Determinants of Super-Schur Functions Lattice Paths, and Dotted Plane Partitions

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Extended Abstract

Let $\mathbf{x} = (x_1, x_2, \ldots)$ and $\mathbf{y} = (y_1, y_2, \ldots)$ be two sequences of independent variables and λ be a partition. We denote by

$$s_\lambda(x_1, x_2, \ldots / y_1, y_2, \ldots)$$

the super-Schur function corresponding to λ in the variables $\mathbf{x}$ and $\mathbf{y}$. These functions arise naturally in the representation theory of Lie superalgebras [6] and were also defined, independently, by Metropolis, Nicoletti, and Rota in [8], under the name of bisymmetric functions. Since then, they have been studied extensively and we refer the reader to [1], [2], or [4] for their definition (they can be defined in several equivalent ways) and further information about them.

The purpose of the present work is to give combinatorial interpretations to the minors of the infinite matrix

$$\mathcal{S}(\mathbf{x}, \mathbf{y}) \stackrel{\text{def}}{=} (s_{(k)}(x_1, \ldots, x_n / y_1, \ldots, y_n))_{n,k \in \mathbf{N}} .$$

Our main results (Theorems 1.1 and 1.3) are proved combinatorially using lattice paths and are stated in terms of dotted and diagonal strict plane partitions, respectively. They also have many applications. As special cases we obtain combinatorial interpretations of determinants of homogeneous, elementary, and Hall-Littlewood symmetric functions, Schur's Q-functions, q-binomial coefficients, and q-Stirling numbers of both kinds. Other applications include the solution of a problem posed by Yahory in [10] and the combinatorial interpretation of a class of symmetric functions first defined, algebraically, by Macdonald in [7]. Many of our results are new even in the case $q = 1$. Others are q-analogues of known results. Our main theorem also has several interesting applications to the theory of total positivity. These are treated in [3].

In order to state the main results we need to define some notation, and terminology. Given an infinite matrix $M = (M_{n,k})_{n,k \in \mathbf{N}}$ (where $M_{n,k}$ is the entry in the n-th *row* and k-th *column* of M) and $\{n_1, \ldots, n_r\}_<$, $\{k_1, \ldots, k_r\}_< \subseteq \mathbf{N}$ we let

$$M \begin{pmatrix} n_1, \ldots, n_r \\ k_1, \ldots, k_r \end{pmatrix} \stackrel{\text{def}}{=} \det \left[(M_{n_i,k_j})_{1 \le i,\, j \le r} \right] .$$

Given an infinite sequence $\{a_i\}_{i\in\mathbf{N}}$ we let

$$\{a_i\}_{i\in\mathbf{N}}\begin{pmatrix} n_1,\ldots,n_r \\ k_1,\ldots,k_r \end{pmatrix} \stackrel{\text{def}}{=} A\begin{pmatrix} n_1,\ldots,n_r \\ k_1,\ldots,k_r \end{pmatrix}$$

where $A \stackrel{\text{def}}{=} (a_{n-k})_{n,k\in\mathbf{N}}$ (and $a_i \stackrel{\text{def}}{=} 0$ if $i<0$).

A *dotted partition* is a partition $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_n)$ where, for each $i\in\mathbf{P}$ such that $m_i(\lambda)>0$, the rightmost occurrence of i in λ may be dotted. Given two (possibly) dotted integers we will write $a \doteq b$ to indicate that they are equal as *dotted* integers, and $a=b$ if they are only equal as integers (so that, for example, $2=\dot{2}$, $2\doteq 2$, $\dot{2}\doteq\dot{2}$). We will also write $(a+b)^{\cdot}$ instead of the more cumbersome $\overset{\cdot}{\overbrace{a+b}}$. Given a partition $\lambda=(\lambda_1,\ldots,\lambda_r)$ a *shifted dotted plane partition* of shape λ is an array of (possibly dotted) positive integers $\pi=(\pi_{i,j})_{1\le i\le r,\, i\le j\le i+\lambda_i-1}$ where each row is a dotted partition and $\pi_{i,j}>\pi_{i+1,j}$ whenever $\pi_{i,j}$ and $\pi_{i+1,j}$ are both defined and $\pi_{i+1,j}$ is *not* dotted. Note that we *do not* require the parts of λ to be distinct. Let π be a shifted dotted plane partition as above. For $k=1,\ldots,\lambda_1$ we let

$$t_k(\pi) \stackrel{\text{def}}{=} \sum_{i=1}^{d_k(\pi)} \pi_{i,i+k-1}\,, \tag{1}$$

where $d_k(\pi) \stackrel{\text{def}}{=} |\{i\in\mathbf{P} \,:\, \pi_{i,i+k-1}>0\}|$, and

$$\dot{d}_k(\pi) \stackrel{\text{def}}{=} |\{i\in\mathbf{P} \,:\, \pi_{i,i+k-1} \text{ is dotted }\}|\,.$$

Also, given π as above we let $\tilde{\pi} \stackrel{\text{def}}{=} (\pi_{ij})_{1\le i\le r\,;\, i\le j\le i+\lambda_i-2}$. We also let

$$t(\pi) \stackrel{\text{def}}{=} (t_1(\pi),\ldots,t_{\lambda_1}(\pi),0,0,\ldots)\,,$$

$$d(\pi) \stackrel{\text{def}}{=} (d_1(\pi),\ldots,d_{\lambda_1}(\pi),0,0,\ldots)\,,$$

and

$$\dot{d}(\pi) \stackrel{\text{def}}{=} (\dot{d}_1(\pi),\ldots,\dot{d}_{\lambda_1}(\pi),0,0,\ldots)\,.$$

Given a set of variables $\mathbf{x}=(x_1,x_2,x_3,\ldots)$ and a vector $d=(d_1,d_2,d_3,\ldots)$ of integers we let

$$\mathbf{x}^d \stackrel{\text{def}}{=} \prod_{i\ge 1} x_i^{d_i}\,.$$

and $S(d) \stackrel{\text{def}}{=} (d_2,d_3,\ldots)$. Finally, we define the *weight* of π to be

$$w(\pi) \stackrel{\text{def}}{=} \mathbf{y}^{\dot{d}(\tilde{\pi})}\,\mathbf{x}^{t(\tilde{\pi})-\dot{d}(\tilde{\pi})-S(t(\pi))}\,.$$

Our main results are the following.

Theorem 1.1 *Let* $\{n_1, \ldots, n_r\}_>$, $\{k_1, \ldots, k_r\}_> \subseteq \mathbf{N}$. *Then*

$$\mathcal{S}(\mathbf{x},\mathbf{y}) \begin{pmatrix} n_r, \ldots, n_1 \\ k_r, \ldots, k_1 \end{pmatrix} = \sum_{\pi} \mathbf{x}^{t(\bar{\pi}) - \dot{d}(\bar{\pi}) - \mathcal{S}(t(\pi))} \mathbf{y}^{\dot{d}(\bar{\pi})} \tag{2}$$

where the sum is over all shifted dotted plane partitions π *of shape* $(n_1+1, \ldots, n_r+1)$ *in which the i-th row has smallest part* $\doteq \dot{1}$ *and largest part* $= k_i + 1$ *for* $i = 1, \ldots, r$.

Theorem 1.2 *Let* $\{m_1, \ldots, m_r\}_<$, $\{k_1, \ldots, k_r\}_< \subseteq \mathbf{N}$ *and* $m, n \in \mathbf{P}$, $m > max\{m_r, k_r\}$. *Then*

$$\{s_{(k)}(x_1, \ldots, x_n / y_1, \ldots, y_n)\}_{k \in \mathbf{N}} \begin{pmatrix} m_1, \ldots, m_r \\ k_1, \ldots, k_r \end{pmatrix} = \sum_{\pi} \mathbf{x}^{t(\bar{\pi}) - \dot{d}(\bar{\pi}) - \mathcal{S}(t(\pi))} \mathbf{y}^{\dot{d}(\bar{\pi})} \tag{3}$$

where the sum is over all shifted dotted plane partitions π *of shape* $((n+1)^r)$ *in which the i-th row has smallest part* $\doteq (m - k_i + 1)^{\cdot}$ *and largest part* $= m - m_i + 1$, *for* $i = 1, \ldots, r$.

Since the minor in the last theorem is just the skew super-Schur function corresponding to the skew shape $(m - m_1 + 1, \ldots, m - m_r + r)/(m - k_1 + 1, \ldots, m - k_r + r)$, this theorem gives a combinatorial interpretation for these skew super-Schur functions. Other combinatorial interpretations have been obtained by Berele and Regev [2], Balantekin and Bars [1], Dondi and Jarvis [4], and Stanley [9].

It is also possible to state the preceding results in terms of diagonal strict plane partitions (i.e., plane partitions in which parts decrease strictly along each diagonal, from upper left to lower right). Let T be a shifted (or skew) plane partition. For $i \in \mathbf{P}$ we let $c_i(T)$ (respectively $r_i(T)$) be the number of columns (respectively rows) of T that contain at least one part equal to i, and $m_i(T)$ be the number of parts of T that are equal to i. We then let

$$c(T) \stackrel{\text{def}}{=} (c_1(T), c_2(T), \ldots),$$

$$r(T) \stackrel{\text{def}}{=} (r_1(T), r_2(T), \ldots),$$

and

$$m(T) \stackrel{\text{def}}{=} (m_1(T), m_2(T), \ldots).$$

Theorem 1.3 *Let* $\{n_1, \ldots, n_r\}_>$, $\{k_1, \ldots, k_r\}_> \subseteq \mathbf{N}$. *Then*

$$\mathcal{S}(\mathbf{x},\mathbf{y}) \begin{pmatrix} n_r, \ldots, n_1 \\ k_r, \ldots, k_1 \end{pmatrix} = \sum_{T} \mathbf{y}^{m(T) - c(T)} \mathbf{x}^{m(T) - r(T)} (\mathbf{y} + \mathbf{x})^{r(T) + c(T) - m(T)}$$

where the sum is over all diagonal strict shifted plane partitions T *of shape* $(k_1, \ldots, k_r)$ *in which the i-th row has largest part* $\leq n_i$ *and* $\geq n_{i+1} + 1$, *for* $i = 1, \ldots, r$ *(where* $n_{r+1} \stackrel{\text{def}}{=} -1$*).*

Theorem 1.4 *Let* $\{m_1,\ldots,m_r\}_<$, $\{k_1,\ldots,k_r\}_< \subseteq \mathbf{N}$ *and* $m,n \in \mathbf{P}$, $m > max\{m_r,k_r\}$. *Then*

$$\{s_{(k)}(x_1,\ldots,x_n/y_1\ldots,y_n)\}_{k\in\mathbf{N}}\begin{pmatrix} m_1,\ldots,m_r \\ k_1,\ldots,k_r \end{pmatrix} = \sum_T \mathbf{y}^{m(T)-c(T)}\mathbf{x}^{m(T)-r(T)}(\mathbf{y}+\mathbf{x})^{r(T)+c(T)-m(T)}$$

where the sum is over all diagonal strict plane partitions T *of shape* $(m-m_1+1,\ldots,m-m_r+r)\setminus(m-k_1+1,\ldots,m-k_r+r)$ *with largest part* $\leq n$.

In the case that $k_i = i$, for $i = 1,\ldots,r$, the preceding theorem first appeared, though without proof, in [9, Theorem 5.2].

In the second part of this work the preceding results are specialized to several interesting cases. In particular, using the fact that

$$e_k(y_1,\ldots,y_n) = s_{(k)}(\mathbf{0}/y_1,\ldots y_n)\,,$$

$$h_k(x_1,\ldots,x_n) = s_{(k)}(x_1,\ldots,x_n/\mathbf{0})\,,$$

$$q_k(x_1,\ldots,x_n;\alpha) = s_{(k)}(x_1,\ldots,x_n/-\alpha x_1,\ldots,-\alpha x_n) \tag{4}$$

and

$$q_k(x_1,\ldots,x_n;-1) = Q_{(k)}(x_1,\ldots,x_n)\,,$$

we can interpret combinatorially several determinants of elementary and complete homogeneous symmetric functions, Hall-Littlewood symmetric functions, and Schur's Q-functions. In some cases we obtain the classical Jacobi-Trudi identity, in others analogs of it. We give two examples of such results here.

Let $T = (T_{i,j})_{1\leq i\leq r,\, i\leq j\leq i+n_i}$ be a shifted plane partition of shape $(n_1+1,\ldots,n_r+1)$. We call a part $T_{i,j}$, of T, *free* if $T_{i-1,j} > T_{i,j} > T_{i,j+1}$ (the inequalities being vacuously satisfied if either one of $T_{i-1,j}$ and $T_{i,j+1}$ are undefined). We let

$$\mathcal{F}(T) \stackrel{\text{def}}{=} \{(i,j)\in sh(T):\ T_{i,j} \text{ is free}\}\,,$$

and call $\mathcal{F}(T)$ the *free set* of T. Given T as above we define

$$\mathcal{U}(T) \stackrel{\text{def}}{=} \{(i,j)\in sh(T):\ (i-1,j)\in sh(T),\ T_{i-1,j} = T_{i,j}\}\,,$$

and call $\mathcal{U}(T)$ the *upper set* of T.

Theorem 1.5 *Let* $\{n_1,\ldots,n_r\}_>$, $\{k_1,\ldots,k_r\}_> \subseteq \mathbf{N}$. *Then*

$$(q_k(x_1,\ldots,x_n;\alpha))_{n,k\in\mathbf{N}}\begin{pmatrix} n_r,\ldots,n_1 \\ k_r,\ldots,k_1 \end{pmatrix} = \sum_T \mathbf{x}^{m(T)}(-\alpha)^{|\mathcal{U}(T)|}(1-\alpha)^{|\mathcal{F}(T)|}, \tag{5}$$

where the sum is over all diagonal strict shifted plane partitions T *of shape* $(k_1,\ldots,k_r)$ *in which the i-th row has largest part* $\leq n_i$ *and* $\geq n_{i+1}+1$, *for* $i = 1,\ldots,r$ *(where* $n_{r+1} \stackrel{\text{def}}{=} -1$*).*

Theorem 1.6 *Let* $\{m_1,\ldots,m_r\}_<$, $\{k_1,\ldots,k_r\}_< \subseteq \mathbf{N}$ *and* $m,n \in \mathbf{P}$, $m > max\{m_r,k_r\}$. *Then*

$$\{q_k(x_1,\ldots,x_n;\alpha)\}_{k\in\mathrm{N}}\begin{pmatrix} m_1,\ldots,m_r \\ k_1,\ldots,k_r \end{pmatrix} = \sum_T \mathbf{x}^{m(T)}(-\alpha)^{|\mathcal{U}(T)|}(1-\alpha)^{|\mathcal{F}(T)|} \qquad (6)$$

where the sum is over all diagonal strict skew plane partitions T *of shape* $(m-m_1+1,\ldots,m-m_r+r)\setminus(m-k_1+1,\ldots,m-k_r+r)$ *with largest part* $\le n$.

Note that the symmetric function on the LHS of (6) is just the symmetric function $S_{\lambda/\mu}(\mathrm{x};\alpha)$ defined (in the case that $\mu=\emptyset$) by Macdonald in [7, p.116, eq. (4.5)], where $\lambda \stackrel{\text{def}}{=} (m-m_1+1,\ldots,m-m_r+r)$ and $\mu \stackrel{\text{def}}{=} (m-k_1+1,\ldots,m-k_r+r)$. Therefore Theorem 1.6 gives a combinatorial interpretation of these symmetric functions.

Finally, by suitably specializing our main results we can give combinatorial interpretations of determinants of q-binomial coefficients, and of q-Stirling numbers of both kinds.

For example, using the fact that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = h_{n-k}(1,q,q^2,\ldots,q^k), \qquad (7)$$

we can obtain the following result (where $B(q)$ denotes the infinite matrix of q-binomial coefficients).

Theorem 1.7 *Let* $\{n_1,\ldots n_r\}_>$, $\{k_1,\ldots,k_r\}_> \subseteq \mathbf{N}$. *Then*

$$B(q)\begin{pmatrix} n_r,\ldots,n_1 \\ k_r,\ldots,k_1 \end{pmatrix} = q^{-n((k_1+1,\ldots,k_r+1)')}\sum_T q^{|\overline{T}|}$$

where the sum is over all row strict shifted plane partitions T *of shape* $(k_1+1,\ldots,k_r+1)$ *in which the i-th row has largest part equal to* n_i+1, *for* $i=1,\ldots,r$ *(where for a partition* $\lambda=(\lambda_1,\lambda_2,\ldots)$, $n(\lambda) \stackrel{\text{def}}{=} \sum_{i\ge 1}(i-1)\lambda_i$*).*

In the case $q=1$ the preceding theorem was first proved (though stated in a slightly different way) by Gessel and Viennot [5, Corollary 11].

Given a permutation $\sigma \in S_n$ having k cycles $C_1,\ldots,C_k$ we let $S(\sigma) \stackrel{\text{def}}{=} \{\min(C_1),\ldots,\min(C_k),n+1\}$, and $\{\sigma^{(1)},\ldots,\sigma^{(k+1)}\}_> \stackrel{\text{def}}{=} S(\sigma)$. We say that σ is written in *normal form* if:

i) each cycle of σ is written with its smallest element first;

ii) the cycles are written in increasing order of their first elements.

The *normal representation* of σ is the word obtained from the normal form of σ by erasing all the parentheses. The number of *inversions* of σ, denoted by $\mathrm{inv}(\sigma)$, is the number of inversions in the normal representation of σ. Given an r-tuple of permutations $(\sigma_1, \ldots, \sigma_r)$ and a partition $\mu = (\mu_1, \ldots, \mu_r)$ we associate to them a shifted skew array, denoted $ST_\mu(\sigma_1, \ldots, \sigma_r)$, by letting $\sigma_i^{(j)}$ be its $(i, i + \mu_i + j - 1)$ entry, for $i = 1, \ldots, r$, $j = 1, \ldots, k_i + 1$ (where k_i is the number of cycles of σ_i, for $i = 1, \ldots, r$). Using the fact that

$$C[n+1, k+1]_q = e_{n-k}([1]_q, [2]_q, \ldots, [n]_q).$$

it is possible to deduce the following result (where $C(q)$ denotes the infinite matrix of q-Stirling numbers of the first kind).

Theorem 1.8 *Let* $\{n_1, \ldots, n_r\}_>$, $\{k_1, \ldots, k_r\}_> \subseteq \mathbf{N}$. *Then*

$$C(q) \begin{pmatrix} n_r, \ldots, n_1 \\ k_r, \ldots, k_1 \end{pmatrix} = \sum_{(\sigma_1, \ldots, \sigma_r)} \prod_{i=1}^{r} q^{inv(\sigma_i)}$$

where the sum is over all r-tuples $(\sigma_1, \ldots, \sigma_r) \in S_{n_1+1} \times \ldots \times S_{n_r+1}$ *such that* $ST(\sigma_1, \ldots, \sigma_r)$ *is a shifted plane partition of shape* $(k_1 + 2, \ldots, k_r + 2)$.

Theorem 1.9 *Let* $\{m_1, \ldots, m_r\}_<$, $\{k_1, \ldots, k_r\}_< \subseteq \mathbf{N}$ *and* $m, n \in \mathbf{P}$, $m > max\{m_r, k_r\}$. *Then*

$$\{c[n+1, k+1]_q\}_{k \in \mathbf{N}} \begin{pmatrix} m_1, \ldots, m_r \\ k_1, \ldots, k_r \end{pmatrix} = \sum_{(\sigma_1, \ldots, \sigma_r)} \prod_{i=1}^{r} q^{inv(\sigma_i)}$$

where the sum is over all r-tuples $(\sigma_1, \ldots, \sigma_r) \in (S_{n+1})^r$ *such that* $ST(\sigma_1, \ldots, \sigma_r)$ *is a skew plane partition of shape* $(m - m_1 + 3, \ldots, m - m_r + r + 2) \setminus (m - k_1 + 1, \ldots, m - k_r + r)$.

Let $m, n \in \mathbf{P}$, $m < n$. Given a partition $\pi = \{B_1, \ldots, B_k\}$ of $[m, n]$ into k blocks we let $S(\pi) \stackrel{\text{def}}{=} \{\max(B_1), \ldots, \max(B_k), m-1\}$ and $\{\pi^{(1)}, \ldots, \pi^{(k+1)}\}_> \stackrel{\text{def}}{=} S(\pi)$. Let now π_i be the (unique) block of π containing $\pi^{(i)}$, for $i = 1, \ldots, k$. We define the *height* of π to be the number

$$ht(\pi) \stackrel{\text{def}}{=} \sum_{i=1}^{k} (i-1)(|\pi_i| - 1).$$

Given an r-tuple of partitions $(\pi_1, \ldots, \pi_r)$ we associate to it a shifted array $ST(\pi_1, \ldots \pi_r)$ by letting the elements of $S(\pi_i)$ (in decreasing order) be the i-th row of it, for $i = 1, \ldots, r$, and then shifting the resulting array. Using the fact that

$$S[n+1, k+1]_q = h_{n-k}([1]_q, [2]_q, \ldots, [k+1]_q).$$

it is possible to deduce the following results (where $S(q)$ denotes the infinite matrix of q-Stirling numbers of the second kind).

Theorem 1.10 *Let* $\{n_1, \ldots, n_r\}_>$, $\{k_1, \ldots, k_r\}_> \subseteq \mathbf{N}$. *Then*

$$S(q)\begin{pmatrix} n_r, \ldots, n_1 \\ k_r, \ldots, k_1 \end{pmatrix} = \sum_{(\pi_1,\ldots,\pi_r)} \prod_{i=1}^{r} q^{ht(\pi_i)},$$

where the sum is over all r-tuples $(\pi_1, \ldots, \pi_r) \in \Pi([n_1+1]) \times \ldots \times \Pi([n_r+1])$ *such that* $ST(\pi_1, \ldots, \pi_r)$ *is a shifted plane partition of shape* $(k_1+1, \ldots, k_r+1)$.

Theorem 1.11 *Let* $\{m_1, \ldots, m_r\}_<$, $\{k_1, \ldots, k_r\}_< \subseteq \mathbf{N}$ *and* $m, k \in \mathbf{P}$, $m > max\{m_r, k_r\}$. *Then*

$$\{S[n+1, k+1]_q\}_{n\in\mathbf{N}}\begin{pmatrix} m_1, \ldots, m_r \\ k_1, \ldots, k_r \end{pmatrix} = \sum_{(\pi_1,\ldots,\pi_r)} \prod_{i=1}^{r} q^{ht(\pi_i)}$$

where the sum is over all r-tuples $(\pi_1, \ldots, \pi_r) \in \Pi([m-k_1+1, m-m_1+1]) \times \ldots \times \Pi([m-k_r+1, m-m_r+1])$ *such that* $ST(\pi_1, \ldots, \pi_r)$ *is a shifted plane partition of shape* $((k+2)^r)$.

The last four results are new even in the case $q = 1$.

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