COMBINATORIAL PROOFS OF HOOK GENERATING FUNCTIONS FOR SKEW PLANE PARTITIONS

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Abstract

We provide combinatorial proofs of two hook generating functions for skew plane partions. One proof involves the Hillman-Grassl algorithm [H-G 76] and the other uses a modification of Schützenberger's jeu de taquin [Scü 63, Scü 76] due to Kadell [Kad pr]. We also provide a bijection showing directly that these two generating functions are equal. Analogous results for skew shifted plane partitions are proved. Finally some open questions are discussed.

1 Preliminaries

Stanley [Stn 71] was the first to derive the hook generating function for reverse plane partitions and a combinatorial proof of this result was given by Hillman and Grassl [H-G 76]. In an earlier paper [Sag 82] we showed how their algorithm could be generalized to give bijective proofs of other generating functions for partially ordered sets with hooklengths. It turns out that there are two hook generating functions for skew plane partitions, also first demonstrated algebraically by Stanley [Stn ms]. We will show that one can be proved using Hillman-Grassl and the other by a modified version of the Schützenberger jeu de taquin [Scü 63, Scü 76] created by Kadell [Kad pr]. We also give a bijection which shows directly that these two product generating functions are equal. These proofs will be found in Section 2.

Similarly, shifted reversed plane partitions are enumerated by a hook generating function, as was first proved by Gansner [Gan 78]. We show that shifted plane partitions also have a pair of generating functions and use analogous techniques to derive the associated bijections. See Section 3. The shifted results as well as their proofs are new. The last section contains some open questions.

Many of these proofs have been discovered independently by Kevin Kadell [private communication]. We appreciate his permission to include them here. First, however, we must give some definitions and notation.

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Consider the plane

$$\Lambda = \{(i,j) \mid i,j \geq 1\}$$

viewed as an infinite array of boxes or cells arranged matrix-style in left-justified rows. Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ be a fixed partition considered as a Ferrers diagram sitting in the upper-left corner of Λ. This gives rise to the skew shape

$$\Lambda/\lambda = \{(i,j) \mid (i,j) \in \Lambda, (i,j) \notin \lambda\}$$

A *skew plane partition of shape* Λ/λ is a filling, P, of Λ/λ with non-negative integers called *parts* such that rows and columns weakly decrease. For example, if $\lambda = (3,1)$ then one such skew plane partition (0 parts omitted) is

$$P = \begin{array}{ccccc} \blacksquare & \blacksquare & \blacksquare & 4 & 4 \\ \blacksquare & 3 & 3 & 2 & \\ 4 & 3 & 3 & 1 & \end{array}$$

If $P_{i,j}$ denotes the part of P in cell (i,j), then we say that P is a *skew plane partition of* n if $\sum_{(i,j)\in\Lambda/\lambda} P_{i,j} = n$. Our example is a skew plane partition of $4+4+3+3+2+4+3+3+1 = 27$. Let

$$pp_{\Lambda/\lambda}(n) = \text{ number of plane partitions of } n \text{ having shape } \Lambda/\lambda.$$

We will be interested product forms for the generating function of $pp_{\Lambda/\lambda}(n)$. For this, we need to define two types of hooks.

If $(i,j) \in \lambda$ then this cell has the usual hook of all cells directly to the right or directly below,

$$H_{i,j} = \{(i,j') \in \lambda \mid j' \geq j\} \cup \{(i',j) \in \lambda \mid i' \geq i\}$$

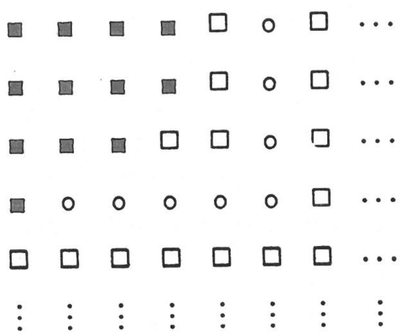
If, instead, $(i,j) \in \Lambda/\lambda$ then we take the reflection of a normal hook in an anti-diagonal $i + j$ =constant , i.e., using those cells to the left or above (i,j),

$$H_{i,j} = \{(i,j') \in \Lambda/\lambda \mid j' \leq j\} \cup \{(i',j) \in \Lambda/\lambda \mid i' \leq i\}$$

In either case, the *hooklength* of cell (i,j) is $h_{i,j} = |H_{i,j}|$, where $|\cdot|$ denotes cardinality. For example, if $\lambda = (4,4,3,1)$ then the cells in the hooks of $(2,2) \in \lambda$ are shown as circles in

$$\begin{array}{cccc} \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \bullet & \bullet & \bullet \\ \blacksquare & \bullet & \blacksquare & \\ \blacksquare & & & \end{array}$$

while those of $(4,6) \notin \lambda$ are the circles in



Thus $h_{2,2} = 4$ and $h_{4,6} = 8$.

2 Plane partitions

We will give combinatorial proofs of the two product formulae for the generating function for skew plane partitions. We will also show by a direct bijection that the two products are equal.

Theorem 2.1 *If λ is a fixed shape, then*

$$\sum_{n\geq 0} pp_{\Lambda/\lambda}(n)x^n = \prod_{(i,j)\in\Lambda/\lambda} \frac{1}{1-x^{h_{i,j}}} \tag{1}$$

$$= \prod_{k\geq 1} \frac{1}{(1-x^k)^k} \prod_{(i,j)\in\lambda} \frac{1}{1-x^{h_{i,j}}} \tag{2}$$

Proof of (1). We merely use a reflection of the normal Hillman-Grassl algorithm in an anti-diagonal. (This corresponds to the fact that the associated algebraic proof derives (1) as a limiting case of the ordinary hook generating function for reverse plane partitions.) Since details of this approach have already appeared in [Sag 82] for the case $\lambda = \emptyset$, and the general case is virtually the same, we will only sketch the proof here for completeness.

It suffices to find a bijection

$$P \longleftrightarrow \kappa = (h_{i_1,j_1}, h_{i_2,j_2}, \ldots)$$

where P is a plane partition of shape Λ/λ and κ is a partition all of whose parts are hooklengths of Λ/λ such that

$$\sum_{(i,j)\in P} P_{i,j} = \sum_k h_{i_k,j_k}$$

We will define a path p in P and then subtract one from every part on the path. The definition of p is as follows.

HG1 Start p at (a,b), the rightmost highest cell of P containing a nonzero entry.

HG2 Continue by iterating

$$(i,j) \in p \Longrightarrow \begin{cases} (i+1,j) \in p & \text{if } P_{i+1,j} = P_{i,j} \\ (i,j-1) \in p & \text{otherwise} \end{cases}$$

In other words, move left unless forced to move down in order not to violate the weakly decreasing condition along the rows (once the ones are subtracted).

HG3 Terminate p when the preceding induction rule fails. At this point we must be at the left end of some row, say row r.

It is easy to see that after subtracting one from the elements in p, the array remains a plane partition and the amount subtracted is $h_{r,b}$.

For example, the following diagram shows an array P with the cells of the path p enclosed in boxes, as well as the resulting plane partition P' after subtraction.

$$P = \begin{array}{cccccc} \blacksquare & \blacksquare & \blacksquare & \blacksquare & 3 & \boxed{3} \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare & \boxed{3} & \boxed{3} \\ \blacksquare & \blacksquare & \blacksquare & \boxed{4} & \boxed{3} & 2 \\ \blacksquare & \boxed{5} & \boxed{4} & \boxed{4} & 2 & \\ 5 & 4 & 3 & 3 & 2 & \end{array} \longrightarrow P' = \begin{array}{cccccc} \blacksquare & \blacksquare & \blacksquare & \blacksquare & 3 & 2 \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare & 2 & 2 \\ \blacksquare & \blacksquare & \blacksquare & 3 & 2 & 2 \\ \blacksquare & 4 & 3 & 3 & 2 & \\ 5 & 4 & 3 & 3 & 2 & \end{array}$$

In this case $(a,b) = (1,6)$ and $r = 4$ so the number of ones subtracted is $h_{4,6} = 8$. Make $h_{r,b}$ the first part of κ and continue the process by finding a path in P', subtracting ones to find the second part of κ, etc. The algorithm terminates when every entry of P has been zeroed out.

To reverse the process, given a partition of hooklengths, we must rebuild the plane partition. First, however, we must know in what order the hooklengths were removed. The following lemma, whose proof is omitted, answers that question

Lemma 2.2 *In the decomposition of P into hooklengths, $h_{i,j}$ was removed before $h_{i',j'}$ if and only if*

$$j > j', \text{ or } j = j' \text{ and } i \le i'. \blacksquare$$

Now arrange the hooklengths in κ according to the total order given in the lemma and start adding them back, starting with the last hooklength and the plane partition of all zeros. In general to add $h_{r,b}$ to P, we construct a reverse path q along which to add ones.

GH1 Start q at the leftmost cell in row r

GH2 Continue by

$$(i,j) \in q \Longrightarrow \begin{cases} (i-1,j) \in q & \text{if } P_{i-1,j} = P_{i,j} \\ (i,j+1) \in q & \text{otherwise} \end{cases}$$

GH3 Terminate q when it passes through the highest cell of Λ/λ in column b.

This is a step-by-step inverse of the construction of the path p, as can be verified in the previous example. Thus to finish the proof it suffices to show that r is well defined—i.e., that it must pass through the highest cell in column b. We leave this verification to the reader.

Proof of (2). First we must describe the modified version of jeu de taquin that we will need. Pick any cell $c = (i,j) \in \lambda$ which is at the end of its row and column. If P is a plane partition of shape Λ/λ, then we can perform a *backward jeu de taquin slide* into cell c using the following algorithm.

B1 **While** $P_{i,j+1} \neq 0$ **or** $P_{i+1,j} \neq 0$ **do**

B2 **if** $P_{i,j+1} \geq P_{i+1,j}$ **then** slide $P_{i,j+1}$ into cell c

else slide $P_{i+1,j} - 1$ into cell c. **fi**

B3 Let $c :=$ the cell of the element that slid in step B2. **od**

Of course, the coordinates (i,j) of c also get changed by the assignment statement in step B3. Note, also, that 1 is subtracted from every element that moves up during the slide. If the result of a slide on P into c is P' and the total amount subtracted is d, then we will write $P' = j^c(P)$ and $d = d^c(P)$. It is easy to verify that P' is still a plane partition.

To illustrate, we have boxed the elements on the path of a slide into $c = (2,2)$ on the following partition and displayed the result after the slide is complete.

$$P = \begin{array}{ccc} \blacksquare & \blacksquare & 4 \\ \blacksquare & \boxed{\blacksquare} & 3 \\ 4 & \boxed{4} & \boxed{3} \\ 3 & 3 & \boxed{2} \\ 2 & 2 & \boxed{1} \\ 1 & 1 & \end{array} \longrightarrow P' = j^{(2,2)}(P) = \begin{array}{ccc} \blacksquare & \blacksquare & 4 \\ \blacksquare & 3 & 3 \\ 4 & 3 & 1 \\ 3 & 3 & \\ 2 & 2 & \\ 1 & 1 & \end{array}$$

In this case $d^c(P) = 3$.

Now to the proof of (2). By theorems of MacMahon [Mac 15] and Stanley [Stn 71] the two products on the right side of the equality count normal plane partitions (those where $\lambda = \emptyset$) and reverse plane partitions of shape λ (arrays obtained by replacing the boxes of λ by non-negative integers such that rows and columns weakly increase), respectively. Thus it suffices to find a bijection

$$P \longleftrightarrow (Q, R)$$

where P is a plane partition of shape Λ/λ, Q is a normal plane partition and R is a reverse plane partition of shape λ, such that

$$\sum_{(i,j)\in\Lambda/\lambda} P_{i,j} = \sum_{(i,j)\in\Lambda} Q_{i,j} + \sum_{(i,j)\in\lambda} R_{i,j}$$

First we discuss the map $P \longrightarrow (Q,R)$. The basic idea is that we will use slides on P to obtain the normal array Q while R keeps track of the amount subtracted at each stage. Specifically, let $c_1,\ldots,c_n$ be the cells of $\lambda = (\lambda_1,\ldots,\lambda_l)$ listed in the order

$$(l,\lambda_l),(l,\lambda_l-1),\ldots,(l,1),(l-1,\lambda_{l-1}),\ldots,(1,1) \tag{3}$$

–i.e., list each row from right to left, starting with the lowest row and working up. Define

$$Q = j^{c_n}(\cdots(j^{c_1}(P)))$$

Further, let p_k be the path corresponding to the slide into cell c_k. Finally, after performing j^c on some intermediate partition P' where $c=(i,j)$, then we let

$$R_{i,\lambda_i-j+1} = d^c(P')$$

–i.e., we fill R by rows from *left to right* starting with the lowest row and working up. Using the previous example for our initial P, we make the following computation.

$$Q: \begin{array}{ccc} \blacksquare & \blacksquare & 4 \\ \blacksquare & \blacksquare & 3 \\ 4 & 4 & 3 \\ 3 & 3 & 2 \\ 2 & 2 & 1 \\ 1 & 1 & \end{array}, \quad \begin{array}{ccc} \blacksquare & \blacksquare & 4 \\ \blacksquare & 3 & 3 \\ 4 & 3 & 1 \\ 3 & 3 & \\ 2 & 2 & \\ 1 & 1 & \end{array}, \quad \begin{array}{ccc} \blacksquare & \blacksquare & 4 \\ 3 & 3 & 3 \\ 3 & 2 & 1 \\ 3 & 1 & \\ 2 & & \\ 1 & & \end{array}, \quad \begin{array}{ccc} \blacksquare & 4 & 2 \\ 3 & 3 & \\ 3 & 2 & \\ 3 & 1 & \\ 2 & & \\ 1 & & \end{array}, \quad \begin{array}{ccc} 4 & 2 & 2 \\ 3 & 1 & \\ 3 & & \\ 3 & & \\ 2 & & \\ 1 & & \end{array}$$

$$R: \begin{array}{cc} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{array}, \quad \begin{array}{cc} \blacksquare & \blacksquare \\ 3 & \blacksquare \end{array}, \quad \begin{array}{cc} \blacksquare & \blacksquare \\ 3 & 4 \end{array}, \quad \begin{array}{cc} 2 & \blacksquare \\ 3 & 4 \end{array}, \quad \begin{array}{cc} 2 & 3 \\ 3 & 4 \end{array}.$$

Thus

$$\begin{array}{ccc} \blacksquare & \blacksquare & 4 \\ \blacksquare & \blacksquare & 3 \\ 4 & 4 & 3 \\ 3 & 3 & 2 \\ 2 & 2 & 1 \\ 1 & 1 & \end{array} \longrightarrow \left(\begin{array}{ccc} 4 & 2 & 2 \\ 3 & 1 & \\ 3 & & \\ 3 & & \\ 2 & & \\ 1 & & \end{array}, \quad \begin{array}{cc} 2 & 3 \\ 3 & 4 \end{array} \right)$$

We must show that this map is well-defined. It is easy to see that Q is a normal plane partition and that R has the right shape. We need to verify that the rows and columns of R are weakly increasing This will follow from Lemmas 2.3 and 2.4, respectively.

Lemma 2.3 *Let $p = p_k$ and $p' = p_{k+1}$ be the paths corresponding to backward slides into adjacent cells c_k and c_{k+1} in the same row. If (i,j) is the rightmost cell of p in row i then the rightmost cell of p' in row i lies in a column $< j$, i.e., p' lies to the left of p.*

Proof. Since c_{k+1} lies directly to the left of c_k, it suffices to verify that if p' reaches $(i, j-1)$, then its next step will be down. Let x and y be the elements in cells $(i+1, j-1)$ and $(i+1, j)$ before the slide into c_k, see Figure 1(a). So $x \geq y$ since this array is a skew plane partition.

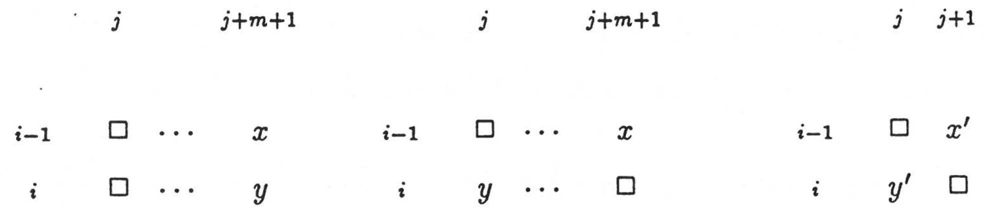
(a) Before p_k (b) After p_k

Figure 1. Slide situations

Now, by the hypothesis on p, after the c_k slide we have x and $y-1$ in cells $(i+1, j-1)$ and (i,j), respectively, see Figure 1(b). Thus when p' reaches $(i, j-1)$ it must continue to $(i+1, j-1)$ since $x > y-1$. ■

Lemma 2.4 *Let $p = p_k$ and $p' = p_l$ be the paths corresponding to forward slides into cells c_k and c_l, respectively, where $c_k = (\lambda_r, \lambda_r - s)$ and $c_l = (\lambda_{r-1}, \lambda_{r-1} - s)$ for some r, s. If (i,j) is the lowest cell of p in column j then the lowest cell of p' in column j lies in a row $< i$, i.e., p' lies above p.*

Proof. We will induct on k. Since c_l lies above and right of c_k, it suffices to verify that if p' reaches $(i-1, j)$, then its next step will be right. Let m be the largest integer such that cell $(i, j+t)$ is the lowest cell on path p_{k-t} for $0 \leq t \leq m$. Let x and y be the elements in cells $(i-1, j+m+1)$ and $(i, j+m+1)$, respectively, just before sliding along the path p_{k-m}. So we have the situation in Figure 2(a). Thus $x \geq y$.



(a) Before p_{k-m} (b) After p_k (c) Before p_l

Figure 2. More slide situations

By our assumption about the p_{k-t}'s, the situation after completion of p_k must look like Figure 2(b). Further slides from the same row as p_k can change the entry in cell (i,j) to some y', but since the elements passing through a given cell weakly decrease, we must have $y' \leq y$. Also, because of the previous lemma, x does not change with such slides.

As for the slides from row λ_{r-1}, those before p_{l-m} cannot change x or y' by induction and Lemma 2.3 applied to p_{k-m+1}. For similar reasons, no slide before p_l can change y'. To see how the slides $p_{l-m}, \cdots, p_{l-1}$ effect the elements in row $i-1$, note that, by the previous lemma, no element that moves up a row during a given slide can be moved again by subsequent slides starting in the same row as the given one. Thus the element x' that occupies the $(i-1, j+1)$ cell before p_l (see Figure 3(c)) must either have come from cell $(i, j+1)$ or from row i. The first case can't happen since p_{l-1} and previous slides from that row are above p_{k-1}. In the second case, since an element can be moved a maximum of t times in t slides, x' must have occupied a cell weakly to the left of x in Figure 2(b). Thus $x' \geq x$. Putting everything together, we have

$$x' \geq x \geq y \geq y'$$

Hence x' will move left into cell (i,j) during the slide p_l and we are done with the proof of the lemma. ■

We now need to create the inverse map

$$(Q, R) \longrightarrow P$$

First we formulate the inverse of a backward slide, called (oddly enough) a *forward slide.* For such a slide we are given a skew plane partition Q of shape Λ/λ and a cell $c = (i,j)$, which is the leftmost zero cell of Q in row i. Now perform the following steps.

F1 **While** $(i, j-1) \in \Lambda/\lambda$ **or** $(i-1, j) \in \Lambda/\lambda$ **do**

F2 **if** $P_{i,j-1} \leq P_{i-1,j}$ **then** slide $P_{i,j-1}$ into cell c

else slide $P_{i-1,j}+1$ into cell c. **fi**

F3 Let $c :=$ the cell of the element that slid in step F2. **od**

If only one of the two elements of the **if** clause above is defined, then that one automatically slides into c (with 1 added if necessary). The reader can check that a forward slide into cell (5,3) of P' in the example after the definition of a backward slide restores P. It is easy to see that, in general, forward slides can be used to reverse backward slides and vice versa.

Now suppose the pair (Q,R) is given. Order the cells of λ as in (3) and perform forward slides on Q associated with $c_n, c_{n-1}, \ldots, c_1$ in turn: if $c_k = (i,j)$ then the associated slide will be into the leftmost zero cell which lies in row $i + R_{i,\lambda_i-j+1}$ in the current version of Q. The final version of Q will be the image of the pair, P.

It is clear that the composition of our previous map with this one is the identity. To make sure that the other composition is too, we need to verify that the forward slides made on Q vacate the cells $c_n, c_{n-1}, \ldots, c_1$ in that order. This is accomplished by analogs of Lemmas 2.3 and 2.4. Since their proofs are similar to what we have already seen, we will merely state the results.

Lemma 2.5 *Let $p = p_k$ and $p' = p_{k-1}$ be the paths of forward slides corresponding to adjacent cells c_k and c_{k-1} in the same row. If (i,j) is the leftmost cell of p in row i then the leftmost cell of p' in row i lies in a column $> j$, i.e., p' lies to the right of p.* ■

Lemma 2.6 *Let $p = p_k$ and $p' = p_l$ be the paths of backward slides corresponding to cells c_k and c_l, respectively, where $c_k = (\lambda_r, \lambda_r - s)$ and $c_l = (\lambda_{r+1}, \lambda_{r+1} - s)$ for some r, s. If (i,j) is the highest cell of p in column j then the highest cell of p' in column j lies in a row $> i$, i.e., p' lies below p.* ■

Proof of (1)=(2). To show directly that the two products are equal, we merely need to demonstrate that the same exponents appear in both denominators. Clearly, it suffices to find an injection

$$f : \lambda \rightarrow \Lambda/\lambda$$

such that

f1 for all cells $(i,j) \in \lambda$ we have $h_{i,j} = h_{f(i,j)}$, and

f2 the multiset ("set" with repetitions) of hooklengths for the cells in Λ and $\Lambda/\lambda - f(\lambda)$ are the same.

To define this injection, it will be convenient to introduce the notion of a row-strip.

The *r-th row strip of* λ is the set of all cells of the shape λ that are i cells from the bottom of their respective columns. For example, we have marked the cells of the r-th row-strip in the following diagram with an r.

$$\begin{array}{cccc} 4 & 3 & 3 & 2 \\ 3 & 2 & 2 & 1 \\ 2 & 1 & 1 & \\ 1 & & & \end{array}$$

Similarly, the *r-th row strip of* Λ/λ is the set of all cells of Λ/λ that are r cells from the top of their respective columns. Marking a skew shape with r's gives the following figure.

$$\begin{array}{cccccccc} \blacksquare & \blacksquare & \blacksquare & \blacksquare & 1 & 1 & 1 & \cdots \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare & 2 & 2 & 2 & \cdots \\ \blacksquare & \blacksquare & \blacksquare & 1 & 3 & 3 & 3 & \cdots \\ \blacksquare & 1 & 1 & 2 & 4 & 4 & 4 & \cdots \\ 1 & 2 & 2 & 3 & 5 & 5 & 5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

Let σ_r and τ_r denote the r-th row-strips of λ and Λ/λ respectively. We will define the injection f by defining it on each row-strip

$$f : \sigma_r \longrightarrow \tau_r$$

Specifically, let the cells of σ_r be $(i_1, 1), (i_2, 2), (i_3, 3), \ldots$ and inductively define

$$f(i_j, j) = (i', j') \quad \text{if } j > 1 \text{ where } (i', j') \text{ is the rightmost box of } \tau_r \text{ to the left of } (i_j + r, \lambda_{i_j}) \text{ which is not already in the image of } f. \tag{4}$$

as j successively takes on the values 1, 2, 3, etc. For example, if $\lambda = (9, 8, 6, 2)$ and $r = 2$ then we have marked $(i, j) \in \sigma_2$ and $f(i, j) \in \tau_2$ with the same letter in the following diagram.

$$\begin{array}{ccccccccc} \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & g & h & \blacksquare \\ \blacksquare & \blacksquare & c & d & e & f & \blacksquare & \blacksquare & \square \\ a & b & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \square & \square & g \\ \blacksquare & \blacksquare & \square & \square & \square & \square & d & c & \square \\ \square & \square & f & e & b & a & \square & \square & \square \\ \square & h & \square & \square & \square & \square & \square & \square & \square \end{array}$$

Note that $f(i_1, 1) = (i_1 + r, \lambda_{i_1})$ is indeed in τ_r and has the same hooklength as (i, j) by construction. We must show that the rest of f is well-defined in that the cell (i', j') exists (in which case f is clearly injective), and that conditions f1 and f2 are satisfied. This will be taken care of by the following lemma and the fact that the r-th row-strip of Λ has hooklengths $\{r, r + 1, r + 2, \ldots\}$.

Lemma 2.7 *The function defined by equation (4) is well-defined and satisfies*

1. *for all cells* $(i,j) \in \sigma_r$ *we have* $h_{i,j} = h_{f(i,j)}$, *and*

2. *the hooklengths of the cells of* $\tau_r - f(\sigma_r)$, *read from left to right, are precisely* $\{r, r+1, r+2, \ldots\}$.

Proof. We induct on the number of rows of λ. If $\lambda = (\lambda_1, \ldots, \lambda_l)$, then let $\bar{\lambda} = (\lambda_2, \ldots, \lambda_l)$. Now the row strips and their images in columns $j \leq \lambda_2$ of $\bar{\lambda}$ and λ (for $i \geq 2$) are exactly the same. So, by induction, f is well-defined and preserves hooklengths there. Also, $|f(\sigma_r \cap \bar{\lambda})| = \lambda_{r+1}$. So there are $\lambda_2 - \lambda_{r+1}$ elements of τ_r in columns $j \leq \lambda_2$ which are not in $f(\sigma_r \cap \bar{\lambda})$. Thus, by induction again, the hooklengths of these cells must be $r, r+1, \ldots, r + \lambda_2 - \lambda_{r+1} - 1$.

As far as the columns j with $\lambda_2 < j \leq \lambda_1$, induction and the previous sentence combine to show that there τ_r has hooklengths from $r + \lambda_2 - \lambda_{r+1}$ to

$$r + \lambda_2 - \lambda_{r+1} + (\lambda_1 - \lambda_2) - 1 = r + \lambda_1 - \lambda_{r+1} - 1$$

Thus the hooklengths available in columns $j \leq \lambda_1$ make $f(\sigma_r \cap \lambda_1)$ well-defined and hooklength preserving if we use the rule (4). Furthermore, there are $\lambda_r - \lambda_{r+1}$ elements of σ_r in row 1. So the hooklengths unused by f in columns $j \leq \lambda_1$ form an interval from r to

$$r + \lambda_1 - \lambda_{r+1} - 1 - (\lambda_r - \lambda_{r+1}) = r + \lambda_1 - \lambda_r - 1$$

Finally, the cells of τ_r in columns $j > \lambda_1$ clearly have hooklengths

$$r + \lambda_1 - \lambda_r,\ r + \lambda_1 - \lambda_r + 1, \ldots$$

so we are done. ∎

3 Shifted plane partitions

Consider the shifted plane

$$\Lambda^* = \{(i,j) \in \Lambda \mid i \leq j\}$$

so that now each row is shifted over one box from the row above. Let $\lambda^* = (\lambda_1^*, \ldots, \lambda_l^*)$ be a strict partition, i.e., one where $\lambda_1^* > \ldots > \lambda_l^*$. Then λ^* can be viewed as a shifted shape in the upper-left corner of Λ^* via

$$\lambda^* = \{(i,j) \in \Lambda^* \mid i \leq j \leq i + \lambda_i - 1\}$$

Now we have a skew shifted shape This gives rise to the skew shape

$$\Lambda^*/\lambda^* = \{(i,j) \mid (i,j) \in \Lambda^*, (i,j) \notin \lambda^*\}$$

A *skew plane partition of n with shape Λ^*/λ^**, P^*, is defined in the obvious way. For example, if $\lambda = (3,1)$ then one such skew shifted plane partition is

$$P^* = \begin{array}{ccccc} \blacksquare & \blacksquare & \blacksquare & 4 & 4 \\ & \blacksquare & 3 & 2 & \\ & & 3 & 1 & \end{array}$$

Let

$$pp_{\Lambda^*/\lambda^*}(n) = \text{ number of shifted plane partitions of } n \text{ having shape } \Lambda^*/\lambda^*.$$

Shifted hooks are defined as follows. If $(i,j) \in \lambda^*$ then

$$H^*_{i,j} = \{(i,j') \mid j' \geq j\} \cup \{(i',j) \mid i' \geq i\} \cup \{(j+1,j') \mid j' \geq j+1\}$$

where all sets are contained in λ^*. This is just the normal hook if $j \geq l =$ the number of parts of λ^* (i.e., (i,j) is not over the left staircase). If $j < l$ then the vertical portion of $H^*_{i,j}$ does a right turn and picks up all elements in row $j+1$. In the case $(i,j) \in \Lambda/\lambda$, we again take reflections to give

$$H^*_{i,j} = \{(i,j') \mid j' \leq j\} \cup \{(i',j) \mid i' \leq i\} \cup \{(i',i-1) \mid i' \leq i-1\}$$

where all sets are now in Λ^*/λ^*. Of course, the *shifted hooklength* of cell (i,j) is $h^*_{i,j} = |H^*_{i,j}|$. For example, if $\lambda^* = (6,5,3,1)$ then the cells in the hooks of $(1,2) \in \lambda^*$ are shown as circles in

$$\begin{array}{cccccc} \blacksquare & \bullet & \bullet & \bullet & \bullet & \bullet \\ & \bullet & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ & & \bullet & \bullet & \bullet & \\ & & & \blacksquare & & \end{array}$$

while those of $(7,8) \notin \lambda^*$ are the circles in

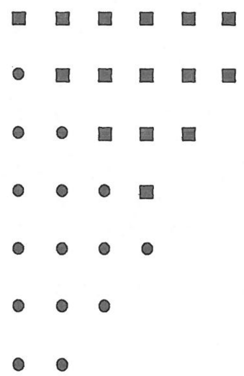
$$\begin{array}{cccccccccc} \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \square & \circ & \square & \cdots \\ & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \square & \circ & \square & \cdots \\ & & \blacksquare & \blacksquare & \blacksquare & \circ & \square & \circ & \square & \cdots \\ & & & \blacksquare & \square & \circ & \square & \circ & \square & \cdots \\ & & & & \square & \circ & \square & \circ & \square & \cdots \\ & & & & & \circ & \square & \circ & \square & \cdots \\ & & & & & & \circ & \circ & \square & \cdots \\ & & & & & & & \ddots & \ddots & \end{array}$$

So $h^*_{1,2} = 9$ and $h^*_{7,8} = 12$.

The way to motivate the definition of these hooks is as follows. Given the shifted shape λ^*, let $\hat{\lambda}$ denote the left-justified shape obtained by gluing together λ^* and its transpose, i.e.,

$$\hat{\lambda} = \{(i,j) \mid i \leq j \leq i + \lambda^*_i - 1 \text{ or } j+1 \leq i \leq j + \lambda^*_j\}$$

To illustrate, if $\lambda^* = (6,5,3,1)$ as before, then

$\hat{\lambda} =$ 

where the circles now indicate the cells of λ^*'s transpose. It's easy to see that if $(i,j) \in \Lambda^*/\lambda^*$ then $h^*_{i,j} = h_{i,j}$ where the normal hooklength is calculated in $\Lambda/\hat{\lambda}$. Similarly if $(i,j) \in \lambda^*$ then $h^*_{i,j} = h_{j+1,i}$ where the normal hooklength is in $\hat{\lambda}$. This is because, in both cases, the shifted hook is just the normal hook with one of its appendages bent.

We can now state the analog of Theorem 2.1.

Theorem 3.1 *If λ^* is a fixed shifted shape, then*

$$\sum_{n\geq 0} pp_{\Lambda^*/\lambda^*}(n)x^n = \prod_{(i,j)\in\Lambda^*/\lambda^*} \frac{1}{1-x^{h^*_{i,j}}} \tag{5}$$

$$= \prod_{k\geq 1} \frac{1}{(1-x^k)^{\lceil k/2 \rceil}} \prod_{(i,j)\in\lambda^*} \frac{1}{1-x^{\hat{h}_{i,j}}} \tag{6}$$

Proof of (5). Again, we are just reflecting the shifted Hillman-Grassl algorithm (see [Sag 82]) in an anti-diagonal. Because of the similarity with the proof of (1), we content ourselves with defining the path p^* along which to subtract ones in a given shifted skew plane partition P^*. The reader who has made it this far will find no difficulty in filling in the details of the rest of the algorithm.

SHG1 Start p^* at (a,b), the rightmost highest cell of P^* containing a nonzero entry.

SHG2 Continue by iterating

$$(i,j) \in p^* \Longrightarrow \begin{cases} (i+1,j) \in p^* & \text{if } P^*_{i+1,j} = P^*_{i,j} \\ (i,j-1) \in p^* & \text{otherwise} \end{cases}$$

SHG3 The induction rule in SHG2 will fail at some cell (r,s) at the left end of a row, so subtact ones along this portion of p^*.

SHG4 **If** $r < s$ **then stop**

else (now r=s) continue p^* by $(r-1,r-1) \in p^*$ and iterate

$$(i,j) \in p^* \Longrightarrow \begin{cases} (i,j+1) \in p^* & \text{if } P^*_{i,j+1} = P^*_{i,j} \\ (i-1,j) \in p^* & \text{otherwise fi} \end{cases}$$

SHG5 Now the induction rule in SHG4 will fail at some cell (t, u) at the top of a column.

It is easy to see that after subtracting one from the elements in p^*, the array remains a shifted plane partition and the amount subtracted is $h^*_{r,b}$ or $h^*_{u+1,b}$ depending on whether the path terminates in step SHG3 or SHG5, respectively. (The crucial observation is that the second half of p^*, if it exists, cannot intersect the first half because of the subtraction in SHG3.) ■

Proof of (5)=(6). We obtain the analog of the map f of the proof that (1)=(2) as follows. Using column-strips (rather than row-strips) define an injection

$$f : \hat{\lambda} \to \Lambda/\hat{\lambda}$$

A simple argument shows that if we restrict the domain of f to Λ^* then the range also becomes included in the shifted plane. Furthermore, those cells in $\Lambda^* - f(\lambda^*)$ have hooklengths given by the first product in (6). ■

4 Open questions

First of all, the reader will have noticed that we gave no direct proof that the product (6) counts shifted skew plane partitions. There is a shifted version of the jeu de taquin [Sag 87, Wor 84], but it is not clear how to apply it in this case.

The main problem is finding a combinatorial interpretation of the second product in (6) since it only contains half the hooklengths in $\hat{\lambda}$. If $\lambda^* = (\lambda^*_1, \ldots, \lambda^*_l)$ then the following two multisets are the same.

$$\{\{\hat{h}_{i,j} \mid (i,j) \in \lambda^*\}\} = \{\{h^*_{i,j} \mid (i,j) \in \lambda^*, j \neq l\}\} \cup \{\{2h^*_{i,l} \mid (i,l) \in \lambda^*\}\}$$

Also the second multiset in the union (which is really just a set) can also be expressed as

$$\{2\lambda^*_i \mid 1 \leq i \leq l\}$$

Stanley [private communication] has suggested factoring the corresponding terms of (6) as a difference of squares and bringing the binomials with positive signs over to the left side of the equation. Maybe this will help.

In [Sag 82] we also consider a third family of partitions with hooklengths: rooted trees. A rooted tree, τ, is a finite partially ordered set with a unique minimal element (called the root) whose Hasse diagram is a tree in the graph-theoretic sense of the term. A reverse τ-partition is an assignment, T, of non-negative integers to the vertices of τ, such that if $v \leq w$ in the partial order in τ then $T(v) \leq T(w)$ as integers. This is the tree analog of reverse plane partition or a reverse shifted plane partition. The hooks in this case are just

$$H_v = \{w \in \tau \mid w \geq v\}$$

In all three cases, the generating function for those reverse partitions summing to n is a finite product in terms of hooklengths. However, we have been unable to define a notion of skewness for trees that will yield a nice generating function for the corresponding (non-reverse) partitions. Perhaps one of our readers will have better luck.

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