# PLANE AND PROJECTIVE MEANDERS

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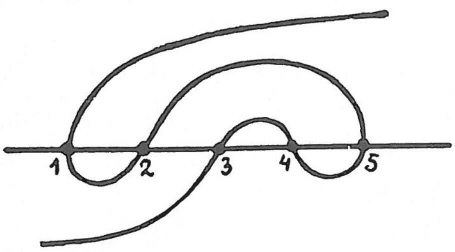
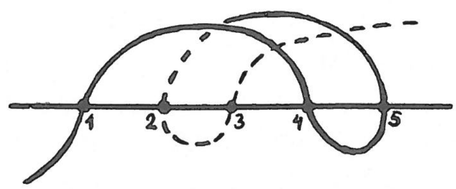
J'errais dans un méandre;
J'avais trop de partis,
trop compliqués, à prendre...
(E.Rostand, *Cyrano de Bergerac*,
act 1 scene 5)

A highway from West to East several times crosses a river flowing from South-West also to East. Enumerate the bridges as they are located along the highway (from West to East). The order of the bridges along the river determines a permutation. Following V.I.Arnol'd, we call the permutation (and a corresponding geometrical image) a *meander*.

Obviously, not any permutation can be obtained in this way. In particular, in meanders even numbers must occupy even positions, odd numbers, odd positions.




meander 34521




not meander 14523

Numerous pictures of meanders can be found in the last paper by Henri Poincaré "Sur un théorème de géométrie" [Poi] where he tried to prove, by means of meanders, that a transformation of a ring into itself preserving the area and shifting border circles into opposite directions has not less than two fixed points. The theorem was proved by Birkhoff in 1913 by a different method, but its generalization on the transformation of a sphere with handles was proved by Ya.M.Eliashberg in 1978 with the help of meanders [Eli]. "Projective meanders" to be defined below were used by V.I.Arnol'd [Ar] as a tool for analyzing differential-geometric properties of the manifold of zeroes of hyperbolic polynomials. In a number of papers meanders appeared not so much as a tool but as an object of investigation. For instance in the paper [Ros] "plane permutations" are introduced and investigated that coincide with "closed meanders" to be defined below. Such permutations prove to be sorted in linear time. In the paper [Ph] a class of mazes is introduced that are in one-to-one correspondence with meanders.

The problem of enumerating meanders proved to be especially complicated. In the paper [Koe] for a similar problem of enumeration for the number of folding a strip of stamps certain recurrent formulas are introduced. They can serve as the basis for constructing an algorithm of computation of corresponding numbers but, unfortunately, they do not yield either explicit formulas or even the information on the asymptotics of the number sequence in question, while the algorithm has exponential complexity and does not allow us to compute a large number of sequence terms. In the paper of present authors [LaZ] the problem of enumerating closed meanders was studied. There were received non-trivial upper estimates for the main term of the asymptotics and the relation between the problem of meanders and the theory of formal languages and that of Feynman diagrams in the quantum field theory was indicated. R.Cori [Cor] attracted the authors' attention to the fact that the problem of enumerating closed meanders was equivalent to that of determining the complexity of a class of hypermaps.

Thus the problem of meanders seams to belong to the simply formulated but fairly difficult problems of combinatorial analysis related to different sections of mathematics and are a touchstone for various methods of enumerative combinatorics.

The present paper examines arithmetical properties of meandric numbers and also introduces and studies projective meanders.

The authors are grateful to V.I.Arnol'd, S.V.Chmutov, D.Ivanov, A.Phillips, R.Cori, and Ph.Flajolet for useful discussions as well as the organizers of the conference "Formal Power Series and Algebraic Combinatorics" for their hospitality and goodwill.

## 1. Arithmetical properties of meandric numbers

1.1. Definition. Denote the number of meanders passing through n bridges by $m_n$, n = 1, 2, 3 ... and call the n-th *meandric number*. Assume $m_0$ = 1. Sequence $m_n$ can be readily shown to increase monotonously when $n \geq 2$.

1.2. Table of meandric numbers.

| n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $m_n$ | 1 | 1 | 1 | 2 | 3 | 8 | 14 | 42 | 81 | 262 | 538 |

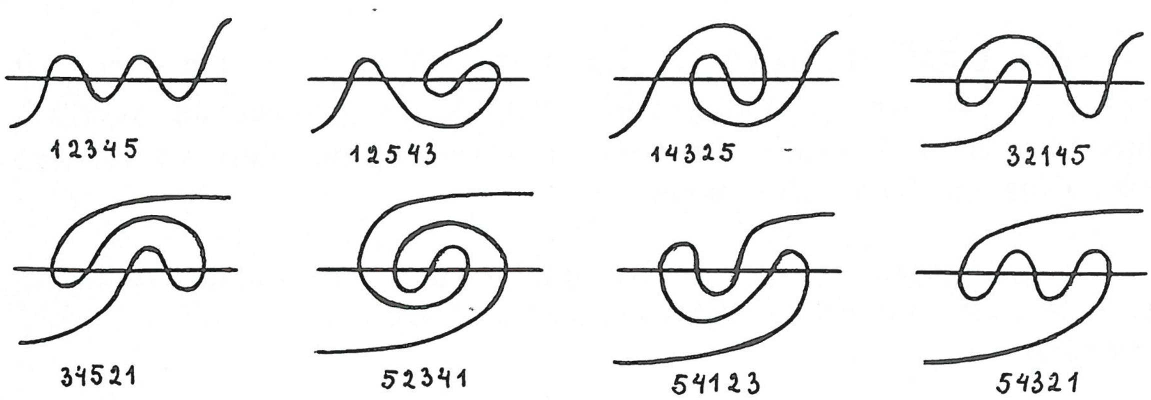
| n | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|
| $m_n$ | 1828 | 3926 | 13820 | 30694 | 110954 | 252939 |

| n | 17 | 18 | 19 | 20 | 21 |
|---|---|---|---|---|---|
| $m_n$ | 933458 | 2172830 | 8152860 | 19304190 | 73424650 |

| n | 22 | 23 | 24 | 25 | 26 | 27 |
|---|---|---|---|---|---|---|
| $m_n$ | ? | 678390116 | ? | 6405031050 | ? | 61606881612 |

The table is based on computational results obtained by the present authors, A.Phillips [Ph], J.Reeds and L.Shepp (ibid.). The algorithm of enumerating closed meanders given in [LaZ] allows with a slight modification to enumerate not closed meanders as well.

The figure below shows all the eight meanders of order 5.

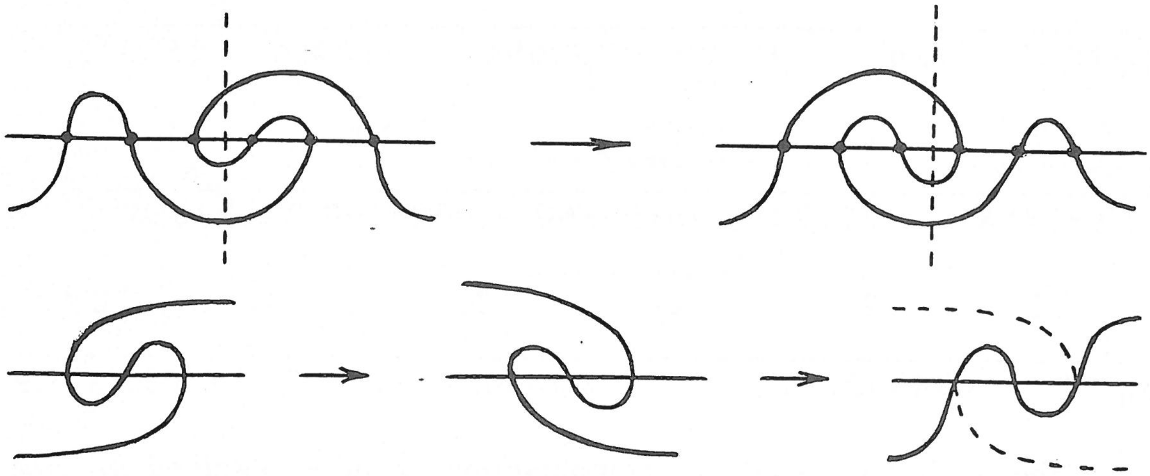


<u>1.3. Statement.</u> The number $m_n$ is odd iff $n = 2^q$, $q = 0, 1, 2, 3$ ...

To prove that we shall need the following

<u>Lemma.</u> $m_{2n} \equiv m_n \pmod 2$, $n = 1, 2, \ldots$

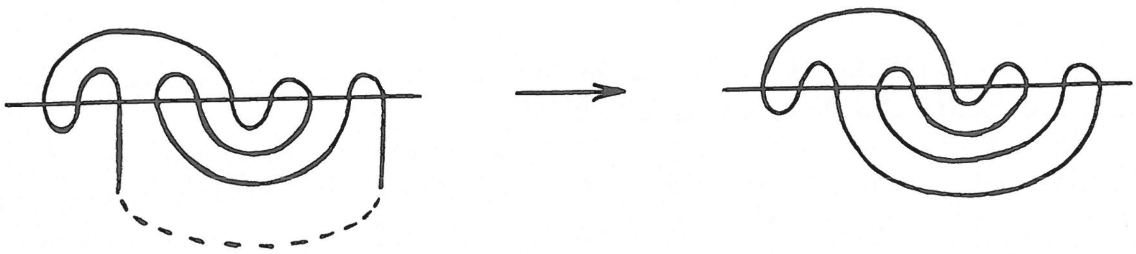
<u>Proof.</u> On a set of meanders of order k define the involution of "reflection" when permutation $(a_1, \ldots, a_n)$ corresponds to permutation $(a'_1, \ldots, a'_n)$, $a'_i = k+1-a_i$, $i = 1, \ldots, k$. Geometrically, this is the reflection with respect to the vertical axis passing through the middle of segment [1,k] (when k is odd it remains to "correct" the directions of the curve ends).



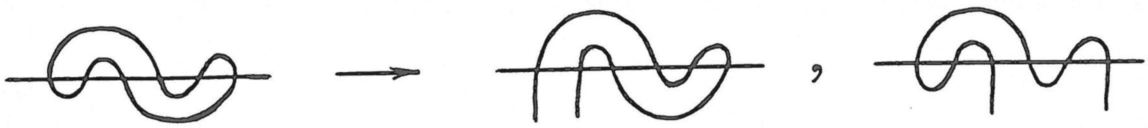
To each symmetric meander of order 2n one can put into correspondence a meander of order n, its left-hand half. As to non symmetric meanders, they are divided into pairs, which proves the lemma.

The <u>proof of statement 1.3</u> now follows from the fact that $m_1=1$; for all the remaining odd orders k the number $m_k$ is even, since the involution of reflection on a set of meanders of an odd order does not have fixed points.

<u>1.4. Closed meanders.</u> By joining the ends of the meandric curve passing through an even number of bridges we obtain a *closed meander*.



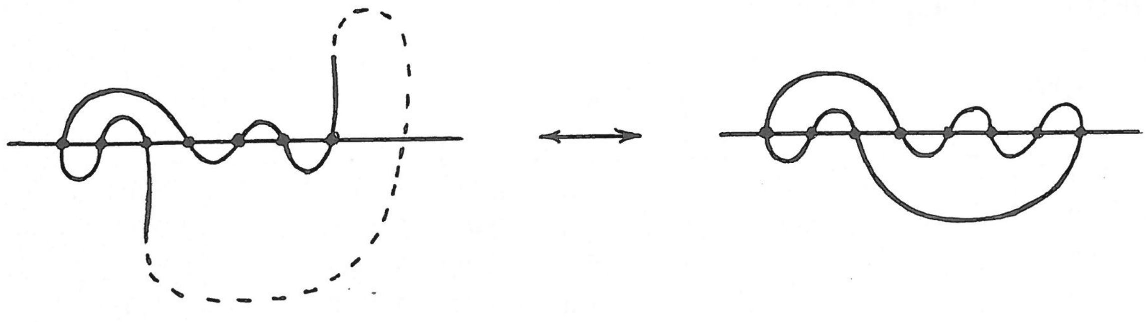
Different meanders passing through 2n bridges may correspond to the same closed meander.



The number of closed meanders passing through 2n bridges will be denoted by $M_n$.

<u>Statement.</u> $M_n = m_{2n-1}$.

Indeed, there is a natural one-to-one correspondence between meanders passing through 2n-1 bridges and closed meanders passing through 2n bridges. It can be seen in the picture below.

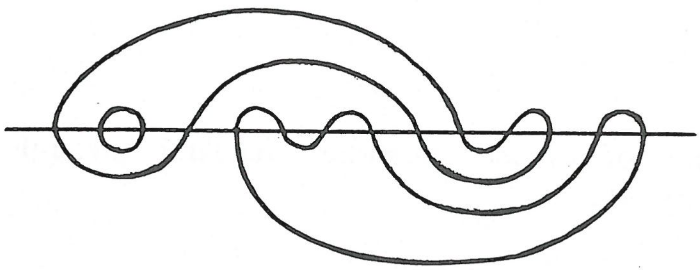


Thus, numbers $M_n$ = 1, 2, 8, 42, ... are actually contained in Table 1.2.

<u>1.5. Statement</u> (v. [LaZ]). If $n = p^q$ where p is a prime, $q > 1$, then $M_n = m_{2n-1} \equiv 2 \pmod{p}$.

1.6. Systems of closed meanders and their distribution according to the number of components.

If we omit the condition of connectedness of the curve in the definition of a closed meander, we will obtain the definition of *a system of meanders*. Note that systems of meanders are in one-to-one correspondence with the pairs of correct parenthesis systems: the set of arcs in the upper half plane corresponds to one parenthesis system, the set of arcs in the lower half plane - to the other parenthesis system. Consequently the number of meander systems of order n is equal to the square of n-th Catalan number. On the figure the meander system of order 7 with 3 components is shown.



In the table below the distribution of meander systems according to the number of components is given (calculations of D.Ivanov). The order of a system is denoted by n, and k denotes number of components.

| k\n | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 8 | 42 | 262 | 1828 | 13820 |
| 2 | | 2 | 12 | 84 | 640 | 5236 | 45164 |
| 3 | | | 5 | 56 | 580 | 5894 | 60312 |
| 4 | | | | 14 | 240 | 3344 | 42840 |
| 5 | | | | | 42 | 990 | 17472 |
| 6 | | | | | | 132 | 4004 |
| 7 | | | | | | | 429 |

1.7. Distribution of meandric numbers according to the number of the first bridge.

Denote by $m_{n,k}$ the number of meanders with n bridges for which the number of the first bridge equals k. It is obvious from the figure in section 1.2 that $m_{5,1} = 3$, $m_{5,3} = 2$, $m_{5,5} = 3$. Below we give the table of values for number $m_{n,k}$ for n = 1, 2, ..., 18.

| k\n | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 2 | 3 | 8 | 14 | 42 | 81 | 262 | 538 |
| 3 | | | 1 | 1 | 2 | 3 | 7 | 14 | 36 | 81 | 221 |
| 5 | | | | | 3 | 3 | 7 | 11 | 28 | 57 | 155 |
| 7 | | | | | | | 14 | 14 | 36 | 57 | 155 |
| 9 | | | | | | | | | 81 | 81 | 221 |
| 11 | | | | | | | | | | | 538 |

| k\n | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|---|---|---|---|---|---|---|---|
| 1 | 1828 | 3926 | 13820 | 30694 | 110954 | 252939 | 933458 |
| 3 | 538 | 1530 | 3926 | 11510 | 30694 | 92114 | 252939 |
| 5 | 353 | 1003 | 2458 | 7214 | 18575 | 55880 | 149183 |
| 7 | 316 | 902 | 2053 | 6059 | 14810 | 44842 | 115009 |
| 9 | 353 | 1003 | 2053 | 6059 | 13827 | 41908 | 102555 |
| 11 | 538 | 1530 | 2458 | 7214 | 14810 | 44842 | 102555 |
| 13 | | 3926 | 3926 | 11510 | 18575 | 55880 | 115009 |
| 15 | | | | 30694 | 30694 | 92114 | 149183 |
| 17 | | | | | | 252939 | 252939 |

<u>Statement.</u> (1) $m_{n+1,1} = m_n$; $m_{2k-1,1} = m_{2k,3}$.

(2) When n is odd, sequence

$m_{n,1}, m_{n,3}, ..., m_{n,n}$

is symmetric, i.e. $m_{n,k} = m_{n,n+1-k}$ when k = 1, ..., n-1.

(3) When n is even, sequence

$m_{n,3}, m_{n,5}, ..., m_{n,n-1}$

is symmetric, i.e. $m_{n,k} = m_{n,n+2-k}$ when k = 3, ..., n-1.

All the above statements are proved by establishing a one-to-one correspondence between the meandric families under consideration. Thus, for instance, in proving statements (2) and (3) the reflection operation defined in section 1.3 is made use of.

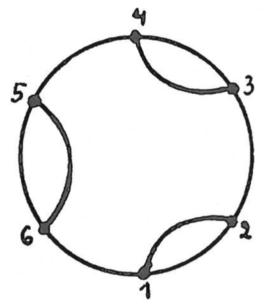
<u>1.8. Conjecture.</u> For any n sequence $m_{n,1}$, ...$m_{n,3}$, ...is unimodal, i.e. there can be found such k that

$m_{n,1} \geq m_{n,3} \geq ... \geq m_{n,k} \leq m_{n,k+2} \leq ...$

## 2. Projective meanders

2.1. Definition. Consider 2n points on a circle that divide it into equal arcs. Enumerate them in succession by numbers 1, 2, ..., 2n. Now divide the points into n pairs so that the chords connecting points in each pair would not intersect. Identify the diametrically opposed points of the circle thus turning the disc into a projective plane. Then the set of chords forms a family of closed non-intersecting curves on a projective plane. We shall call the set of curves a *system of projective meanders* of order n. If the family consists of a single curve, we shall call the latter a *projective meander*. The number of projective meanders of order n will be denoted by $pm_n$.

As is known, the number of ways of drawing n non-intersecting chords and, consequently, the number of systems of projective meanders of order n is equal to the n-th Catalan number $Cat_n = \frac{1}{n+1}\binom{2n}{n}$. In the figure you can see all the five systems of meanders of order 3; the two upper ones are projective meanders.



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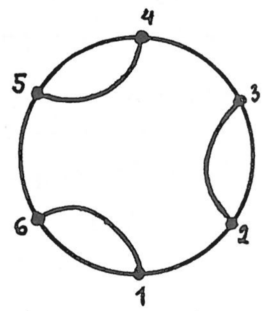

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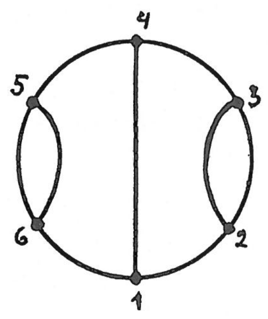

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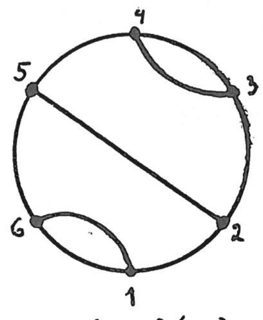
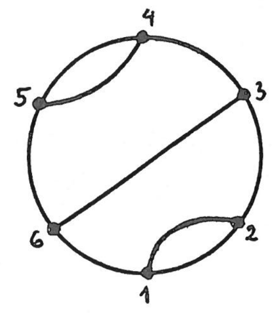


(1634)(25)



(1254)(36)

The number of projective meanders in the system is actually equal to the number of cycles in a permutation on the set of 2n elements, the permutation being defined by means of two involutions without fixed points: one involution is given by a system of chords, the other one, by central symmetry $k \mapsto k+n \pmod{2n}$.

2.2. Table. Below the values of projective meandric numbers $pm_n$ for n = 0, 1, ..., 15 are given (computed by the present authors).

| n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $pm_n$ | 1 | 1 | 2 | 2 | 8 | 12 | 52 | 86 | 400 | 710 | 3404 |

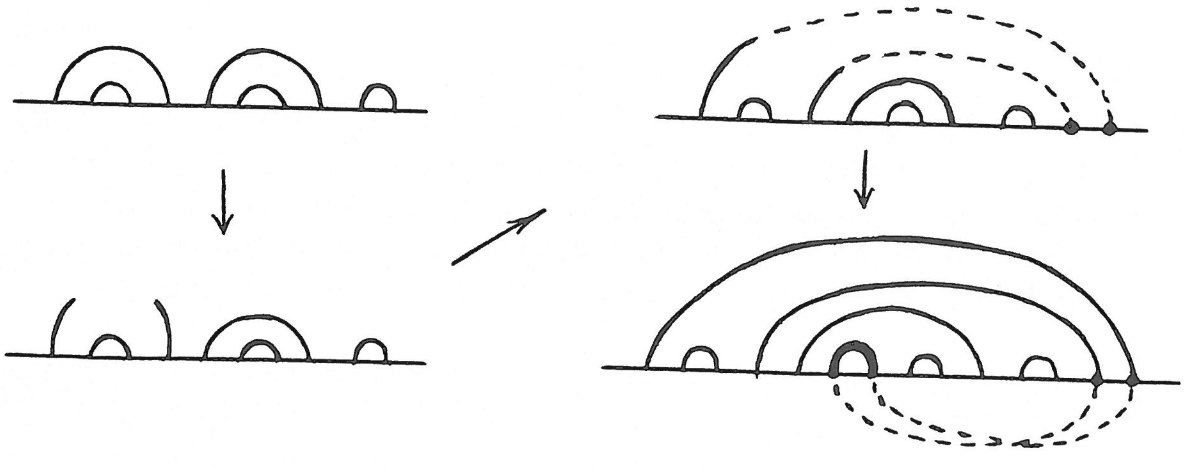
| n | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|
| $pm_n$ | 6316 | 30888 | 59204 | 293192 | 576018 |

The algorithms of polynomial complexity for computing meandric and projective meandric numbers are not yet known.

2.3. Statement. Sequences $pm_0$, $pm_2$, $pm_4$ ... and $pm_1$, $pm_3$, $pm_5$, ... increase monotonously.

The proof is based on the fact that it is possible to make of any projective meander of order n one or more projective meanders of order n+2 by means of an operation of "stretching" to be defined below. The projective meanders of order n+2 thus obtained will be different.

For convenience we shall further draw an infinitely distant straight line on a projective plane as a horizontal one. The operation of stretching consists in the following: (1) cut one of the "upper" arcs; (2) add points 2n+1, 2n+2 and join them to the ends of the cut arc; (3) add two points between points n and n+1 and join them by an arc; re-enumerate the points.



2.4. Action of group $\mathbb{Z}_{2n}$ and its orbits.

Group $\mathbb{Z}_{2n}$ acts on a set of systems of closed meanders of order n and on a set of systems of projective meanders. Its generator is given by the cyclic shift $k \mapsto k+1 \pmod{2n}$. The action preserves the number of components. Investigation of the orbits of the action allows us to receive some congruences for meandric numbers, v. e.g. Statement 1.5.

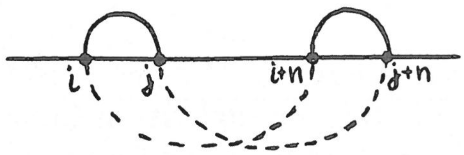
Statement. (1) If $n = p^q$ is a power of an odd prime then $pm_n \equiv 2 \pmod{2p}$.

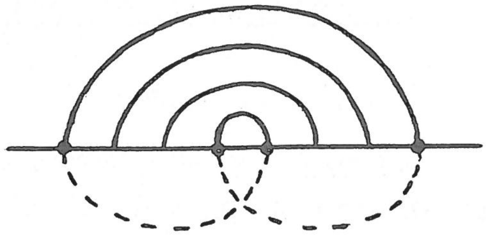
(2) If $n = 2^q$, then $pm_n \equiv 0 \pmod{2n}$.

In order to prove the above statement we shall need the following lemma.

Lemma. Let $n > 2$. Then the order of the orbit of group $\mathbb{Z}_{2n}$ action on the set of projective meanders of order n does not divide n.

Proof. If the orbit order divides n, the system of arcs over the straight line passes into itself under the shift $k \mapsto k+n \pmod{2n}$. If there is at least one arc whose beginning and end are among the first n points, it is isolated into a separate meander together with the arc shifted for n (v. Fig.). If there is no such arc, the farthest outside and inside arcs are isolated into a separate meander (v. Fig.).



Proof of the statement. (1) Orbits orders divide $2n = 2p^q$. Therefore, they are either equal to 2 or are divided by $2p$. There exists the only orbit of order 2 forming a projective meander (v. Fig.).

(2) Order of any orbit divides $2n = 2^{q+1}$ and does not divide $2^q$. Therefore it equals $2^{q+1}$.

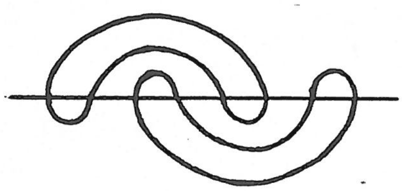
It is well-known that systems of non-intersecting chords are in one-to-one correspondence with rooted plane trees, and their orbits under the action of $\mathbb{Z}_{2n}$ correspond to (non-rooted) plane trees. However, this correspondence does not allow us to determine to which plane trees projective meanders correspond, and to which, only their systems do.

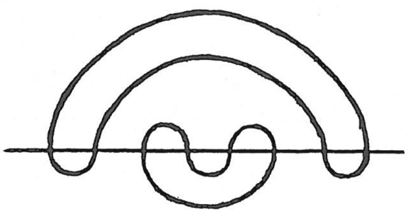
## 3. Estimates of asymptotics

### 3.1. Closed meanders.

We call a system of closed meanders *irreducible* if there is no such subsegment $[a, a+1, \ldots, b] \subset [1, 2, \ldots, 2n]$ that through its points there passes an independent system of meanders. Denote the number of irreducible systems of closed meanders passing through $2n$ points by $N_n$.

Any single meander obviously forms an irreducible system so $M_n \leq N_n$. The left-hand figure shows an irreducible system, the right-hand figure, a reducible one.





In the paper [LaZ] of the present authors the following results have been obtained.

Theorem.

1) Generating function $N(x) = \sum_{n=0}^{\infty} N_n x^n$ for the number of irreducible systems of meanders satisfies functional equation

$$B(x) = N(xB^2(x)), \tag{1}$$

where $B(x)$ is a generating function for square Catalan numbers:

$$B(x) = \sum_{n=0}^{\infty} (Cat_n)^2 x^n$$

2) Function B(x) is expressed by the formula

$$B(t^2) = \frac{1}{4t^2}\left(-1 + \frac{1}{2\pi}\int_0^{2\pi}\sqrt{1-8t\cos\phi + 16t^2}\, d\phi\right)$$

3) Convergence radius of series N(x) equals $\left(\frac{4-\pi}{\pi}\right)^2 = \frac{1}{13.3923...}$.

Empirical estimation of ratio $M_{n+1}/M_n$ obtained by means of Padé approximation yields value 12.26...

Equation (1) allows us to construct a polynomial algorithm for computing numbers $N_n$. We give below a few initial values:

| n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $N_n$ | 1 | 1 | 2 | 8 | 46 | 322 | 2546 | 21870 | 199494 |

If $B$ (x,u) is a generating function for the system of meanders classifying them by the number of irreducible components, then by the methods similar to [LaZ] it is easy to obtain equation

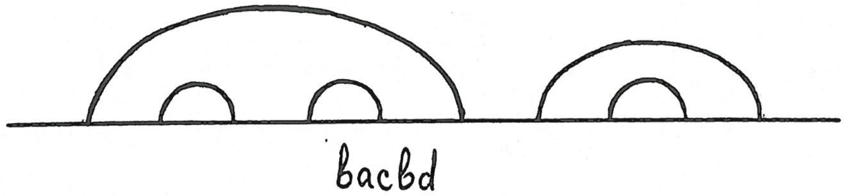
(2) $$B(x,u) = N(xu\, B^2(x,u)).$$

3.2. Projective meanders.

We shall encode a system of projective meanders of order n or, which is the same, a system of n non-intersecting arcs in upper semi-plane by a word of n letters in the alphabet {a, b, c, d} according to the following rule. Each point is either the beginning or the end of an arc. Consider points i and i+n; the i-th letter of the word will correspond to them; this letter is defined by the following rule:

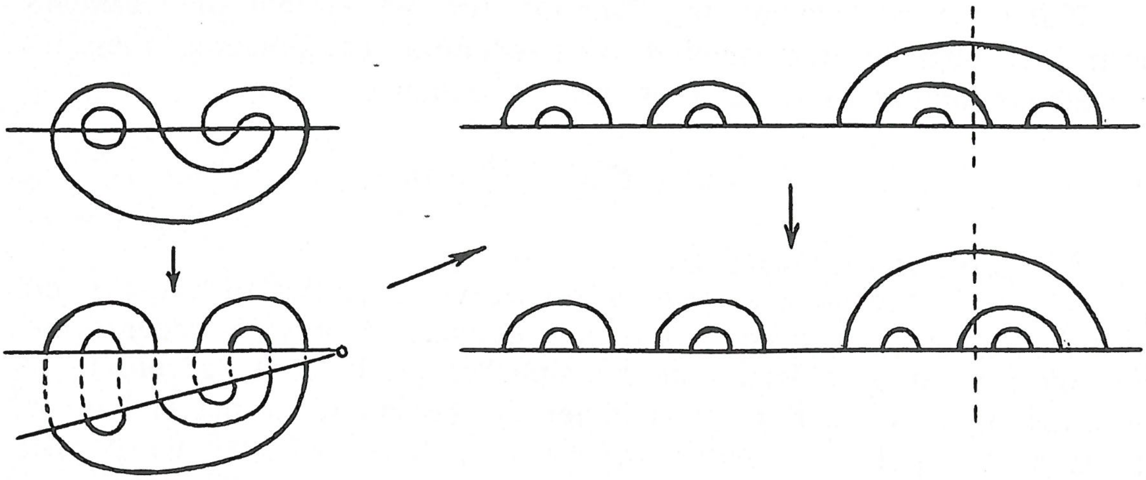
(beginning, beginning) → a
(beginning, end) → b
(end, beginning) → c
(end,end) → d

The example is given in the figure.




It can be seen that the number of all words of length n in the alphabet {a, b, c, d} equals $4^n$ while the number of systems of projective meanders of order n, i.e. the n-th Catalan number is asymptotically equal to $\frac{1}{\sqrt{\pi}} 4^n n^{-3/2}$.

It is possible to put into correspondence to each system of closed meanders of order n a system of projective meanders of order 2n with the same number of components. To achieve that, the following operations should be made with the system of closed meanders: (1) "cut off" the lower system of arcs and transfer it by



the turn of 180° into the upper semiplane; (2) apply reflection operation to the right-hand half of the system of arcs thus obtained. Therefore we receive out of closed meanders of order n projective meanders of order 2n, though not all of them but only those whose system of arcs is divided into two halves: one "inhabits" set of points [1, 2, ..., n], the other, [n, n+1, ..., 2n]. We have proved the following statement.

<u>Statement.</u> $pm_{2n} \geq M_n$.

Note that encoding of systems of closed meanders by words in the alphabet {a, b, c, d} adopted in paper [LaZ] and encoding of systems of projective meanders adopted in the present paper are consistent.

We shall further need the following lemma (v. e.g. [GJ], s. 2.8.8).

<u>Lemma.</u> Let the set A of words in the alphabet of k letters possess the property that any two words $u, v \in A$, $u \neq v$ "do not overlap", i.e. none of them is a subword of another one and there do not exist such three words $\alpha$, $\beta$, $\gamma$ with a non-empty $\beta$ that $u = \alpha\beta$, $v = \beta\gamma$. Let $f_A = \sum a_n x^n$ be a generating function for the words of the set A, i.e. $a_n$ is the number of words of length n belonging to the set A. Then generating function F(x) for the set of all words in the same alphabet not containing a single subword from A equals

$$F_A(x) = (1 - kx + f_A)^{-1}.$$

For instance, generating function for the words in the alphabet {a, b, c, d} not containing a single subword ad equals $(1 - 4x + x^2)^{-1}$.

<u>Theorem.</u> For all n large enough

$$pm_{2n} \leq \left(\frac{1}{R_0} + \varepsilon\right)^{2n},$$

where $\varepsilon > 0$ is arbitrary and $R_0$ is the least positive root of function $1 - 4x + (N(x^2) - 1)$, where N(x) is a generating function for irreducible systems of closed meanders.

<u>Proof.</u> We have to estimate from below the convergence radius of the series $\sum_{n=0}^{\infty} pm_n x^n$. The radius will obviously remain unchanged if we change in an arbitrary way a few initial coefficients of the series.

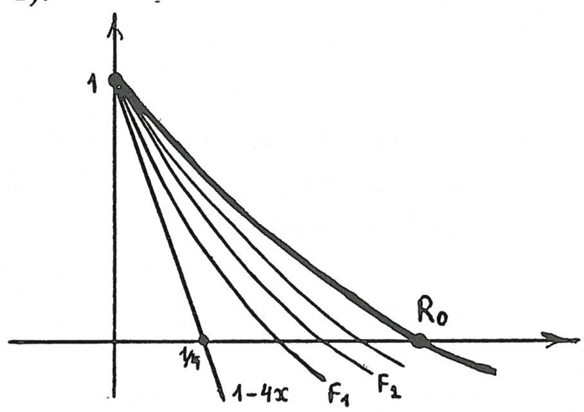
For obvious geometrical reasons the set of words in the alphabet {a, b, c, d} describing irreducible systems of closed meanders satisfy the condition of non-overlapping formulated in the lemma. Let $A_m$ be a set of such words of length not more than 2m, except the empty word. Then a set of all words in the same alphabet, not containing a single subword from $A_m$ and with a length more than 2m contains all the words describing projective meanders (as well as many unnecessary words). Thus, coefficients of the corresponding generating function

$$F_m(x) = (1 - 4x + f_{A_m})^{-1}$$

grow not slower than numbers $pm_n$, the convergence radius of the series $F_m(x)$ being equal to the least positive root $R_m$ of the polynomial

$$1 - 4x + f_{A_m}(x).$$

Increasing m, we increase the number of forbidden subwords and thus improve our estimate leaving asymptotically fewer words of a big length. Polynomials $1 - 4x + f_{A_m}(x)$ converge coefficient-wise to the series $1 - 4x + (N(x^2) - 1)$ ($x^2$ instead of x appears because meanders of order n are described by words of length 2n; unity is subtracted from $N(x^2)$ due to the fact that the empty word does not enter the set of forbidden ones). Since the coefficients of the series N(x) are positive, $R_m \uparrow R_0$, where $R_0$ is the root of function $1 - 4x + (N(x^2) - 1)$.




The theorem is proved.

A specific feature of the method is the more terms of the series N(x) we compute, the more accurate is the estimate of the rate of growth for numbers $pm_n$.

Our calculations give the value $(\frac{1}{R_0})^2 = 13.42...$

## REFERENCES


[Ar] Arnol'd V.I. *Ramified covering* $\mathbb{CP}^2 \to \mathbb{S}^4$, *hyperbolicity and projective topology,* Sib. Math J., 1988, v. 29, N 5 (in Russian)

[Cor] Cori R. *Private communication,* November 1990

[Eli] Eliashberg Ya.M. *Estimates of number of fixed points of area preserving transformations,* VINITI, Syktyvkar, 1979, 104 p.

[GJ] Goulden I.P., Jackson D.M. *Combinatorial Enumeration,* Wiley, 1983

[Koe] Koehler J.E. *Folding a strip of stamps,* J. Comb. Theory, 1968, v. 5, N 2, p. 135-152

[LaZ] Lando S.K., Zvonkin A.K. *Meanders,* to appear in Selecta Mathematica Sovetica

[Ph] Phillips A. *Simple alternating transit mazes,* preprint, June 1988; Addenda & Errata, September 1988

[Poi] Poincaré H. *Sur un théorème de géométrie,* Rendiconti del Circolo matematico di Palermo, 1912, v.33, p.375-407 (Oeuvres, t.VI, p.499-538)

[Ros] Rosenstiel P. *Planar permutations defined by two intersecting Jordan curves,* in "Graph Theory and Combinatorics", Academic Press, London, 1984



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