NEW PLETHYSM OPERATION, CHERN CHARACTERS OF EXTERIOR AND SYMMETRIC POWERS WITH APPLICATIONS TO STIEFEL-WHITNEY CLASSES OF GRASSMANNIANS

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ABSTRACT. In this paper a new plethysm operation is proposed and a technique for coefficient extraction for a fairly general class of symmetric power series (e.g. multiplicative sequences of the theory of characteristic classes) is developed, together with various applications.

1. INTRODUCTION

Many problems in combinatorics, representation theory of symmetric and linear groups, K–theory and topology can be stated purely in terms of the theory of symmetric functions: counting 0–1 matrices with given row and column sums corresponds to expressing products of elementary symmetric functions in terms of monomials; the decomposition of the composition of exterior powers $\Lambda^i(\Lambda^j)$ of vector bundles corresponds to the plethysm of elementary symmetric functions and is one of the most difficult problems left in the theory of the symmetric and linear groups. The problem of describing the Chern classes of $\Lambda^i(\Lambda^j), \Lambda^i(S^j), S^i(\Lambda^j), S^i(S^j)$ is even more difficult. For example, $\sum S^n(S^2) = \prod_{i\leq j}(1-x_ix_j)^{-1}$ is equal to the sum of all Schur functions indexed by even partitions (c.f.[M], I.5.Ex5). The corresponding function for Chern classes $\sum c_n(S^2)$ is $\prod_{i\leq j}(1-(x_i+x_j))^{-1}$ which now involves multiplicites which are binomial determinants instead od being 0 or 1 (see[L]).

The general problem for describing $\sum_n c_n(\Lambda^r)$, the Chern classes of the r-th exterior power, leads to the following purely combinatorial problem: express the function $\prod_{1\leq i_1<i_2<\ldots<i_r}(1+(x_{i_1}+x_{i_2}+\cdots+x_{i_r}))$ in an appropriate basis of symmetric functions. Instead of Chern classes, we will use Chern character $\mathrm{ch}(\Lambda^r)=\sum_n \mathrm{ch}_n(\Lambda^r)$, which is more convenient in K–theory, corresponding to the following symmetric function involving exponentials od x_i: $\sum_{1\leq i_1<i_2<\ldots<i_r}\exp(x_{i_1}+x_{i_2}+\cdots+x_{i_r})=r$-th elementary summetric function of the power series $f(x)=\exp(x)$. A difficulty arising from nonstability of the operations Λ^r (or S^r) (in formulas coefficients may depend

on the number of variables) can be easily circumvented by considering so called K–theoretic Chern classes $c_r(\xi) = \sum_{i=0}^{r}(-1)^i\binom{N-i}{r-i}\Lambda^i\xi$, $N = \mathrm{rank}(\xi)$ for which $\mathrm{ch}(c_r(\xi)) = r$-th elementary symmetric function of the power series $f_0(x) = 1 - \exp(x)$. Note that f_0 has compositional inverse $f_0^{-1}(x) = \ln(1-x)$. Now let f be any (invertible) formal power series. The expansion of the associated elementary symmetric functions $\sum_{i_1<\ldots<i_r} f(x_{i_1})\ldots f(x_{i_r})$ in the power sum basis of symmetric functions is given compactly (Main Theorem) in terms of the powers of the compositional inverse of f. As a consequence, for $\mathrm{ch}(c_r(\xi))$ we get a formula involving Stirling numbers of the second kind (arising from the Taylor coefficients of the (negative) powers of $\ln(1+x)$). The formula for $\mathrm{ch}(\Lambda^r\xi)$ is then obtained by the binomal inversion.

2. SYMMETRIC FUNCTIONS

We recall some basic definitions and facts about symmetric functions, with notation and terminology following Macdonald's treatise on symmetric functions [M]. We work mainly with symmetric functions (power series) in the infinitely many indeterminates $x_1, x_2, \ldots$. We shall be concerned with the following particular symmetric functions:

The *elementary symmetric function* e_n is the sum of all products of n distinct variables x_i so that $e_0 = 1$ and

$$e_n = \sum_{i_1<i_2<\ldots<i_n} x_{i_1}x_{i_2}\ldots x_{i_n}$$

for $n \geq 1$. If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$ is a partition, i.e., a nonincreasing sequence of nonnegative integers we define $e_\alpha = e_{\alpha_1}e_{\alpha_2}\ldots e_{\alpha_l}$.

The *complete homogeneous symmetric function* h_n defined by

$$h_n = \sum_{i_1\leq i_2\leq\ldots\leq i_n} x_{i_1}x_{i_2}\ldots x_{i_n}$$

is the sum of all monomials of total degree n in the variables $x_1, x_2, \ldots$ ($h_0 = 1$, $h_1 = e_1$.) It is convenient to define h_α to be $h_{\alpha_1}h_{\alpha_2}\ldots h_{\alpha_l}$ for any sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$ of nonnegative integers not necessarily a partition. We set $h = \sum_{n=0}^{\infty} h_n$ and $e = \sum_{n=0}^{\infty} e_n$. We also define h_n to be 0 for $n < 0$.

The generating function for e_n's and h_n's are

$$e(t) = \sum_{n\geq 0} e_n t^n = \prod_{i\geq 1}(1 + x_i t), \quad h(t) = \sum_{n\geq 0} h_n t^n = \prod_{i\geq 1}(1 - x_i t)^{-1} \tag{2.1}$$

The *monomial symmetric function* m_α is the sum of all distinct monomials of the form $s_{i_1}^{\alpha_1} s_{i_2}^{\alpha_2} \dots s_{i_l}^{\alpha_l}$, where $i_1, \dots, i_l$ are distinct.

The *power sum symmetric function* p_n is defined by

$$p_n = \sum_i x_i^n$$

If $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_l)$ is a partition, we define $p_\alpha = p_{\alpha_1} p_{\alpha_2} \dots p_{\alpha_l}$.

The *Schur function* s_α is the determinant $\det(h_{\alpha_i - i + j})_{1 \le i,j \le k} = \det(e_{\alpha'_i - i + j})$, where α' is the partition conjugate to α.

Let us only mention that e_n, h_n, s_α correspond to exterior power (Λ^n), symmetric power (S^n) and irreducible (S^α) representations of the general linear groups respectively, while p_n corresponds to a (virtual) representation Ψ^n (or the Adams operation in K–theory).

It is known that each of the sets $\{e_\alpha\}$, $\{h_\alpha\}$, $\{m_\alpha\}$ and $\{s_\alpha\}$, where α ranges over all partitions of n, form a Z-basis, and $\{p_\alpha\}$ form a Q-basis of the homogenous symmetric functions of degree n.

It is convenient to use notation $(1^{a_1} 2^{a_2} \dots n^{a_n})$ for the partition with a_i parts equal to i. If $\alpha = (1^{a_1} 2^{a_2} \dots n^{a_n})$ then we define z_α to be $1^{a_1} 2^{a_2} \dots n^{a_n} \cdot a_1! \cdot a_2! \dots a_n!$, $\varepsilon_\alpha =$ sign of a permutation of the cycle type α. (α in ε_α may also be a multiindex of naturals) We also identify partitions which differ only in the number of zero parts. The empty partition we denote by 0.

There is a symmetric Z-valued (nondegenerate) bilinear form $\langle u, v \rangle$ defined on symmetric functions by requiring that the bases $\{h_\alpha\}$ and $\{m_\alpha\}$ should be dual to each other

$$\langle h_\alpha, m_\beta \rangle = \delta_{\alpha\beta} \tag{2.2}$$

Then $\langle p_\alpha, p_\beta \rangle = z_\alpha \delta_{\alpha\beta}$, $\langle s_\alpha, s_\beta \rangle = \delta_{\alpha\beta}$ i.e. $\{p_\alpha\}$ is an orthogonal, and $\{s_\alpha\}$ an orthonormal basis. These facts are equivalent to the following identities:

$$\prod_{i,j} (1 - x_i y_j)^{-1} = \sum_\alpha z_\alpha^{-1} p_\alpha(x) p_\alpha(y); \quad \prod_{i,j} (1 + x_i y_j) = \sum_\alpha \varepsilon_\alpha z_\alpha^{-1} p_\alpha(x) p_\alpha(y) \tag{2.3}$$

$$\prod_{i,j} (1 - x_i y_j)^{-1} = \sum_\alpha s_\alpha(x) s_\alpha(y) \quad \text{(Cauchy determinant identity)} \tag{2.4}$$

Finally we recall the operation of *composition* (also called *plethysm*) for symmetric functions. To motivate the general definition, first suppose that f is a symmetric function which can be expressed in the form $t_1 + t_2 + \cdots$ where each term t_j is of the form $x_1^{i_1} x_2^{i_2} \dots x_k^{i_k}$ (the terms t_j need not be distinct). Then for any symmetric function $g = g(x_1, x_2, \dots)$ we define the composition $g \circ f = g(f)$ to be $g(t_1, t_2, \dots)$, called the plethysm of g and f.

3. NEW PLETHYSM OPERATION ON SYMMETRIC FUNCTIONS

Let f be a symmetric function with separated variables (and zero constant term) i.e. f can be written as $t_1' + t_2' + \cdots$ where each term t_i' depends only on the i-th variable x_i, $t_i' = \varphi_i(x_i)$ for some power series φ_i. The function f being symmetric implies that $\varphi_1 = \varphi_2 = \ldots =: \varphi$. Observe that $\varphi(x) = f(x_1 = x, 0, 0, \ldots)$ is uniquely determined by f, and is called the *characteristic power series* of f; such f is called *primitive*.

Definition. For any symmetric functions g and f, f primitive, we define a *new plethysm* $g \bullet f$ by

$$g \bullet f := g(t_1', t_2', \ldots) = g(\varphi(x_1), \varphi(x_2), \ldots)$$

where φ is the characteristic power series of f, $f(x_1, x_2, \ldots) = \varphi(x_1) + \varphi(x_2) + \cdots$. We shall also write $g \bullet \varphi$ instead of $g \bullet f$.

In the sequel we shall see the relevance of such an operation to several different problems.

For g we take now the elementary symmetric functions $e = \sum_{n \geq 0} e_n = \prod_{i \geq 1}(1 + x_i)$, and let $\varphi(x) = a_1 x + a_2 x^2 + \cdots \in K[[x]]$ be any invertible formal power series ($a_1 \neq 0$). Then the generating function for $e_n \bullet \varphi$, $n \geq 0$, can be written as

$$\begin{aligned} (e \bullet \varphi)(t) &= \sum_{n \geq 0} (e_n \bullet \varphi)\, t^n \\ &= \prod_{i \geq 1}(1 + \varphi(x_i) t) = \prod_i \Phi(x_i) \quad \text{where} \quad \Phi(x) := 1 + \varphi(x) t. \end{aligned} \tag{3.1}$$

By interpreting the coefficients of $\Phi(x)$ w.r.t. variable x as elementary symmetric functions of some "formal roots" η_j^Φ of Φ i.e. by writing a formal factorization:

$$\Phi(x) = 1 + t a_1 x + t a_2 x^2 + \cdots = \prod_j (1 + \eta_j^\Phi x) \tag{3.2}$$

we can continue (3.1):

$$\begin{aligned} (e \bullet \varphi)(t) &= \prod_{i,j} (1 + \eta_j^\Phi x_i) \\ &= \sum \varepsilon_\alpha z_\alpha^{-1} p_\alpha^\Phi p_\alpha(x) \quad \text{(by 2.3)} \end{aligned} \tag{3.3}$$

The following Lemma is fundamental and seems to be new:

KEY LEMMA. *The power sums* $p_n^\Phi = \sum_{j\geq 1}(\eta_j^\Phi)^n$ *associated to the power series* $\Phi(x) = 1+\varphi(x)t$ *are given by the following formula*

$$p_n^\Phi = \operatorname{res}(1-tz)^{-1}(-\varphi^{-1}(-z))^{-n}t\,dz \tag{3.4}$$

where φ^{-1} *is the compositional inverse of* φ, *and where* $\operatorname{res} f(z)\,dz$ *denotes the residue of* f *at* $z=0$ *(i.e. the coefficient of* z^{-1} *in* $f(z)$*).*

Proof: By applying the logarithmic derivative to both sides od (3.3) we have:

$$\begin{aligned} p_n^\Phi &= (-1)^{n-1}[x^{n-1}]\,\frac{d}{dx}\log\Phi(x) \quad \text{(Cauchy's identity)} \\ &= (-1)^{n-1}\operatorname{Res} x^{-n}\frac{t\varphi'(x)}{1+t\varphi(x)}\,dx \quad \text{(chain rule)} \end{aligned} \tag{3.5}$$

where $\varphi'(x)$ denotes the derivative of φ. Since φ is invertible, we can use a new variable $z=-\varphi(x)$ ($\Rightarrow x=\varphi^{-1}(-z)$, $dz=-\varphi'(x)\,dx$) in (3.5) and the Lemma follows.

Combining (3.3) and the Key Lemma we get the following:

MAIN THEOREM. *Let* φ *be any invertible formal power series, and* $e=\sum_{n\geq 0}e_n$ *the elementary symmetric function. Then the new plethysms* $e_n \bullet \varphi = e_n(\varphi(x_1),\dots,\varphi(x_n))$ *decompose in the power sum basis* $\{p_\alpha \mid \alpha = (\alpha_1 \geq \dots \geq \alpha_l > 0)$ *a partition* $\}$ *with the coefficients given by the following formula:*

$$[p_\alpha]\,(e_n \bullet \varphi) = \varepsilon_\alpha z_\alpha^{-1}\operatorname{res} t^{n-1}\Big(\prod_{i=1}^{l}\operatorname{res}(1-tz)^{-1}(-\varphi^{-1}(-z))^{-\alpha_i}t\,dz\Big)\,dt$$

where φ^{-1} *is the compositional inverse of* φ, ($\varepsilon_\alpha = (-1)^{|\alpha|-l_\alpha}$, $z_\alpha = 1^{a_1}2^{a_2}\dots k^{a_k}\cdot a_1!a_2!\dots a_k!$, $a_i = \operatorname{Card}\{j \mid \alpha_j = i\}$, $l_\alpha =$ *length of* α*).*

Proof: By pluging $p_\alpha^\Phi = p_{\alpha_1}^\Phi \cdot p_{\alpha_2}^\Phi \cdots p_{\alpha_l}^\Phi$ into (3.3) and using (3.4).

4. APPLICATIONS

4.1. Chern character of exterior and symmetric powers

We recall that the Chern character $\operatorname{ch}(\xi) = \sum \operatorname{ch}_k(\xi)$ of an $U(N)$-bundle ξ over a (paracompact) topological space X is defined by $\operatorname{ch}(\xi) = \sum \exp(x_i) \in H^{\text{even}}(X;\mathbb{Q})$, where $\sum c_i(\xi)t^i =$

$\prod(1+x_it)$ is a "formal" factorization of a generating function for Chern classes $c_i(\xi) \in H^{2i}(X)$ in terms of the formal "Chern roots" $x_1, x_2, \dots x_N$ (lying in an extension of the cohomology of the base space and corresponding to line bundles via splitting principle). This link between characteristic classes and symmetric functions is known as Borel – Hirzebruch formalism. Via this formalism one has various formal factorization formulas for the characteristic classes of the associated bundles, like exterior powers, symmetric powers: (c.f. [H])

$$\sum c_i(\Lambda^r\xi)t^i = \prod_{1\leq i_1<i_2<\dots<i_r\leq N} (1+(x_{i_1}+x_{i_2}+\dots+x_{i_r})t) \tag{4.1}$$

$$\sum c_i(S^r\xi)t^i = \prod_{1\leq i_1\leq i_2\leq\dots\leq i_r\leq N} (1+(x_{i_1}+x_{i_2}+\dots+x_{i_r})t) \tag{4.2}$$

$$\sum_{r\geq 0} \mathrm{ch}(\Lambda^r\xi)t^r = \prod_i(1+\exp(x_i)t) \tag{4.3}$$

$$\sum \mathrm{ch}(\Lambda^r\xi)t^r = \prod_i(1-\exp(x_i)t)^{-1} \tag{4.4}$$

This formalism solves the problem only in principle, since everything expressed only in terms of "Chern roots" may require even in small cases a lot of computational work. Hence it arises a natural

GENERAL PROBLEM: Find a "reasonable" (or "satisfactory") formulas for the characteristic classes of the associated bundles in terms of the characteristic classes of the original bundle.

In order to attack the problem of finding formulas for the Chern character of exterior/symmetric powers we first recall so called K–theoretic Chern classes (c.f. [K],p.253)

$$\mathrm{c}_r(\xi) = \sum_{i=0}^{r}(-1)^i\binom{N-i}{r-i}\Lambda^i(\xi),\ N = \mathrm{rank}\,(\xi) \tag{4.5}$$

Using the fact that

$$\mathrm{ch}(\mathrm{c}_r(\xi)) = r\text{-th elementary symmetric function of } 1-\exp(x_1),\dots,1-\exp(x_N)$$

we can take $\varphi(x) = 1-\exp(x) \in \mathbb{Q}[[x]]$ and by the Main Theorem we get:

THEOREM 1. *The Chern character of the r-th K–theoretic Chern class of ξ expressed in terms of the components of the Chern character of ξ is given by*

$$\mathrm{ch}(\mathrm{c}_r(\xi)) = \sum_\alpha \frac{(-1)^l}{\|\alpha\|} C_\alpha \mathrm{ch}_\alpha(\xi) \tag{4.6}$$

where for a partition $\alpha = (\alpha_1 \geq \dots \geq \alpha_l > 0)$, $\|\alpha\| = a_1!a_2!\dots$, $a_i = \mathrm{Card}\{j \mid \alpha_j = i\}$, $C_\alpha = \sum_{\varrho\in\mathbf{N}^l,\varrho\leq\alpha,|\varrho|=r} \overleftarrow{\varrho}!S(\alpha,\varrho)$, where $S(\alpha,\varrho) = \prod_{i=1}^l S(\alpha_i,\varrho_i)$ is the product of the Stirling numbers of the second kind, $\overleftarrow{\varrho} = (\varrho_1-1,\varrho_2-1,\dots,\varrho_l-1)$, $\mathrm{ch}_\alpha(\xi) = \prod_{i=1}^l \mathrm{ch}_{\alpha_i}(\xi)$.

Proof. Immediate from the Main Theorem by using the identity: (c.f. [G-K-P], (7.51))

$$\left(\frac{z}{\ln(1+z)}\right)^m = \sum_{n\geq 0} \frac{z^n}{n!} S(m, m-n) \Big/ \binom{m-1}{n} \tag{4.7}$$

The relation (4.5) in $K(X)$ is invertible so we can express exterior powers in terms of K–theoretic Chern classes:

$$\Lambda^r \xi = \sum (-1)^i \binom{N-i}{r-i} \mathrm{c}_i(\xi) \tag{4.8}$$

Now, additivity of the Chern character together with (4.6) gives the formula for $\mathrm{ch}(\Lambda^r\xi)$ in terms of $\mathrm{ch}(\xi)$.

THEOREM 2. *The Chern character of the r-th exterior power of the $U(N)$-bundle ξ is given by the following formula*

$$\mathrm{ch}(\Lambda^r\xi) = \sum_\alpha \frac{1}{\|\alpha\|} \Big(\sum_{\varrho \leq \alpha} \varepsilon_\varrho \binom{N-|\varrho|}{N-r} \overleftarrow{\varrho}\; S(\alpha, \varrho) \Big) \mathrm{ch}_\alpha(\xi) \tag{4.9}$$

where $\varrho \in \mathrm{N}^{l_\alpha}$. ($l_\alpha$ = length of α)

Let us now illustrate the Theorem 1 and Theorem 2 as identities in symmetric functions. Let $P_k^{r,N}$ and $C_k^{r,N}$ be the following symmetric functions:

$$P_k^{r,N} = \begin{cases} \sum\limits_{1\leq i_1<\ldots<i_r\leq N} (x_{i_1} + x_{i_2} + \cdots + x_{i_r})^k \\ \delta_{k,0}\,, \quad r = 0 \end{cases} \Big(= k!\mathrm{ch}_k(\Lambda^r(\xi^N)) \Big) \tag{4.10}$$

$$C_k^{r,N} = \sum_{i=0}^{r} (-1)^{r-i} \binom{N-i}{N-r} P_k^{i,N} \Big(= k!(-1)^r \mathrm{ch}_k(\mathrm{c}_r(\xi)) \Big) \tag{4.11}$$

Then by the binomial inversion we get

$$P_k^{r,N} = \sum_{i=0}^{r} \binom{N-i}{N-r} C_k^{i,N} \tag{4.12}$$

The Teorem 1 gives the following identity:

$$C_k^{r,N} = \sum_{\alpha\vdash k} \begin{bmatrix} k \\ \alpha \end{bmatrix} \Big(\sum_{\beta\in \mathrm{N}^{l_\alpha}, \beta\leq\alpha, |\beta|=r} \varepsilon_\beta\, \overleftarrow{\beta}! S(\alpha,\beta) \Big) p_\alpha \tag{4.13}$$

where $\overleftarrow{\beta} = (\beta_1 - 1, \beta_2 - 1, \ldots)$. Substituing in (4.12) we get the identity:

$$P_k^{r,N} = \sum_{\alpha\vdash k} \begin{bmatrix} k \\ \alpha \end{bmatrix} \Big(\sum_{i=1}^{r} \binom{N-i}{r-i} \sum_{\beta\in \mathrm{N}^{l_\alpha}, \beta\leq\alpha, |\beta|=i} \varepsilon_\beta\, \overleftarrow{\beta}! S(\alpha,\beta) \Big) p_\alpha + \binom{N}{r} \delta_{k,0} \tag{4.14}$$

where $\alpha \vdash k$ means α is a partition of k, and $\begin{bmatrix} k \\ \alpha \end{bmatrix} := \frac{k!}{\prod \alpha_i! \cdot a_i!}$, $a_i = \mathrm{Card}\{j \mid \alpha_j = i$ (the incidence coefficient) equivalent to the Theorem 2.

For example:

$$P_k^{2,N} = (N - 2^{k-1})p_k + \frac{1}{2} \sum_{i,j>0, i+j=k} \binom{k}{i} p_i p_j \tag{4.15}$$

$$P_k^{3,N} = \left[\binom{N}{2} - N \cdot 2^{k-1} + 3^{k-1}\right] p_k + \frac{1}{2} \sum_{i,j>0, i+j=k} \binom{k}{i} (N - 2^{i-1} - 1) p_i p_j + + \frac{1}{6} \sum_{i,j,l>0, i+j+l=k} \binom{k}{i, j, l} p_i p_j p_l \tag{4.16}$$

COROLLARY 1. ([S–S]) *Suppose X is a finite complex and ξ an $U(N)$-bundle over X. Suppose that all products in $\widetilde{H}^*(X, \mathbb{Q})$ are zero (e.g. X a suspension). Then*

$$\mathrm{ch}_N(\mathrm{c}_r(\xi)) = (-1)^{r-1}(r-1)! S(N,r) \mathrm{ch}_N(\xi), \tag{4.17}$$

where $S(N,r) = \frac{1}{r!}\sum_{i=1}^{r}(-1)^{r-i}\binom{r}{i} i^N$ is the Stirling number of the second kind.

Note that in [S–S] a sign $(-1)^{i-1}$ is missing in the Theorem 1.1., because $(-1)^{n-1}$ is missing in the second line of page 210.

COROLLARY 2. *The coefficient of the primitive part in $\mathrm{ch}_k(\Lambda_r)$ is given by*

$$\left[\frac{p_k}{k!}\right] \mathrm{ch}_k(\Lambda^r) = A(k,r,N) = [x^r](1+x)^{N-k} A_{k-1}(-x) = \sum_{j=1}^{r} (-1)^{j-1} \binom{N}{r-j} j^{k-1}, \tag{4.18}$$

where $A_{k-1}(x)$ is the Eulerian polynomial.

Now we can add one more formula to our list of $P_k^{r,N}$:

$$P_k^{4,N} = A(k,4,N)p_k + \sum_{\alpha=(\alpha_1,\alpha_2)\vdash k} \begin{bmatrix} k \\ \alpha \end{bmatrix} (A(\alpha_1, 3, N-1) + (2^{\alpha_1 - 1} - 1)(2^{\alpha_2 - 1} - 1)) p_{\alpha_1} p_{\alpha_2} + + \sum_{\alpha=(\alpha_1,\alpha_2,\alpha_3)\vdash k} \begin{bmatrix} k \\ \alpha \end{bmatrix} A(\alpha_1, 2, N-2) p_{\alpha_1} p_{\alpha_2} p_{\alpha_3} + \sum_{\alpha=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\vdash k} \begin{bmatrix} k \\ \alpha \end{bmatrix} p_{\alpha_1} p_{\alpha_2} p_{\alpha_3} p_{\alpha_4} \tag{4.19}$$

with $A(k,r,N)$ defined in (4.18).

THEOREM 3. *Let $\Lambda(\xi) = \oplus_{r=0}^{N} \Lambda^r(\xi)$ be the total exterior power of a vector bundle ξ of rank N. Then for the Chern character we hawe the following formula*

$$\mathrm{ch}(\Lambda(\xi)) = \sum_{\alpha} \frac{2^{N-|\alpha|}}{\|\alpha\|} (1 + \tanh)^{(\alpha-1)}(0) \mathrm{ch}_\alpha(\xi) \tag{4.20}$$

where $(1+\tanh)^{(\alpha-1)} := f^{(\alpha_1-1)}(0) \cdot f^{(\alpha_2-1)}(0) \cdots f^{(\alpha_l-1)}(0)$ denotes the product of derivatives of $f(x) = 1 + \tanh(x)$ at 0.

COROLLARY 3. *If $H^2(X,\mathbb{Q}) = 0 = H^{4k}(X,\mathbb{Q}) = 0$, $k \geq 1$, then for any ξ the total exterior power $\Lambda(\xi)$ represents a torsion element of $K(X)$. In particular, if $K(X)$ has no torsion then the inverse of ξ in K–theory is given by $-[\xi] = \oplus_{r\geq 2}[\Lambda^r\xi]$.*

COROLLARY 4. *For any $U(N)$-bundle ξ over a $(4k+2)$-sphere S^{4k+2}, $k \geq 1$, $\Lambda\xi$ is stably trivial.*

In a similar manner, as in Theorem 2, we get the following results for the symmetric powers $S^r\xi$:

THEOREM 4. *The Chern character of the r-th symmetric power $S^r\xi$ of an $U(N)$ bundle ξ is given by*

$$\mathrm{ch}(S^r\xi) = \sum_\alpha \frac{1}{\|\alpha\|} \sum_{\beta\leq\alpha} \binom{N+r-1}{r-|\beta|} \overleftarrow{\beta}! S(\alpha,\beta)\mathrm{ch}_\alpha(\xi) \tag{4.21}$$

COROLLARY 5.

$$\left[\frac{p_k}{k!}\right] \mathrm{ch}_k(S^r\xi^N) = \overline{A}(r,k,N) = [x^r](1-x)^{-N-k}A_{k-1}(x) \tag{4.22}$$

where

$$\overline{A}(r,k,N) = \sum_{j=1}^{r} \binom{N-1+r-j}{N-1} j^{k-1}$$

and $A_{k-1}(x)$ is the Eulerian polynomial.

The Theorem 4 is equivalent to the following identity for symmetric functions:

$$\overline{P}_k^{r,N} = \sum_{i=1}^{r} \binom{N+r-1}{r-1} \sum_{\alpha\vdash} \begin{bmatrix} k \\ \alpha \end{bmatrix} \Big(\sum_{\beta\in\mathbb{N}^{l_\alpha},\beta\leq\alpha,|\beta|=i} \overleftarrow{\beta}! S(\alpha,\beta) \Big) p_\alpha \tag{4.23}$$

where

$$\overline{P}_k^{r,N} = \sum_{1\leq i_1\leq\ldots\leq i_r\leq N} (x_{i_1}+x_{i_2}+\cdots+x_{i_r})^k (= k!\mathrm{ch}_k(S^r\xi^N)) \tag{4.24}$$

In particular

$$\overline{P}_k^{2,N} = (N+2^{k-1})p_k + \frac{1}{2} \sum_{i,j>0,i+j=k} \binom{k}{i} p_i p_j \tag{4.25}$$

$$\overline{P}_k^{3,N} = \left[\binom{N+1}{2} + N\cdot 2^{k-1} + 3^{k-1}\right] p_k + \frac{1}{2} \sum_{i,j>0,i+j=k} \binom{k}{i} (N+2^{i+1}+1) p_i p_j + $$

$$+ \frac{1}{6} \sum_{i,j,l>0,i+j+l=k} \binom{k}{i,j,l} p_i p_j p_l \tag{4.26}$$

4.2. Chern classes and Stiefel-Whitney classes of the second exterior power $\Lambda^2\xi$

Let $E_k^{2,N}$ be the polynomial corresponding to the k-th Chern class of the second exterior power $\Lambda^2\xi$ i.e.

$$E_k^{2,N} = e_k(x_i + x_j \mid 1 \le i < j \le N) \quad (= c_k(\Lambda^2\xi)) \tag{4.27}$$

where e_k is the k-th elementary symmetric function. By using the Newton relations between $E_k^{2,N}$ and the $P_k^{2,N}(= 2!\mathrm{ch}_k(\Lambda^2(\xi^N)))$ and the Girard formula, which expresses power sums p_k's in terms of the elementary symmetric functions, we obtain the following results.

PROPOSITION 1. *1) Let $0 \le a < k/2$. Then for the Chern classes of $\Lambda^2\xi$, $c_k(\Lambda^2\xi) = E_k^{2,N}$ we have:*

$$\begin{aligned}[c_1^a c_{k-a}]\, c_k(\Lambda^2\xi) &= \sum_{r=0}^{a}(-1)^{a-r}\left[N - 2^{k-r-1} + \sum_{t=1}^{a-r}\binom{k-r-1}{t}\right]\binom{N-1}{r} \\ &= (N-1)\binom{N-2}{a} + (-1)^a \sum_{r=0}^{a}\left[\binom{k-N}{r} - (-1)^r 2^{k-r-1}\binom{N-1}{r}\right]\end{aligned} \tag{4.28}$$

where $c_i = c_i(\xi)$.

2) For Stiefel–Whitney classes $w_r(\Lambda^2\xi)$ we have

$$\begin{aligned}[w_1^a w_{k-a}]\, w_k(\Lambda^2\xi) &\equiv (N-1)\binom{N-2}{a} + \binom{N-k+a}{a} \pmod 2 \\ [w_2^a w_{k-a}]\, w_k(\Lambda^2\xi) &\equiv N\cdot\binom{N-3}{a} \pmod 2 \quad (\textit{for } 0 \le a < \tfrac{k}{3})\end{aligned} \tag{4.29}$$

Proof. The proof consists of four steps:

Step 1. Use the formula (4.15).

Step 2. Use the Girard formula in the following form:

$$p_k = k \sum_{a,b>0,\, ia+jb=k} (-1)^{k-a-b}\frac{1}{a+b}\binom{a+b}{a} e_i^a e_j^b + \cdots \tag{4.30}$$

Step 3. Use the Newton formulas (for $E_k^{2,N}$)

$$e_k = \sum_{\alpha\vdash k} \varepsilon_\alpha z_\alpha^{-1} p_\alpha = \sum_{m,t\ge 1,\, 0\le r\le m} \frac{(1)^{(m-1)t}}{t!\, m^t} p_m^t e_r \cdot e_k' \tag{4.31}$$

where e_k' corresponds to terms p_α such that $\alpha_1 \le \sum\{\alpha_j \mid \alpha_j < \alpha_1\}$.

Step 4. Use the identity (for part *2)*)

$$\sum_{t=0}^{k}(-1)^t\binom{n}{t} = (-1)^k\binom{n-1}{k} \tag{4.32}$$

and diagonal summation in getting the second formula in (4.28).

Remark. Our result for $[w_1^2 w_{k-2}]w_k(\Lambda^2\xi)$ settles the following Conjecture of Korbaš [K2]:

CONJECTURE (Korbaš): Let $\sigma_k = w_k(\gamma_N^n)$, $\overline{\sigma}_k = w_k(\Lambda^2\gamma_N^n)$. Then

$$\begin{aligned}
\overline{\sigma}_{4l} &= n_0(1+n_1)\sigma_1^2\sigma_{4l-2} + \text{other terms}\\
\overline{\sigma}_{4l+1} &= (1+n_0n_1)\sigma_1^2\sigma_{4l-1} + \cdots\\
\overline{\sigma}_{4l+2} &= (1+n_0(1+n_1))\sigma_1^2\sigma_{4l} + \cdots\\
\overline{\sigma}_{4l+3} &= n_0n_1\sigma_1^2\sigma_{4l+1} + \cdots
\end{aligned}$$

where $n_0 = n \bmod 2$, $n_1 = \frac{(n-n_0)}{2} \bmod 2$ are the last two binary digits of n.

This conjecture implies the following formulas for the Stiefel–Whitney classes of the Grassmann manifold

COROLLARY 6. If $k \geq 1$ then

$$\begin{aligned}
w_{8k}(G_{n,N}) &= (N_1+n_0)(N_2+n_0+n_1)w_1^4w_{4k-2}^2 + N_0w_{8k} + (N_1+n_0)w_{4k}^2 +\\
&+ (1+N_1+n_0)w_1^2w_{8k-2} + \cdots\\
w_{8k+2}(G_{n,N}) &= [(1+(N_1+n_0)(N_1+N_2+n_1)]w_1^4w_{4k-1}^2 + N_0w_{8k+2} + \cdots\\
w_{8k+4}(G_{n,N}) &= [(1+(N_1+n_0)(N_2+n_0+n_1)]w_1^4w_{4k}^2 + \cdots\\
w_{8k+6}(G_{n,N}) &= (N_1+n_0)(N_1+N_2+n_1)]w_1^4w_{4k+1}^2 + \cdots
\end{aligned}$$

Note that the four casses of the conjecture can be unified in a single formula

$$\overline{\sigma}_k = (n_0(n_1+k_0+1)+k_0+k_1)\sigma_1^2\sigma_{k-2} + \cdots \tag{4.33}$$

following easily from (4.29).

Let us recall that the Stiefel–Whitney classes of the Grassmann manifold $G_{n,N} = G_n(\mathbb{R}^N)$ can be written as

$$\Big[\prod_{i=1}^{n}(1+x_i)\Big]^N \Big/ \Big(\prod_{1\leq i<j\leq n}(1+x_i+x_j)^2\Big) \tag{4.34}$$

corresponding to the Whitney sum decompsition

$$\tau \oplus \mathrm{Hom}(\gamma,\gamma) \cong N\cdot\gamma^*, \quad \mathrm{Hom}(\gamma,\gamma)\cong\gamma\otimes\gamma, \;\; \gamma^*\cong\gamma \tag{4.35}$$

(τ = the tangent bundle, γ = the canonical bundle). By using the Whitney sum formulas for $N\gamma$ and by observing that the Stiefel – Whitney classes of $\gamma \otimes \gamma$ are the sum of squares of those of $\Lambda^2\gamma$, one gets a number of informations like in the Corollary 6 about the coefficients of the S–W classes of grassmannians. In [K1] a number of formulas is obtained by using Steenrod squares and an algorithm which deals with formal roots, what makes computations quite difficult.

Remark. The research reported here was motivated by the Problem of Bredon [B], for which an explicit solution is given in the autor's Ph.D Thesis [S] and which led to the Theorem 3.

The Theorem 3 was first proved in [S] directly by a long argument which used Hopf algebra structure of the ring Λ of symmetric functions, Möbius inversion on partition lattices and which required solving a system of partial differential equations. In [S] several recursions for the Chern character polynomials are also obtained.

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