Sorting twice through a Stack

Julian West
Dept. of Math. Sciences, Lakehead U.
Thunder Bay, Ontario P7B 5E1

In the following, we describe in terms of a simple card game the problem of sorting a permutation on a stack, and obtain the well-known result for the number of stack-sortable permuations of length n. This problem has been generalized in a number of ways; some of these were collected by Knuth. [2] Our description of the problem in terms of a game suggests a natural-sounding generalization to k stacks which has apparently not previously been considered.

Consider playing the following game with a deck of n cards, numbered $1, 2, 3, \ldots, n$. Shuffle the deck and hold it faceup in your left hand. You can see only the top card of the deck, which we will call the *new card.* On the table in front of you, you will maintain two piles. Every card must first be placed faceup on the first pile, called the *stack*, then later moved facedown onto the second pile, called the *output.*

The first card is placed onto the stack, becoming the *top stack card*, thus exposing the second card. From now on, there may be two choices of move: (1) if there are still cards in your hand, you may place the *new card* on top of the stack, or (2) if there are cards on the stack, you can move the *top stack card* to the output.

You consider that you have won the game if all the cards are moved to the output pile *in order.* It should be clear that you will not be able to win if you cover any card in the stack with a higher numbered card. For the larger card on top would be moved to the output before the card it is covering. Therefore, adopt the following strategy: compare the card in your hand (if any) to the card on top of the stack (if any). Always place the new card on the stack if it is smaller than the top stack card. If the new card is greater than the top stack card, move the top stack card to the output.

This simple strategy is the best one for playing this game, as it simply avoids making losing moves. It is always a losing move to cover a smaller card by a larger one. Similarly, it is always a losing move to move a larger card to the output if there is a smaller one yet to come.

If our deck contains n cards, how many of the $n!$ starting positions for this game result in winning games using this strategy? The answer has been well known for at least 20 years, and appears in [2]. Let the cards be represented by the permutation π, so that the top card is card number $\pi(1)$, etc. If we can win the game from the starting position π, we will say π is *stack-sortable.*

Let a subsequence $\pi(i), \pi(j), \pi(k)$, with $i < j < k$, of the permutation π be called a *wedge* if $\pi(j) > \pi(i) > \pi(k)$

Lemma 1. *A permutation π is stack-sortable if and only if π contains no wedge.*

Proof: If $i < j$ and $\pi(i) < \pi(j)$ then $\pi(i)$ must be removed from the stack before $\pi(j)$ is put on. If $i < k$ and $\pi(i) > \pi(k)$ then $\pi(i)$ must be remain on the stack until after $\pi(k)$ is added.

So if $i < j < k$ and $\pi(k) < \pi(i) < \pi(j)$, $\pi(i)$ must be removed before the addition of $\pi(j)$ but after that of $\pi(k)$. But this is impossible, as $\pi(j)$ must be added before $\pi(k)$. So a stack-sortable permutation cannot contain a wedge. Conversely, if a permutation avoids 231, it can be sorted according to the strategy above. The algorithm will fail to sort only if it forces us to remove an element from the top of the stack which is not the largest element which has yet to be removed. Then the top element of the stack is smaller than the next element to be added, but larger than some later element. These three elements constitute a wedge. □

It remains to count the permutations which contain no wedges. Let c_n be the number of wedge-free permutations of length n.

Assume by induction that we have enumerated c_m for $m < n$, and consider an arbitrary wedge-free permutation. Let j be the position such that $\pi(j) = n$. Then the substring $\pi^L = (\pi(1), \pi(2), \ldots, \pi(j-1))$ must consist of the elements $(1, 2, \ldots, j-1)$. For if not, it must contain some element $\pi(i) \geq j$, while the substring $\pi^R = (\pi(j+1), \pi(j+2), \ldots, \pi(n))$ would contain some $\pi(k) < j$. But then we would have a wedge, since $i < j < k$ and $\pi(k) < \pi(i) < \pi(j) = n$.

So the elements of the left substring and the right substring are determined by the position of n. But the permutations π^L and π^R, being subsequences of π, must themselves be wedge free. It is also sufficient that they are, since if all the elements of π^L are less than all those of π^R there cannot be any subsequence of type 231 with elements in both the left and right substrings. But since an admissible left substring is just a wedge-free permutation of j cards, and the admissible right substrings are permutations sequences likewise counted by the wedge-free permutations of $n-j$ cards, we can invoke the induction hypothesis.

Using the induction hypothesis and summing over j, we thus establish that

$$c_n = c_{n-1} + \sum_{j=2}^{n-1} c_{j-1}c_{n-j} + c_{n-1} = \sum_{j=1}^{n} c_{j-1}c_{n-j}, \tag{1}$$

where we set $c_0 = 1$.

This is the famous recurrence relation for the Catalan numbers [1]. That is, $c_n = \frac{1}{n+1}\begin{pmatrix} 2n \\ n \end{pmatrix}$.

In the context of formal power series, if we let $C = \sum_{i=0}^{\infty} c_i x^i$, then C satisfies the equation $C^2 = \frac{C-1}{x}$.

We present a second, more direct, way to enumerate the stack-sortable permutations. This is of interest both for its elegance and because it provides a natural derivation of the Catalan numbers in terms of a difference.

If a given permutation can be sorted, then we have seen that there is a unique procedure for sorting it: if the next element to be added to the stack is larger than the element on top of the stack, remove the top element from the stack; if it is smaller, add it to the stack; if the stack is empty, add to it; if p_n has been added clear the stack. Consider the sequence of operations which must be performed to sort a given permutation, writing '(' if an element is added to the stack and ')' if one is removed.

Then we have a sequence of n open and n closed parentheses, since each p_i must be added to the stack once and removed once. Also this sequence must be well-formed in the sense that, working from left to right, there will always be a surplus of open parentheses, so whenever we encounter a ')' we will be able to supply it with a mate somewhere to its left. This is so because we can never remove more elements from the stack than have been added to it.

André gave, in 1878, an enumeration of the well-formed sequences. To enumerate the well-formed sequences of n open and n closed parentheses, observe that there is a bijection between *all* sequences of $n-1$ open and $n+1$ closed parentheses and those sequences of n open and n closed parentheses which are *not* well-formed. If a sequence is not well-formed, there must be a leftmost occurence of a ')' which has equally many '('s and ')'s to its left. Replace each '(' to the right of this by a ')' and each ')' to its right by a '('. We thus obtain a sequence of $n-1$ open and $n+1$ closed parentheses. Similarly, given a sequence of $n-1$ open and $n+1$ closed parentheses, there must be a leftmost ')' which has as many '('s as ')'s to its left. Invert each parenthesis to the right of this location to obtain a sequence with n of each type of symbol, which is not well-formed.

The number of well-formed sequences of n open and n closed parentheses is simply the total number of sequences of n '('s and n ')'s, less the number of such sequences which are not well-formed. This number is seen to be, as before in (1),

$$c_n = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}.$$

The idea of the first proof above was to characterize those permutations which can be sorted with a stack, by showing a pattern which appears in a permutation if and only if it cannot be sorted. This idea, of characterizing with a list of forbidden subpatterns the set of permutations which can be sorted by a given procedure, is quite general.

For $\tau = (\tau(1), \tau(2), \ldots, \tau(k)) \in S_k$, a permutation $\pi = (\pi(1), \pi(2), \ldots, \pi(n)) \in S_n$ is τ-avoiding iff there is no $1 \leq i_{\tau(1)} < i_{\tau(2)} < \cdots < i_{\tau(k)} \leq n$ such that $\pi(i_1) < \pi(i_2) < \cdots < \pi(i_k)$. Such a $(\pi(i_{\tau(1)}), \pi(i_{\tau(2)}), \ldots, \pi(i_{\tau(k)}))$ is called a subsequence of type τ.

For instance, a wedge is simply a subsequence of type $2,3,1$ and our theorem above characterized the stack-sortable permutations as those which are $2,3,1$-avoiding.

Given $\tau \in S_k$, let us write $S_n(\tau)$ for the set of τ-avoiding permutations of length n.

As an example of the generality of forbidden-subsequence classifications, the permutations which can be sorted using a double-ended queue are characterized by Pratt [4], who finds an infinite family of restrictions, with 4 restrictions of each odd length greater than or equal to 5.

Now consider the following extension of our card game. Shuffle the cards and play exactly as before, using the same simple strategy. This time, when all the cards have been placed on the output pile, pick up the output, turn it faceup and begin the game again. Now how many of the $n!$ starting positions result in wins after the second pass?

Our first thought is to procede, as in Knuth's treatment of a single stack or in Pratt's treatment of a double-ended queue, to characterize the winning positions in terms of forbidden subsequences. This we can do, after making a few more definitions. First, if we play the game with π as our input, let us call the resulting output $\Pi(\pi)$. We ask for how many permutations π of length n is $\Pi(\Pi(\pi))$ equal to the identity. Let us call a permutation π *two-stack sortable* if $\Pi(\Pi(\pi))$ is the identity permtation.

We will say that an element p precedes an element q in a permutation ρ if $\rho^{-1}(p) < \rho^{-1}(q)$. For instance, in $\rho = (3,5,2,4,1)$, the element 5 precedes 4 because $\rho^{-1}(5) = 2$ and $\rho^{-1}(4) = 4$.

Lemma 2. *If $\pi \in S_n$, and $1 \leq a < b \leq n$, and if a precedes b in π then a precedes b in $\Pi(\pi)$.*

Proof: Since a precedes b in π, a enters the stack before b. When b is processed, either a has already been removed from the stack, in which case a will precede b in $\Pi(\pi)$, or a must be removed from the stack to accomodate the larger element b. □

Lemma 3. *If $\pi \in S_n$, and $1 \leq a < b \leq n$, and if b precedes a in π, then b precedes a in $\Pi(\pi)$ if there exists $c > b$ such that b precedes c and c precedes a in π. If there is no such c, a precedes b in $\Pi(\pi)$.*

Proof: If there is a c satisfying the given conditions, then b, c and a form a wedge. In this case, b must be removed from the stack before c is placed on. Since c is placed on the stack before a, this will cause b to precede a in $\Pi(\pi)$.

On the other hand, if there is *no* c satisfying the given conditions, then b remains on the stack until a is processed. Since $a < b$, a will be placed on the stack above b, and so a precedes b in $\Pi(\pi)$. □

Lemma 4. *If b and a form an inversion in $\Pi(\pi)$, that is if b precedes a in $\Pi(\pi)$ but $b > a$ then there is a wedge b, c, a in π for some $c > b$.*

Proof: An easy consequence of the two previous lemmata. If $b > a$, either b precedes a in π or *vice versa*. Only in the case that b precedes a and a larger element c is interposed between the two might b precede a in $\Pi(\pi)$. □

We are now ready to present a characterization of the two-stack sortable permutations. This theorem appeared in [9] as part of a more general result.

Theorem. *A permutation $\pi \in S_n$ fails to be two-stack sortable if it contains a subsequence of type 2341, or a subsequence of type 3241 which is not part of a subsequence of type 35241. If it contains no such subsequence, π is two-stack sortable.*

Proof: The proof is an exercise in the application of the basic lemmata.

First suppose π has a subsequence of either of the given forms, and consider $\Pi(\pi)$. First consider a subsequence of type 2341, consisting of the elements b, c, d, a where $a < b < c < d$. Since b precedes c in π and $b < c$, it follows that b will precede c in $\Pi(\pi)$, regardless of the other elements of π. Also, because c, d, a form a wedge in π, c will precede a in $\Pi(\pi)$. Therefore, the elements b, c, a appear in that order in $\Pi(\pi)$, where they form a wedge. Since therefore $\Pi(\pi)$ is not one-stack sortable by lemma 1, it follows that π is not two-stack sortable.

Second, consider a subsequence of type 3241, say consisting of the elements c, b, d, a, where there is no element larger than d which follows c but precedes b. There are two cases: either there is an element $x > c$ which follows c but precedes b, or there is no such element. If there is such an x, by assumption $c < x < d$, and consequently c, x, d, a is of type 2341 and we are back in the case of the preceding paragraph. Otherwise, if there is no such element $x > c$, then b precedes c in $\Pi(\pi)$. And since c, d, a is a wedge in π, c precedes a in $\Pi(\pi)$. Once again, b, c, a form a wedge in $\Pi(\pi)$.

It follows that if π has one of the forbidden subsequences, then π fails to be two-stack sortable.

Conversely, we can show that if $\Pi(\pi)$ fails to be one-stack sortable; that is, if it contains a wedge, then π must contain one of the two forbidden subsequences.

Suppose that b, c, a form a wedge in $\Pi(\pi)$. We look at two cases; either b precedes c in π or *vice versa.*

First suppose b precedes c in π. Lemma 4 guarantees a wedge c, x, a in π. But then b, c, x, a is a subsequence of type 2341.

If c precedes b in π, then there can be no $x > c$ such that c precedes x and x precedes b in π. But since c precedes a in $\Pi(\pi)$, there is some wedge c, y, a in π. Since $y > c$, y cannot precede b in π, by the remark in the first sentence of this paragraph. Therefore b precedes y, and c, b, y, a is a sequence of type 3241. Again by the remark in the first sentence, this subsequence is not part of a subsequence of type 35241.

Thus if b, c, a is a wedge in $\Pi(\pi)$, we see that π has a subsequence of one of the two forms given in the statement of the theorem. So if π is not two-stack sortable, it has one of the forbidden subsequences. □

The above theorem does not, strictly speaking, give a characterization in terms of forbidden subsequences, in the usual sense. That is, it does not permit us to write the class of two-stack sortable permutations as an intersection of sets of the form $S_n(\tau)$, because of the unusual restriction that "forbidden" subsequences of type 3241 are permitted if each is mitigated by being part of a 35241. Nevertheless, it has much the same flavour of our characterization of one-stack-sortable permutations as $S_n(231)$.

We should like to exploit this characterization to enumerate the two-stack-sortable permutations. Unfortunately, we have not been able to do so. Enumerative problems involving forbidden subsequences are notoriously hard. [5] [9] But there is a simple closed form which is consistent with the known data. In the table below we give known values of the number of two-stack-sortable permutations (2) and, for comparison, the Catalan numbers (1).

n	1	2	3	4	5	6	7	8	9	10	11
1	1	2	5	14	42	132	429	1430	4862	16 796	58 786
2	1	2	6	22	91	408	1938	9614	49 335	260 130	1 402 440

The data in this table permit us to make the following conjecture.

Conjecture. *The number of permutations of length n which are two-stack-sortable is* $\frac{2}{(n+1)(2n+1)}\binom{3n}{n}$.

The form of this conjecture strongly suggests a proof in terms of formal power series. In fact, the sequence $b(n) = \frac{2}{(n+1)(2n+1)}\binom{3n}{n}$ has appeared before as the result of an argument involving power series. This was in a paper of Tutte counting the number of non-separable planar graphs. [8] [6] In what follows, we give a brief summary of Tutte's work as it pertains to our conjecture.

Following an intricate graph-theoretic argument in [7], Tutte reveals the following formula for *the number a_n of rooted maps with n edges:*

$$\frac{2(2n)!3^n}{n!(n+2)!}$$

He shows, by an application of Lagrange's theorem [10], that the generating function $A(x) = \sum_{n=1}^{\infty} a_n x^n$ satisfies the following parametric equations:

$$\begin{aligned} \xi &= 1 + 3x\xi^2 \\ A(x) &= \frac{1}{3}(3-\xi)(\xi-1). \end{aligned}$$

He next lets $B(x) = \sum_{n=1}^{\infty} b_n x^n$ be the generating function for the *non-separable rooted maps with n edges*, and concludes, after showing how each planar map has a uniquely determined nonseparable core, and can be built up from this core by a process of edge splitting, that $A(x)$ and $B(x)$ satisfy the following functional equation:

$$A(x) = B(x\{1 + A(x)\}^2)$$

Tutte solves this equation by writing $u = x\{1 + A(x)\}^2$ and performing the following algebraic manipulations:

$$\begin{aligned} u &= x\{1 + a(x)\}^2 \\ u &= \frac{\xi - 1}{3\xi^2}\{1 + \frac{1}{3}(3-\xi)(\xi-1)\}^2 \\ u &= \frac{\xi - 1}{3\xi^2}\{1 + \frac{2}{3}(3-\xi)(\xi-1) + \frac{1}{9}(\xi-3)^2(\xi-1)^2\} \\ 27u &= \frac{\xi - 1}{\xi^2}\{9 + 6(3-\xi)(\xi-1) + (\xi-3)^2(\xi-1)^2\} \\ 27u &= \tfrac{\xi-1}{\xi^2}\{9 - 6\xi^2 + 24\xi - 6 + \xi^4 - 8\xi^3 + 22\xi^2 - 24\xi + 1\} \\ 27u &= \frac{\xi - 1}{\xi^2}\{16\xi^2 - 8\xi^3 + \xi^4\} \\ 27u &= -(1-\xi)(4-\xi)^2 \end{aligned}$$

He then sets $\eta = 1 - \xi$, so that

$$\begin{aligned} A(x) = B(u) &= -\tfrac{1}{3}\eta(2+\eta) \\ \eta &= \frac{-27u}{(3+\eta)^2}, \end{aligned}$$

and finally applies Lagrange's theorem again, to obtain:

$$\begin{aligned} B(u) &= \sum_{n=1}^{\infty} \frac{(-27u)^n}{n!}\left[\frac{d^{n-1}}{da^{n-1}}\left\{\frac{1}{(3+a)^{2n}}\left(\frac{-2(1+a)}{3}\right)\right\}\right]_{a=0} \\ &= -\frac{2}{3}\sum_{n=1}^{\infty} \frac{(-27u)^n}{n!}\left[\frac{d^{n-1}}{da^{n-1}}\left\{\frac{3+a-2}{(3+a)^{2n}}\right\}\right]_{a=0} \\ &= -\frac{2}{3}\sum_{n=1}^{\infty} \frac{(-27u)^n}{n!}\left[\frac{d^{n-1}}{da^{n-1}}\left\{\frac{1}{(3+a)^{2n-1}} - \frac{2}{(3+a)^{2n}}\right\}\right]_{a=0} \end{aligned}$$

$$= \frac{2}{3}\sum_{n=1}^{\infty}\frac{(27u)^n}{n!}\left\{\frac{(2n-1)(2n)\cdots(3n-3)}{3^{3n-2}} - \frac{2(2n)(2n+1)\cdots(3n-2)}{3^{3n-1}}\right\}$$

$$= 2\sum_{n=1}^{\infty}\frac{u^n}{n!}\frac{(3n-3)!}{(2n-1)!}\{3(2n-1)-2(3n-2)\}$$

$$= 2\sum_{n=1}^{\infty}\frac{(3n-3)!}{n!(2n-1)!}u^n$$

The coefficients of this final power series are the terms of our conjecture. Can any application of these generating function results be made to our stack-sorting problem?

At a simple level, it might be possible to use the identity

$$\frac{2}{n+1}\cdot\frac{1}{(n+1)(2n+1)}\binom{3n}{n} = \frac{2}{n+1}\left[3\binom{3n}{2n} - 2\binom{3n+1}{2n+1}\right]$$

implicit in the last step of the Tutte derivation, as an analogue for

$$\frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1}$$

by making appropriate definitions about well-formed and not-well-formed sorting operations with two stacks. But a complete understanding of the relevance of generating functions would give more insight into the problem, and might suggest information about the general case of repeated sorting through k stacks.

The data for k stacks, through permutations of length 11, is given in the table below. The number in the k-th row and n-th column is the number of permutations of length n which sort after k passes but not after $k-1$. Let us call such permutations *exactly k-sortable.* Thus the two rows of the previous table represent the partial sums down each column of respectively the first two, and the first three rows.

$k\backslash n$	1	2	3	4	5	6	7	8	9	10	11
0	1	1	1	1	1	1	1	1	1	1	1
1		1	4	13	41	131	428	1429	4861	16 795	58 785
2			1	8	49	276	1509	8184	44 473	243 334	1 343 654
3				2	23	198	1556	11 812	88 566	662 732	4 975 378
4					6	90	982	9678	91 959	863 296	8 093 662
5						24	444	5856	68 820	775 134	8 618 740
6							120	2640	40 800	555 828	7 201 188
7								720	18 360	325 200	5 033 952
8									5040	1461 60	2 918 160
9										40 320	1 310 400
10											362 880

It is evident from this table that no permutation of length n requires n passes to sort, but that some do require $n-1$ passes. The number of these appears to be $(n-2)!$. We prove this result by characterizing these permutations.

If $\pi \in S_n$ is given by $\pi = (a_1, a_2, \ldots, a_{k-1}, n, b_1, b_2, \ldots, b_{n-k})$, and $\pi^L = (a_1, a_2, \ldots, a_{k-1})$, $\pi^R = (b_1, b_2, \ldots, b_{n-k})$, then we write $\pi = \pi^L n \pi^R$. In abbreviating a permutations in this fashion, we use greek letters for permutations and permutation sequences, and reserve roman letters and arabic numerals for individual elements of a permutation.

Lemma 5. *If for $\pi \in S_n$, $\pi = \pi^L n \pi^R$, then $\Pi(\pi) = \Pi(\pi^L)\Pi(\pi^R)n$.*

Proof: Consider the application of the sorting algorithm to π. When the element n is reached, all the elements of π^L and none of π^R have been processed. Some may remain on the stack. The element n is larger than every element on the stack, and so the stack is cleared. Thus the elements of π^L are output as $\Pi(\pi^L)$, exactly as though an end-of-input had been reached. Next the element n is entered onto the stack. As it is larger than every element of π^R, n remains on the stack until the end-of-input is reached. So n does not interfere with the processing of π^R, which is output as $\Pi(\pi^R)$. Finally, an end-of-input is reached, and n is removed from the stack. □

As evidently the largest element must appear in *some* position of the permutation π, the sorted permutation $\Pi(\pi)$ has the largest element occupying the final position. Since after one pass, the largest element has been shifted to the end, two passes will shift the largest two elements to the end, and so on. We prove this in the following lemma.

Lemma 6. *If $\rho = \Pi^k(\pi)$ for any $\pi \in S_n$, then $\rho(j) = j$, for $n-k+1 \leq j \leq n$.*

Proof: The proof is by induction. The statement is vacuously true for $k = 0$, and true for $k = 1$ by lemma 5.

If $\rho = \Pi^{k+1}(\pi)$, then $\rho = \Pi(\Pi^k(\pi))$. By the induction hypothesis, $\Pi^k(\pi)$ has its k largest elements in order in the final k positions. When the first of these is encountered, it will clear the stack, being larger than any previous input. The rest of the elements are encountered in increasing order, and so are simply passed through.

So if $\Pi^k(\pi) = (a_1, a_2, \ldots, a_{n-k}, n-k+1, \ldots, n)$, we can take $\alpha = (a_1, a_2, \ldots, a_{n-k}) \in S_{n-k}$. By the remarks of the previous paragraph, $\Pi^{k+1}(\pi) = \Pi(\Pi^k(\pi)) = \Pi(\alpha(n-k+1)\ldots(n-1)(n)) = \Pi(\alpha)(n-k+1)\ldots(n-1)(n)$. By lemma 5, $\Pi(\alpha)$ has $n-k$ for its final element. Hence $\Pi^{k+1}(\pi)$ has $(n-k, n-k+1, \ldots, n-1, n)$ for its final elements. □

These observations lead to the observation that the sorting process always terminates, after at most n iterations. Actually, $n-1$ iterations will do, for if the last $n-1$ positions of a permutation of length n are occupied by $2, 3, \ldots, n$, then clearly the first element is 1.

Conversely, we prove that for every n there are some permutations of length n which actually require $n-1$ passes through the stack. At the same time, we find the number of these permutations.

We have noted that a pass through the stack moves the largest element to the end of the permutation. In other words, for all $\pi \in S_n$ it is the case that $\Pi(\pi) = \alpha n$ for some $\alpha \in S_{n-1}$. Since then $\Pi^k(\pi) = \Pi^{k-1}(\Pi(\pi)) = \Pi^{k-1}(\alpha n) = \Pi^{k-1}(\alpha)n$, it is true that π is exactly k-stack sortable if and only if α is exactly $(k-1)$-stack sortable.

We use this observation in the following inductive proof.

Theorem. *A permutation $\pi \in S_n$ is exactly $(n-1)$-stack sortable if and only if $\pi = \rho n 1$ for some $\rho \in S_{n-2}$.*

Proof: The statement is true for $n = 3$ as the only permutation in S_3 which is exactly two-stack sortable is 231. (It is also true for $n = 2$.)

Now assume the truth of the given statement for $n-1$. A permutation $\pi \in S_n$ is exactly $(n-1)$-stack sortable if and only if $\Pi(\pi) = \alpha n$ where $\alpha \in S_{n-1}$ is exactly $(n-2)$-stack sortable.

We check that likewise π has the form $\rho n1$ if and only if α has the form $\beta, (n-1), 1$.

The proof will then follow by induction. The two classes, of permutations having the given form, and of permutations requiring the maximum number of passes to sort, are equivalent for $n-1$ by the induction hypothesis. The arguments of the previous paragraph will show them also to be equivalent for n.

First let $\pi = \rho n1$. Then $\Pi(\pi) = \Pi(\rho)1n$. Since $\Pi(\rho)$ will have the form $\sigma, n-1$, we can write $\Pi(\pi) = \alpha n$, where $\alpha = \sigma, n-1, 1$. This is the desired form.

Conversely, suppose

$$\Pi(\pi) = \sigma, n-1, 1, n \tag{2}$$

Write π in the form $\pi^L n \pi^R$, so that

$$\Pi(\pi) = \Pi(\pi^L)\Pi(\pi^R)n \tag{3}$$

Since both $\Pi(\pi^L)$ and $\Pi(\pi^R)$ must end with an ascent if they have length greater than 1, a comparison of the forms 2 and 3 reveals that $\Pi(\pi^R) = 1$. Then $\pi = \pi^L n 1$, the desired form. $\square$

It is immediate from this classification theorem that the number of exactly $(n-1)$-sortable permutations of length n is $(n-1)!$.

By similar techniques, we have also been able to enumerate the permutations of length n which are exactly $(n-2)$-sortable. There are $\frac{7}{2}(n-2)! + (n-3)!$ of these.

References

[1] EUGENE CATALAN, Note sur une Équation aux différences finies, J. de Mathématiques pures et appliquées, 3 (1838), 508-516

[2] D. E. KNUTH, "The art of computer programming, vol. 1," Addison-Wesley, Reading MA, 1973 (second edition), p. 239

[3] LAMÉ, Extrait d'une lettre de M. Lamé à M. Liouville sur cette question: Un polygone convexe étant donné, de combien de manieères peut-on le partager en triangles au moyen de diagonales?, J. de Mathématiques pures et appliquées, 3 (1838), 505-507

[4] V. PRATT, Proc. ACM Symp. Theory of Computing 5 (1973), 268-277.

[5] RODICA SIMION AND FRANK W. SCHMIDT, Restricted Permutations, European Journal of Combinatorics 6 (1985), 383-406

[6] N.J.A. SLOANE, "A Handbook of Integer Sequences," Academic Press (1973)

[7] W. T. TUTTE, A census of slicings, Canadian Jornal of Mathematics 14 (1962), 708-722

[8] W. T. TUTTE, A census of planar maps, Canadian Journal of Mathematics 15 (1963), 249-271

[9] JULIAN WEST, "Permutations with forbidden subsequences; and, stack-sortable permutations", doctoral thesis, M.I.T. 199

[10] E.T. WHITTAKER AND G.N.WATSON, "A course of modern analysis," Cambridge (1940)