From algebraic sets to monomial linear bases by means of combinatorial algorithms.

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1.1 Let $\mathbf{N}$ be the monoid of non-negative integers. Denote by $\mathbf{i} := (i_1, \ldots, i_n)$ an arbitrary element in the power $\mathbf{N}^n$. The usual order on $\mathbf{N}$, as well as the partial order it induces on $\mathbf{N}^n$, will be denoted by $\leq$.

Define an *n-dimensional Ferrers diagram* to be any ideal of the poset $\mathbf{N}^n$, i.e. any non-empty subset $\mathcal{F} \subseteq \mathbf{N}^n$ such that $\mathbf{j} < \mathbf{i} \in \mathcal{F} \Longrightarrow \mathbf{j} \in \mathcal{F}$. An element $\mathbf{i} = (i_1, \ldots, i_n) \notin \mathcal{F}$ is said to be a *co-minimal element* for the Ferrers diagram $\mathcal{F}$ if it is a minimal element of the complementary filter $\mathbf{N}^n \setminus \mathcal{F}$, i.e. if $(i_1, \ldots, i_{r-1}, i_r - 1, i_{r+1}, \ldots, i_n) \in \mathcal{F}$ for each r such that $i_r \geq 1$. Of course, $\mathbf{i} \notin \mathcal{F}$ is a co-minimal element iff $\mathcal{F}' := \{\mathbf{i}\} \cup \mathcal{F}$ is a Ferrers diagram.

We will write $\preceq$ for any *term-ordering* on $\mathbf{N}^n$, i.e. a linear ordering which is compatible with the monoid structure on $\mathbf{N}^n$:

$$\mathbf{0} \prec \mathbf{i} \qquad \text{for every } \mathbf{i} \neq \mathbf{0} \text{ in } \mathbf{N}^n$$

$$\mathbf{i} \preceq \mathbf{j} \Longrightarrow \mathbf{i} + \mathbf{r} \preceq \mathbf{j} + \mathbf{r} \qquad \text{for every } \mathbf{i}, \mathbf{j}, \mathbf{r} \in \mathbf{N}^n.$$

It is well known that any term-ordering on $\mathbf{N}^n$ is also a well-ordering.

1.2 Let K be a field and let $X := \{x_1, x_2, \ldots, x_n\}$ be a given set of indeterminates. Let us consider the usual polynomial algebra $K[X] := K[x_1, \ldots, x_n]$. Denote by $M_X \subseteq K[X]$ the free abelian monoid on X. The elements of M_X (i.e. the monic monomials) will be called *terms* of $K[X]$ and denoted by $\mathbf{x}^{\mathbf{i}} := x_1^{i_1} \cdots x_n^{i_n}$ with $\mathbf{i} := (i_1, \ldots, i_n) \in \mathbf{N}^n$. The orders $\leq$ and $\preceq$ on $\mathbf{N}^n$, as well as the notion of Ferrers diagram, extend to M_X in an obvious way.

An ideal J of the algebra $K[X]$ is said to be *cofinite* if

$$\operatorname{codim}(J) := \dim(K[X]/J) < \infty$$

Given a finite set $\mathcal{P} := \{P_1, \ldots, P_N\} \subseteq K^n$, the ideal

$$\Im(\mathcal{P}) := \{p \in K[X] \mid (\forall P \in \mathcal{P})(p(P) = 0)\}$$

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MB2. [Put the first i points of the list $\mathcal{P}$ in $\mathcal{Q}$.]
Set $\mathcal{Q} \leftarrow \{P_j \mid 1 \leq j \leq i\}$.

MB3. [Find which coordinate of $\mathbf{d}_{i+1}$ has to be changed, say $d_{i+1,s}$.]
Set $s \leftarrow \max\{k \geq 1 \mid \pi_{k-1}(P_j) = \pi_{k-1}(P_{i+1})$, for some $P_j \in \mathcal{Q}\}$.
($s-1$ is the length of the longest initial segment shared by P_{i+1} and some point $P_j \in \mathcal{Q}$. If $s > 1$, then in successive steps this decreases.)

MB4. [Find the points that determine the s-th coordinate of $\mathbf{d}_{i+1}$.]
Set $\mathcal{E} \leftarrow \{j \mid P_j \in \mathcal{Q},\ \ \pi_{s-1}(P_j) = \pi_{s-1}(P_{i+1}),\ \ \pi^{s+1}(\mathbf{d}_j) = \pi^{s+1}(\mathbf{d}_{i+1})\}$.
(Indices of the points of $\mathcal{Q}$ which have the first $s-1$ coordinates equal to those of P_{i+1} and whose corresponding elements in $\underline{\mathcal{F}}$ have the $n-s$ rightmost coordinates equal to those of $\mathbf{d}_{i+1}$. $\mathcal{E}$ is always not-empty.)

MB5. [Assign the value to the s-th coordinate of $\mathbf{d}_{i+1}$.]
Set $d_{i+1,s} \leftarrow (1 + \max\{d_{j,s} \mid j \in \mathcal{E}\})$.

MB6. [Did you determine the first coordinate of $\mathbf{d}_{i+1}$?]
If $s > 1$

MB.6.1. [Find the points that determine another coordinate of $\mathbf{d}_{i+1}$.]
Set $\mathcal{Q} \leftarrow \{P_j \mid 1 \leq j \leq i,\ \pi^s(\mathbf{d}_j) = \pi^s(\mathbf{d}_{i+1}) = (d_{i+1,s}, \ldots, d_{i+1,n})\}$.
(Points of $\underline{\mathcal{P}}$ whose corresponding elements in $\underline{\mathcal{F}}$ have the $n-s+1$ rightmost coordinates equal to those of $\mathbf{d}_{i+1}$.)

MB6.2. [Is $\mathcal{Q}$ empty?]
If $\mathcal{Q} \neq \emptyset$, return to step **MB3.**

MB7. [Increase i]
Set $i \leftarrow i + 1$. If $i < N$, return to step **MB2.**

MB8. [Done] Terminate the algorithm.

We put: $\mathcal{MB}(\mathcal{P}) := \{\mathbf{d}_1, \ldots, \mathbf{d}_N\}$; $\delta_{\underline{\mathcal{P}}}: \mathcal{P} \to \mathcal{MB}(\mathcal{P})$, $P_i \mapsto \mathbf{d}_i$. One could think that $\mathcal{MB}(\mathcal{P})$ is ill-defined, that is it depends on the order which has been used for arranging the points $P_1, \ldots, P_N$ when starting **Algorithm MB** (i.e. on the list $\underline{\mathcal{P}}$) rather than on the set $\mathcal{P} = \{P_1, \ldots, P_N\}$ itself. Well, this is not true. In fact, it is possible to prove the following propositions.

Prop. 1 *Let $\underline{\mathcal{F}'} := (\mathbf{d}'_1, \ldots, \mathbf{d}'_N)$ be the list associated with the list of points $\underline{\mathcal{P}'} = (P_{\sigma(1)}, \ldots, P_{\sigma(N)})$, where $\sigma \in S_N$, by* **Algorithm MB**. *Then, for some $\tau \in S_N$, $(\mathbf{d}'_1, \ldots, \mathbf{d}'_N) = (\mathbf{d}_{\tau(1)}, \ldots, \mathbf{d}_{\tau(N)})$.* □

Corollary 1 *If $\mathcal{P} \subseteq \mathcal{Q}$, then $\mathcal{MB}(\mathcal{P}) \subseteq \mathcal{MB}(\mathcal{Q})$.* □

of the algebraic set $\mathcal{P}$ is cofinite. More generally, when K is algebraically closed, from the *Nullstellensatz* it follows that the ideal $J \subseteq K[X]$ is a cofinite ideal iff the algebraic set of J, i.e.

$$\mathcal{V}(J) := \{P \in K^n \mid (\forall p \in J)(p(P) = 0)\},$$

is finite. In this case we have $\#\mathcal{V}(J) \leq \text{codim}(J)$ (cfr. [3] p.23).

For a given ideal J, any linear basis $\mathcal{L}_J$ of the quotient algebra $K[X]/J$ whose elements are of the form $[\mathbf{x}^{\mathbf{i}}]_J := \mathbf{x}^{\mathbf{i}} + J$ will be called a *monomial basis*. If $\mathcal{L}_J = \{\mathbf{x}^{\mathbf{i}} + J \mid \mathbf{i} \in L \subseteq \mathbf{N}^n\}$ is a monomial basis, we shall say that $\{\mathbf{x}^{\mathbf{i}} \mid \mathbf{i} \in L\}$ is a *system of representatives* for the monomial basis $\mathcal{L}_J$. Obviously, if $\mathcal{L}_J = \{\mathbf{x}^{\mathbf{i}} + J \mid \mathbf{i} \in L\}$ is a monomial basis of $K[X]/J$, then any polynomial $p \in K[X]$ is congruent modulo J to exactly one polynomial of the form $\sum_{\mathbf{i} \in L} a_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}$, $a_{\mathbf{i}} \in K$. Given an arbitrary term-ordering $\preceq$ on M_X, a monomial basis $\mathcal{L}_J = \{\mathbf{x}^{\mathbf{i}_1} + J, \ldots, \mathbf{x}^{\mathbf{i}_N} + J\}$ with $\mathbf{x}^{\mathbf{i}_1} \prec \ldots \prec \mathbf{x}^{\mathbf{i}_N}$ is said to be *minimal* with respect to $\preceq$ if for any other monomial basis $\mathcal{L}'_J = \{\mathbf{x}^{\mathbf{i}'_1} + J, \ldots, \mathbf{x}^{\mathbf{i}'_N} + J\}$ with $\mathbf{x}^{\mathbf{i}'_1} \prec \ldots \prec \mathbf{x}^{\mathbf{i}'_N}$ we have $\mathbf{x}^{\mathbf{i}_j} \preceq \mathbf{x}^{\mathbf{i}'_j}$ for $j = 1, \ldots, N$. Of course, both $\mathcal{L}_J$ and $\mathcal{L}'_J$, $\mathcal{L}_J \neq \mathcal{L}'_J$, could be minimal with respect to different term-orderings. It is not hard to prove that if $\mathcal{L}_J = \{\mathbf{x}^{\mathbf{i}} + J \mid \mathbf{i} \in L\}$ is a minimal monomial basis then $L \subseteq \mathbf{N}^n$ is an n-dimensional Ferrers diagram.

1.3 In the search for a minimal monomial basis $\mathcal{L}_{\mathcal{P}}$, we present a purely combinatorial algorithm to get it from $\mathcal{P}$. In fact, making use of the **Algorithm MB** below, we associate a Ferrers diagram $\mathcal{MB}(\mathcal{P}) = \{\mathbf{d}_1, \ldots, \mathbf{d}_N\} \subseteq \mathbf{N}^n$ to any finite set $\mathcal{P} := \{P_1, \ldots, P_N\} \subseteq K^n$. The Ferrers diagram $\mathcal{MB}(\mathcal{P})$ gives the monomial basis $\mathcal{L}_{\mathcal{P}} = \{\mathbf{x}^{\mathbf{d}} + \Im(\mathcal{P}) \mid \mathbf{d} \in \mathcal{MB}(\mathcal{P})\}$ which is minimal with respect to the inverse lexicographical ordering $\preceq_{i.l.}$ with $x_1 \prec_{i.l.} x_2 \prec_{i.l.} \cdots \prec_{i.l.} x_n$.

Put:

$\mathcal{P} := \{P_1, \ldots, P_N\} \subseteq K^n$
$\underline{\mathcal{P}} := (P_1, \ldots, P_N)$
$\mathbf{d}_j = (d_{j,1}, \ldots, d_{j,n}) \in \mathbf{N}^n$
$\pi_s \colon K^n \longrightarrow K^s, \quad (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_s)$
($\pi_0(P)$ is assumed to be the empty sequence.)
$\pi^s \colon K^n \longrightarrow K^{n-s+1}, \quad (x_1, \ldots, x_n) \mapsto (x_s, \ldots, x_n)$

ALGORITHM MB. Given a list $\underline{\mathcal{P}} := (P_1, \ldots, P_N)$ of points, we determine an ordered Ferrers diagram $\underline{\mathcal{F}} := (\mathbf{d}_1, \ldots, \mathbf{d}_N)$.

MB1. [Initialize.]
Set $\mathbf{d}_1 \leftarrow \mathbf{d}_2 \leftarrow \ldots \mathbf{d}_N \leftarrow (0, \ldots, 0)$; $i \leftarrow 1$.
(In the beginning the coordinates of all the elements of $\underline{\mathcal{F}}$ *are zero.)*

Prop. 2 *It is possible to arrange the points $P_1, \dots, P_N$ in a suitable list $\underline{\mathcal{P}'} = (P_{\sigma(1)}, \dots, P_{\sigma(N)})$ such that the elements in the corresponding list $\underline{\mathcal{F}'} = (\mathbf{d}_{\tau(1)}, \dots \dots, \mathbf{d}_{\tau(N)})$ are arranged according to the inverse lexicographical order $\preceq_{i.l.}$: $s < t \Rightarrow \mathbf{d}_{\tau(s)} \prec_{i.l.} \mathbf{d}_{\tau(t)}$* □

Lemma 1 *Let $\underline{\mathcal{F}} := (\mathbf{d}_1, \dots, \mathbf{d}_N)$ be the Ferrers diagram associated to $\underline{\mathcal{P}} := (P_1, \dots, P_N)$ by* **Algorithm MB**; *let $\mathbf{d}_N = (d_{N,1}, \dots, d_{N,n})$. If $d_{N,i} \neq 0$, then there is some $k < N$ such that $\mathbf{d}_k = (d_{N,1}, \dots, d_{N,i-1}, d_{N,i}-1, d_{N,i+1}, \dots, d_{N,n})$.*

Proof. In order to calculate the first $i-1$ coordinates $d_{N,1}, \dots, d_{N,i-1}$ of $\mathbf{d}_N = \delta_{\underline{\mathcal{P}}}(P_N)$, we have to consider the set $\{P_{j_1}, \dots, P_{j_s}, P_{j_{s+1}} = P_N\} \subseteq \mathcal{P}$ $j_1 < \dots < j_s < N$, of all points $P_{j_r} \in \mathcal{P}$ such that $\pi^i(\delta_{\underline{\mathcal{P}}}(P_{j_r}) = (d_{N,i}, \dots, d_{N,n})$. Putting $\tilde{\mathcal{P}} := \{\pi_{i-1}(P_{j_1}), \dots, \pi_{i-1}(P_{j_s}), \pi_{i-1}(P_N)\}$, we have $\pi_{i-1}(\delta_{\underline{\mathcal{P}}}(P_N)) = \delta_{\underline{\tilde{\mathcal{P}}}}(\pi_{i-1}(P_N)) = (d_{N,1}, \dots, d_{N,i-1})$. For every $r \in \{1, \dots, s+1\}$, there is a point $P_{j'_r} \in \mathcal{P}$, $j'_r < j_r$, such that $d_{j'_r,i} = d_{j_r,i} - 1 = d_{N,i} - 1, d_{j'_r,i+1} = d_{j_r,i+1} = d_{N,i+1}, \dots, d_{j'_r,n} = d_{j_r,n} = d_{N,n}$ and $\pi_{i-1}(P_{j'_r}) = \pi_{i-1}(P_{j_r})$. Up to a suitable rearrangement of the points in $\underline{\mathcal{P}}$, we may assume that $j'_1 < \dots < j'_{s+1}$. When calculating $\pi_{i-1}(\delta_{\underline{\mathcal{P}}}(P_{j'_{s+1}}))$ we have to consider the set $\mathcal{Q} := \{P_l \in \mathcal{P} \mid l \leq j'_{s+1}, \pi^i(\mathbf{d}_l) = (d_{j'_{s+1},i}, \dots, d_{j'_{s+1},n}) = (d_{N,i} - 1, d_{N,i+1} \dots, d_{N,n})\}$. Of course $P_{j'_r} \in \mathcal{Q}$. Let $\tilde{\mathcal{Q}} := \{\pi_{i-1}(P) \mid P \in \mathcal{Q}\} \subseteq K^{i-1}$; we have $\tilde{\mathcal{P}} \subseteq \tilde{\mathcal{Q}}$. Hence $\mathcal{MB}(\tilde{\mathcal{P}}) \subseteq \mathcal{MB}(\tilde{\mathcal{Q}}) \subseteq \mathbf{N}^{i-1}$; in particular $(d_{N,1}, \dots, d_{N,i-1}) \in \mathcal{MB}(\tilde{\mathcal{Q}})$. It follows that there exists a point $P_k \in \mathcal{Q}$ such that $\delta_{\underline{\mathcal{P}}}(P_k) = (d_{N,1}, \dots, d_{N,i-1}, d_{N,i} - 1, d_{N,i+1} \dots, d_{N,n})$. □

As a straightforward consequence of **Lemma 1** we get:

Prop. 3 *The set $\mathcal{MB}(\mathcal{P})$ is an n-dimensional Ferrers diagram.* □

Prop. 4 *The set $\mathcal{L}_{\mathcal{P}} = \{\mathbf{x}^{\mathbf{d}} + \Im(\mathcal{P}) \mid \mathbf{d} \in \mathcal{MB}(\mathcal{P})\}$ is a monomial linear basis for $K[X]/\Im(\mathcal{P})$.*

Proof. By induction on the dimension n of K^n.

For $n = 1$, we have $\mathcal{P} = \{\rho_1, \dots, \rho_n\}$, $\rho_i \in K$, and $\Im(\mathcal{P}) = (g)$, with $g = \Pi_{i=1}^N (x - \rho_i)$. **Algorithm MB** gives $\mathcal{MB}(\mathcal{P}) = \{0, 1, \dots, N-1\}$; hence $\mathcal{L}_{\mathcal{P}} = \{\mathbf{x}^{\mathbf{d}} + \Im(\mathcal{P}) \mid \mathbf{d} \in \mathcal{MB}(\mathcal{P})\} = \{1 + (g), x + (g), \dots, x^{N-1} + (g)\}$, which is a minimal monomial basis for $K[X]/(g)$.

Suppose now that the statement is true for every finite subset of $K^{n'}$, $n' < n$, and prove it for $\mathcal{P} = \{P_1, \dots, P_N\} \subset K^n$. As $\dim K[X]/\Im(\mathcal{P}) = N = \#\mathcal{MB}(\mathcal{P})$, it remains to prove that the residue classes mod.$\Im(\mathcal{P})$ of the monomials $\mathbf{x}^{\mathbf{d}}$, $\mathbf{d} \in \mathcal{MB}(\mathcal{P})$, are linearly independent over $K[X]/\Im(\mathcal{P})$; in other words, we have to prove that any polynomial of the form

$$(1) \qquad p(x_1, \dots, x_n) = \sum_{\mathbf{d} \in \mathcal{MB}(\mathcal{P})} \alpha_{\mathbf{d}} x^{\mathbf{d}} \in \Im(\mathcal{P}), \qquad \alpha_{\mathbf{d}} \in K$$

is the zero polynomial.

Putting $D := \mathcal{MB}(\mathcal{P})$, $D_r := \{\mathbf{d} = (d_1, \ldots, d_n) \in D \mid d_n = r\}$ and $\mathcal{P}_r := \{P \in \mathcal{P} \mid \delta_{\underline{\mathcal{P}}}(P) \in D_r\}$, it is easy to check that

$$(2) \qquad \mathcal{MB}(\pi_{n-1}(\mathcal{P}_r)) = \pi_{n-1}(D_r).$$

Let us write down polynomial (1) in the form

$$(3) \qquad p(x_1, \ldots, x_n) = \sum_{r=0}^{h} p_r(x_1, \ldots, x_{n-1}) x_n^r$$

where $h = \max\{d_n \mid \mathbf{d} = (d_1, \ldots, d_n) \in D\}$ and

$$(4) \qquad p_r(x_1, \ldots, x_{n-1}) = \sum_{(d_1, \ldots, d_{n-1}) \in \pi_{n-1}(D_r)} \alpha_{(d_1, \ldots, d_{n-1}, r)} x_1^{d_1} \cdots x_{n-1}^{d_{n-1}}.$$

The polynomial $p(x_1, \ldots, x_n) \in \Im(\mathcal{P})$ has to vanish at every point in $\mathcal{P}$. Consider a point $P = (a_1, \ldots, a_n) \in \mathcal{P}_h \subseteq \mathcal{P}$; there exist in $\mathcal{P}$ exactly $h + 1$ points that have the same first $n - 1$ coordinates as $P = (a_1, \ldots, a_n)$. It follows that the polynomial

$$p(a_1, \ldots, a_{n-1}, x_n) = \sum_{r=0}^{h} p_r(a_1, \ldots, a_{n-1}) x_n^r$$

vanishes identically. In particular $p_h(a_1, \ldots, a_{n-1}) = 0$ for every $(a_1, \ldots, a_{n-1}) \in$ $\in \mathcal{Q} := \pi_{n-1}(\mathcal{P}_h)$. Hence

$$(5) \qquad p_h(x_1, \ldots, x_{n-1}) \in \Im(\mathcal{Q}) \subseteq K[x_1, \ldots, x_{n-1}].$$

By (2) and (4) we have

$$(6) \qquad p_h(x_1, \ldots, x_{n-1}) = \sum_{(d_1, \ldots, d_{n-1}) \in \mathcal{MB}(\mathcal{Q})} \alpha_{(d_1, \ldots, d_{n-1}, h)} x_1^{d_1} \cdots x_{n-1}^{d_{n-1}}$$

Because of the inductive hypothesis, the set $\{x_1^{d_1} \cdots x_{n-1}^{d_{n-1}} + \Im(\mathcal{Q}) \mid (d_1, \ldots$ $\ldots, d_{n-1}) \in \mathcal{MB}(\mathcal{Q})\}$ is a monomial basis for $K[x_1, \ldots, x_{n-1}]/\Im(\mathcal{Q})$. ¿From this and from (5) we deduce that polynomial (6) vanishes identically. Hence

$$(7) \qquad p(x_1, \ldots, x_n) = \sum_{r=0}^{h-1} p_r(x_1, \ldots, x_{n-1}) x_n^r$$

Arguing for $r = h - 1, h - 2, \ldots, 1, 0$ as for $r = h$, we conclude that $p_r(x_1, \ldots, x_{n-1})$ is the zero polynomial for every $r \in \{0, \ldots, h\}$. □

Prop. 5 *The monomial linear basis $\mathcal{L}_{\mathcal{P}} = \{\mathbf{x^d} + \Im(\mathcal{P}) \mid \mathbf{d} \in \mathcal{MB}(\mathcal{P})\}$ for $K[X]/\Im(\mathcal{P})$ is minimal with respect to the inverse lexicographical order $\preceq_{i.l.}$ on M_X.*

Proof. Let $\underline{\mathcal{P}} = (P_1, \dots, P_N)$, $\underline{D} := \mathcal{MB}(\mathcal{P}) = (\mathbf{d}_1, \dots, \mathbf{d}_N)$; $\mathbf{d}_i = (d_{i,1}, \dots$ $\dots, d_{i,n})$, $P_i = (a_{i,1}, \dots, a_{i,n})$ for $i = 1, \dots, N$. Let $h := \max\{d_{i,n} \mid i = 1, \dots, N\}$, and $\mathcal{P}_h = \{P_{j_1}, \dots, P_{j_m}\} \subseteq \mathcal{P}$; for any P_{j_r} there are in $\mathcal{P}$ exactly $h+1$ points which have the same $n-1$ coordinates as P_{j_r}; let us denote them by $Q_{j_r,0}, \dots, Q_{j_r,h} = P_{j_r}$ Up to a suitable rearrangement we may assume that they are the last $(h+1)m$ elements in the list $\underline{\mathcal{P}}$, i.e.

$$(Q_{j_1,0}, \dots, Q_{j_1,h}, \dots, Q_{j_m,0}, \dots, Q_{j_m,h}) = (P_{N-(h+1)m+1}, P_{N-(h+1)m+2}, \dots, P_N),$$

so that

$$d_{N-(h+1)(m-1),n} = d_{N-(h+1)(m-2),n} = \cdots = d_{N-(h+1),n} = d_{N,n} = h$$

We have to prove that for any $\mathbf{d}' = (d'_1, \dots, d'_n)$ such that $\mathbf{d}' \prec_{i.l} \mathbf{d}_N$ there exists in $\Im(\mathcal{P})$ a polynomial of the form

$$\sum_{i=1}^{N-1} \alpha_i \mathbf{x}^{\mathbf{d}_i} + \alpha \mathbf{x}^{\mathbf{d}'} \in \Im(\mathcal{P}). \tag{8}$$

Because of **MB.6.1** of **Algorithm MB**, without loss of generality we may assume that $d'_n < d_{N,n}$. Observe that (8) is equivalent to

$$\sum_{i=1}^{N-1} \alpha_i a_{s,1}^{d_{i,1}} \cdots a_{s,n}^{d_{i,n}} + \alpha a_{s,1}^{d'_1} \cdots a_{s,n}^{d'_n} = 0, \qquad P_s = (a_{s,1}, \dots, a_{s,n}) \in \underline{\mathcal{P}} \tag{9}$$

Hence, it is enough to prove that the N by N matrix A whose $s-th$ row is

$$a_{s,1}^{d_{1,1}} \cdots a_{s,n}^{d_{1,n}} \quad \dots\dots \quad a_{s,1}^{d_{N-1,1}} \cdots a_{s,n}^{d_{N-1,n}} \quad a_{s,1}^{d'_1} \cdots a_{s,n}^{d'_n}$$

(that is, the evaluation at P_s of the list of monomials $\mathbf{x}^{\mathbf{d}_1}, \dots, \mathbf{x}^{\mathbf{d}_{N-1}}, \mathbf{x}^{\mathbf{d}'}$) is a singular matrix. We shall prove this by showing that the submatrix A' consisting of the last $(h+1)m$ rows of A has rank less than $(h+1)m$.

Let X be any minor of order $(h+1)m$ of A'; notice that X can be given the form

$$X = \sum M_1 \cdot M_2 \cdot \dots \cdot M_m \tag{10}$$

where M_i is a minor of A' which consists of the $h+1$ rows whose indices are $N-(h+1)(m-i+1)+1, N-(h+1)(m-i+1)+2, \dots, N-(h+1)(m-i)$. Since all the points $P_{N-(h+1)(m-i+1)+1}, P_{N-(h+1)(m-i+1)+2}, \dots, P_{N-(h+1)(m-i)}$ have the same $(n-1)$ first coordinates, the minor M_i is different from zero only if its $(h+1)$ columns correspond to monomials $\mathbf{x}^{\mathbf{i}} = x_1^{i_1} \cdots x_{n-1}^{i_{n-1}} x_n^{i_n}$ such that their exponents i_n's have all the possible values $0, 1, \dots, h$. On the other hand, there are no more than $m-1$ such $(h+1)$-tuples of different columns (since only $m-1$ among the monomials $\mathbf{x}^{\mathbf{d}_1}, \dots, \mathbf{x}^{\mathbf{d}_{N-1}}, \mathbf{x}^{\mathbf{d}'}$ have h as last exponent);

it follows that at least one of the m minors M_i's in (10) is zero. Hence $X = 0$.
□

1.4 In the case where $n = 2$ **Algorithm MB** can be given the following simplified form.

Assume that

$$\mathcal{P} = \{(a_1, b_{11}), \ldots, (a_1, b_{1h_1}), \ldots, (a_m, b_{m1}), \ldots, (a_m, b_{mh_m})\}$$

with $h_1 + \ldots + h_m = N$ and $i \neq j \Rightarrow a_i \neq a_j$. Then,

$$\mathcal{MB}(\mathcal{P}) = \{(p, q) \mid 0 \leq p < m,\ 0 \leq q < h_p\}.$$

1.5 All that we have stated up to now can be generalized to what we might call *algebraic multisets* in the following sense.

Consider the linear map

$$\begin{aligned} \mathbf{D}_{\mathbf{i}}\colon K[X] &\longrightarrow K[X] \\ \mathbf{x}^{\mathbf{h}} &\longmapsto \binom{\mathbf{h}}{\mathbf{i}} \mathbf{x}^{\mathbf{h}-\mathbf{i}} \end{aligned}$$

where $\mathbf{h} = (h_1, \ldots, h_n)$, $\mathbf{i} = (i_1, \ldots, i_n) \in \mathbf{N}$ and $\binom{\mathbf{h}}{\mathbf{i}} := \binom{h_1}{i_1} \cdot \ldots \cdot \binom{h_n}{i_n}$. Observe that when the field K has characteristic zero, then

$$\mathbf{D}_{\mathbf{i}} = \frac{1}{\mathbf{i}!}\mathbf{D}^{\mathbf{i}} := \frac{1}{\mathbf{i}!} \frac{\partial^{i_1 + \cdots + i_n}}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}$$

where $\mathbf{i} = (i_1, \ldots, i_n)$ and $\mathbf{i}! = i_1! \cdots i_n!$.
Let v_P be the evaluation map at the point P:

$$\begin{aligned} v_P\colon K[X] &\longrightarrow K \\ q &\longmapsto q(P) \end{aligned}$$

Define the linear map $v_P^{\mathbf{i}}$ as the composition $v_P \circ \mathbf{D}_{\mathbf{i}}$, i.e.

$$\begin{aligned} v_P^{\mathbf{i}}\colon K[X] &\longrightarrow K \\ q &\longmapsto (\mathbf{D}_{\mathbf{i}} q)(P). \end{aligned}$$

For every ideal J of $K[x_1, \ldots, x_n]$ and every $P \in \mathcal{V}(J)$, put

$$\mathcal{F}_J(P) := \left\{ \mathbf{i} \in \mathbf{N}^n \mid (\forall p \in J) \left(v_P^{\mathbf{i}}(p) = 0 \right) \right\}$$

Prop. 6 *(i) $\mathcal{F}_J(P)$ is a Ferrers diagram; (ii) if $(g_1, \ldots, g_s)$ is a system of generators of J, then $\mathcal{F}_J(P)$ is the largest Ferrers diagram contained in the set*

$$\left\{\mathbf{i} \in \mathbf{N}^n \,\middle|\, v_P^{\mathbf{i}}(g_j) = 0 \textit{ for every } 1 \le j \le s\right\}.$$

We define a *finite n-dimensional algebraic multiset*, or simply an *algebraic multiset*, to be a set $\wp := \{(P_1, \mathcal{F}_1), \ldots, (P_N, \mathcal{F}_N)\}$; each element $(P_j, \mathcal{F}_j) \in \wp$ consists of a point P_j of K^n together with a Ferrers diagram $\mathcal{F}_j \subset \mathbf{N}^n$, which will be called the *algebraic diagram* of the point P_j. We shall freely make use of the notation $(P, \mathbf{i}) \in \wp$, or also $P \in \wp$, to mean that for some $j \in \{1, \ldots, N\}$, $P = P_j$ and $\mathbf{i} \in \mathcal{F}_j$. With every algebraic multiset $\wp = \{(P_1, \mathcal{F}_1), \ldots, (P_N, \mathcal{F}_N)\}$ we associate the set

$$\Im(\wp) := \left\{p \in K[X] \,\middle|\, (\forall j)(\forall \mathbf{i} \in \mathcal{F}_j)\left(v_{P_j}^{\mathbf{i}}(p) = 0\right)\right\}.$$

It is not difficult to prove that $\Im(\wp)$ is a cofinite ideal of $K[X]$ and that $\mathcal{F}_{\Im(\wp)}(P_j) = \mathcal{F}_j$ for every $j \in \{1, \ldots, N\}$. Moreover, one can prove that

$$\operatorname{codim} \Im(\wp) = \#\wp := \sum_{j=1}^{N} \#\mathcal{F}_j.$$

The question now is: how do we get a monomial linear basis for $K[X]/\Im(\wp)$? Once more the problem can be solved by applying a slightly modified version of **Algorithm MB** (in fact **Algorithm MB** itself with a few obvious changes) to a suitable set $\mathcal{R}(\wp) \subset (K \times \mathbf{N})^n$ associated with the algebraic multiset $\wp$. $\mathcal{R}(\wp)$ will be called the *umbral representation* of $\wp$. To be precise, consider the bijection

$$\begin{array}{rcl} u\colon K^n \times \mathbf{N}^n & \longrightarrow & (K \times \mathbf{N})^n \\ ((a_1, \ldots, a_n), (i_1, \ldots, i_n)) & \longmapsto & ((a_1, i_1), \ldots, (a_n, i_n)). \end{array}$$

Put

$$\mathcal{R} = \mathcal{R}(\wp) := \{u(P, \mathbf{i}) \mid (P, \mathbf{i}) \in \wp\}.$$

and

$$\mathcal{MB}(\wp) := \mathcal{MB}(\mathcal{R}).$$

(the symbol $\mathcal{MB}$ on the right-hand side represents the operator defined by **Algorithm MB**). It is possible to prove that $\mathcal{MB}(\wp)$ satisfies properties analogous to those of $\mathcal{MB}(\mathcal{P})$; in particular, (i) $\mathcal{MB}(\wp)$ is a Ferrers diagram and (ii) the set $B := \{\mathbf{x}^{\mathbf{i}} + \Im(\wp) \mid \mathbf{i} \in \mathcal{MB}(\wp)\}$ is a monomial linear basis of $K[X]/\Im(\wp)$.

1.6 The above algorithms may come in handy for solving various problems. Let us examine a few of them.

I. First of all, let us see how to determine a system of generators $(\gamma_1, \dots, \gamma_r)$ for the ideal $\Im(\wp)$ of a finite algebraic multiset $\wp := \{(P_1, \mathcal{F}_1), \dots, (P_N, \mathcal{F}_N)\}$. In fact, the set $\{\gamma_1, \dots, \gamma_r\}$ we shall obtain is also a *reduced Gröbner basis* of $\Im(\wp)$. It goes without saying that the same procedure works also when $\wp$ is a finite algebraic set.

Let B be the monomial linear basis obtained in **1.5** and let $Y = \{\mathbf{x}_{\mathbf{r}_1}, \dots, \mathbf{x}_{\mathbf{r}_r}\} \subset M_X$ be the minimal set of terms such that $B = M_X \setminus \sum_{i=1}^{r} \mathbf{x}_{\mathbf{r}_i} \cdot M_X$. For each $\mathbf{x}_{\mathbf{r}_i} \in Y$ determine a polynomial $\gamma_i \in K[X]$ in the form of a determinant in the following way. The first row of γ_i is the list $(\mathbf{x}_{\mathbf{d}_1}, \dots, \mathbf{x}_{\mathbf{d}_m}, \mathbf{x}_{\mathbf{r}_i})$ where $\mathbf{x}_{\mathbf{d}_j} \in B$ and $m = \#B = \text{codim}(\Im(\wp))$; the succesive rows are lists of the form $(v^{\mathbf{i}}_{P_j}(\mathbf{x}_{\mathbf{d}_1}), \dots, v^{\mathbf{i}}_{P_j}(\mathbf{x}_{\mathbf{d}_m}), v^{\mathbf{i}}_{P_j}(\mathbf{x}_{\mathbf{r}_i}))$, one for each $(P_j, \mathbf{i}) \in \wp$. It can be proved that the list $(\gamma_1, \dots, \gamma_r)$ is a reduced Gröbner basis of $\Im(\wp)$.

II. Consider a linear form $f \in K[X]^* \cong K[[X]]$. If $\text{Ker}(f)$ contains a cofinite ideal J of $K[X]$, then f is said to be an *n-linearly recursive function* and J is called a *characteristic ideal* of f. This notion has been introduced in [4] as a generalization of that of *linearly recursive sequence*, to which it reduces when $n = 1$. n-linearly recursive functions may also be regarded as elements of the dual bialgebra of the usual polynomial bialgebra on $K[X]$. When working on these subjects, it may happen that examples (perhaps, *suitable* examples) of n-linearly recursive functions are needed. How to construct them? It is convenient to divide the answer to this question into two parts.

(A) Let us first suppose that we know a system of generators $(g_1, \dots, g_s)$ of the characteristic ideal J of the n-linearly recursive functions we are considering. In this case we may calculate a reduced Gröbner basis $G_{\preceq} := RGB(g_1, \dots, g_s)$ of $(g_1, \dots, g_s)$ with respect to some term-ordering $\preceq$ on M_X. Let $G_{\preceq} = (\gamma_1, \dots, \gamma_r)$, $\gamma_j \in K[X]$, and let $\xi_j \in M_X$ be the leading term (with respect to $\preceq$) of the polynomial γ_j. Then the set $B = M_X \setminus \sum_{j=1}^{r} \xi_j \cdot M_X$ is a monomial linear basis for $K[X]/J$, which is minimal with respect to $\preceq$. It follows that any n-linearly recursive function whose characteristic polynomial is J is uniquely determined by the set of its *initial values* $\{f(\mathbf{x}^{\mathbf{d}}) \mid \mathbf{x}^{\mathbf{d}} \in B\}$, all the other values $f(\mathbf{x}^{\mathbf{i}})$, $\mathbf{x}^{\mathbf{i}} \notin B$, being calculated by making use of the polynomials $\gamma_j \in G_{\preceq}$ as *scales of recurrence*.

(B) If instead the characteristic ideal is not given, we are quite unlikely to obtain one of them (remember it must be cofinite!) just choosing at random a set of generators: most of the times we would get a non-cofinite ideal or, when cofinite, a trivial one. The correct way to do this consists instead in choosing a finite algebraic multiset (possibly, a finite algebraic set) $\wp$ and then determining both the monomial linear basis B of $K[X]/\Im(\wp)$ and a set of generators for $\Im(\wp)$ by means of the machinery described in the previous sections.

III. Lastly, consider the following interpolation problem: given a finite n-dimensional algebraic multiset $\wp := \{(P_1, \mathcal{F}_1), \dots, (P_N, \mathcal{F}_N)\}$ and a set of values $\{\alpha_{j,\mathbf{i}} \mid j = 1, \dots, N \text{ and } \mathbf{i} \in \mathcal{F}_j\} \subset K$ determine the *unique* polynomial p of the form $\sum_{\mathbf{x}^{\mathbf{i}} \in B} a_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}$ ($a_{\mathbf{i}} \in K$ and B is a monomial linear basis of $K[X]/\Im(\wp)$) for which we have $v^{\mathbf{i}}_{P_j}(p) = \alpha_{j,\mathbf{i}}$.

This problem appears to be an n-dimensional natural generalization of the unidimensional one which is solved by means of Lagrange interpolation formula (though a thorough analysis of these two shows that in some respects the analogy necessarily fails). Once more, the key point for solving this problem is to determine the monomial basis B. Let $B = \{\mathbf{x}_{\mathbf{d}_1}, \ldots, \mathbf{x}_{\mathbf{d}_m}\}$ and let $\mathcal{A}$ be the $m \times m$ matrix whose rows are of the form $(v^{\mathbf{i}}_{P_j}(\mathbf{x}_{\mathbf{d}_1}), \ldots, (v^{\mathbf{i}}_{P_j}(\mathbf{x}_{\mathbf{d}_m}))$, one for each $(P_j, \mathbf{i}) \in \wp$. Consider the vector $\overline{\alpha}$ whose components are the values $\alpha_{j,\mathbf{i}} = v^{\mathbf{i}}_{P_j}(p)$ (arranged according to the order that has been used for the rows of $\mathcal{A}$). The components of the vector $\overline{\beta} := \mathcal{A}^{-1} \cdot \overline{\alpha}$ are the coefficients of the desired polynomial.

Appendix.

1) Example of **Algorithm MB** for a 4-dimensional set.

$\mathcal{P}$	$\longleftrightarrow$	$\mathcal{F}$
$P_1 = (2,3,9,4)$	$\longleftrightarrow$	$\mathbf{d}_1 = (0,0,0,0)$
$P_2 = (2,5,7,3)$	$\longleftrightarrow$	$\mathbf{d}_2 = (0,1,0,0)$
$P_3 = (2,3,3,2)$	$\longleftrightarrow$	$\mathbf{d}_3 = (0,0,1,0)$
$P_4 = (2,5,5,1)$	$\longleftrightarrow$	$\mathbf{d}_4 = (0,1,1,0)$
$P_5 = (6,1,1,3)$	$\longleftrightarrow$	$\mathbf{d}_5 = (1,0,0,0)$
$P_6 = (2,3,3,6)$	$\longleftrightarrow$	$\mathbf{d}_6 = (0,0,0,1)$
$P_7 = (8,3,4,0)$	$\longleftrightarrow$	$\mathbf{d}_7 = (2,0,0,0)$
$P_8 = (6,5,6,3)$	$\longleftrightarrow$	$\mathbf{d}_8 = (1,1,0,0)$
$P_9 = (4,7,6,6)$	$\longleftrightarrow$	$\mathbf{d}_9 = (3,0,0,0)$
$P_{10} = (4,1,7,7)$	$\longleftrightarrow$	$\mathbf{d}_{10} = (2,1,0,0)$
$P_{11} = (2,5,7,8)$	$\longleftrightarrow$	$\mathbf{d}_{11} = (0,1,0,1)$
$P_{12} = (4,3,0,3)$	$\longleftrightarrow$	$\mathbf{d}_{12} = (0,2,0,0)$
$P_{13} = (1,1,2,5)$	$\longleftrightarrow$	$\mathbf{d}_{13} = (4,0,0,0)$
$P_{14} = (2,1,9,0)$	$\longleftrightarrow$	$\mathbf{d}_{14} = (1,2,0,0)$
$P_{15} = (4,1,7,6)$	$\longleftrightarrow$	$\mathbf{d}_{15} = (1,0,0,1)$
$P_{16} = (8,1,8,0)$	$\longleftrightarrow$	$\mathbf{d}_{16} = (3,1,0,0)$
$P_{17} = (2,1,7,6)$	$\longleftrightarrow$	$\mathbf{d}_{17} = (0,2,1,0)$
$P_{18} = (4,3,3,3)$	$\longleftrightarrow$	$\mathbf{d}_{18} = (1,0,1,0)$
$P_{19} = (4,1,1,0)$	$\longleftrightarrow$	$\mathbf{d}_{19} = (1,1,1,0)$
$P_{20} = (8,1,2,4)$	$\longleftrightarrow$	$\mathbf{d}_{20} = (2,0,1,0)$
$P_{21} = (6,3,4,8)$	$\longleftrightarrow$	$\mathbf{d}_{21} = (2,2,0,0)$
$P_{22} = (2,9,6,7)$	$\longleftrightarrow$	$\mathbf{d}_{22} = (0,3,0,0)$
$P_{23} = (2,3,7,1)$	$\longleftrightarrow$	$\mathbf{d}_{23} = (0,0,2,0)$
$P_{24} = (2,1,7,5)$	$\longleftrightarrow$	$\mathbf{d}_{24} = (0,2,0,1)$
$P_{25} = (4,1,1,2)$	$\longleftrightarrow$	$\mathbf{d}_{25} = (0,0,1,1)$
$P_{26} = (2,7,6,5)$	$\longleftrightarrow$	$\mathbf{d}_{26} = (0,4,0,0)$
$P_{27} = (2,5,5,4)$	$\longleftrightarrow$	$\mathbf{d}_{27} = (1,0,1,1)$
$P_{28} = (2,3,7,7)$	$\longleftrightarrow$	$\mathbf{d}_{28} = (0,1,1,1)$
$P_{29} = (2,3,0,2)$	$\longleftrightarrow$	$\mathbf{d}_{29} = (0,0,3,0)$
$P_{30} = (2,1,1,1)$	$\longleftrightarrow$	$\mathbf{d}_{30} = (0,1,2,0)$
$P_{31} = (4,3,3,7)$	$\longleftrightarrow$	$\mathbf{d}_{31} = (1,1,0,1)$
$P_{32} = (8,5,6,4)$	$\longleftrightarrow$	$\mathbf{d}_{32} = (3,2,0,0)$
$P_{33} = (4,5,5,2)$	$\longleftrightarrow$	$\mathbf{d}_{33} = (1,3,0,0)$
$P_{34} = (2,1,1,8)$	$\longleftrightarrow$	$\mathbf{d}_{34} = (0,2,1,1)$
$P_{35} = (8,1,1,1)$	$\longleftrightarrow$	$\mathbf{d}_{35} = (1,0,2,0)$
$P_{36} = (4,5,5,5)$	$\longleftrightarrow$	$\mathbf{d}_{36} = (1,2,0,1)$
$P_{37} = (6,1,8,8)$	$\longleftrightarrow$	$\mathbf{d}_{37} = (3,0,1,0)$
$P_{38} = (2,7,8,4)$	$\longleftrightarrow$	$\mathbf{d}_{38} = (0,3,1,0)$

The system of representatives for the corresponding monomial basis is given by

1	x_1	x_1^2	x_1^3	x_1^4
x_2	x_1x_2	$x_1^2x_2$	$x_1^3x_2$	
x_2^2	$x_1x_2^2$	$x_1^2x_2^2$	$x_1^3x_2^2$	
x_2^3	$x_1x_2^3$			
x_2^4				

x_3	x_1x_3	$x_1^2x_3$	$x_1^3x_3$
x_2x_3	$x_1x_2x_3$	$x_1^3x_3$	
$x_2^2x_3$			

x_3^2	$x_1x_3^2$
$x_2x_3^2$	

x_3^3

x_4	x_1x_4
x_2x_4	$x_1x_2x_4$
$x_2^2x_4$	$x_1x_2^2x_4$

x_3x_4	$x_1x_3x_4$
$x_2x_3x_4$	
$x_2^2x_3x_4$	

2) Example of **Algorithm MB** for a 3-dimensional algebraic multiset.

Consider the algebraic multiset $\wp$ given by

$$P_1 = (0,0,0) \quad \mathcal{F}_1 = \begin{bmatrix} (0,0,0) & (1,0,0) & & (0,0,1) \\ (0,1,0) & & & \end{bmatrix}$$

$$P_2 = (0,0,1) \quad \mathcal{F}_2 = \begin{bmatrix} (0,0,0) & (1,0,0) & (2,0,0) & (0,0,1) & (1,0,1) \end{bmatrix}$$

$$P_3 = (1,1,0) \quad \mathcal{F}_3 = \begin{bmatrix} (0,0,0) & (1,0,0) & (0,0,1) & (1,0,1) & (0,0,2) \\ (0,1,0) & (1,1,0) & (0,1,1) & & (0,1,2) \end{bmatrix}$$

$$P_4 = (1,1,1) \quad \mathcal{F}_4 = \begin{bmatrix} (0,0,0) \\ (0,1,0) \end{bmatrix}$$

The umbral representation $\mathcal{R} = \mathcal{R}(\wp)$ of $\wp$ (whose elements are intentionally arranged at random) as well as the corresponding diagram $\mathcal{MB}(\mathcal{R})$ are

$\wp$		$\mathcal{R}$		$\mathcal{MB}(\mathcal{R})$
$(1,1,0),(0,0,2)$	$\longleftrightarrow$	$R_1 = (10,\ 10,\ 02)$	$\longleftrightarrow$	$\mathbf{d}_1 = (0,\ 0,\ 0)$
$(1,1,0),(0,1,2)$	$\longleftrightarrow$	$R_2 = (10,\ 11,\ 02)$	$\longleftrightarrow$	$\mathbf{d}_2 = (0,\ 1,\ 0)$
$(0,0,1),(1,0,1)$	$\longleftrightarrow$	$R_3 = (01,\ 00,\ 11)$	$\longleftrightarrow$	$\mathbf{d}_3 = (1,\ 0,\ 0)$
$(1,1,0),(1,0,1)$	$\longleftrightarrow$	$R_4 = (11,\ 10,\ 01)$	$\longleftrightarrow$	$\mathbf{d}_4 = (2,\ 0,\ 0)$
$(0,0,1),(0,0,1)$	$\longleftrightarrow$	$R_5 = (00,\ 00,\ 11)$	$\longleftrightarrow$	$\mathbf{d}_5 = (3,\ 0,\ 0)$
$(1,1,0),(0,0,1)$	$\longleftrightarrow$	$R_6 = (10,\ 10,\ 01)$	$\longleftrightarrow$	$\mathbf{d}_6 = (0,\ 0,\ 1)$
$(0,0,1),(2,0,0)$	$\longleftrightarrow$	$R_7 = (02,\ 00,\ 10)$	$\longleftrightarrow$	$\mathbf{d}_7 = (4,\ 0,\ 0)$
$(0,0,0),(1,0,0)$	$\longleftrightarrow$	$R_8 = (01,\ 00,\ 00)$	$\longleftrightarrow$	$\mathbf{d}_8 = (1,\ 0,\ 1)$
$(1,1,0),(0,0,0)$	$\longleftrightarrow$	$R_9 = (10,\ 10,\ 00)$	$\longleftrightarrow$	$\mathbf{d}_9 = (0,\ 0,\ 2)$
$(0,0,0),(0,1,0)$	$\longleftrightarrow$	$R_{10} = (00,\ 01,\ 00)$	$\longleftrightarrow$	$\mathbf{d}_{10} = (1,\ 1,\ 0)$
$(0,0,0),(0,0,0)$	$\longleftrightarrow$	$R_{11} = (00,\ 00,\ 00)$	$\longleftrightarrow$	$\mathbf{d}_{11} = (2,\ 0,\ 1)$
$(0,0,0),(0,0,1)$	$\longleftrightarrow$	$R_{12} = (00,\ 00,\ 01)$	$\longleftrightarrow$	$\mathbf{d}_{12} = (1,\ 0,\ 2)$
$(1,1,0),(0,1,0)$	$\longleftrightarrow$	$R_{13} = (10,\ 11,\ 00)$	$\longleftrightarrow$	$\mathbf{d}_{13} = (0,\ 1,\ 1)$
$(1,1,0),(1,0,0)$	$\longleftrightarrow$	$R_{14} = (11,\ 10,\ 00)$	$\longleftrightarrow$	$\mathbf{d}_{14} = (3,\ 0,\ 1)$
$(0,0,1),(0,0,0)$	$\longleftrightarrow$	$R_{15} = (00,\ 00,\ 10)$	$\longleftrightarrow$	$\mathbf{d}_{15} = (0,\ 0,\ 3)$
$(1,1,0),(0,1,1)$	$\longleftrightarrow$	$R_{16} = (10,\ 11,\ 01)$	$\longleftrightarrow$	$\mathbf{d}_{16} = (0,\ 1,\ 2)$
$(1,1,0),(1,1,0)$	$\longleftrightarrow$	$R_{17} = (11,\ 11,\ 00)$	$\longleftrightarrow$	$\mathbf{d}_{17} = (2,\ 1,\ 0)$
$(0,0,1),(1,0,0)$	$\longleftrightarrow$	$R_{18} = (01,\ 00,\ 10)$	$\longleftrightarrow$	$\mathbf{d}_{18} = (2,\ 0,\ 2)$
$(1,1,1),(0,1,0)$	$\longleftrightarrow$	$R_{19} = (10,\ 11,\ 10)$	$\longleftrightarrow$	$\mathbf{d}_{19} = (1,\ 0,\ 3)$
$(1,1,1),(0,0,0)$	$\longleftrightarrow$	$R_{20} = (10,\ 10,\ 10)$	$\longleftrightarrow$	$\mathbf{d}_{20} = (0,\ 1,\ 3)$

Therefore, a monomial linear basis of $K[X]/\Im(\wp)$ consists of the equivalence classes (modulo $\Im(\wp)$) of the monomials

1	x_1	x_1^2	x_1^3	x_1^4
x_2	x_1x_2	$x_1^2x_2$		

x_3	x_1x_3	$x_1^2x_3$	$x_1^3x_3$
x_2x_3			

x_3^2	$x_1x_3^2$	$x_1^2x_3^2$
$x_2x_3^2$		

x_3^3	$x_1x_3^3$
$x_2x_3^3$	

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