setminus(\beta,\varphi\alpha)$ have become irreducible and can be removed. Continuing from top to bottom, all elements of $(\beta,i\alpha)\setminus(\beta,\varphi\alpha)$ eventually become irreducible (being covered only by $i\alpha$) and can successively be removed. At the end of this process only the subposet $(\beta,\varphi\alpha)\cup\{\varphi\alpha,i\alpha,\alpha',\alpha''\}$ remains (see Figure 2). □

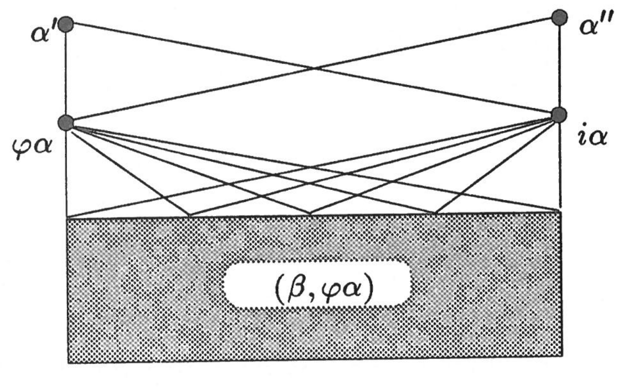


**Figure 2.**

The *join* of two posets $P$ and $Q$ , denoted $P * Q$, is the poset on the set $P \cup Q$ in which $P$ and $Q$ retain their internal orders and all elements of $P$ are below all elements of $Q$. Let $A_2$ denote the 2-element antichain, and $A_2^k$ the join of $k$ copies of $A_2$. (For example, Figure 2 shows a poset isomorphic to $(\beta,\varphi\alpha) * A_2^2$.)

**Lemma 3.** *Suppose that $\beta < \alpha$. If $s(\beta,\alpha) \geq 0$, then $(\beta,\alpha)$ is dismantlable to a subposet isomorphic to $A_2^{s(\beta,\alpha)+1}$. If $s(\beta,\alpha) = -\infty$, then $(\beta,\alpha)$ is dismantlable to a point.*

**Proof.** We will use induction on $l(\beta,\alpha) \geq 2$. If $l(\beta,\alpha) = 2$ then $(\beta,\alpha)$ is either a 2-element antichain or a singleton (since $\alpha$ covers at most 2 elements). These two cases correspond exactly to whether $s(\beta,\alpha) = 0$ or $s(\beta,\alpha) = -\infty$, according to the definition (iii) of the function $s$.

Suppose that $l(\beta,\alpha) > 2$ and that $\alpha$ is not trivial. If $\beta < \varphi\alpha \not\leq i\alpha$ we have by definition (iv) that $s(\beta,\alpha) = s(\beta,\varphi\alpha) + 2$, and Lemma 2 shows that $(\beta,\alpha)$ is dismantlable to a subposet isomorphic to $(\beta,\varphi\alpha) * A_2^2$. By induction, if $s(\beta,\varphi\alpha) \geq 0$ then $(\beta,\varphi\alpha)$ is dismantlable to a subposet isomorphic to $A_2^{s(\beta,\varphi\alpha)+1}$. It follows that $(\beta,\alpha)$ is dismantlable to a copy of $A_2^{s(\beta,\varphi\alpha)+1} * A_2^2 = A_2^{s(\beta,\alpha)+1}$. If, on the other hand, $s(\beta,\varphi\alpha) = -\infty$ then by induction $(\beta,\varphi\alpha)$ is dismantlable to a point. It follows that $(\beta,\alpha)$ is dismantlable to a copy of $\{pt\} * A_2^2$, which is further dismantlable to a point (being a cone). The degenerate case when $s(\beta,\varphi\alpha) = -1$, i.e. $l(\beta,\varphi\alpha) = 1$ and $(\beta,\varphi\alpha) = \emptyset$, is easily checked to be consistent.

Keep the assumptions from the preceding paragraph, except let $\beta = \varphi\alpha$. Then $s(\beta,\alpha) = s(\beta,\beta) + 2 = 0$, and by Lemma 2 $(\beta,\alpha)$ is dismantlable to $\{\alpha',\alpha''\} \simeq A_2$.

Suppose now that $l(\beta,\alpha) > 2$, and that $\alpha$ is trivial, or $\beta \not\leq \varphi\alpha$, or $\varphi\alpha \leq i\alpha$. In each of these cases $s(\beta,\alpha) = -\infty$, by definition. If $\alpha$ is trivial then $(\beta,\alpha)$ is a chain, and hence dismantlable to a point. For the other two cases the conclusion follows from Lemma 2. □

**Proof of Theorem 2.** It is well-known, and easy to see, that if $x$ is an irreducible element in a poset $P$ then $P - \{x\}$ is a strong deformation retract of $P$ (the retraction is the simplicial map that sends $x$ to the unique element covering or covered by $x$ and all other elements to themselves, cf. Corollary 10.12 of [3].) Hence, if $P$ is dismantlable to $Q$ then $Q$ is a strong deformation retract of $P$, and in particular $P$ and $Q$ are homotopy equivalent.

The theorem is therefore a direct consequence of Lemma 3. For this, note that the order complex of $A_2^{k+1}$ is homeomorphic to the $k$-sphere, being the $k$-fold suspension of the 0-sphere $A_2$. □

## 3. The Möbius function of subword order.

We start with a few definitions. Given a word $\alpha = a_1a_2\cdots a_n \in A^*$, its *repetition set* is $\mathcal{R}(\alpha) = \{i : a_{i-1} = a_i\} \subseteq \{2,\cdots,n\}$. An *embedding* of $\beta$ in $\alpha$ is a sequence $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $\beta = a_{i_1}a_{i_2}\cdots a_{i_k}$. It is a *normal embedding* if $\mathcal{R}(\alpha) \subseteq \{i_1,\cdots,i_k\}$. For $\alpha,\beta \in A^*$ let

$$\binom{\alpha}{\beta}_n = \text{number of normal embeddings of } \beta \text{ in } \alpha.$$

For instance, $\binom{aabac}{aac}_n = 2$.

The following combinatorial rule for the Möbius function of subword order was given in [2]. The original proof using lexicographic shellability, as well as a later algebraic proof in [4], are not as simple and elementary as the formula itself. However, both these proofs yield additional information. Here a short and elementary proof (giving no additional information) will be given.

**Theorem 3.** *The Möbius function of subword order is given by:*

$$\mu(\beta,\alpha) = (-1)^{|\alpha|+|\beta|}\binom{\alpha}{\beta}_n,$$

*for all $\alpha$, $\beta \in A^*$.*

**Proof.** Suppose that $\gamma \leq \alpha = a_1 a_2 \cdots a_n$, and let

$$S = \{1 \leq i_1 < i_2 < \cdots < i_k \leq n : \mathcal{R}(\alpha) \subseteq \{i_1, \cdots, i_k\} \text{ and } \gamma \leq a_{i_1} \cdots a_{i_k}\}.$$

(In this section "$\leq$" of course denotes subword order.) Then,

$$\sum_{\gamma \leq \beta \leq \alpha} (-1)^{|\alpha|+|\beta|} \binom{\alpha}{\beta}_n = (-1)^n (\sharp S_{\text{even}} - \sharp S_{\text{odd}}).$$

Thus, if we show for $\gamma < \alpha$ that $S$ has as many members of even as of odd length (so that t sum equals zero), we will have verified the defining recursion for the Möbius function. To this we construct a simple parity-changing involution $\varphi$ on $S$.

Given $I = (i_1, \cdots, i_k) \in S$ let $f = f_I$ be the minimal number in $\{1, \cdots, n\}$ such that $a_f$ not in the final embedding of $\gamma$ in $a_{i_1} \cdots a_{i_k}$. The final embedding of $\gamma$ in $\delta$ is the embeddi $(j_1, \cdots, j_g)$ uniquely characterized by $j'_e \leq j_e$, $1 \leq e \leq g$, for every other embedding $(j'_1, \cdots, j$ of $\gamma$ in $\delta$. Then define:

$$\varphi(I) = \begin{cases} I \cup \{f_I\}, & \text{if } f_I \notin I, \\ I - \{f_I\}, & \text{if } f_I \in I. \end{cases}$$

It is clear that $\varphi(I) \in S$ in the first case. In the second case we see that $\gamma$ is a subwo of $a_{i_1} \cdots a_{i_k}$ also after $a_f$ is erased (the final embedding remains), and that $\mathcal{R}(\alpha) \subseteq \varphi(I)$ $f \in \mathcal{R}(\alpha)$ then $a_{f-1} = a_f$, which is impossible if $a_{f-1}$ but not $a_f$ lies in the final embeddin so that also here $\varphi(I) \in S$.

Along the same lines one sees that $f_{\varphi(I)} = f_I$, because the final embedding of $\gamma$ remains t same after adding or deleting $a_f$. This implies that $\varphi^2(I) = I$, for all $I \in S$.

As an illustration of the involution $\varphi$ constructed in the proof, let $\gamma = ab$ and $\alpha = abc$ Then

$$\begin{array}{ccccccccccc} \cdot & \cdot & \cdot & a & b & \longleftrightarrow & a & \cdot & \cdot & a & b \\ a & \cdot & c & \cdot & b & \longleftrightarrow & a & b & c & \cdot & b \\ a & b & \cdot & a & b & \longleftrightarrow & \cdot & b & \cdot & a & b \end{array}$$

## 4. Final remarks.

**A.** The Möbius number of a poset is the number of odd cardinality chains minus the numb of even cardinality chains (see [13], p.119). It is easy to see directly that this difference doesn change when an irreducible is removed. Therefore, if $P$ is dismantlable to $Q$ it follows th $\mu(P) = \mu(Q)$.

Consequently, Theorem 1 can be directly deduced from Lemma 3 with no mention of topolog One needs only to check that $\mu(A_2^{k+1}) = (-1)^k$ and $\mu(pt.) = 0$.

**B.** If a poset $P$ is dismantlable to a subposet $Q$, then $Q$ is a strong deformation retract of in the "ideal topology", a finite topology studied by Stong [14], Farmer [5] and others. Henc one can from Lemma 3 deduce an "ideal topology" version of Theorem 2, which with know implications is strictly stronger than the stated "order complex topology" version. Farmer [ takes this point of view in his study of the $\beta = \lambda$ case.

**C.** Kahn [8] uses the method of "non-evasiveness" to prove that $\mu(x, y) = 0$ in certain poset It is known that "dismantlable to a point" implies "non-evasive" (see [3]), so Kahn's meth could be used also here.

**D.** In [4] it is shown that for subword order the formal power series $\sum_{\beta \leq \alpha} \alpha$ and $\sum_{\beta \leq \alpha} \mu(\beta, \alpha)$ are rational for all $\beta \in A^*$. For factor order the first series is rational (a finite automaton ca recognize the language of all words containing $\beta$ as a factor), but the rationality of the secon series seems doubtful.

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