A NEW PROGRESS OF CHROMATIC AND DICHROMATIC SUM EQUATIONS FOR PLANAR MAPS

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Abstract

In this paper, we firstly introduce the enlarged problems on chromatic and dichromatic sums for general graphs and maps. Then, the Tutte's contribution to the problem for planar triangulations is briefly surveyed. In the main part, the recent progress made by the author for outerplanar, cubic and general maps is explained. Meanwhile, a new result on dichromatic sums of outerplanar maps is also provided. Finally, some problems for further research on this topic are proposed.

1. A few words on graphs

Two kinds of polynomials of graphs or maps as their embeddings on a surface are particularly concerned in this paper. The first one is the chromatic polynomials which were firstly created by G.D. Birckhoff [1] in 1912 for the intention of solving the Four Colour Problem. And, the other is the dichromatic polynomials of graphs or maps, which were firstly discovered by W.T. Tutte as a new development of the first. [8] More interestingly, the later can be applied to finding a topological invarant from which a new progress has be made for the knot problem in topology. [2]

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2. The statements of the problems

The most general problem is as follows. Let $\mathcal{G}$ be a set of graphs, $R(G)$ be a polynomial of graph G. Our purpose is to find

$$\sum_{\substack{G\in\mathcal{G}\\ n_i(G)=n_i\geq 0\\ i\geq 1}} R(G) \tag{2.1}$$

which is called the R-sum of G, for a set of given values $n_1, n_2, \cdots$ of the corresponding members in a fixed set of combinatorially invariants $\{n_1(G), n_2(G), \cdots\}$ of G.

In order to do this, we have to investigate the following function as a whole

$$r_{\mathcal{G}}(x_1, x_2, \cdots; R) = \sum_{G\in\mathcal{G}} R(G) \prod_{i\geq 1} x_i^{n_i(G)} \tag{2.2}$$

which is said to be the R-sum function of the set $\mathcal{G}$ of graphs.

Our problems in what follows are divided into two stages: first, to find the functional equation satisfied by $r_{\mathcal{G}}(x_1, x_2, \cdots; R)$; then, to solve the equation by determining all the terms, or all the coefficients of the terms in the power seies of $r_{\mathcal{G}}(x_1, x_2, \cdots; R)$.

Here, we are only restricted ourselves to discuss the cases when R represents the chromatic polynomial and the dichromatic polynomial of a map. By a map, we shall mean an embedding of a graph on a surface. Naturally, the chromatic and dichromatic polynomials of a map are defined to be the same as the ones of the underlying graph of the map. Further, we also only consider the surface to be the sphere, or equivalently the plane in the whole paper.

Because of the symmetry which has to be explored for each map, we are not allowed now to investigate general maps for this problem directly without consideration of automorphic groups. Moreover, we have known a number of kinds of planar maps including 3 - connected ones in each of which almost all maps are asymmetric.

That means that it is allowed to treat all those kinds of planar maps without symmetry. What could we do for asymmetric maps? One might think of choosing a suitable rooting rule. That is surely right. We may mark an edge by a direction as the root-edge in a map. Meanwhile, we also have to define the vertex which is incident to the starting end of the root-edge to be the root-vertex, and the face which is incident to the root-edge on the right hand side when one moves along the root-edge in the direction to be the root-face. All the maps which have been rooted by the rooting rule are said to be rooted maps. For rooted maps, we may prove that all of them are without symmetry. And for an asymmetric map, we may also show that there are exactly four times its edge number distinct rooted ones. This leads us to observe rooted maps for the problem first.

Even the maps which have been selected as rooted ones, the problem is also by no means accessible at least by now for any choice of the set of combinatorial, or topological for maps, invariants $\{n_1(G),\ n_2(G), \cdots\}$. We now take $\{n_1(G),\ n_2(G), \cdots\} = \{p(G),\ q(G),\ r(G), s(G)\}$ where $p(G), q(G), r(G)$ and $s(G)$ are the number of non-root-vertices, the number of non-root-faces, the valency of the root-face, and the valency of the root-vertex of G, a rooted map respectively. Thus, our restricted problem here can be described precisely as follows.

Let S be a set of rooted planar maps.

$$F_S(p,q,r,s;R) = \sum_{\substack{M\in S\\ p(M)=p,q(M)=q,r(M)=r,s(M)=s}} R(M) \tag{2.3}$$

is said to be the chromatic sum, or dichromatic sum of S for $p(M) = p,\ q(M) = q,\ r(M) = r$, and $s(M) = s$ according as R is chosen to be the chromatic polynomial P, or dichromtic polynomial Q of maps. Of course, p, q, r and s are chosen to be natural numbers.

Our purpose in this paper is to find the functional equation satisfied by the chromatic sum function, or dichromatic sum function

$$f_S(x,y,z,t;R) = \sum_{M\in S} R(M)\ x^{p(M)} y^{q(M)} z^{r(M)} t^{s(M)} \tag{2.4}$$

according as R is the chromatic, or dichromatic polynomial of M, and to solve the equations for a number of kinds of rooted planar maps.

3. Tutte's contributions to the problem

The earliest work of Professor Tutte related to the problem is on dichromatic sums for rooted planar maps. It was published in 1971 [8]. The dichromatic polynomial of a map M, which is denoted by $Q(M;\mu,\nu)$, can be defined as

$$Q(M;\mu,\nu) = \sum_{S \in E} (\mu - 1)^{c_o(M:\, S) - c_o(M)} (\nu - 1)^{c_1(M:\, S)} \tag{3.1}$$

where E is the edge set of the underlying graph of M, $M : S$ is the resultant map of deleting all the edges not in S except for their ends; $c_o(N)$ and $c_1(N)$ represent the number of connected components and the number of inner faces of a map N.

Characterizing a polynomial of two variables to be a dichromatic one is no doubt a very difficult problem. Only a few kinds of maps whose dichromatic polynomials have been known. However, the following two relations are useful for finding the dichromatic polynomial of a map. The first one is that for a map M if e is an edge which is neither an isthmus nor a loop, then we have

$$Q(M;\mu,\nu) = Q(M - e;\mu,\nu) + Q(M \cdot e;\mu,\nu) \tag{3.2}$$

where $M - e$ and $M \cdot e$ are the resultant maps of deleting e except for the two ends and contracting e into a vertex in M. The second one is that for a map M, if $M = M_1 \dot{+} M_2$ that means that $M = M_1 \cup M_2$ provided $M_1 \cap M_2 = \{v\}, v$ is a vertex, then we have

$$Q(M;\mu,\nu) = Q(M_1;\mu,\nu)\, Q(M_2;\mu,\nu). \tag{3.3}$$

Let $\mathcal{U}$ be the set of all general root planar maps, of course, connected. The adjective "general" means that loops, multi-edges, isthmus and cut vertices are allowed in a map of $\mathcal{U}$.

For a function $\phi = \phi(x,y,z,t)$, let us write

$$\hat{\phi} = \phi(x,y,1;1);\ \hat{\phi}^{*} = \phi(x,y,z,1);\ {}^{*}\hat{\phi} = \phi(x,y,1,t). \tag{3.4}$$

And, for a variable $u = x, y, z$, or t, let us write

$$\partial_u \phi = \frac{u\phi_{u=1} - \phi}{1-u};\quad \delta_u \phi = \frac{\phi_{u=1} - \phi}{1-u}. \tag{3.5}$$

Then, we may state the Tutte's result as follows.

Theorem 3.1 *For rooted general planar maps, the dichromatic sum function*

$$f_{\mathcal{U}} = f_{\mathcal{U}}(x,y,z,t;Q) = f_{\mathcal{U}}(x,y,z,t;\mu,\nu) \tag{3.6}$$

satisfies the functional equation as

$$\phi + xzt(\phi\,\hat{\phi}^{*} - \delta_t(t\phi)) + yzt(\phi\,{}^{*}\hat{\phi} - \delta_z(z\phi)) = 1 + \mu x z^2 t + \nu y z t^2 \tag{3.7}$$

where ϕ is the unknown function.

Since 1973, Tutte published a series of papers to discuss the chromatic sums for rooted planar triangulations. As known, triangulations suffice to the Four Colour Problem. The chromatic polynomial of a map, as mentioned above, was created by G. D. Birkhoff. For a map M, let $P(M;\lambda)$ be the chromatic polynomial of M. That is to say that $P(M;\lambda)$ is the number of ways of colouring the vertices of M such that adjacent vertices have different colours by using at most λ different colours. Here, two ways of colouring are said to be different if there is a vertex which has different colours. That means without the effect of the symmetry even for non-rooted maps. The topic on chromatic polynomials is still very active today.

Many absorbing results have been found. Of course, the characterization of chromatic polynomial of a map is difficult as well[1]. There is the unimodal conjecture on chromatic polynomials which is also well known in graph theory. We similarly have two kinds of recursive relations which are useful for finding the chromatic polynomial of a map. The first one is that for any edge e of M, isthmus is allowed here, we always have

$$P(M;\lambda) = P(M-e;\lambda) - P(M\cdot e;\lambda). \tag{3.8}$$

The second one is that if $M = M_1 \cup M_2$ provided $M_1 \cap M_2 = K_l$, the complete graph of order l, for planar maps, only $1 \leq l \leq 4$ are allowed, then we have

$$P(M;\lambda) = \frac{P(M_1;\lambda)\ P(M_2;\lambda)}{\lambda(\lambda-1)\cdots(\lambda-l+1)}. \qquad (3,9)$$

Let $\mathcal{T}$ be the set of all rooted nearly planar triangulations, that means that only the root-face is allowed to be not 3-valent, which are, of course, assumed to be non-separable. Tutte introduced the chromatic sum function as

$$f_{\mathcal{T}} = f_{\mathcal{T}}(x,y,z;\lambda) = \sum_{M\in\mathcal{T}} P(M.\lambda)x^{r(M)}y^{s(M)}z^{q(M)} \tag{3.10}$$

and found the following general result [9].

Theorem 3.2 *The function given by (3.10) satisfies the functional equation as follows:*

$$x\Psi + x^2y^2z\delta_y\Psi = \lambda(\lambda-1)x^3y + \frac{1}{\lambda}yz\hat{\Psi}^* + yz(\Psi - x^2\Psi_2) \tag{3.11}$$

where Ψ is the unknown function, $\hat{\Psi}^* = \Psi_{y=1}(x,y,z)$ and Ψ_2 is the coefficient of the term with x^2 in the power series form of Ψ.

We usually employ the notation that $[x^i]f$ represents the coefficient of x^i, $i \geq 1$, in a function f which is expressed as a power series of x. By this notation, we have $\Psi_2 = [x^2]\Psi$.

Because of the complicatedness of the equation (3.11) to be treated, Tutte could not solve the equation directly at that time.

He investigated a number of specific cases. Especially, for $\lambda = 2, 3$ and $\tau + 1, \tau$ is the golden ratio, he obtained perfect successful results by showing the explicit formulae of the chromatic sum functions with only one variable which is related the number of non-root-faces. Then, he concentrated to observe the equation (3.11) itself [10]. Let $h(z;\lambda) = \Psi_2(1, z; \lambda)$ for λ being a real or a complex and introduce

$$\gamma = \gamma(z;\lambda) = \lambda^{-1}\nu u^2 h(z;\lambda) + \nu^2 u + \frac{3}{2}u^2 \tag{3.12}$$

where $u = z^2$, $\nu = 4(sin^2\frac{\pi}{n})^{-1}$. We may describe the Tutte's results as follows.

Theorem 3.3 *For $\lambda \neq 4$, γ satisfies the differential equation as*

$$(1 - \frac{d^2}{du^2}\Psi)(-\lambda\nu^2 u + 12u^2 + 10\Psi) = -6u\frac{d\Psi}{du}\frac{d^2}{du^2}\Psi \tag{3.13}$$

where Ψ is the unknown function.

Theorem 3.4 *For $\lambda = 4$, h satisfies the differential equation with Ψ as the unknown function as follows:*

$$\frac{d^2}{du^2}(u^2\Psi)(2u + 5u^2\Psi - 3u\frac{d}{du}(u^2\Psi)) = 48u. \tag{3.14}$$

Because there is a 1- to -1 correspondence between nearly planar triangulations with $2i$ non-root-faces when the valency of the root-face is 2 and planar triangulations with $2i$ faces including the root-face, we may see that h is surely the chromatic sum function of rooted planar triangulations with the face number as a parameter.

This quadratic differential equation looks simpler. However, we have not found the explicit solution up to now yet. I have only known that the first fifty terms of the solution have been determined by using computers.

4. Recent progress on the problem

In 1984, the Tutte's theory on the chromatic sums for rooted planar triangulations was further developed to the more general case: for rooted non-separable planar maps [3].

Theorem 4.1 *Let $\mathcal{U}_{nons}$ be the set of all rooted nonseparable planar maps. Then the chromatic sum function $f_{\mathcal{U}_{nons}}$ defined by (2.4) when $S = \mathcal{U}_{nons}$ and $R = P$, the chromatic polynomial, satisfies the following functional equation:*

$$\begin{aligned}(\Psi - \lambda(\lambda-1)xz^2t)(1 - \frac{1}{\lambda}\,\partial_z\hat{\Psi}^*)(1 - \frac{1}{\lambda}\partial_t\,{}^*\,\hat{\Psi}) \\ = yzt\partial_z\Psi(1 - \frac{1}{\lambda}\partial_t\,{}^*\hat{\Psi}) - xzt\partial_t\Psi(1 - \frac{1}{\lambda}\partial_z\hat{\Psi}^*)\end{aligned} \tag{4.1}$$

where Ψ is the unkuown function.

Because $q = 2p - r$ from the Euler formula in the case of planar tringulations, if we introduce the substitution of variables

$$\begin{cases} u = & x^{\frac{1}{2}}\, y; \\ v = & x^{\frac{1}{2}}\, z \end{cases} \tag{4.2}$$

then $f_{\mathcal{U}_{nons}}$ will be transformed into a function of u, v, and t, which is just the chromatic sum function $f_{\mathcal{T}}$ defined by (3.10). Therefore, the equation (3.11) is surely a special case which can directiy be derived from the equation (4.1).

One might think that for rooted cubic planar maps which also suffice to the Four Colour Problem, the functional equation satisfied by the corresponding chromatic sum function can be deduced without much complication. However, it is not the case. Here, we can see that it is surely rather complicated. In this case, we have to introduce the chromatic sum function as

$$f_C = \sum_{M \in C} p(M;\lambda)\; x^{p(M)} y^{r(M)} z^{s(M)} \tag{4.3}$$

where C is the set of all rooted nearly cubic planar maps in each of which only the root-vertex is with the degree which is allowed to be rather than 3. If we treat f_C as a power series of z as

$$f_C = \sum_{i \geq 1} F_i\; z^i \tag{4.4}$$

then F_3 is just the chromatic sum function of the set of all rooted cubic planar maps. Of course, F_i, $i \geq 1$, are all functions of two variables: x and y which are related to the number of non-root-vertices and the valency of the root-face respectively.

Theorem 4.2 *The following functional equation system with* $\alpha = \alpha(x,y)$, $\beta = \beta(x,y)$, $\xi = \xi(x,y)$, $\nu = \nu(x,y)$, *and* $\varsigma = \varsigma(x,y)$ *as unkuown functions has a solution as* $\alpha = F_2$, $\beta = F_3$, $\xi = F_4$, $\eta = F_5$, *and* $\varsigma = F_6$:

$$\alpha\left(1 - \frac{\lambda-1}{\lambda} x^2y^2\hat{\alpha}\right) = (\lambda-2)xy\beta + (\lambda-1)x^2y^2\xi; \tag{4.5}$$

$$\begin{aligned}
&\alpha + xy\beta - (\lambda-1)(\lambda-2)x^2y^2\left(y + \frac{1}{\lambda}\alpha + \frac{xy}{\lambda}\beta\right)\delta_y\alpha \\
=&(\lambda-1)^2x^2y^2\left\{\lambda y + xy(\hat{\beta}+2\beta) + \left[1 + \frac{x\hat{\alpha}}{\lambda(\lambda-1)} + \frac{x}{\lambda}\beta\right.\right. \\
&\left. + \frac{x^2y}{\lambda}\left(\xi + \frac{1}{\lambda}\hat{\alpha}^2\right)\right] + \frac{x}{\lambda}\Big(\alpha + \hat{\alpha} + xy(\hat{\beta}+2\hat{\beta})\Big)\beta \\
&\left. + \frac{2x^2y}{\lambda}\hat{\alpha}\xi + x\eta + x^2y\varsigma\right\};
\end{aligned} \tag{4.6}$$

$$\xi + xy\left(\eta + \frac{1}{\lambda}\hat{\beta}\alpha + \frac{1}{\lambda}\hat{\alpha}\beta\right) = (y + \alpha + xy\beta)\partial_y\alpha; \tag{4.7}$$

$$\begin{aligned}
&\alpha + \frac{\lambda-1}{\lambda}x^2y^2\left(\lambda y + \left(1 + \frac{xy}{\lambda}\hat{\alpha}\right)\alpha + xy\beta\right)\delta_y\alpha \\
=&(\lambda-2)xy\left(\beta + \frac{1}{\lambda}\hat{\alpha}\alpha\right) + (\lambda-1)\left\{\left(2x^2y^2 + \frac{x^2y}{\lambda}\alpha\right)\alpha\right. \\
&+ \frac{x^3y^2}{\lambda}(\alpha - \hat{\alpha})\beta + x^2y(1+y)\xi + \frac{x^2y^2}{\lambda}\hat{\alpha}\alpha \\
&+ x^3y^2\left(\eta + \frac{3}{\lambda}\hat{\beta}\alpha\right) + \frac{x^3y^2}{\lambda(\lambda-1)}\Big((\lambda-2)\alpha + (\lambda-1)xy\beta\Big) \\
&\left.\left[(\lambda-2)(\hat{\beta}+\beta) + (\lambda-1)x\left(\hat{\xi} + y\xi + \frac{1}{\lambda}\hat{\alpha}(\hat{\alpha} - y\alpha)\right)\right]\right]\right\};
\end{aligned} \tag{4.8}$$

$$
\begin{aligned}
&\beta + xy^2\left(1+\frac{\lambda-1}{\lambda}x^2y(\xi+\frac{1}{\lambda}\hat{\alpha}\alpha)\right)\delta_y\alpha \\
=&(\lambda-2)xy(\xi+\frac{1}{\lambda}\hat{\alpha}\alpha)+(\lambda-1)xy\left\{\lambda y+y\hat{\xi}\right. \\
&+xy\hat{\beta}+x(1+y)\eta+\frac{x(1+2y)}{\lambda}\hat{\alpha}\beta+\frac{x(1+y)}{\lambda}\hat{\beta}\alpha \\
&+x^2y(\varsigma+\frac{3}{\lambda}\hat{\varsigma}\alpha+\frac{2}{\lambda}\hat{\beta}\beta)+\frac{x^2y}{\lambda(\lambda-1)}\Big((\lambda-2)\beta \\
&+(\lambda-1)xy(\xi+\frac{1}{\lambda}\hat{\alpha}\alpha)\Big)\Big[(\lambda-2)(\hat{\beta}+\beta)+(\lambda-1)x \\
&\left.\left(\hat{\xi}+y\xi+\frac{1}{\lambda}\hat{\alpha}(\hat{\alpha}+y\alpha)\right)\Big]\right\}
\end{aligned}
\tag{4.9}
$$

where $\hat{\alpha}=\alpha(x,1)$, $\hat{\beta}=\beta(x,1), \xi=\xi(x,1)$, $\hat{\eta}=\eta(x,1)$ *and* $\hat{\varsigma}=\varsigma(x,1)$.

In fact, for finding the equation system (4.5-9), the first difficulty we encountered is that we have to find a new recursive relation on chromatic polynomials, which is closed in the set $\mathcal{C}$ considered here. Before explaining the formula, we have to introduce five kinds of operations which are closed in $\mathcal{C}$, denoted by $\Pi_{(+,+)}$, $\Pi_{(\cdot,+)}$, $\Pi_{(+,\cdot)}$, $\Pi_{(\cdot,\cdot)}$ and $\Pi_{(\uparrow,\uparrow)}$, as shown in the following manner.

For a map $M \in \mathcal{C}$, let $A = <\theta,\ \alpha>$ be the root-edge of M; θ be the root-vertex; β, γ be the two vertices adjacent to α; and $<\alpha, \gamma>$ be on the boundary of the outer face, the root-face, with the same direction as A. And, let $\hat{\dot{M}} = M - \alpha$ be the resultant map of deleting α with all its incident edges but remaining all ends rather than α from M. Now, we define the operations $\Pi_{(\kappa,\eta)}$ on $\hat{\dot{M}}$ in accordance with κ and η, where $\kappa =$ "$+$", or "$\cdot$" denotes that θ and β are joined by a new edge, or identified with one another, and $\eta =$ "$+$", or "$\cdot$" denotes the same as for κ except that γ replaces β here. Moreover, $\Pi_{(\uparrow,\uparrow)}\hat{\dot{M}}$ stands for the resultant map of joining an edge (β, γ) on $\hat{\dot{M}}$. Thus, the recursive formula for the chromatic polynomial on $M \in \mathcal{C}$ can be written as

$$
\begin{aligned}
P(M;\lambda) = &(\lambda-1)[P(\Pi_{(\cdot,+)}\hat{\dot{M}};\lambda)+P(\Pi_{(+,\cdot)}\hat{\dot{M}};\lambda) \\
&+P(\Pi_{(\cdot,\cdot)}\hat{\dot{M}};\lambda)]+(\lambda-2)P(\Pi_{(+,+)}\hat{\dot{M}};\lambda) \\
&-P(\Pi_{(\uparrow,\uparrow)}\hat{\dot{M}};\lambda).
\end{aligned}
\tag{4.10}
$$

Now, the main difficulty is how to decompose $\mathcal{C}$ such that (4.10) can be applied for finding the equations. Here, we need severeal dozens of lemmas which were shown in [4,5].

We may see that all the chromatic sum equations mentioned above have not been solved to find an explicit expression of the corresponding chromatic sum function for each case. However, very recently, we have found the chromatic sum equations for rooted outerplanar maps, non-separable of course, by which the explicit expressions of several chromatic sum functions have been determined[6].

Let O be the set of all rooted non-separeble outerplanar simple maps. And for convenience, let the chromatic sum function for O be as

$$f_O = \sum_{M \in O} P(M;\lambda) x^{m(M)} y^{r(M)} z^{s(M)} \tag{4.11}$$

where $m(M)$ is the number of edges in M.

Theorem 4.3 *The following functional equation with* $\Psi = \Psi(x,y,z;\lambda)$ *as the unknown function and* $\hat{\Psi}^* = (x,y,1;\lambda)$,

$$\begin{aligned}&\left[1 - xz\left(\frac{\hat{\Psi}^*}{\lambda y - \hat{\Psi}^*} - \frac{\hat{\Psi}^*}{\lambda(\lambda-1)y} + \frac{y}{1-z}\right)\right]\Psi \\ =&\lambda(\lambda-1)xy^2 z - \frac{xyz^2}{1-z}\hat{\Psi}^*\end{aligned} \tag{4.12}$$

has a solution which is the chromatic sum function defined by (4.11).

Although Eq (4.12) has a linear form, there is another unknown function $\hat{\Psi}^*$, which is related to Ψ, involved in it. However, we may imagine that $z = \xi(x,y) = \xi$ is a function of x and y. If the following two equations

$$\begin{cases} 1 - x\xi\left[\frac{\hat{\Psi}^*}{\lambda y - \hat{\Psi}^*} - \frac{\hat{\Psi}^*}{\lambda(\lambda-1)y} + \frac{y}{1-\xi}\right] = 0; \\ \lambda(\lambda-1)xy^2\xi - \frac{xy\xi^2}{1-\xi}\hat{\Psi}^* = 0 \end{cases} \tag{4.13}$$

can be solved simultaneously to find the expressions of x and $\hat{\Psi}^*$ with the parameter ξ and the variable y, then we may find $\hat{\Psi}^*$, which is a function of x and y, from the expressions by employing the Lagrangian inversion. This is just the case. In consequence, we obtain the following theorem.

Theorem 4.4 *The chromatic sum function* $\hat{f}_O^*$ *and* $h_O = \hat{f}_O = f_O(x,1,1;\lambda)$ *for rooted non -separable outerplanar maps have the following explicit forms:*

$$
\begin{aligned}
f_O^* = & \lambda(\lambda-1)y\Big\{xy + \sum_{m\geq 2}\sum_{s=\lceil\frac{m+1}{2}\rceil}\sum_{j=0}^{2s-m-1} \\
& \times (-1)^j \frac{(m-1)!(s-2)!}{s!(m-s)!j!(2s-m-j-1)!(s-j-2)!} \\
& \times A_{m-s}^{m-s}(2s-m-j-1;\lambda)y^n x^m\Big\};
\end{aligned}
\tag{4.14}
$$

$$
\begin{aligned}
h_O = & \lambda(\lambda-1)\Big(x + \sum_{m\geq 2}\sum_{s=\lceil\frac{m+1}{2}\rceil}\sum_{j=0}^{2s-m-1} \\
& \times (-1)^j \frac{\lambda(\lambda-1)(m-1)!(s-2)!}{s!(m-s)!(s-j-2)!(2s-m-j-1)!} \\
& \times A_{m-s}^{m-s}(2s-m-j-1;\lambda)x^m\Big)
\end{aligned}
\tag{4.15}
$$

where

$$
\begin{aligned}
A_q^p(r;\lambda) = & \sum_{j=0}^{r} \lambda^{r-j}(\lambda-1)^{p-q}(\lambda-2)^{p-j} \\
& \times \frac{p!(q+r-j-1)!r!}{(p-j)!(q-1)!j!(r-j)!}.
\end{aligned}
\tag{4.16}
$$

On the dichromatic sum equations, we obtain the result for rooted non-separable planar maps which is independent of the one for rooted general planar maps as shown by Eq (3.7).

Theorem 4.5 *The functional equation with ϕ as the unknown function*

$$
\begin{aligned}
& (\phi - \mu x z^2 t - \nu y z t^2)(1-\partial_z \hat{\phi}^{*})(1-\partial_t {}^{*}\hat{\phi}) \\
& = yzt\partial_z\phi(1-\partial_t {}^{*}\hat{\phi}) + xzt\partial_t\phi(1-\partial_z\hat{\phi}^{*})
\end{aligned}
\tag{4.17}
$$

has a solution which is the chromatic sum function for rooted nonseparable planar maps $f_{\mathcal{U}_{nons}}(x,y,z,t;Q)$ defined by (2.4) when $\mathcal{S} = \mathcal{U}_{nons}$ and $R = Q$, the dichromatiac polynomial [7].

Let $\mathcal{O}_n$ be the set of all nonseparable outerplanar maps. The link map and the loop map are defined in $\mathcal{O}_n$ excluding the vertex map. And, let

$$
f_{\mathcal{O}_n} = f_{\mathcal{O}_n}(x,y,z;Q) = \sum_{M\in\mathcal{O}_n} Q(M;\mu,\nu)x^{m(M)}y^{r(M)}z^{s(M)}
\tag{4.18}
$$

be the dichromatic sum function for O_n. Then, by decomposing the set O_n such that (3.2) and (3.3) can be employed, we may find

Theorem 4.6 *The functional equation of* $\psi = \psi(x, y, z; \mu, \nu)$ *with* $\psi^* = \psi(x, y, 1; \mu, \nu)$ *as unknown as*

$$\psi = xyz(\nu z\mu y + \frac{\psi - \nu xyz^2}{y + \nu xy - \psi^*} + \frac{\partial_z\psi}{1 - \nu xz}) \tag{4.19}$$

is well defined on the domain of all the power series with polynomials of μ *and* ν *as coefficients of the monoids of undeterminates* x, y *and* z. *And, the solution is* $\psi = f_{O_n}$.

The form of Eg.(4.19) suggests that the characteristic method as described in (4.13) for the Lagrangian inversion can be applied to finding an explicit expression of $f_{O_n}(x, 1, 1, Q)$ and then $f_{O_n}(x, y, 1; Q)$. The detail will be seen in the forthcoming paper.

5. Discussions

Here, we have to discuss a few points on the topic.

1. The crucial step here for determining the functional equation satisfied by the chromatic, or dichromatic sum function of a set of maps, especially planar maps given is to find a suitable way to decompose the set of maps into parts such that each part can be generated from the given set of maps.

However, there is no universal way to do so. This is one of the main difficulties. Generally speaking, the equation obtained is usually dependent on not only the unknown function itself but also a partial function of the unknown. If the equation can be transformed into an equation which has the only one unknown, the partial function. Then, we are allowed to apply the methods in equation theory. In this case, by no means the equation can always be solved. Sometime, it is still very difficult to solve. Eqs. (3.13) and (3.14) are such examples.

2. The general clue for solving the equations appearing here is to find parametric expressions of the partial function of the unknown and the variable involved in the partial function in order to employ the Lagrangian inversion.

However, by no means we can always find the parametric expressions. Moreover, even the parametric expressions available for the inversion, the calculation is usually very complicated. Therefore, we have to find suitable expressions by experience for the inversion simpler enough if it does exist.

This is another main difficuty we meet here for solving the equations such as Eqs. (4.1), (4.5-9), and (4.17).

3. For the case of rooted non-separable outerplanar maps, the chromatic sum equation is solved by following the clue described in **2** from showing the chromatic sum functions in explicit expressions as power series as (4.14-15)

Of course, there is still a problem of detemining all the coefficents in (4.14) and (4.15) by summation free forms, or the summation forms with all the terms positive if possible.

4. For planar triangulations, the dichromatic sum equation can probably be found without much difficulty.

However, it can be imagined that the equation is also difficult to solve. Similarly, for rooted outerplanr maps, we may find the functional equation satisfied by the dichromatic sum function and solve it in the way of showing the explicit expression in the power series form.

It seems that the main thing we have to do is to find a new kind of recursive formula for the dichromatic polynomial in the 2-separable case.

5. All the equations we have found above can be shown to be well defined in the domain which consists of all the power series with non-negative integer coefficients by constructing a recursive relation determined by the first few terms.

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