YOUNG STRAIGHTENING IN A QUOTIENT S_N-MODULE

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ABSTRACT. We describe a straightening algorithm for the action of S_n on a certain graded ring $\mathcal{R}_\mu$. The ring $\mathcal{R}_\mu$ appears in the work of C. de Concini and C. Procesi [1], and T.Tanisaki [7] and more recently in the work of A. Garsia and C. Procesi[3]. This ring is a graded version of the permutation representation resulting from the action of S_n on the left cosets of a Young subgroup. As a corollary of our straightening algorithm we obtain a combinatorial proof of the fact that the top degree component of $\mathcal{R}_\mu$ affords the irreducible representation of S_n indexed by μ.

Introduction

This paper is concerned with certain graded S_n−modules $\mathcal{R}_\mu$ studied by H. Kraft in [5], C. de Concini and C. Procesi in [1] and more recently by A. Garsia and C. Procesi in [3].

Given $\mu = (\mu_1, \ldots \mu_k)$ a partition of n let p^μ denote the character of the permutation representation resulting from the action of S_n on the left cosets of the Young subgroup

$$S_{\mu_1} \times S_{\mu_2} \times \ldots \times S_{\mu_k}$$

It is well known [8] that p^μ contains the irreducible character χ^μ (in the Young-Frobenius indexation) with multiplicity 1. It follows by combining the results of [5] and [1] that $\mathcal{R}_\mu$ yields a graded version of this representation. In particular its graded character is given by a polynomial $p^\mu(q)$ which reduces to p^μ for $q = 1$. We refer to [3] for an elementary presentation of the background material. It was conjectured by H. Kraft in [5] that χ^μ is given by the top degree component of $p^\mu(q)$. This conjecture was first proved by de Concini and Procesi in their 1981 paper [1]. Another proof can be found in the 1991 paper of Garsia and Procesi [3]. In this paper the authors describe the graded character $p^\mu(q)$ as a linear combination of irreducibles χ^λ with coefficient equal to the cocharge version of the Kostka-Foulkes polynomials $K_{\lambda\mu}(q)$ [6]. Garsia and Procesi also construct a homogeneous basis $\mathcal{B}_\mu$ for the ring $\mathcal{R}_\mu$, and one of their questions was to know if the action of S_n on the homogeneous component of highest degree of $\mathcal{B}_\mu$ *naturally* yields the irreducible representation of S_n indexed by μ. Our objective in this paper is to answer this question by developing a straightening law for the action of S_n on the basis $\mathcal{B}_\mu^{top}$ of $\mathcal{R}_\mu^{top}$, the homogeneous component of highest degree of $\mathcal{R}_\mu$. As a corollary we obtain an alternative, direct combinatorial proof of Kraft's conjecture.

One way to obtain the irreducible representation of S_n indexed by μ is through the action of S_n on Young's natural set of units $\{E_{T_i,T_1}\}_{i=1}^{\eta_\mu}$ (where T_i for $i = 1, \ldots n$ are the standard tableaux

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of shape μ). More precisely, one obtains the matrices of the representation by finding the images $\sigma E_{T_i,T_1}$ for each of the permutations σ of S_n. For a given standard tableau T_i let

$$\sigma E_{T_i,T_1} = \sum_{j=1}^{\eta_\mu} a_j(T_i) E_{T_j,T_1}.$$

We shall show that the action of S_n on a basis $\mathcal{B}_\mu^{top}$ of $\mathcal{R}_\mu^{top}$ (the homogeneous component of highest degree of $\mathcal{R}_\mu$) is an action identical to the one on Young's natural units. For this we shall calculate the images of the elements of $\mathcal{B}_\mu^{top}$ under the action of the permutations σ of S_n. As we shall see in section 1 the elements $m(T_i)$ of $\mathcal{B}_\mu^{top}$ are monomials indexed by the set of standard tableaux of shape μ. Thus if we let

$$\sigma m(T_i) = \sum_{j=1}^{\eta_\mu} b_j(T_i) m(T_j)$$

our result will be that

$$a_j(T_i) = b_j(T_i)$$

for all i and j.

This paper is divided in four sections. In section 1 we give the description of the ring $\mathcal{R}_\mu$ as a quotient of the polynomial ring $Q[x_1, x_2, \ldots, x_n]$. We also describe the basis $\mathcal{B}_\mu^{top}$. These results are due to T. Tanisaki, A. Garsia, and C. Procesi. In section 2 we develop some of the congruence relations that will be needed for working on the quotient ring $\mathcal{R}_\mu$. Section 3 is devoted to the construction of a straightening algorithm. More precisely we give a rule for finding the coefficients $b_j(T_i)$ in the expansion $\sigma m(T_i) = \sum_{j=1}^{\eta_\mu} b_j(T_i) m(T_j)$. In section 4 we show that this is the same as Young's straightening law.

1. The ring $\mathcal{R}_\mu$.

Some of the results we need have been recently described in a paper of A. Garsia and C. Procesi [3]. We shall adopt here their notation. The partitions of n will be represented by $n-$vectors:

$$\mu = (\mu_1, \mu_2, \ldots, \mu_n) \quad (0 \le \mu_1 \le \mu_2 \ldots \le \mu_n)$$

or by their corresponding Ferrers' diagrams drawn according to the French notation. Figure $1(a)$ gives the Ferrers' diagram associated to the partition $\mu = (0,0,1,2,2)$.

$\mu = (0,0,1,2,2)$ (a) $\qquad$ $\mu = (0,0,0,2,3)$ (b)

Figure 1

The number of positive components of μ is the height of μ and is denoted by $h(\mu)$. The conjugate of a partition μ is the partition μ' whose Ferrers' diagram is the transpose of the diagram of μ. In our previous example $h(\mu) = 3$, and $\mu' = (0,0,0,2,3)$ (see figure $1(b)$). Let $X_n = \{x_1, x_2, \ldots, x_n\}$ be an ordered set of commuting variables. For any integer r ($\le n$) and for any subset $S \subseteq X_n$ such that $r \leq |S|$, let $e_r(S)$ be the r^{th} elementary symmetric function of the variables in S, that is

$$e_r(S) = \sum_{\substack{i_1 < i_2 < \ldots < i_r \\ x_{i_j} \in S}} x_{i_1} x_{i_2} \ldots x_{i_r}$$

Define $d_k(\mu) = \sum_{i=1}^{k} \mu_i{}'$ for all $k = 1, \ldots, n$. Let $\mathbf{Q}[x_1, x_2, \ldots, x_n]$ be the ring of polynomials in the variables $x_1, \ldots x_n$ with rational coefficients. Let $\mathcal{I}_\mu$ be the ideal (in $\mathbf{Q}[x_1, x_2, \ldots, x_n]$) generated by the collection of partial elementary symmetric functions:

$$\mathcal{C}_\mu = \{e_r(S) \mid k - d_k(\mu) < r \leq k,\ |S| = k,\ S \subseteq X_n\}\ . \tag{1.1}$$

The following presentation of the rings $\mathcal{R}_\mu$ is due to T. Tanisaki (see [7]). For a given μ, the ring $\mathcal{R}_\mu$ is given by the quotient

$$\mathcal{R}_\mu = \mathbf{Q}[x_1, x_2, \ldots, x_n]/\mathcal{I}_\mu\ .$$

For example when $\mu = (0,0,1,2,2)$ we have $\mu' = (0,0,0,2,3)$ while $(d_1(\mu), \ldots, d_n(\mu)) = (0,0,0,2,5)$ and $(1 - d_1(\mu), \ldots, 5 - d_5(\mu)) = (1,2,3,2,0)$. Schematically we can represent the pairs (k, r) satisfying the condition $k - d_k(\mu) < r \leq k$ by the diagram of figure 2.

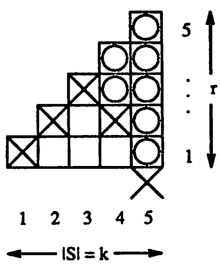

Figure 2

The squares with coordinates given by $(k, k - d_k(\mu))$ for $k = 1, \ldots n$ are marked with an X. From equation (1.1) it is easy to see that the partial elementary symmetric functions $e_r(S)$ belonging to $\mathcal{C}_\mu$ are the ones for which the points $(|S|, r)$ are given by the coordinates of all the squares above the one marked with an X. Thus in our example, $\mathcal{I}_{122}$ is generated by the following 15 partial elementary symmetric functions:

$$\begin{gathered} e_1(X_5),\ e_2(X_5),\ e_3(X_5),\ e_4(X_5),\ e_5(X_5), \\ e_3(x_1,x_2,x_3,x_4),\ e_3(x_1,x_2,x_3,x_5),\ e_3(x_1,x_2,x_4,x_5), \\ e_3(x_1,x_3,x_4,x_5),\ e_3(x_2,x_3,x_4,x_5),\ x_1x_2x_3x_4, \\ x_1x_2x_3x_5,\ x_1x_2x_4x_5,\ x_1x_3x_4x_5,\ x_2x_3x_4x_5\ . \end{gathered}$$

It is easy to see that there is, in general, redundancy among the generators of $\mathcal{I}_\mu$. For example, consider

$$e_4(X_5) = x_1x_2x_3x_4 + x_1x_2x_3x_5 + x_1x_2x_4x_5 + x_1x_3x_4x_5 + x_2x_3x_4x_5\ . \tag{1.2}$$

Note that each of the monomials of the right hand side of equation (1.2) is also included in $\mathcal{I}_\mu$. Thus it is unnecessary to add $e_4(X_5)$. We have studied which partial elementary symmetric functions are sufficient to generate the ideals $\mathcal{I}_\mu$. But before we summarize these results in the next two lemmas, observe first that:

Remark : for any partition μ of n it is always true that $n - d_n(\mu) = 0$; thus for all $i = 1, \ldots, n$, $e_i(X_n)$ will always belong to $\mathcal{C}_\mu$.

Lemma 1.1

Let μ be a partition of n and let S be a proper subset of X_n . Let $1 \leq k < n$ $(k \neq n)$ and $r + 1 \leq k$. If for all subsets S of cardinality k we have that $e_r(S) \in \mathcal{I}_\mu$ then

$$e_{r+1}(S) \in \mathcal{I}_\mu$$

for all subsets S of cardinality k .

Proof

Let $S = \{x_{s_1}, x_{s_2}, \ldots x_{s_k}\}$. Since k is strictly less than n there exists a variable say x_* that belongs to X_n but not to S . Define S_i to be the set obtained from S by removing the variable x_{s_i}, and define S_i^* to be the one obtained from S by replacing the variable x_{s_i} by the variable x_*. We shall prove that for any S satisfying the conditions of the lemma the following equation holds

$$(r+1)e_{r+1}(S) \;=\; \sum_{i=1}^{k} x_{s_i} e_r(S_i^*) \;-\; (r)x_* e_r(S)$$

Note that the left hand side of equation (1.3) is by definition

$$(r+1)e_{r+1}(S) = (r+1) \sum_{\substack{i_1 < \cdots < i_{r+1} \\ x_{s_{i_j}} \in S}} x_{s_{i_1}} \cdots x_{s_{i_{r+1}}}$$

On the other hand the right hand side of equation (1.3) is equal to:

$$= \sum_{i=1}^{k} x_{s_i} \Big[x_* e_{r-1}(S_i) + e_r(S_i) \Big] - (r)x_* e_r(S)$$

$$= \sum_{i=1}^{k} x_{s_i} x_* e_{r-1}(S_i) + \sum_{i=1}^{k} x_{s_i} e_r(S_i) - (r)x_* e_r(S) \tag{1.4}$$

A moment of thought reveals that in equation (1.4) the terms containing x_* cancel out, leaving

$$\sum_{i=1}^{k} x_{s_i} \sum_{\substack{i_1 < \cdots < i_r, \\ x_{s_{i_j}} \in S_i}} x_{s_{i_1}} \cdots x_{s_{i_r}}$$

which is precisely $(r+1)e_{r+1}(S)$. This completes the proof of the lemma since for all $i = 1, \ldots k$ we have that $e_r(S_i^*)$ belongs to $\mathcal{I}_\mu$. ♣

Lemma 1.2

Let μ be a partition of n, and let k be a fixed integer such that $k \leq n$. If for every subset S of X_n, of cardinality $k - 1$, there exists an r such that $e_r(S)$ belongs to $\mathcal{I}_\mu$, then for all subsets S^+ of X_n of cardinality k we have:

$$e_r(S^+) \in \mathcal{I}_\mu \; .$$

Proof

For any given set S^+, let $\{S_j\}_{j=1}^k$ be the collection of all subsets of S^+ of cardinality $k-1$. The proof follows from the fact that:

$$(k-r)e_r(S^+) = \sum_{j=1}^{k} e_r(S_j) \tag{1.5}$$

Indeed it is easy to see that every monomial $x_{s_{i_1}} \ldots x_{s_{i_r}}$ of $e_r(S^+)$ appears exactly $(k-r)$ times on the right hand side of equation (1.5). Thus since each $e_r(S_j)$ belongs to $\mathcal{I}_\mu$ so does $e_r(S^+)$. ♣

We remark that a similar argument shows that if for all $S \subseteq X_n$ of cardinality k, $e_r(S) \in \mathcal{I}_\mu$ then $e_{r+1}(S') \in \mathcal{I}_\mu$ for all $S' \subseteq X_n$ of cardinality $k+1$. Note that this also follows from the two previous lemmas, except for the case where $r = k = n-1$.

Applying these two lemmas to our previous example yields that $\mathcal{I}_{122}$ is generated by the following seven partial elementary symmetric functions:

$$\mathcal{C}^*_\mu = \{e_1(X_5),\ e_2(X_5),\ e_3(x_1,x_2,x_3,x_4),\ e_3(x_1,x_2,x_3,x_5),\ e_3(x_1,x_2,x_4,x_5),\\ e_3(x_1,x_3,x_4,x_5),\ e_3(x_2,x_3,x_4,x_5)\}\ .$$

The ideals $\mathcal{I}_\mu$ are generated by homogeneous polynomials, thereby inducing a natural grading on each ring $\mathcal{R}_\mu$. The symmetric group S_n acts naturally on $\mathbf{Q}[x_1,\ldots,x_n]$ by simply permuting the variables. That is:

$$\sigma P(x_1,\ldots,x_n) = P(x_{\sigma_1},x_{\sigma_2},\ldots,x_{\sigma_n})$$

for each $\sigma \in S_n$ and each polynomial $P(x_1,\ldots,x_n) \in \mathbf{Q}[x_1,x_2,\ldots,x_n]$. Clearly this action leaves each of the ideals $\mathcal{I}_\mu$ invariant. Thus one can define an action of S_n on each of the quotient rings $\mathcal{R}_\mu$. One also observes that the natural grading of $\mathcal{R}_\mu$ is preserved by this action of S_n. As we mentioned in the introduction we shall be concerned here with the action of S_n on the homogeneous component of highest degree of a given $\mathcal{R}_\mu$. A. Garsia and C. Procesi showed in [3, proposition 4.2] that each ring $\mathcal{R}_\mu$ has a basis of homogeneous monomials that can be constructed as follow. First recall that there is a Ferrers diagram associated to any partition μ of n. If one fills the diagram using all the integers 1 to n the resulting diagram is called an *injective tableau*. If the integers of each row and column of an injective tableau are in increasing order (from left to right, and from bottom to top) the tableau is said to be standard . Since we are only concerned here with injective tableaux we shall (by abuse of language) use the term tableau for injective tableau. Given a partition μ, for each tableau T of shape μ we shall associate the following monomial :

$$m(T) = \prod_{i \in T} x_i^{h(i,T)-1}$$

where $h(i,T)$ denotes the height of the letter i in T. (Recall that we draw the tableaux according to the French notation.) Let $\mathcal{B}_\mu$ be the lower order ideal of monomials whose maximal elements are the monomials of the standard tableaux of shape μ. Proposition 4.2 of [3] states that indeed the monomials of $\mathcal{B}_\mu$ form a basis for the corresponding ring $\mathcal{R}_\mu$. In particular the standard tableau monomials $\{m(T) | T\ \textit{is a standard tableau of shape}\ \mu\}$ are a basis for the homogeneous component of highest degree:

$$n(\mu) = \sum_{i=1}^{h(\mu)} (i-1)\mu_i$$

where for convenience we let $\mu = (\mu_1, \ldots, \mu_{h(\mu)})$ with $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_{h(\mu)} > 0$. Denote the highest homogeneous component of $\mathcal{R}_\mu$ by $\mathcal{R}_\mu^{top}$. Therefore in our previous example a basis for $\mathcal{R}_{122}^{top}$ is given by

$$\mathcal{B}_{122}^{top} = \{x_2x_5x_3^2,\ x_2x_5x_4^2,\ x_2x_4x_5^2,\ x_3x_5x_4^2,\ x_3x_4x_5^2\} .$$

Observe that the number of elements of $\mathcal{B}_\mu^{top}$ is equal to the number of standard tableaux of shape μ denoted by η_μ. We claim (theorem 4.1) that the matrices obtained by acting with the permutations of S_n on $\mathcal{B}_\mu^{top}$, are the same as the matrices of Young's natural representation. In order to prove this result we have to express any element of the module $\mathcal{R}_\mu^{top}$ as a linear combination of the elements of the basis $\mathcal{B}_\mu^{top}$. The next section is devoted to the construction of a straightening algorithm that will (step by step) express any monomials of $\mathcal{R}_\mu^{top}$ as a linear combination of the monomials in $\mathcal{B}_\mu^{top}$.

2. Congruence relations in $\mathcal{R}_\mu$.

We first need to determine which monomials of a given ring $\mathcal{R}_\mu$ are congruent to zero modulo $\mathcal{I}_\mu$. The following result can be found in a paper by A. Garsia and N. Bergeron [2]. Let μ be a partition of n, and let $X = \{x_1, \ldots x_n\}$. For any sequence of n integers $p_1, \ldots, p_n$ let $x^p = x_1^{p_1} \ldots x_n^{p_n}$. Define $x^p << x^q$ if and only if $p_i \leq q_i$, for all $i = 1, \ldots, n$.

Lemma 2.1 [2, proposition 4.5]

In $\mathcal{R}_\mu$,

$$x^p \not\equiv 0 \Longleftrightarrow x^p << m(T)$$

We mention here that this is an equivalent form of proposition 4.5 of [2]. Indeed their proposition states that a monomial of $\mathcal{R}_\mu$ is not congruent to zero if and only if it is an S_n−image of an element of $\mathcal{B}_\mu$. Here is a schematic interpretation of this result that will be useful later on. For any tableau T of shape $\mu = (\mu_1, \ldots, \mu_k)$ with $\mu_1 \geq \ldots \geq \mu_k > 0$, $(\mu \vdash n,\ k = h(\mu))$, we shall represent the sequence of exponents of $m(T)$ by the diagram of figure 3.

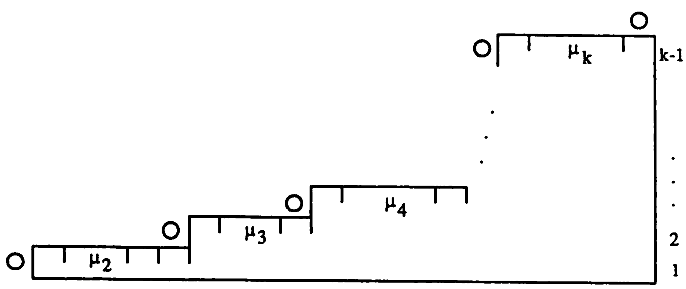

Figure 3

With this in mind, one can reformulate lemma 2.1 as follows: any monomial x^p of $\mathcal{R}_\mu$ whose sequence of powers $(p_1, \ldots, p_n)$ fills at least one of the corner marked with a *circle*, is congruent to zero modulo $\mathcal{I}_\mu$. (In other words if a monomial of $\mathcal{R}_\mu$ has μ_2 variables of degree 1, μ_3 variables of degree $2, \ldots, \mu_k$ variables of degree $h(\mu) - 1$, it is not congruent to zero (and indeed belongs to $\mathcal{R}_\mu^{top}$).) Note that two distinct tableaux T_1 and T_2 can yield the same monomial $m(T)$. For example if

Figure 4

then $m(T_1) = m(T_2) = x_3 x_4 x_5^2$. Therefore we shall only consider row-increasing tableaux. (That is, the ones for which the rows are in increasing order from left to right). A few more observations are needed before we can give a description of our straightening algorithm. For this we shall look at an example. Let T be the following tableau of shape $\mu = (3,3,3)$:

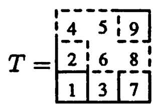

Figure 5

Since T is not a standard tableau, its corresponding monomial $m(T) = x_2 x_6 x_8 x_4^2 x_5^2 x_9^2$ belongs to $\mathcal{R}_{333}^{top}$ but not to the basis $\mathcal{B}_{333}^{top}$. Thus we shall expand $m(T)$ as a linear combination of monomials of $\mathcal{B}_{333}^{top}$. For this notice that the first break of standardness occurs in position $(2,3)$. We shall draw a ribbon as in figure 5, delimiting the position $(2,3)$. Let $X = \{1,\ldots,9\}$, $A = \{x_4, x_5\}$, and $B = \{x_6, x_8\}$. The first step of our algorithm will be to transform the monomial $m(T)$ into a new monomial $m'(T)$ as follows:

1. subtract one from all the exponents of the variables of $m(T)$ belonging to the set A.

This yields the new monomial $m'(T) = x_2 x_6 x_8 x_4 x_5 x_9^2$. The second step consits of multiplying $m'(T)$ by $e_2(A \cup B)$, to get a polynomial :

2. $p(T) = m'(T) e_2(A \cup B)$.

If $e_2(A \cup B)$ beolnged to $\mathcal{C}_{333}$ we would be done. Indeed we would then have that

$$x_2 x_6 x_8 x_4 x_5 x_9^2 e_2(A \cup B) \equiv 0 \ in \ \mathcal{R}_{333} \qquad (*)$$

But $(*)$ simply means that in $\mathcal{R}_{333}$ we have:

$$\begin{aligned} x_2 x_6 x_8 x_4^2 x_5^2 x_9^2 = -\big(& x_2 x_6^2 x_8 x_4^2 x_5 x_9^2 + x_2 x_6 x_8^2 x_4^2 x_5 x_9^2 + x_2 x_6^2 x_8 x_4 x_5^2 x_9^2 \\ & + x_2 x_6 x_8^2 x_4 x_5^2 x_9^2 + x_2 x_6^2 x_8^2 x_4 x_5 x_9^2 \big) \ . \end{aligned} \qquad (**)$$

Now looking at the right hand side of equation $(**)$ one sees that each of the monomials corresponds to a standard tableau of shape $(3,3,3)$. Thus we have expressed $x_2 x_6 x_8 x_4^2 x_5^2 x_9^2$ as a linear combination of the elements of the basis $\mathcal{B}_{333}^{top}$. Unfortunately $e_2(A \cup B)$ does not belong to $\mathcal{C}_{333}$, as one can easily see from figure 6, which depicts the admissible pairs $(|S|, r)$.

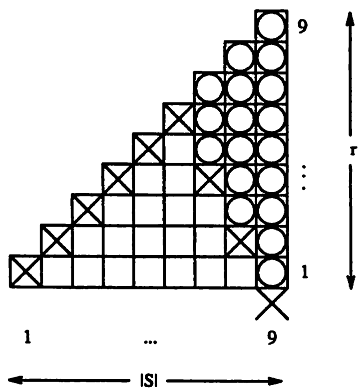

Figure 6

On the other hand according to our remark of section 1, $e_2(X)$ and $e_1(X)$ are certainly present in $\mathcal{C}_{333}$. We claim here that :

$$x_2x_6x_8x_4x_5x_9^2(e_2(X) - x_2e_1(X)) \equiv x_2x_6x_8x_4x_5x_9^2e_2(A \cup B) \text{ in } \mathcal{R}_{333} \qquad (2.1).$$

But the left hand side of equation 2.1 is certainly congruent to zero in $\mathcal{R}_{333}$, therefore yielding $(*)$ and consequently $(**)$. We shall prove this assertion in two steps. First observe that the sequence of exponents of a monomial corresponding to any tableau of shape $(3,3,3)$ is given by figure 7(a).

Figure 7

Thus for any $i \in \{1,3,7,9\}$ we have that

$$x_2x_6x_8x_4x_5x_9^2x_i \equiv 0. \qquad (***)$$

Indeed, it is clear that the corresponding sequence of exponents for these monomials will fill at least one of the *forbidden* squares introduced in lemma 2.1 (see figure 7 b, for the case $x_2x_6x_8x_4x_5x_9^2x_3$). Thus we now have that

$$x_2x_6x_8x_4x_5x_9^2e_2(X) \equiv x_2x_6x_8x_4x_5x_9^2e_2(S)$$

where $S = \{x_2, x_4, x_5, x_6, x_8\}$. One realizes that S differs from the set $A \cup B$ only in the variable x_2. Therefore our next step is to eliminate x_2 of S. For this notice that $e_2(S) - x_2e_1(S)$ has the double effect of removing all pairs x_2x_j (for $j = 4,5,6,8$) from $e_2(S)$ and of adding to it the monomial $-x_2^2$. Thus

$$e_2(S) - x_2e_1(S) = e_2(4,5,6,8) - x_2^2$$

But clearly

$$x_2x_6x_8x_4x_5x_9^2x_2^2 \equiv 0 \ in \ \mathcal{R}_{333}.$$

We are therefore left with

$$x_2x_6x_8x_4x_5x_9^2e_2(4,5,6,8),$$

which yields equation 2.1 as desired. Observe that it is not always the case that one application of steps 1 and 2 yields a set of *standard monomials* (monomials corresponding to standard tableaux). But as we shall show in theorem 3.1 a recursive application of these two steps will eventually lead to a set of standard monomials. We are now ready to generalize this construction.

Let T be a tableau where the first break of standardness occurs in position k of its $(i+1)^{th}$ row. We shall depict the elements of rows i and $i+1$ as in figure 8.

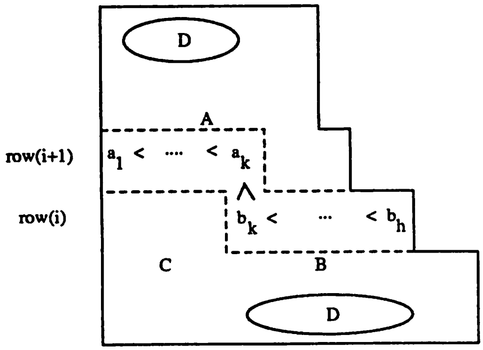

Figure 8

Let D be the set of variables corresponding to the entries of the first $(i-1)$ rows, together with $a_{k+1}, \ldots, a_l$ and together with rows $(i+2)$, $(i+3)$ $, \ldots, h(\mu)$. Let $m(D)$ be the monomial obtained from $m(T)$ by removing all the variables not belonging to D. Let $A = \{a_1, \ldots, a_k\}$, $B = \{b_k, \ldots, b_m\}$ and $C = \{b_1, \ldots, b_{k-1}\}$. (See figure 8).

To avoid confusion with indices denote the height of the i^{th} row by h. With this notation $m(T)$ is given by:

$$m(T) = m(D)(\prod_{i \in A} x_i)^h (\prod_{i \in B} x_i)^{h-1} (\prod_{i \in C} x_i)^{h-1}.$$

As in the previous example define a new monomial $m'(T)$ by subtracting one from all the exponents of the variables of $m(T)$ belonging to A. More precisely:

1. $$m'(T) = m(D)(\prod_{i \in A} x_i)^{h-1} (\prod_{i \in B} x_i)^{h-1} (\prod_{i \in C} x_i)^{h-1}.$$

Now using the principle of inclusion-exclusion we define the polynomial $p(T)$ as follows:

2. $$p(T) = m'(T)\left(\sum_{i=0}^{k-1}(-1)^i\left[\left(\sum_{\substack{j_1<\cdots<j_i \\ j_l \in C}} x_{j_1}\cdots x_{j_i}\right)e_{k-i}(X)\right]\right)$$

where $x_{j_1}\cdots x_{j_i} = 1$ when $i = 0$. Observe that $p(T)$ is congruent to zero in $\mathcal{R}_\mu$. The next proposition is the last step we shall need to prove theorem (3.1)

Proposition 2.2.

$$p(T) \equiv m'(T)e_k(A \cup B) \text{ in } \mathcal{R}_\mu.$$

Proof

Observe first that

$$\sum_{\substack{j_1<\dots<j_i \\ j_l \in C}} x_{j_1} \dots x_{j_i} = e_i(C).$$

Thus $p(T)$ is nothing more than :

$$p(T) = m'(T) \sum_{i=0}^{k-1} (-1)^i e_i(C) e_{k-i}(X)$$

where $e_0(C) = 1$. On the other hand we have that

$$\sum_{i=0}^{k-1} (-1)^i e_i(C) e_{k-i}(X) = \sum_{i=0}^{k-1} \left(\prod_{j \in C} (1 - tx_j) \Big|_{t^i} \prod_{j \in A \cup B \cup C \cup D} (1 + tx_j) \Big|_{t^{k-i}} \right)$$
$$= \left(\prod_{j \in C} (1 - tx_j) \prod_{j \in A \cup B \cup C \cup D} (1 + tx_j) \right) \Big|_{t^k}$$
$$= \left(\prod_{j \in C} (1 - t^2 x_j^2) \prod_{j \in A \cup B \cup D} (1 + tx_j) \right) \Big|_{t^k}$$

Therefore we have that,

$$p(T) = m(D) \big(\prod_{i \in A} x_i\big)^{h-1} \big(\prod_{i \in B} x_i\big)^{h-1} \big(\prod_{i \in C} x_i\big)^{h-1} \left(\prod_{j \in C} (1 - t^2 x_j^2) \prod_{j \in A \cup B \cup D} (1 + tx_j) \right) \Big|_{t^k}. \quad (2.2)$$

It is not too difficult to realize that in the right-hand side of equation (2.2) the coefficients of t^k involving variables in D or C will give rise to monomials congruent to zero in $\mathcal{R}_\mu$. Indeed the corresponding sequence of exponents will be outside the permissible squares of the diagram of lemma 2.1. See figure (9), where A' corresponds to the set of variables which had their exponent diminished by one.

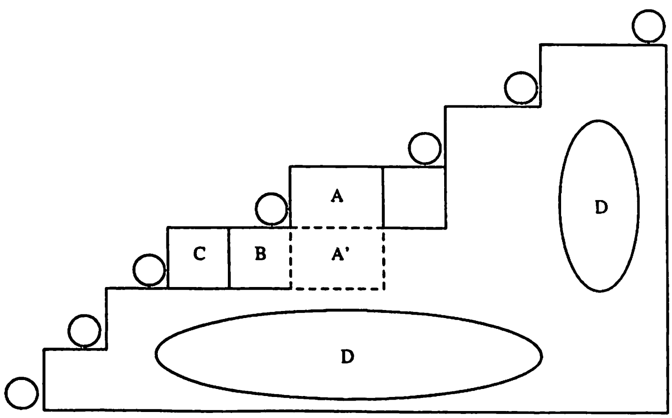

Figure 9

Thus we are left with

$$p(T) \equiv m(D)\Big(\prod_{i\in A} x_i\Big)^{h-1}\Big(\prod_{i\in B} x_i\Big)^{h-1}\Big(\prod_{i\in C} x_i\Big)^{h-1}\Big(\prod_{j\in A\cup B}(1+tx_j)\Big|_{t^k}\Big)$$
$$\equiv m'(T)e_k(A\cup B). \quad \clubsuit$$

As we observed earlier on, $p(T) \equiv 0$ in $\mathcal{R}_\mu$. Thus

$$m'(T)e_k(A\cup B) \equiv 0 \ in\ \mathcal{R}_\mu\ . \tag{2.3}$$

We can now finalize the straightening algorithm.

3. Straightening algorithm in $\mathcal{R}_\mu^{top}$.

We first need a total order on the set of row-increasing tableaux. Define the row-word $w(T)$ of a tableau T, to be the word obtained by reading the successive rows of T from left to right and from bottom to top. This given, order all the row-increasing tableaux of shape μ according to the lexicographic order of their corresponding row-words. For a given partition μ of n let $\{T_1,\ldots,T_{n_\mu}\}$ be the set of standard tableaux of that shape, and let $\{m(T_i)\}_{i=1}^{n_\mu}$ be the set of corresponding monomials. As we saw in section 1, a basis for $\mathcal{R}_\mu^{top}$ is given by:

$$\mathcal{B}_\mu^{top} = \{m(T_1),\ldots m(T_{n_\mu})\}$$

Let T be a non-standard injective tableau, filled with the integers 1 to n. The monomial $m(T)$ associated to this tableau is an element of $\mathcal{R}_\mu^{top}$, and we want to express it as a linear combination of the elements of $\mathcal{B}_\mu^{top}$. For this we shall give an algorithm (*straightening algorithm*) that will explicitly produce the coefficients $a_i(T)$ in the expansion of $m(T)$:

$$m(T) \ \equiv \sum_{i=1}^{n_\mu} a_i(T)m(T_i) \tag{3.1}$$

This algorithm is, in its essence, similar to Young's straightening law. For more details the reader can consult the original work of A. Young, [8, QSA II p. 95] and [8,QSA III p. 356] or the more recent work of A. Garsia and M. Wachs [GW, proposition 2.2] . We shall adopt here the same notation as in [GW]. Consider the set of row-words of all the row-increasing tableaux of shape μ, ordered according to the lexicographic order of their corresponding row-words. Clearly the first row-word in this ordered set is the one corresponding to the super standard tableau of shape μ. A super standard tableau is a tableau obtained by filling its rows (from bottom to top and from left to right), with the consecutive integers $1, 2, \ldots n$. For this super-standard tableau T^* equation 3.1 reduces to

$$m(T^*) \ = \ m(T_1)$$

i.e.: $a_i(T) = 0$, for all $i \neq 1$ and $a_1 = 1$. Therefore we shall proceed by induction on this order, and assume that T is the first row-increasing tableau for which equation 3.1 has not yet been established . Assume that the first break of standardness in T occurs in position k of the $(i+1)^{th}$ row where the element in this position is smaller (rather than larger) than the element directly below it. As in section 2, let $A = \{a_1,\ldots,a_k\}$ and $B = \{b_k,\ldots,b_h\}$. Observe that all the letters of A are smaller than any letter of B. We claim that:

$$m(T) = -\sum_{\tau\neq\epsilon} m(\tau T) \tag{3.2}$$

where the sum is over all permutations τ (distinct from the identity) in $S_{A\cup B}$ where the first $|A|$ letters and the next $|B|$ letters are in increasing order. In two line notation, this simply means that τ has the form:

$$\tau = \begin{bmatrix} A & B \\ \tau_A & \tau_B \end{bmatrix}$$

with τ_A, τ_B a pair of subsets (such that $|A| = |\tau_A|$, and $|B| = |\tau_B|$) partioning $A \cup B$. Observe first that upon proving equation (3.2) our argument will be completed. Indeed each monomial on the right hand side of equation (3.2) corresponds to a tableau τT whose row word is lexicographically smaller than the row word of T. To see this, one should observe the following two facts:

1. the elements of rows $1, 2, \ldots i-1$ have not changed.
2. Acting with any of the permutations τ ($\tau \neq \epsilon$) on T will bring at least one of the $a's$ down to the i^{th} row. But as we observed earlier, all of the $a's$ are smaller than all of the $b's$. This makes the row word $w(\tau T)$ of τT, lexicographically smaller than $w(T)$.

We shall now prove equation (3.2) . Using the construction introduced in section 2, let

$$p(T) = m'(T) \sum_{i=0}^{k-1} (-1)^i e_i(C) e_{k-i}(X)$$

where $m'(T)$ is the monomial obtained from $m(T)$ by subtracting one from the exponents of all the variables of $m(T)$belonging to A. By proposition 2.2 we have that :

$$p(T) \equiv m'(T) e_k(a \cup B),$$

Observe that $p(T)$ is a sum of monomials, among which we find $m(T)$. Indeed in

$$e_k(A \cup B) = \sum_{\substack{j_1 < \cdots < j_i \\ j_k \in A\cup B}} x_{j_1} \ldots x_{j_i},$$

choosing the k elements $a_1, \ldots a_k$ of A, yields that the monomial $m'(T) x_{a_1} \ldots x_{a_k} \in p(T)$. But this monomial is precisely $m(T)$. Thus equation 2.3 yields that

$$m(T) = -\big(m'(T) e'_k(A \cup B)\big)$$

where $e'_k(A \cup B) = e_k(A \cup B) - x_{a_1} \ldots x_{a_k}$. We claim that

$$-\big(m'(T) e'_k(A \cup B)\big) = -\sum_{\tau \neq \epsilon} m(\tau T) . \tag{3.3}$$

Indeed a moment of thought reveals that for each choice of k variables $x_{i_1} < \ldots < x_{i_k}$ among $A \cup B$, the corresponding monomial $m'(T) x_{i_1} \ldots x_{i_k}$ of $m'(T) e_k(A \cup B)$ is precisely $m(\tau T)$ where $\tau_A = \{x_{i_1}, \ldots, x_{i_k}\}$ and $\tau_B = \{A \cup B\} \backslash \{x_{i_1}, \ldots x_{i_k}\}$. ♣

We have now shown:

Theorem 3.1 (Straightening algorithm)

For any (injective) tableau T of shape μ the coefficients $a_i(T)$, described in the algorithm above, satisfy :

$$m(T) \equiv \sum_{i=1}^{n_\mu} a_i(T) m(T_i)$$

in $\mathcal{R}_\mu^{top}$ where the sum is over all standard tableaux of shape μ, ordered according to the lexicographic order of their corresponding row-words.

4. Action of S_n on $\mathcal{R}_\mu^{top}$.

Let $\mu = (\mu_1, \ldots, \mu_k)$ be a partition of n. As we mentioned earlier (section 1), there is a natural action of the symmetric group S_n on $\mathcal{R}_\mu$ which presrves the natural grading of $\mathcal{R}_\mu$. Briefly, for any polynomial $P(x_1, \ldots x_n)$ of $\mathcal{R}_\mu$ and for any permutation σ of S_n

$$\sigma P(x_1, \ldots, x_n) = P(x_{\sigma_1}, \ldots, x_{\sigma_n}).$$

We want to discuss here the action of S_n on $\mathcal{R}_\mu^{top}$ via its basis $\mathcal{B}_\mu^{top}$. More precisely for any monomials $m(T)$ of $\mathcal{B}_\mu^{top}$ we are interested in finding its image under the action of a permutation σ of S_n. It is not difficult to see that the images $\sigma m(T)$ can be obtained by acting directly with σ on the standard tableaux T themselves: replace every entry i of T by its image $\sigma(i)$. In this manner for any given standard tableau T of shape μ,

$$\sigma m(T) = m(\sigma T).$$

We shall show that the matrices obtained from the action of S_n on the elements of $\mathcal{B}_\mu^{top}$ are exactly the same ones resulting from the action of S_n on the so called Young's natural units.

We shall first recall the definition of Young's natural units. Again we will use the notation found in [GW]. For a set A of integers define $[A]$ to be the formal sum of all permutations of A. That is,

$$[A] = \sum_{\sigma \in S_A} \sigma$$

if S_A denotes the symmetric group of A. Also let $[A']$ be given by:

$$[A'] = \sum_{\sigma \in S_A} sign(\sigma)\sigma.$$

Note that $[A]$ and $[A']$ can be interpreted as elements of the group algebra of S_n, denoted here by $A(S_n)$. For a given tableau T of shape μ, let $R_1, \ldots, R_k$ denote the rows of T and $C_1, \ldots, C_h$ denote the columns of T. In $A(S_n)$, define the row group of T to be

$$P(T) = [R_1][R_2] \ldots [R_k]$$

and the signed column group of T :

$$N(T) = [C_1][C_2] \ldots [C_h].$$

Next, let $E(T) = P(T)N(T)$ and for any two tableaux T_1 and T_2 of the same shape, define

$$E_{T_1,T_2} = P(T_1)\sigma_{T_1,T_2}N(T_2)$$

where σ_{T_1,T_2} is the permutation which sends T_2 to T_1. Let $T_1, \ldots, T_{n_\mu}$ be the standard tableaux of shape μ ordered according to the lexicographic order of their row-words (see section 3). A. Young showed that for any given partition μ of n, and for each $s = 1, 2, \ldots, n_\mu$ the elements (now called Young's natural units)

$$E_{T_1,T_s},\ E_{T_2,T_s},\ \ldots, E_{T_{n_\mu},T_s}$$

span a subspace of the group algebra $A(S_n)$ which is invariant under left multiplication. Moreover the matrices expressing this action in terms of this basis are the *same* for each s and they give the irreducible representaiton of S_n usually indexed by μ. From now on set $s = 1$. We plan to show that for any given shape μ, the matrices resulting from the action of S_n on $\mathcal{B}_\mu^{top}$ are exactly the same as the one resulting from the action of S_n on

$$E_{T_1,T_1},\ E_{T_2,T_1},\ \ldots, E_{T_{n_\mu},T_1}$$

Thus for any $\sigma \in S_n$, we shall express $\sigma E_{T_i,T_1}$ as a linear combination of the elements of the set $\{E_{T_1,T_1},\ E_{T_2,T_1},\ \ldots, E_{T_{n_\mu},T_1}\}$. For this we shall need a few relations between E_{T_i,T_1} and $E(T)$ (for more details see [GW, eq. 2.3]). Given any two tableaux T_1, T_2 of shape μ, one has

$$\begin{aligned}\sigma_{T_1,T_2}P(T_2) &= P(T_1)\sigma_{T_1,T_2}\\ \sigma_{T_1,T_2}N(T_2) &= N(T_1)\sigma_{T_1,T_2}\\ E_{T_1,T_2} &= E(T_1)\sigma_{T_1,T_2} = \sigma_{T_1,T_2}E(T_2)\end{aligned}$$

Therefore $\sigma E_{T_i,T_1}$ becomes:

$$\begin{aligned}\sigma\sigma_{T_i,T_1}E(T_1) &= \sigma_{T_*,T_1}E(T_1)\\ &= \sigma_{T_*,T_1}P(T_1)N(T_1) = P(T_*)\sigma_{T_*,T_1}N(T_1)\\ &= P(T_*)N(T_*)\sigma_{T_*,T_1} = E(T_*)\sigma_{T_*,T_1}\end{aligned}$$

where T_* is the tableau σT_i. The reason for replacing $\sigma E_{T_i,T_1}$ with $E(T_*)\sigma_{T_*,T_1}$, is that Young also give an algorithm for expanding any element $E(T)$ (for any injective tableau T) as a linear combination of elements of $\{E_{T_i,T},\ E_{T_2,T},\ldots, E_{T_{n_\mu},T}\}$. More precisely, this procedure, called Young's straightening formula, is stated as follows in [GW, prop. 2.2]

Lemma 4.1 Young's straightening algorithm

For any (injective) tableau T of shape μ there are some coefficients $a_i(T)$ giving

$$E(T) = \sum_{i=1}^{n_\mu} a_i(T)E_{T_i,T}. \tag{4.1}$$

First observe that using equation 4.1 together with equation 4.2 we get

$$\begin{aligned}\sigma E_{T_i,T_1} &= E(T_*)\sigma_{T_*,T_1}\\ &= \sum_{i=1}^{n_\mu} a_i(T_*)E_{T_i,T_*}\sigma_{T_*,T_1}\\ &= \sum_{i=1}^{n_\mu} a_i(T_*)E_{T_i,T_1}.\end{aligned}$$

Therefore in order to expand $\sigma E_{T_i,T_1}$ as a linear combination of Young's natural units $\{E_{T_i,T_1}\}_{i=1}^{n_\mu}$ we need only to find the coefficients $a_i(T_*)$ in the expansion of $E(T_*)$ given in equation (4.1). The remarkable fact is that Young's proof of lemma 4.1 is algorithmic and analogous to our own straightening algorithm given in the proof of theorem 3.1. But thanks to A. Garsia and M. Wachs, Young's straightening algorithm has been eloquently reproduced (as well as generalized to skew-shaped tableaux) in [GW, prop. 2.2]. Thus a glance at the proof of proposition 2.2 of [GW] will convince the reader that the coefficients $a_i(T)$ of equation 4.1 are precisely the same ones appearing in our previous equation 3.1. Thus keeping in mind the remark following the definitions of Young's natural units we have proved:

Theorem 4.2

The action of S_n on the basis $\mathcal{B}_\mu^{top}$ of $\mathcal{R}_\mu^{top}$ yields the same matrices as the action of S_n on Young's basis of natural units for the irreducible representation indexed by μ.

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