ALGEBRAIC, COMBINATORIAL AND SYNTACTIC TECHNIQUES IN NONLINEAR CONTROL

By

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Introduction

Several techniques have recently been introduced in order to compute approximate solutions to forced nonlinear diffrential equations. The aim of this paper is to briefly review two of these methods which are of different nature but leads to the same approximations. The first method is based on the combinatorial notion of $\mathbb{L}$-species introduced by Leroux and Viennot [13-15] and the second is based on automata representations due to Hespel and Jacob [8,9]. Both of these methods are intimately related to the algebraic appraoch of nonlinear functional expansions introduced by Fliess[4,5] . Strong connections should also appear with the work of Grossman[7].

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The relevance of these combinatorial and syntactic approaches is that they provide a clear iterative scheme in order to find the functional expansion of the solution. They lead to efficient computer tools for analyzing the behavior of the solution around equilibrium points. See Martin [16] for a first attempt. Moreover any choice of an iterated truncation procedure provides a family of bilinear approximations, that can be viewed as a family of noncommutative Padé-type approximants. For example, *bounding the possible widths of trees and hedges at the order p* in the first approach (see Lamnabhi-Lagarrigue, Leroux and Viennot [12]) or *truncating structural $\mathbb{R}$–automaton at height p* in the second approach (see Hespel and Jacob [10]) lead to the well known Brockett's approximations by bilinear systems based on a Carleman linearization [3]. More precisely, in this case, the Volterra series associated with the nonlinear forced differential equation coïncides up to order p with the Volterra series of the ouput of a bilinear system.

The algebraic approach is briefly recalled in the first part of the paper . Also the derivation of the Volterra kernels using Brockett bilinear approximations is translated in this context. The second part is devoted to the combinatorial representation and the last part to the syntactic approach.

I. Algebraic representation: The generating power series approach

Let us recall first some definitions and results from the Fliess algebraic approach [4, 5]. Let $u_1(t), u_2(t), \ldots, u_m(t)$ be some inputs and $X = \{x_0, x_1, \ldots, x_m\}$ be a finite set called alphabet. We denote by X^* the set of words generated by X. The algebraic approach introduced by Fliess may be sketched as follows. Let us consider the letter x_0 as an operator which codes the integration with respect to time and the letter x_i, $i = 1, \ldots, m$, as an operator which codes the integration with respect to time after multiplying by the input $u_i(t)$. In this way, any word $w \in X^*$ gives rise to an iterated integral, denoted by $\mathrm{I}^t\{w\}$, which can be defined recursively as follows: $\mathrm{I}^t\{\emptyset\} = 1$ and for $w = x_\alpha v \in X^*$, $\mathrm{I}^t\{w\} = \int_0^t d\tau\, \mathrm{I}^\tau\{v\}$ if $\alpha = 0$ and $\mathrm{I}^t\{w\} = \int_0^t u_i(\tau) d\tau \mathrm{I}^\tau\{v\}$ if $\alpha = i$.

Now, let us consider the control system

$$\begin{cases} \dot{q}(t) = f_0(q) + \sum_{i=1}^{m} u_i(t) f_i(q) \,, \\ y(t) = h(q) \,, \end{cases} \tag{1.1}$$

where the state q belongs to a finite-dimensional $\mathbb{R}$-analytic manifold M, the vector fields f_0, $f_1, \ldots, f_m : M \to M$ and the output function $h : M \to \mathbb{R}$ are analytic and defined in a neighborhood of the initial state q(0). Using a local coordinates chart, $q = (q^1, \ldots, q^N)^T$, (1.1) can be written in the following form

$$\begin{cases} \dot{q}^k(t) = f_0^k(q^1, \ldots, q^N) + \sum_{i=1}^{m} u_i(t) f_i^k(q^1, \ldots, q^N), \quad 1 \le k \le N \,, \\ y(t) = h(q^1, \ldots, q^N), \end{cases} \tag{1.2}$$

where the functions $f_i^k : \mathbb{R}^N \to \mathbb{R}$ are analytic in a neighborhood of $q(0) = (\gamma^1, \ldots, \gamma^N)$.

The solution y(t) of the control system is given by [4]

$$y(t) = y(0) + \sum_{\nu \ge 0} \sum_{j_0, \ldots, j_\nu = 0}^{m} f_{j_\nu} \ldots f_{j_0} \cdot h(q) \Big|_{q=\gamma} \mathbb{I}^t\{x_{j_0} \ldots x_{j_\nu}\} \,. \tag{1.3}$$

This functional expansion is called the *Fliess expansion* of the solution. The associated power series

$$s = y(0) + \sum_{\nu \ge 0} \sum_{j_0, \ldots, j_\nu = 0}^{m} f_{j_\nu} \ldots f_{j_0} \cdot h(q) \Big|_{q=\gamma} x_{j_0} \ldots x_{j_\nu} \tag{1.4}$$

is called the *Fliess series* associated with the control system (1.1).

Let us now consider the Carleman bilinearization technique introduced by Brockett. Let us express the analytic functions f_i^k and h as Taylor expansions

$$f_i^k(q^1, \ldots, q^N) = \sum_{j_1, \ldots, j_N \ge 0} a^{k,i}_{j_1, \ldots, j_N} (q^1)^{j_1} \ldots (q^N)^{j_N} \,, \tag{1.5a}$$

$$h(q^1, \ldots, q^N) = \sum_{j_1, \ldots, j_N \ge 0} \mathfrak{h}_{j_1, \ldots, j_N} (q^1)^{j_1} \ldots (q^N)^{j_N} \,, \tag{1.5b}$$

in a neighborhood of $\gamma^i = q^i(0)$, $i = 1,\ldots, N$. If γ is an equilibrium point of the system (1.1) then the Brockett bilinear system which has the same Volterra series up to order p than the Volterra series of the nonlinear system (1.1) is obtained by introducing new states

$$q^{<p>}_{j_1,\ldots,j_N} = (q^1)^{j_1}\ldots(q^N)^{j_N}, \qquad j_1 + \ldots + j_N \leq p,$$

with initial conditions $q^{<p>}_{j_1,\ldots,j_N}(0) = (\gamma^1)^{j_1}\ldots(\gamma^N)^{j_N}$. If $q^{<p>}_{j_1,\ldots,-1,\ldots,j_N} = 0$ for all $j_1,\ldots,j_N \geq 0$ and $q^{<p>}_{j_1,\ldots,j_N} = 0$ if $j_1 + \ldots + j_N > p$, we obtain

$$\begin{cases} \dot{q}^{<p>}_{j_1,\ldots,j_N} = \sum_{k=1}^{N} j_k \Big(\sum_{i=0}^{m} u_i \sum_{i_1,\ldots,i_N \geq 0} a^{k,i}_{i_1,\ldots,i_N} \, q^{<p>}_{j_1+i_1,\ldots,j_k-1+i_k,\ldots,j_N+i_N} \Big), \\ y^{<p>} = \sum_{j_1,\ldots,j_N \geq 0} h_{j_1,\ldots,j_N} \, q^{<p>}_{j_1,\ldots,j_N}, \end{cases} \tag{1.6}$$

This system may be interpreted in the algebraic context by defining the generating power series $g^{<p>}_{j_1,\ldots,j_N}$ associated with $q^{<p>}_{j_1,\ldots,j_N}$:

$$\begin{cases} g^{<p>}_{j_1,\ldots,j_N} = \sum_{k=1}^{N} j_k \Big(\sum_{i=0}^{m} \sum_{i_1,\ldots,i_N \geq 0} a^{k,i}_{i_1,\ldots,i_N} \, x_i \, g^{<p>}_{j_1+i_1,\ldots,j_k-1+i_k,\ldots,j_N+i_N} \Big), \\ g^{<p>} = \sum_{j_1,\ldots,j_N \geq 0} h_{j_1,\ldots,j_N} \, g^{<p>}_{j_1,\ldots,j_N}, \end{cases} \tag{1.7}$$

The rational power series $g^{<p>}$ may be seen as a *non commutative Padé-type approximant* for the forced differential system (1.1) which generalizes the notion of Padé-type approximant obtained by Brezinski [2] for free differential systems. The effective computation of the rational power series $g^{<p>}$ has been derived using formal languages and applied to the analysis of nonlinear electronics circuits [1, 11]. However the computations become quikly unwieldy. The two representations described in the next parts lead a better understanding of these powers series and therefore allow to carry further the study of this wide class of nonlinear systems. For instance equation (1.7) will receive a clear interpretation using one method or the other. Moreover any choice of an iterated truncation procedure of these representations will produce a new family of bilinear approximations.

For the sake of simplicity we will consider in the following the two dimensional nonlinear system

$$\begin{cases} \dot{Y}(t) = A(Y, Z) + u(t)B(Y, Z), & Y(0) = \gamma, \\ \dot{Z}(t) = C(Y, Z) + u(t)D(Y, Z), & Z(0) = \delta, \end{cases} \qquad (1.8)$$

with only one input u(t). All the results below may be generalized to nonlinear system (1.1) without adding any complexity.

II. Combinatorics representations: $\mathbb{L}$-species.

2.1 Generalities[13].

Let $\mathbb{E}$ and $\mathbb{L}$ denote respectively the category of finite sets and functions and the category of finite linearly ordered sets and order preserving bijections. Recall that a species of structures M called a $\mathbb{L}$-species is a functor $M : \mathbb{L} \to \mathbb{E}$. This means that to each linearly ordered finite set l, M associates a finite set, denoted by $M[l]$, whose elements are called *M-structures on l* and to each order preserving bijection $\varphi : l_1 \to l_2$, a function

$$M[\varphi]: M[l_1] \to M[l_2]$$

called *the transport of structures*, in a functorial way, that is such that

$$M[\varphi \circ \phi] = M[\phi] \circ M[\phi] \quad \text{and} \quad M[1_l] = 1_{M[l]}$$

A convenient and useful graphical representation of a generic or typical M-structure on a linearly ordered set is given by figure 2.1, where the curved arrow indicates the linear order on the set of points and the label M represents the M-structure.

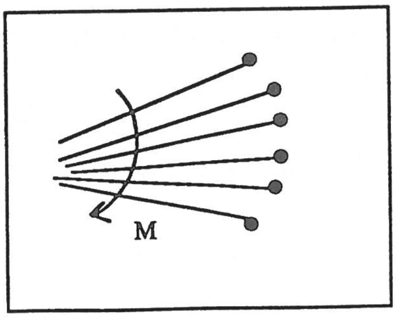


Figure 2.1. Generic M-structure

Definition 2.1: Two $\mathbb{L}$-species M and N are isomorphic if there exists a natural isomorphism of functors $\alpha : M \to N$. In other words there should exist a bijection $\alpha_\ell : M[\ell] \to N[\ell]$, for each $\ell \in \mathbb{L}$, such that for any increasing bijection $\varphi : \ell \to h$, the following diagram commutes:

$$\begin{tikzcd} M[\ell] \arrow[r, "\alpha_\ell"] \arrow[d] & N[\ell] \arrow[d] \\ M[h] \arrow[r, "\alpha_h"] & N[h] \end{tikzcd}$$

Let us define now some operations on M-structures. In the following, the operations + (and $\sum$) and $\times$ (and $\prod$) on sets are the disjoint union and cartesian product respectively. We let min(ℓ) denote the new minimum element to ℓ and $1 + \ell$ denote the ordered set obtained by adjunction of a new minimum element.

Let M and N be species, and ℓ be a linearly ordered set. The following operations are defined,

addition: M + N by $\quad (M + N)[\ell] = M[\ell] + N[\ell]$,

product: M.N by $\quad (M.N)[\ell] = \sum_{\ell_1 + \ell_2 = \ell} M[\ell_1] \times M[\ell_2]$,

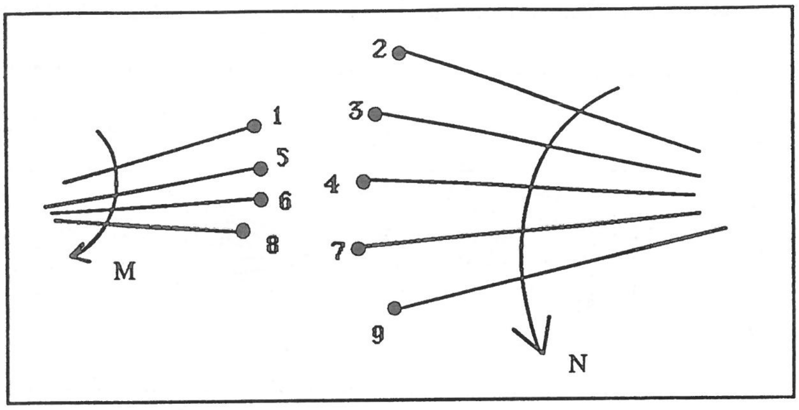


Figure 2.2 : Generic M.N-structure

substitution: M(N) , when $N[\varnothing] = \varnothing$, by

$$(M(N))[l] = \sum_{\rho \in N[l]} M[l/\rho] \times \prod_{c \in l/\rho} M[c]$$

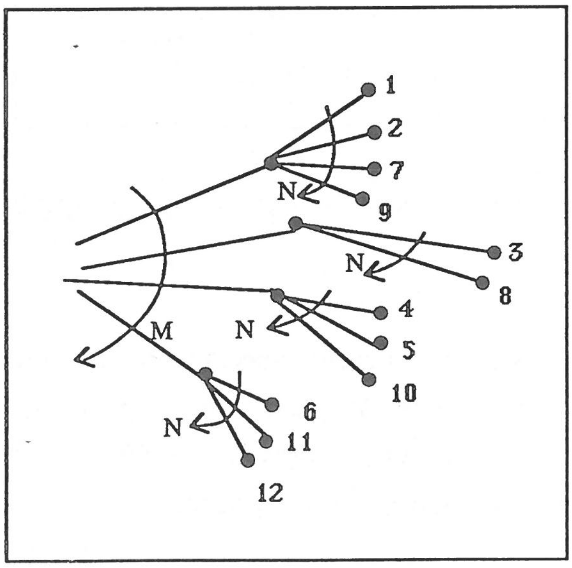


Figure 2.3 : Generic M(N) - structure

The operation of substitution for $\mathbb{L}$-species is closely related to the concept of *composé partitionnel* introduced by Foata and Schützenberger[6].

Derivation: M' (also denoted by dM/dT) by $\quad M'[\ell] = M[\ell + 1]$

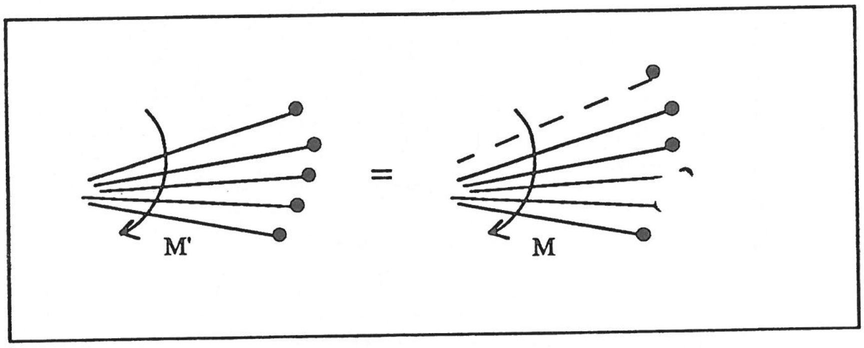


Figure 2.4 : Generic M'-structure.

Integration: $\quad F(T) = F = \int_0^T M(X)dX \quad$ by

$F[\varnothing] = \varnothing$ and $\quad F[\ell] = M[\ell \setminus \min\{\ell\}]$, for $\ell \neq \varnothing$.

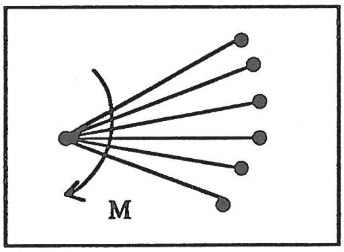


Figure 2.5 : Generic $\int_0^T M(X)dX$ **-structure.**

All the elementary properties, associativity, commutativity, distributivity, linearity, etc., of the operations are true at the combinatorial level, including the **Leibnitz rule** and the **Chain rule** for the derivative, $(M.N)' = M.N' + M'.N$ and $M(N)' = M'(N).N'$ where equality means isomorphism of $\mathbb{L}$-species.

Let us now consider the generating power series

$$F(t) = \sum_{n \geq 0} | F[n] | \frac{t^n}{n!} \quad .$$

The following properties are easily verified,

$$(F+G)(t) = F(t) + G(t), \quad (F.G)(t) = F(t)\,G(t), \quad (F \circ G)(t) = F(G(t))$$

and $F'(t) = \dfrac{d}{dt} F(t)$.

Finally if $\quad F = \int_0^T M(X)dX \quad$ then $\quad F(t) = \int_0^t M(x)dx$.

2.2 Arborescences.

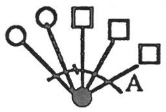
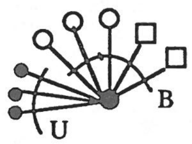
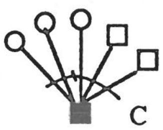
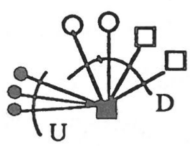
Eclosions			Type
$\bigcirc$ γ $\longrightarrow$		$TA(\gamma,\delta)\dfrac{\partial}{\partial\gamma}$	1
		$\int_0^T U(X)dX.B(\gamma,\delta)\,\dfrac{\partial}{\partial\gamma}$	2
$\square$ δ $\longrightarrow$		$TC(\gamma,\delta)\dfrac{\partial}{\partial\delta}$	3
		$\int_0^T U(X)dX.D(\gamma,\delta)\,\dfrac{\partial}{\partial\delta}$	4

Figure 2.6: Eclosions associated with (1.8).

Let us consider now equation (1.8) in its integral form,

$$\begin{cases} Y(t) = \gamma + \int_0^t u(\tau)A(Y(\tau),Z(\tau))d\tau + \int_0^t u(\tau)B(Y(\tau),Z(\tau))d\tau \\ Z(t) = \delta + \int_0^t u(\tau)C(Y(\tau),Z(\tau))d\tau + \int_0^t u(\tau)D(Y(\tau),Z(\tau))d\tau \end{cases} \qquad (2.1)$$

.

By virtue of the definition of the integration and substitution, the integral equation (1.9) leads to the definition of the eclosions visualized in figure 2.6.

It now suffices to iterate this process to obtain a canonical combinatorial solution of (1.8), that is the $\mathbb{L}$-species of enriched increasing trees, generically described by figure 2.7.

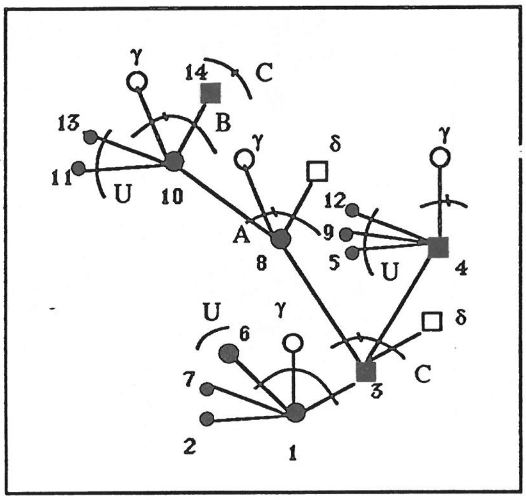


Figure 2.7: Generic Y(T)-structure.

We denote by $\mathcal{T}$ the set of all these trees and by $\mathcal{T}_Y$ (resp. $\mathcal{T}_Z$) the set of trees whose root is a Y-vertex (resp. a Z-vertex). With each tree $T \in \mathcal{T}_Y$, we associate the **weight** $v(T)$ defined as the product of the weights of the vertices of T, and the **word** $w(T) = w_1w_2...w_n$, where $w_i = x_i$ if the vertex labelled i is a leaf and $w_i = x_0$ otherwise. For example the tree T of figure 2.7 has weight $v(T) = \gamma^4\delta^2$ and the associated word is $w(T) = x_1x_0x_1x_0x_1x_1x_0$.

From the general theory [14], the solution Z(t) of the equation (1.8) is given by

$$y(t) = \sum_{d \in \mathcal{T}_Z} v(d)\mathbb{I}^t\{w(d)\} \qquad (2.2)$$

where $\mathrm{I}^t\{w(d)\}$ denotes the iterated integral associated with the word w(d) (see §1).

The formal power series

$$g = \sum_{d \in \mathfrak{T}_Z} v(d) w(d) \tag{2.3}$$

in non commuting variables x_0, x_1, is the *Fliess series* of the system and (2.2), the *Fliess expansion* of the solution Z(t).

2.3 Bilinear approximants.

We now describe a truncation process of the arborescences which leads to an approximation process for the solution of (1.8). To do this we need to introduce the concept of width of trees. The **width** of a tree T, denoted by wit(T) is defined as the maximun number of buds of the trees that appear at any stage of the growth of T. For example, the tree T of figure 2.7 has wit(T) = 7, see figure 2.8 where the "squelette" of T is represented. This notion can be extended to the notion of **hedges of trees**. We say that a row of weighted plane rooted trees $\bar{T}$ is a hedge H of trees T if

i) the labels (1,2,..., n) for the n labelled vertices are distributed freely among the trees but should be increasing from the root to the leaves within each tree and,

ii) the trees with a Y-root precede the trees with a Z-root.

Let us denote by $\mathfrak{T}_{i,j}$ the set of hedges consisting of a row of i trees with Z-roots followed by j trees with Y-roots. In particular we have

$$\mathfrak{T}_Y = \mathfrak{T}_{1,0} \quad \text{and} \quad \mathfrak{T}_Z = \mathfrak{T}_{0,1} \quad .$$

The weight v(H) and the word w(H) of a hedge H are defined in exactly the same way as for trees. Similarly, the width wit(H) of $H \in \mathfrak{T}_{i,j}$ is defined as the maximum number of buds that appear at any stage in the growth of H, using the same eclosions as before, but starting with a row of i Y-buds followed by j Z-buds, if $H \in \mathfrak{T}_{i,j}$. The proposed approximants are then obtained by imposing a bound on the possible hedges of trees. More precisely, we denote by $\mathfrak{T}^{<p>}$ the set of trees T such that wit(T) $\leq$ p and $\mathfrak{T}^{<p>}_{i,j}$ the set of hedges H such that wit(H) $\leq$ p. We define the p^{th}-approximants $Z^{<p>}(t)$ and $Z^{<p>}_{i,j}(t)$ as the corresponding Fliess expansions, with $\mathfrak{T}^{<p>}_Z = \mathfrak{T}^{<p>}_{1,0}$.

$$Z^{<p>}(t) = \sum_{T\in \mathfrak{T}_Z^{<p>}} v(T)\, \mathbb{I}^t\{w(T)\} \quad , \tag{2.4}$$

$$q_{i,j}^{<p>}(t) = \sum_{H\in \mathfrak{T}_{i,j}^{<p>}} v(H)\, \mathbb{I}^t\{w(H)\} \quad . \tag{2.5}$$

Now let consider the Taylor expansion of each function appearing in (1.8).

$$A(Y,Z) = \sum_{r,s\geq 0} a_{r,s}(Y)^r\,(Z)^s \; ; \qquad B(Y,Z) = \sum_{r,s\geq 0} b_{r,s}(Y)^r\,(Z)^s \; ;$$

$$C(Y,Z) = \sum_{r,s\geq 0} c_{r,s}(Y)^r\,(Z)^s \; ; \qquad D(Y,Z) = \sum_{r,s\geq 0} d_{r,s}(Y)^r\,(Z)^s \; ;$$

The bilinear state system satisfied by the functions $q_{i,j}^{<p>}(t)$, for $1 \leq i + j \leq p$ is simply obtained by examining what can happen when the vertex with minimum label is removed from a hedge, i.e. what is the nature of the very first eclosion.

Proposition 2.1: Given $p \geq 1$, for $1 \leq i + j \leq p$, the p^{th} - approximants $q_{i,j}^{<p>}(t)$ satisfy the differential equations,

$$\dot{q}_{i,j}^{<p>} = i \sum_{r,s\geq 0} (a_{r,s} + u b_{r,s}) q_{i+r-1,j+s}^{<p>}(t) + j \sum_{r,s\geq 0} (c_{r,s} + u d_{r,s}) q_{i+r,j+s-1}^{<p>}(t)$$

(2.6)

with initial conditions $q_{ij}(0) = \gamma^i \delta^j$ where we have set $q_{-1,j} = q_{i,-1} = 0$ for all i, j, and $q_{i,j} = 0$ if $i + j > p$.

Proof: The initial condition $q_{ij}(0)$ corresponds to a row of buds, which gives $g^i d^j$. The first term of the right hand side of (2.6), $i \sum_{r,s\geq 0} (a_{r,s} + u b_{r,s}) q_{i+r-1,j+s}^{<p>}$, corresponds to the case where the minimum label is a Y-vertex: there are i possibilities for its position, it appeared as a type 1 and 2 eclosion and hence its removal leaves a $q_{i+r-1,j+s}^{<p>}$ -structure. A similar discussion, where eclosions of type 3 and 4 can occurr, explains the second term.

Proposition 2.2: Assume that the point (0, 0) is an equilibrium point. Then the functionals $(Y(t))^i(Z(t))^j$ and $q_{i,j}^{<p>}(t)$ have the same Volterra kernels up to order p.

Remarks: i) Note the proposed combinatorics appraoch allows to derive equation (2.6) directly from (1.8).

ii) Any other process of truncature of the arborescences would lead to a different type of approximations.

iii) This appraoch allows also to pick easily a particular coefficient in the Fliess series or in the formal equivalence of the associated Volterra series expansion[16].

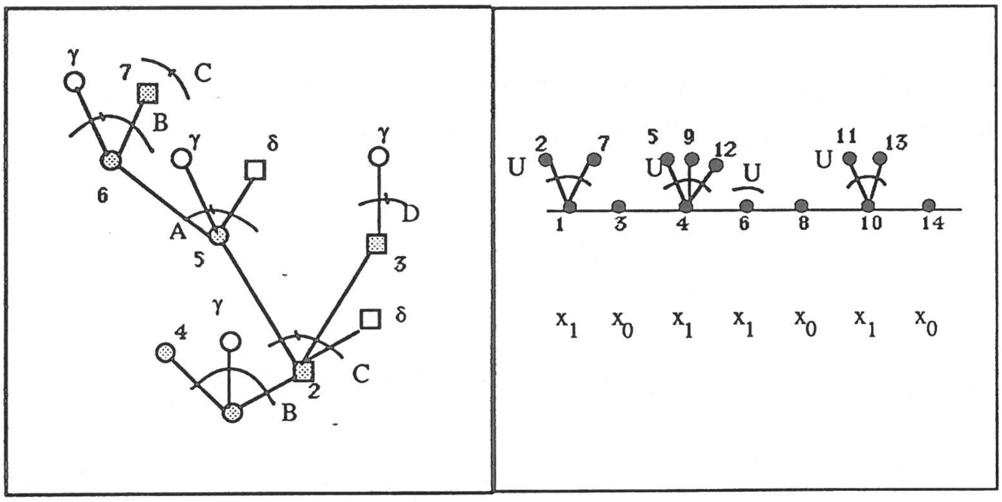


Figure 2.8

III. Automata representation: A geometric $\mathbb{R}$- automata.

3.1 Generalities[8,9].

Let us associate to the system (1.1) the following infinite geometrical $\mathbb{R}$-automaton $\mathcal{G}$

$$\mathcal{G} = (\mathcal{E}, V, \mu, \lambda) \qquad (3.1)$$

where $\mathcal{E}$ is the vector space generated by $(q^1)^{j1} \dots (q^N)^{jN}$, $j_1, \dots, j_N \geq 0$, V is the initial vector associated to the observation h,

$$h(q^1, \ldots, q^N) = \sum_{j_1,\ldots,j_N \geq 0} h_{j_1,\ldots,j_N}(q^1)^{j_1}\ldots(q^N)^{j_N} \quad ,$$

μ is the linear mapping $\mathcal{E} \to \mathcal{E}$, such that

$$\mu(x_j) \;:\; (q^1)^{j1}\ldots(q^N)^{jN} \;\to\; f_j(\,(q^1)^{j1}\ldots(q^N)^{jN}\,)$$

where f_j is a vector field, see (1.3). Finally λ is a linear form defined by the row vector whose components are the evaluations of the final states evaluated at $t = 0$.

From this definition it not difficult to see that the power series associated with the $\mathbb{R}$-automaton (3.1) is precisely the Fliess power series (1.4b),

$$g = \sum_{w \in X^*} \lambda(\mu(w)V)\; w$$

Endeed, if w is the word $x_{j0} \ldots x_{j\nu}$ then $\lambda(\mu(w)V)$ is equal to $f_{j\nu} \ldots f_{j0}.h(\gamma)$.

In particular for the system (1.8), we can associate the infinite geometric $\mathbb{R}$-automaton

$$\mathcal{B} = (\;\mathcal{E},\; V,\; \mu,\; \lambda\;)$$

with the vector fields, f_0 and f_1 given respectively by,

$$[\sum_{r,s\geq 0} a_{r,s}(Y)^r\,(Z)^s]\,\frac{\partial}{\partial Y} + [\sum_{r,s\geq 0} c_{r,s}(Y)^r\,(Z)^s]\,\frac{\partial}{\partial Z}$$

and

$$[\sum_{r,s\geq 0} b_{r,s}(Y)^r\,(Z)^s]\,\frac{\partial}{\partial Y} + [\sum_{r,s\geq 0} d_{r,s}(Y)^r\,(Z)^s]\,\frac{\partial}{\partial Z}\,.$$

The cell of this automaton is represented on the figure 3.1

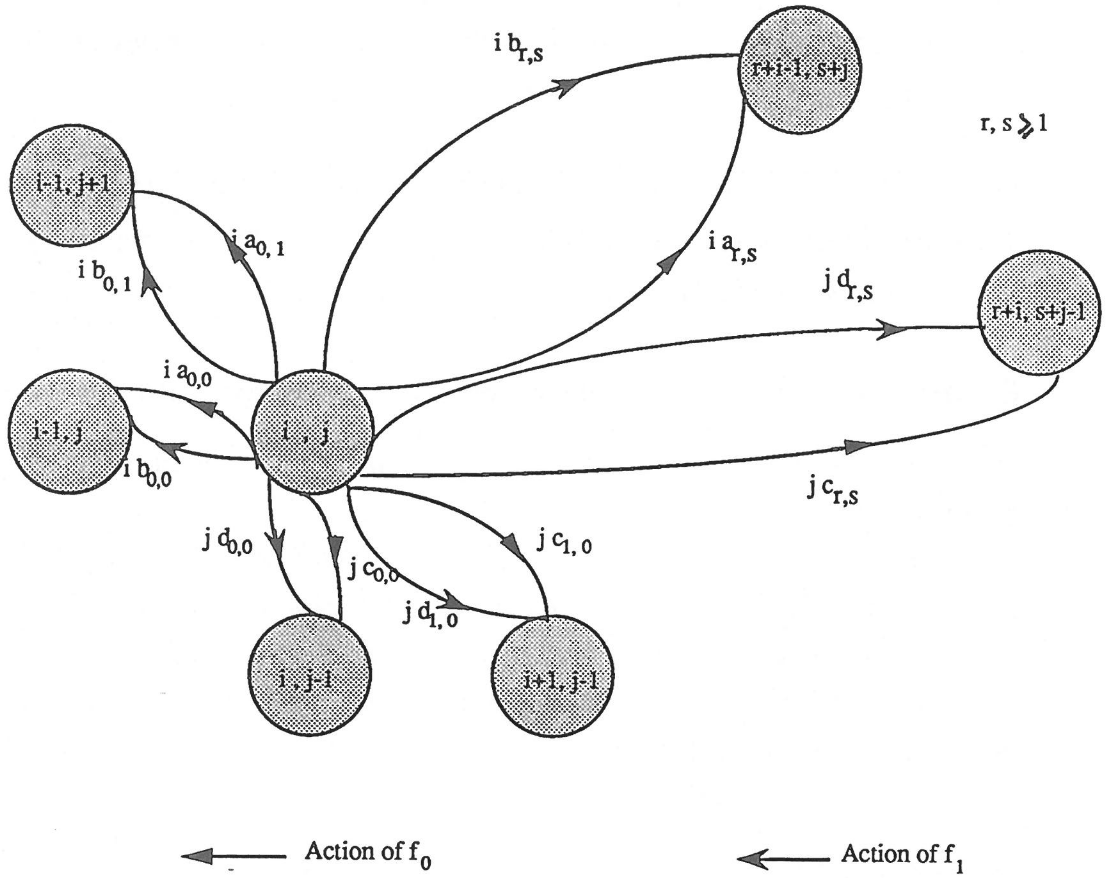


Figure 3.1: Cell of an automaton.

3.2 Bilinear approximations.

Now from the theorem of Schützenberger[17], recall that a power series $g \in R\langle\langle X^*\rangle\rangle$ is said to be rational, (or regular) if and only if it can be recognized, i.e. if there exists a finite dimensional $\mathbb{R}$-automaton $\mathcal{A}$ over X,

$$\mathcal{A} = (Q, \gamma, \mu, \lambda)$$

where Q is a finite dimensional $\mathbb{R}$-vector space, γ is the initial state , $\mu(x_i)$ is a linear map: Q $\rightarrow$ Q and the observation λ is a linear map: Q $\rightarrow \mathbb{R}$, satisfying

$$g = \sum_{x_{j0} \dots x_{jv} \in X^*} \lambda\mu(x_{j0})\dots\mu(x_{jv})\gamma\, x_{j0} \dots x_{jv}.$$

Now let us consider again the infinite geometric $\mathbb{R}$-automaton $\mathcal{B}$. Let us truncate this automaton to the states Y^iZ^j of degree not more than p, $i+j \leq p$. We obtain a finite $\mathbb{R}$-automaton $\mathcal{A}^{<p>}$. Then $\mathcal{A}^{<p>}$ is represented by a rational power series and the state equations satisfy a bilinear system.

Proposition 3.1: Assume that the point (0, 0) is an equilibrium point. In this case $\mathcal{A}^{<p>}$ is the Volterra automaton at the order p, i.e., the associated bilinear system corresponds to the Brockett bilinear system (1.7).

Remarks: i) Note that $\mathcal{T}^{<p>}_{i,j}$ from the combinatorial approach and $\mathcal{A}^{<p>}$ from the syntactic approach are two equivalent representations of the same differential equation (1.6) or in its algebraic form (1.7) .

ii) Any finite truncation of the infinite automaton $\mathcal{G}$ furnishes a finite automaton, describing a bilinear system approximating the given nonlinear analytical system. For instance one may consider only the states Y^iZ^j , $i+j \leq p$ and $i \leq j$. See [10].

iii) In this context also, the derivation of the associated bilinear system is evident from the differential equations.

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