S-series and Plethysm of Hook-shaped Schur Functions with Power Sum Symmetric Functions

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ABSTRACT

We present a simple combinatorial rule to expand the plethysm $s_{(1^a,b)}[p_n]$ of a Schur function of hook shape $s_{(1^a,b)}$ and a power symmetric function p_n as a sum of Schur functions. As an application of our rule, we derive explicit formulas for the expansion of the the plethysms $s_2[s_{(1^a,b)}]$ and $s_{1^2}[s_{(1^a,b)}]$ as a sum of Schur functions.

One of the fundamental problems in the theory of symmetric functions is to find a combinatorial rule to find the expansion of the plethysm of two Schur functions $s_\lambda[s_\mu]$ as a sum of Schur functions. Let Λ^n denote the space of homogeneous symmetric polynomials of degree n. Then given symmetric polynomials with integer coefficients, $P \in \Lambda^n$ and $Q \in \Lambda^m$, we can formally define the plethysm $P[Q]$ as follows. First write $Q = \sum_\alpha a_\alpha x^\alpha$ where a_α is an integer and if $\alpha = (\alpha_1, \alpha_2, ...)$, then $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots$. Then define

$$e_r^Q(x) = \prod_\alpha (1 + tx^\alpha)^{a_\alpha} \; |_{t^r} \tag{1}$$

and for $\lambda = (\lambda_1, \ldots, \lambda_k)$,

$$e_\lambda^Q(x) = e_{\lambda_1}^Q(x) \ldots e_{\lambda_k}^Q(x). \tag{2}$$

Here given a series $f(t)$, $f(t)\, |_{t^r}$ denotes the coefficient of t^r in $f(t)$. Next since P is symmetric, we can express P in terms of the elementary symmetric functions $e_\lambda(x)$,

$$P(x) = \sum_{\lambda \vdash n} c_\lambda e_\lambda(x). \tag{3}$$

Then by definition,

$$P[Q] = \sum_{\lambda \vdash n} c_\lambda e_\lambda^Q(x) \tag{4}$$

It is easy to see that if $Q = \sum_\alpha a_\alpha x^\alpha$ and a_α is nonnegative for all α, then $P[Q]$ is nothing but the symmetric polynomial which results by substituting the monomials of Q for the variables of P. Thus for example, let $CS(\mu) = \{T_1(\mu), T_2(\mu), ...\}$ denote the set of column strict tableaux of shape μ and use the usual combinatorial definition of s_μ as

$$s_\mu(x) = \sum_{T \in CS(\lambda)} x^T \tag{5}$$

where $x^T = \prod_i x_i^{m_i(T)}$ and $m_i(T)$ equals the number of occurences of i in T. Then

$$s_\lambda[s_\mu](x) = s_\lambda(x^{T_1(\mu)}, x^{T_2(\mu)}, \ldots) \tag{6}$$

The notion of plethysm goes back to Littlewood (ref. [6]). The basic problem of plethysm is to find the coefficients $a_{\lambda,\mu,\nu}$ where

$$s_\lambda[s_\mu] = \sum a_{\lambda,\mu,\nu} s_\nu \tag{7}$$

It is known that $a_{\lambda,\mu,\nu}$ are nonnegative integers. That is, let S_n denote the symmetric group on n letters and given a partition λ of n, let U_λ denote the irreducible S_n-module corresponding to λ. Let $U_\mu^{\otimes n}$ denote the n-fold tensor product of U_μ where μ is a partition of m. Then the wreath product of S_n with S_m, $S_n(S_m)$, acts naturally on $U_\lambda \otimes U_\mu^{\otimes n}$ and $a_{\lambda,\mu,\nu}$ is the multiplicity of U_ν in the $S_{n\cdot m}$-module which results by inducing the action of $S_n(S_m)$ on $U_\lambda \otimes U_\mu^{\otimes n}$ to a representation of $S_{n\cdot m}$. See (ref. [5]) and (ref. [8]) for details.

The problem of computing the coefficients $a_{\lambda,\mu,\nu}$ has proved to be very difficult. Essentially, there are explicit formulas for $a_{\lambda,\mu,\nu}$ only when λ is a partition of 2, 3, or 4 and $\mu = (m)$. Most algorithms for the computation of $a_{\lambda,\mu,\nu}$ reduce the problem to the problem of multiplying Schur functions and finding explicit expansions of $s_\mu[p_k]$ where $p_k = \sum_i x_i^k$ is the power symmetric function. That is, the following properties of plethysm hold.

$$(P_1 \pm P_2)[Q] = P_1[Q] \pm P_2[Q] \tag{8}$$

$$(P_1 \cdot P_2)[Q] = P_1[Q]P_2[Q] \tag{9}$$

$$p_n[Q] = Q[p_n] \tag{10}$$

$$s_n[P \cdot Q] = \sum_\lambda s_\lambda[P]s_\lambda[Q] \tag{11}$$

$$s_\lambda[P+Q] = \sum_{\mu\subseteq\lambda} s_\mu[P]s_{\lambda/\mu}[Q] \tag{12}$$

$$s_\lambda[s_\mu]' = \begin{cases} s_\lambda[s_{\mu'}] & \text{if } |\mu| \text{ is even} \\ s_{\lambda'}[s_{\mu'}] & \text{if } |\mu| \text{ is odd} \end{cases} \tag{13}$$

where for any sum $\sum c_\nu s_\nu$, $(\sum c_\nu s_\nu)'$ denotes the sum $\sum c_\nu s_{\nu'}$ and ν' denotes the conjugate of ν. Given (8)-(10), it follows that since

$$s_\lambda = \sum_{\alpha_1+2\alpha_2+\cdots+n\alpha_n=n} \frac{\chi^\lambda_{(1^{\alpha_1}2^{\alpha_2}\cdots n^{\alpha_n})}}{\alpha_1!\cdots\alpha_n!}\left(\frac{p_1}{1}\right)^{\alpha_1}\cdots\left(\frac{p_n}{n}\right)^{\alpha_n} \tag{14}$$

where $\chi^\lambda_{(1^{\alpha_1}2^{\alpha_2}\cdots n^{\alpha_n})}$ denotes the value of the irreducible character of S_n corresponding to λ at the conjugacy class corresponding to the partition $(1^{\alpha_1}2^{\alpha_2}\cdots n^{\alpha_n})$, then

$$s_\lambda[s_\mu] = \sum_{\alpha_1+2\alpha_2+\cdots+n\alpha_n=n} \frac{\chi^\lambda_{(1^{\alpha_1}2^{\alpha_2}\cdots n^{\alpha_n})}}{\alpha_1!\cdots\alpha_n!}\left(\frac{s_\mu[p_1]}{1}\right)^{\alpha_1}\cdots\left(\frac{s_\mu[p_n]}{n}\right)^{\alpha_n} \tag{15}$$

Hence since we can compute $\chi^\lambda_{(1^{\alpha_1}2^{\alpha_2}\cdots n^{\alpha_n})}$, we can reduce the calculation of $s_\lambda[s_\mu]$ to the problem of multiplying Schur functions if we had a way to compute

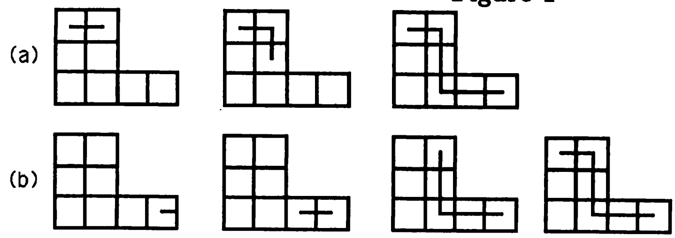
$$s_\mu[p_k] = \sum_\nu d^\nu_{\mu,k} s_\nu \tag{16}$$

or even the special case of (16),

$$s_{(n)}[p_k] = \sum_\nu d^\nu_{(n),k} s_\nu. \tag{17}$$

Now there are algorithms to compute the coefficients $d^\nu_{\mu,k}$ given by Chen, Garsia, and Remmel (ref. [2]) which we shall describe later. Moreover, there is a particularly simple algorithm due to Chen (ref.[1]) to compute the coefficients $d^\nu_{(n),k}$. (A similar algorithm for computing $d^\nu_{(n),k}$ had been given by Duncan (ref. [3])). To describe Chen's algorithm, we need to define the notion of special and transposed special rim hook tabloids. Given a Ferrers diagram λ, a *rim hook* h of λ is a consecutive sequence of cells along the North-East boundary of λ such that any two consecutive cells of h share and edge and the removal of the cells of h from λ results in a Ferrers diagram corresponding to another partition. We let $r(h)$ denote the number of rows of h and $c(h)$ denote the number of columns of h. We say that h is *special* if h has a cell in the first column of λ and h is *transposed special* (t-special) if h has cells in the first row of λ. For example, figure 1(a) pictures all special rim hooks of $\lambda = (2,2,4)$ and figure 1(b) pictures all t-special rim hooks of $\lambda = (2,2,4)$.

Figure 1



This given, a *rim hook tabloid* T of shape λ and type $\mu = (\mu_1, \ldots, \mu_k)$ is a filling of the Ferrers diagram of μ with rim hooks $(h_1, ..., h_k)$ such that $(|h_1|, ..., |h_k|)$ is a rearrangement of $(\mu_1, \ldots, \mu_k)$ where $|h_i|$ denotes the number of cells of h_i. To be more precise, one can think of a rim hook tabloid T as a sequence of shapes $\{\phi = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(k)} = \lambda\}$ such that for all $i \geq 1$, $\lambda^{(i)}/\lambda^{(i-1)}$ is a rim hook of $\lambda^{(i)}$ and $(|\lambda^{(1)}/\lambda^{(0)}|, |\lambda^{(2)}/\lambda^{(1)}|, \ldots, |\lambda^{(k)}/\lambda^{(k-1)}|)$ is a rearrangement of μ. T is called a *special (t-special) rim hook tabloid* if for all $i \geq 1$, $\lambda^{(i)}/\lambda^{(i-1)}$ is a special (t-special) rim hook of $\lambda^{(i)}$. The sign of T, $sgn(T)$, is defined by

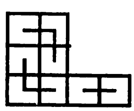
$$sgn(T) = \prod_i sgn(\lambda^{(i)}/\lambda^{(i-1)}) \tag{18}$$

where if h is a rim hook,

$$sgn(h) = (-1)^{r(h)-1}. \tag{19}$$

We emphasize however that the rim hook tabloid T of shape λ is the filling of the Ferrers diagram and is not the sequence of shapes $\{\phi = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(k)} = \lambda\}$. That is, figure 2 pictures a rim hook tabloid T of shape $\lambda = (2,2,4)$ and type (2,3,3) whose sign is $(-1)^{2-1}(-1)^{2-1}(-1)^{1-1} = 1$ and gives the two sequence of shapes that can be associated to it. Of course, if T is a special rim hook tabloid or a t-special rim hook tabloid, then there is a unique sequence of shapes $\{\phi = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(k)} = \lambda\}$ that can be associated to T.

Figure 2

T =  =

$\{\phi \subset (1,2) \subset (2,2,2) \subset (2,2,4)\} = \{\phi \subset (1,2) \subset (1,4) \subset (2,2,4)\}$.

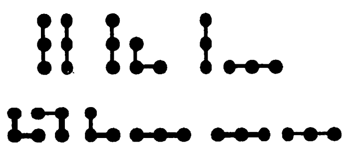
Let $SRHT(\lambda, \mu)$ (t-$SRHT(\lambda, \mu)$) denote the set of special (t-special) rim hook tabloids of shape λ and type μ. Note that if $\mu = (k^r)$ for some r, then there can be at most 1 special (t-special) rim hook of shape λ. This given, Chen's algorithm can be stated as

$$d^{\nu}_{(n),k} = \begin{cases} sgn(T) & \text{if there is a } t\text{-special rim hook tabloid T of shape } \nu \text{ and type } (k^n) \\ 0 & \text{otherwise} \end{cases} \tag{20}$$

Thus to compute $s_n[p_k]$, we need only generate all t-special rim hook tabloids T of type (k^n) and replace each such T by $sgn(T)s_{sh(T)}$ where $sh(T)$ denotes the shape of T.

For example, figure 3 pictures all t-special rim hook tabloids of type (3^2) where instead of drawing a Ferrers diagram, we have indicated the cells of the Ferrers diagram by dots.

Figure 3



Thus $s_2[p_3] = s_{(2^3)} - s_{(1,2,3)} + s_{(1^2,4)} + s_{(3,3)} - s_{(1,5)} + s_{(6)}$.

The main purpose of this paper is to give an extension of Chen's algorithm to compute the plethysm of a power symmetric function and a Schur function of hook shape. That is, we shall give an algorithm to compute $s_{(1^a,b)}[p_k]$ where $a + b = n$. To this end, we define a rim hook tabloid $T = \{\phi \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(k)} = \lambda\}$ to be *bispecial* if for all $i \geq 1$, $\lambda^{(i)}/\lambda^{(i-1)}$ is either a special rim hook of $\lambda^{(i)}$ or $\lambda^{(i)}/\lambda^{(i-1)}$ is a t-special rim hook of $\lambda^{(i)}$. We then say that T is a *$(1^a, b)$-bispecial rim hook tabloid of type (k^n)* if among the rim hooks $\lambda^{(2)}/\lambda^{(1)}, \lambda^{(3)}/\lambda^{(2)} \ldots \lambda^{(k)}/\lambda^{(k-1)}$, there are exactly a special rim hook and *b-1* transposed special rim hooks. That is, in T, the number of special rim hooks is the length of the first column of $(1^a, b)$ and the number of t-special rim hooks is the length of the first row of $(1^a, b)$ where we count the first rim hook $\lambda^{(1)}/\lambda^{(0)}$ as both special and t-special rim hook. Then our main result is the following.

Theorem 1

$$d^{\nu}_{(1^a,b),k} = \sum_T sgn(T) \tag{21}$$

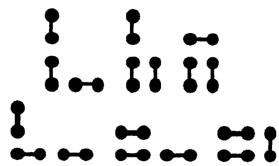
where T runs over all $(1^a, b)$-bispecial rim hook tabloids of shape ν and type (k^{a+b}).

We shall also prove that

Proposition 2 *If T_1 and T_2 are bispecial rim hook tabloids of shape ν and type k^{a+b}, then $sgn(T_1) = sgn(T_2)$.*

Thus there is no cancellation on the right hand side of (21) so that as with Chen's algorithm, to compute $s_{(1^a,b)}[p_k]$ we need only generate all $(1^a, b)$-bispecial rim hook tabloids T of type (k^n) and replace each such T by $sgn(T)s_{sh(T)}$. For example, figure 4 pictures all (1,2)-bispecial rim hook tabloids of type (2^3).

Figure 4



Thus $s_{(1,2)}[p_2] = s_{(1^3,3)} - s_{(1^2,2^2)} + s_{(2^3)} - s_{(1^2,4)} + s_{(2,4)} - s_{(3,3)}$.

The motivation for our result is related to the study of Schur function series, although the proof is through a different channel. Consider the Schur function series of the form:

$$\prod_i \frac{1 + a_1x_i + a_2x_i^2 + \cdots + a_px_i^p}{1 + b_1x_i + b_2x_2^2 + \cdots + b_qx_i^q} = 1 + \sum_\alpha d_\alpha s_\alpha(x) \tag{22}$$

where p, q are positive integers, a_i, b_j are real numbers, and $\alpha = (\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_\ell)$ is a partition in increasing order. In [10], we have developed a combinatorial method for evaluating the coefficients d_α of Schur functions $s_\alpha(x)$ in the above expansion. Briefly speaking, the coefficients are calculated through the construction of certain bispecial rim hook tabloids of shape α with restricted hook lengths which depend on the generating function on the left hand side of (22). It is known that the series of hook-shaped Schur functions is generated by:

$$\prod_i \frac{1 + x_i}{1 - x_i} = 1 + 2 \sum_{a+b\geq 0} s_{(1^b,a+1)}. \tag{23}$$

Hence the plethysm of the sum of hook-shaped Schur functions with the power sum symmetric function $p_k(x)$ is the series generated by

$$\prod_i \frac{1 + x_i^k}{1 - x_i^k} = 1 + 2\sum_\alpha k_\alpha s_\alpha(x), \tag{24}$$

where if $|\alpha|$ denotes the size of α, then

$$k_\alpha = \sum_{a+b+1=|\alpha|} s_{(1^b,a+1)}(x^k)|_{s_\alpha(x)}. \tag{25}$$

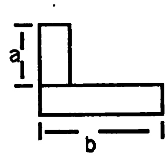
The coefficient k_α can easily be obtained using the general combinatorial construction in [10]. This lead us to conjecture Theorem 1 as a combinatorial algorithm for the Schur function expansion for the plethysm of a single hook-shaped Schur function with the power sum symmetric function.

We do not have the space in these proceedings to give a full proof of Theorem 1. Basically, the proof of Theorem 1 relies on two rules which can be used to compute the general case of plethysm of Schur functions $s_\lambda[s_\mu]$ as outlined in (ref. [2]), namely a version of the Littlewood-Richardson rule for multiplying Schur functions due to Remmel and Whitney (ref. [9]) and the SXP algorithm for computing $s_\lambda[p_k]$ due to Chen, Garsia, and Remmel (ref. [2]). We shall end this paper with a description of the SXP algorithm and a brief indication of how it can be used to prove Theorem 1. However before that we give a nice application of Theorem 1 by explicitly computing the plethysm $s_2[s_{(1^a,b)}]$ and $s_{1^2}[s_{(1^a,b)}]$.

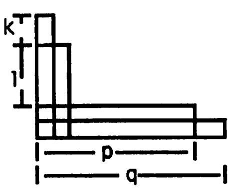
We say λ is of *hook shape* if $\lambda = (1^a, b)$ for some a and b and λ is of *double hook shape* if $\lambda = (1^k, 2^l, p, q)$ where $2 \le p \le q$. These shapes are pictured in figure 5.

Figure 5

Hook shapes $(1^a, b)$



Double hook shapes $(1^k, 2^l, p, q)$



Let $\langle , \rangle$ denote the Hall inner product on symmetric functions. Thus $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}$.

Theorem 3 *Let $a + b = n$, λ be a partition of $2n$, $u_\lambda = \langle s_2[s_{(1^a,b)}], s_\lambda \rangle$, and $v_\lambda = \langle s_{1^2}[s_{(1^a,b)}], s_\lambda \rangle$.*
Then

(a) (i) $u_\lambda = 0$ *if λ is not a hook shape or double hook shape*

(ii) *If $\lambda = (1^k, 2n-k)$ is a hook shape,*

$$u_\lambda = \begin{cases} 1 & \text{if } k=2a \text{ and } a \text{ is even} \\ 1 & \text{if } k = 2a+1 \text{ and } a \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

(iii) *If $\lambda = (1^k, 2^l, p, q)$ where $2 \le p \le q$*

$$u_\lambda = \begin{cases} 1 & \text{if } q+p \in \{2b, 2b+2\} \text{ and } \frac{k}{2}+p \text{ is even} \\ 1 & \text{if } q+p=2b+1 \\ 0 & \text{otherwise} \end{cases}$$

(b) (i) $v_\lambda = 0$ *if* λ *is not of hook shape or double hook shape*

(ii) *If* $\lambda = (1^k, 2n-k)$ *is a hook shape,*

$$v_\lambda = \begin{cases} 1 & \textit{if } k{=}2a \textit{ and } a \textit{ is odd} \\ 1 & \textit{if } k = 2a{+}1 \textit{ and } a \textit{ is even} \\ 0 & \textit{otherwise} \end{cases}$$

(iii) *If* $\lambda = (1^k, 2^l, p, q)$ *where* $2 \le p \le q$ *is a double hook shape, then*

$$u_\lambda = \begin{cases} 1 & \textit{if } q{+}p \in \{2b, 2b{+}2\} \textit{ and } \frac{k}{2}{+}p \textit{ is odd} \\ 1 & \textit{if } q{+}p{=}2b{+}1 \\ 0 & \textit{otherwise} \end{cases}$$

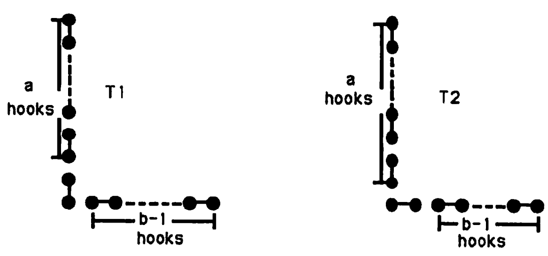
Proof: First we use the fact that $s_2 = \frac{1}{2}(p_1^2 + p_2)$ and $s_{1^2} = (p_1^2 - p_2)$. Moreover, clearly $p_1[s_\lambda] = s_\lambda$ for all λ so that

$$s_2[s_{(1^a,b)}] = \frac{1}{2}(s_{(1^a,b)}s_{(1^a,b)} + s_{(1^a,b)}[p_2]) \tag{26}$$

$$s_{1^2}[s_{(1^a,b)}] = \frac{1}{2}(s_{(1^a,b)}s_{(1^a,b)} - s_{(1^a,b)}[p_2]) \tag{27}$$

Thus we need to compute $s_{(1^a,b)}s_{(1^a,b)}$ and $s_{(1^a,b)}[p_2]$. By Theoreom 1, to compute $s_{(1^a,b)}[p_2]$, we must generate all $(1^a,b)$-bispecial rim hook tabloids T of type (2^{a+b}). Let $B(1^a,b)$ denote the set of all $(1^a,b)$-bispecial rim hook tabloids T of type (2^{a+b}). It is easy to see that if $T \in B(1^a,b)$, then T must be of hook shape or double hook shape. Now consider a $T \in B(1^a,b)$ of hook shape. Other than the initial rim hook of T, we must have a special rim hooks of size 2 all of which lie in the first column of T and $(b-1)$ special rim hooks of size 2 all of which lie in the first row of T. Since the initial rim hook of T is either horizontal or vertical, it is easy to see that there are two posibilities for such T which are pictured in figure 6.

Figure 6



$sh(T_1) = (1^{2a+1}, 2b-1)$ $\quad$ $sh(T_2) = (1^{2a}, 2b)$

$sgn(T_1) = (-1)^{a+1}$ $\quad$ $sgn(T_2) = (-1)^a$

Thus if we let $l_\lambda = \langle s_{(1^a,b)}[p_2], s_\lambda \rangle$, then if $\lambda = (1^k, 2n-k)$ is a hook shape,

$$l_\lambda = \begin{cases} (-1)^{a+1} & \text{if k=2a+1} \\ (-1)^a & \text{if k=2a} \\ 0 & \textit{Otherwise} \end{cases} \tag{28}$$

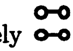
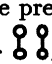
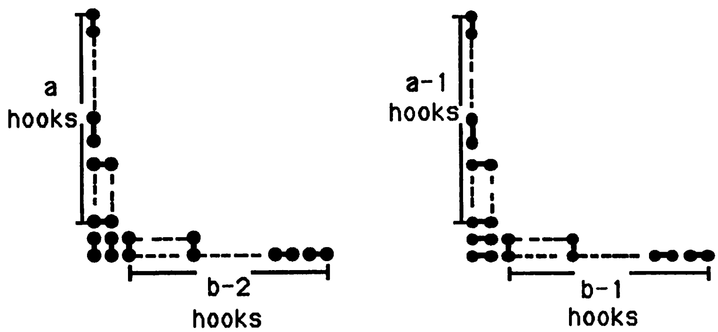
Next consider the $T \in B(1^a, b)$ of double hook shape. There are precisely two ways to fill the 2×2 corner square of T with hooks of size 2, namely  or . Thus any such T have one of the two forms pictured in figure 7.

Figure 7



It follows that the number of squares in the first two rows of T is either *2b* or *2b + 2*. Using this it is easy to show that if $\lambda = (1^k, 2^l, p, q)$ where $2 \le p \le q$ is a double hook shape, then

$$l_\lambda = \begin{cases} (-1)^{(k/2)+p} & \text{if p+q = 2b} \\ (-1)^{(k/2)+p-2} & \text{if p+q = 2b + 2} \\ 0 & \textit{otherwise} \end{cases} \tag{29}$$

Note the fact that λ is a partition of $2n$ and $p + q \in \{2b, 2b+2\}$ automatically forces that k is even.

Next let $t_\lambda = \langle s_{(1^a,b)} s_{(1^a,b)}, s_\lambda \rangle$. One can show by a careful analysis of the Remmel-Whitney rule to expand $s_{(1^a,b)} s_{(1^a,b)}$ as a sum of Schur functions that we have the following. First t_λ=0 unless λ is a hook shape or a double hook shape. If $\lambda = (1^k, 2n-k)$ is a hook shape, then

$$t_\lambda = \begin{cases} 1 & \text{if k=2a or k=2a+1} \\ 0 & \textit{otherwise} \end{cases} \tag{30}$$

Finally if $\lambda = (1^k, 2^k, p, q)$ where $2 \le p \le q$ is a double hook shape, then

$$t_\lambda = \begin{cases} 1 & \text{if p+q} \in \{2b,2b+2\} \\ 2 & \text{if p+q} = 2b+1 \\ 0 & \textit{otherwise} \end{cases} \tag{31}$$

If we combine the results of (24)-(27) using equations (22) and (23), then one can easily derive the explicit formulas for u_λ and v_λ. □

Next we present the SXP-algorithm of Chen, Garsia and Remmel (ref. [2]) to compute $s_\mu[p_k]$ from which Theorem 1 can be derived. They show that if μ is a partition of n, then

$$s_\mu[p_k] = \sum_{|I_0|+\cdots+|I_{k-1}|=n} c^\mu_{I_0,\ldots,I_{k-1}} SS_{I_0,\ldots,I_{k-1}}(x) \tag{32}$$

where

(a) the sum is to be carried out over all k-tuples of partitions $I_0,\ldots,I_{k-1}$ whose diagrams are contained in μ and whose sum of parts add up to n,

(b) we have

$$c^\mu_{I_0,\cdots,I_{k-1}} = \langle s_{I_0}\cdots s_{I_{k-1}}, s_\mu\rangle \tag{33}$$

and

(c) the expression $SS_{I_0,\ldots,I_{k-1}(x)}$ denotes certain signed Schur function indexed by a partition with empty k-core whose construction is best explained through an example.

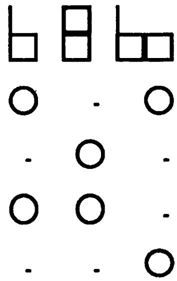
For instance, in the expansion of $s_{113}(x^3)$, since n=5 and k=3, one of the terms in (a) is that which corresponds to the triple of partitions $I_0 = (1), I_1 = (1^2), I_2 = (2)$. By using the Remmel-Whitney rule for multiplying Schur functions (ref. [9]), we obtain

$$c^{(113)}_{(1),(1^2),(2)} = 2.$$

To construct

$$SS_{(1),(1^2),(2)}(x),$$

we proceed as follows. First of all we represent $(1),(1^2)$, and (2) as partitions with a equal number of parts, that is, we write $(0,1)$ instead of (1), (1^2) since it already has two parts, and $(0,2)$ instead of (2). This given, we construct the *circle diagram* given below

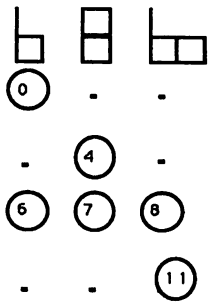


(34)

The precise rule for putting together a column of this diagram from a partition $(i_1, i_2, \ldots, i_m)$ is that the distance (in dots) between the s^{th} and $(s+1)^{th}$ circle is given by the difference $i_{s+1} - i_s$, or equivalently the s^{th} circle is at distance $(i_s + s - 1)$ from the top. That is pictorially we have

$$
\begin{array}{l}
(i_1, i_2, \ldots, i_m) \\
\cdot \\
\cdot \\
\mathrm{o} \leftarrow i_1 \\
\cdot \\
\cdot \\
\mathrm{o} \leftarrow i_2 + 1 \\
\cdot \\
\cdot \\
\cdot \\
\mathrm{o} \leftarrow i_m + m - 1
\end{array}
$$

Accordingly, in the column labelled by (1,1) the first circle is at distance 1 from the top and the second circle is at distance one from the first. Proceeding in the same manner for the other two partitions we obtain the circle diagram given in (34). This done, assign to the positions in the diagram (indicated by dots when not by circles) the labels 0,1,2,3,... sucessively from left to right and from top to bottom, and record the label only when it falls in one of the circles. This gives the labelled diagram



(35)

In the case of a general k-tuple $I_0, I_1, \dots, I_{k-1}$ we obtain a circle diagram with m circles in each column, where m is the maximum number of non-zero parts appearing in any of the partitions $I_0, I_1, \dots, I_{k-1}$.

Let $b_1 < b_2 < b_3 < \cdots < b_{m\cdot k}$ be the labels placed on the circles and $q_{s,1} < q_{s,2} < \cdots < q_{s,m}$ be the labels appearing in the column corresponding to the partition I_s. Finally, let $inv(I_0, \dots, I_{k-1})$ denote the number of inversions of the permutation

$$q_{0,1}q_{0,2}\cdots q_{0,m}q_{1,1}q_{1,2}\cdots q_{1,m}\cdots q_{k-1,1}q_{k-1,2}\cdots q_{k-1,m} \tag{36}$$

and let $sh(I_0, \dots, I_{k-1}) = (b_1, b_2 - 1, b_3 - 2, \dots, b_{m\cdot k} - mk + 1)$. This given, we set

$$SS_{I_0,\dots,I_{k-1}} = (-1)^{\binom{m}{2}\binom{k}{2}+inv(I_0,\dots,I_{k-1})} s_{sh(I_0,\dots,I_k)} \tag{37}$$

We note that $I_0, \dots, I_{k-1}$ is also called the k-quotient of $sh(I_0, \dots, I_{k-1})$, see (ref. [5]). Going back to our particular example, we can easily see that to calculate the number of inversion of the permutation

$$0, 6, 4, 7, 2, 11,$$

we need only count, for each circle in the diagram of (35), the number of circles that are North-East of it, and add all these counts. This gives

$$inv((1), (1^2), (2)) = 0 + 0 + 1 + 2 + 1 + 0 = 4$$

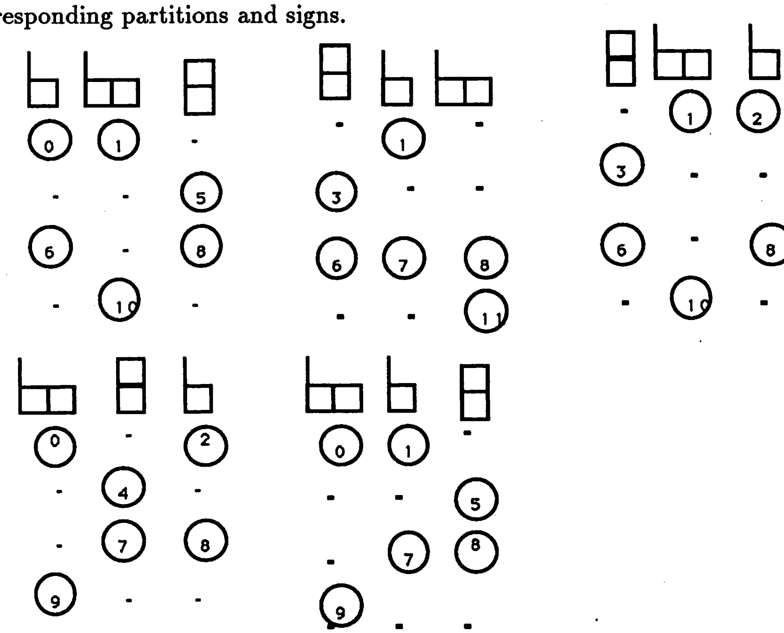
at the same time, we have $\binom{m}{2}\binom{k}{2} = \binom{2}{2}\binom{3}{2} = 3$ and

$$I(I_0, \cdots, I_{k-1}) = (0-0, 2-1, 4-2, 6-3, 7-4, 11-5) = (0, 1, 2, 3, 3, 6)$$

So we finally obtain in this case

$$SS_{(1)(1^2)(2)} = -s_{12336}(x).$$

It is worthwhile noting at this point that, in view of (33), the coefficient $c^{\mu}_{I_0,\dots,I_{k-1}}$ does not depend on the order in which $I_0, \dots, I_{k-1}$ are given. This means that in our particular example we can take advantage of the result that $c^{(113)}_{(1),(1^2),(2)} = 2$ and obtain 5 additional terms in the expansion of $s_{113}[p_3]$ by carrying out the above process for each of the remaining permutations of the triplet $((1),(1^2),(2))$. In the table below we give the resulting diagrams and the corresponding partitions and signs.



We thus obtain that the contribution to the expansion of $s_{113}[p_3]$ coming from the triplet $((1),(1^2),(2))$ is the expression

$$-2(s_{123^26}(x) + s_{3^245}(x) - s_{1^33^26}(x) + s_{1^345}(x) - s_{124^3}(x) + s_{34^3}(x)).$$

Taking all of this into account, we can easily see that the SXP algorithm decomposes into successive applications of the following 3 basic steps. Namely, to calculate the expansion of $s_\mu[p_k]$ when μ is a partition of n, we proceed as follows.

Step 1. We pick a k-tuple of partitions $I_0, I_1, \ldots, I_{k-1}$ satisfying the 3 conditions

(a) The Ferrers' diagram of each I_s is contained in that of μ.

(b) $| I_0 | \leq | I_1 | \leq \cdots \leq | I_{k-1} |$

(c) $| I_0 | + | I_1 | + \cdots + | I_{k-1} | = n$.

This done we calculate the coefficent $c^{\mu}_{I_0,\ldots,I_{k-1}}$ by the Schur function multiplication algorithm. If this coeficient is not zero we proceed to the next step. Otherwise we repeat step 1.

Step 2. Pick a permutation $I_{\sigma_1}, I_{\sigma_2}, \ldots, I_{\sigma_k}$ of $I_0, I_1, \ldots, I_{k-1}$ and construct the labelled circle diagram whose s^{th} column is indexed by I_{σ_s}.

Step 3. Calculate the partition $sh(I_{\sigma_1}, I_{\sigma_2}, \ldots, I_{\sigma_k})$and the corresponding sign.

Step 2 and 3 are to be repeated over all distinct permutations of $I_0, I_1, \ldots, I_{k-1}$. This done we go back to step 1 and repeat the process over all posible choices of $I_0, I_1, \ldots, I_{k-1}$ satisfying (a) (b) and (c).

We prove Theorem 1 by showing that our algorithm to compute $s_{(1^a,b)}[p_n]$ is equivalent to the SXP algorithm. To do this we need to prove two things. First, it is well known that $c^{\mu}_{I_0,\ldots,I_{k-1}} = 0$ unless $I_j \subseteq \mu$ for all j. Thus when μ is a hook, it must be the case that I_j is hook for $j = 0, \cdots, k-1$. In our case, we must show that if ν is a partition such that there is a $(1^a, b)$-bispecial rim hook tabloid of shape ν and type (k^{a+b}), then ν is a partition with empty k-core and the k-quotient of $\nu = (I_0, \ldots, I_{k-1})$ where I_j is hook for $j = 0, \cdots, k-1$. Second, we must show that if ν is a partition with empty k-core and the k-quotient of $\nu = (I_0, \ldots, I_{k-1})$ where I_j is hook for $j = 0, \cdots, k-1$, then then number of $(1^a, b)$-bispecial rim hook tabloids T of shape ν and type (k^{a+b}) is precisely $c^{(1^a,b)}_{I_0,\ldots,I_{k-1}}$ and the sign of any such T agrees with the sign given in (37). These two facts are proved by studying how the movement of circles in a circle diagram D effects the shape of the partition ν associated with D and the k-quotient of ν.

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